On a threshold descent method for quasi-equilibria

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Abstract

We describe a special class of quasi-equilibrium problems in metric spaces and propose a novel simple threshold descent method for solving these problems. Due to the framework, the convergence of the method cannot be established with the usual convexity or generalized convexity assumptions. Under mild conditions, the iterative procedure gives solutions of the quasi-equilibrium problem. We apply this method to scalar and vector generalized quasi-equilibrium problems and to some classes of relative optimization problems.

Key words: Quasi-equilibrium problems; Brezis pseudomonotonicity; threshold descent method; existence results; convergence

MSC: 90C33, 65K05

1 Introduction

A quasi-equilibrium problem is an equilibrium problem where the feasible set is moving depending on the considered point. This kind of problems encompasses many relevant problems like quasi-variational inequalities, Nash equilibrium problems, quasi-optimization problems, and also relative optimization problems, recently introduced in [9], as special cases. It should be noticed that investigation of quasi-equilibrium problems is usually restricted to various existence results, since existing solution methods are usually very complicated. Even the solution of equilibrium problems without the usual convexity or monotonicity assumptions meets serious difficulties, so that most solution methods concerning equilibrium problems involve these assumptions (see e.g. [1, 7]).

In this paper, we describe a special class of quasi-equilibrium problems, which admits simple descent solution methods. The proposed method is an extension of the procedure suggested by Konnov in [11]. In particular, the method creates suitable trajectories converging to an equilibrium state by solving approximating problems, and the convergence of the approximating solutions is proved under mild conditions on the data. This approach also gives existence results for these problems under mild conditions. The framework is a

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metric space and therefore convexity or generalized convexity are banned, but only topological conditions will be considered. A prominent role is played by the property of Brezis (or topological) pseudomonotonicity of the bifunction involved, which provides a weakening of the upper semicontinuity in the first variable; in the recent paper [2] this property has been fruitfully applied to the study of existence for quasi-equilibria. In case the metric space is complete, an interesting outcome is that any feasible sequence built up via the threshold descent method fully converges to a solution of the quasi-equilibrium problem. We apply this method to scalar and vector generalized quasi-equilibrium problems and to some classes of relative optimization problems.

The paper is organized as follows. In Section 2 we introduce the threshold descent method for quasi-equilibrium problems and its preliminary properties. The method is applied in Section 3 to scalar and vector generalized quasi-equilibrium problems, respectively. We prove its convergence that under mild conditions gives also existence of solutions for the quasi-equilibrium problems analysed.

Applications to relative optimization problems are considered in Section 4.

2 The threshold descent method

In the recent paper [11], the author introduced a *threshold descent method (TDM)* to solve a particular quasi-equilibrium problem related to the so called relative optimization problems. This method takes into consideration the solutions of suitable approximate problems that admit as limit points solutions of the original problem.

The aim of this section is to describe this method for quasi-equilibrium problems and give its preliminary properties. In particular, we will provide the basic conditions under which the sequence generated by the method happens to be finite or, otherwise, it admits limit points that are solutions of the quasi-equilibrium problem.

Given a metric space (X, d), let $F : X \times X \to \mathbb{R}$ be a bifunction, $D : X \rightrightarrows X$ a set-valued map, and consider the quasi-equilibrium problem:

$$\text{find } x \in D(x) : \; F(x, y) \geq 0, \quad \forall y \in D(x). \tag{\textbf{QEP}}$$

We will denote by $\mathrm{gr}(D)$ the graph of D. A sequence $\{z_k\} \subseteq X$ (finite, or infinite) will be called *feasible* if $(z_k, z_{k+1}) \in \mathrm{gr}(D)$ for all k.

If the set X involves all the possible states of some system modelled by (**QEP**), then $D(x)$ is treated as the set of feasible states at $x \in X$. Therefore, the system can move from x to any $y \in D(x)$. It seems natural to suppose that the system can also stay at x. In the sequel we will therefore assume that $x \in D(x)$ for all $x \in X$ (see e.g. [10, 11]).

Let us now describe the threshold descent method for (**QEP**).

Method (TDM).
Initialization: Take a point $x_0 \in X$, choose a sequence $\{\delta_l\} \searrow 0$. Set $l = 1$, $k = 0$, $z_0 = x_0$.

Step 1: Find a point $z_{k+1} \in D(z_k)$ such that

$$F(z_k, z_{k+1}) < -\delta_l, \tag{1}$$

set $k = k + 1$ and go to the beginning of Step 1. Otherwise, if this point does not exist, go to Step 2.

Step 2: Set $x_l = z_k$, $l = l + 1$ and go to Step 1.

