

On Finite-dimensional Generalized Variational Inequalities*

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Abstract. Our aim is to provide a short analysis of the generalized variational inequality (GVI) problem from both theoretical and algorithmic point of view. First, we show connections among some well known existence theorems for GVI and for inclusions. Then, we recall the proximal point approach and a splitting algorithm for solving GVI. Finally, we propose a class of differentiable gap functions for GVI, which is a natural extension of a well known class of gap functions for variational inequalities (VI).

Keywords. Variational Inequality, Generalized Variational Inequality, equilibrium point, gap function.

1 Introduction

Competitive phenomena in diverse disciplines are often characterized by the specific equilibrium state. Some well known equilibrium problems are general economic equilibrium problems, Nash equilibria for noncooperative games, traffic network equilibrium problems and so on (see [3, 16, 18] and reference therein). In recent years VI and GVI have emerged as very useful tool for the qualitative analysis and computation of various equilibrium problems. This paper aims to study some theoretical and algorithmic topics of GVI.

The paper is organized as follows. In section 2 we present definitions and notations needed in addressing our study. Section 3 describes some connections among two well-known existence results for GVI [10, 8] and two well-known existence theorems for inclusions [1, 21]. In section 4 we briefly recall some known computational schemes in order to solve GVI. In section 5 we extend some gap functions for VI [4, 23] to GVI.

2 Preliminaries

In this section we give some definitions and results which will be needed in the following. Let X be a subset of \mathbb{R}^n , we denote by $\text{int}(X)$, $\text{ri}(X)$, and $\text{cl}(X)$ the interior, the relative interior, and the closure of X , respectively. The recession cone of X is defined by

$$X_\infty = \{d \in \mathbb{R}^n : x + td \in \text{cl}(X) \quad \forall x \in X, \forall t \geq 0\},$$

the (negative) polar cone of X is a convex closed cone defined by

$$X^- = \{d \in \mathbb{R}^n : \langle d, x \rangle \leq 0 \quad \forall x \in X\},$$

the tangent cone $T_X(x)$ to X at $x \in X$ is the closed cone spanned by $X - x$, i.e.

$$T_X(x) = \text{cl} \left(\bigcup_{h>0} \frac{X - x}{h} \right),$$

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the normal cone to X at x is

$$N_X(x) = \begin{cases} \{d \in \mathbb{R}^n : \langle d, y - x \rangle \leq 0 \quad \forall y \in X\} & \text{if } x \in X \\ \emptyset & \text{if } x \notin X \end{cases}$$

Let $A : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a set-valued map, i.e. an operator which associates with each $x \in \mathbb{R}^n$ a set $A(x) \subseteq \mathbb{R}^n$. The image of X under A is

$$A(X) = \bigcup_{x \in X} A(x),$$

the inverse of A is defined by $A^{-1}(u) = \{x : u \in A(x)\}$, the domain and the range of A are, respectively,

$$\text{dom}(A) = \{x : A(x) \neq \emptyset\}, \quad \text{range}(A) = \{u : \exists x \text{ with } u \in A(x)\},$$

the graph of A is

$$\text{graph}(A) = \{(x, u) : u \in A(x)\},$$

the recession function of A is

$$f_\infty^A(d) = \sup_{u \in \text{range}(A)} \langle u, d \rangle.$$

The map A is said to be upper semicontinuous (u.s.c.) at x if for each open set $V \supseteq A(x)$ there exists a neighborhood U of x such that $A(x) \subseteq V$ for all $x \in U$; A is u.s.c. on X if it is u.s.c. at each point of X .

We say that A is pseudomonotone on X if

$$\langle u, y - x \rangle \geq 0 \implies \langle v, y - x \rangle \geq 0 \quad \forall x, y \in X, \quad \forall u \in A(x), v \in A(y);$$

it is monotone on X if

$$\langle u - v, x - y \rangle \geq 0 \quad \forall x, y \in X, \quad \forall u \in A(x), v \in A(y),$$

it is strictly monotone on X if the above inequality is strict for $x \neq y$; it is maximal monotone on X if it is monotone on X and its graph is not properly contained in the graph of any other monotone operator on X .

