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Thermoelasticity with temperature and microtemperatures with fading memory

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Abstract
In this paper, we investigate a model of poro-thermoelasticity with microtemperatures, where the behavior of the body is influenced by the history of both temperature and microtemperatures. Mathematically, this translates into a system of partial integro-differential equations. Under suitable condition on the tensors appearing in the model, we prove that the resulting system is well posed. In the one-dimensional case, the exponential decay of the energy is proved.

Keywords
Fading memory, microtemperatures, porous elasticity, exponential decay

1. Introduction

1.1. Thermoelasticity of porous materials

The classical theory of thermoelasticity is especially well-suited for the description of macroscopic phenomena related to elastic deformations. Notwithstanding, there are many physical situations in which microscopic phenomena play a big role and, therefore, cannot be ignored. From a modeling perspective, this requires to take into account the microstructure of the material. Perhaps the first to allow for such effects were the Cosserat brothers, who proposed micropolar theories at the beginning of the 20th century [1]. However, it was not until the sixties that materials with microstructure started to be investigated in a significant way. For a thorough description of these models, we refer to Eringen [2] and Ieşan [3].

Among the several theories that appeared during this period, we want to focus on the theory of materials with voids (also known as porous materials), first introduced by Cowin and Nunziato in the previous studies [4–6]. The fundamental concept underlying this model is the decomposition of the bulk density as the product of two fields, namely, the density field of the matrix material and the volume fraction field. The latter expresses the idea that the material point might have some small voids, and ultimately introduces an additional degree of freedom in the model. Let aside its undisputed mathematical interest, porous materials have soon found application in many fields of technology, ranging from

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the building industry, where they are used for their appealing properties of lightness and resistance, to medicine, to repair injuries in bones. For some extensive comments regarding the applicability of this theory, we suggest to look at [7, pp. 307–308]. Nowadays, porous materials have been considered and studied in such a large number of situations that it would not be possible to mention all of the contributions in the field. For an introduction to the subject and its applications, we direct the interested reader to the previous studies [8–11] and references therein, while for some works concerning the dynamical aspect of the theory we refer to the previous studies [12–18], but the list is far from being exhaustive.

Between all the aspects which have been considered when studying deformations at the microstructural level (micropolar, microstretch, etc.), we are mostly interested in the concept of microtemperatures, which is related to the temperature distribution in porous materials. Materials with a microstructure are usually thought as composed of microelements. In turn, each of these microelements is modeled itself as a material with deformations and temperature. If we denote by **x** the center of mass of a microelement and denote by $\tilde{\theta}$ the absolute temperature, we can consider the approximation:

$$\tilde{\theta}(\mathrm{x}', t) = \tilde{\theta}(\mathrm{x}, t) + T_i(x_i' - x_i) + O(d^2),$$

where $O(d^2)$ is a second-order term in the diameter d of the microelement. The terms T_i determine the temperature variation in the microelement and are what we call microtemperatures. Historically, this notion was proposed for the first time in the works of Grot, Riha and Verma [19–22], even though it did not receive much attention until the article [23] was published in 2000. The latter represented a turning point in the study of materials with microtemperatures and sparked a lot of interest in the subject. Today, we can say that there is an important amount of scientific work related to this phenomenon (see, e.g., [24–29]).

1.2. A causality issue

Most of the studies carried out on the topic of thermoelasticity with microtemperatures over the last decade assume both the temperature and the microtemperatures to follow the parabolic structure related to the Fourier law of heat conduction. It has been verified that, similarly to how the usual thermal dissipation acts as a damping mechanism on the deformations, the microthermal dissipation has the same effect on the microstructure. Although this behavior is certainly significant from a physical standpoint, from a mathematical perspective this is somewhat expected. Indeed, the regularizing nature of the Fourier law is well known, and usually endows a physical system of good dissipative properties. Nevertheless, the Fourier law has a strong disadvantage, since it predicts an instantaneous propagation of thermal waves. This fact is incompatible with the causality principle, and has prompted physicists and mathematicians alike to propose alternative laws for the description of heat conduction in the theory of thermoelasticity. For these reasons, the notion of microtemperatures has been recently extended to the case in which the Fourier law is replaced, first, by the (hyperbolic) Cattaneo law [30], and then by Tzou's theory [31]. In both situations, the authors have observed similar behaviors and dissipative properties to the Fourier case.

1.3. Main results

Another classical way to get rid of the paradox of infinite speed of propagation is to relax the constitutive law for the thermal flux by means of a convolution integral. This makes the dynamics nonlocal in time, meaning that the evolution of the heat flux at time t depends also on its history up time t. This idea was originally introduced by Gurtin and Pipkin [32]. An interesting study about materials with memory (also on the thermal variables) can be found at Amendola et al. [33]. Today, the literature on the subject is quite rich and active, and we refer the interested reader to the works of the previous studies [34–36], just to name a few. The present work fits into the above setting. Indeed, our goal is threefold. First and foremost, we want to define a theory for poro-thermoelasticity with microtemperatures that considers the history of both the temperature and that of the microtemperatures. To be more specific, we will start from the model of poro-thermoelasticity with microtemperatures proposed in Bazarra et al. [30], and show how this can be interpreted (and generalized) by means of the theory of materials with memory. This extension makes it possible to consider a wider range of problems, depending on the choice of the different memory kernels. Second, we want to propose some adequate conditions that will allow us to say that the problem is well posed in the Hadamard sense (i.e., existence, uniqueness and continuous dependence of solutions). Finally, we will restrict ourselves to the one-dimensional case and demonstrate (under suitable conditions) the exponential decay of solutions.

The mathematical tool best suited to treat partial differential equations with memory terms is the well-known *past history framework*, first introduced by Dafermos in the seminal work [37]. This setting will allow us to exploit results from the theory of linear semigroups. More in detail, we will prove the well-posedness of our system by means of the Lumer–Phillips corollary to the Hille–Yosida theorem, and use the classical characterization of exponentially stable semigroups due to Gearhart, Greiner, Huang, and Prüss (see, e.g., [38]) to demonstrate the exponential decay of the solutions in the one-dimensional case. The main mathematical difficulty of the problem at hand resides in the fact that we have to handle more than one memory term. As we will see, this requires some form of uniform control over the memory kernels, along the lines of the work [36].

1.4. Plan of the paper

In the next section, we propose the new model that we are going to work with as well as the general assumptions on the constitutive fields. In section 3, we propose the abstract setting for our problem, and in section 4, we rephrase the equations in the past history framework. The existence and uniqueness theorem is stated and proved in section 5. In the final section, we restrict our attention to the one-dimensional case and obtain the exponential stability of the solutions.

2. The model system

We consider a nonhomogeneous porous material occupying a smooth, bounded domain $\Omega \subset \mathbb{R}^3$. First, let us state the evolution equations for the theory of poro-thermoelasticity with microtemperatures for a centrosymmetric material. These equations are as follows:

$$\rho \ddot{u}_i = t_{ij,j}, \tag{1}$$

$$J\ddot{\phi} = h_{j,j} + g, \tag{2}$$

$$\rho\dot{\eta} = q_{j,j}, \tag{3}$$

$$\rho\dot{\epsilon}_i = q_{ji,j} + q_i - Q_i. \tag{4}$$

The first two equations represent, respectively, the balances of the linear momentum and of the first stress moment. Here, ρ is the mass density, u_i is the displacement vector, t_{ij} is the stress tensor, J is the equlibrated inertia, ϕ is the volume fraction, h_j is the equilibrated stress, and g is the equilibrated body force. Next, we have the balances of the energy and of its first moment, where η is the entropy, q_i is the heat flux vector, ϵ_i is the first moment of the energy vector, q_{ij} is the first heat flux moment tensor, and Q_i is the microheat flux average vector.