The sequence $\{z_k\}$ is clearly feasible and, by construction, each point $x_l \in D(x_l)$ is a solution of the *approximate equilibrium problem* $(\mathbf{EP}_{\delta_l})$

$$F(x_l, y) \geq -\delta_l \quad \forall y \in D(x_l). \tag{2}$$

The sequences $\{z_k\}$ and $\{x_l\}$ can be finite or infinite. We will say that the sequence $\{x_l\}$ is *fixed* if it is infinite and there exists an index l' such that

$$\tilde{x} = x_l = x_{l+1} \text{ for each } l \geq l'. \tag{3}$$

If the sequence $\{x_l\}$ is fixed and $x_{l'} \in D(x_{l'})$, then (2) gives that $\tilde{x} = x_{l'}$ is a solution of (**QEP**). Besides, the sequence $\{z_k\}$ is finite and stops at $\tilde{x}$. Otherwise, if the sequence $\{x_l\}$ is infinite and not fixed, it may admit a limit point. Under suitable assumptions on the data, this limit point is a solution of the original quasi-equilibrium problem. Hence, we need some assumptions for the sequence $\{x_l\}$ to be well defined.

Lemma 1. Assume that $\{z_k\}$ and $\{x_l\}$ are sequences generated by (TDM), and
(C1) for any feasible sequence $\{u_k\}_k$ we have

$$\sum_{k=0}^{N} F(u_k, u_{k+1}) \geq C > -\infty, \quad \forall N \in \mathbb{N}.$$

Then for each l there exists an index $k = k(l)$ such that (1) does not hold.

Proof. If there exist $\delta = \delta_l > 0$ and a feasible sequence $\{z_k\}$ in (TDM) such that

$$F(z_k, z_{k+1}) < -\delta, \quad \forall k \geq k',$$

adding these inequalities from $k = k'$ up to $k = k' + N$, we get

$$\sum_{k=k'}^{k'+N} F(z_k, z_{k+1}) < -(N + 1)\delta, \quad \forall N \geq 0,$$

thus contradicting **(C1)**. □

This means that, under assumption **(C1)**, the sequence $\{x_l\}$ is well defined and always infinite since, for any δ_l, after a finite number of steps in k the point z_{k+1} satisfying (1) does not exists and (TDM) goes to Step 2. Therefore, we have to consider the case where the sequence $\{x_l\}$ is not fixed.

Let us first recall the following

Definition 1. A set-valued map $D : X \rightrightarrows X$ is said to be lower semicontinuous at $x \in X$ if for every $x_n \to x$, and for every $y \in D(x)$, there exists a subsequence $\{x_{n_k}\}$ and $y_k \in D(x_{n_k})$ such that $y_k \to y$.

Lemma 2. Let **(C1)** hold and let $\{x_l\}$ be a sequence generated by (TDM). Assume that the sequence $\{x_l\}$ is not fixed,

i. the sequence $\{x_l\}$ admits a limit point $\overline{x}$,

ii. for every $(u_l, v_l) \in \text{gr}(D)$, $(u_l, v_l) \to (u, v)$, $\epsilon_l \searrow 0$, and $F(u_l, v_l) \geq -\epsilon_l$ then $F(u, v) \geq 0$,

iii. D is a lower semicontinuous map such that $x \in D(x)$ for all $x \in X$.

Then $\overline{x}$ is a solution of (**QEP**).

Proof. If the sequence $\{x_l\}$ is not fixed, let us assume, without loss of generality, that $x_l \to \overline{x}$. By assumption, $\overline{x} \in D(\overline{x})$. Take now any $y \in D(\overline{x})$; from the lower semicontinuity of the map D, there exists a subsequence $\{x_{l_k}\}$ and $y_k \in D(x_{l_k})$ such that $y_k \to y$. By (2) we have

$$F(x_{l_k}, y_k) \geq -\delta_{l_k}.$$

Letting $k \to \infty$, by assumption ii., the assertion follows. □

We now discuss the conditions in Lemmas 1 and 2, which create a basis for convergence of (TDM) and existence of solutions for some classes of quasi-equilibrium problems presented in the next sections.

We first discuss assumption **(C1)**. Clearly, it gives a kind of coercivity for F along any feasible sequence, but it is not easy to check. In the following we mention some conditions on F which ensure **(C1)**. Let us first recall the following definitions:

Definition 2. A bifunction $F : X \times X \to \mathbb{R}$ is said

- to be *cyclically antimonotone* if for any $m \in \mathbb{N}$ and any sets of points $x_0, x_1, \cdots, x_m$

$$F(x_0, x_1) + F(x_1, x_2) + \cdots + F(x_m, x_0) \geq 0;$$

- to satisfy the *triangle inequality* if $F(x, y) \leq F(x, z) + F(z, y)$ for all $x, y, z \in X$.

This last condition turns out to be useful in different settings; it was introduced by Blum and Oettli in their seminal paper [4], and exploited by two of the authors to prove an Ekeland-type principle for equilibrium problems (see [3]).

It can be verified that **(C1)** is satisfied if one of the following conditions on F are fulfilled:

a. F is cyclically antimonotone and $F(\cdot, x)$ is upper bounded for all $x \in X$;

b. F satisfies the triangle inequality and there exists $\hat{x}$ such that $F(\hat{x}, \cdot)$ is bounded from below.

