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Localised analytic torsion and relative analytic torsion for non compact Lie groups of type I

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Let G be a (non compact) connected, simply connected, locally compact, second countable Lie group, either abelian or unimodular of type I, and let ρ be an irreducible unitary representation of G. Then, we define the analytic torsion of G localised at the representation ρ. The idea of considering localised invariants is due to Brodzki, Niblo, Plymen and Wright [5], and was exploited in [31] to define a localised eta function. Next, let Γ be a discrete co compact subgroup of G. We use the localised analytic torsion to define the relative analytic torsion of the pair (G, Γ), and we prove that the last coincides with the Lott L^2 analytic torsion of a covering space. We illustrate these constructions analysing in some details two examples: the abelian case, and the case $G = H$, the Heisenberg group.

Contents

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1. Introduction

The aim of this note is to discuss some possible generalisation of analytic torsion [26] to non compact Lie groups. First, we introduce and investigate the concept of localised analytic torsion for a non compact Lie group G. In brief, localised analytic torsion is the analytic torsion of the field of operators defined by localising the natural Hodge Laplace operator on G at some representation of G. This approach is suggested by work of J. Brodzki, G.A. Niblo, R. Plymen, and N. Wright [5], where the idea of operators localised at some representation of a non compact Lie group was originally introduced. The key point, for what we are concerned with, is that the spectrum of such operators turns out to be discrete. Following this approach, we introduced and studied in [31] the eta function of the Dirac operator localised at a representation of the universal covering group of $SL(2,\mathbb{R})$. Second, assuming that G has a discrete co-compact subgroup Γ, we define what we call the relative analytic torsion of the pair (G,Γ). This is the natural analog of the L^2 analytic torsion of a covering space introduced by J. Lott [19], and indeed we prove their equivalence. In order to illustrate these constructions, we present a detailed analysis of two particular examples: the abelian case, $G=\mathbb{R}$, and the case of $G=H$, the Heisenberg group. We conclude by proposing a "geometric" interpretation of relative analytic torsion in terms of the classical analytic torsion. This interpretation works nicely in the abelian case, though it is less natural in the case of the Heisenberg group.

Now, let us look the plan of the work in more details. The Reidemeister torsion (R torsion) of a finite combinatorial complex K is a kind of determinant in the Whitehead group $Wh(\mathbb{Z}\pi_1(K))$, which describes the way in which the 'cells' of the universal covering complex are fitted together with respect to the action of the fundamental group [21]. When K is a triangulation of a compact, connected, oriented Riemannian manifold (M,g) (for simplicity without boundary), it is natural to 'change' the ring by some orthogonal representation $\alpha : \pi_1(M) \to O(V)$ of the fundamental group. Then, R torsion is a topological invariant of the triple (M, h, α), where h is a basis for the rational homology determinant line bundle of M. In this situation, Ray and Singer introduced an analytic object called analytic torsion, constructed from the de Rham complex of M twisted by α, and conjectured its equivalence with R torsion, when the homology basis h is the one fixed by an orthonormal graded basis of harmonic forms determined by g [26]. J. Cheeger [7] and W. Müller [22] independently proved this conjecture. The analytic torsion is defined as follows. Let $(\Omega^\bullet(M, E_\alpha), d^\bullet)$ be the de Rham complex of the forms on M with coefficients in the real vector bundle $E_\alpha = \widetilde{M} \times_\alpha V$, associated to α. The Riemannian metric on M provides an inner product and makes the de Rham complex a complex of Hilbert spaces. Let $\Delta_\alpha^\bullet$ be the associated Hodge Laplace operator. Whence, $\Delta_\alpha^\bullet$ will have a pure point spectrum, with unique accumulation point at infinity. Then, for complex s, with $\mathrm{Re}(s)$ large, the zeta function of Δ_α^q is defined by [26]

$$\zeta(s, \Delta_\alpha^q) = \sum_{0 \neq \lambda \in \mathrm{Sp}\Delta_\alpha^{(q)}} \lambda^{-s}.$$

Setting (that we call analytic torsion zeta function)

$$t(s; M, \alpha) = \frac{1}{2} \sum_{q=0}^{\dim M} (-1)^q q \zeta(s, \Delta_\alpha^{(q)}),$$

we define the analytic torsion of M in the representation α:

$$T(M, \alpha) = t'(0; M, \alpha).$$

If M is not compact, it is not clear what torsion would be. In [31], we proposed an analog of the eta function (that is a spectral invariant defined in terms of some zeta function for compact manifold, closely similar to analytic torsion), called localised eta function for non compact Lie groups. Here we consider the corresponding definition for analytic torsion. For let G be a (non compact) connected, simply connected, locally compact, second countable Lie group, either abelian or unimodular of type I, equipped with a fixed Haar measure. Then G is a smooth Riemannian manifold with a particular metric determined by left invariance. For this reason we will omit explicit reference to the metric in the notation in the following. Let Δ denote (some self adjoint extension of the formal) Hodge Laplace operator on square integrable forms on G associated to this metric. Let ρ be an irreducible unitary representation of G. There exists a measurable field of Hilbert spaces $h \mapsto \mathcal{H}_h$, called the canonical field, and a field of measurable representations $h \mapsto \rho_h$, such that ρ_h belongs to the class $h \in \hat{G}$, the dual group of G [11, 8.6.1, 8.6.2]. In this setting, Δ determines a continuous field of self adjoint operators, $d\rho_h \Delta$. Assuming that for each fixed h the spectrum $\mathrm{Sp} d\rho_h \Delta$ of $d\rho_h \Delta$ is a discrete sequence with some suitable properties (such sequences are called of spectral type in [30], see also [32]), we may associate to $\mathrm{Sp} d\rho_h \Delta$ some spectral functions, and in particular the zeta function.

If one is able to prove that the analytic extension of this zeta function is regular at the origin, one can define the localised analytic torsion of $d\rho_h\Delta$ as the derivative at the origin of the zeta function. All this is in Section 3.

Next, assuming that G has a discrete co-compact subgroup Γ, we may introduce some analogue of analytic torsion for the pair (G,Γ). The construction is inspired by the one proposed by Lott in [19] (see also [20] and [6]), for a covering space. This was largely based on the properties of the heat kernel. In fact, using the Mellin transform, the zeta function of the Hodge Laplace operator may be restated by

$$\zeta(s,\Delta_\alpha^q)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}e^{-t\Delta_\alpha^{(q)}}dt,$$

and the knowledge of the behaviour of the trace of the heat operator for large and small t are the main tools to study the analytic properties of the zeta function. Hence, a key point in the construction of the analogue of the analytic torsion for covering is to replace the trace of the heat operator with a suitable alternative. This alternative is given by the Diximier trace. Our definition of what we call relative torsion of the pair (G,Γ) follows the alternative approach of exploiting the localised analytic torsion and gluing it along the dual group. For we need some technical result about traces and fields of operator that we develop in Section 2. The second part of Section 3 contains our definition of relative torsion for a pair (G,Γ).

Finally, we consider two examples to illustrate our results. The first is the abelian case, developed in Section 4. In the last Section 5, we study the case $G=H$ the three dimensional Heisenberg group. Working out the abelian case, it turns out that the relative analytic torsion of the pair $(\mathbb{R},\mathbb{Z})$ has a natural description as the continuous sum with respect to the Plancherel measure of the analytic torsion of the quotient space $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ over the irreducible representation of $\mathbb{Z}$. We try to find out a similar interpretation for the relative torsion of the pair (H,Γ) (where Γ is the discrete integral Heisenberg group). Unfortunately, we succeed partially: specifically, the spectral invariants appearing are not so natural as in the abelian case, and do not have a geometric interpretation.

2. Traces and fields of operators

In this work we assume that G is a connected, simply connected, locally compact, second countable Lie group, either abelian or unimodular of type I, with a discrete co-compact subgroup Γ, and a fixed Haar measure dg. These assumptions may appear quite overwhelming, however they cover the cases of interest in this work. A comprehensive approach for a larger family of Lie groups seems extremely unlikely, but other families may be tackled with the suitable technical arrangements.

Note that in the following we will use results of [11], given there for separable postliminal groups. However, for second countable groups type I and postliminal are equivalent [11, 13.9.4].

Let $\hat{G}$ denote the dual space of equivalence classes of unitary representations of G. There exists a measurable field of Hilbert spaces $h \mapsto \mathcal{H}_h$, called the canonical field, and a field of measurable representations $h \mapsto \rho_h$, such that ρ_h belongs to the class $h \in \hat{G}$ [11, 8.6.1, 8.6.2]. Then, the Plancherel theorem states that there exists a unique positive measure $\mathrm{d}\mu(h)$ on $\hat{G}$ (the Plancherel measure), and an isomorphism [11, 18.8.1,2,3]

$$\mathcal{F} : L^2(G) \to \int_{\hat{G}}^{\oplus} \mathcal{H}_h \otimes \bar{\mathcal{H}}_h d\mu(h),$$

that extends to the following isomorphisms (all denoted by the same symbol)

$$\mathcal{F} : L \to \int_{\hat{G}}^{\oplus} \rho_h \otimes 1 d\mu(h), \qquad \mathcal{F} : R \to \int_{\hat{G}}^{\oplus} 1 \otimes \bar{\rho}_h d\mu(h),$$

$$\mathcal{F} : \mathcal{L} \to \int_{\hat{G}}^{\oplus} B(\mathcal{H}_h) \otimes \mathbb{C} d\mu(h), \qquad \mathcal{F} : \mathcal{R} \to \int_{\hat{G}}^{\oplus} \mathbb{C} \otimes B(\bar{\mathcal{H}}_h) d\mu(h),$$

where $\mathcal{L}$ and $\mathcal{R}$ are the von Neumann algebras on $L^2(G)$ generated respectively by the left and right regular representations L and R of G, and $B(\mathcal{H}_h)$ denotes the space of the bounded operators on $\mathcal{H}_h$.

In particular, if $u \in L^1(G) \cap L^2(G)$,

$$\hat{u}(h) = \mathcal{F}(u)(h) = \int_G u(g)\rho_h(g^{-1})dg,$$

the group Fourier transform, and we have the inversion formula [14, (7.38)]

$$u(g) = \mathcal{F}^{-1}\mathcal{F}(u)(g) = \int_{\hat{G}}^{\oplus} \mathrm{Tr}\left(\rho_h(g)\mathcal{F}(u)(h)\right) d\mu(h).$$

The last point, that is the more relevant for our analysis, is the isomorphism

$$\mathcal{F} : \mathrm{Tr}_G \to \int_{\hat{G}}^{\oplus} \mathrm{Tr} \quad d\mu(h),$$

where Tr_G is the natural Diximier trace on $L^2(G)$ (we described it in more details in the following), and Tr the standard trace of a trace class operator in $(B(\mathcal{H}_h) \otimes \mathbb{C})^+$ (the subspace of positive operators).

To proceed, take a positive operator T in $\mathcal{L}$ (or in $\mathcal{R}$). Then, we have a measurable field of operators $h \to \mathcal{F}A\mathcal{F}^{-1}(h)$ on $\int_{\hat{G}}^{\oplus} \mathcal{H}_h \otimes \mathbb{C} d\mu(h)$ (or $\int_{\hat{G}}^{\oplus} \mathbb{C} \otimes \bar{\mathcal{H}}_h d\mu(h)$), and

$$\mathrm{Tr}_G T = \int_{\hat{G}}^{\oplus} \mathrm{Tr}\mathcal{F}T\mathcal{F}^{-1}(h)d\mu(h). \tag{2.1}$$

We explore this construction in some details, specialising to the family of operators we will be interested in.

2.0.1. The Diximier trace

We detail the definition of the traces Tr_G and Tr_Γ, and we provide a local description, main reference for this part are [1] [2] [10]. To begin with, take $f \in L^1(G) \cap L^2(G)$. By the Plancherel Theorem,

$$f(e) = \int_{\hat{G}}^{\oplus} \mathrm{Tr}\hat{f}(h)d\mu(h),$$

is finite since the operator $\hat{f}(h)$ is Hilbert Schmidt, and therefore defines a trace on $L^1(G) \cap L^2(G)$. Since, $C_0^\infty(G) \subseteq L^1(G) \cap L^2(G)$, we have that $C_0^\infty(G) \subseteq \langle R(L^1(G) \cap L^2(G))\rangle$ as subalgebras (the first with convolution product), and the inclusion is dense, and hence we may extend by closure and linearity, to have a trace on the von Neumann algebra $\langle R(L^1(G) \cap L^2(G))\rangle$, that, viewed as a subalgebra of $R(L^1(G))$, coincides with the subalgebra of the von Neumann algebra generated by $R(G)$, i.e. $\mathcal{R}_G$. Therefore, we have a function with real values, and restricting to the family $\mathcal{R}_G^+$ of the operators such that this value is positive, we have a trace:

$$\mathrm{Tr}_G : \mathcal{R}_G^+ \to \mathbb{R}^+,$$
$$\mathrm{Tr}_G : A \mapsto \mathrm{Tr}_G A.$$

In particular, if $A = R(f) \in \mathcal{R}_G^+$,

$$R(f)(u)(h) = \int_G f(g)R(g)(u)(h)dg = \int_G f(g)u(hg)dg = u \star f(h),$$

then,

$$\mathrm{Tr}_G R(f) = f(e).$$

Next, since restriction of left multiplication defines a unitary action ℓ of Γ on $L^2(G)$, we have the identification

$$L^2(G) \cong l^2(\Gamma) \otimes L^2(Q) \cong l^2(\Gamma) \otimes L^2(\Gamma\backslash G),$$

where Q is a fundamental domain of ℓ. Denote by $\mathcal{B}$ the von Neumann algebra of the bounded operators on $L^2(G)$ that commute with Γ, $\mathcal{B} = \{T \in B(L^2(G)) \mid T\ell(\gamma) = \ell(\gamma)T, \forall \gamma \in \Gamma\}$.