In order to obtain the final model, we complement the above relations with the constitutive equations in the case of the Lord–Shulman theory. These are given by (see [30]):

$$\begin{aligned}
&t_{ij} = A_{ijrs}u_{r,s} + D_{ij}\phi - a_{ij}\theta,\\
&h_i = A_{ij}\phi_{,j} - N_{ij}T_j,\\
&g = -D_{ij}u_{i,j} - \xi\phi + F\theta,\\
&\rho\eta = a_{ij}u_{i,j} + F\phi + a\theta,\\
&\rho\epsilon_i = -N_{ji}\phi_{,j} - B_{ij}T_j,\\
&\tau\dot{q}_i + q_i = k_{ij}\theta_{,j} + H_{ij}T_j,\\
&\tau\dot{q}_{ij} + q_{ij} = -P_{ijrs}T_{r,s},\\
&\tau\dot{Q}_i + Q_i = (k_{ij} - K_{ij})\theta_{,j} + (H_{ij} - \Lambda_{ij})T_j,
\end{aligned}$$

where we recall that θ is the temperature and T_i are the microtemperatures. It is understood that all the tensors appearing in the above equations might depend on the space variable $\boldsymbol{x}$ and on time. However, to simplify the notation, we will omit this dependence for the forthcoming computations. We can now formally solve the constitutive equations for q_i, q_{ij}, and Q_i. Multiplying by $e^{t/\tau}$ the constitutive equation for q_i, we get:

$$\frac{d}{dt}(q_i e^{t/\tau}) = \frac{1}{\tau} e^{t/\tau}\left(k_{ij}\theta_{,j} + H_{ij}T_j\right).$$

Integrating and making the reasonable assumption that:

$$\lim_{t\to-\infty} q_i(t)e^{t/\tau} = 0,$$

we have:

$$\begin{aligned} q_i(t) &= \int_{-\infty}^{t} \frac{e^{-(t-s)/\tau}}{\tau}\left(k_{ij}\theta_{,j}(s) + H_{ij}T_j(s)\right)ds \\ &= \int_{-\infty}^{t} \frac{e^{-(t-s)/\tau}}{\tau}\left(k_{ij}\dot{\alpha}_{,j}(s) + H_{ij}\dot{R}_j(s)\right)ds \\ &= \int_{0}^{\infty} \frac{e^{-s/\tau}}{\tau}\left(k_{ij}\dot{\alpha}_{,j}(t-s) + H_{ij}\dot{R}_j(t-s)\right)ds, \end{aligned}$$

where we denote by (see [39,40]):

$$\alpha(t) = \alpha(0) + \int_0^t \theta(s)ds, \quad R_i(t) = R_i(0) + \int_0^t T_i(s)ds,$$

respectively, the thermal displacement and the microthermal displacement. Assuming now:

$$\lim_{t\to-\infty} \alpha_{,i}(t)e^{t/\tau} = \lim_{t\to-\infty} R_i(t)e^{t/\tau} = 0,$$

we can integrate by parts to obtain:

$$q_i(t) = \frac{1}{\tau}\left(k_{ij}\alpha_{,j}(t) + H_{ij}R_j(t)\right) - \frac{1}{\tau^2}\int_0^\infty e^{-s/\tau}\left(k_{ij}\alpha_{,j}(t-s) + H_{ij}R_j(t-s)\right)ds.$$

Calling:

$$\begin{aligned} k^*_{ij}(s) &= \frac{e^{-s/\tau}}{\tau}k_{ij}, \\ H^*_{ij}(s) &= \frac{e^{-s/\tau}}{\tau}H_{ij}, \end{aligned}$$

and substituting into the above equation, we finally arrive to:

$$q_i(t) = k^*_{ij}(0)\alpha_{,j}(t) + H^*_{ij}(0)R_j(t) + \int_0^\infty \left(\frac{\partial}{\partial s}k^*_{ij}(s)\alpha_{,j}(t-s) + \frac{\partial}{\partial s}H^*_{ij}(s)R_j(t-s)\right)ds.$$

Now we can follow the same procedure for the constitutive equations of q_{ij} and Q_i. This yields:

$$q_{ij}(t) = -P^*_{ijrs}(0)R_{r,s}(t) - \int_0^\infty \frac{\partial}{\partial s}P^*_{ijrs}(s)R_{r,s}(t-s)ds,$$

and

$$Q_i(t) = \Big(k^*_{ij}(0) - K^*_{ij}(0)\Big)\alpha_{,j}(t) + \Big(H^*_{ij}(0) - \Lambda^*_{ij}(0)\Big)R_j(t)$$
$$\int_0^\infty \left(\frac{\partial}{\partial s}\Big(k^*_{ij}(s) - K^*_{ij}(s)\Big)\alpha_{,j}(t-s) + \frac{\partial}{\partial s}\Big(H^*_{ij}(s) - \Lambda^*_{ij}(s)\Big)R_j(t-s)\right)ds.$$

We note that q_i, q_{ij}, and Q_i are given in terms of the history of the thermal displacement and the microthermal displacement. This represents an advantage with respect to consider the history of the temperature and the microtemperatures since we can define a larger class of materials (see Remark 1). In fact, we can recover the materials proposed at Conti et al. [34] as a sub-class when the microtemperatures are not present.

Plugging the newly derived constitutive equations for q_i, q_{ij}, and Q_i into those of poro-thermoelasticity, we have the system of field equations:

$$\rho\ddot{u}_i = (A_{ijrs}u_{r,s} + D_{ij}\phi - a_{ij}\theta)_{,j}, \tag{5}$$

$$J\ddot{\phi} = (A_{ij}\phi_{,j} - N_{ij}T_j)_{,i} - D_{ij}u_{i,j} - \xi\phi + F\theta, \tag{6}$$

$$\begin{aligned} a\ddot{\alpha} = &- a_{ij}\dot{u}_{i,j} - F\dot{\phi} + \big(k_{ij}(0)\alpha_{,j} + H_{ij}(0)R_j\big)_{,i} \\ &+ \int_0^\infty \Big(k'_{ij}(s)\alpha_{,j}(t-s) + H'_{ij}(s)R_j(t-s)\Big)_{,i} ds, \end{aligned} \tag{7}$$

$$\begin{aligned} B_{ij}\ddot{R}_j = &- N_{ji}\dot{\phi}_{,j} + (P_{ijrs}(0)R_{r,s})_{,j} + \int_0^\infty \Big(P'_{ijrs}(s)R_{r,s}(t-s)\Big)_{,j} ds \\ &- K_{ij}(0)\alpha_{,j}(t) - \Lambda_{ij}(0)R_j(t) - \int_0^\infty \Big(K'_{ij}(s)\alpha_{,j}(t-s) + \Lambda'_{ij}(s)R_j(s)\Big)ds, \end{aligned} \tag{8}$$

where we have omitted the star to simplify the notation and used the standard writing $f'(s)$ to indicate the derivative $\frac{\partial f}{\partial s}$. Our goal is to study the well-posedness and asymptotic dynamics of systems (5)–(8), supplemented with the Dirichlet boundary conditions:

$$u_i(\boldsymbol{x},t)_{|\boldsymbol{x}\in\partial\Omega} = \phi(\boldsymbol{x},t)_{|\boldsymbol{x}\in\partial\Omega} = \alpha(\boldsymbol{x},t)_{|\boldsymbol{x}\in\partial\Omega} = R_i(\boldsymbol{x},t)_{|\boldsymbol{x}\in\partial\Omega} = 0.$$

2.1. General assumptions

In greater generality, we will consider the system of equations (5)–(8) with general memory kernels:

$$k_{ij} = k_{ij}(s), \quad K_{ij} = K_{ij}(s), \quad H_{ij} = H_{ij}(s), \quad \Lambda_{ij} = \Lambda_{ij}(s), \quad P_{ijrs} = P_{ijrs}(s),$$

which we will assume independent of $\boldsymbol{x} \in \Omega$. This assumption, albeit non-necessary, greatly simplifies the exposition. Furthermore, we require that:

i. There exist positive constants $\rho_0, J_0, \alpha_0, B_0$ such that:

$$\rho(\boldsymbol{x}) \geqslant \rho_0, \; J(\boldsymbol{x}) \geqslant J_0, \; a(\boldsymbol{x}) \geqslant a_0, \; B_{ij}(\boldsymbol{x})T_iT_j \geqslant B_0T_iT_i.$$

ii. There exists a positive constant A_0 such that:

$$A_{ijrs}\eta_{ij}\eta_{rs} + 2D_{ij}\eta_{ij}\phi + \xi\phi^2 \geqslant A_0(\eta_{ij}\eta_{ij} + \phi^2).$$

for every $\boldsymbol{\eta} = (\eta_{ij})$, and $\phi \in \mathbb{R}$.