The proof follows easily taking into account [6], in particular Remark 2.5 and Theorems 2.13 and 2.14. Indeed, both conditions a. and b. above imply

$$F(x,y) \geq g(y) - g(x) \quad \forall x, y \in X, \tag{4}$$

where g is a suitable function bounded from below. In particular, in case b., inequality (4) holds with $g(x) = F(\hat{x}, x)$.

As to condition i. in Lemma 2, it should be clearly replaced with a suitable coercivity property which will be implemented in the next section. With regard to condition ii. of Lemma 2, it is trivially satisfied if F is upper semicontinuous. Another sufficient condition in order to fulfill condition ii. can be given in terms of Brezis pseudomonotonicityof the bifunction f. Let us first recall the following definition:

Definition 3. A bifunction $F : X \times X \to \mathbb{R}$ is called *Brezis pseudomonotone* with respect to a set $D \subseteq X$ if for every $x_n \to x$ in D such that $\liminf_{n\to\infty} F(x_n, x) \geq 0$ it follows that

$$F(x,y) \geq \limsup_{n\to\infty} F(x_n, y) \quad \forall y \in D.$$

Note that Brezis pseudomonotonicity is a weakening of the upper semicontinuity of the functions $F(\cdot, y)$ for all $y \in D$. Suppose now that the bifunction $F : X \times X \to \mathbb{R}$ satisfies the conditions:

a. F is Brezis pseudomonotone on X, and $F(x,x) = 0$ for all $x \in X$;

b. $|F(x,y) - F(x,z)| \leq h(x)d(y,z)$, where $h : X \to \mathbb{R}$ is nonnegative and locally bounded.

Then F satisfies condition ii. Indeed, suppose that $(x_l, y_l) \to (x,y)$, $\epsilon_l \searrow 0$, and $F(x_l, y_l) \geq -\epsilon_l$, for every l. We have by a. that

$$\liminf_{l\to+\infty}(F(x_l, x) - F(x_l, x_l)) \geq \liminf_{l\to+\infty} -h(x_l)d(x_l, x) = 0,$$

and therefore, by a. and b.,

$$F(x,y) \geq \limsup_{l\to+\infty} F(x_l, y) = \limsup_{l\to+\infty}(F(x_l, y_l) + F(x_l, y) - F(x_l, y_l)) \geq 0$$

for all $y \in X$.

3 Convergence properties for quasi-equilibrium problems

In this section we apply Method (TDM) to solve quasi-equilibrium problems with a particular structure, both in the scalar and vector cases. We stress again that our setting is a metric space (X, d) and therefore the standard results of convergence and existence of solutions for equilibrium problems cannot be applied since no convexity/generalized convexity assumptions will be employed.

We have to give some coercivity condition which provides the existence of a limit point of the sequence $\{x_l\}$, i.e. condition i. in Lemma 2. Let us recall the following definition:

Definition 4. A function $\mu : X \to \mathbb{R}$ is said to be *coercive with respect to a set* $Y \subseteq X$ if for every $r \in \mathbb{R}$ there exists a compact set K_r such that

$$B_\mu(r) \cap Y \subseteq K_r$$

where $B_\mu(r) = \{x \in X : \mu(x) \le r\}$. If $Y = X$, the function μ is simply said to be coercive.

If the function μ is coercive with respect to a closed set Y and lower semicontinuous, the sets $B_\mu(r) \cap Y$ are closed, and therefore compact. Besides, if Y is compact, then the function μ is coercive with respect to Y if it is lower semicontinuous. It is well known that, in a metric space, a set K is compact if and only if it is sequentially compact, i.e. for every sequence $\{x_n\} \subseteq K$ there exists a subsequence $x_{n_k} \to x \in K$.

3.1 The scalar case

Let $f : X \times X \to \mathbb{R}$ be a bifunction, $\mu : X \to \mathbb{R}$, and $D : X \rightrightarrows X$, and consider the *generalized quasi-equilibrium problem*:

$$\text{find } x \in D(x) : \ f(x,y) + \mu(y) - \mu(x) \ge 0, \quad \forall y \in D(x). \tag{\textbf{GQEP}}$$

It coincides with (**QEP**) when $F(x,y) = f(x,y) + \mu(y) - \mu(x)$. Hence we can apply Method (TDM) to this problem.

Theorem 1. Suppose that the following assumptions hold:

i. f satisfies condition **(C1)**;

ii. f is Brezis pseudomonotone on X and $f(x,x) = 0$ for every $x \in X$;

iii. $|f(x,y) - f(x,z)| \le h(x)d(y,z)$, where $h : X \to \mathbb{R}$ is positive and locally bounded;

iv. μ is coercive, bounded from below and continuous;

v. D is a lower semicontinuous map such that $x \in D(x)$ for all $x \in X$.

If the sequence $\{x_l\}$ in Method (TDM) is fixed, then $\tilde{x}$ in (3) is a solution of (**GQEP**). Otherwise, the sequence $\{x_l\}$ generated by Method (TDM) has limit points and all these limit points are solutions of problem (**GQEP**).