When X is a closed and convex set, the projection of $x \in \mathbb{R}^n$ on X is defined as

$$P_X(x) = \arg \min_{y \in X} \|y - x\|,$$

furthermore it is well known [13] that the set-valued map $x \mapsto N_X(x)$ is maximal monotone on \mathbb{R}^n and its graph is closed.

3 Generalized Variational Inequalities and Inclusions

Let K be a nonempty subset of \mathbb{R}^n and $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ a set-valued map. The GVI problem consists in finding a vector $x^* \in K$ such that there exists $u^* \in F(x^*)$ such that

$$\langle u^*, y - x^* \rangle \geq 0 \quad \forall y \in K.$$

We denote by \mathcal{S} the set of solutions of GVI. It is well known that the problem GVI is equivalent to the inclusion

$$\text{find } x^* \in K \text{ such that } 0 \in F(x^*) + N_K(x^*),$$

i.e. find a zero of the set-valued map $F + N_K$ in the domain K .

In this section, we recall some classical existence theorems for GVI and for inclusions and then we investigate the relationships among them.

Theorem 1. [10] Assume that:

- K is compact and convex,
- F is u.s.c. on K ,
- $F(x)$ is nonempty, compact and convex $\forall x \in K$.

Then \mathcal{S} is nonempty.

This theorem is a generalization of a classical existence result for VI [14] and it is proved by reformulating GVI as a fixed point problem of a suitable set-valued map.

We state below a known existence theorem for inclusions with a general set-valued map A .

Theorem 2. [1] Assume that:

- K is compact and convex,
- A is u.s.c. on K ,
- $A(x)$ is nonempty, closed and convex $\forall x \in K$,
- $A(x) \cap T_K(x) \neq \emptyset \quad \forall x \in K$ (viability condition).

Then there exists $x^* \in K$ such that $0 \in A(x^*)$.

We now show an alternative proof of Theorem 1 by exploiting Theorem 2. We remark that, under the assumptions of Theorem 1, the map $F + N_K$ is not necessarily u.s.c., thus we can not apply Theorem 2 with $A = F + N_K$ to deduce the existence of a solution to GVI. However, we can overcome this drawback applying Theorem 2 with a suitable operator and noticing that any zero of $-F - N_K$ is a zero of $F + N_K$.

An alternative proof of Theorem 1. Let $m(x)$ denotes any element of $-F(x)$, set $c = \sup_{x \in K} \sup_{y \in -F(x)} \|y\|$ and we notice that c is finite. We now define the map

$$A(x) = -F(x) - (B(0, c) \cap N_K(x)),$$

where $B(0, c) = \{x \in \mathbb{R}^n : \|x\| \leq c\}$. Since $A(x) \subseteq -F(x) - N_K(x)$, in order to prove that $\mathcal{S} \neq \emptyset$ it is sufficient to have a solution of the inclusion $0 \in A(x)$. Thus, it is sufficient to prove that A satisfies the hypotheses of Theorem 2:

- K is compact and convex,
- A is u.s.c. on K :
 $x \mapsto B(0, c)$ and $x \mapsto N_K(x)$ are two set valued maps such that:
 - $B(0, c) \cap N_K(x) \neq \emptyset, \quad \forall x \in K$,
 - $x \mapsto B(0, c)$ is u.s.c.,
 - $B(0, c)$ is compact,
 - $\text{graph}(N_K)$ is closed.

We have that the map $x \mapsto B(0, c) \cap N_K(x)$ is u.s.c. on K [2, Proposition 1.4.9], since F is u.s.c. on K , one has A is u.s.c. on K .

- $A(x)$ is nonempty, closed and convex $\forall x \in K$:
for all $x \in K$ we have $F(x)$ is nonempty, compact and convex, $B(0, c)$ and $N_K(x)$ are nonempty, closed and convex.

- $A(x) \cap T_K(x) \neq \emptyset \quad \forall x \in K$:
since $m(x) = P_{T_K(x)}(m(x)) + P_{N_K(x)}(m(x))$, then

$$P_{T_K(x)}(m(x)) = m(x) - P_{N_K(x)}(m(x)) \in T_K(x) \cap (-F(x) - N_K(x)).$$

Since the projection operator is nonexpansive, we have

$$\|P_{N_K(x)}(m(x))\| = \|P_{N_K(x)}(m(x)) - P_{N_K(x)}(0)\| \leq \|m(x)\| \leq c,$$

so,

$$P_{T_K(x)}(m(x)) \in T_K(x) \cap (-F(x) - (B(0, c) \cap N_K(x))) = T_K(x) \cap A(x).$$

□

If the domain K is not bounded, in order to establish the existence of a solution to GVI, some additional conditions as monotonicity and coercivity for F have to be considered.