iii. The functions $k_{ij}, \Lambda_{ij}, P_{ijrs}$ are symmetric in the sense that:

$$k_{ij} = k_{ji}, \quad \Lambda_{ij} = \Lambda_{ji}, \quad P_{ijrs} = P_{rsij}.$$

Furthermore, we assume that:

$$K_{ij} = H_{ji}. \tag{9}$$

iv. There exists a positive constant g_0 such that for every $\boldsymbol{\xi} = (\xi_i)$, $\boldsymbol{\zeta} = (\zeta_i)$ and $\boldsymbol{\eta} = (\eta_{ij})$,

$$\begin{aligned} &k_{ij}(\infty)\xi_i\xi_j + \big(K_{ij}(\infty) + H_{ji}(\infty)\big)\zeta_i\xi_j + \Lambda_{ij}(\infty)\zeta_i\zeta_j + P_{ijrs}(\infty)\eta_{ij}\eta_{rs} \\ &\qquad \geqslant g_0(\xi_i\xi_i + \zeta_i\zeta_i + \eta_{ij}\eta_{ij}), \end{aligned}$$

where:

$$k_{ij}(\infty) = \lim_{s\to\infty} k_{ij}(s),$$

and similarly for the other kernels.

v. There exists a positive decreasing continuous and integrable scalar function $\ell(s)$ and a constant $\kappa \geqslant 1$ such that:

$$\begin{aligned} &\ell(s)(\xi_i\xi_i + \zeta_i\zeta_i + \eta_{ij}\eta_{ij}) \\ &\leqslant -k'_{ij}(s)\xi_i\xi_j - \Big(K'_{ij}(s) + H'_{ji}(s)\Big)\zeta_i\xi_j - \Lambda'_{ij}(s)\zeta_i\zeta_j - P'_{ijrs}(s)\eta_{ij}\eta_{rs} \\ &\leqslant \kappa\ell(s)(\xi_i\xi_i + \zeta_i\zeta_i + \eta_{ij}\eta_{ij}), \end{aligned} \tag{10}$$

for every $\boldsymbol{\xi} = (\xi_i)$, $\boldsymbol{\zeta} = (\zeta_i)$, and $\boldsymbol{\eta} = (\eta_{ij})$. We denote by:

$$\varkappa = \int_0^\infty \ell(s)ds,$$

the resultant of ℓ.

vi. It holds:

$$k''_{ij}(s)\xi_i\xi_j + \Big(K''_{ij}(s) + H''_{ji}(s)\Big)\zeta_i\xi_j + \Lambda''_{ij}(s)\zeta_i\zeta_j + P''_{ijrs}(s)\eta_{ij}\eta_{rs} \geqslant 0,$$

for every $\boldsymbol{\xi} = (\xi_i)$, $\boldsymbol{\zeta} = (\zeta_i)$, and $\boldsymbol{\eta} = (\eta_{ij})$.

Assumptions (i)–(iii) are natural in the context of thermoelasticity. Indeed, the meaning of (i) is clear, while (ii) is saying that the mechanical energy of the system is positive definite. This hypothesis plays a critical role in the context of elastic stability. However, Assumptions (iv)–(vi) arise naturally in the study of equations with memory terms (see, e.g., [41]).

Remark 1. The observant reader will have noticed that Assumption (iv) is in contrast with the exponential memory kernels that we have found integrating the constitutive equations, where, for instance, $k^*_{ij}(\infty) = 0$. However, in order to consider the general problem, we must allow for the case $k_{ij}(\infty) \neq 0$ (and the same for the other kernels). In this way, e.g., we recover the model analyzed in Conti et al. [34]. The case of kernels vanishing at infinity will be the object of future works.

Assumption (9) is related with *Onsager's postulate* in the case of the classical theory. From now on, we will always write K_{ij} instead of H_{ji}. We note that $K_{ij}+H_{ji}=2K_{ij}$. We conclude this section with a technical lemma, which will be useful in the sequel.

Lemma 1. Let Assumptions (iii) and (v) hold. Then, for every $i,j=1, 2, 3$, we have:

$$-k'_{ij}(s)\leqslant \kappa\ell(s) \quad \forall s\in\mathbb{R}^+,$$

and the same holds for $-\Lambda'_{ij}$, $-K'_{ij}$, and $-P'_{ijrs}$.

Proof. Setting in equation (10):

$$\xi_1=1 \text{ and } \xi_2=\xi_3=\zeta_i=\eta_{ij}=0 \quad \text{for } i,j=1,2,3,$$

we see at once that:

$$-k'_{11}(s)\leqslant \kappa\ell(s).$$

In a similar fashion, we can show that the same holds for $-k'_{22}$, $-k'_{33}$ and $-\Lambda'_{ii}$, $-P'_{ijij}$ for every i,j. Now consider the matrix:

$$\begin{pmatrix} -k'_{11}(s) & -k'_{12}(s) \\ -k'_{21}(s) & -k'_{22}(s) \end{pmatrix}.$$

By Assumption (iii), we have $-k'_{12}(s)=-k'_{21}(s)$. Moreover, it is easy to see that, by Assumption (v), this matrix is actually positive definite. Therefore:

$$k'_{12}(s)k'_{21}(s)=\left(k'_{12}(s)\right)^2=\left(k'_{21}(s)\right)^2\leqslant k'_{11}(s)k'_{22}(s),$$

from which we infer:

$$-k'_{12}(s)\leqslant \kappa\ell(s).$$

By the same token, one can show that all the off-diagonal entries of $-k'_{ij}$, $-\Lambda'_{ij}$, and $-P'_{ijrs}$ are also bounded by $\kappa\ell(s)$. Finally, let us turn to $-K'_{ij}$. Observe first that it is not difficult to prove that $-k'_{ii}$, $-\Lambda'_{ii}$, and $-P'_{ijij}$ are the positive functions, by choosing $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\eta}$ in a suitable way in equation (10). Now let us take:

$$\xi_1=\zeta_1=1 \text{ and } \xi_2=\xi_3=\zeta_2=\zeta_3=\eta_{ij}=0 \quad \text{for } i,j=1,2,3.$$

Then, in view of equation (9), we have:

$$-k'_{11}(s)-2K'_{11}(s)-\Lambda'_{11}(s)\leqslant 2\kappa\ell(s).$$

By the positivity of $-k'_{11}$ and $-\Lambda'_{11}$, we finally get:

$$-K'_{11}(s)\leqslant \kappa\ell(s).$$

With the same reasoning, we can show that the same holds for $-K'_{ij}$ for every i,j, and this concludes the proof. □

3. Functional setting and notation

We indicate by $(H,\langle\cdot,\cdot\rangle,\ \|\cdot\|\ )$ the usual Hilbert space $L^2(\Omega)$ and by $(V,\langle\cdot,\cdot\rangle_1,\ \|\cdot\|_1)$ the standard Sobolev space $H^1_0(\Omega)$ of functions in H^1 vanishing on $\partial\Omega$. We denote by:

$$\boldsymbol{H}=[L^2(\Omega)]^3, \quad \boldsymbol{V}=[H^1_0(\Omega)]^3,$$

the corresponding vectorial versions, keeping the same scalar notation for their norms. We would like to rephrase equations (5)–(8) in the so-called past history framework. To this end, let us preliminarily introduce the Hilbert spaces:

$$\mathcal{M} = L^2_\ell(\mathbb{R}^+, V), \quad \boldsymbol{\mathcal{M}} = L^2_\ell(\mathbb{R}^+, \boldsymbol{V}),$$

of square summable functions with respect to the measure $\ell(s)ds$, endowed with the scalar products:

$$\langle \omega, \omega^* \rangle_{\mathcal{M}} = \int_0^\infty \int_\Omega \ell(s)\omega_{,i}(\boldsymbol{x}, s)\omega^*_{,i}(\boldsymbol{x}, s) d\boldsymbol{x} ds,$$
$$\langle \eta_i, \eta_i^* \rangle_{\mathcal{M}} = \int_0^\infty \int_\Omega \ell(s)(\eta_i(\boldsymbol{x}, s)\eta_i^*(\boldsymbol{x}, s) + \eta_{i,j}(\boldsymbol{x}, s)\eta^*_{i,j}(\boldsymbol{x}, s)) d\boldsymbol{x} ds,$$

and norms:

$$\|\omega\|^2_{\mathcal{M}} = \int_0^\infty \int_\Omega \ell(s)|\omega_{,i}(\boldsymbol{x}, s)|^2 \boldsymbol{x} ds,$$
$$\|\eta_i\|^2_{\mathcal{M}} = \int_0^\infty \int_\Omega \ell(s)(|\eta_i(\boldsymbol{x}, s)|^2 + |\eta_{i,j}(\boldsymbol{x}, s)|^2)\, d\boldsymbol{x} ds.$$