Under the identification above, the action of Γ on $L^2(G)$ corresponds to the left regular representation L_Γ of Γ on $l^2(\Gamma)$ extended by the identity on $L^2(\Gamma\backslash G)$: $\ell \cong L_\Gamma \otimes 1$. But then, the von Neumann algebra $\mathcal{B}$ is the von Neumann algebra generated by $R_\Gamma(\Gamma)$ tensor the space of all bounded operators,

$$\mathcal{B} \cong \mathcal{R}_\Gamma \otimes B(L^2(\Gamma\backslash G)).$$

We may now define a trace on $\mathcal{B}$ as follows. A trace on $\mathcal{R}_\Gamma$ is defined on the generators, i.e. on $R(\Gamma)$, by

$$\begin{aligned} \mathrm{tr}_\Gamma &: R(\Gamma) \to \mathbb{R}^+, \\ \mathrm{tr}_\Gamma &: R(\gamma) \mapsto \delta_{\gamma,e}, \end{aligned}$$

and extended by closure and linearity. A trace on $B(L^2(\Gamma\backslash G))$ is the Hilbert Schmidt trace Tr, so we put

$$\begin{aligned} \mathrm{Tr}_\Gamma &: \mathcal{R}_\Gamma \otimes B(L^2(\Gamma\backslash G)) \to \mathbb{R}^+, \\ \mathrm{Tr}_\Gamma &: S \otimes T \mapsto \mathrm{tr}_\Gamma S \ \mathrm{Tr} T, \end{aligned}$$

and we say that $S \otimes T$ is of Γ-trace class if its trace is finite.

There is a local description, more suitable to work in the smooth category, of this trace as follows [1, pg. 58 and Proposition 4.16, pg. 63]. Any bounded operator A on $L^2(G)$ has a Schwartz kernel $k(a, b; A)$ that is a distribution on $G \times G$. If A is left Γ invariant, then

$$k(\gamma a, \gamma b; A) = k(a, b; A),$$

and viceversa, and therefore $k(a, b; \mathcal{A})$ may be viewed as a distribution on $(G \times G)/\Gamma$. We have the following result [1]. Suppose that $A \in \mathcal{B}$ has a smooth kernel $k(a, b; A)$, and is positive and self adjoint. Then, A is of Γ-trace class, and

$$\mathrm{Tr}_\Gamma A = \int_{\Gamma\backslash G} k(g, g; A) dg.$$

These two traces, Tr_G and Tr_Γ, are indeed equivalent, up to a scalar factor. For observe that the von Neumann algebra $\mathcal{B} = \mathcal{R}_\Gamma \otimes B(L^2(\Gamma\backslash G))$ contains the von Neumann algebra $\mathcal{R}_G$. Suppose that A is an G-left invariant operator on $L^2(G)$. Whence, $A \in L(G)' = \mathcal{R}_G \subseteq \mathcal{B}$, whence the Γ-trace of A is defined. Note that $R(L^1(G)) \subseteq \mathcal{R}_G$, so we have in $\mathcal{R}_G$ the operators $R(f)$, with $f \in L^1(G) \cap L^2(G)$. Since,

$$R(f)(u)(a) = \int_G f(g)R(g)(u)(a)dg = \int_G f(g)u(ag)dg = \int_G f(a^{-1}t)u(t)dt,$$

$R(f)$ is the integral operator with kernel $k(a, g; R(f)) = f(a^{-1}g)$. The Γ trace of $R(f)$ is well defined and

$$\mathrm{Tr}_\Gamma R(f) = k(e, e; R(f)) \int_{\Gamma\backslash G} dg = \mathrm{Vol}(\Gamma\backslash G) f(e) = \mathrm{Vol}(\Gamma\backslash G)\mathrm{Tr}_G R(f).$$

Since $R(L^1(G))$ is dense in $\mathcal{R}_G$, it follows that for all operators $A \in \mathcal{R}_G$:

$$\mathrm{Tr}_\Gamma A = \mathrm{Vol}(\Gamma\backslash G)\mathrm{Tr}_G A.$$

2.0.2. The localised trace

We pass to give a suitable interpretation of the right side of equation (2.1). The key point is a suitable description of the group Fourier transform and of the dual Haar measure. This may depend strongly on the particular group. However, we pursue the general approach as far as possible.

We start by observing that the representation ρ_h has an associated representation [16] [29, 3.1] [25]

$$d\rho_h : \mathfrak{g} \to \mathcal{L}(\mathcal{H}),$$

where $\mathcal{L}(\mathcal{H})$ is the Lie algebra of the skew symmetric operators on $\mathcal{H}$, and formally (whenever the limit exists)

$$d\rho_h(V)(u) = \lim_{t\to 0} \frac{\rho_h(\mathrm{e}^{tV})(u) - u}{t} = \frac{d}{dt}\rho_h(\mathrm{e}^{tV})(u)\bigg|_{t=0}.$$

This representation extends to a representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, and we will use this fact implicitly, without further comment or variation of the notation. In particular, this means that we will apply $d\rho_h$ to polynomials on elements of $\mathfrak{g}$ (at most of degree two with real coefficients) just by linearity [16, Theorem 3.1].

We recall a few important properties of $d\rho_h$. Consider the subspace of $\mathcal{H}$ of the compactly supported $\mathcal{C}^\infty$ vector fields:

$$\mathcal{H}_0 = \{v \in \mathcal{H} \mid v = \rho_h(\varphi)(u),\ \varphi \in \mathcal{C}_0^\infty(G), u \in \mathcal{H}\}.$$

Then, if $V \in \mathfrak{g}$, we have the following facts:

(1) $\mathcal{H}_0$ is dense in $\mathcal{H}$,
(2) $\mathcal{H}_0 \leq \mathsf{D}(d\rho_h(V))$, for all $V \in \mathfrak{g}$,
(3) $d\rho_h(V)(\mathcal{H}_0) \leq \mathcal{H}_0$, for all $V \in \mathfrak{g}$,

(4) $\rho_h(g)(\mathcal{H}_0) \leq \mathcal{H}_0$, for all $g \in G$,
(5) the minimal closed extension of the restriction of $d\rho_h(V)$ to $\mathcal{H}_0$ is $d\rho_h(V)$.

Now, take a basis $\{X_k\}$ of $\mathfrak{g}$, and let $\{x_k(t) = \gamma_{V_k,e}(t) = \mathrm{e}^{tX_k}\}$ denote the corresponding local coordinate system on G near e, i.e. the integral curves of the X_k at e. The infinitesimal generators $d\rho_h(X_k)$ of ρ_h at g are defined by

$$\mathrm{e}^{td\rho_h(X_k)} = \rho_h(\mathrm{e}^{tX_k}).$$

Thus,

$$d\rho_h(X_k) = \frac{d}{dt}\mathrm{e}^{td\rho_h(X_k)}\bigg|_{t=0} = \frac{d}{dt}\rho_h(\mathrm{e}^{tX_k})\bigg|_{t=0} = \partial_{x_k}\rho_h(x)|_{x(0)} = X_k(\rho_h)(e). \tag{2.2}$$

Consider the operator valued smooth function $\varphi(g)\rho(g^{-1})$, $\varphi \in \mathcal{C}_0^\infty(G)$. Then

$$\begin{aligned}\partial_{x_k}(\varphi(x(g))\rho_h^\dagger(x(g))) &= \partial_{x_k}(\varphi(x(g)))\rho_h^\dagger(x(g)) + \varphi(x(g))\partial_{x_k}\rho_h^\dagger(x(g))\\ &= \partial_{x_k}(\varphi(x(g)))\rho_h^\dagger(x(g)) + \varphi(x(g))\partial_{x_k}\rho_h^\dagger(x(g))\\ &= X_k(\varphi)(g)\rho_h^\dagger(x(g)) + \varphi(g)X_k(\rho_h^\dagger)(g).\end{aligned}$$

Since

$$d(\varphi(x(g))\rho_h^\dagger(x(g)) = \sum_k \partial_{x_k}(\varphi(x(g))\rho_h^\dagger(x(g)))dx_k,$$

and by the Stokes theorem on G,

$$\int_G d(\varphi(g)\rho_h^\dagger(g)) = 0,$$

we find that

$$\int_G X_k(\varphi)(g)\rho_h^\dagger(g)dg = \int_G \varphi(g)X_k(\rho_h^\dagger)(g)dg. \tag{2.3}$$

We use these facts as follows. First, observe that the left side of equation (2.3) is the group Fourier transform of $X_k(\varphi)$:

$$\mathcal{F}(X_k(\varphi))(h) = \int_G X_k(\varphi)(g)\rho_h^\dagger(g)dg.$$

Second, about the right side, note that as a function of t,

$$\frac{d}{dt}\rho_h(\mathrm{e}^{tX_k}) = X_k(\rho_h)(g);$$

but then, with $g = \mathrm{e}^{sX_k}$,

$$X_k(\rho_h)(g) = \frac{d}{dt}\rho_h(\mathrm{e}^{(s+t)X_k})\Big|_{t=0} = \rho_h(g)\frac{d}{dt}\rho_h(\mathrm{e}^{tX_k})\Big|_{t=0} = \rho_h(g)X_k(\rho_h)(e).$$

Whence

$$\int_G \varphi(g)X_k(\rho_h^\dagger)(g)dg = X_k(\rho_h^\dagger)(e)\int_G \varphi(g)\rho(g^{-1}dg = -d\rho_h(X_k)\mathcal{F}(\varphi),$$

and we have proved that

$$\mathcal{F}X_k\mathcal{F}^{-1}(h) = d\rho_h(X_k). \tag{2.4}$$

Proposition 2.0.3. *Let T be a positive operator in $\mathcal{R}$, then*

$$\mathrm{Tr}_\Gamma T = \mathrm{Vol}(\Gamma\backslash G)\int_{\hat{G}}^{\oplus} \mathrm{Tr} d\rho_h(T)d\mu(h).$$

Furthermore, if T is a self adjoint integral operator with smooth kernel $k(h, g; T)$, then

$$\int_{\Gamma\backslash G} k(g, g; T)dg = \mathrm{Vol}(\Gamma\backslash G)\int_{\hat{G}}^{\oplus} \mathrm{Tr} d\rho_h(T)d\mu(h).$$

3. Localised analytic torsion and relative analytic torsion

We are now ready to introduce the definitions of the main objects of interest in this work. Given an adjointable operator S acting on $L^2(G)$, and a representation $(\rho_h, \mathcal{H}_h)$, $h \in \hat{G}$, we call the localisation of S at ρ_h the fibre $S(h)$ of the field of operators $h \mapsto \mathcal{F}T\mathcal{F}^{-1}(h)$ acting on $\int_{\hat{G}}^{\oplus} \mathcal{H}_h \otimes \bar{\mathcal{H}}_h d\mu(h)$. We call localised spectrum of S the spectrum of $S(h)$ (these definitions are a particular instance of [5, 3.1]). Thus the localised spectrum of S is the fibre of a field of spectra, in particular in the cases of interest, the fibre of a continuous field of sequences of real numbers with unique accumulation point at infinity. Note that, by equation (2.4), $S(h) = d\rho_h(S)$.

In order to proceed, we fix the left invariant Riemannian metric on G that makes the basis $\{X_k\}$ orthonormal, and denote by $\star$ the induced Hodge operator. Since G is a Lie group, we can fix global bases, and we have the decomposition

$$L^2(\gamma(G, \Lambda^\bullet T^*G)) = L^2(G) \otimes \Lambda^\bullet T^*G.$$

We denote by $\Omega^\bullet(G) = \Gamma(G, \Lambda^\bullet T^*G))$ the space of smooth sections, and by $\Omega_0^\bullet(G)$ the space of the smooth sections with compact support. We denote by d the minimal closed extension of the exterior derivative operator on $\Omega_0^\bullet(G)$, and by $(\mathsf{D}(d^\bullet), d^\bullet)$ the associated de Rham complex of Hilbert spaces and closed operators. We denote by δ the adjoint of d, and by $\Delta = d\delta + \delta \mathrm{d}$ the associated Hodge Laplace operator. This is a self adjoint non negative operator with maximal domain the Sobolev space $H^2(G)$ (equivalently we may construct the formal Hodge Laplace operator on the $\Omega_0^\bullet(G)$, that is essentially self adjoint [8] [24]). In particular, observe that Δ is left invariant by construction.

Since G is the universal cover of the compact manifold $\Gamma\backslash G$, the heat operator $\mathrm{e}^{-t\Delta}$, $t > 0$, is a bounded integral operator with smooth kernel by a form in $G \times G$ [8]. Whence, it is in $\mathcal{R}_G$ and it has finite G trace [1]. In particular, by the Plancherel Theorem, this means that $d\rho_h(\mathrm{e}^{-t\Delta^{(q)}})$ is of trace class, and hence $\Delta^q(h)$ has discrete spectrum, denoted by $\mathrm{Sp}\Delta^{(q)}(h)$. Therefore,

$$\mathrm{Tr}_\Gamma \mathrm{e}^{-t\Delta^{(q)}} = \mathrm{Vol}(\Gamma\backslash G) \int_{\hat{G}}^{\oplus} \mathrm{Tr} d\rho_h(\mathrm{e}^{-t\Delta^{(q)}}) d\mu(h), \tag{3.1}$$

and

$$\mathrm{Tr} d\rho_h(\mathrm{e}^{-t\Delta^{(q)}}) = \mathrm{Tr}\mathrm{e}^{-t d\rho_h \Delta^{(q)}} = \sum_{\lambda^{(q)}(h) \in \mathrm{Sp}\Delta^{(q)}(h)} \mathrm{e}^{-t\lambda^{(q)}(h)}. \tag{3.2}$$

We set

$$\mathrm{Tr}\mathrm{e}^{-t d\rho_h \Delta} = \sum_{q=0}^{m} (-1)^q q \mathrm{Tr}\mathrm{e}^{-t d\rho_h \Delta^{(q)}},$$

and

$$\mathrm{Tr}_\Gamma \mathrm{e}^{-t\Delta} = \sum_{q=0}^{m} (-1)^q q \mathrm{Tr}_\Gamma \mathrm{e}^{-t\Delta^{(q)}}.$$

The main analytic properties of the function $\mathrm{Tr}_\Gamma \mathrm{e}^{-t\Delta^{(q)}}$ as a function of t were investigated in [19]. In particular, it was proved that the behaviour for small t is the same as that of the trace of the heat kernel of the Hodge Laplace operator on the compact quotient $\Gamma\backslash G$.

