International Journal of Approximate Reasoning 145 (2022) 187–209

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Graded cubes of opposition in fuzzy formal concept analysis

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ARTICLE INFO

Article history:
Received 5 May 2021
Received in revised form 14 March 2022
Accepted 16 March 2022
Available online 18 March 2022

Keywords:
Aristotle's square of opposition
Cube of opposition
Graded structures of opposition
Fuzzy formal concept analysis
Quantifier-based operators

ABSTRACT

We recently introduced special fuzzy quantifiers named *quantifier-based operators* to form extended fuzzy concept lattices and to construct graded squares, hexagons, octagons and decagons of oppositions. This article aims to extend our previous works by organizing quantifier-based operators in more general structures of oppositions: the so-called *graded cubes of opposition* and *5-graded cubes of opposition*.

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1. Introduction

The main objective of this paper is a follow-up to a previous publication [8], in which we focused on the construction of graded polygons of opposition in fuzzy formal concept analysis. In our previous publication, we proposed graded polygons of opposition with quantifier-based operators that are monotonous and are defined based on evaluative linguistic expressions. In this article, we aim to study the properties between quantifier-based operators to form generalized graded cubes of opposition.

1.1. Formal concept analysis

Formal Concept Analysis (FCA) is a mathematical theory originally introduced by Rudolf Wille and widely applied in several areas such as machine learning, semantic web, software development, chemistry and biology [19,20,39,42,43].

Formal concept analysis is principally based on the notions of *formal context* and *formal concept*. A formal context is a triple composed of a finite set of objects, a finite set of attributes, and a binary relation between objects and attributes. A formal concept is a pair (A, B) where "*A is the set of all objects having all attributes of B*" and "*B is the set of all attributes being satisfied by all objects of A*". Analogously to the tradition, A and B are respectively named *extent* and *intent* of the concept.

FCA techniques extract a collection of formal concepts from each formal context. Additionally, formal concepts equipped with the *subconcept-superconcept relation*, which expresses that a concept can be more specific than another, form an algebraic structure called *concept lattice* [20].

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https://doi.org/10.1016/j.ijar.2022.03.006
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Fuzzy Formal Concept Analysis (FFCA) extends the formal concept analysis by ideas taken from mathematical fuzzy logic to be able to capture also vague information. Several approaches have been presented to develop fuzzy formal concept analysis. Bělohlávek and Pollandt in [3,37] considered a pair of functions called *concept-forming operators*, where the fundamental tool to generate concepts is based on the quantifier *All*.

The first operator transforms a fuzzy set of objects A into a fuzzy set of attributes B such that, given an attribute y, $B(y)$ is the truth degree to which "*y is shared by all objects of A*". Dually, the second one transforms a fuzzy set of attributes B into a fuzzy set of objects A such that, given an object x, $A(x)$ is the truth degree to which "*x has all attributes of B*". Therefore, the *fuzzy formal concept* is a pair composed of a fuzzy set of objects and a fuzzy set of attributes, which are obtained from each other using the concept-forming operators. Our idea is to generalize these concepts by means of *intermediate quantifiers* which are natural language expressions such as "almost all, many, most, a few", and other ones.

1.2. Fuzzy quantifier-based operators

We recently [7,8] introduced the so-called *fuzzy quantifier-based operators* in fuzzy formal concept analysis to be able to extract more detailed information from the initial dataset. Fuzzy quantifier-based operators are special fuzzy (intermediate) quantifiers. Their definition is based on the *standard Łukasiewicz MV-algebra*. We consider expressions of natural language such as *extremely big*, *very big*, and *not small*, etc. that belong to the class of *evaluative linguistic expressions* whose semantics is formalized in the higher-order fuzzy logic (see [29]). The fuzzy quantifier-based operators are viewed as interpretations in a model of special formulas called *intermediate quantifiers* of the *formal theory of intermediate generalized quantifiers* developed in [27,30], and elsewhere.

Extended fuzzy concepts were formed using the fuzzy quantifier-based operators. If a fuzzy concept (A, B) is given then an extended concept has the form $(A, (B, A^*))$ or $((A, B^*), B)$, where A^* and B^* are fuzzy sets generated by one of our quantifier-based operators. Extended fuzzy concepts have the vantage to capture *positive*, *negative*, and *intermediate information*, namely information based on the presence or the absence of a certain amount of properties in objects.

As an example, given an object x and an attribute y. Then $B^*(x)$ and $A^*(y)$ represent truth degrees to which "*x has almost all attributes of B*" and "*y is shared by almost all objects of A*", respectively.

Eventually, fuzzy quantifier-based operators were employed to construct graded squares, hexagons, octagons, and decagons of opposition, which are extensions of Aristotle's square. This article mainly aims to extend our previous results by organizing fuzzy quantifier-based operators into more general structures of the opposition called *graded cube of opposition* and *5-graded cube of opposition*.

1.3. Aristotle's square of opposition

In this subsection, we will remind the reader the well-known classical square of opposition. Aristotle's square was analyzed in propositional logic in [23,33] and more deeply elaborated in first-order predicate logic by many other authors, for example, [10,35,40]. It comprises relations between classical (universal and existential) quantifiers. We will consider a formula **A** of the form "All B's are A" and its negation **O** "Some B's are not A". Another considered formula is **E** of the form "No B's are A", which is negated by **I** "Some B's are A".

Definition 1.1. We say that

- two formulas are *contradictory* if in any model they cannot both be true and they cannot both be false;
- two formulas are *contrary* if in any model they cannot both be true, but they can both be false;
- two formulas are *subcontrary* if in any model they cannot both be false, but they can both be true;
- a formula is *subaltern* of another one (called *superaltern*) if in any model, it must be true if its superaltern is true. At the same time, the superaltern must be false if the subaltern is false.

The relations of the previous definition are schematically depicted in Fig. 1.

A graded generalization of Aristotle's square (from the point of view of many-valued fuzzy logic) was proposed by several authors (see [15,26]). They proposed generalized definition of contraries, contradictories, subalterns, and sub-contraries properties to form a generalized Aristotle's square of opposition.

The Peterson's square of opposition generalizes Aristotle's one by adding intermediate quantifiers [34]. Its formalization using the tools of higher-order fuzzy logic (fuzzy type theory) was introduced in [26].

Fig. 2 represents a square of opposition formed by four operators defined in [13].[1]

They were introduced as an analogy of the formal concept analysis and possibility theory [14]. Similarly, we can construct a hexagon and a cube of opposition induced by additional operators underlying formal concept analysis and possibility theory.

[1] Consider $Y \subseteq X$, $x \in X$, and the class $R(x)$ of x w.r.t. an equivalence relation R on X. Then, $R^{\Pi}(Y) : \exists y, y \in R(x) \wedge y \in Y$, $R^{\mathrm{N}}(Y) : \forall y, y \in R(x) \rightarrow y \in Y$, $R^{\nabla}(Y) : \exists y, y \notin R(x) \wedge y \notin Y$, and $R^{\triangle}(Y) : \forall y, y \notin R(x) \rightarrow y \notin Y$.

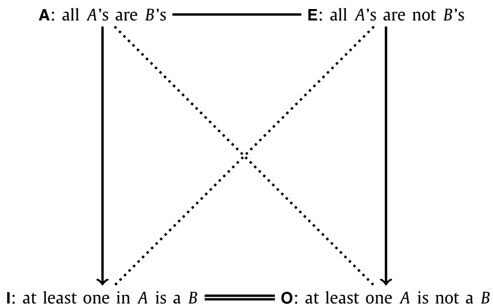

Fig. 1. Aristotle's square. The lines ——, ·····, ⟶, ═══ denote that the corresponding propositions are contraries, contradictories, subalterns, and sub-contraries, respectively.

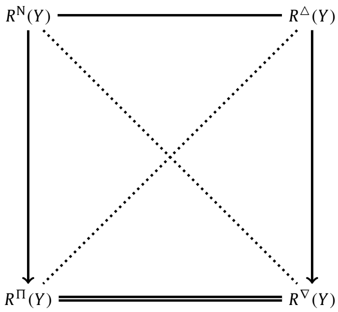

Fig. 2. Aristotle's square with $R^{\Pi}(Y)$, $R^{\mathrm{N}}(Y)$, $R^{\nabla}(Y)$, and $R^{\triangle}(Y)$.

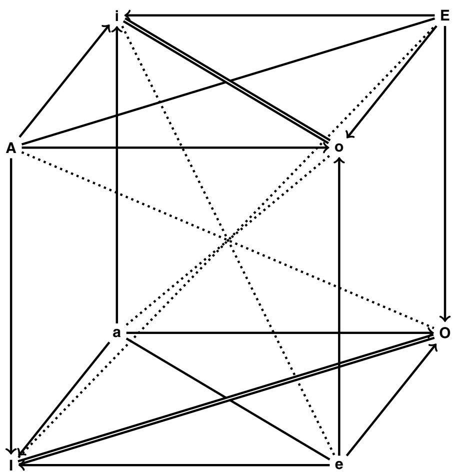

Fig. 3. Moretti's cube of opposition.

1.4. Cube of opposition

In the related literature, we can find several variants of cubes that generalize classical square of opposition. Among the most known are Moretti's [24,25] and Johnson-Keynes cubes [41].

The Aristotle's square of opposition is formed by two positive and two negative quantifiers that are in the relation of contraries, contradictories, sub-contraries, and sub-alterns. If we change B and A into their negation, $\neg B$ and $\neg A$ respectively, we obtain another similar square of opposition **aeio**. Hence, formulas **A, E, I, O, a, e, i**, and **o** form a *cube of opposition*, which require, however, $A, B, \neg A, \neg B$ to be non-empty. We can observe that the new logical square of opposition (**aeio**) follows from Aristotle's square (**AEIO**) by replacing formulas B and A by $\neg B$ and $\neg A$, respectively.

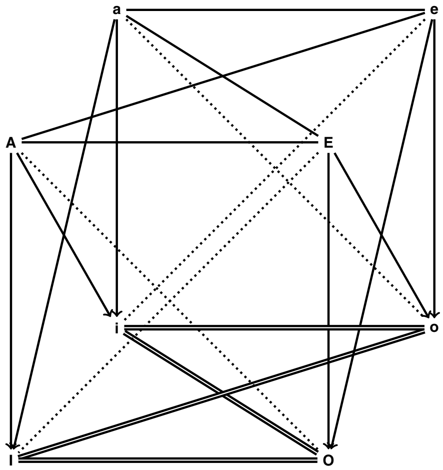

Fig. 4. Johnson and Keynes cube of opposition.

Fig. 3 contains Moretti's cube of opposition, which can be obtained from two squares of opposition **AEOI** and **aeoi**. The other relations require the following constraint to hold:

$$(\mathbf{A} \vee \mathbf{E}) \rightarrow (\mathbf{i} \wedge \mathbf{o}).$$

It means that Moretti's cube of opposition is based on four vertices **A**, **E**, **a**, **e**. The obtained cube of opposition is then formed by six squares of opposition **AEIO**, **AaOo**, **AeOi**, **aEoI**, **eEiI** and **aeoi**.

The graded extension of Moretti's cube of opposition is defined by assuming that formulas **A,E,O, I, a,e,o,i** can be only partially true. The precise definition can be found in [17].