Proof. Let us consider only the case of $\{x_l\}_l$ not fixed. To prove the result it is sufficient to verify the applicability of Lemma 2 to the bifunction $F(x,y) = f(x,y) + \mu(y) - \mu(x)$. First of all, **(C1)** is satisfied for F: indeed, taking into account i. and iv., we have

$$\sum_{k=0}^{N} F(u_k, u_{k+1}) = \sum_{k=0}^{N} f(u_k, u_{k+1}) + \mu(u_{N+1}) - \mu(u_0) \ge C' > -\infty$$

for each $N \in \mathbb{N}$.

Let us now show that assumption i. of Lemma 2 is fulfilled. By construction

$$f(z_k, z_{k+1}) + \mu(z_{k+1}) - \mu(z_k) < 0, \quad \forall k,$$

and, again, adding both sides from $k = 0$ to $k = n$, we get

$$\sum_{k=0}^{n} f(z_k, z_{k+1}) + \mu(z_{n+1}) < \mu(z_0), \quad \forall n. \tag{5}$$

Using assumptions i. and iv. in (5), we obtain that there exists $M \in \mathbb{R}$ such that

$$\mu(z_{n+1}) < M, \quad \forall n,$$

i.e. $z_n \in B_\mu(M)$. From the coercivity of μ, there exists a compact set K_M such that $\{z_n\} \subseteq K_M$. In particular, $\{x_l\} \subset K_M$, and there exists a subsequence $\{x_{l_k}\}$ such that $x_{l_k} \to \overline{x}$ and assumption i. of Lemma 2 is fulfilled.

Let us finally show that ii. of Lemma 2 holds for the bifunction $F(x, y) = f(x, y) + \mu(y) - \mu(x)$. Assumptions ii. and iii. entail that

$$\liminf_k f(x_{l_k}, \overline{x}) = \liminf_k (f(x_{l_k}, \overline{x}) - f(x_{l_k}, x_{l_k}) \geq \liminf_k -h(x_{l_k})d(x, x_{l_k}) = 0;$$

then, the Brezis pseudomonotonicity of f gives

$$f(\overline{x}, y) \geq \limsup_k f(x_{l_k}, y) \quad \forall y \in X.$$

Consequently, by the continuity of μ we get

$$\begin{aligned} F(\overline{x}, y) = f(\overline{x}, y) + \mu(y) - \mu(\overline{x}) &\geq \limsup_h \left(f(x_{l_{k_h}}, y) + \mu(y) - \mu(x_{l_{k_h}})\right) \\ &= \limsup_h F(x_{l_{k_h}}, y), \end{aligned} \tag{6}$$

for all $y \in X$ and for every subsequence $\{x_{l_{k_h}}\}$ of $\{x_{l_k}\}$. By construction, we have that

$$f(x_{l_k}, y) + \mu(y) - \mu(x_{l_k}) \geq -\delta_{l_k}, \qquad \forall y \in D(x_{l_k}). \tag{7}$$

Take now any $y \in D(\overline{x})$; from the lower semicontinuity of the map D, there exists $y_{l_{k_h}} \in D(x_{l_{k_h}})$ such that $y_{l_{k_h}} \to y$. Thus from (7) it follows that

$$\begin{aligned} f(x_{l_{k_h}}, y) + &\mu(y) - \mu(x_{l_{k_h}}) \\ &= f(x_{l_{k_h}}, y) - f(x_{l_{k_h}}, y_{l_{k_h}}) + f(x_{l_{k_h}}, y_{l_{k_h}}) + \mu(y_{l_{k_h}}) - \mu(x_{l_{k_h}}) + \mu(y) - \mu(y_{l_{k_h}}) \\ &\geq f(x_{l_{k_h}}, y) - f(x_{l_{k_h}}, y_{l_{k_h}}) + \mu(y) - \mu(y_{l_{k_h}}) - \delta_{l_{k_h}}. \end{aligned} \tag{8}$$

Therefore for all $y \in D(\overline{x})$, from assumption iii. we have that

$$\begin{aligned}\limsup_h \,(f(x_{l_{k_h}}, y) + &\mu(y) - \mu(x_{l_{k_h}})) \\ &\geq \limsup_h(-h(x_{l_{k_h}})d(y, y_{l_{k_h}}) + \mu(y) - \mu(y_{l_{k_h}}) - \delta_{l_{k_h}}) \\ &= \lim_h(-h(x_{l_{k_h}})d(y, y_{l_{k_h}})) + \limsup_h(\mu(y) - \mu(y_{l_{k_h}})) - \lim_h(\delta_{l_{k_h}}) \\ &= \lim_h(-h(x_{l_{k_h}})d(y, y_{l_{k_h}})) - \liminf_h(\mu(y_{l_{k_h}}) - \mu(y)) - \lim_h(\delta_{l_{k_h}}) \\ &\geq 0,\end{aligned}$$

thereby proving, thanks to (6), that $F(\overline{x}, y) \geq 0$ for all $y \in D(\overline{x})$. □

We observe that the existence of solutions for (**GQEP**) follows immediately from the convergence property of Method (TDM).