In this setting, we introduce the following definitions. We call zeta function of $\Delta^{(q)}$ localised at the representation ρ_h, $h \in \hat{G}$, the function of the complex variable s defined for large $\mathrm{Re}(s)$ by the series

$$\zeta(s, \Delta^{(q)}(h)) = \sum_{\lambda^{(q)}(h) \in \mathrm{Sp}_+ \Delta^{(q)}(h)} \lambda^{(q)}(h)^{-s},$$

and by analytic continuation elsewhere. Thus, $\zeta(s, \Delta^{(q)}(h))$ is the fibre of a field of functions on $\hat{G}$. The analytic properties of the localised zeta function are determined by the asymptotic for small and large t of the trace of the localised heat operator by the Mellin transform

$$\zeta(s,\Delta^{(q)}(h))=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}e^{-t\Delta_+^{(q)}(h)}dt.$$

We call analytic torsion zeta function of the group G localised at the representation ρ_h, $h\in\hat{G}$, the graded sum

$$\mathfrak{t}(s;G,h)=\sum_{q=0}^{m}(-1)^q q\zeta(s,\Delta^{(q)}(h)),\tag{3.3}$$

and assuming that this is regular at $s=0$, we call localised analytic torsion of G the complex vector field on $\hat{G}$

$$\mathfrak{T}(G;h)=\left.\frac{d}{ds}\mathfrak{t}(s;G,h)\right|_{s=0}.\tag{3.4}$$

Next, let $\hat{G}_0$ denote the subspace of $\hat{G}$ determined by the following requirement: $h\in\hat{G}-\hat{G}_0$ if and only if there exist real numbers K_h and K such that

$$\mathrm{Sp}d\rho_h(\Delta^{(q)})>K_h>K>0,$$

for all q. Then, we call relative analytic torsion of the pair (G,Γ) the (possibly infinite) number

$$\mathfrak{T}_\Gamma(G)=\mathrm{Vol}(\Gamma\backslash G)\left.\frac{d}{ds}\int_{\hat{G}-\hat{G}_0}^{\oplus}\mathfrak{t}(s;G,h)d\mu(h)\right|_{s=0}+\mathrm{Vol}(\Gamma\backslash G)\int_{\hat{G}_0}^{\oplus}\mathfrak{T}(G;h)d\mu(h).\tag{3.5}$$

It will be useful to introduce also the relative analytic torsion zeta function

$$\mathfrak{t}_\Gamma(s;G)=\int_{\hat{G}-\hat{G}_0}^{\oplus}\mathfrak{t}(s;G,h)d\mu(h),\tag{3.6}$$

such that

$$\mathfrak{T}_\Gamma(G)=\mathrm{Vol}(\Gamma\backslash G)\left.\frac{d}{ds}\mathfrak{t}_\Gamma(s;G)\right|_{s=0}+\mathrm{Vol}(\Gamma\backslash G)\int_{\hat{G}_0}^{\oplus}\mathfrak{T}(G;h)d\mu(h).$$

This definition is clearly inspired by Definition 2 of [19],

$$\mathcal{T}_\Gamma(G) = \frac{d}{ds} \int_0^\epsilon t^{s-1} \frac{1}{\Gamma(s)} \mathrm{Tr}_\Gamma \mathrm{e}^{-t\Delta_+} dt \Big|_{s=0} + \int_\epsilon^\infty \frac{1}{t} \mathrm{Tr}_\Gamma \mathrm{e}^{-t\Delta_+} dt.$$

where

$$\mathrm{Tr}_\Gamma \mathrm{e}^{-t\Delta_+} = \sum_{q=0}^m (-1)^q q \mathrm{Tr}_\Gamma \mathrm{e}^{-t\Delta_+^{(q)}},$$

and in fact we prove that they are equivalent for a significant family of groups, see Proposition 3.0.4. This family is defined by the assumptions introduced below.

Assumption 3.0.1. In the following we omit the dependence on q in the constants.

(1) The space $\hat{G}$ is homeomorphic to a subset A of $\mathbb{R}^k$, for some k.
(2) The Plancherel measure in A reads $d\mu(h(a)) = f(a)da$, $a \in A$, where da is the classical Lebesgue measure, and $f(a)$ some continuous (smooth) non negative function integrable on A.
(3) The eigenvalues $\lambda^{(q)}(h(a))$ of $d\rho_{h(a)}\Delta^{(q)}$ are continuous (smooth) functions of a, monotone in $\|a\|$ (by this we mean that if $\|a_1\| \leq \|a_2\|$, then $\lambda_n^{(q)}(h(a_1)) \leq \lambda_n^{(q)}(h(a_2))$, recall these eigenvalues are non negative real numbers).
(4) For all $a \in A$:

$$\lambda_n^{(q)}(h(a)) > Cn^\alpha \|a\|^\beta,$$

for some positive constants $C, \alpha \geq 1, \beta$.
(5) For all $a \in A$:

$$f(a) \leq \|a\|^\kappa,$$

with $\kappa \geq 1$.
(6) For $0 < t \leq 1$:

$$\mathrm{Tr}\mathrm{e}^{-td\rho_{h(a)}\Delta^{(q)}} = \sum_{j=0}^J c_j^{(q)}(a) t^{j-j_0} + c^{(q)}(a,t),$$

for integers $j_0 \leq J$, $c_j^{(q)} \in L^1(A, f(a)da)$, $c^{(q)} \in L^1(A \times (0,1], f(a)dadt)$, with $c^{(q)}(a,t) \geq 0$, for $a \in A$, and

$$c^{(q)}(a,t) \leq c^{(q)} t^{J-j_0},$$

for some positive constant $c^{(q)}$.

Note that the requirement on the large t behaviour of $\mathrm{Tr}_\Gamma \mathrm{e}^{-t\Delta}$ assumed in [19], Note 1.3, follows by these assumptions.

Note also that the measurable fields introduced above are continuous under assumptions (1) and (2).

Proposition 3.0.2. *The localised torsion zeta function $\mathfrak{t}(s; G, h)$ of G is analytic for $s > j_0$, and has an analytic continuation to $\mathbb{C}$ with possible simple poles located at the integers points of type $j - j_0$, $j = 0, 1, 2, \ldots$, $j \neq j_0$.*

Proof. This follows by assumption (4), using classical methods (see for example [18] or [31]). By definition

$$\zeta(s, \Delta^{(q)}(h(a))) = \sum_{\lambda^{(q)}(h(a)) \in \mathrm{Sp}_+ \Delta^{(q)}(h(a))} \lambda^{(q)}(h(a))^{-s}.$$

with $a \in A$. Since $\mathrm{Sp}_+ \Delta^{(q)}(h(a))$ is a discrete set of positive numbers, we write $\mathrm{Sp}_+ \Delta^{(q)}(h(a)) = \{\lambda_n^{(q)}(h(a))\}_{n=1}^{\infty}$, where the $\lambda_n^{(q)}$ are smooth functions of a by Assumption (3). Thus,

$$\zeta(s, \Delta^{(q)}(h(a))) = \sum_{n=1}^{\infty} \lambda_n^{(q)}(h(a))^{-s}.$$

By the Mellin transform

$$\begin{aligned}\zeta(s, \Delta^{(q)}(h(a))) =& \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} dt \\ =& \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} dt + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} dt \quad \square\end{aligned}$$

Proposition 3.0.3. *Let G_1 and G_2 be two Lie groups as above, and $[\rho_{1,h_1}] \in \hat{G}_1$, $[\rho_{2,h_2}] \in \hat{G}_2$. Then, (with a little obvious change of notation)*

$$\mathfrak{T}(G_1 \times G_2, \rho_{1,h_1} \otimes \rho_{2,h_2}) = \chi(G_1, \rho_{1,h_1}) \mathfrak{T}(G_2, \rho_{2,h_2}) + \chi(G_2, \rho_{2,h_2}) \mathfrak{T}(G_1, \rho_{1,h_1}),$$

where $\chi(G, \rho_h) = \sum_{q=0}^{\dim G} (-1)^q \dim \ker d\rho_h \Delta^{(q)}$.

Proof. This follows as in the proof of Theorem 2.3 of [26] or point (2) pg. 268 of [22]. $\square$

Proposition 3.0.4. *If the Assumptions 3.0.1 are satisfied, then, when both defined, $\mathfrak{T}_\Gamma(G) = \mathcal{T}_\Gamma(G)$.*

Proof. Let A be the subset of $\mathbb{R}^k$ in Assumption 3.0.1(1). Let denote by A_0 the intersection of A with an open ball of 0 with radius ϵ in $\mathbb{R}^k$, by A_∞ the intersection of A with an open ball of infinity of radius δ. By Assumptions (1) and (3), it follows that $\hat{G}_0$ is homeomorphic to A_0 (up to readjusting some constants).

We want to prove that

$$I_1 = \frac{d}{ds} \int\limits_{A-A_0} \sum_{n=1}^{\infty} \lambda_n^{(q)}(h(a))^{-s} f(a)da \Bigg|_{s=0} + \int\limits_{A_0} \frac{d}{ds} \sum_{n=1}^{\infty} \lambda_n^{(q)}(h(a))^{-s} \Bigg|_{s=0} f(a)da$$

coincides with

$$\begin{aligned} I_2 = {} & \frac{d}{ds} \frac{1}{\Gamma(s)} \int\limits_0^1 t^{s-1} \int\limits_A \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} f(a)dadt \Bigg|_{s=0} \\ & + \int\limits_1^{\infty} \frac{d}{ds} \frac{1}{\Gamma(s)} t^{s-1} \Bigg|_{s=0} \int\limits_A \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} f(a)dadt. \end{aligned}$$

Using the Mellin transform in first equation

$$\begin{aligned} I_1 = {} & \frac{d}{ds} \int\limits_{A-A_0} \frac{1}{\Gamma(s)} \int\limits_0^{\infty} t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} dt f(a)da \Bigg|_{s=0} \\ & + \int\limits_{A_0} \frac{d}{ds} \frac{1}{\Gamma(s)} \int\limits_0^{\infty} t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} dt \Bigg|_{s=0} f(a)da, \end{aligned}$$

whence the proof essentially consists in proving that we can change the order of integration and derivation.

Split the t-integrals in I_1:

$$\begin{aligned} I_{1,1} &= \int\limits_{A_0} \frac{d}{ds} \frac{1}{\Gamma(s)} \int\limits_0^1 t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} dt \Bigg|_{s=0} f(a)da, \\ I_{1,2} &= \int\limits_{A_0} \frac{d}{ds} \frac{1}{\Gamma(s)} \int\limits_1^{\infty} t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} dt \Bigg|_{s=0} f(a)da, \\ I_{1,3} &= \frac{d}{ds} \int\limits_{A-A_0} \frac{1}{\Gamma(s)} \int\limits_0^1 t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} dt f(a)da \Bigg|_{s=0}, \\ I_{1,4} &= \frac{d}{ds} \int\limits_{A-A_0} \frac{1}{\Gamma(s)} \int\limits_1^{\infty} t^{s-1} \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} dt f(a)da \Bigg|_{s=0}, \end{aligned}$$

and the a-integral in I_2:

$$I_{2,1} = \frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\int_{A_0}\sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} f(a)dadt\Bigg|_{s=0},$$

$$I_{2,2} = \int_1^{\infty} \frac{d}{ds}\frac{1}{\Gamma(s)}t^{s-1}\Bigg|_{s=0}\int_{A_0}\sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} f(a)dadt,$$

$$I_{2,3} = \frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\int_{A-A_0}\sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} f(a)dadt\Bigg|_{s=0},$$

$$I_{2,4} = \int_1^{\infty} \frac{d}{ds}\frac{1}{\Gamma(s)}t^{s-1}\Bigg|_{s=0}\int_{A-A_0}\sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^{(q)}(h(a))} f(a)dadt.$$

Then, we verify that $I_{1,j} = I_{2,j}$. In particular, we use Assumptions 3.0.1(3) and (4) to verify this for $j = 2$ and $j = 4$, and Assumption 3.0.1(6) for $j = 1$ and $j = 3$. The verifications consist in proving that the classical conditions for changing the order of integration in multiple integrals and for changing the order of derivation and integration are satisfied. □