Theorem 3. [8] *Assume that:*

- K is closed and convex,
- F is u.s.c. on K ,
- $F(x)$ is nonempty, compact and convex $\forall x \in K$,
- F is pseudomonotone on K .

Then \mathcal{S} is nonempty and compact if and only if $K_\infty \cap (F(K))^- = \{0\}$.

For the inclusions the following theorem holds.

Theorem 4. [21] *If A is maximal monotone on \mathbb{R}^n , then:*

- $A^{-1}(0)$ is closed and convex,
- $A^{-1}(0)$ is nonempty and compact if and only if $0 \in \text{int}(\text{range}(A))$.

Now we prove a direct relation between Theorem 3 for GVI and Theorem 4 for inclusions. Exploiting Theorem 4, we prove the following existence result for GVI, which is similar to Theorem 3: we require neither the upper semicontinuity of F , nor the compactness, nor the convexity of $F(x)$, instead we need the maximal monotonicity of F .

Theorem 5. *Assume that:*

- K is closed and convex,
- F is maximal monotone on \mathbb{R}^n ,
- $F(x)$ is nonempty $\forall x \in K$,

Then \mathcal{S} is nonempty and compact if and only if $K_\infty \cap (F(K))^- = \{0\}$.

Proof. We observe that $\mathcal{S} = (F + N_K)^{-1}(0)$. Since F is maximal monotone and $K \subseteq \text{dom}(F)$, we have [21, Chapter 12, Example 12.48] that $F + N_K$ is maximal monotone on \mathbb{R}^n . Moreover from Theorem 4 it follows that \mathcal{S} is nonempty and compact if and only if $0 \in \text{int}(\text{range}(F + N_K))$, which is equivalent [20, Theorem 13.1] to

$$f_\infty^{F+N_K}(d) > 0, \quad \forall d \neq 0.$$

By Proposition 2.2 in [5] we have

$$f_\infty^{F+N_K}(d) = \begin{cases} \sup_{u \in F(K)} \langle u, d \rangle & \text{if } d \in K_\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Then \mathcal{S} is nonempty and compact if and only if there is not $d \neq 0$ such that $d \in K_\infty$ and $\langle d, u \rangle \leq 0, \forall u \in F(K)$, i.e.

$$K_\infty \cap (F(K))^\circ = \{0\}.$$

□

4 Solution methods for GVI

In this section we give a summary of some known algorithms which are useful to solve GVI. Throughout this section we will consider the domain K explicitly defined by

$$K = \{x \in \mathbb{R}^n : g(x) \leq 0\},$$

where $g(x) = (g_1(x), \dots, g_m(x))$ with $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ given continuously differentiable convex functions.

We have just remarked that any GVI can be seen as the problem of finding the zeros of a suitable operator in the variables x ; this is called primal formulation of GVI. Another equivalent formulation is the so called primal-dual formulation [5]:

Theorem 6. *Assume that a constraint qualification holds for K (e.g. Slater's condition or Mangasarian-Fromovitz's condition). Then x^* is a solution of GVI if and only if there exists $\lambda^* \in \mathbb{R}_+^m$ such that*

$$(0, 0) \in H(x^*, \lambda^*),$$

where

$$H(x, \lambda) = \{(u, \xi) \in \mathbb{R}^n \times \mathbb{R}_+^m : u \in F(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x), \xi \in -g(x) + N_{\mathbb{R}_+^m}(\lambda)\}$$

if $(x, \lambda) \in \text{dom}(F) \times \mathbb{R}_+^m$ and $H(x, \lambda) = \emptyset$ otherwise.