Corollary 1. Suppose that the following assumptions hold:

i. f satisfies the triangle inequality, $f(x, x) = 0$ for any $x \in X$ and there exists $x_0 \in X$ such that $f(x_0, \cdot)$ is bounded from below on X;

ii. $f(x, \cdot)$ and $f(\cdot, y)$ are upper semicontinuous on X for any x and y in X, respectively;

iii. μ is coercive, bounded from below and continuous;

iv. D is a lower semicontinuous map such that $x \in D(x)$ for all $x \in X$.

If the sequence $\{x_l\}$ in Method (TDM) is fixed, then $\tilde{x}$ in (3) is a solution of (**GQEP**). Otherwise, the sequence $\{x_l\}$ generated by Method (TDM) has limit points, and all these limit points are solutions of problem (**GQEP**).

Proof. We can follow all the steps of the proof of Theorem 1, until (8). Then, thanks to the triangle inequality,

$$\begin{aligned}f(x_{l_{k_h}}, y) + &\mu(y) - \mu(x_{l_{k_h}}) \\ &\geq -f(y, y_{l_{k_h}}) + \mu(y) - \mu(y_{l_{k_h}}) - \delta_{l_{k_h}}.\end{aligned}$$

Therefore, for all $y \in D(\overline{x})$,

$$\begin{aligned}\limsup_h \,(f(x_{l_{k_h}}, y) + &\mu(y) - \mu(x_{l_{k_h}})) \\ &\geq \limsup_h(-f(y, y_{l_{k_h}}) + \mu(y) - \mu(y_{l_{k_h}}) - \delta_{l_{k_h}}) \\ &\geq \liminf_h(-f(y, y_{l_{k_h}})) + \lim_h(\mu(y) - \mu(y_{l_{k_h}})) - \lim_h \delta_{l_{k_h}} \\ &= -\limsup_h f(y, y_{l_{k_h}}) + \lim_h(\mu(y) - \mu(y_{l_{k_h}})) - \lim_h \delta_{l_{k_h}} \\ &\geq 0,\end{aligned}$$

thereby proving, thanks to (6), that $F(\overline{x}, y) \geq 0$ for all $y \in D(\overline{x})$. □

Let us now suppose that X is a complete metric space. In this framework, the assumptions on μ can be weakened, since its coercivity is no longer required. Furthermore, we will get as outcome the convergence of any feasible sequence to a solution of the problem (**GQEP**).

In the following, given $a \in \mathbb{R}$, we will set

$$[a]_+ = \max\{a, 0\}, \qquad [a]_- = \min\{a, 0\}.$$

Let us first recall the following auxiliary property:

Lemma 3. (see Lemma 1 in [8, Chapter III]) Let a numerical sequence $\{\alpha_k\}$ be bounded from below and

$$\alpha_{k+1} \leq \alpha_k + \epsilon_k, \quad \epsilon_k \geq 0, \ k = 0, 1, \ldots$$

If $\sum_{k=0}^{+\infty} \epsilon_k < +\infty$, then $\{\alpha_k\}$ converges to a number $\overline{\alpha} \in \mathbb{R}$.

Theorem 2. Suppose that the following assumptions hold:

i. f satisfies the triangle inequality and $f(x, x) = 0$;

ii. $f(x, \cdot)$ and $f(\cdot, y)$ are upper semicontinuous on X for any x and y in X, respectively;

iii. μ is bounded from below and continuous;

iv. $\sum_{k=1}^{+\infty} [f(z_k, z_{k+1})]_-$ converges for every feasible sequence $\{z_k\}$; in addition, there exists an increasing and continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, with $\theta(0) = 0$, and $g : \mathbb{R} \to \mathbb{R}_+$, with $g(0) = 0$ and continuous at 0, such that

$$\theta(d(x, y)) \leq [f(x, y)]_+ + g(\mu(x) - \mu(y)), \text{ for every } x, y \in X;$$

v. D is a lower semicontinuous map such that $x \in D(x)$ for all $x \in X$.

If the sequence $\{x_l\}$ in Method (TDM) is fixed, then $\tilde{x}$ in (3) is a solution of (**GQEP**). Otherwise, the sequence $\{x_l\}$ generated by Method (TDM) converges to a solution of problem (**GQEP**).