4. The abelian case

In this section we apply our construction to the simplest case of the real number field $G = \mathbb{R}$ considered as a Lie group with respect to the addition. This is a connected simply connected locally compact second countable space, and an abelian group (actually it is contractible and separable). The discrete subgroup $\Gamma = \mathbb{Z}$ has compact quotient $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The irreducible representations are one dimensional, i.e. the characters

$$\chi_h : \mathbb{R} \to U(\mathbb{C}) = U(1), \qquad \chi_h : g \to \mathrm{e}^{2\pi i h g},$$

with $h \in \mathbb{R}$. The action χ_h restricts to an action of $\mathbb{Z}$ with fundamental domain $[0, 1]$. The dual $\hat{\mathbb{R}}$ of $\mathbb{R}$ is isomorphic to $\mathbb{R}$ by the paring $\hat{g}_h(g') = \mathrm{e}^{2\pi i h g'}$. The Plancherel measure is the Lebesgue measure dh, the direct integrals are Lebesgue integrals on $\mathbb{R}$, and the group Fourier transform is the classical Fourier transform

$$\hat{f}(h) = \int_{\mathbb{R}} f(g)\mathrm{e}^{-2\pi i h g}dg, \qquad f(g) = \int_{\mathbb{R}} \hat{f}(h)\mathrm{e}^{2\pi i h g}dh.$$

As a differentiable manifold, we have a global coordinate $\{x\}$, with coordinate basis of the tangent space $\{\partial_x\}$, with dual $\{dx\}$. These are left invariant vector fields. The formal exterior differential on $\mathcal{C}^{\infty}(\mathbb{R})$ is $du = \frac{d}{dx}udx$, and as usual we use the same notation for the minimal closed extension of its restriction on $\mathcal{C}_0^\infty(\mathbb{R})$. A left invariant Riemannian metric is $dx \otimes dx$, and with respect to it the previous bases are orthonormal. The Hodge star $\star dx = 1$, the volume form dx. The inner product $\langle \omega, \varphi \rangle = \int_{\mathbb{R}} \omega \wedge \star\varphi dx$, and the Hodge Laplace operator (we just need it on functions)

$$\Delta^{(0)}\omega = -\frac{d^2}{dx^2}\omega.$$

The heat operator $\mathrm{e}^{-t\Delta^{(0)}}$ is the integral operator with smooth kernel [27, pg. 6]

$$k(x, y; \mathrm{e}^{-t\Delta^{(0)}}) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{|x-y|^2}{4t}}.$$

The $\Gamma = \mathbb{Z}$ trace is

$$\mathrm{Tr}_{\mathbb{Z}}\mathrm{e}^{-t\Delta^{(0)}} = \int_0^1 k(x, x; \mathrm{e}^{-t\Delta^{(0)}})dx = \int_0^1 \frac{1}{\sqrt{4\pi t}}dx = \frac{1}{\sqrt{4\pi t}}.$$

In order to compute the $\mathcal{C}^\infty$ vectors, using equation (2.2), we need to isolate the case $h = 0$. For $h \neq 0$: $d\chi_h(\partial_x) = 2\pi i h$. Thus,

$$d\chi_h(\Delta^{(0)}) = 4\pi^2 h^2,$$

and

$$\mathrm{Sp}d\chi_h(\Delta^{(0)}) = \mathrm{Sp}d\chi_h(\Delta^{(1)}) = \{4\pi^2 h^2\}.$$

When $h = 0$, $\chi_0 = 1$, the constant representation, so the tangent vector is the zero vector, and the Laplace operator is multiplication by 0, that has one eigenvalue 0 with infinite multiplicity. However, the space $\{0\}$ has measure zero, whence we ignore this representation, and we proceed assuming $h \neq 0$.

The heat operator localised at χ_h, $h \neq 0$ is

$$d\chi_h(\mathrm{e}^{-t\Delta^{(0)}}) = \mathrm{e}^{-4\pi^2 h^2 t},$$

and its trace is

$$\mathrm{Tr}d\chi_h(\mathrm{e}^{-t\Delta^{(0)}}) = \mathrm{e}^{-4\pi^2 h^2 t}.$$

The $\Gamma = \mathbb{Z}$ equivariant trace is

$$\mathrm{Tr}_{\mathbb{Z}}\mathrm{e}^{-t\Delta^{(0)}} = \mathrm{Vol}(\Gamma\backslash G) \int_{\mathbb{R}-\{0\}} \mathrm{e}^{-4\pi^2 h^2 t} dh = 2\int_0^\infty \mathrm{e}^{-4\pi^2 h^2 t} dh = \frac{1}{2\sqrt{\pi t}},$$

since $\mathrm{Vol}(\Gamma\backslash G) = 1$. Note that this gives the Novikov Shubin invariant of G, that is $\alpha_0(G) = 1$, known by Proposition 43 of [19]. The localised analytic torsion zeta function is $(h \neq 0)$

$$\mathfrak{t}(s; \mathbb{R}, h) = -(2\pi h)^{-2s},$$

that is holomorphic for all s. Whence

$$\mathfrak{T}(\mathbb{R}, h) = 2\log 2\pi|h|.$$

According to equation (3.6), the relative torsion zeta function, for large s, is

$$\mathfrak{t}_{\mathbb{Z}}(s; \mathbb{R}) = -2\int_{\epsilon}^{\infty}(2\pi h)^{-2s}dh = 2\frac{(2\pi\epsilon)^{1-2s}}{2\pi(1-2s)},$$

that gives

$$\mathfrak{T}_{\mathbb{Z}}(\mathbb{R}) = \mathfrak{t}'_{\mathbb{Z}}(0; \mathbb{R}) + 2\int_0^{\epsilon} 2\log(2\pi h)dh = 4\epsilon - 4\epsilon\log 2\pi\epsilon + 4\epsilon\log 2\pi\epsilon - 4\epsilon = 0.$$

On the other side (where the first term is defined for large s)

$$\mathcal{T}_\Gamma = \frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^{\epsilon} t^{s-1}\frac{1}{2\sqrt{\pi t}}dt\bigg|_{s=0} + \int_{\epsilon}^{\infty}\frac{1}{t}\frac{1}{2\sqrt{\pi t}}dt = 0.$$

Next, we compare this with the classical analytic torsion. The irreducible representations $\chi_\alpha : \pi_1(\mathbb{T}) = \mathbb{Z} \to U(1)$ of the fundamental group of the circle are the characters $\mathrm{e}^{2\pi i\alpha n}$, with $0 \leq \alpha < 1$. Fix $\alpha \neq 0$. The Hodge Laplace operator on function with values in E_α is positive definite, and its spectrum is $\mathrm{Sp}\Delta^{(0)} = \{(n+\alpha)^2\}_{n\in\mathbb{Z}}$. The torsion zeta function of $\mathbb{T}$ with coefficients twisted by χ_α, is

$$\begin{aligned} t(s; \mathbb{T}, \alpha) &= \sum_{q=0}^{1}(-1)^q q\zeta(s, \Delta_+^{(q)}) = -\zeta(s, \Delta_+^{(1)}) = -(2\pi)^{-2s}\sum_{n\in\mathbb{Z}}(n+\alpha)^{-2s} \\ &= -(2\pi)^{-2s}\zeta_H(2s, \alpha) - (2\pi)^{-2s}\zeta_H(2s, 1-\alpha). \end{aligned} \tag{4.1}$$

Recalling that $\zeta_H(0, q) = \frac{1}{2} - q$, and $\zeta'_H(0, q) = \log\Gamma(q) - \frac{1}{2}\log 2\pi$, we compute the analytic torsion of $\mathbb{T}$ with coefficients twisted by χ_α:

$$T(\mathbb{T}, \alpha) = t'(0, \alpha) = 2\log 2\sin\pi\alpha.$$

Reconsider the definition of the relative zeta function of $\mathbb{R}$, with $\delta = 1$:

$$\begin{aligned}
\mathfrak{t}_{\mathbb{Z}}(s;\mathbb{R}) &= -2\int_1^{\infty}(2\pi h)^{-2s}dh = -\int_{-\infty}^{-1}(2\pi h)^{-2s}dh - \int_1^{\infty}(2\pi h)^{-2s}dh \\
&= -\sum_{n=-\infty}^{-2}\int_0^1(2\pi(n+\alpha))^{-2s}d\alpha - \sum_{n=1}^{\infty}\int_0^1(2\pi(n+\alpha))^{-2s}d\alpha \\
&= -\int_0^1\sum_{n=2}^{\infty}(2\pi(n-\alpha))^{-2s}d\alpha - \int_0^1\sum_{n=1}^{\infty}(2\pi(n+\alpha))^{-2s}d\alpha \\
&= \int_0^1 t(s;\mathbb{T},\alpha)d\alpha + 2\int_0^1(2\pi\alpha)^{-2s}d\alpha, \\
&= \int_0^1 t(s;\mathbb{T},\alpha)d\alpha - 2\int_0^1 \mathfrak{t}(s;\mathbb{R},\alpha)d\alpha,
\end{aligned}$$

where the last equivalence follows because the series of function converges uniformly in α for $\alpha \in [0,1)$ when $\mathrm{Re}(s)$ is large. Whence, collecting and using equation (4.1),

$$\mathfrak{t}_{\mathbb{Z}}(s;\mathbb{R}) = \int_0^1 t(s;\mathbb{T},\alpha)d\alpha + 2\int_0^1(2\pi\alpha)^{-2s}d\alpha = \int_0^1 t(s;\mathbb{T},\alpha)d\alpha + \frac{2(2\pi)^{-2s}}{1-2s}.$$

Thus,

$$\mathfrak{t}_{\mathbb{Z}}(s;\mathbb{R}) = -\int_0^1(2\pi)^{-2s}\zeta_H(2s,1-\alpha)d\alpha - \int_0^1(2\pi)^{-2s}\zeta_H(2s,\alpha)d\alpha + 2\int_0^1(2\pi\alpha)^{-2s}d\alpha.$$

Using for example the Hermite representation for the Hurwitz zeta function, we find that

$$-\int_0^1(2\pi)^{-2s}\zeta_H(2s,1-\alpha)d\alpha = -(2\pi)^{-2s}\int_0^1\left(\frac{(1-\alpha)^{1-2s}}{2s-1} + f(s,\alpha)\right)d\alpha,$$

where $f(s,\alpha)$ is a regular analytic function of s for all s, smooth and bounded in α for $\alpha \in [0,1]$. Therefore, for s near $s = 0$,

$$-\frac{d}{ds}\int_0^1(2\pi)^{-2s}\zeta_H(2s,1-\alpha)d\alpha = -\int_0^1\frac{d}{ds}(2\pi)^{-2s}\zeta_H(2s,1-\alpha)d\alpha.$$

According to the definition, equation (3.5), this gives

$$\begin{aligned}\mathfrak{T}_{\mathbb{Z}}(\mathbb{T}) &= \frac{d}{ds}\mathfrak{t}_{\mathbb{Z}}(s;\mathbb{R})\Big|_{s=0} + 2\int_0^1 \mathfrak{T}(\mathbb{R},\alpha)d\alpha \\ &= \int_0^1 \frac{d}{ds}t(s;\mathbb{T},\alpha)\Big|_{s=0} d\alpha - 4\log 2\pi + 4 + 2\int_0^1 \mathfrak{T}(\mathbb{R},\alpha)d\alpha \\ &= \int_0^1 T(\mathbb{T},\alpha)d\alpha.\end{aligned}$$

Recalling the multiplicative property of all the torsions, we have proved the following result (where we are assuming that G has trivial compact factor).

Proposition 4.0.1. *Let G be a simply connected abelian real Lie group, with discrete co compact subgroup Γ, then*

$$\mathfrak{T}_{\Gamma}(G) = \int_0^1 T(G,\alpha)d\alpha.$$

5. The Heisenberg group

This section is devoted to our main application, i.e. the three dimensional Heisenberg group.

5.1. The Heisenberg group

There exist several equivalent definitions of the Heisenberg group, we chose the more suitable for our purpose. We call Heisenberg group H the three dimensional real space with the Lie group operation (this group is called polarised Heisenberg group by Folland [13, pg. 19], see also [9, pg. 47])

$$(a,b,t)(a',b',t') = (a+a', b+b', t+t'+ab').$$

The discrete subgroup Γ is

$$\Gamma = \{(l,m,n) \in H \mid l,m,n \in \mathbb{Z}\}.$$

Topologically, the Heisenberg group is a contractible space, locally compact and second countable, and as a Lie group is nilpotent unimodular of type I [14, Ex. 3, pg. 229]. As a real smooth manifold, H has a global coordinate system $\{a,b,t\}$, with coordinate basis of T_eH: $\{\partial_a, \partial_b, \partial_t\}$. The corresponding basis of left invariant vector fields in the Lie algebra $\mathfrak{h}$ is

$$X(g) = \partial_a, \qquad Y(g) = \partial_b + a\partial_t, \qquad T(g) = \partial_t,$$

with $[X, Y] = T$, and dual basis $\{da, db, \theta\}$, where $\theta = dt - adb$. The formal exterior derivative operator on functions ω and one forms $\omega = \omega_a da + \omega_b db + \omega_\theta \theta$ is

$$\begin{aligned}
d\omega &= \partial_a \omega da + \partial_b \omega db + \partial_t \omega dt = X\omega da + Y\omega db + T\omega\theta,\\
d\omega &= (X\omega_b - Y\omega_a - \omega_\theta)\, da \wedge db + (X\omega_\theta - T\omega_a)\, da \wedge \theta + (Y\omega_\theta - T\omega_b)\, db \wedge \theta.
\end{aligned}$$

We fix the Riemannian structure on H determined by the left invariant Riemannian metric making the bases above orthonormal, that reads

$$da \otimes da + \left(1 + a^2\right) db \otimes db + dt \otimes dt - ada \otimes db - adb \otimes da,$$

with volume form $dvol = da \wedge db \wedge \theta$. The Hodge star $\star$ follows easily.