In order to find zeros of a maximal monotone operator, Rockafellar [19] proposed a proximal point algorithm (PPA). Conditions under which the corresponding operators of the two above formulations are maximal monotone can be found in [5]. For the problem of finding zeros of an operator T , the PPA generates a sequence $\{x_k\} \subseteq \mathbb{R}^n$ such that

$$0 \in T_k(x_{k+1})$$

where

$$T_k(x) = T(x) + \lambda_k(x - x_k),$$

and $\lambda_k \geq 0$. It is shown in [19] that $\{x_k\}$ converges to a zero of T , provided that λ_k is bounded away from zero and the set of zeros of T is nonempty. In [5] a proximal-type algorithm is applied to the primal-dual formulation of GVI.

Alternatively, we note that the primal and primal-dual formulations of GVI involve the sum of two operators and so GVI can be solved using a splitting algorithm. We shall consider the case where $T = A + B$ with A and B maximal monotone. The algorithm generates a sequence $\{x_k\} \subseteq \mathbb{R}^n$ in the following way: start with any $x_0 \in \mathbb{R}^n$, choose $a_0 \in A(x_0)$ and set $w_0 = x_0 - \lambda a_0$, then $x_1 = (I + \lambda B)^{-1} w_0$, i.e. $w_0 \in x_1 + \lambda B(x_1)$, and then, given x_k ,

$$x_{k+1} \in (I + \lambda B)^{-1}(I - \lambda A)x_k.$$

In order to improve the rate of convergence we can vary the parameter λ at each step and we can rescaling the operators with a matrix H_k . Now we start with any $x_0 \in \mathbb{R}^n$, choose $a_0 \in A(x_0)$ and set $w_0 = H_0 x_0 - \lambda_0 a_0$, then $x_1 = (H_0 + \lambda_0 B)^{-1} w_0$, i.e. $w_0 \in H_0 x_1 + \lambda_0 B(x_1)$, and then, given x_k ,

$$x_{k+1} \in (I + \lambda_k H_k^{-1} B)^{-1}(I - \lambda_k H_k^{-1} A)x_k.$$

When $A = F$ and $B = N_k$, letting, for example, $H_k = 0$ and $\lambda_k = \lambda$ we get

$$x_{k+1} \in (I + \lambda N_K)^{-1}(I + \lambda F)x_k.$$

We note that $(I + \lambda N_K)^{-1} = P_K$, indeed $(I + \lambda N_K)^{-1}y = x$ means $y \in x + \lambda N_K(x)$, i.e. $\langle y - x, v - x \rangle \leq 0$, for all $v \in K$ so $x = P_K(y)$. Then $x_{k+1} \in P_K(x_k + \lambda F(x_k))$.

Remark 1. If $A = \nabla f$ and $B = N_k$, then $x_{k+1} = P_K(x_k - \lambda \nabla f(x_k))$, which is the projected gradient method.

If $A = \nabla f$ and $B = 0$, then $x_{k+1} = (I - \lambda \nabla f)x_k$, which is the steepest decent method.

If $A = 0$, then $x_{k+1} \in (I + \lambda T)^{-1}x_k$, which is the PPA.

5 Gap functions for GVI

In this section we propose a new approach for solving GVI, introducing a class of gap functions.

First, we consider the following function defined on the domain K :

$$\phi(x) = \inf_{u \in F(x)} \sup_{y \in K} \langle u, x - y \rangle.$$

It is easy to see that, if $F(x)$ is a nonempty and compact set for each $x \in K$, then

- $\phi(x) \geq 0$, $\forall x \in K$;
- x^* is a solution of GVI if and only if $x^* \in K$ and $\phi(x^*) = 0$.

Thus GVI is equivalent to the constrained optimization problem

$$\begin{cases} \min \phi(x) \\ x \in K \end{cases}$$

where, generally, the function ϕ is neither finite, nor differentiable on K .

In order to overcome this drawback, we can see the problem in a different but equivalent way. We say that a function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a gap function for GVI if

- $\varphi(x, u) \geq 0 \quad \forall (x, u) \in \text{graph}(F)$;
- x^* is a solution of GVI if and only if $x^* \in K$ and there is $u^* \in F(x^*)$ such that $\varphi(x^*, u^*) = 0$.