Proof. First let us prove that for every l the number of points z_k of the sequence satisfying

$$f(z_k, z_{k+1}) + \mu(z_{k+1}) - \mu(z_k) < -\delta_l \tag{9}$$

is finite. Indeed, assume that there exists l and k_0 such that

$$f(z_k, z_{k+1}) + \mu(z_{k+1}) - \mu(z_k) < -\delta_l, \quad \forall k \geq k_0.$$

In particular,

$$[f(z_k, z_{k+1})]_- + \mu(z_{k+1}) - \mu(z_k) < -\delta_l, \quad \forall k \geq k_0.$$

Adding both sides from $k = k_0$ until $k = k_0 + N$, we have that

$$\sum_{k=k_0}^{k_0+N} [f(z_k, z_{k+1})]_- + \mu(z_{k+N+1}) - \mu(z_k) < -(N+1)\delta_l,$$

and, taking $N \to +\infty$, we get a contradiction with assumptions iii. and iv. Therefore the sequence $\{x_l\}$ is well defined. If the set $\{x_l\}$ is finite, as in Theorem 1 problem (**GQEP**) is solvable. Otherwise, if the set $\{x_l\}$ is infinite, we show that $\{x_l\} \to \overline{x}$ for some $\overline{x} \in X$. Indeed, let us now show that the whole sequence $\{z_k\}$ converges.

From (9) we have that

$$\mu(z_{k+1}) \leq \mu(z_k) - [f(z_k, z_{k+1})]_- \qquad \forall k.$$

By applying Lemma 3 we argue that the sequence $\mu(z_k)$ converges to some $\mu' \in \mathbb{R}$. Then, from the second part of assumption iv. and from (9), and taking into account that the bifunction $[f(x, y)]_+$ inherits from f the triangle inequality property, we have that

$$\begin{aligned}\theta(d(z_k, z_{k+p})) &\leq [f(z_k, z_{k+p})]_+ + g(\mu(z_k) - \mu(z_{k+p})) \\ &\leq \sum_{i=0}^{p-1}[f(z_{k+i}, z_{k+i+1})]_+ + g(\mu(z_k) - \mu(z_{k+p})) \\ &\leq \sum_{i=0}^{p-1}(-\mu(z_{k+i+1}) + \mu(z_{k+i})) - \sum_{i=0}^{p-1}[f(z_{k+i}, z_{k+i+1})]_- + g(\mu(z_k) - \mu(z_{k+p})) \\ &= -\mu(z_{k+p+1}) + \mu(z_k) - \sum_{i=0}^{p}[f(z_{k+i}, z_{k+i+1})]_- + g(\mu(z_k) - \mu(z_{k+p})).\end{aligned}$$

From the first part of iv., the convergence of $\mu(z_k)$ and the continuity of g at 0, for every positive ϵ there exists k_0 such that for every $k > k_0$ and for every p

$$\theta(d(z_k, z_{k+p})) \leq -\mu(z_{k+p+1}) + \mu(z_k) - \sum_{i=0}^{p}[f(z_{k+i}, z_{k+i+1})]_- + g(\mu(z_k) - \mu(z_{k+p})) < \theta(\epsilon),$$

thereby implying that $\{z_k\}$ is a Cauchy sequence. Since X is complete, $\{z_k\} \to \overline{x} \in X$. In particular $\{x_l\} \to \overline{x}$. By the continuity of μ and assumption ii. we have

$$f(\overline{x}, y) + \mu(y) - \mu(\overline{x}) \geq \limsup_l \left(f(x_l, y) + \mu(y) - \mu(x_l)\right)$$

for all $y \in X$. By construction, we have that

$$f(x_l, y) + \mu(y) - \mu(x_l) \geq -\delta_{l_k}, \qquad \forall y \in D(x_l). \tag{10}$$

Take now any $y \in D(\overline{x})$; from the lower semicontinuity of the map D, there exists $y_{l_k} \in D(x_{l_k})$ such that $y_{l_k} \to y$.

The proof that $\overline{x}$ is a solution of problem (**GQEP**) follows now the same line of the proof of the last part of Corollary 1. □

3.2 The vector case

Given m bifunctions $f_i : X \times X \to \mathbb{R}$, $i = 1, \cdots, m$ and $\mu_i : X \to \mathbb{R}$, $i = 1, \cdots m$, define

$$F(x, y) = \max_{i=1,\cdots,m} (f_i(x, y) + \mu_i(y) - \mu_i(x)) \tag{11}$$

and consider the quasi-equilibrium problem (**QEP**).

Note that with this choice of the bifunction F, (**QEP**) is equivalent to a *vector quasi-equilibrium problem*. Indeed, given the Pareto cone $\mathbb{R}^m_+$, problem (**QEP**) requires to find a point $x \in D(x)$ such that

$$\mathbf{f}(x, y) \notin -\mathbb{R}^m_+ \quad \forall y \in D(x) \qquad (\mathbf{VQEP})$$

where $\mathbf{f} = (f_1 + \mu_1(y) - \mu_1(x), f_2 + \mu_2(y) - \mu_2(x), \ldots, f_m + \mu_m(y) - \mu_m(x))$.

By applying the technique exploited in Corollary 1, via Method (TDM) we can provide one more time existence of solutions for (**VQEP**). It is easy to realise that some of the assumptions are required only for one index $\hat{i}$.