A more general is Johnson and Keynes cube (see [17]) **AEIOaeio**, which consists of 4 squares **AEIO**, **aeio**, **AeOi** and **aEoI** as depicted in Fig. 4. This requires the following additional condition to hold:

$$(\mathbf{A} \vee \mathbf{a}) \rightarrow (\neg\mathbf{E} \wedge \neg\mathbf{e}).$$

A graded Johnson and Keynes cube can be formed similarly to Moretti's one. The details (examples of cubes, precise definitions, etc.) can be found in [17].

Another graded extension of the cube in the possibility theory with fuzzy events was proposed by Dubois, Prade and Rico in [16]. The presented cube includes also the possibility to work with Sugeno and Choquet integrals using the Łukasiewicz operations.

1.5. Structure of the paper

The structure of this paper is the following: in Section 2, we first recall the construction of graded squares of opposition using the concept-forming operators that satisfy the generalized properties of contrary, contradictory, sub-contrary, and subalterns introduced in [8]. Then, we repeat definitions of quantifier-based operators with which we form graded decagons, octagons, and hexagons of opposition in fuzzy formal concept analysis.

Important and new results are given in Sections 3 and 4, in which we construct graded cubes of opposition and generalized graded cubes opposition by employing the concept-forming operators and the quantifier-based operators. Moreover, we introduce several figures describing the relationships between the considered operators.

Similarly as in the Aristotle's square, the existential import (presupposition) plays a significant role in the construction of the graded cube of opposition. It was introduced by Peterson in [36] and is discussed below.

Finally, we underline that the structures of opposition introduced in this paper can be viewed as generalizations of the cube of opposition constructed using the concept-forming operators of the standard formal concept analysis [12,14].

2. Preliminaries

In this section we introduce notation, preliminary notions and some results, which will be used further in this article.

2.1. Algebraic structures of truth values

In this subsection, we will remind few of the main algebras of truth values, namely complete residuated lattices and MV-algebras.

Definition 2.1. A lattice $\langle L, \vee, \wedge\rangle$ is called *complete* if and only if all subsets of L have both a supremum and an infimum.

Definition 2.2. A *residuated lattice* is an algebra

$$\langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$$

where

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice,
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid,
(iii) $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, for each $a, b, c \in L$ (adjunction property).

We will denote the complete residuated lattice $\langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$ with its support L.

Definition 2.3 *([11,32]).* An *MV-algebra* is a residuated lattice

$$\langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$$

where $a \vee b = (a \rightarrow b) \rightarrow b$, for each $a, b \in L$. Furthermore, the following operations are defined on L. Let $a, b \in L$,

(i) $\neg a = a \rightarrow 0$ (negation),
(ii) $a \oplus b = \neg(\neg a \otimes \neg b)$ (strong summation),
(iii) $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ (biresiduation).

The results of this paper are based on the standard Łukasiewicz MV-algebra defined as follows.

Definition 2.4. The *standard Łukasiewicz MV-algebra* is a special MV-algebra

$$\langle [0, 1], \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$$

where for each $a, b \in [0, 1]$

(i) $a \wedge b = \min(a, b)$,
(ii) $a \vee b = \max(a, b)$,
(iii) $a \otimes b = \max(0, a + b - 1)$,
(iv) $a \rightarrow b = \min(1, 1 - a + b)$.

Derived operations are negation $\neg a = 1 - a$ and Łukasiewicz disjunction $a \oplus b = \min\{1, a + b\}$.

In the following lemma, we list some properties of the standard Łukasiewicz MV-algebra that will be used further.

Lemma 2.5. *Let $\langle [0, 1], \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$ be the standard Łukasiewicz MV-algebra and let $I = \{1, \dots, n\}$. Then, the following hold for all $a_1, \dots, a_n, b_1, \dots, b_n, a, b, c, d, e \in [0, 1]$:*

(a) $\neg\neg a = a$ *(double negation law).*
(b) *for each* $k \in I$, $\bigwedge_{i\in I} a_i \leq a_k$ *and* $a_k \leq \bigvee_{i\in I} a_i$.
(c) *if* $a_i \leq b_i$ *for each* $i \in I$, *then* $\bigwedge_{i\in I} a_i \leq \bigwedge_{i\in I} b_i$ *and* $\bigvee_{i\in I} a_i \leq \bigvee_{i\in I} b_i$.
(d) $\bigvee_{i\in I}(a \otimes b_i) = a \otimes \bigvee_{i\in I} b_i$.
(e) $a \oplus \neg a = 1$ *and* $a \otimes \neg a = 0$.
(f) *if* $a \otimes b \leq e$, *then* $(a \wedge c) \otimes (b \wedge d) \leq e$.
(g) *if* $a \leq b$ *and* $c \leq d$, *then* $a \otimes c \leq b \otimes d$ *and* $a \oplus c \leq b \oplus d$.
(h) $a \otimes b = \neg(a \rightarrow \neg b)$ *and* $a \rightarrow b = \neg(a \otimes \neg b)$.
(i) $a \rightarrow b \leq \neg b \rightarrow \neg a$.
(j) $\neg(a \otimes b) = \neg a \oplus \neg b$ *and* $\neg(a \oplus b) = \neg a \otimes \neg b$.

Lemma 2.6. *Let $\langle [0, 1], \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$ be the standard Łukasiewicz MV-algebra, and let $I = \{1, \dots, n\}$. Then, the following hold for all $a_1, \dots, a_n, b_1, \dots, b_n \in [0, 1]$:*

(a) $\bigwedge_{i\in I}(a_i \rightarrow b_i) \leq \bigwedge_{i\in I} a_i \rightarrow \bigwedge_{i\in I} b_i$,
(b) $\bigwedge_{i\in I}(a_i \rightarrow b_i) \leq \bigvee_{i\in I} a_i \rightarrow \bigvee_{i\in I} b_i$.

From now on, we write $A \underset{\sim}{\subset} X$ to denote that A is a fuzzy set in the universe X, i.e. A is a function $A : X \rightarrow [0, 1]$.

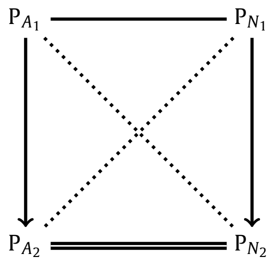

Fig. 5. Graded square of opposition.

2.2. Graded square of opposition in Fuzzy Formal Concept Analysis

In this subsection, we recall results from [7,8] where we constructed graded squares of opposition with fuzzy concept-forming operators. The structure of truth degrees is assumed to be the standard Łukasiewicz MV-algebra.

In our explanation we must consider properties of elements of some universe, for example *tall, young, shallow*, etc. In the *predicate first-order fuzzy logic* they are represented by special symbols of its language that are in a model interpreted by fuzzy sets (cf. [21]).

To simplify readability of this paper, in the sequel we will introduce predicates $P_A, P_B, \dots$ and assume that they are interpreted by the respective fuzzy sets $A, B, \dots \underset{\sim}{\subset} X$, where X is a universe.[2] For example, if X is a set of roses and P_A is the property *fragrant*, then it is interpreted by a fuzzy set $A \underset{\sim}{\subset} X$. If $x \in X$ is a rose then $A(x)$ expresses how much fragrant the rose x is.

The graded square of opposition arises on the basis of the following relations between predicates (i.e. properties).

Definition 2.7. Let P_A and P_B be predicates. Then, we say:

(i) P_A and P_B are *contraries* if and only if $A(x) \otimes B(x) = 0$ for each $x \in X$,
(ii) P_A and P_B are *sub-contraries* if and only if $A(x) \oplus B(x) = 1$ for each $x \in X$,
(iii) P_B is *subaltern* of P_A if and only if $A(x) \to B(x) = 1$, i.e. $A(x) \le B(x)$, for each $x \in X$,
(iv) P_A and P_B are *contradictories* if and only if $A(x) = \neg B(x)$ for each $x \in X$.

The graded square of opposition is a diagram that shows existing relations between the predicates on its vertices.

Definition 2.8. Let P_{A_1}, P_{N_1}, P_{A_2} and P_{N_2} be predicates. The *graded square of opposition* is a square whose vertices are the predicates P_{A_1}, P_{N_1}, P_{A_2}, P_{N_2} and the following conditions hold:

(i) P_{A_1} and P_{N_1} are contraries,
(ii) P_{A_2} and P_{N_2} are sub-contraries,
(iii) P_{A_2} is subaltern of P_{A_1} and P_{N_2} is subaltern of P_{N_1},
(iv) P_{A_1} and P_{N_2} as well as P_{A_2} and P_{N_1} are contradictories.

A graded square of opposition is depicted in Fig. 5.

Remark 2.9. Let $A_1, N_1, A_2, N_2 \underset{\sim}{\subset} X$, suppose that the predicates P_{A_1}, P_{N_1}, P_{A_2}, and P_{N_2} form a graded square of opposition. Then, given $x \in X$, the truth values $A_1(x)$, $N_1(x)$, $A_2(x)$, and $N_2(x)$ form a graded square of opposition as defined in [18].

In the previous articles, we proved that a collection of graded squares is associated with some fuzzy formal context (X, Y, I). Indeed, given a fuzzy set $A \underset{\sim}{\subset} X$, a square is determined by applying the following concept-forming operators to A: Let $H \in \{I, \neg I\}$[3] then we put

$$A^{\uparrow}_H(y) = \bigwedge_{x \in X} (A(x) \to H(x, y)) \quad \text{and} \quad A^{\cap}_H(y) = \bigvee_{x \in X} (A(x) \otimes H(x, y)) \tag{1}$$

for all $y \in Y$.

[2] More precisely, we first must consider a language of a predicate fuzzy logic which contains the predicate symbols $P_A, P_B, \dots$ and then a model with the support X. Each predicate is in this model interpreted by the corresponding fuzzy set in $A, B, \dots \underset{\sim}{\subset} X$.

[3] Given $I : X \times Y \to [0, 1]$, $\neg I$ is a fuzzy relation given by the membership function $\neg I(x, y) = 1 - I(x, y)$ for all $x \in X$ and $y \in Y$.

Table 1
Verbal description of $A_I^{\uparrow}(y)$, $A_{\neg I}^{\uparrow}(y)$, $A_I^{\cap}(y)$, and $A_{\neg I}^{\cap}(y)$.

Truth degree	*Statement*
$A_I^{\uparrow}(y)$	y is shared by *all* objects of A
$A_{\neg I}^{\uparrow}(y)$	y is *not* shared by *all* objects of A
$A_I^{\cap}(y)$	y is shared by *at least* one object of A
$A_{\neg I}^{\cap}(y)$	y is *not* shared by *at least* one object of A

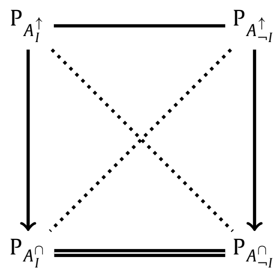

Fig. 6. Graded square of opposition with concept-forming operators.

Let us recall that ${}^{\uparrow}_{H}$ and ${}^{\cap}_{H}$ were introduced in [3,37,38] to extract two different types of fuzzy concept lattices from (X, Y, H). Moreover, for a given $y \in Y$, the values $A_I^{\uparrow}(y)$, $A_{\neg I}^{\uparrow}(y)$, $A_I^{\cap}(y)$ and $A_{\neg I}^{\cap}(y)$ correspond to the truth degrees of the statements listed in Table 1.