The (formal) Hodge Laplace operator on q-forms, $\Delta^{(q)} = \delta d + d\delta$, where $\delta = - \star d \star$, in terms of the orthogonal basis has the following explicit description (in the relevant degrees) [28] [23] [24]:

$$\begin{aligned}
\Delta^{(0)}\omega =& - (X^2 + Y^2 + T^2)\omega,\\
\Delta^{(1)}(\omega_a da + \omega_b db + \omega t\theta) =&(\Delta^{(0)}\omega_a - T\omega_b - Y\omega_t)da + (\Delta^{(0)}\omega_b + T\omega_a - X\omega_t)db\\
&+ (\Delta^{(0)}\omega_t + Y\omega_a - X\omega_b + \omega_t)\theta.
\end{aligned}$$

5.2. Dual group, irreducible representations and localised operators

The dual space $\hat{G}$ (with the Fell topology [12]) is homeomorphic with the set of the co adjoint orbits of $\mathfrak{h}^*$, with the natural quotient topology, where the last may be identified with real line private by the origin [14, Theorem 7.9, and example 7.6.1]. Thus, assumption (1) is satisfied with $A = \mathbb{R} - \{0\}$. The Plancherel measure is $d\mu(h) = |h|dh$, $h \in A$, and thus also assumption (2) is satisfied. The group Fourier transform on H may be described explicitly as [13, pg. 43]:

$$\mathrm{Tr}\,\hat{f}(h)\rho_h(a, b, t) = \frac{1}{|h|} \int_{\mathbb{R}} f(a, b, t)\mathrm{e}^{-2\pi i h(t-s)} ds.$$

The irreducible (infinite dimensional, the finite dimensional ones do not matter since they determine a set of measure zero in $\hat{G}$) unitary representations of the Heisenberg group are the Schrödinger representations $\rho_h : H \to U(\mathcal{S}) \subseteq U(L^2(\mathbb{R}))$,

$$\rho_h(a, b, t)(f)(x) = \mathrm{e}^{2\pi i h t + 2\pi i b x} f(x + ha),$$

on $L^2(\mathbb{R})$, where $h \in \mathbb{R} - \{0\}$ is the parameter described above for the dual group $\hat{G}$ [13, 3(1.25), pg. 22, Th. 1.59, pg. 37] (see also equation (4.1), pg. 47 of [9]). The infinitesimal generators of the associated representations of universal enveloping algebra $\mathcal{U}(\mathfrak{h})$ are

$$d\rho_h(X) = h\partial_x, \qquad d\rho_h(Y) = 2\pi i x, \qquad d\rho_h(T) = 2\pi i h,$$

that gives the (continuous) field of operators $d\rho_h \Delta^{(q)}$. Our aim is to give an explicit description of the spectrum of these operators. For functions, this is quite easy, see next section. A similar calculation for $\Delta^{(1)}$ is more involved, so we prefer to follow the easier alternative approach delineated in [23] and based on the Bergmann Fock representation, described in Section 5.4.

5.3. The localised Hodge Laplace operator on functions and its spectrum

A direct calculation gives

$$d\rho_h(\Delta^{(0)}) = -h^2\partial_x^2 + 4\pi^2 x^2 + 4\pi^2 h^2.$$

We aim to determine a spectral resolution of Δ^0. For consider the eigenvalues equation

$$-\partial_x^2 f + \frac{4\pi^2}{h^2} x^2 f = \left(\frac{\lambda}{h^2} - 4\pi^2\right) f. \tag{5.1}$$

Let $u_n(x) = H_n(x)$ the Hermite polynomial. It satisfies the differential equation

$$u_n'' - 2x u_n' + 2n u_n = 0.$$

By the Liouville transform $u_n(x) = \mathrm{e}^{\frac{x^2}{2}} v_n(x) = k(x) v_n(x)$, with

$$u = kv, \qquad u' = k'v + kv', \qquad u'' = k'' + 2k'v' + kv,$$

we end up with

$$-v_n'' + x^2 v_n = (2n+1) v_n.$$

Set $x = \sqrt{a} t$, $f_n(t) = v_n(\sqrt{a} t)$, and $a = \frac{2\pi}{|h|}$, then f satisfies

$$-f_n'' - \frac{4\pi^2}{h^2} t^2 f_n = (2n+1)\frac{2\pi}{|h|} f_n.$$

Comparing the last equation with equation (5.1), we find:

$$\begin{aligned} f(x) &= v_n(\sqrt{2\pi/h} x) = \mathrm{e}^{-\frac{\pi x^2}{|h|}} H_n(\sqrt{2\pi/|h|} x) = (-1)^n \mathrm{e}^{\frac{\pi x^2}{|h|}} \frac{d^n}{dx^n} \mathrm{e}^{-\frac{2\pi x^2}{|h|}}, \\ \lambda &= \lambda_n = 2\pi|h|(2n+1) + 4\pi^2 h^2, \end{aligned}$$

with $n = 0, 1, 2, \dots$, and $h \in \mathbb{R} - \{0\}$ (compare with [13, Section 7, pg. 51]).

5.4. The Bargmann Fock representation and the spectrum of the localised Hodge Laplace operator on one forms

Fix $h > 0$, and let

$$\|f\|^2_{\mathcal{F}_h} = h \int_{\mathbb{C}} |f(z)|^2 \mathrm{e}^{-\pi h|z|^2} dz,$$

and

$$\mathcal{F}_h = \{f : \mathbb{C} \to \mathbb{C},\ f \ \textit{entire},\ \|f\|_{\mathcal{F}_h} < \infty\},$$

then, the Bargmann transform is the map [13, pg. 47]

$$B_h : L^2(\mathbb{R}, dx) \to B_h(L^2(\mathbb{R})) = \mathcal{F}_h \subseteq L^2(\mathbb{C}, h\mathrm{e}^{-\pi h|z|^2} dz), \qquad B_h : f \mapsto B_h(f),$$

where

$$B_h(f)(z) = \left(\frac{2}{h}\right)^{\frac{1}{4}} \int_{-\infty}^{+\infty} f(x) \mathrm{e}^{2\pi xz - \frac{\pi}{h}x^2 - \frac{\pi h}{2}z^2} dx,$$

and is a unitary map intertwining the Shrodinger representation with the Bargmann representation $\beta_h B_h = B_h \rho_h$. To see this is better to complexify all the construction. Setting $z = a + ib$, we find

$$\beta_h(z,t)(f)(w) = \mathrm{e}^{2\pi iht - \frac{\pi h}{2}|z|^2 - \frac{\pi h}{4}(z^2 - \bar{z}^2) - \pi hw\bar{z}} f(z + w).$$

If $h < 0$, the representation space is

$$\bar{\mathcal{F}}_h = \{f \mid fj \in \mathcal{F}_{-h}\},$$

where j denotes conjugation, i.e. $j(z) = \bar{z}$; the Bargmann transform is the map

$$\bar{B}_h : L^2(\mathbb{R}, dx) \to \bar{\mathcal{F}}_h \subseteq L^2(\mathbb{C}, |h|\mathrm{e}^{\pi h|z|^2} dz), \qquad \bar{B}_h : f \mapsto B(f),$$

where

$$\bar{B}_h(f)(z) = \left(\frac{2}{|h|}\right)^{\frac{1}{4}} \int_{-\infty}^{+\infty} f(x) \mathrm{e}^{-2\pi xz + \frac{\pi}{h}x^2 + \frac{\pi h}{2}z^2} dx.$$

and a direct calculation as the previous one gives

$$\bar{\beta}_h(z,t)(f)(w) = \mathrm{e}^{2\pi i h t + \frac{\pi h}{2}|z|^2 - \frac{\pi h}{4}(z^2 - \bar{z}^2) + \pi h w z} f(\bar{z} + w).$$

We compute the infinitesimal generators of the coordinate basis vectors:

$$\begin{aligned}
&h > 0: \qquad && d\beta_h(\partial_z) = \partial_w, \qquad && d\beta_h(\partial_{\bar{z}}) = -\pi h w, \qquad && d\beta_h(\partial_t) = 2\pi i h,\\
&h < 0: \qquad && d\bar{\beta}_h(\partial_z) = h\pi w, \qquad && d\bar{\beta}_h(\partial_{\bar{z}}) = \partial_w, \qquad && d\bar{\beta}_h(\partial_t) = 2\pi i h.
\end{aligned}$$

An orthonormal basis of left invariant vector fields is

$$Z = \sqrt{2}\partial_z - \frac{i}{2\sqrt{2}}(z + \bar{z})\partial_t, \qquad \bar{Z} = \sqrt{2}\partial_{\bar{z}} + \frac{i}{2\sqrt{2}}(z + \bar{z})\partial_t, \qquad T,$$

with $[Z, \bar{Z}] = i[X, Y] = iT$, and

$$X = \frac{1}{\sqrt{2}}(Z + \bar{Z}), \qquad Y = \frac{i}{\sqrt{2}}(Z - \bar{Z});$$

the dual basis is

$$\begin{aligned}
&\zeta = \frac{1}{\sqrt{2}}dz = \frac{1}{\sqrt{2}}(da + idb), \qquad && \bar{\zeta} = \frac{1}{\sqrt{2}}d\bar{z} = \frac{1}{\sqrt{2}}(da - idb),\\
&\beta = dt + \frac{i}{2\sqrt{2}}(z + \bar{z})(\zeta - \bar{\zeta}) = \theta,
\end{aligned}$$

Then, we compute

$$\Delta^{(0)} = -X^2 - Y^2 - T^2 = -Z\bar{Z} - \bar{Z}Z - T^2.$$

and

$$\begin{aligned}
\Delta^{(1)} =& (-(Z\bar{Z} + \bar{Z}Z + T^2)\omega_z - iT\omega_z - iZ\omega_t)\zeta\\
&+ (-(Z\bar{Z} + \bar{Z}Z + T^2)\omega_{\bar{z}} + iT\omega_{\bar{z}} + i\bar{Z}\omega_t)\bar{\zeta}\\
&+ (-(Z\bar{Z} + \bar{Z}Z + T^2 + 1)\omega_t - i\bar{Z}\omega_z + iZ\omega_{\bar{z}})\theta.
\end{aligned}$$

The infinitesimal generators read

$$\begin{aligned}
&h > 0: \qquad && d\beta_h(Z) = \sqrt{2}\partial_w, \qquad && d\beta_h(\bar{Z}) = -\sqrt{2}\pi h w, \qquad && d\beta_h(\partial_t) = 2\pi i h,\\
&h < 0: \qquad && d\bar{\beta}_h(Z) = \sqrt{2}h\pi w, \qquad && d\bar{\beta}_h(\bar{Z}) = \sqrt{2}\partial_w, \qquad && d\beta_h(\partial_t) = 2\pi i h,
\end{aligned}$$

and, setting $k = 2\pi h$, the localised Hodge Laplace operator is

$$d\beta_h(\Delta^{(1)}) = \begin{pmatrix} 2kw\partial_w + k^2 + 2k & 0 & -\sqrt{2}i\partial_w\\ 0 & 2kw\partial_w + k^2 & -\frac{ik}{\sqrt{2}}w\\ \frac{ik}{\sqrt{2}}w & \sqrt{2}i\partial_w & 2kw\partial_w + k + k^2 + 1 \end{pmatrix}$$

and

$$d\bar{\beta}_h(\Delta^{(1)}) = \begin{pmatrix} -2kw\partial_w + k^2 & 0 & -\frac{ik}{\sqrt{2}}w \\ 0 & -2kw\partial_w + k^2 - 2k & \sqrt{2}i\partial_w \\ -\sqrt{2}i\partial_w & \frac{ik}{\sqrt{2}}w & -2kw\partial_w - k + k^2 + 1 \end{pmatrix}.$$

We want to compute the spectrum of these operators. Since the analysis for positive and negative h is analogous, we give details for $h > 0$. First, observe that

$$\begin{aligned} d\beta_h(\Delta^{(1)}) - xI = &\begin{pmatrix} 2kw\partial_w + k^2 + k & 0 & 0 \\ 0 & 2kw\partial_w + k^2 + k & 0 \\ 0 & 0 & 2kw\partial_w + k^2 + k \end{pmatrix} \\ &+ \begin{pmatrix} k - x & 0 & -\sqrt{2}i\partial_w \\ 0 & -k - x & -\frac{ik}{\sqrt{2}}w \\ \frac{ik}{\sqrt{2}}w & \sqrt{2}i\partial_w & 1 - x \end{pmatrix}. \end{aligned}$$

Next, observe that the homogeneous polynomials belong to $\mathcal{F}_h$, more precisely the set of the monomials

$$\chi_l(w) = \frac{(h\pi)^{\frac{l}{2}}}{\sqrt{l!}} w^l,$$

$l = 0, 1, 2, \ldots$, is an orthonormal basis of $\mathcal{F}_h$ [13, (1.63), pg. 40]. Thus, apply the Laplacian to the vectors $(A\chi_{l_1}(w), B\chi_{l_2}(w), C\chi_{l_3}(w))$. We end up with the system of equations

$$\begin{cases} (2kl_1 + k^2 + 2k)\frac{(h\pi)^{\frac{l_1}{2}}}{\sqrt{l_1!}} w^{l_1} A - \sqrt{2}i\frac{(h\pi)^{\frac{l_3}{2}}}{\sqrt{l_3!}} l_3 C w^{l_3 - 1} = x\frac{(h\pi)^{\frac{l_1}{2}}}{\sqrt{l_1!}} w^{l_1} A, \\ (2kl_2 + k^2)\frac{(h\pi)^{\frac{l_2}{2}}}{\sqrt{l_2!}} w^{l_2} B - \frac{ik}{\sqrt{2}}\frac{(h\pi)^{\frac{l_3}{2}}}{\sqrt{l_3!}} C w^{l_3+1} = x\frac{(h\pi)^{\frac{l_2}{2}}}{\sqrt{l_2!}} w^{l_2} B, \\ \frac{ik}{\sqrt{2}}\frac{(h\pi)^{\frac{l_1}{2}}}{\sqrt{l_1!}} A w^{l_1+1} + \sqrt{2}i\frac{(h\pi)^{\frac{l_2}{2}}}{\sqrt{l_2!}} l_2 w^{l_2-1} B + (2kl_3 + k^2 + k + 1)\frac{(h\pi)^{\frac{l_3}{2}}}{\sqrt{l_3!}} w^{l_3} C \\ \quad = x\frac{(h\pi)^{\frac{l_3}{2}}}{\sqrt{l_3!}} w^{l_3} C. \end{cases}$$