Therefore a gap function allows to formulate GVI as an equivalent constrained optimization problem

$$\begin{cases} \min \varphi(x, u) \\ (x, u) \in \text{graph}(F). \end{cases}$$

An example of gap function for GVI is the following:

$$\varphi(x, u) = \sup_{y \in K} \langle u, x - y \rangle. \tag{1}$$

It generalizes the well known Auslender's gap function [4] and, in general, it is neither finite nor differentiable on $\text{graph}(F)$. However, it represents a duality gap in the Mosco's duality scheme for GVI. In [15] the following more general GVI is considered:

$$\begin{aligned} & \text{find } x^* \in \mathbb{R}^n \text{ and } u^* \in F(x^*) \text{ such that} \\ & \langle u^*, x - x^* \rangle \geq f(x^*) - f(x) \quad \forall x \in \mathbb{R}^n, \end{aligned} \tag{2}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, lower semicontinuous convex function; the dual problem of (2) is defined as:

$$\begin{aligned} & \text{find } v^* \in \mathbb{R}^n \text{ and } y^* \in -F^{-1}(-v^*) \text{ such that} \\ & \langle y^*, v - v^* \rangle \geq f^*(v^*) - f^*(v) \quad \forall v \in \mathbb{R}^n, \end{aligned} \tag{3}$$

where $f^*(v) = \sup_{x \in \mathbb{R}^n} \{\langle v, x \rangle - f(x)\}$ is the Fenchel conjugate of f .

Theorem 7. *The gap function (1) measures the duality gap of the Mosco's duality scheme.*

Proof. It was proved in [15] that (x^*, u^*) solves (2) if and only if $(-u^*, -x^*)$ solves (3). Moreover, by definition of f^* one has

$$f(x) + f^*(-u) + \langle u, x \rangle \geq 0 \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and given $(x^*, u^*) \in \text{graph}(F)$, by Theorem 1 in [15], we have

$$f(x^*) + f^*(-u^*) + \langle u^*, x^* \rangle = 0 \iff (x^*, u^*) \text{ solves (2) and } (-u^*, -x^*) \text{ solves (3),}$$

hence $f(x) + f^*(-u) + \langle u, x \rangle$ represents a duality gap for (2) and (3).

We note that if we choose the function f as the indicator of the set K , i.e.

$$f(x) = I_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{otherwise} \end{cases}$$

then the problem (2) coincides with GVI and (3) reduces to:

$$\text{find } v^* \in \mathbb{R}^n \text{ such that } 0 \in -F^{-1}(-v^*) + N_K^{-1}(v^*),$$

furthermore, the duality gap can be written as

$$f(x) + f^*(-u) + \langle u, x \rangle = I_K(x) + \sup_{y \in K} \langle u, -y \rangle + \langle u, x \rangle = \begin{cases} \varphi(x, u) & \text{if } x \in K \\ +\infty & \text{otherwise} \end{cases}$$

□

Now, we introduce for GVI a class of continuously differentiable gap functions, which generalizes the one introduced in [23] for VI. Let us consider a function $\Omega : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that:

- $\Omega(x, y) \geq 0 \quad \forall (x, y) \in K \times K$;
- Ω is continuously differentiable on $K \times K$;
- $\forall x \in K$, $\Omega(x, \cdot)$ is strongly convex on K ;
- $\Omega(x, x) = 0$ and $\nabla_y \Omega(x, x) = 0$, $\forall x \in K$, where $\nabla_y \Omega$ is the gradient of Ω with respect to the second variable.

Define

$$h(x, u, y) = \langle u, x - y \rangle - \Omega(x, y),$$

and

$$\psi(x, u) = \max_{y \in K} h(x, u, y) = h(x, u, H(x, u)), \tag{4}$$

where $H(x, u)$ is the unique maximizer since h is strongly concave with respect to y .