Next, we define the Hilbert space:

$$\mathcal{N} = \mathcal{M} \times \boldsymbol{\mathcal{M}}.$$

endowed with the standard product norm. Observe that, omitting for simplicity the explicit dependence of the involved functions on s and $\boldsymbol{x}$, in view of Assumption (v),

$$\|(\omega, \eta_i)\|^2_{\mathcal{N}} = -\int_0^\infty \int_\Omega k'_{ij}\omega_{,i}\omega_{,j} + 2K'_{ij}\eta_i\omega_{,j} + \Lambda'_{ij}\eta_i\eta_j + P'_{ijrs}\eta_{i,j}\eta_{r,s} d\boldsymbol{x} ds,$$

is an equivalent norm on $\mathcal{N}$, with corresponding scalar product:

$$\langle (\omega, \eta_i), (\omega^*, \eta_i^*) \rangle_{\mathcal{N}} = -\int_0^\infty \int_\Omega k'_{ij}\omega_{,i}\omega^*_{,j} + K'_{ij}(\eta_i\omega^*_{,j} + \eta_i^*\omega_{,j}) + \Lambda'_{ij}\eta_i\eta_j^* + P'_{ijrs}\eta_{i,j}\eta^*_{r,s} d\boldsymbol{x} ds.$$

Finally, we introduce the phase space associated with our problem,

$$\mathcal{H} = \boldsymbol{V} \times \boldsymbol{H} \times V \times H \times V \times H \times \boldsymbol{V} \times \boldsymbol{H} \times \mathcal{N},$$

endowed with the norm:

$$\begin{aligned}\|\boldsymbol{u}\|^2_{\mathcal{H}} = &\int_\Omega \Big(A_{ijrs}u_{i,j}u_{r,s} + 2D_{ij}u_{i,j}\phi + \xi|\phi|^2 + A_{ij}\phi_{,i}\phi_{,j} + \rho|v_i|^2 + J|\psi|^2 + a|\theta|^2 + B_{ij}T_iT_j\Big) d\boldsymbol{x} \\ &+ \int_\Omega \big(k_{ij}(\infty)\alpha_{,i}\alpha_{,j} + 2K_{ij}(\infty)R_i\alpha_{,j} + \Lambda_{ij}(\infty)R_iR_j + P_{ijrs}(\infty)R_{i,j}R_{r,s}\big) d\boldsymbol{x} \\ &- \int_0^\infty \int_\Omega \Big(k'_{ij}\omega_{,i}\omega_{,j} + 2K'_{ij}\eta_i\omega_{,j} + \Lambda'_{ij}\eta_i\eta_j + P'_{ijrs}\eta_{i,j}\eta_{r,s}\Big) d\boldsymbol{x} ds,\end{aligned}$$

where:

$$\boldsymbol{u} = (u_i, v_i, \phi, \psi, \alpha, \theta, R_i, T_i, \omega, \eta_i).$$

Thanks to Assumptions (i), (ii), (iv), and (v), this is equivalent to the standard product norm defined on $\mathcal{H}$. We will also consider the infinitesimal generator of the right-translation semigroup on $\mathcal{N}$, i.e., the linear operator $\mathcal{T}$ given by:

$$\mathcal{T}(\omega, \eta_i) = -(\omega', \eta_{i'}),$$

with domain,

$$\mathfrak{D}(\mathcal{T}) = \{(\omega, \eta_i) \in \mathcal{N} : (\omega', \eta_{i'}) \in \mathcal{N},\ (\omega, \eta_i)(0) = 0\}.$$

In light of Assumption (vi), a straightforward integration by parts yields the dissipative estimate:

$$\begin{aligned}\langle \mathcal{T}(\omega, \eta_i), (\omega, \eta_i)\rangle_{\mathcal{N}} &= -\frac{1}{2}\int_0^\infty \int_\Omega k''_{ij}\omega_{,i}\omega_{,j} + 2K''_{ij}\eta_i\omega_{,j} + \Lambda''_{ij}\eta_i\eta_j + P''_{ijrs}\eta_{i,j}\eta_{r,s} d\boldsymbol{x} ds \\ &\leqslant 0,\end{aligned} \tag{11}$$

for every $(\omega, \eta_i) \in \mathfrak{D}(\mathcal{T})$. We refer the interested reader to Pata [41] for a thorough discussion on the mathematical properties of $\mathcal{T}$ and of the semigroup of right translation on memory spaces.

4. Basic equations in linear heat conduction with memory

In the same spirit of Dafermos [37], we introduce the variables (omitting the dependence on $\boldsymbol{x}$):

$$\begin{aligned}\omega^t(s) &= \alpha(t) - \alpha(t-s),\\ \eta_i^t(s) &= R_i(t) - R_i(t-s),\end{aligned}$$

modeling the histories of the thermal and microthermal displacements. Then, we can rewrite equations (5)–(8) as:

$$\rho\ddot{u}_i = (A_{ijrs}u_{r,s} + D_{ij}\phi - a_{ij}\theta)_{,j}, \tag{12}$$

$$J\ddot{\phi} = (A_{ij}\phi_{,j} - N_{ij}T_j)_{,i} - D_{ij}u_{i,j} - \xi\phi + F\theta, \tag{13}$$

$$\begin{aligned}a\ddot{\alpha} = &-a_{ij}\dot{u}_{i,j} - F\dot{\phi} + \big(k_{ij}(\infty)\alpha_{,j} + K_{ji}(\infty)R_j\big)_{,i}\\ &-\int_0^\infty \Big(k'_{ij}(s)\omega_{,j}(s) + K'_{ji}(s)\eta_j(s)\Big)_{,i} ds,\end{aligned} \tag{14}$$

$$\begin{aligned}B_{ij}\ddot{R}_j = &-N_{ji}\dot{\phi}_{,j} + (P_{ijrs}(\infty)R_{r,s})_{,j} - \int_0^\infty \Big(P'_{ijrs}(s)\eta_{r,s}(s)\Big)_{,j} ds\\ &- K_{ij}(\infty)\alpha_{,j} - \Lambda_{ij}(\infty)R_j + \int_0^\infty \Big(K'_{ij}(s)\omega_{,j}(s) + \Lambda'_{ij}(s)\eta_j(s)\Big) ds,\end{aligned} \tag{15}$$

$$(\dot{\omega}, \dot{\eta}_i) = \mathcal{T}(\omega, \eta_i) + (\theta, T_i). \tag{16}$$