Additionally, the following theorem states that the predicates interpreted by the fuzzy sets $A_I^{\uparrow}$, $A_{\neg I}^{\uparrow}$, $A_I^{\cap}$, and $A_{\neg I}^{\cap}$ can be organized in a graded square. Its proof can be found in [8].

Theorem 2.10. *Let (X, Y, I) be a fuzzy formal context and let $A \underset{\sim}{\subset} X$. Then, the predicates $\mathrm{P}_{A_I^{\uparrow}}$, $\mathrm{P}_{A_I^{\cap}}$, $\mathrm{P}_{A_{\neg I}^{\cap}}$ and $\mathrm{P}_{A_{\neg I}^{\uparrow}}$ respectively interpreted by $A_I^{\uparrow}$, $A_I^{\cap}$, $A_{\neg I}^{\cap}$ and $A_{\neg I}^{\uparrow}$, form a graded square of opposition as shown in Fig. 6.*

Therefore, Theorem 2.10 ensures that $\mathrm{P}_{A_I^{\uparrow}}$ and $\mathrm{P}_{A_{\neg I}^{\uparrow}}$ are contraries, $\mathrm{P}_{A_I^{\cap}}$ and $\mathrm{P}_{A_{\neg I}^{\cap}}$ are sub-contraries, $\mathrm{P}_{A_I^{\cap}}$ is subaltern of $\mathrm{P}_{A_I^{\uparrow}}$, $\mathrm{P}_{A_{\neg I}^{\cap}}$ is sub-altern of $\mathrm{P}_{A_{\neg I}^{\uparrow}}$, and $\mathrm{P}_{A_{\neg I}^{\uparrow}}$ and $\mathrm{P}_{A_I^{\cap}}$ as well as $\mathrm{P}_{A_{\neg I}^{\cap}}$ and $\mathrm{P}_{A_I^{\uparrow}}$ are contradictories (see Definition 2.8). On the other hand, the formulas expressing the relations of contrary, sub-contrary and sub-altern carry out an existential import, which allows us to assume that the universe of quantification must be no-empty.

More precisely, putting $A^2(x) = A(x) \otimes A(x)$, we proved that

(i) $A_I^{\uparrow}(y) \otimes A_{\neg I}^{\uparrow}(y) \otimes \bigvee_{x \in X} A^2(x) = 0$, i.e. $\mathrm{P}_{A_I^{\uparrow}}$ and $\mathrm{P}_{A_{\neg I}^{\uparrow}}$ are contraries.
(ii) $A_I^{\cap}(y) \oplus A_{\neg I}^{\cap}(y) \oplus \neg \bigvee_{x \in X} A^2(x) = 1$, i.e. $\mathrm{P}_{A_I^{\cap}}$ and $\mathrm{P}_{A_{\neg I}^{\cap}}$ are sub-contraries.
(iii) $A_I^{\uparrow}(y) \otimes \bigvee_{x \in X} A^2(x) \leq A_I^{\cap}(y)$, i.e. $\mathrm{P}_{A_I^{\cap}}$ is sub-altern of $\mathrm{P}_{A_I^{\uparrow}}$.
(iv) $A_{\neg I}^{\uparrow}(y) \otimes \bigvee_{x \in X} A^2(x) \leq A_{\neg I}^{\cap}(y)$, i.e. $\mathrm{P}_{A_{\neg I}^{\cap}}$ is sub-altern of $\mathrm{P}_{A_{\neg I}^{\uparrow}}$.

Let us underline that our approach generalizes that proposed in [17] where the hypothesis of *normality* of A (there is an element x of X such that $A(x) = 1$) appears instead of the existential import, which here is captured by $\bigvee_{x \in X} A(x)$.

2.3. Graded decagon of opposition in Fuzzy Formal Concept Analysis

In [7,8], we also introduced a new figure of opposition named *graded decagon of opposition* and defined as follows. Before we introduce the definition, let us remark the following.

In the sequel, we will consider 5 positive and 5 negative predicates. Why just 5? The main reason consists in the fact that our concept of quantifier-based operators is closely related to the theory of intermediate quantifiers. Peterson in his book [34] introduced 5 of them, namely “all, almost all, most, many, some”; two are classical (∀, ∃) and three non-classical. This paper stems from the formalization of their theory suggested in [30] and in detail elaborated in several other papers (see the citations). Therefore, we naturally consider also just these five quantifiers. Of course, it is possible to introduce more of them but their number is limited by the linguistic considerations because the definition of intermediate quantifiers

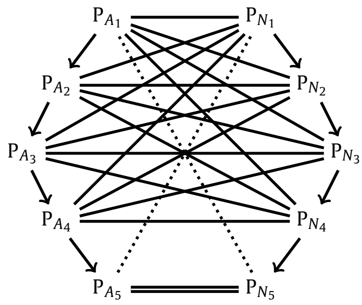

Fig. 7. Graded decagon of opposition.

(and thus also the quantifier-based operators) is based on special evaluative linguistic expressions. From it follows that other quantifiers must be introduced carefully w.r.t. the latter.[4]

Definition 2.11. Let $\mathrm{P}_{A_1}, \dots, \mathrm{P}_{A_5}$, $\mathrm{P}_{N_1}, \dots, \mathrm{P}_{N_5}$ be predicates respectively.[5]

A *graded decagon of opposition* is a decagon where the vertices are the predicates $\mathrm{P}_{A_1}, \dots, \mathrm{P}_{A_5}$, $\mathrm{P}_{N_1}, \dots, \mathrm{P}_{N_5}$ and the following conditions hold:

(i) P_{A_i} and P_{N_j} are contraries, for each $i, j \in \{1, 2, 3, 4\}$,
(ii) P_{A_5} and P_{N_5} are sub-contraries,
(iii) $\mathrm{P}_{A_{i+1}}$ is subaltern of P_{A_i} and $\mathrm{P}_{N_{i+1}}$ is subaltern of P_{N_i}, for each $i \in \{1, 2, 3, 4\}$,
(iv) P_{A_1} and P_{N_5} as well as P_{A_5} and P_{N_1} are contradictories.

A graded decagon of opposition is depicted in Fig. 7.

Let us observe that the notion of graded decagon of opposition generalizes that of graded square of opposition. Indeed, observing Fig. 7, it is clear that the predicates P_{A_1}, P_{A_5}, P_{N_1}, and P_{N_5} form the graded square of opposition depicted in Fig. 5.

Remark 2.12. Sub-diagrams of the decagon in Fig. 7 can be obtained by considering two or three intermediate levels less. More precisely, for each $i \in \{2, 3, 4\}$, P_{A_1}, P_{A_i}, P_{A_5}, P_{N_1}, P_{N_i}, and P_{N_5} form a graded hexagon of opposition, and for each $i, j \in \{2, 3, 4\}$ with $i \neq j$, P_{A_1}, P_{A_i}, P_{A_j}, P_{A_5}, P_{N_1}, P_{N_i}, P_{N_j}, and P_{N_5} form a graded octagon of opposition.

A collection of graded decagons of opposition is determined starting from a fuzzy formal context (X, Y, I). In achieving this goal, we employed the operators given by (1) and the quantifier-based operators introduced in our previous papers to create extended fuzzy concepts.[6]

A special operation called *cut of a fuzzy set* is fundamental for defining quantifier-based operators:

Let $A, B \underset{\sim}{\subset} X$. The *cut* of A with respect to B is the fuzzy set

$$(A|B)(x) = \begin{cases} A(x), & \text{if } A(x) = B(x), \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The latter is motivated by the need to form a new fuzzy set from a given one by extracting several elements together with their membership degrees and putting the other membership degrees equal to 0.

For defining quantifier-based operators, we also need to consider a special *measure* that expresses *how large the size of a fuzzy set is w.r.t. the size of another one*.

[4] In strict mathematical sense, we might consider even infinite number of the quantifier-based operators because the semantics of evaluative expressions is defined over the interval $[0, 1]$. However, this would hardly have any sense for two reasons: first, we would have to forget natural language and so, such operators would have no meaning, they would be just mathematical construction. Second, the concept analysis is the technique dealing with data which are never infinite.

[5] To consider a decagon of opposition (so, a figure having ten vertices and five sides), we need the fuzzy sets $A_1, \dots, A_5$ and $N_1, \dots, N_5$. We use two types of letters (i.e. A_i and N_i) because we have to distinguish the left vertices of the decagon (i.e. $\mathrm{P}_{A_1}, \dots, \mathrm{P}_{A_5}$) from those on the right (i.e. $\mathrm{P}_{N_1}, \dots, \mathrm{P}_{N_5}$). Indeed, at the end of this section, we will exhibit a concrete decagon of opposition where vertices on the left side and the right side have a different meaning: the first ones correspond to *positive operators* (they describe the presence of a property in the objects), and the second ones correspond to *negative operators* (they describe the absence of a property in the objects).

[6] The next notions are based on the standard Łukasiewicz MV-algebra. Moreover, in our approach, the universe X is the collection of the objects of a fuzzy formal context. Hence, it must be finite.

Table 2
Verbal description of $A^{\uparrow}_{I,\mathsf{BiEx}}(y)$, $A^{\uparrow}_{I,\mathsf{BiVe}}(y)$, $A^{\uparrow}_{I,\neg\mathsf{Sm}}(y)$, $A^{\uparrow}_{\neg I,\mathsf{BiEx}}(y)$, $A^{\uparrow}_{\neg I,\mathsf{BiVe}}(y)$, and $A^{\uparrow}_{\neg I,\neg\mathsf{Sm}}(y)$.

Truth degree	*Statement*
$A^{\uparrow}_{I,\mathsf{BiEx}}(y)$	y is shared by *almost all* objects of A
$A^{\uparrow}_{I,\mathsf{BiVe}}(y)$	y is shared by *most* objects of A
$A^{\uparrow}_{I,\neg\mathsf{Sm}}(y)$	y is shared by *many* objects of A
$A^{\uparrow}_{\neg I,\mathsf{BiEx}}(y)$	y is not shared by *almost all* objects of A
$A^{\uparrow}_{\neg I,\mathsf{BiVe}}(y)$	y is not shared by *most* objects of A
$A^{\uparrow}_{\neg I,\neg\mathsf{Sm}}(y)$	y is not shared by *many* objects of A

Let $A, B \underset{\sim}{\subseteq} X$,

$$\mu_B(A) = \begin{cases} 1, & \text{if } B = \emptyset \text{ or } A = B, \\ \dfrac{|A|}{|B|}, & B \neq \emptyset \text{ and } A \subseteq B, \\ 0, & \text{otherwise}, \end{cases}$$

where $|A| = \sum_{x\in X} A(x)$ and $|B| = \sum_{x\in X} B(x)$ represent the *cardinalities* of the fuzzy sets A and B, respectively.

The quantifier-based operators are defined as follows by using the evaluative linguistic expressions $\neg\mathsf{Sm}$, BiVe, and BiEx, which respectively model the linguistic expressions *not small*, *very big*, and *extremely big*. Their meaning (extensions) are fuzzy sets on $[0, 1]$ in the standard context $\langle 0, 0.5, 1\rangle$. Formulas for their computation can be found in [8], and in more details also in [29,31].