Theorem 3. Suppose that the following assumptions hold:

i. for every $i = 1, 2, \ldots, m$, f_i satisfies the triangle inequality, $f_i(x, x) = 0$ for every $x \in X$, and there exists $\hat{i}$ and $x_0 \in X$ such that $f_{\hat{i}}(x_0, \cdot)$ is bounded from below;

ii. $f_i(x, \cdot)$ and $f_i(\cdot, y)$ are upper semicontinuous on X for any x and y in X, respectively, and for every $i = 1, 2, \ldots, m$;

iii. μ_i is continuous for every $i = 1, 2, \ldots, m$,; moreover, $\mu_{\hat{i}}$ is coercive and bounded from below;

iv. D is a lower semicontinuous map such that $x \in D(x)$ for all $x \in X$.

If the sequence $\{x_l\}$ in Method (TDM) is fixed, then $\tilde{x}$ in (3) is a solution of (**VQEP**). Otherwise, the sequence $\{x_l\}$ generated by Method (TDM) has limit points, and all these limit points are solutions of problem (**VQEP**).

Proof. The proof follows the same line of the proof of Corollary 1 taking into account that the bifunction F defined in (11) satisfies the triangle inequality, and $F(x, x) = 0$ for all $x \in X$; in addition, by assumptions i. and ii., $F(x, \cdot)$ and $F(\cdot, y)$ are upper semicontinuous on X, for any x and y in X, respectiveley, as the maximum of a finite number of upper semicontinuous functions. □

In case the metric space is complete, also in the vector case we can get rid of the coercivity assumption. We state the following theorem whose proof can be carried on along the same lines of the previous results and is therefore omitted:

Theorem 4. Suppose that the following assumptions hold:

i. for every $i = 1, 2, \dots, m$ f_i satisfies the triangle inequality, and $f_i(x, x) = 0$ for every $x \in X$;

ii. $f_i(\cdot, y)$ and $f_i(x, \cdot)$ are upper semicontinuous for every $y \in X$, and $x \in X$, respectively, and for any $i = 1, 2, \dots, m$;

iii. there exists $\hat{i}$ such that $\sum_{k=1}^{+\infty}[f_{\hat{i}}(z_k, z_{k+1})]_-$ converges for every feasible sequence $\{z_k\}$; in addition, there exists an increasing and continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, with $\theta(0) = 0$, and $g : \mathbb{R} \to \mathbb{R}_+$, with $g(0) = 0$ and continuous at 0, such that

$$\theta(d(x, y)) \leq [f_{\hat{i}}(x, y)]_+ + g(\mu(x) - \mu(y)), \text{ for every } x, y \in X;$$

iv. for every $i = 1, 2, \dots, m$ μ_i is continuous; moreover, $\mu_{\hat{i}}$ is bounded from below;

v. D is a lower semicontinuous map such that $x \in D(x)$ for all $x \in X$.

If the sequence $\{x_l\}$ in Method (TDM) is fixed, then $\tilde{x}$ in (3) is a solution of (**VQEP**). Otherwise, the sequence $\{x_l\}$ generated by Method (TDM) converges to a solution of problem (**VQEP**).

4 Application to relative optimization problems

In this section we consider relative or subjective optimization problems, that is problems where the goal function and the feasible set are dependent of the current state of the system under consideration. In [9], Konnov proposed an equilibrium formulation for these problems that leads to general (quasi-)equilibrium problems. As a matter of fact, finding their solutions may be rather troublesome. In order to overcome this difficulty, in [11] the author proposed the technique (TDM) to solve problems in a rather general class of relative optimization problems in metric spaces.

Our purpose is to apply the existence results of the previous sections to the particular case of the relative optimization problem introduced in [11]. We will remind a description of the general model studied in [11]. Let X be the metric space of the possible states of the system. Denote by $D : X \rightrightarrows X$ the set-valued map of the *feasible states*, i.e., $D(x)$ represents the set of points where the system can move to, starting from x and therefore it is reasonable to assume that $x \in D(x)$ for all $x \in X$. A sequence $\{x_k\}_k \subset X$ is said to be a *feasible trajectory* if $x_{k+1} \in D(x_k)$, for every k. Let us consider the following functions:

- $\varphi(x, \cdot) : D(x) \to \mathbb{R}$, that represents the *utility estimate* for the system to move from x to any state of the trust region $D(x)$;

- $c(x, \cdot) : D(x) \to \mathbb{R}$, that represents the expense required to move from x to a point of $D(x)$;

- $u : X \to \mathbb{R}$, defined as $u(x) = \varphi(x, x)$, that represents the *real utility* at the point x.

The *estimate of pure expenses* is defined via the bifunction $F : \mathrm{gph}(D) \to \mathbb{R}$, where

$$F(x, y) := c(x, y) - (\varphi(x, y) - \varphi(x, x)). \tag{12}$$

The bifunction F can be split as follows:

$$F(x, y) = c(x, y) - (\varphi(x, y) - \varphi(y, y)) - \varphi(y, y) + \varphi(x, x) \tag{13}$$

and, with the notations of the previous section, we set

- $f(x, y) = c(x, y) - (\varphi(x, y) - \varphi(y, y))$
- $\mu(x) = -\varphi(x, x)$.