Introducing the state vector:

$$\boldsymbol{u}(t) = (u_i(t), \dot{u}_i(t), \phi(t), \dot{\phi}(t), \alpha(t), \dot{\alpha}(t), R_i(t), \dot{R}_i(t), \omega, \eta_i),$$

we view systems (12)–(15) as the ordinary differential equation (ODE) on $\mathcal{H}$:

$$\frac{d}{dt}\boldsymbol{u}(t) = \mathbb{A}\boldsymbol{u}(t).$$

here, $\mathbb{A}$ is the linear operator defined as:

$$\mathbb{A}\begin{pmatrix} u_i \\ v_i \\ \phi \\ \psi \\ \alpha \\ \theta \\ R_i \\ T_i \\ \omega \\ \eta_i \end{pmatrix} = \begin{pmatrix} v_i \\ \frac{1}{\rho}\left(A_{ijrs}u_{r,s} + D_{ij}\phi - a_{ij}\theta\right)_{,j} \\ \psi \\ \frac{1}{J}\left[(A_{ij}\phi_{,j} - N_{ij}T_j)_{,i} - D_{ij}u_{i,j} - \xi\phi + F\theta\right] \\ \theta \\ a^{-1}\mathrm{M} \\ T_i \\ C_{ij}\mathrm{N}_j \\ \mathcal{T}\omega + \theta \\ \mathcal{T}\eta_i + T_i \end{pmatrix}, \tag{17}$$

where C_{ij} is the inverse matrix of B_{ij} (which certainly exists in view of Assumption (i)) and:

$$\begin{aligned} \mathrm{M} = & - a_{ij}v_{i,j} - F\psi + \left(k_{ij}(\infty)\alpha_{,j} + K_{ji}(\infty)R_j\right)_{,i} \\ & - \int_0^\infty \left(k'_{ij}(s)\omega_{,j}(s) + K'_{ji}(s)\eta_j(s)\right)_{,i} ds, \end{aligned} \tag{18}$$

while,

$$\begin{aligned} \mathrm{N}_i = & - N_{ji}\psi_{,j} + (P_{ijrs}(\infty)R_{r,s})_{,j} - K_{ij}(\infty)\alpha_{,j} - \Lambda_{ij}(\infty)R_j \\ & - \int_0^\infty \left(P'_{ijrs}(s)\eta_{r,s}(s)\right)_{,j} ds + \int_0^\infty \left(K'_{ij}(s)\omega_{,j}(s) + \Lambda'_{ij}(s)\eta_j(s)\right) ds. \end{aligned} \tag{19}$$

The operator $\mathbb{A}$ has dense domain $\mathfrak{D}(A)$ defined by:

$$\mathfrak{D}(\mathbb{A}) = \left\{ \boldsymbol{u} \in \mathcal{H} \;\middle|\; \begin{matrix} v_i, \psi, \theta, T_i \in V \\ (A_{ijrs}u_{r,s} + D_{ij}\phi - a_{ij}\theta)_{,j} \in H \\ (A_{ij}\phi_{,j} - N_{ij}T_j)_{,i} \in H \\ \mathrm{M}, \mathrm{N}_i \in H \\ (\omega, \eta) \in \mathfrak{D}(\mathcal{T}) \end{matrix} \right\}.$$

5. Existence and uniqueness

This section is devoted to the proof of the generation of a solution semigroup for systems (12)–(16). Let us state the main result.

Theorem 1. The operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous linear semigroup $S(t)$ on the phase space $\mathcal{H}$. Besides, $S(t)$ is contractive with respect to the norm of $\mathcal{H}$.

The proof of Theorem 1 is obtained exploiting the well-known Lumer–Phillips theorem. In turn, this amounts in proving the following two lemmas. Before delving into details, we remark that since we are dealing with real Banach spaces, in what follows $\mathbb{A}$ will actually denote the complexification of the infinitesimal generator $\mathbb{A}$, i.e., the operator acting on the complex Hilbert space $\mathcal{H} + i\mathcal{H}$ by the rule:

$$\boldsymbol{u} + i\boldsymbol{v} \mapsto \mathbb{A}\boldsymbol{u} + i\mathbb{A}\boldsymbol{v}.$$

Lemma 2. The operator $\mathbb{A}$ is dissipative, i.e.,

$$\mathfrak{Re}\,\langle \mathbb{A}\boldsymbol{u}, \boldsymbol{u}\rangle_{\mathcal{H}} \leqslant 0, \quad \forall \boldsymbol{u} \in \mathfrak{D}(\mathbb{A}).$$

Proof. By means of the divergence theorem and exploiting the boundary conditions, a direct computation reveals that:

$$\begin{aligned}\langle \mathbb{A}\mathcal{U},\mathcal{U}\rangle_{\mathcal{H}} &= -\int_0^\infty \int_B \Big[k'_{ij}\mathcal{T}\omega_{,i}\omega_{,j} + K'_{ij}(\mathcal{T}\eta_i\omega_{,j} + \mathcal{T}\omega_{,j}\eta_i) + \Lambda'_{ij}\mathcal{T}\eta_i\eta_j + P'_{ijrs}(\mathcal{T}\eta_{i,j}\eta_{r,s})\Big]dsdv \\ &= \langle \mathcal{T}(\omega,\eta_i),(\omega,\eta_i)\rangle_{\mathcal{N}} \\ &\leqslant 0,\end{aligned}$$

where the inequality follows from equation (11). Therefore, the operator $\mathbb{A}$ is dissipative. □

Lemma 3. The operator $\mathbb{I}-\mathbb{A}$ is onto from $\mathfrak{D}(\mathbb{A})$ into $\mathcal{H}$.

Proof. For every vector,

$$\boldsymbol{f} = \left(f_i^{(0)}, f_i^{(1)}, f^{(2)}, f^{(3)}, f^{(4)}, f^{(5)}, f_i^{(6)}, f_i^{(7)}, f^{(8)}, f_i^{(9)}\right) \in \mathcal{H},$$

we look for a unique solution $\boldsymbol{u} \in \mathfrak{D}(\mathbb{A})$ to the resolvent equation:

$$(\mathbb{I}-\mathbb{A})\boldsymbol{u} = \boldsymbol{f}.$$

Equivalently, we try to solve in $\mathfrak{D}(\mathbb{A})$ the following system:

$$u_i - v_i = f_i^{(0)}, \tag{20}$$

$$\rho v_i - (A_{ijrs}u_{r,s} + D_{ij}\phi - a_{ij}\theta)_{,j} = \rho f_i^{(1)}, \tag{21}$$

$$\phi - \psi = f^{(2)}, \tag{22}$$

$$J\psi - (A_{ij}\phi_{,j} - N_{ij}T_i)_{,j} + D_{ij}u_{i,j} + \xi\phi - F\theta = Jf^{(3)}, \tag{23}$$

$$\alpha - \theta = f^{(4)}, \tag{24}$$

$$a\theta - \mathrm{M} = af^{(5)}, \tag{25}$$

$$R_i - T_i = f_i^{(6)}, \tag{26}$$

$$B_{ij}T_j - \mathrm{N}_i = B_{ij}f_i^{(7)}, \tag{27}$$

$$\omega - \mathcal{T}\omega - \theta = f^{(8)}, \tag{28}$$

$$\eta_i - \mathcal{T}\eta_i - T_i = f_i^{(9)}. \tag{29}$$

where M and N_i were defined at equations (18) and (19). Integrating equations (28) and (29), we obtain:

$$\begin{aligned}\omega(s) &= \int_0^s \mathrm{e}^{-(s-y)}f^{(8)}(y)dy + (1-\mathrm{e}^{-s})\theta = (E * f^{(8)})(s) + (1-\mathrm{e}^{-s})\theta, \\ \eta_i(s) &= \int_0^s \mathrm{e}^{-(s-y)}f_i^{(9)}(y)dy + (1-\mathrm{e}^{-s})T_i = (E * f_i^{(9)})(s) + (1-\mathrm{e}^{-s})T_i,\end{aligned}$$

where $E(s) = \mathrm{e}^{-s}$ and $*$ denotes the convolution product on $(0,s)$. Making use of the standard properties of the convolution, we have:

$$\|\omega\|^2_{\mathcal{M}} \leqslant 2\,\|E * f^{(8)}\|^2_{\mathcal{M}} + 2\kappa\,\|\theta\|_1^2 \leqslant 2\,\|f^{(8)}\|^2_{\mathcal{M}} + 2\varkappa\,\|\theta\|_1^2\,,$$

so that $\omega \in \mathcal{M}$. In a similar way, one is able to show that $\boldsymbol{\eta} = (\eta_i) \in \mathbf{M}$. Substituting equations (20), (22), (24), and (26) into the main system, we arrive at:

$$\begin{aligned}
&\rho u_i - (A_{ijrs} u_{r,s} + D_{ij}\phi - a_{ij}\alpha)_{,j} = \Psi_i^{(1)},\\
&J\phi - (A_{ij}\phi_{,j} - N_{ij}R_i)_{,j} + D_{ij}u_{ij} + \xi\phi - F\alpha = \Psi^{(2)},\\
&a\alpha + a_{ij}u_{i,j} + F\phi - \big(k_{ij}(\infty)\alpha_{,i} + K_{ji}(\infty)R_i\big)_{,j} + \widehat{k'_{ij}}\alpha_{,ij} + \widehat{K'_{ji}}R_{i,j} = \Psi^{(3)},\\
&B_{ij}R_j + N_{ij}\phi_{,j} - \big(P_{ijrs}(\infty)R_{r,s}\big)_{,j} + K_{ij}(\infty)\alpha_{,j} + \Lambda_{ij}(\infty)R_j, + \widehat{P'_{ijrs}}R_{r,sj} - \widehat{K'_{ij}}\alpha_{,j} - \widehat{\Lambda'_{ij}}R_j = \Psi_i^{(4)},
\end{aligned} \tag{30}$$

here:

$$\widehat{k'_{ij}} = \int_0^\infty k'_{ij}(s)(1 - \mathrm{e}^{-s})ds,$$

and in the same way, we define $\widehat{K'_{ij}}$, $\widehat{P'_{ijrs}}$ and $\widehat{\Lambda'_{ij}}$. Moreover,

$$\begin{aligned}
\Psi_i^{(1)} &= \rho f_i^{(0)} + \rho f_i^{(1)} + a_{ij} f_{,j}^{(5)},\\
\Psi^{(2)} &= J f^{(2)} + J f^{(3)} + N_{ij} f_{i,j}^{(6)} - F f^{(4)},\\
\Psi^{(3)} &= a f^{(4)} + a f^{(5)} + a_{ij} f_{i,j}^{(0)} + F f^{(2)} + \widehat{k'_{ij}} f_{,ij}^{(4)}\\
&\quad - \int_0^\infty k'_{ij}(s)(E * f^{(8)})_{,ij}(s)ds - \int_0^\infty K'_{ji}(s)(E * f_i^{(9)})_{,j}(s)ds,\\
\Psi_i^{(4)} &= B_{ij}(f_j^{(6)} + f_j^{(7)}) + N_{ij} f_{,j}^{(3)} + \widehat{K'_{ij}} f_{,j}^{(4)}\\
&\quad - \int_0^\infty K'_{ij}(s)(E * f^{(8)})_{,j}(s)ds - \int_0^\infty P'_{ijrs}(s)(E * f_r^{(9)})_{,sj}(s)ds - \int_0^\infty \Lambda'_{ij}(s)(E * f_j^{(9)})(s)ds.
\end{aligned}$$

In order to prove the existence of $\boldsymbol{u} \in \mathfrak{D}(\mathbb{A})$ satisfying the resolvent equation, we make use of the Lax–Milgram theorem. To this end, we define the following bilinear form:

$$\begin{aligned}
\mathbf{a}\big((u_i, \phi, \alpha, R_i), (\tilde{u}_i, \tilde{\phi}, \tilde{\alpha}, \tilde{R}_i)\big) &= \rho u_i \tilde{u}_i + (A_{ijrs}u_{r,s} + D_{ij}\phi - a_{ij}\alpha)\tilde{u}_{i,j} + J\phi\tilde{\phi} + (A_{ij}\phi_{,j} - N_{ij}R_i)\tilde{\phi}_{,j}\\
&+ D_{ij}u_{ij}\tilde{\phi} + \xi\phi\tilde{\phi} - F\alpha\tilde{\phi} + a\alpha\tilde{\alpha} - a_{ij}u_i\tilde{\alpha}_{,j} + F\phi\tilde{\alpha} + \big(k_{ij}(\infty)\alpha_{,i}\tilde{\alpha} + K_{ji}(\infty)R_i\big)\tilde{\alpha}_{,j} - \widehat{k'_{ij}}\alpha_{,i}\tilde{\alpha}_{,j}\\
&- \widehat{K'_{ji}}R_i\tilde{\alpha}_{,j} + B_{ij}R_j\tilde{R}_i + N_{ij}\phi_{,j}\tilde{R}_i + \big(P_{ijrs}(\infty)R_{r,s}\big)\tilde{R}_{i,j} + K_{ij}(\infty)\alpha_{,j}\tilde{R}_i + \Lambda_{ij}(\infty)R_j\tilde{R}_i\\
&- \widehat{P'_{ijrs}}R_{r,s}\tilde{R}_{i,j} - \widehat{K'_{ij}}\alpha_{,j}\tilde{R}_i - \widehat{\Lambda'_{ij}}R_j\tilde{R}_i.
\end{aligned}$$

In particular, $\mathbf{a} : V^8 \times V^8 \to \mathbb{R}$. We need to show that $\mathbf{a}$ is continuous and coercive. Moreover, we need to prove that $\Psi^i \in V^{-1}$ for every $i = 1, \ldots, 4$, where V^{-1} is the dual space of V. Continuity is a straightforward consequence of the Cauchy–Schwarz and Young inequalities. For what concerns coercivity, by direct computations and making use of Assumption (v), we have:

$$\mathbf{a}\big((u_i, \phi, \alpha, R_i), (u_i, \phi, \alpha, R_i)\big) \geqslant \rho \parallel (u_i, \phi, \alpha, R_i) \parallel_{V^4}^2 .$$

Finally, with the help of Lemma 1, we have:

$$\begin{aligned}
\| - \int_0^\infty k'_{ij}(s)(E * f^{(8)})(s)ds\|_1 &\leqslant \int_0^\infty -k'_{ij}(s)(E* \| f^{(8)}\|_1)(s)ds\\
&\leqslant \kappa \int_0^\infty \ell(s)(E* \| f^{(8)}\|_1)(s)ds\\
&\leqslant \kappa \int_0^\infty \sqrt{\ell(s)}(E * \sqrt{\ell} \, \| f^{(8)}\|_1)(s)ds\\
&\leqslant \kappa\sqrt{\varkappa} \parallel E * \sqrt{\ell} \, \| f^{(8)}\|_1\|_{L^2(\mathbb{R}^+)}\\
&\leqslant \kappa\sqrt{\varkappa} \parallel \sqrt{\ell} \, \| f^{(8)}\|_1\|_{L^2(\mathbb{R}^+)}
\end{aligned}$$

Similarly, we can show that:

$$-\int_0^\infty K'_{ji}(s)F_i(s)ds \in H^1.$$

Therefore, $\Psi_3 \in V^{-1}$. By the same token, we have $\Psi_4 \in V^{-1}$. An application of the Lax–Milgram theorem yields $u_i, \phi, \alpha, R_i \in V$ satisfying equation (30). Thanks to equations (20), (22), (24), and (26), we immediately find also v_i, ψ, θ, and T_i. The final step to conclude the proof consists in showing that the solution we have found belongs to $\mathfrak{D}(\mathbb{A})$. The only thing we need to check is that $(\omega, \eta_i) \in \mathfrak{D}(\mathcal{T})$. However, using the fact that $\omega \in \mathcal{M}$ and $\boldsymbol{\eta} \in \boldsymbol{\mathcal{M}}$, we see at once that:

$$(\mathcal{T}\omega, \mathcal{T}\eta_i) = (\omega, \eta_i) - (\theta, T_i) - (f^{(8)}, f_i^{(9)}) \in \mathcal{N}.$$

Besides, it is straightforward to check that $\omega(s), \eta_i(s) \to 0$ in V as $s \to 0$. Hence, $(\omega, \eta_i) \in \mathfrak{D}(\mathcal{T})$ and the proof is finished. □