Theorem 8. *The function ψ defined in (4) is a gap function for GVI. Furthermore, ψ is continuously differentiable and its gradient is given by*

$$\nabla_x \psi(x, u) = u - \nabla_x \Omega(x, H(x, u)), \quad \nabla_u \psi(x, u) = x - H(x, u).$$

Proof. Since $h(x, u, x) = 0$, one has that $\psi(x, u) \geq 0$, $\forall (x, u) \in \text{graph}(F)$. Now, if x^* is a solution of GVI, then there exists $u^* \in F(x^*)$ such that

$$\langle u^*, y - x^* \rangle \geq 0 \quad \forall y \in K,$$

then

$$h(x^*, u^*, y) = \langle u^*, x^* - y \rangle - \Omega(x^*, y) \quad \forall y \in K,$$

thus

$$\psi(x^*, u^*) = 0.$$

Conversely, suppose that there is $(x^*, u^*) \in \text{graph}(F)$ such that $\psi(x^*, u^*) = 0$, i.e.

$$h(x^*, u^*, y) \leq 0 \quad \forall y \in K,$$

then x^* solves the optimization problem $\max_{y \in K} h(x^*, u^*, y)$, hence x^* satisfies the following optimality condition

$$\langle \nabla_y h(x^*, u^*, x^*), y - x^* \rangle \leq 0 \quad \forall y \in K,$$

which is equivalent to

$$\langle -u^* - \nabla_y \Omega(x^*, x^*), y - x^* \rangle \leq 0 \quad \forall y \in K.$$

Since $\nabla_y \Omega(x^*, x^*) = 0$, we obtain

$$\langle u^*, y - x^* \rangle \geq 0 \quad \forall y \in K,$$

i.e. x^* solves GVI.

Since h is continuously differentiable and $\max_{y \in K} h(x, u, y)$ is uniquely attained at $y = H(x, u)$, it follows from [4, Chapter 4, Theorem 1.7] that ψ is continuously differentiable and its gradient is given by

$$\nabla \psi(x, u) = \nabla_{(x, u)} h(x, u, H(x, u)) = (u - \nabla_x \Omega(x, H(x, u)), x - H(x, u)).$$

□

We remark that if we choose

$$\Omega(x, y) = \frac{1}{2} \|x - y\|_G^2 = \frac{1}{2} \langle x - y, G(x - y) \rangle,$$

where G is any symmetric positive definite matrix, then

$$\psi(x, u) = \max_{y \in K} \left[\langle u, x - y \rangle - \frac{1}{2} \|x - y\|_G^2 \right]$$

is an extension to GVI of the gap function introduced in [11]. In this case, the problem

$$\max_{y \in K} \left[\langle u, x - y \rangle - \frac{1}{2} \|x - y\|_G^2 \right]$$

is equivalent to

$$\min_{y \in K} \|y - (x - G^{-1}u)\|_G^2,$$

therefore we have $H(x, u) = P_{K, G}(x - G^{-1}u)$, i.e. the projection of $x - G^{-1}u$ on the set K with respect to the norm $\|\cdot\|_G$, and the gradient of ψ is

$$\nabla_x \psi(x, u) = u + G(P_{K, G}(x - G^{-1}u) - x), \quad \nabla_u \psi(x, u) = x - P_{K, G}(x - G^{-1}u).$$

From the above theorem it follows that GVI is equivalent to the following differentiable constrained optimization problem

$$\begin{cases} \min & \psi(x, u) \\ & (x, u) \in \text{graph}(F) \end{cases} \quad (5)$$

In the special case when $\Omega(x, y) = \frac{1}{2} \|x - y\|_G^2$, we generalize Theorem 3.3 in [11] to GVI.

Theorem 9. Assume that the gap function ψ is defined with $\Omega(x, y) = \frac{1}{2}\|x - y\|_G^2$. If F is strictly monotone on K and $(x^*, u^*) \in \text{graph}(F)$ is such that

$$\langle \nabla\psi(x^*, u^*), (x, u) - (x^*, u^*) \rangle \geq 0 \quad \forall (x, u) \in \text{graph}(F), \quad (6)$$

then x^* is a solution of GVI.

Proof. The condition (6) can be written as

$$\begin{aligned} & \langle u^* + G(P_{K,G}(x^* - G^{-1}u^*) - x^*), x - x^* \rangle + \\ & + \langle x^* - P_{K,G}(x^* - G^{-1}u^*), u - u^* \rangle \geq 0 \quad \forall (x, u) \in \text{graph}(F) \end{aligned}$$