Due to (13), we obtain also (**GQEP**). Therefore, we intend to find a feasible trajectory without positive estimates of pure expenses, which converges in some sense to a solution of (**QEP**) where F is defined in (12), or a solution of the equivalent (**GQEP**).

The following result holds:

Theorem 5. Let φ, c and D satisfy the following assumptions

i. $\varphi(x, y) + \varphi(y, z) \leq \varphi(x, z) + \varphi(y, y)$ for all $x, y, z \in X$;

ii. there exists $x_0 \in X$ such that $\varphi(y, y) - \varphi(x_0, y)$ is bounded from below for all $y \in X$;

iii. $\varphi(\cdot, y)$ is lower semicontinuous for all $y \in X$, and $\varphi(x, \cdot)$ is lower semicontinuous for all $x \in X$;

iv. $-\varphi(x, x)$ is coercive on X, bounded from below and continuous;

v. c is a nonnegative function satisfying the triangle inequality, $c(x, x) = 0$, $c(\cdot, y)$ is upper semicontinuous for all $y \in X$ and $c(x, \cdot)$ is upper semicontinuous for all $x \in X$;

vi. D is a lower semicontinuous map such that $x \in D(x)$ for all $x \in X$.

If the sequence $\{x_l\}$ in Method (TDM) applied to (**QEP**), where F is defined in (12), is fixed, then $\tilde{x}$ in (3) is a solution of this problem. Otherwise, the sequence $\{x_l\}$ generated by Method (TDM) has limit points, and all these limit points are solutions of problem (**QEP**) where F is defined in (12).

Proof. We apply Corollary 1 to the bifunction $f(x, y) = c(x, y) - (\varphi(x, y) - \varphi(y, y))$ and the function $\mu(x) = -\varphi(x, x)$. Note that f satisfies the triangle inequality by assumptions i. and v. Moreover, $f(x, x) = 0$ by v., and, by iii. and v., $f(x, \cdot)$ is upper semicontinuous. From ii. and v. there exists $x_0 \in X$ such that $f(x_0, \cdot)$ is bounded from below. □

Remark 1. We list below some comments on the conditions in Theorem 5:

a. Condition i. is fulfilled if φ is non negative on the diagonal and $-\varphi$ satisfies the triangle inequality.

b. From condition i. it follows that condition ii. is indeed equivalent to require that, fixed $x \in X$, $\varphi(y,y) - \varphi(x,y)$ is bounded from below for all $y \in X$, i.e. for every $x \in X$, there exists M_x such that

$$\varphi(x,y) \leq \varphi(y,y) - M_x$$

for every $y \in X$, where $M_x = M_0 + \varphi(x_0, x) - \varphi(x,x)$ and M_0 is a lower bound in condition ii., i.e. the utility estimates $\varphi(x,y)$ to move from x to y is controlled by the real utility $\varphi(y,y)$ plus a constant. Since $\varphi(y,y)$ is bounded from above, this means, in particular, that the utility estimates to move from x to y is bounded from above.

c. Condition i. is equivalent to

$$(\varphi(x,y) - \varphi(y,y)) + (\varphi(y,z) - \varphi(z,z)) \leq (\varphi(x,z) - \varphi(z,z))$$

for all $x, y, z \in X$, i.e. the triangle inequality for the bifunction $\varphi(y,y) - \varphi(x,y)$.

The difference $(\varphi(x,y) - \varphi(y,y))$ measures the overestimation error of the utility at x when y is reached. The inequality means that errors decrease when we have a longer walk from x to z, i.e. by increasing the length of the walk we learn to estimate better the utility.

d. The assumptions of Theorem 5 are different from those in Theorems 3.1 and 4.1 and Corollaries 4.1–4.3 from [11].

Example 1. A bifunction satisfying conditions i. and ii. is given, for instance, by

$$\varphi(x,y) = h(y) - k(d(x,y)),$$

where $h : X \to \mathbb{R}$, and $k : [0, +\infty) \to \mathbb{R}$ is increasing and subadditive, with $k(0) = 0$. In this case we have always a pessimistic estimate, i.e. the real utility in y is always greater that the estimate evaluated in x., i.e. there are no overestimates of the utilities.

Another example, is given by

$$\varphi(x,y) = h(y) + [k(y) - k(x)]$$

where $k : X \to \mathbb{R}$ is bounded from above in X. In this case the overestimates $\varphi(x,y) - \varphi(y,y)$ have not a fixed sign.

Acknowledgement

The work of the second author in this paper has been supported by the Kazan Federal University Strategic Academic Leadership Program ("PRIORITY-2030"). The research of the first and third author has been partially supported by GNAMPA-INdAM, Project GNAMPA 2021.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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