We have the particular solutions: if $A = C = 0$, then $l_2 = 0$ and we have the equation

$$k^2 B = xB,$$

that gives $\{k^2; (0, \chi_0(w), 0)\}$; if $A = 0$, and $C \neq 0$, we have $l_3 = 0$, $l_2 = 1$, that gives

$$\begin{cases} (2k + k^2)\sqrt{\pi h} w B - \frac{ik}{\sqrt{2}} w C = x\sqrt{\pi h} w B, \\ \sqrt{2}i\sqrt{\pi h} B + (k^2 + k + 1)C = xC, \end{cases}$$

with eigenvalues $k^2 + k$, and $(k + 1)^2$. Otherwise, assuming $ABC \neq 0$, $l_1 = l_3 - 1$, $l_2 = l_3 + 1$, and we find

$$\begin{cases} ((2l_3+1)k+k^2-k-x)A - i\sqrt{2\pi h l_3}C = 0, \\ ((2l_3+1)k+k^2+k-x)B - ik\sqrt{\frac{l_3+1}{2\pi h}}C = 0, \\ ik\sqrt{\frac{l_3}{2\pi h}}A + i\sqrt{2\pi h(l_3+1)}B + ((2l_3+1)k+k^2+1-x)C = 0. \end{cases}$$

The matrix of the coefficients of this system reads

$$\begin{pmatrix} (2l_3+1)k+k^2-k-x & 0 & -i\sqrt{2\pi h l_3} \\ 0 & (2l_3+1)k+k^2+k-x & -ik\sqrt{\frac{l_3+1}{2\pi h}} \\ ik\sqrt{\frac{l_3}{2\pi h}} & i\sqrt{2\pi h(l_3+1)} & (2l_3+1)k+k^2+1-x \end{pmatrix}.$$

Setting $x = y + (2l_3+1)k + k^2$,

$$\begin{pmatrix} -k-y & 0 & -i\sqrt{2\pi h l_3} \\ 0 & k-y & -ik\sqrt{\frac{l_3+1}{2\pi h}} \\ ik\sqrt{\frac{l_3}{2\pi h}} & i\sqrt{2\pi h(l_3+1)} & 1-y \end{pmatrix}.$$

The characteristic equation is

$$y(y^2 - y - (2l+1)k - k^2) = 0,$$

with solutions

$$y = 0, \qquad y = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4(2l+1)k + 4k^2},$$

that give respectively

$$x = (2l+1)k + k^2, \qquad x = \left(\sqrt{k(2l+1) + k^2 + \frac{1}{4}} \pm \frac{1}{2}\right)^2$$

with $l = 0, 1, 2, \ldots$. Note that when $l = 0$ we find the eigenvalue $k + k^2$, with the same eigenvector as found in the particular case $A = 0$ considered above, thus we do not list this eigenvalue separately. This completes the determination of the spectrum of the Hodge Laplace operator.

Proposition 5.4.1. *The spectrum of the Hodge Laplace operator $\Delta^{(q)}$ on H localised at the representation ρ_h is as follows ($k = 2\pi h \in \mathbb{R} - \{0\}$):*

$$\begin{aligned} \mathrm{Sp} d\rho_h(\Delta^{(0)}) =& \{(2m+1)|k| + k^2\}_{m=0}^{\infty}, \\ \mathrm{Sp} d\rho_h(\Delta^{(1)}) =& \{k^2, (|k|+1)^2\} \cup \{(2m+1)|k| + k^2\}_{m=0}^{\infty} \\ & \cup \left\{ \left(\sqrt{|k|(2m+1) + k^2 + \frac{1}{4}} \pm \frac{1}{2} \right)^2 \right\}_{m=0}^{\infty}. \end{aligned}$$

Moreover, $\mathrm{Sp}d\rho_h(\Delta^{(2)}) = \mathrm{Sp}d\rho_h(\Delta^{(1)})$*, and* $\mathrm{Sp}d\rho_h(\Delta^{(3)}) = \mathrm{Sp}d\rho_h(\Delta^{(0)})$*. Each eigenvalue has multiplicity one.*

5.5. The localised heat operator

The localised heat operator in degree q is

$$d\rho_h \mathrm{e}^{-t\Delta^{(q)}} = \mathcal{F}\mathrm{e}^{-t\Delta^{(q)}}\mathcal{F}^{-1}(h).$$

This is a trace class operator, and

$$\begin{aligned}
\mathrm{Tr} d\rho_h \mathrm{e}^{-t\Delta^{(0)}} &= \sum_{m=0}^{\infty} \mathrm{e}^{-t((2m+1)k+k^2)},\\
\mathrm{Tr} d\rho_h \mathrm{e}^{-t\Delta^{(1)}} &= \mathrm{e}^{-tk^2} + \mathrm{e}^{-t(k+1)^2} + \sum_{m=0}^{\infty} \mathrm{e}^{-t((2m+1)k+k^2)}\\
&\quad + \sum_{m=0}^{\infty} \mathrm{e}^{-t\left(\sqrt{k(2m+1)+k^2+\frac{1}{4}}+\frac{1}{2}\right)^2} + \sum_{m=0}^{\infty} \mathrm{e}^{-t\left(\sqrt{k(2m+1)+k^2+\frac{1}{4}}-\frac{1}{2}\right)^2}.
\end{aligned}$$

By some estimates we verify that

$$\mathrm{Tr}\mathrm{e}^{-td\rho_h\Delta^{(0)}} = \frac{\mathrm{e}^{-t(k+k^2)}}{1-\mathrm{e}^{-2kt}},$$

and

$$\begin{aligned}
\mathrm{Tr}\mathrm{e}^{-td\rho_h\Delta^{(1)}} &\geq \mathrm{e}^{-tk^2} + \mathrm{e}^{-t(k+1)^2} + \frac{\mathrm{e}^{-t(k+k^2)}}{1-\mathrm{e}^{-2kt}},\\
\mathrm{Tr}\mathrm{e}^{-td\rho_h\Delta^{(1)}} &\leq \mathrm{e}^{-tk^2} + \mathrm{e}^{-t(k+1)^2} + \frac{\mathrm{e}^{-t(k+k^2)} + \mathrm{e}^{-t(k^2+2k+1)}}{1-\mathrm{e}^{-2kt}} + \frac{1}{k}\frac{1}{t} + \frac{\pi}{2k}\frac{1}{\sqrt{t}}.
\end{aligned}$$

This shows that the condition in Assumption 3.0.1(6) is satisfied. Note that these estimates may be used to determine the Novikov Shubin invariants of H (computed in Proposition 53 of [19]): $\alpha_0(H) = 4$, $\alpha_1(H) = 2$.

5.6. The localised analytic torsion

According to the definition, the analytic torsion of the Heisenberg group H localised at the representation ρ_h, is $\mathfrak{T}(h) = \mathfrak{t}'(0; H, h)$, where the localised analytic torsion zeta function is

$$\mathfrak{t}(s; H, h) = \sum_{q=0}^{3} (-1)^q q \frac{d}{ds}\, \zeta(s, d\rho_h(\Delta^{(q)}))\Big|_{s=0},$$

and

$$\zeta(s, d\rho_h \Delta^{(q)}) = \sum_{\lambda(h) \in \mathrm{Sp}_+ d\rho_h(\Delta^{(q)})} \lambda(h)^{-s}.$$

The last function is clearly defined by the uniformly convergent series when $\mathrm{Re}(s) > 1$, and otherwise by its analytic extension. Note that, by direct substitution, cancellations appear in the function $\mathfrak{t}$ that reduces to

$$\mathfrak{t}(s; H, h) = k^{-2s} + (1+k)^{-2s} - 2t_0(s) + t_1(s),$$

where

$$\begin{aligned}
t_0(s) &= \sum_{m=0}^{\infty} ((2m+1)k + k^2)^{-s}, \\
t_1(s) &= \sum_{m=0}^{\infty} \left(\sqrt{k(2m+1) + k^2 + \frac{1}{4}} + \frac{1}{2} \right)^{-2s} + \sum_{m=0}^{\infty} \left(\sqrt{k(2m+1) + k^2 + \frac{1}{4}} - \frac{1}{2} \right)^{-2s}.
\end{aligned}$$

In order to study the analytic extension of $\mathfrak{t}$ and to compute the localised analytic torsion, we proceed as follows. First, the case of $t_0(s) = \zeta(s, \Delta^{(0)}, \pi_h)$ is quite easy. Indeed,

$$t_0(s) = k^{-s} \zeta_H(s, k) - (2\lambda)^{-s} \zeta_H(s, k/2),$$

where ζ_H is the Hurwitz zeta function. It follows that the analytic extension of $t_0(s)$ has a unique simple pole at $s = 1$, it is regular at $s = 0$, and we compute

$$\begin{aligned}
t_0(0) &= -\frac{k}{2}, \\
t_0'(0) &= \frac{1}{2} k \log k + \left(\frac{1}{2} - \frac{k}{2} \right) \log 2 + \log \Gamma(k) - \log \Gamma(k/2).
\end{aligned}$$

Next, we consider t_1. This requires a bit more work. With

$$a_m = \sqrt{k(2m+1) + k^2 + \frac{1}{4}},$$

and $b = k + \frac{1}{4k}$, we define

$$z(s) = \sum_{m=0}^{\infty} a_m^{-2s} = \sum_{m=0}^{\infty} \left(k(2m+1) + k^2 + \frac{1}{4} \right)^{-s} = k^{-s} \sum_{m=0}^{\infty} (2m + 1 + b)^{-s},$$

and

$$\zeta_\pm(s) = \sum_{m=0}^{\infty} \left(a_m \pm \frac{1}{2}\right)^{-2s} = \sum_{m=0}^{\infty} \left(\sqrt{k(2m+1)+k^2+\frac{1}{4}} \pm \frac{1}{2}\right)^{-2s}.$$

Proceeding as above, we write:

$$z(s) = k^{-s}\sum_{n=0}^{\infty}(n+b)^{-s} - k^{-s}\sum_{n=0}^{\infty}(2n+b)^{-s} = k^{-s}\zeta_H(s,b) - (2k)^{-s}\zeta_H(s,b/2).$$

It follows that the analytic extension of z has a unique pole at $s=1$ with residuum

$$\begin{aligned}\operatorname*{Res_1}_{s=1} z(s) &= \frac{1}{2k},\\ \operatorname*{Res_0}_{s=1} z(s) &= \frac{1}{k}\left(\frac{1}{2}\psi(b/2) - \psi(b)\right),\end{aligned}$$

and

$$\begin{aligned}z(0) &= -\frac{b}{2} = -\frac{k}{2} - \frac{1}{8k},\\ z'(0) &= \log\frac{\Gamma(b)}{\Gamma(b/2)} + \frac{1}{2}(1-b)\log 2 + \frac{b}{2}\log k.\end{aligned}$$

We use this information as follows. Expanding (for $\mathrm{Re}(s)$ large)

$$\zeta_\pm(s) = \sum_{m=0}^{\infty} a_m^{-2s}\sum_{j=0}^{\infty}\binom{-2s}{j}(\pm 1)^j 2^{-j} a_m^{-j},$$

we have

$$\begin{aligned}t_1(s) =&\zeta_+(s) + \zeta_-(s)\\ =&2\sum_{m=0}^{\infty} a_m^{-2s} - \frac{s}{2}\sum_{m=0}^{\infty} a_m^{-2s-2} + 2\sum_{j=2}^{\infty}\binom{-2s}{2j}2^{-2j}\sum_{m=0}^{\infty} a_m^{-2s-2j}\\ =&2z(s) + \frac{1}{2}s(1+2s)z(s+1) + 2\sum_{j=2}^{\infty}\binom{-2s}{2j}2^{-2j}z(2s+2j).\end{aligned}$$

Note that this means that the possible poles of (the analytic continuation of) $t_1(s)$ are simple and are located at $s=1$ and at the negative half integers $s = -\frac{3}{2}, -\frac{5}{2}, \ldots$.