Definition 2.13. Let $A \underset{\sim}{\subseteq} X$ and let $E \in \{\neg\mathsf{Sm}, \mathsf{BiVe}, \mathsf{BiEx}\}$. Then

$$A^{\uparrow}_{I}(y) = \bigvee_{Z \underset{\sim}{\subset} X} \Big(\bigwedge_{x\in X} ((A|Z)(x) \to I(x,y)) \wedge E(\mu_A(A|Z))\Big) \tag{3}$$

$$A^{\uparrow}_{\neg I,E}(y) = \bigvee_{Z \underset{\sim}{\subset} X} \Big(\bigwedge_{x\in X} ((A|Z)(x) \to \neg I(x,y)) \wedge E(\mu_A(A|Z))\Big) \tag{4}$$

for each $y \in Y$.

Let us notice that the quantifier-based operators given by (3) and (4) are constructed by applying $^{\uparrow}_{I}$ and $^{\uparrow}_{\neg I}$ on the cuts of A. The latter represent universes of quantification smaller than A and their size is characterized by an evaluative expression from $\{\neg\mathsf{Sm}, \mathsf{BiVe}, \mathsf{BiEx}\}$. Hence, it is easy to see that $A^{\uparrow}_{I,E}(y)$ is the truth degree to which:

There exists a cut of A such that "all its objects have the attribute y" and "its size is E (not small, very big or extremely big) w.r.t. the size of A",

and $A^{\uparrow}_{\neg I,E}(y)$ is the truth degree to which:

There exists a cut of A such that "all its objects do not have the attribute y" and "its size is E (not small, very big or extremely big) w.r.t. the size of A".

The existing correlation between evaluative linguistic expressions and intermediate quantifiers suggests us to rewrite the meaning of $A^{\uparrow}_{I,\mathsf{BiEx}}(y)$, $A^{\uparrow}_{I,\mathsf{BiVe}}(y)$, $A^{\uparrow}_{I,\neg\mathsf{Sm}}(y)$, $A^{\uparrow}_{\neg I,\mathsf{BiEx}}(y)$, and $A^{\uparrow}_{\neg I,\mathsf{BiVe}}(y)$ as in Table 2.

Using the operators defined by (1), (3), and (4), we obtain predicates that can be organized in a graded decagon of opposition, as shown by the following theorem whose proof is found in [8].

Theorem 2.14. *Let (X, Y, I) be a fuzzy context and let $A \underset{\sim}{\subseteq} X$. Then, the predicates $\mathrm{P}_{A^{\uparrow}_{I}}$, $\mathrm{P}_{A^{\uparrow}_{I,\mathsf{BiEx}}}$, $\mathrm{P}_{A^{\uparrow}_{I,\mathsf{BiVe}}}$, $\mathrm{P}_{A^{\uparrow}_{I,\neg\mathsf{Sm}}}$, $\mathrm{P}_{A^{\cap}_{I}}$, $\mathrm{P}_{A^{\uparrow}_{\neg I}}$, $\mathrm{P}_{A^{\uparrow}_{\neg I,\mathsf{BiEx}}}$, $\mathrm{P}_{A^{\uparrow}_{\neg I,\mathsf{BiVe}}}$, $\mathrm{P}_{A^{\uparrow}_{\neg I,\neg\mathsf{Sm}}}$, and $\mathrm{P}_{A^{\cap}_{\neg I}}$ form a graded decagon of opposition as depicted in Fig. 8.*

The formulas involving the quantifier-based operators and corresponding to the relations of contrary, sub-contrary, and subaltern established by Theorem 2.14 carry out an existential import. Indeed, we proved that

(i) $\bigvee_{Z \underset{\sim}{\subset} X}((A|Z)^{\uparrow}_{I}(y) \wedge E(\mu_A(A|Z)) \otimes \bigvee_{x\in X}(A|Z)^2(x)) \leq A^{\cap}_{I}(y)$, i.e. $\mathrm{P}_{A^{\cap}_{I}}$ is subaltern of $\mathrm{P}_{A^{\uparrow}_{I,E}}$.

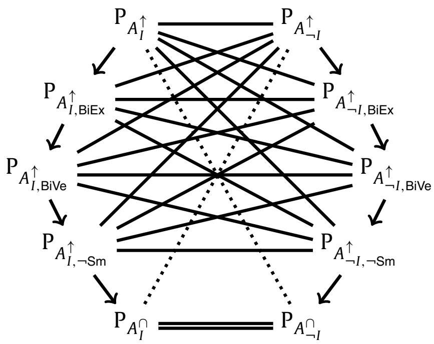

Fig. 8. Graded decagon of opposition with quantifier-based and concept-forming operators.

(ii) $\bigvee_{Z \underset{\sim}{\subset} X}((A|Z)^{\uparrow}_{\neg I}(y) \wedge E(\mu_A(A|Z)) \otimes \bigvee_{x\in X}(A|Z)^2(x)) \leq A^{\cap}_{\neg I}(y)$, i.e. $P_{A^{\cap}_{\neg I}}$ is subaltern of $P_{A^{\uparrow}_{\neg I,E}}$.

(iii) $\bigvee_{Z_1 \underset{\sim}{\subset} X} \bigvee_{Z_2 \underset{\sim}{\subset} X} \left((A|Z_1)^{\uparrow}_{I}(y) \wedge E_1(\mu_A(A|Z_1))\right) \otimes \left(((A|Z_2)^{\uparrow}_{\neg I}(y) \wedge E_2(\mu_A(A|Z_2))\right) \otimes \bigvee_{x\in X}((A|Z_1)(x) \otimes (A|Z_2)(x)) = 0$, i.e. $P_{A^{\uparrow}_{I,E_1}}$ and $P_{A^{\uparrow}_{\neg I,E_2}}$ are contraries.

(iv) $\bigvee_{Z \underset{\sim}{\subset} X}(((A|Z)^{\uparrow}_{I}(y) \wedge E(\mu_A(A|Z))) \otimes \bigvee_{x\in X}(A|Z)^2(x)) \otimes A^{\uparrow}_{\neg I}(y) = 0$, i.e. $P_{A^{\uparrow}_{I,E}}$ and $P_{A^{\uparrow}_{\neg I}}$ are contraries.

(v) $\bigvee_{Z \underset{\sim}{\subset} X}(((A|Z)^{\uparrow}_{\neg I}(y) \wedge E(\mu_A(A|Z))) \otimes \bigvee_{x\in X}(A|Z)^2(x)) \otimes A^{\uparrow}_{I}(y) = 0$., i.e. $P_{A^{\uparrow}_{I}}$ and $P_{A^{\uparrow}_{\neg I,E}}$ are contraries.

Remark 2.15. According to Remark 2.12, let $E \in \{\mathit{BiEx}, \mathit{BiVe}, \neg\mathit{Sm}\}$, then $P_{A^{\uparrow}_{I}}$, $P_{A^{\uparrow}_{I,E}}$, $P_{A^{\cap}_{I}}$, $P_{A^{\uparrow}_{\neg I}}$, $P_{A^{\uparrow}_{\neg I,E}}$, and $P_{A^{\cap}_{\neg I}}$ form a graded hexagon of opposition. Moreover, let $E_1, E_2 \in \{\mathit{BiEx}, \mathit{BiVe}, \neg\mathit{Sm}\}$ with $E_1 \neq E_2$, then $P_{A^{\uparrow}_{I}}$, $P_{A^{\uparrow}_{I,E_1}}$, $P_{A^{\uparrow}_{I,E_2}}$, $P_{A^{\cap}_{I}}$, $P_{A^{\uparrow}_{\neg I}}$, $P_{A^{\uparrow}_{\neg I,E_1}}$, $P_{A^{\uparrow}_{\neg I,E_2}}$, and $P_{A^{\cap}_{\neg I}}$ form a graded octagon of opposition.

3. Graded cube of opposition with concept-forming operators

In this section, we construct graded cubes of opposition using the operators provided in Subsection 2.2. Note that a graded cube of opposition can be viewed as an extension of the square given by Definition 2.8.

Definition 3.1. Let P_{A_1}, P_{A_2}, P_{N_1}, P_{N_2}, P_{a_1}, P_{a_2}, P_{n_1}, P_{n_2} be predicates. A *graded cube of opposition* is a cube where the vertices are the predicates P_{A_1}, P_{A_2}, P_{N_1}, P_{N_2}, P_{a_1}, P_{a_2}, P_{n_1}, P_{n_2} and the following conditions hold:

(i) P_{A_1} and P_{N_1} are contraries as well as P_{a_1} and P_{n_1}, P_{a_1} and P_{N_1}, P_{A_1} and P_{n_1}.
(ii) P_{A_2} and P_{N_2} are sub-contraries as well as P_{a_2} and P_{n_2}, P_{a_2} and P_{N_2}, P_{A_2} and P_{n_2}.
(iii) P_{A_2} is subaltern of P_{A_1}, P_{N_2} is subaltern of P_{N_1}, P_{A_2} is subaltern of P_{a_1}, P_{a_2} is subaltern of P_{A_1}, P_{N_2} is subaltern of P_{n_1}, P_{a_2} is subaltern of P_{a_1}, P_{n_2} is subaltern of P_{n_1}, and P_{n_2} is subaltern of P_{N_1}.
(iv) P_{A_1} and P_{N_2} are contradictories as well as P_{A_2} and P_{N_1}, P_{a_1} and P_{n_2}, P_{a_2} and P_{n_1}.

A graded cube of opposition is depicted in Fig. 9.

Let us observe that the notion of cube of opposition generalizes the graded square of opposition. Indeed, observing Fig. 9, it is clear that the predicates P_{A_1}, P_{A_2}, P_{N_1}, P_{N_2} as well as P_{a_1}, P_{a_2}, P_{n_1}, P_{n_2} form a graded square of opposition (see Fig. 5).

In the sequel, we prove a list of lemmas, which will allows us to construct graded cubes of opposition with the concept-forming operators of Subsection 2.2.

Lemma 3.2. *Let (X, Y, I) be a fuzzy formal context and let $A \underset{\sim}{\subset} X$. Then, the predicates $P_{(\neg A)^{\uparrow}_{\neg I}}$, $P_{(\neg A)^{\cap}_{\neg I}}$, $P_{(\neg A)^{\cap}_{I}}$, $P_{(\neg A)^{\uparrow}_{I}}$ form a graded square of opposition as shown in Fig. 10.*[7]

Proof. This follows from Theorem 2.10. Indeed, we can obtain a new square of opposition by substituting, in the square of Fig. 6, A and I by $\neg A$ and $\neg I$, respectively. □

[7] If $A \underset{\sim}{\subset} X$ then $\neg A$ is a new fuzzy set on X with the membership function $\neg A(x) = 1 - A(x)$ for each $x \in X$.

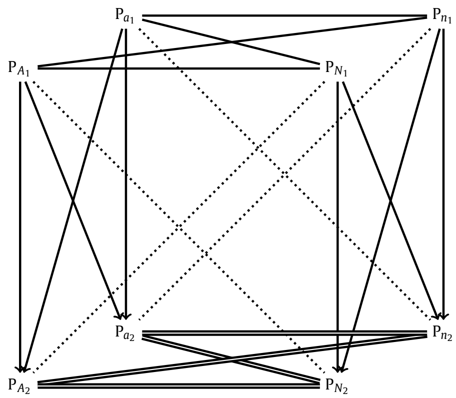

Fig. 9. Graded cube of opposition.