6. Exponential stability: the one-dimensional system

In this section, we focus on the exponential stability of equations (12)–(16) in one space dimension. In particular, the system becomes:

$$\rho u_{tt} = A u_{xx} + D\phi_x - a^*\alpha_{tx}, \tag{31}$$

$$J\phi_{tt} = A^*\phi_{xx} - NR_{tx} - Du_x - \xi\phi + F\alpha_t, \tag{32}$$

$$a\alpha_{tt} = k_\infty\alpha_{xx} - a^*u_{tx} - F\phi_t + K_\infty R_x - \int_0^\infty (k'(s)\omega_{xx}(s) + K'(s)\eta_x(s))ds, \tag{33}$$

$$\begin{aligned} BR_{tt} = {} & P_\infty R_{xx} - N\phi_{tx} - K_\infty\alpha_x - \Lambda_\infty R \\ & + \int_0^\infty \big(K'(s)\omega_x(s) + \Lambda'(s)\eta(s) - P'(s)\eta_{xx}(s)\big)ds, \end{aligned} \tag{34}$$

$$(\omega_t, \eta_t) = \mathcal{T}(\omega, \eta) + (\alpha_t, R_t). \tag{35}$$

To obtain the exponential stability, we need an additional hypothesis. Namely, we assume there exists $\delta > 0$, such that:

$$\begin{aligned} & (k''(s) + \delta k'(s))\xi^2 + 2(K''(s) + \delta K'(s))\zeta\xi \\ & + \big(\Lambda''(s) + \delta\Lambda'(s)\big)\zeta^2 + (P''(s) + \delta P'(s))\eta^2 \geqslant 0, \end{aligned} \tag{36}$$

for every $s \geqslant 0$ and $\xi, \zeta, \eta \in \mathbb{R}$.

Remark 2. Assumption (36) plays the same role of the well-known Dafermos inequality, which is usually stated for a generic memory kernel $\mu(s)$ as:

$$\mu'(s) + \delta\mu(s) \leqslant 0, \quad \forall s \geqslant 0. \tag{37}$$

For many equations with memory, equation (37) is sufficient to obtain the exponential stability. In our case, upon choosing ξ, ζ, and η in a suitable way, it is not difficult to show that the kernels $-k'(s)$, $-\Lambda'(s)$, and $-P'(s)$ satisfy equation (37).

The following theorem holds.

Theorem 2. Under Assumption (36), the semigroup $S(t)$ is exponentially stable.

The proof of Theorem 2 relies on the following abstract result, which is a simplified version of the famous characterization of Gearhart, Greiner, Huang, and Prüss. We refer the interested reader to Giorgi et al. [35] for the proof.

Proposition 1. Let $\mathbb{A}$ be the infinitesimal generator of a linear contraction semigroup $S(t) = e^{\mathbb{A}t}$ on a Banach space $\mathcal{X}$. Then, $S(t)$ is exponentially stable if and only if there exists $\sigma > 0$ such that:

$$\inf_{\lambda \in \mathbb{R}} \| (i\lambda - \mathbb{A})x\|_{\mathcal{X}} \geqslant \sigma \| x\|_{\mathcal{X}}, \quad \forall x \in \mathfrak{D}(\mathbb{A}).$$

We are now in position to prove the main result of this section. We proceed by contradiction and assume that $S(t)$ does not decay exponentially. On account of Proposition 1, this means that there exist sequences $\lambda_n \in \mathbb{R}$ and,

$$\boldsymbol{u}_n = (u_n, v_n, \phi_n, \psi_n, \alpha_n, \theta_n, R_n, T_n, \omega_n, \eta_n) \in \mathfrak{D}(\mathbb{A}),$$

such that:

$$\| \boldsymbol{u}_\mathbf{n} \|^2_{\mathcal{H}} = 1, \tag{38}$$

and

$$\| i\lambda_n \boldsymbol{u}_n - \mathbb{A}\boldsymbol{u}_n\|_{\mathcal{H}} \to 0. \tag{39}$$

Without loss of generality, we set all coefficients to be equal to 1. In components, equation (39) reads:

$$i\lambda_n u_n - v_n \to 0 \quad \text{in } V, \tag{40}$$

$$i\lambda_n v_n - \partial_{xx} u_n - \partial_x \phi_n + \partial_x \theta_n \to 0 \quad \text{in } H, \tag{41}$$

$$i\lambda_n \phi_n - \psi_n \to 0 \quad \text{in } V, \tag{42}$$

$$i\lambda_n \psi_n - \partial_{xx}\phi_n + \partial_x T_n + \partial_x u_n + \phi_n - \theta_n \to 0 \quad \text{in } H, \tag{43}$$

$$i\lambda_n \alpha_n - \theta_n \to 0 \quad \text{in } V, \tag{44}$$

$$i\lambda_n \theta_n - \mathrm{M}_n \to 0 \quad \text{in } H, \tag{45}$$

$$i\lambda_n R_n - T_n \to 0 \quad \text{in } V, \tag{46}$$

$$i\lambda_n T_n - \mathrm{N}_n \to 0 \quad \text{in } H, \tag{47}$$

$$i\lambda_n \omega_n - \mathcal{T}\omega_n - \theta_n \to 0 \quad \text{in } \mathcal{M}, \tag{48}$$

$$i\lambda_n \eta_n - \mathcal{T}\eta_n - T_n \to 0 \quad \text{in } \mathcal{M}, \tag{49}$$

where we recall that:

$$\mathrm{M}_n = -\partial_x v_n - \psi_n + \partial_{xx}\alpha_n + \partial_x R_n - \int_0^\infty (k'(s)\partial_{xx}\omega_n(s) + H'(s)\partial_x \eta_n(s))ds,$$

and

$$\mathrm{N}_n = -\partial_x \psi_n + \partial_{xx} R_n - \partial_x \alpha_n - R_n - \int_0^\infty \big(P'(s)\partial_{xx}\eta_n(s) - H'(s)\partial_x \omega_n(s) - \Lambda'(s)\eta_n(s)\big)ds.$$

The contradiction will be obtained by showing that $\|\boldsymbol{u}_n\|^2 \to 0$. First, we observe that, by the dissipativity of $\mathbb{A}$:

$$\langle \mathbb{A}\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\mathcal{H}} = \langle \mathcal{T}(\omega_n, \eta_n), (\omega_n, \eta_n)\rangle_{\mathcal{N}} \leqslant -\delta \, \|(\omega_n, \eta_n)\|^2_{\mathcal{N}} \,,$$

where the inequality follows from assumption (36). Then, since,

$$\Re\mathfrak{e}\, \langle i\lambda_n \boldsymbol{u}_n - \mathbb{A}\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\mathcal{H}} = -\Re\mathfrak{e}\, \langle \mathbb{A}\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\mathcal{H}} \to 0,$$

we have:

$$\delta \parallel (\omega_n, \eta_n) \parallel^2_{\mathcal{N}} \leqslant -\Re\mathfrak{e}\langle \mathbb{A}\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\mathcal{H}} \to 0.$$

In the same spirit as Pata [42], we now distinguish two cases.

Case 1: $\lambda_n \nrightarrow 0$. Up to a subsequence, we can assume that:

$$\inf_n |\lambda_n| > 0. \tag{50}$$

The proof will be carried out with the help of some technical lemmas.