Near $s=0$

$$\frac{1}{2}s(1+2s)z(s+1) = \frac{1}{2}s(1+2s)\left(r_0 + \frac{r_1}{s} + O(s^2)\right) = \frac{r_1}{2} + \frac{1}{2}(r_0+2r_1)s + O(s^2),$$

where r_k denote the residues, and therefore

$$
\begin{aligned}
t_1(s) =&2z(s) + \frac{1}{2}\operatorname*{Res_1}_{s=1} z(s) + \frac{1}{2}\left(\operatorname*{Res_0}_{s=1} z(s) + 2\operatorname*{Res_1}_{s=1} z(s)\right)s + O(s^2)\\
&+ 2\sum_{j=2}^{\infty}\binom{-2s}{2j}2^{-2j}a_m^{-2s-2j}.
\end{aligned}
$$

This gives

$$
t_1(0) = 2z(0) + \frac{1}{2}\operatorname*{Res_1}_{s=1} z(s) = -k.
$$

In order to compute the derivative, note that

$$
\frac{d}{ds}\binom{-2s}{2j}\Big|_{s=0} = \frac{1}{j},
$$

then

$$
t_1'(0) =2z'(0) + \frac{1}{2}\left(\operatorname*{Res_0}_{s=1} z(s) + 2\operatorname*{Res_1}_{s=1} z(s)\right) + 2\sum_{j=2}^{\infty}\frac{1}{j}\sum_{m=0}^{\infty}2^{-2j}a_m^{-2j}.
$$

We deal with the last term as follows. Observing that:

$$
\sum_{j=2}^{\infty}\frac{1}{j}\left(\frac{1}{2a_m}\right)^{2j} = \sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{1}{2a_m}\right)^{2j} - \left(\frac{1}{2a_m}\right)^{2} = -\log\left(1-\frac{1}{4a_m^2}\right)\mathrm{e}^{\frac{1}{4a_m^2}},
$$

we compute

$$
\begin{aligned}
2\sum_{j=2}^{\infty}\frac{1}{k}\sum_{m=0}^{\infty}2^{-2j}a_m^{-2j} =& -2\log\prod_{m=0}^{\infty}\left(1-\frac{1/4k}{(2m+1)+k+\frac{1}{4k}}\right)\mathrm{e}^{\frac{1/4k}{(2m+1)+k+\frac{1}{4k}}}\\
&+\frac{1}{2k}\sum_{m=0}^{\infty}(2m+1)^{-1} - \frac{1}{2k}\sum_{m=0}^{\infty}(2m+1)^{-1}\\
=& -2\log\prod_{m=0}^{\infty}\left(1-\frac{1/4k}{(2m+1)+b}\right)\mathrm{e}^{\frac{1/4k}{2m+1}}\\
&-\frac{1}{2k}\sum_{m=0}^{\infty}((2m+1)+b)^{-1} + \frac{1}{2k}\sum_{m=0}^{\infty}(2m+1)^{-1}.
\end{aligned}
$$

We tackle the two terms separately. For the second one

$$
\begin{aligned}
&\frac{1}{2k}\left(\sum_{m=0}^{\infty}(2m+1)^{-1} - \sum_{m=0}^{\infty}((2m+1)+b)^{-1}\right)\\
&\quad= \frac{1}{2k}\left(\zeta_R(s) - 2^{-s}\zeta_R(s) - \zeta_H(s,b) + 2^{-s}\zeta_H(s,b/2)\right)\Big|_{s=1}.
\end{aligned}
$$

Near $s = 1$,

$$\zeta_H(s, b) = -\psi(b) + \frac{1}{s-1},$$

and therefore ($\psi(1) = \gamma$)

$$\frac{1}{2k}\left(\sum_{m=0}^{\infty}(2m+1)^{-1} - \sum_{m=0}^{\infty}((2m+1)+b)^{-1}\right) = \frac{1}{2k}\left(\frac{\gamma}{2} + \psi(b) - \frac{1}{2}\psi(b/2)\right).$$

For the first term, we recall the definition of the Euler Gamma function,

$$\begin{aligned}
&-2\log\prod_{m=0}^{\infty}\left(1 - \frac{1/4k}{(2m+1)+b}\right)\mathrm{e}^{\frac{1/4k}{2m+1}} \\
&\quad = -2\log\prod_{n=0}^{\infty}\left(1 - \frac{1/(4k)}{n+b}\right)\mathrm{e}^{\frac{1/(4k)}{n}} + 2\log\prod_{n=0}^{\infty}\left(1 - \frac{1/(8k)}{n+b/2}\right)\mathrm{e}^{\frac{1/(8k)}{n}} \\
&\quad = -2\log\frac{\mathrm{e}^{\gamma/(4k)}\Gamma(b+1)}{\Gamma(k+1)} + 2\log\frac{\mathrm{e}^{\gamma/(8k)}\Gamma(b/2+1)}{\Gamma(k/2+1)}.
\end{aligned}$$

Thus,

$$t_1'(0) = (1-b)\log 2 + b\log k + \frac{1}{2k} + 2\log\Gamma(k) - 2\log\Gamma(k/2).$$

We have proved the following results.

Proposition 5.6.1. *The analytic torsion zeta function $\mathfrak{t}(s; H, h)$ of the Heisenberg group H localised at the representation ρ_h, $0 \neq h \in \hat{H}$, is a regular analytic function of s for all s with a simple poles at $s = 1$, and possible simple poles at $s = -\frac{3}{2}, -\frac{5}{2}, \ldots$. Near $s = 0$, we have the expansion*

$$\mathfrak{t}(s; H, h) = \mathfrak{t}(0; H, h) + \mathfrak{t}'(0; H, h)s + O(s^2),$$

where

$$\begin{aligned}
&\mathfrak{t}(0; H, h) = 2, \\
&\mathfrak{t}'(0; H, h) = -2\log k - 2\log(1+k) + \frac{1}{4k}\log k - \frac{1}{4k}\log 2 + \frac{1}{2k}.
\end{aligned}$$

Corollary 5.6.2. *The analytic torsion of the three dimensional Heisenberg group H localised at the representation ρ_h, $h \neq 0$, is*

$$\mathfrak{T}(H, h) = -2\log 2\pi h(1 + 2\pi h) + \frac{1}{8\pi h}\log \pi h + \frac{1}{4\pi h}.$$

5.7. The relative analytic torsion I

By Assumption 3.0.1 (1), (2), and according to equation (3.5), the relative analytic torsion of (H, Γ) is (recall that $\mathfrak{t}(s; H, h) = \mathfrak{t}(s; H, -h)$, and that $\mathrm{Vol}(\Gamma\backslash G) = 1$)

$$\mathfrak{T}_\Gamma(H) = \frac{d}{ds}\mathfrak{t}_\Gamma(s; H)\bigg|_{s=0} + 2\int_0^\delta \mathfrak{T}(H, h)|h|dh, \tag{5.2}$$

where the relative analytic torsion zeta function (equation (3.6) is

$$\mathfrak{t}_\Gamma(s; H) = 2\int_\delta^\infty \mathfrak{t}(s; H, h)|h|dh. \tag{5.3}$$

We would like to find a geometric interpretation of this invariant as in Section 4 for the abelian case. Take $\delta = 1$, then, for large $\mathrm{Re}(s)$, by uniform convergence of the series of function for $\alpha \in (0, 1)$ when $\mathrm{Re}(s)$ is large, this gives

$$\begin{aligned}\mathfrak{t}_\Gamma(s; H) &= \int_{-\infty}^{-1} \mathfrak{t}(s; H, h)|h|dh + \int_1^\infty \mathfrak{t}(s; H, h)hdh \\ &= \int_0^1 \sum_{n=-\infty}^{-2} \mathfrak{t}(s; H, n+\alpha)|n+\alpha|d\alpha + \int_0^1 \sum_{n=1}^{\infty} \mathfrak{t}(s; H, n+\alpha)(n+\alpha)d\alpha.\end{aligned}$$

After some simplifications, we find

$$\begin{aligned}\mathfrak{t}_\Gamma(s; H) = &\int_0^1 \sum_{n\in\mathbb{Z}} \mathfrak{t}(s; H, n+\alpha)|n|d\alpha - \int_0^1 \mathfrak{t}(s; H, 1-\alpha)d\alpha \\ &+ \int_0^1 \sum_{n=1}^{\infty} \mathfrak{t}(s; H, n+\alpha)\alpha d\alpha - \int_0^1 \sum_{n=-\infty}^{-2} \mathfrak{t}(s; H, n+\alpha)\alpha d\alpha.\end{aligned} \tag{5.4}$$

We would like to identify the integrands in the previous formula with the analytic torsion of some smooth Riemannian manifold. Unfortunately, we are only able to partially accomplish this purpose. This is the subject of the next sections.

5.8. Some quotients of the Heisenberg group: H_{red} and H_{cpt}

Following the line indicated in the abelian case, we would like to rewrite the integrands appearing in equation (5.4) as the analytic torsion of some Riemmanian manifold. Unfortunately, this does not work so nicely: the spectral invariant appearing are not the analytic torsion but some less natural ones. In order to proceed we need first some quotient spaces, and second to identify the spectrum of the Hodge Laplace operator on these quotient spaces.

We start by describing the quotient spaces. The first is the reduced Heisenberg group, $H_{\rm red} = Z\backslash H$, that is the quotient space of H by the action of the centre of Γ, i.e. the subgroup $Z = \{0\} \times \{0\} \times \mathbb{Z}$, see for example [13, pg. 23]. This is a complete smooth Riemannian manifold, with the quotient metric, universal covering space H, and fundamental group $\pi_1(H_{\rm red}) = \mathbb{Z}$.

The second quotient is $H_{\rm cpt} = \Gamma\backslash H$ obtained by taking the quotient of H by the left action of Γ, and is usually called compact Heisenberg space [13, pg. 68] (see also [15] for the complete classification of the lattices of H). $H_{\rm cpt}$ is a compact smooth Riemannian manifold, with the quotient metric, universal covering space H and fundamental group $\pi_1(H_{\rm cpt}) = \Gamma$, homeomorphic to a circle bundle over the torus $\mathbb{T}^2$ (more precisely such bundle are classified by $H^1(H_{\rm cpt}) = [H_{\rm cpt}, B\mathbb{Z}] = \mathbb{Z}$, and $H_{\rm cpt}$ corresponds to the generator).

In both cases, since we are taking left actions, invariant vector fields, exterior derivative and Hodge Laplace operator descend to the quotient manifolds.

Accomplished the geometric first step, we pass to analysis: we would like to identify the spectrum of the Hodge Laplace operator on $H_{\rm red}$ and on $H_{\rm cpt}$ and to use it to define some spectral invariants. In order to identify the spectrum we proceed adapting the approach of Folland, Auslander and Tolimieri for the compact Heisenberg group [13, pg. 68] [15] [3, Section 1] and [4].

5.9. The spectrum of the Hodge Laplace operator and a spectral invariant on $H_{\rm red}$

Since $H_{\rm red}$ is not compact, in order to develop spectral analysis we need to work with square integrable forms. In particular, since $H_{\rm red}$ is complete, we work with the unique self adjoint extension in the space of the square integrable forms of the restriction of the formal Hodge Laplace operator on the space of smooth forms with compact support [17].

Consider the representation π of H on $L^2(H_{\rm red})$ determined by right translation:

$$\pi(g)(f)(Zx) = f(Zxg),$$

and the π invariant subspaces of $L^2(H_{\rm red})$, $n \in \mathbb{Z}$,

$$\mathcal{H}_n = \{f \in L^2(H_{\rm red}) \mid \pi(0,0,t)f = \mathrm{e}^{2\pi i n t} f\}.$$

According to the Stone Von Neumann Theorem, the restriction π_n of π to $\mathcal{H}_n$ is a direct sum of ρ_n. This proves that the eigenvalues of the Hodge Laplace operator on $H_{\rm red}$ coincide with those of the Hodge Laplace operator on H, with $h = n \neq 0$ (this is known by [23]). Next, we consider multiplicity. Let see in some details the case of functions when $h = n = 1$. We define the function

$$\begin{aligned}\Phi_1 :&L^2(\mathbb{R}) \to \mathcal{H}_1 \subseteq L^2(H_{\mathrm{red}}),\\ \Phi_1 :&f \mapsto \Phi_1(f),\end{aligned}$$

where

$$\begin{aligned}\Phi_1(f)(p,q,t) &= \mathrm{e}^{2\pi it} \int_{-\infty}^{\infty} f(p+s)\mathrm{e}^{2\pi iqs} ds = \mathrm{e}^{2\pi it}\mathrm{e}^{-2\pi iqp} \int_{\mathbb{R}} f(v)\mathrm{e}^{2\pi iqv} dv\\ &= \mathrm{e}^{2\pi it}\mathrm{e}^{-2\pi iqp}\mathcal{F}(f)(q) = \mathrm{e}^{2\pi it}\mathcal{F}(f(\_ - p))(q).\end{aligned}$$

Recall that, as observed above, we are thinking to function in $\mathcal{S}$, and to the usual completion to square integrable function. Also note that Φ_1 is obviously Z invariant. It is clear that Φ_1 preserves the L^2 norms, and is therefore and isometry of $L^2(\mathbb{R})$ into $L^2(\mathcal{H}_1)$.