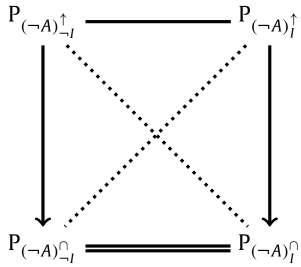

Fig. 10. Graded square of opposition with the predicates $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I}}$, $\mathrm{P}_{(\neg A)^{\cap}_{\neg I}}$, $\mathrm{P}_{(\neg A)^{\cap}_{I}}$, and $\mathrm{P}_{(\neg A)^{\uparrow}_{I}}$.

Lemma 3.3. *Let (X, Y, I) be a fuzzy formal context. Let $A \underset{\sim}{\subset} X$ and let $\mathrm{P}_{A^{\uparrow}_{I}}$ $\mathrm{P}_{(\neg A)^{\uparrow}_{I}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I}}$, $\mathrm{P}_{A^{\uparrow}_{\neg I}}$ be predicates. Then the following holds true:*

(a) $\mathrm{P}_{A^{\uparrow}_{I}}$ *and* $\mathrm{P}_{(\neg A)^{\uparrow}_{I}}$ *are contraries,*
(b) $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I}}$ *and* $\mathrm{P}_{A^{\uparrow}_{\neg I}}$ *are contraries.*

Proof. (a) Let $y \in Y$, by Lemma 2.5, (i)

$$A(x) \to I(x, y) \leq \neg I(x, y) \to \neg A(x)$$

as well as

$$\neg A(x) \to I(x, y) \leq \neg I(x, y) \to A(x)$$

for each $x \in X$.

By the adjunction property, we have

$$(A(x) \to I(x, y)) \otimes \neg I(x, y) \leq \neg A(x)$$

as well as

$$(\neg A(x) \to I(x, y)) \otimes \neg I(x, y) \leq A(x).$$

Then, by Lemma 2.5 (g) and by the associativity and the commutativity of $\otimes$,

$$(A(x) \to I(x, y)) \otimes (\neg A(x) \to I(x, y)) \otimes (\neg I(x, y))^2 \leq A(x) \otimes \neg A(x)$$

where $(\neg I(x, y))^2 = \neg I(x, y) \otimes \neg I(x, y)$.

This means that

$$(A(x) \to I(x, y)) \otimes (\neg A(x) \to I(x, y)) \otimes (\neg I(x, y))^2 \leq 0$$

from Lemma 2.5 (e).

Also, we get

$$(A(x) \to I(x, y)) \le (\neg A(x) \to I(x, y)) \to ((\neg I(x, y))^2 \to 0),$$

for each $x \in X$, by applying the adjunction property twice.

By Lemma 2.5 (c),

$$\bigwedge_{x\in X} (A(x) \to I(x, y)) \le \bigwedge_{x\in X} ((\neg A(x) \to I(x, y)) \to ((\neg I(x, y))^2 \to 0)).$$

By Lemma 2.6 (a), (b), we obtain

$$\bigwedge_{x\in X} (A(x) \to I(x, y)) \le \bigwedge_{x\in X} (\neg A(x) \to I(x, y)) \to (\bigvee_{x\in X} (\neg I(x, y))^2 \to 0).$$

Using the adjunction property again, we conclude that

$$\bigwedge_{x\in X} (A(x) \to I(x, y)) \otimes \bigwedge_{x\in X} (\neg A(x) \to I(x, y)) \otimes \bigvee_{x\in X} (\neg I(x, y))^2 = 0.$$

Namely, $A_I^{\uparrow}(y) \otimes (\neg A)_I^{\uparrow}(y) \otimes \bigvee_{x\in X} (\neg I(x, y))^2 = 0$.

(b) Analogous to item (a), we can prove that

$$\bigwedge_{x\in X} (\neg A(x) \to \neg I(x, y)) \otimes \bigwedge_{x\in X} (A(x) \to \neg I(x, y)) \otimes \bigvee_{x\in X} I(x, y)^2 = 0.$$

Namely, $(\neg A)_{\neg I}^{\uparrow}(y) \otimes A_{\neg I}^{\uparrow}(y) \otimes \bigvee_{x\in X} I(x, y)^2 = 0$.

Remark 3.4. Observe that, in every model of classical predicate logic, when the set B interpreting the formula $B(x)$ is equal to the universe X, i.e. its complement $X \setminus B$ interpreting $\neg B(x)$ is empty, the formulas $(\forall x)(A(x) \Rightarrow B(x))$ and $(\forall x)(\neg A(x) \Rightarrow B(x))$ are both true. Consequently, these are not contraries. For such reason, the contrary requires to assume *the existence of at least an element in* $X \setminus B$, i.e. *the existence of at least an element that does not belong to* B. In mathematical logic, as we mentioned above, this is called *existential import* and it can be captured by the formula $(\exists x)\neg B(x)$. Since in fuzzy logic the existential quantifier $\exists$ is interpreted by the supremum $\vee$, in order to prove that $\mathrm{P}_{A_I^{\uparrow}}$ and $\mathrm{P}_{(\neg A)_I^{\uparrow}}$ are contraries, we need to verify that

$$A_I^{\uparrow}(y) \otimes (\neg A)_I^{\uparrow}(y) \otimes \bigvee_{x\in X} (\neg I(x, y))^2 = 0 \quad \text{for each } y \in Y.$$

Let us recall that we have already addressed the presupposition in the last publication [8]. Furthermore, another group of authors elaborated a presupposition, not only from a logical but also from a philosophical point of view (see [1,2,22,36]).

Analogously, the following condition

$$(\neg A)_{\neg I}^{\uparrow}(y) \otimes A_{\neg I}^{\uparrow}(y) \otimes \bigvee_{x\in X} I(x, y)^2 = 0,$$

where $\bigvee_{x\in X} I(x, y)^2$ plays the role of existential import, must hold to ensure that $\mathrm{P}_{(\neg A)_{\neg I}^{\uparrow}}$ and $\mathrm{P}_{A_{\neg I}^{\uparrow}}$ are contraries. □

Lemma 3.5. *Let (X, Y, I) be a fuzzy formal context. Let $A \subseteq X$ and let $\mathrm{P}_{A_{\neg I}^{\cap}}$, $\mathrm{P}_{(\neg A)_{\neg I}^{\cap}}$, $\mathrm{P}_{A_I^{\cap}}$, and $\mathrm{P}_{(\neg A)_I^{\cap}}$ be the predicates interpreted by $A_{\neg I}^{\cap}$, $(\neg A)_{\neg I}^{\cap}$, $A_I^{\cap}$, and $(\neg A)_I^{\cap}$, respectively. Then,*

(a) $\mathrm{P}_{A_{\neg I}^{\cap}}$ and $\mathrm{P}_{(\neg A)_{\neg I}^{\cap}}$ are sub-contraries,
(b) $\mathrm{P}_{A_I^{\cap}}$ and $\mathrm{P}_{(\neg A)_I^{\cap}}$ are sub-contraries.

Proof. (a) Let $y \in Y$, by Lemma 3.3 (a),

$$\bigwedge_{x\in X} (A(x) \to I(x, y)) \otimes \bigwedge_{x\in X} (\neg A(x) \to I(x, y)) \otimes \bigvee_{x\in X} (\neg I(x, y))^2 = 0.$$

Then,

$$\neg(\bigwedge_{x\in X} (A(x) \to I(x, y)) \otimes \bigwedge_{x\in X} (\neg A(x) \to I(x, y)) \otimes \bigvee_{x\in X} (\neg I(x, y))^2) = 1.$$

By Lemma 2.5 (j),

$$\neg \bigwedge_{x\in X}(A(x) \rightarrow I(x,y)) \oplus \neg \bigwedge_{x\in X}(\neg A(x) \rightarrow I(x,y)) \oplus \neg \bigvee_{x\in X}(\neg I(x,y))^2 = 1.$$

Finally,

$$\bigvee_{x\in X}(A(x) \otimes \neg I(x,y)) \oplus \bigvee_{x\in X}(\neg A(x) \otimes \neg I(x,y)) \oplus \neg \bigvee_{x\in X}(\neg I(x,y))^2 = 1$$

from Lemma 2.5 (h) and De Morgan's law ($\neg(a \wedge b) = \neg a \vee \neg b$, for each $a, b \in [0,1]$).
Namely, $A^{\cap}_{\neg I}(y) \oplus (\neg A)^{\cap}_{\neg I}(y) \oplus \neg \bigvee_{x\in X}(\neg I(x,y))^2 = 1$.

(b) Analogous to item (a), we can prove that

$$\bigvee_{x\in X}(A(x) \otimes I(x,y)) \oplus \bigvee_{x\in X}(\neg A(x) \otimes I(x,y)) \oplus \neg \bigvee_{x\in X} I(x,y)^2 = 1.$$

Namely, $A^{\cap}_{I}(y) \oplus (\neg A)^{\cap}_{I}(y) \oplus \neg \bigvee_{x\in X} I(x,y)^2 = 1$. □

Remark 3.6. In a model of classical predicate logic, when the set $X \setminus B$ interpreting the formula $\neg B(x)$ is empty then the formulas $(\exists x)(A(x) \wedge \neg B(x))$ and $(\exists x)(\neg A(x) \wedge \neg B(x))$ are both false. Consequently they are not sub-contraries. Therefore, also in this case the existential import is necessary. Since we deal with fuzzy sets, we have shown that

$$A^{\cap}_{\neg I}(y) \oplus (\neg A)^{\cap}_{\neg I}(y) \oplus \neg \bigvee_{x\in X}(\neg I(x,y))^2 = 1 \text{ for each } y \in Y,$$

in order to prove that $\mathrm{P}_{A^{\cap}_{\neg I}}$ and $\mathrm{P}_{(\neg A)^{\cap}_{\neg I}}$ are sub-contraries.
Analogously, the relation of sub-contrary of $\mathrm{P}_{A^{\cap}_{I}}$ and $\mathrm{P}_{(\neg A)^{\cap}_{I}}$ is expressed by

$$A^{\cap}_{I}(y) \oplus (\neg A)^{\cap}_{I}(y) \oplus \neg \bigvee_{x\in X} I(x,y)^2 = 1$$

where the subformula $\neg \bigvee_{x\in X} I(x,y)^2$ plays the role of existential import.

Lemma 3.7. *Let (X, Y, I) be a fuzzy formal context. Let $A \underset{\sim}{\subset} X$ and let $\mathrm{P}_{(\neg A)^{\cap}_{\neg I}}$, $\mathrm{P}_{A^{\uparrow}_{I}}$, $\mathrm{P}_{A^{\cap}_{I}}$, and $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I}}$ be predicates. Then*

(a) $\mathrm{P}_{(\neg A)^{\cap}_{\neg I}}$ *is subaltern of* $\mathrm{P}_{A^{\uparrow}_{I}}$,
(b) $\mathrm{P}_{A^{\cap}_{I}}$ *is subaltern of* $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I}}$.