Lemma 4. Up to a subsequence, we have that:

$$\lim_{n\to\infty} \parallel \theta_n \parallel_H = 0,$$

and

$$\lim_{n\to\infty} \parallel T_n \parallel_H = 0.$$

Proof. We will prove the lemma only for θ_n. The proof for T_n is identical and therefore omitted. We preliminary show that:

$$\sup_{n\in\mathbb{N}} |\lambda_n| \parallel \theta_n \parallel_{V^{-1}} < \infty,$$

where V^{-1} is the dual space of V. Henceforth, we will denote by $\|\cdot\|_{-1}$ the norm in V^{-1}, coherently with the notation used for V. We can write:

$$i\lambda_n\theta_n = i\lambda_n\theta_n + \mathrm{M}_n - \mathrm{M}_n.$$

Hence,

$$\parallel i\lambda_n\theta_n \parallel_{-1} \leqslant \parallel i\lambda_n\theta_n + \mathrm{M}_n \parallel_{-1} + \parallel \mathrm{M}_n \parallel_{-1}.$$

The first term of the sum is clearly bounded, being infinitesimal. However,

$$\parallel \mathrm{M}_n \parallel_{-1} \leqslant \parallel v_n \parallel + \parallel \psi_n \parallel_{-1} + \parallel \alpha_n \parallel_1 + \parallel R_n \parallel + \left\| \int_0^\infty -k'(s)\partial_{xx}\omega_n(s)ds \right\|_{-1} + \left\| \int_0^\infty -H'(s)\partial_x\eta_n(s)ds \right\|_{-1}.$$

We can bound the final two terms on the right-hand side in the following way:

$$\begin{aligned}\left\| \int_0^\infty -k'(s)\partial_{xx}\omega_n(s)ds \right\|_{-1} &\leqslant \int_0^\infty k'(s) \parallel \omega_n(s) \parallel_1 ds \\ &\leqslant \kappa \int_0^\infty \ell(s) \parallel \omega_n(s) \parallel_1 ds \\ &= \kappa \int_0^\infty \sqrt{\ell(s)}\sqrt{\ell(s)} \parallel \omega_n(s) \parallel_1 ds \\ &\leqslant \kappa\sqrt{\varkappa} \parallel \omega_n \parallel_{\mathcal{M}}.\end{aligned}$$

By the same token, one can show that the other integral term is also bounded. We rephrase equation (48) as:

$$i\lambda_n\omega_n - \mathcal{T}\omega_n - \theta_n = \varepsilon_n,$$

with $\varepsilon_n \to 0$ in $\mathcal{M}$. Since $\omega_n \in \mathfrak{D}(\mathcal{T})$, we can solve the above equation to obtain the explicit representation:

$$\omega_n(s) = \frac{1}{i\lambda_n}(1 - e^{-i\lambda_n s})\theta_n + \int_0^s e^{-i\lambda_n(s-y)}\varepsilon_n(y)dy. \tag{51}$$

Now observe that:

$$|i\lambda_n \langle \omega_n, A^{-1}\theta_n \rangle_{\mathcal{M}}| \leqslant |\lambda_n| \parallel \theta_n \|_{-1} \int_0^\infty \ell(s) \parallel \omega_n \|_1 ds \to 0,$$

since $\omega_n \to 0$ in $\mathcal{M}$ and $\parallel \theta_n \|_{-1}$ was bounded. Hence, we have:

$$|i\lambda_n \langle \omega_n, A^{-1}\theta_n \rangle_{\mathcal{M}}| = a_n \parallel \theta_n \|^2 + b_n \to 0, \tag{52}$$

having set,

$$a_n = \int_0^\infty \ell(s)(1 - e^{-i\lambda_n s})ds,$$
$$b_n = i\lambda_n \int_0^\infty \ell(s)\left(\int_0^s e^{-i\lambda_n(s-y)}\langle \varepsilon_n(y), A^{-1}\theta_n \rangle_V dy\right)ds.$$

Following exactly the same reasoning of Pata [42, Lemma 5.5], we see that $b_n \to 0$. For what concerns a_n, we consider two separate cases. Let $\lambda_\star$ be a limit point of the sequence λ_n. From equation (50), we have:

$$\lambda_\star \in [-\infty, \infty]\backslash\{0\}.$$

If $\lambda_\star \in \{-\infty, \infty\}$, then by the Riemann–Lebesgue lemma, we have the convergence (up to a subsequence):

$$a_n \to \int_0^\infty \ell(s)ds > 0.$$

However, if $\lambda_\star \in \mathbb{R}\backslash\{0\}$,

$$a_n \to \int_0^\infty \ell(s)(1 - e^{-i\lambda_\star s})ds,$$

and

$$\mathfrak{Re}\int_0^\infty \ell(s)(1 - e^{-i\lambda_\star s})ds = \int_0^\infty \ell(s)(1 - \cos\lambda_\star s)ds > 0.$$

In both cases, in order for equation (52) to hold, it must be $\parallel \theta_n \parallel \to 0$. □

Lemma 5. Up to a subsequence,

$$\lim_{n\to\infty} \parallel R_n \|_1 = \lim_{n\to\infty} \parallel \alpha_n \|_1 \to 0.$$

Proof. Define,

$$\rho_n(s) = \frac{1}{i\lambda_n}(1 - e^{-i\lambda_n s})(\theta_n - i\lambda_n \alpha_n).$$

In view of equation (44), it is clear that $\rho_n \to 0$ in $\mathcal{M}$. We can then rewrite equation (51) as:

$$\omega_n(s) = (1 - \mathrm{e}^{-i\lambda_n s})\alpha_n + \int_0^s \mathrm{e}^{-i\lambda_n(s-y)}\varepsilon_n(y)dy + \rho_n(s),$$

which, on account of Step 1, entails:

$$\langle \omega_n, \alpha_n \rangle_{\mathcal{M}} = a_n \parallel \alpha_n \parallel_1^2 + c_n + \langle \rho_n, \alpha_n \rangle_{\mathcal{M}} \to 0,$$

with a_n as above and:

$$c_n = \int_0^\infty \ell(s) \left(\int_0^s \mathrm{e}^{i\lambda_n(s-y)} \langle \varepsilon_n(y), \alpha_n \rangle_1 dy \right) ds.$$

Clearly,

$$\langle \rho_n, \alpha_n \rangle_{\mathcal{M}} \to 0.$$

Besides, with the same reasoning of Lemma 4, $c_n \to 0$. Hence, we obtain that $\parallel \alpha_n \parallel_1 \to 0$. This proof can then be repeated to show that $\parallel R_n \parallel_1 \to 0$. □

Conclusion of the proof. At this point, we proceed as in Bazarra et al. [30]. We multiply equation (47) by $\lambda_n^{-1}\partial_x\phi_n$. In view of equation (42), and exploiting the convergences obtained above, we get:

$$i \parallel \phi_n \parallel_1^2 + \langle \partial_x R_n, \frac{\partial_{xx}\phi_n}{\lambda_n} \rangle \to 0.$$

Thanks to equation (43), we see that $\partial_{xx}\phi_n/\lambda_n$ is bounded. In turn, this yields that:

$$\parallel \phi_n \parallel_1^2 \to 0.$$

In a similar fashion, using equations (45) and (41), it is possible to show that $\parallel u_n \parallel_1 \to 0$ too, as $n \to \infty$. Finally, a straightforward application of equations (40) and (42) yields the convergence of $v_n, \psi_n \to 0$ in H.

Case 2: $\lambda_n \to 0$. In this case, in light of equations (38), (40), (44), and (46), we have:

$$v_n \to 0 \quad \text{in } V,$$
$$\theta_n \to 0 \quad \text{in } V,$$
$$T_n \to 0 \quad \text{in } V.$$

In turn, due to equations (41) and (43), this entails:

$$-\partial_{xx}u_n - \partial_x\phi_n \to 0 \quad \text{in } H, \tag{53}$$

$$-\partial_{xx}\phi_n + \partial_x u_n + \phi_n \to 0 \quad \text{in } H. \tag{54}$$

Multiplying equation (53) by u_n, equation (54) by ϕ_n, and summing up the two, we get:

$$\parallel \partial_x u_n \parallel^2 + \langle \phi_n, \partial_x u_n \rangle + \langle \partial_x u_n, \phi_n \rangle + \parallel \phi_n \parallel^2 + \parallel \partial_x \phi_n \parallel^2 \to 0. \tag{55}$$

Since,

$$\parallel \partial_x u_n \parallel^2 + \langle \phi_n, \partial_x u_n \rangle + \langle \partial_x u_n, \phi_n \rangle + \parallel \phi_n \parallel^2 = \parallel \partial_x u_n + \phi_n \parallel^2 \geqslant 0,$$

by equation (55), we have $\parallel \phi_n \parallel \to 0$ in V. In turn, this gives us the convergence of $u_n \to 0$ in V. An almost identical reasoning yields the convergence of R_n and α_n to 0 in the space V.

Remark 3. If we do not assume all the constants to be equal to 1, we do not obtain a perfect square in equation (55). However, the thesis follows in the same way thanks to Assumption (ii).

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