If we choose $x = P_{K,G}(x^* - G^{-1}u^*)$, then we have that for all

$$u \in F(P_{K,G}(x^* - G^{-1}u^*))$$

one has

$$\begin{aligned} & \langle u - u^*, P_{K,G}(x^* - G^{-1}u^*) - x^* \rangle \leq \\ & \leq \langle u^* + G(P_{K,G}(x^* - G^{-1}u^*) - x^*), P_{K,G}(x^* - G^{-1}u^*) - x^* \rangle = \\ & = \langle G^{-1}u^* + (P_{K,G}(x^* - G^{-1}u^*) - x^*), G(P_{K,G}(x^* - G^{-1}u^*) - x^*) \rangle = \\ & = \langle P_{K,G}(x^* - G^{-1}u^*) - (x^* - G^{-1}u^*), G(P_{K,G}(x^* - G^{-1}u^*) - x^*) \rangle \leq \\ & \leq 0. \end{aligned}$$

thus

$$\langle u - u^*, P_{K,G}(x^* - G^{-1}u^*) - x^* \rangle \leq 0 \quad \forall u \in F(P_{K,G}(x^* - G^{-1}u^*)). \quad (7)$$

Since F is strictly monotone, it follows that the inequality (7) holds only if

$$x^* = P_{K,G}(x^* - G^{-1}u^*),$$

i.e. $\psi(x^*, u^*) = 0$, hence, by Theorem 8, x^* solves GVI. \square

Much more research is needed in order to provide algorithmic tools to effectively solve GVI. In this regard we feel to deserve further investigations in the gap functions for GVI.

References

- [1] J. P. Aubin and A. Cellina, *Differential inclusions*, Springer-Verlag, Berlin, 1984.
- [2] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.
- [3] J. P. Aubin, *Optima and Equilibria*, Springer-Verlag, Berlin, 1993.
- [4] A. Auslender, *Optimisation: Mthodes Numriques*, Masson, Paris, 1976.
- [5] A. Auslender and M. Teboulle, *Lagrangian duality and related multiplier methods for variational inequality problems*, Siam Journal of Optimization, **10** (2000), 1097–1115.
- [6] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Application to Free-Boundary Problems*, John Wiley & Sons, New York, 1984.
- [7] R. S. Burachik and A. N. Iusem, *A Generalized Proximal Point Algorithm for the Variational Inequality Problem in a Hilbert Space*, SIAM Journal of Optimization, **8** (1998), 197–216.
- [8] J. P. Crouzeix, *Pseudomonotone Variational Inequality Problems: Existence of solutions*. Mathematical Programming, **78** (1997), 305–314.
- [9] S. Dafermos, *An iterative scheme for variational inequalities*, Mathematical Programming, **26** (1983), 40–47.

- [10] S. C. Fang and E. L. Peterson, *Generalized Variational Inequalities*, Journal of Optimization Theory and Applications, **38** (1982), 363–383.
- [11] M. Fukushima, *Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems*, Mathematical Programming, **53** (1992), 99–110.
- [12] M. Fukushima, *Merit functions for variational inequality and complementarity problems*, in Non-linear optimization and applications, edited by Di Pillo G. and Giannessi F., Plenum Press, 1996, 155–170.
- [13] J. B. Hiriart-Urruty and C. Lemarechal, *Convex analysis and minimization algorithms*, Springer-Verlag, Berlin, 1993.
- [14] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequality and Their Application*, Academic Press, New York, 1980.
- [15] U. Mosco, *Dual Variational Inequalities*, Journal of Mathematical Analysis and Applications, **40** (1972), 202–206.
- [16] A. Nagurney, *Network economics: a variational inequality approach*, Kluwer, Dordrecht, 1993.
- [17] M. Pappalardo, *Equilibrium in Variational Inequalities, Games and Dynamical Systems*, in “Optimization in Economics, Finance and Industry”, Datanova, 47–60.
- [18] M. Patriksson, *The Traffic Assignment Problem: Models and Methods*, VSP, Utrecht, 1994.
- [19] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM Journal of Control and Optimization, **14** (1976), 877–898.
- [20] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.
- [21] R. T. Rockafellar and R. J-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.
- [22] J.-C. Yao, *Variational Inequalities with Generalized Monotone Operators*, Mathematics of Operations Research, **19** (1994), 691–705.
- [23] D. L. Zhu and P. Marcotte, *An Extended Descent Framework for Variational Inequalities*, Journal of Optimization Theory and Applications, **80** (1994), 349–366.