Proof. (a) Let $y \in Y$, by Lemma 2.5(i),

$$A(x) \rightarrow I(x,y) \le \neg I(x,y) \rightarrow \neg A(x)$$

for each $x \in X$.
By Lemma 2.5(b),

$$A^{\uparrow}_{I}(y) \le \neg I(x,y) \rightarrow \neg A(x).$$

From the adjunction property, we have $A^{\uparrow}_{I}(y) \otimes \neg I(x,y) \le \neg A(x)$.
By Lemma 2.5 (g),

$$A^{\uparrow}_{I}(y) \otimes \neg I(x,y)^2 \le \neg A(x) \otimes \neg I(x,y).$$

By Lemma 2.5 (c), (d),

$$A^{\uparrow}_{I}(y) \otimes \bigvee_{x\in X} \neg I(x,y)^2 \le \bigvee_{x\in X}(\neg A(x) \otimes \neg I(x,y)).$$

Namely, $A^{\uparrow}_{I}(y) \otimes \bigvee_{x\in X} \neg I(x,y)^2 \le (\neg A)^{\cap}_{\neg I}(y)$.

(b) Analogous to item (a), we can prove that

$$\bigwedge_{x\in X}(\neg A(x) \rightarrow \neg I(x,y)) \otimes \bigvee_{x\in X} I(x,y)^2 \le \bigvee_{x\in X}(A(x) \otimes I(x,y)).$$

Namely, $(\neg A)^{\uparrow}_{\neg I}(y) \otimes \bigvee_{x\in X} I(x,y)^2 \le A^{\cap}_{I}(y)$. □

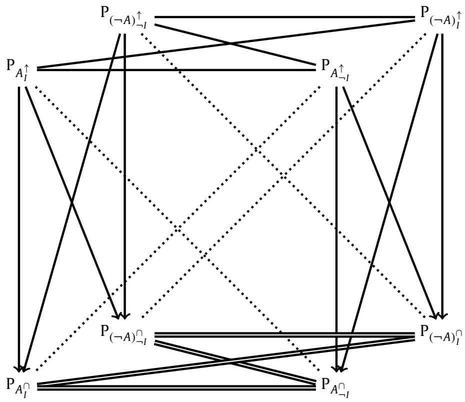

Fig. 11. Graded cube of opposition with the concept-forming operators.

Lemma 3.8. *Let (X, Y, I) be a fuzzy formal context. Let $A \underset{\sim}{\subset} X$ and let $\mathrm{P}_{A^{\cap}_{\neg I}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{I}}$, $\mathrm{P}_{(\neg A)^{\cap}_{I}}$, and $\mathrm{P}_{A^{\uparrow}_{\neg I}}$ be the predicates interpreted by $A^{\cap}_{\neg I}$, $(\neg A)^{\uparrow}_{I}$, $(\neg A)^{\cap}_{I}$, and $A^{\uparrow}_{\neg I}$, respectively. Then,*

(a) $\mathrm{P}_{A^{\cap}_{\neg I}}$ *is subaltern of* $\mathrm{P}_{(\neg A)^{\uparrow}_{I}}$,
(b) $\mathrm{P}_{(\neg A)^{\cap}_{I}}$ *is subaltern of* $\mathrm{P}_{A^{\uparrow}_{\neg I}}$.

Proof. (a) Analogous to the proof of Lemma 3.7 (a), we can prove that

$$\bigwedge_{x\in X}(\neg A(x) \to I(x,y)) \otimes \bigvee_{x\in X}(\neg I(x,y))^2 \le \bigvee_{x\in X}(A(x) \otimes \neg I(x,y)).$$

Namely, $(\neg A)^{\uparrow}_{I}(y) \otimes \bigvee_{x\in X}(\neg I(x,y))^2 \le A^{\cap}_{\neg I}(y)$.

(b) Analogous to the proof of Lemma 3.7 (a), we can prove that

$$\bigwedge_{x\in X}(A(x) \to \neg I(x,y)) \otimes \bigvee_{x\in X} I(x,y)^2 \le \bigvee_{x\in X}(\neg A(x) \otimes I(x,y)).$$

Namely, $A^{\uparrow}_{\neg I}(y) \otimes \bigvee_{x\in X} I(x,y)^2 \le (\neg A)^{\cap}_{I}(y)$. $\square$

Remark 3.9. Also for the subaltern relations given by Lemma 3.7 and Lemma 3.8, an existential import is necessary. This is represented by the subformula $\bigvee_{x\in X}(\neg I(x,y))^2$ for item (a), and by the subformula $\bigvee_{x\in X} I(x,y)^2$ for item (b).

We now have enough tools to construct graded cubes of opposition using the concept-forming operators. Indeed, the following theorem can be proved.

Theorem 3.10. *Let (X, Y, I) be a fuzzy context and let $A \underset{\sim}{\subset} X$. Then, the predicates $\mathrm{P}_{A^{\uparrow}_{I}}$, $\mathrm{P}_{A^{\uparrow}_{\neg I}}$, $\mathrm{P}_{A^{\cap}_{I}}$, $\mathrm{P}_{A^{\cap}_{\neg I}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{I}}$, $\mathrm{P}_{(\neg A)^{\cap}_{\neg I}}$, and $\mathrm{P}_{(\neg A)^{\cap}_{I}}$ form a graded cube of opposition as shown in Fig. 11.*

Proof. The theorem immediately follows from Theorems 2.10 and 2.14, and Lemmas 3.2, 3.3, 3.5, and 3.8. $\square$

Remark 3.11. It is important to underline that the cube from Fig. 11 can be understood as an extension of the cube of opposition constructed using the concept-forming operators in the standard formal concept analysis. In fact, let (X, Y, I) be a formal context such that $I \subseteq X \times Y$ and $A \subseteq X$. Then, $A^{\uparrow}_{I}$, $A^{\uparrow}_{\neg I}$, $A^{\cap}_{I}$, $A^{\cap}_{\neg I}$, $(\neg A)^{\uparrow}_{\neg I}$, $(\neg A)^{\uparrow}_{I}$, $(\neg A)^{\cap}_{\neg I}$, and $(\neg A)^{\cap}_{I}$, are classical sets. More precisely, we get

$$A^{\uparrow}_{I} = \{y \in Y \mid \forall x \in A, \quad (x,y) \in I\} \text{ and } A^{\cap}_{I} = \{y \in Y \mid \exists x \in A \mid (x,y) \in I\}.$$

Moreover, if $(X \times Y) \setminus I = \{(x,y) \in X \times Y \mid (x,y) \notin I\}$ and $X \setminus A$ denotes the complement of A w.r.t. X, then $A^{\uparrow}_{\neg I} = A^{\uparrow}_{(X\times Y)\setminus I}$, $A^{\cap}_{\neg I} = A^{\cap}_{(X\times Y)\setminus I}$, $(\neg A)^{\uparrow}_{\neg I} = (X \setminus A)^{\uparrow}_{(X\times Y)\setminus I}$, $(\neg A)^{\cap}_{I} = (X \setminus A)^{\cap}_{I}$, $(\neg A)^{\cap}_{\neg I} = (X \setminus A)^{\cap}_{(X\times Y)\setminus I}$, and $(\neg A)^{\uparrow}_{I} = (X \setminus A)^{\uparrow}_{I}$. Then, our cube of opposition coincides with that presented in [12,14].

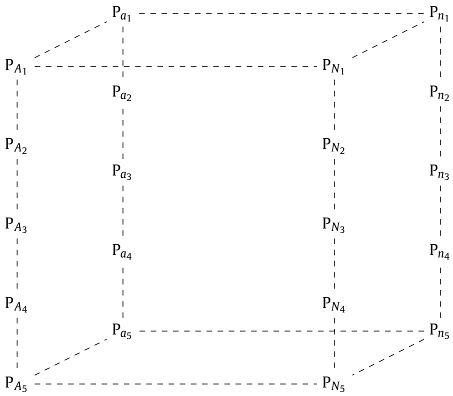

Fig. 12. Predicates on the surface of a 5-graded cube of opposition.

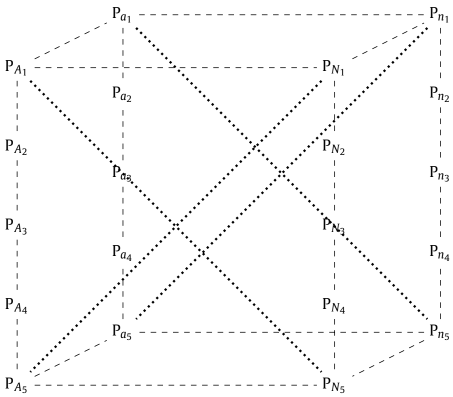

Fig. 13. Relations of contradictory on a 5-graded cube of opposition.

Remark 3.12. Fixing $y \in Y$ and assuming the so-called *normalization conditions* (there exist $x_1, x_2, x_3, x_4 \in X$ such that $A(x_1) = 1$, $A(x_2) = 0$, $I(x_3, y) = 1$, and $I(x_4, y) = 0$), we obtain an example of graded cube of opposition as defined in [16]. Therefore, we here adopt a more generic approach, where the normalization condition is replaced by the "existential import".

4. 5-Graded cube of opposition with concept-forming and quantifier-based operators

In this section, we construct a 5-graded cube of opposition with the concept-forming and quantifier-based operators. A 5-graded cube of opposition can be understood as an extension both of the graded cube and the graded decagon described above.

Definition 4.1. Let $\mathrm{P}_{A_1}, \ldots, \mathrm{P}_{A_5}$, $\mathrm{P}_{N_1}, \ldots, \mathrm{P}_{N_5}$, $\mathrm{P}_{a_1}, \ldots, \mathrm{P}_{a_5}$, $\mathrm{P}_{n_1}, \ldots, \mathrm{P}_{n_5}$ be predicates. A *5-graded cube of opposition* is a cube such that the predicates $\mathrm{P}_{A_1}, \ldots, \mathrm{P}_{A_5}$, $\mathrm{P}_{N_1}, \ldots, \mathrm{P}_{N_5}$, $\mathrm{P}_{a_1}, \ldots, \mathrm{P}_{a_5}$, $\mathrm{P}_{n_1}, \ldots, \mathrm{P}_{n_5}$ are organized on its surface as shown in Fig. 12. Moreover, the following properties hold:

(i) P_{A_1} and P_{N_5} are contradictories as well as P_{N_1} and P_{A_5}, P_{a_1} and P_{n_5}, P_{a_5} and P_{n_1} (see Fig. 13).
(ii) P_{A_5} and P_{N_5} are sub-contraries as well as P_{a_5} and P_{n_5}, P_{A_5} and P_{n_5}, P_{N_5} and P_{a_5} (see Fig. 14).
(iii) P_{A_5} is subaltern of P_{a_1}, P_{a_5} is subaltern of P_{A_1}, P_{N_5} is subaltern of P_{n_1}, P_{n_5} is subaltern of P_{N_1}, $\mathrm{P}_{A_{i+1}}$ is subaltern of P_{A_i} and $\mathrm{P}_{N_{i+1}}$ subaltern of P_{N_i}, for each $i \in \{1, 2, 3, 4\}$ (see Fig. 15).
(iv) P_{A_i} and P_{N_j} are contraries for each $i, j \in \{1, 2, 3, 4\}$ (see Fig. 16).
(v) P_{a_i} and P_{n_j} are contraries for each $i, j \in \{1, 2, 3, 4\}$ (see Fig. 17).
(vi) P_{A_i} and P_{n_j} are contraries for each $i, j \in \{1, 2, 3, 4\}$ (see Fig. 18).
(vii) P_{a_i} and P_{N_j} are contraries for each $i, j \in \{1, 2, 3, 4\}$ (see Fig. 19).

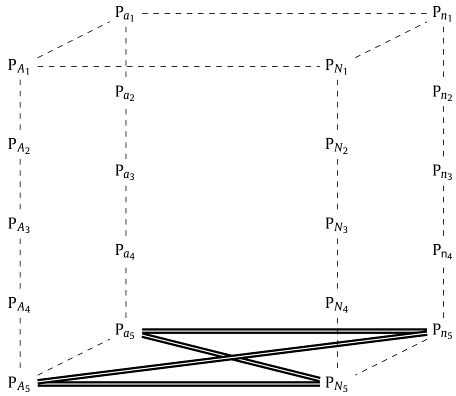

Fig. 14. Relations of sub-contrary on a 5-graded cube of opposition.

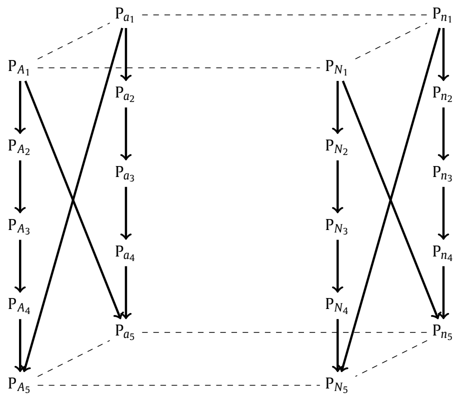

Fig. 15. Relations of subaltern on a 5-graded cube of opposition.

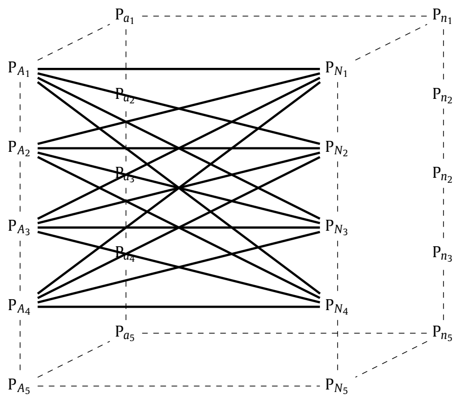

Fig. 16. Relations of contrary on a 5-graded cube of opposition.

Remark 4.2. Notice that Definition 4.1 implies that the predicates $\mathrm{P}_{A_1}, \ldots, \mathrm{P}_{A_5}, \mathrm{P}_{N_1}, \ldots, \mathrm{P}_{N_5}$ as well as $\mathrm{P}_{a_1}, \ldots, \mathrm{P}_{a_5}, \mathrm{P}_{n_1}, \ldots, \mathrm{P}_{n_5}$ form a graded decagon of opposition (see Definition 2.11). Furthermore, the predicates P_{A_1}, P_{A_5}, P_{A_5}, P_{N_5}, P_{a_1}, P_{n_1}, P_{a_5}, P_{n_5} form a graded cube of opposition (see Definition 3.1).

The following lemmas will be used below.

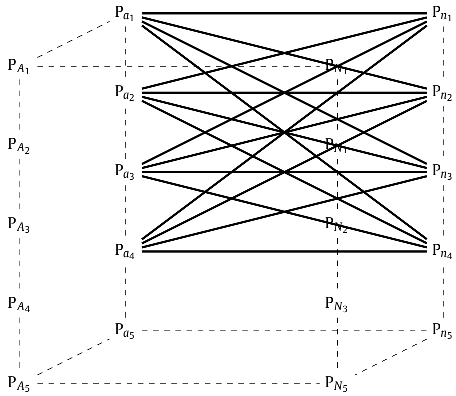

Fig. 17. Relations of contrary on a 5-graded cube of opposition.

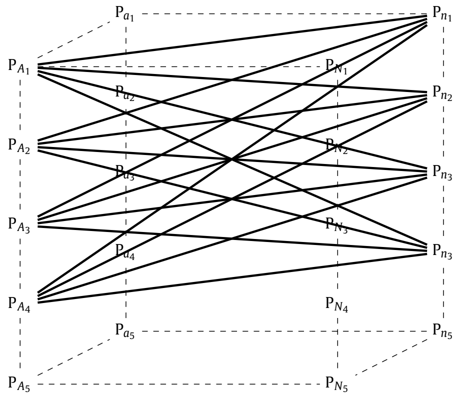

Fig. 18. Relations of contrary on a 5-graded cube of opposition.

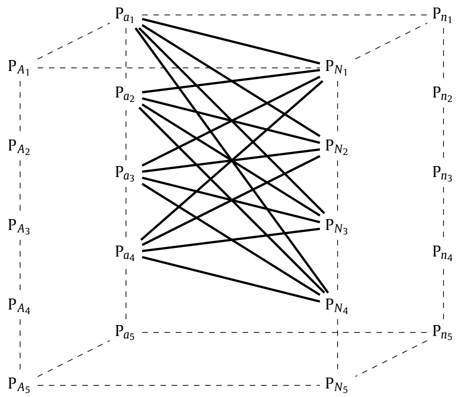

Fig. 19. Relations of contrary on a 5-graded cube of opposition.

Lemma 4.3. *Let (X, Y, I) be a fuzzy formal context. Let $A \underset{\sim}{\subset} X$ and let $E_1, E_2 \in \{\neg\mathsf{Sm}, \mathsf{BiVe}, \mathsf{BiEx}\}$. Let $\mathrm{P}_{A^{\uparrow}_{I,E_1}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{I,E_2}}$, $\mathrm{P}_{A^{\uparrow}_{\neg I,E_1}}$, and $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I,E_2}}$ be predicates. Then*

(a) $\mathrm{P}_{A^{\uparrow}_{I,E_1}}$ *and* $\mathrm{P}_{(\neg A)^{\uparrow}_{I,E_2}}$ *are contraries,*
(b) $\mathrm{P}_{A^{\uparrow}_{\neg I,E_1}}$ *and* $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I,E_2}}$ *are contraries.*

Proof. (a) Let $Z_1, Z_2 \underset{\sim}{\subset} X$ and let $y \in Y$. Then, by Lemma 2.5 (b), (i),

$$(A|Z_1)_I^{\uparrow}(y) \leq \neg I(x,y) \to \neg(A|Z_1)(x) \text{ and}$$
$$(\neg A|Z_2)_I^{\uparrow}(y) \leq \neg I(x,y) \to \neg(\neg A|Z_2)(x)$$

for all $x \in X$. Hence, the inequality

$$(A|Z_1)_I^{\uparrow}(y) \otimes (\neg A|Z_2)_I^{\uparrow}(y) \otimes (\neg I(x,y))^2 \leq \neg(A|Z_1)(x) \otimes \neg(\neg A|Z_2)(x)$$

holds by the adjunction property and Lemma 2.5 (g).

Note that $\neg(A|Z_1)(x) \otimes \neg(\neg A|Z_2)(x) = 0$ from (2) and Lemma 2.5 (e). Hence,

$$(A|Z_1)_I^{\uparrow}(y) \otimes (\neg A|Z_2)_I^{\uparrow}(y) \otimes (\neg I(x,y))^2 = 0.$$

Then, by Lemma 2.5 (f), (d),

$$((A|Z_1)_I^{\uparrow}(y) \wedge E_1(\mu_A(A|Z_1))) \otimes ((\neg A|Z_2)_I^{\uparrow}(y) \wedge E_2(\mu_A(\neg A|Z_2))) \otimes \bigvee_{x\in X} (\neg I(x,y))^2 = 0.$$

Finally, using Lemma 2.5 (c), we obtain

$$\bigvee_{Z_1 \underset{\sim}{\subset} X} \bigvee_{Z_2 \underset{\sim}{\subset} X} ((A|Z_1)_I^{\uparrow}(y) \wedge E_1(\mu_A(A|Z_1))) \otimes ((\neg A|Z_2)_I^{\uparrow}(y) \wedge E_2(\mu_A(\neg A|Z_2))) \otimes \bigvee_{x\in X} (\neg I(x,y))^2 = 0. \tag{5}$$

(b) Analogously to item (a), we can prove that

$$\bigvee_{Z_1 \underset{\sim}{\subset} X} \bigvee_{Z_2 \underset{\sim}{\subset} X} ((A|Z_1)_{\neg I}^{\uparrow}(y) \wedge E_1(\mu_A(A|Z_1))) \otimes ((\neg A|Z_2)_{\neg I}^{\uparrow}(y) \wedge E_2(\mu_A(\neg A|Z_2))) \otimes \bigvee_{x\in X} I(x,y)^2 = 0. \quad \square \tag{6}$$

Remark 4.4. Lemma 4.3 has important consequences on the behavior of the corresponding quantifier-based operators. Note the subformula

$$\bigvee_{x\in X} (\neg I(x,y))^2 \tag{7}$$

in (5). We can understand it as a *common presupposition*[8] of the quantifier-based operators $A_{I,E_1}^{\uparrow}$ and $(\neg A)_{I,E_2}^{\uparrow}$. Analogously, the subformula $\bigvee_{x\in X} I(x,y)^2$ in (6) can be viewed as a common presupposition of the quantifier-based operators $A_{\neg I,E_1}^{\uparrow}$ and $(\neg A)_{\neg I,E_2}^{\uparrow}$.

Lemma 4.5. *Let (X, Y, I) be a fuzzy formal context. Let $A \underset{\sim}{\subset} X$ and let $E \in \{\neg\mathrm{Sm}, \mathrm{BiVe}, \mathrm{BiEx}\}$. Let $\mathrm{P}_{A_I^{\uparrow}}$, $\mathrm{P}_{(\neg A)_{I,E}^{\uparrow}}$, $\mathrm{P}_{A_{\neg I}^{\uparrow}}$, and $\mathrm{P}_{(\neg A)_{\neg I,E}^{\uparrow}}$ be predicates. Then the following holds true:*

(a) $\mathrm{P}_{A_I^{\uparrow}}$ and $\mathrm{P}_{(\neg A)_{I,E}^{\uparrow}}$ are contraries,
(b) $\mathrm{P}_{A_{\neg I}^{\uparrow}}$ and $\mathrm{P}_{(\neg A)_{\neg I,E}^{\uparrow}}$ are contraries.

Proof. (a) Let $y \in Y$, by Lemma 2.5 (b), (i),

$$A_I^{\uparrow}(y) \leq \neg I(x,y) \to \neg A(x) \text{ and } (\neg A|Z)_I^{\uparrow}(y) \leq \neg I(x,y) \to \neg(\neg A|Z)(x)$$

for all $x \in X$. Then, the inequality

$$A_I^{\uparrow}(y) \otimes (\neg A|Z)_I^{\uparrow}(y) \otimes (\neg I(x,y))^2 \leq \neg A(x) \otimes \neg(\neg A|Z)(x)$$

holds from the adjunction property and Lemma 2.5 (g).

Note that $\neg A(x) \otimes \neg(\neg A|Z)(x) = 0$ follows from (2) and Lemma 2.5 (e).

Hence,

$$A_I^{\uparrow}(y) \otimes (\neg A|Z)_I^{\uparrow}(y) \otimes (\neg I(x,y))^2 = 0.$$

[8] Recall that the concept of common presupposition was introduced by the authors in [28].

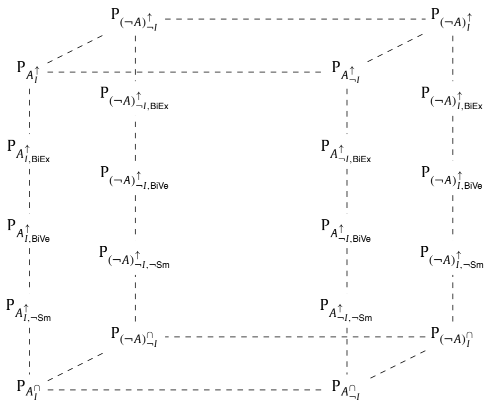

Fig. 20. Predicates related to concept-forming and quantifier-based operators on a 5-graded cube of opposition.

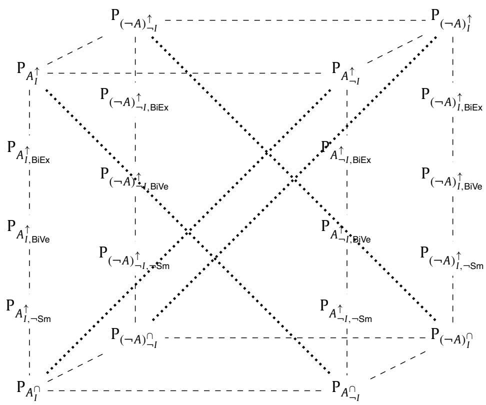

Fig. 21. Relations of contradictory with the concept-forming operators and the quantifier-based operators.

Then, by Lemma 2.5 (f), (d),

$$A^{\uparrow}_{I}(y) \otimes \bigvee_{Z \underset{\sim}{\subset} X} ((\neg A|Z)^{\uparrow}_{I}(y) \wedge E(\mu_A(\neg A|Z))) \otimes \bigvee_{x\in X} (\neg I(x,y))^2 = 0. \tag{8}$$

(b) Analogously to item (a), we can prove that

$$A^{\uparrow}_{\neg I}(y) \otimes \bigvee_{Z \underset{\sim}{\subset} X} ((\neg A|Z)^{\uparrow}_{\neg I}(y) \wedge E(\mu_A(\neg A|Z))) \otimes \bigvee_{x\in X} I(x,y)^2 = 0. \quad \square \tag{9}$$

Remark 4.6. The subformula $\bigvee_{x\in X}(\neg I(x,y))^2$ related to (8) represents the existential import, which is essential to be able to show the contrary of $P_{A^{\uparrow}_{I}}$ and $P_{(\neg A)^{\uparrow}_{I,E}}$. Analogously, the subformula $\bigvee_{x\in X} I(x,y)^2$, appearing in Equation (9), is the existential import related to the relation of contrary between the predicates $P_{A^{\uparrow}_{\neg I}}$ and $P_{(\neg A)^{\uparrow}_{\neg I,E}}$.

The previous lemmas allow us to obtain the main result of this paper.

Theorem 4.7. *Let (X, Y, I) be a fuzzy formal context and let $A \underset{\sim}{\subset} X$. Then, the predicates*

$$P_{A^{\uparrow}_{I}}, P_{A^{\uparrow}_{I,\mathrm{BiEx}}}, P_{A^{\uparrow}_{I,\mathrm{BiVe}}}, P_{A^{\uparrow}_{I,\neg\mathrm{Sm}}}, P_{A^{\cap}_{I}}, P_{A^{\uparrow}_{\neg I}}, P_{A^{\uparrow}_{\neg I,\mathrm{BiEx}}}, P_{A^{\uparrow}_{\neg I,\mathrm{BiVi}}}, P_{A^{\uparrow}_{\neg I,\neg\mathrm{Sm}}}, P_{A^{\cap}_{\neg I}}, P_{(\neg A)^{\uparrow}_{\neg I}}, P_{(\neg A)^{\uparrow}_{\neg I,\mathrm{BiEx}}}, P_{(\neg A)^{\uparrow}_{\neg I,\mathrm{BiVe}}}, P_{(\neg A)^{\uparrow}_{\neg I,\neg\mathrm{Sm}}}, P_{(\neg A)^{\cap}_{\neg I}},$$
$$P_{(\neg A)^{\uparrow}_{I}}, P_{(\neg A)^{\uparrow}_{I,\mathrm{BiEx}}}, P_{(\neg A)^{\uparrow}_{I,\mathrm{BiVi}}}, P_{(\neg A)^{\uparrow}_{I,\neg\mathrm{Sm}}}, \textit{ and } P_{(\neg A)^{\cap}_{I}}$$

form a 5-graded cube of opposition as shown in Figs. 21–27 (see also Fig. 20).

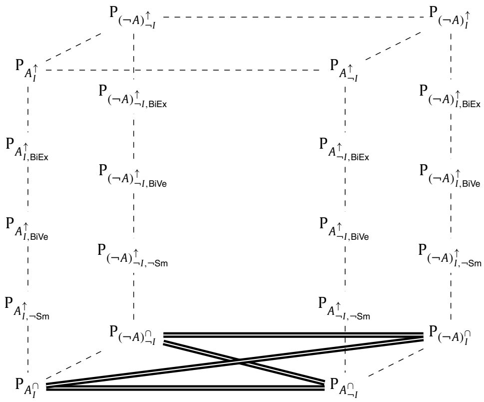

Fig. 22. Relations of sub-contrary with the concept-forming operators and the quantifier-based operators.

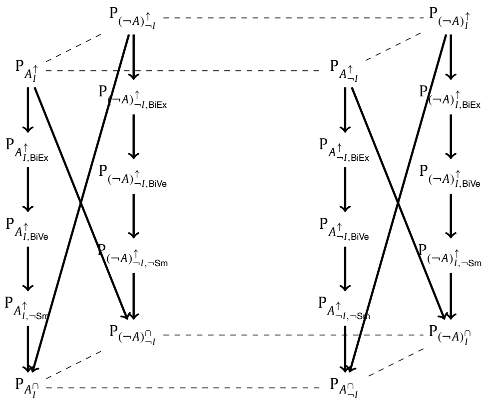

Fig. 23. Relations of subaltern with the concept-forming operators and the quantifier-based operators.

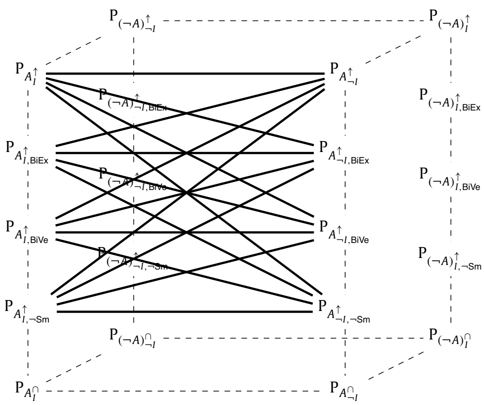

Fig. 24. Relations of contrary with the concept-forming operators and the quantifier-based operators.

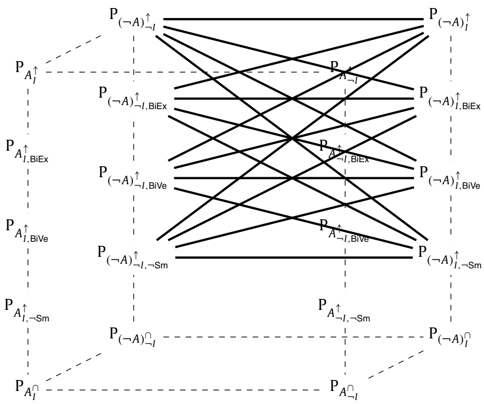

Fig. 25. Relations of contrary with the concept-forming operators and the quantifier-based operators.

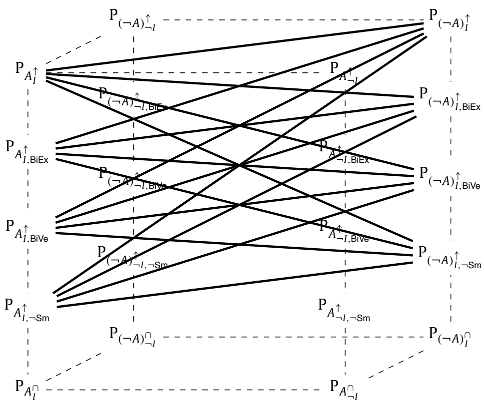

Fig. 26. Relations of contrary with the concept-forming operators and the quantifier-based operators.

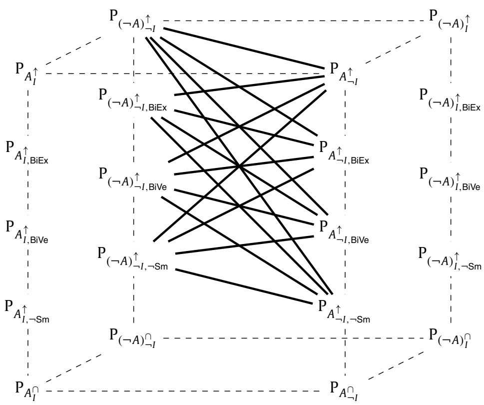

Fig. 27. Relations of contrary with the concept-forming operators and the quantifier-based operators.

Proof. The theorem follows from Theorem 3.10, Theorem 2.14,[9] and Lemmas 4.3 and 4.5. □

5. Conclusion and future directions

This article follows previous publication where we constructed graded polygons of opposition using the quantifier-based operators, which we defined as extensions of the fuzzy concept-forming operators.

Additionally, the graded cubes and 5-graded cubes of opposition here are presented as extensions of graded polygons of opposition, which in turn are generalizations of Aristotle's square.

As a future work, we plan to address the construction of a generalized hexagon ([4–6]) and to define new quantifier-based operators in the fuzzy formal concept analysis. Moreover, we intend to construct new structures of opposition with the so-called *fuzzy scaling quantifiers*, which are introduced in [9] to extend the Relational Concept Analysis using fuzzy logic.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The work was supported from ERDF/ESF by the project "Centre for the development of Artificial Intelligence Methods for the Automotive Industry of the region" No. CZ.02.1.01/0.0/0.0/17-049/0008414. The paper relates to the project COST CA17124 Methods of fuzzy modeling in the analysis of forensic digital data (10/2018-12/2020).

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[9] Theorem 2.14 guarantees that additional relations of opposition hold for the predicates $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I,\mathrm{BiEx}}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I,\mathrm{BiVe}}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{\neg I,\neg\mathrm{Sm}}}$, $\mathrm{P}_{(\neg A)^{\cap}_{\neg I}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{I}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{I,\mathrm{BiEx}}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{I,\mathrm{BiVe}}}$, $\mathrm{P}_{(\neg A)^{\uparrow}_{I,\neg\mathrm{Sm}}}$, and $\mathrm{P}_{(\neg A)^{\cap}_{I}}$ by substituting A and I with $\neg A$ and $\neg I$ in the decagon of Fig. 8.

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