We show that this map intertwines ρ_1 restricted to $\mathcal{H}_1$ with π. For compute on one side

$$\rho_1(a,b,c)(f)(x) = \mathrm{e}^{2\pi ic}\mathrm{e}^{2\pi ibx} f(x+a),$$

and

$$\Phi_1\rho_1(a,b,c)(f)(p,q,t) = \mathrm{e}^{2\pi it}\mathrm{e}^{2\pi ic} \int f(p+s+a)\mathrm{e}^{2\pi ib(p+s)}\mathrm{e}^{2\pi iqs} ds,$$

and on the other

$$\Phi_1(f)(p,q,t) = \mathrm{e}^{2\pi it} \int_{-\infty}^{\infty} f(p+s)\mathrm{e}^{2\pi iqs} ds,$$

and

$$\pi(a,b,c)\Phi_1(f)(p,q,t) = \mathrm{e}^{2\pi it}\mathrm{e}^{2\pi ic}\mathrm{e}^{2\pi ipb} \int f(p+s+a)\mathrm{e}^{2\pi i(q+b)s} ds.$$

To conclude we need to verify that Φ_1 is onto. For take $f \in \mathcal{H}_1$, then we write

$$f(p,q,t) = \mathrm{e}^{2\pi it} g(p,q),$$

and let

$$F(x) = \int_{\mathbb{R}} \mathrm{e}^{2\pi i(p-x)t} g(p,t) dt = \mathcal{F}_2(g(p, p - \_))(x) = \mathcal{F}_2^{-1}(g(p, \_ - p))(x).$$

Then,

$$\Phi_1(F)(p,q,t) = \mathrm{e}^{2\pi it}\mathcal{F}(\mathcal{F}_2^{-1}(g(p, \_ - p))(\_ - p))(q) = \mathrm{e}^{2\pi it} g(p,q) = f(p,q,t).$$

This construction extends to $h = n$ as follows. Define

$$\begin{aligned}\Phi_n :& L^2(\mathbb{R}) \to \mathcal{H}_n \subseteq L^2(H_{\mathrm{red}}),\\ \Phi_n :& f \mapsto \Phi_n(f),\end{aligned}$$

where

$$\Phi_n(f)(p,q,t) = \mathrm{e}^{2\pi int} \int_{-\infty}^{\infty} f(n(p+s))\mathrm{e}^{2\pi inqs} ds.$$

Also Φ_n is Z invariant. We show that this map intertwines π restricted to $\mathcal{H}_n$ with ρ_n. For compute on one side

$$\rho_n(a,b,c)(f)(x) = \mathrm{e}^{2\pi inc}\mathrm{e}^{2\pi ibx} f(x+na),$$

and

$$\Phi_n\rho_n(a,b,c)(f)(p,q,t) = \mathrm{e}^{2\pi int}\mathrm{e}^{2\pi inc} \int f(n(p+s)+na)\mathrm{e}^{2\pi inb(p+s)}\mathrm{e}^{2\pi inqs} ds,$$

and on the other

$$\Phi_n(f)(p,q,t) = \mathrm{e}^{2\pi int} \int_{-\infty}^{\infty} f(n(p+s))\mathrm{e}^{2\pi inqs} ds,$$

and

$$\pi(a,b,c)\Phi_n(f)(p,q,t) = \mathrm{e}^{2\pi int}\mathrm{e}^{2\pi inc}\mathrm{e}^{2\pi inpb} \int f(n(p+a+s))\mathrm{e}^{2\pi in(q+b)s} ds.$$

To conclude we need to verify that Φ_n is onto. For take $f \in \mathcal{H}_n$, then we write

$$f(p,q,t) = \mathrm{e}^{2\pi int} g(p,q),$$

and expand g in its Fourier transform

$$f(p,q,t) = \mathrm{e}^{2\pi int} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{g}(\hat{p},\hat{q})\mathrm{e}^{2\pi i\hat{p}p}\mathrm{e}^{2\pi i\hat{q}q} d\hat{p}d\hat{q}.$$

Define the function $f \in L^2(\mathbb{R})$:

$$F(n(x+\hat{q})) = n \int \hat{g}(\hat{p}, n\hat{q})\mathrm{e}^{2\pi i\hat{p}x} d\hat{p},$$

then we verify that $\Phi_n(F) = f$, for compute

$$\begin{aligned}\Phi_n(F)(p,q,t) &= \mathrm{e}^{2\pi i n t}\int_{-\infty}^{\infty} F(n(p+\hat{q}))\mathrm{e}^{2\pi i n q\hat{q}}d\hat{q}\\ &= \mathrm{e}^{2\pi i n t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{g}(\hat{p}, n\hat{q})\mathrm{e}^{2\pi i \hat{p}p}d\hat{p}\mathrm{e}^{2\pi i n q\hat{q}}nd\hat{q}\\ &= \mathrm{e}^{2\pi i n t}g(p,q).\end{aligned}$$

This proves that the eigenvalues of the Hodge Laplace operator on H_{red} all have multiplicity one.

The last point is to twist the coefficients. For consider the finite irreducible representation of Z: $\chi_\alpha(0,0,n) = \mathrm{e}^{2\pi i\alpha n}$, and the induced bundle E_α over H_{red}. The smooth sections of this bundle may be identified with the function on H_{red} satisfying the conditions

$$f((0,0,1)(x,y,t)) = \mathrm{e}^{2\pi i\alpha}f(x,y,t).$$

It follows that the space of the square integrable sections of H_{red} with values in E_α, $L^2(H_{\mathrm{red}}, E_\alpha)$ decomposes in the π invariant subspaces

$$\mathcal{H}_{n+\alpha} = \{f \in L^2(H_{\mathrm{red}}, E_\alpha) \mid \pi(0,0,t)f = \mathrm{e}^{2\pi i(n+\alpha)t}f\}.$$

We may repeat the previous construction and show that π restricted to $\mathcal{H}_{n+\alpha}$ is equivalent to $\rho_{n+\alpha}$; this concludes the determination of the spectrum and the proof of the following proposition.

Proposition 5.9.1. *The spectrum of the Hodge Laplace operator $\Delta_\alpha^{(q)}$ on forms on H_{red} with values in E_α, $0 < \alpha < 1$, is as follows:*

$$\begin{aligned}\mathrm{Sp}\Delta_\alpha^{(0)} =&\{(2m+1)|n+\alpha| + (n+\alpha)^2\}_{m=0,n=-\infty}^{\infty},\\ \mathrm{Sp}\Delta_\alpha^{(1)} =&\{(n+\alpha)^2, (|n+\alpha|+1)^2\} \cup \{(2m+1)|n+\alpha| + (n+\alpha)^2\}_{m=0,n=-\infty}^{\infty}\\ &\cup\left\{\left(\sqrt{|n+\alpha|(2m+1) + (n+\alpha)^2 + \frac{1}{4}} \pm \frac{1}{2}\right)^2\right\}_{m=0,n=-\infty}^{\infty}.\end{aligned}$$

Moreover, $\mathrm{Sp}\Delta_\alpha^{(2)} = \mathrm{Sp}\Delta_\alpha^{(1)}$, *and* $\mathrm{Sp}\Delta_\alpha^{(3)} = \mathrm{Sp}\Delta_\alpha^{(0)}$. *Each eigenvalue has multiplicity one.*

Then, a direct verification gives the following result.

Lemma 5.9.2. *The spectrum* $\mathrm{Sp}\Delta_\alpha^{(q)}$ *of the Hodge Laplace operator* $\Delta^{(q)}$ *on forms on* H_{red} *with values in* E_α, $0 < \alpha < 1$, *is a sequence of spectral type of genus 2. In particular, the associated zeta functions have analytic expansion regular at* $s = 0$.

In order to proceed with our interpretation of the second line of equation (5.4), we need to introduce a suitable spectral invariant on H_{red}. This invariant "measures" the spectral asymmetry of the Fourier group decomposition of $L^2(H_{\mathrm{red}}, E_\alpha)$ into the subspaces $\mathcal{H}_{n+\alpha}$, and is defined as follows. Let $\lambda_{m,n}^{(q)}(\alpha)$ denote the eigenvalue of the Hodge Laplace operator $\Delta_\alpha^{(q)}$ on H_{red} with indices n and m given in Proposition 5.9.1. Consider the function of the complex variable s, defined for $\mathrm{Re}(s)$ large by the series

$$e(s; H_{\mathrm{red}}, \alpha) = \sum_{q=0}^{3} (-1)^q \sum_{m\in\mathbb{N}, n\in\mathbb{Z}} \mathrm{sgn}(n)(\lambda_{m,n}^{(q)}(\alpha))^{-s},$$

and by analytic extension elsewhere. By Lemma 5.9.2, $e(s; H_{\mathrm{red}}, \alpha)$ is regular at $s = 0$, so we define

$$E(H_{\mathrm{red}}, \alpha) = e'(0; H_{\mathrm{red}}, \alpha).$$

5.10. The spectrum of the Hodge Laplace operator and a spectral invariant on H_{cpt}

We generalise the approach in Section 10 of [13], see also [15, 3] and [23]). Consider the representation P of H on $L^2(H_{\mathrm{cpt}})$, determined by the right translation,

$$\begin{aligned} &\mathrm{P} : H \to U(L^2(H_{\mathrm{cpt}})), \\ &\mathrm{P} : g \mapsto \mathrm{P}(g)(f)(\Gamma x) = f(\Gamma x g), \end{aligned}$$

and the P invariant subspaces of $L^2(H_{\mathrm{cpt}})$

$$\mathcal{H}_n = \{f \in L^2(H_{\mathrm{cpt}}) \mid \mathrm{P}(0,0,c)f = \mathrm{e}^{2\pi i n c} f\}.$$

According to the Stone Von Neumann Theorem, the restriction P_n of P to $\mathcal{H}_n$ is a direct sum of ρ_n. This proves that the eigenvalues of the Hodge Laplace operator on H_{cpt} coincide with those of the Hodge Laplace operator on H, with $h = n \neq 0$.

Next, we consider multiplicity. We proceed adapting the construction in Chapter 1 of [3]. Define the function

$$\begin{aligned} \Psi_n :& L^2(\mathbb{R}) \to \mathcal{H}_1 \subseteq L^2(H_{\mathrm{cpt}}), \\ \Psi_n :& f \mapsto \Psi_n(f), \end{aligned}$$

where

$$\Psi_n(f)(p,q,t) = \mathrm{e}^{2\pi int} \sum_{k\in\mathbb{Z}} \mathrm{e}^{2\pi ikq} f(np+k).$$

Here (p,q,t) is in H, but we verify that $\Psi_n(f)$ is Γ invariant, and therefore defines a function on H_{cpt} as claimed.

We show that Ψ_n intertwines P_n with ρ_n. For compute on one side

$$\rho_n(a,b,c)(f)(x) = \mathrm{e}^{2\pi inc}\mathrm{e}^{2\pi ibx} f(x+na),$$

and

$$\Psi_n\rho_n(a,b,c)(f)(p,q,t) = \mathrm{e}^{2\pi in(t+c)} \sum_{k\in\mathbb{Z}} \mathrm{e}^{2\pi ikq}\mathrm{e}^{2\pi ib(np+k)} f(n(p+a)+k),$$

and on the other

$$\Psi_n(f)(p,q,t) = \mathrm{e}^{2\pi int} \sum_{k\in\mathbb{Z}} \mathrm{e}^{2\pi ikq} f(np+k),$$

and

$$\mathrm{P}(a,b,c)\Psi_n(f)(p,q,t) = \mathrm{e}^{2\pi in(t+c+pb)} \sum_{k\in\mathbb{Z}} \mathrm{e}^{2\pi ik(q+b)} f(n(p+a)+k).$$

We verify that Ψ_n is unitary:

$$\begin{aligned}
\|\Psi_n(f)\|_{L^2(M)} &= \int_0^1\int_0^1\int_0^1 \left| \mathrm{e}^{2\pi int} \sum_{k\in\mathbb{Z}} \mathrm{e}^{2\pi ikq} f(np+k)\right|^2 dpdqdt \\
&= \int_0^1 \sum_{k\in\mathbb{Z}} |f(np+k)|^2\, dp \\
&= \frac{1}{n}\sum_{k\in\mathbb{Z}} \int_0^n |f(p+k)|^2\, dp \\
&= \|f\|_{L^2(\mathbb{R})}.
\end{aligned}$$

The next step is to compute multiplicity. Take $f \in \mathcal{H}_n$, then we write

$$f(p,q,t) = \mathrm{e}^{2\pi int} g(p,q),$$

and expand g in its Fourier transform

$$f(p,q,t) = \mathrm{e}^{2\pi int} \sum_{l,k\in\mathbb{Z}} g_{l,k}\mathrm{e}^{2\pi ilp}\mathrm{e}^{2\pi ikq}.$$

Define the functions $F_j \in L^2(\mathbb{R})$, $1 \le j \le n$, $0 \le u \le 1$, $m \in \mathbb{Z}$:

$$F_j(ju + jm) = \sum_{l \in \mathbb{Z}} g_{l,jm} \mathrm{e}^{2\pi i \frac{lju}{n}},$$

then, we verify that $\Psi_n(F_j) = f$, for all j. For identifying $ju + jm = np + k$

$$\begin{aligned}\Psi_n(F_j)(p, q, t) &= \mathrm{e}^{2\pi int} \sum_{k \in \mathbb{Z}} \mathrm{e}^{2\pi ikq} F_j(np + k) \\ &= \mathrm{e}^{2\pi int} \sum_{k \in \mathbb{Z}} \mathrm{e}^{2\pi ikq} \sum_{l \in \mathbb{Z}} g_{l,k} \mathrm{e}^{2\pi ilp}.\end{aligned}$$

This proves that the representation $\mathrm{P}_n = \mathrm{P}|_{\mathcal{H}_n}$ is equivalent to the sum of n copies of ρ_n, and therefore the eigenvalue with index n of the Hodge Laplace operator on H_{red} has multiplicity n, and therefore the following proposition.

Proposition 5.10.1. *The spectrum of the Hodge Laplace operator on forms $\Delta^{(q)}$ over H_{cpt}, is as follows:*

$$\begin{aligned}\mathrm{Sp}\Delta^{(0)} =& \{(2m+1)|n| + n^2\}_{m=0,n=-\infty}^{\infty}, \\ \mathrm{Sp}\Delta^{(1)} =& \{n^2, (|n|+1)^2\} \cup \{(2m+1)|n| + n^2\}_{m=0,n=-\infty}^{\infty} \\ & \cup \left\{ \left(\sqrt{|n|(2m+1) + n^2 + \frac{1}{4}} \pm \frac{1}{2} \right)^2 \right\}_{m=0,n=-\infty}^{\infty},\end{aligned}$$

plus a number of eigenvalues when $n = 0$ that are not of interest here. $\mathrm{Sp}\Delta^{(2)} = \mathrm{Sp}\Delta^{(1)}$, *and* $\mathrm{Sp}\Delta^{(3)} = \mathrm{Sp}\Delta^{(0)}$. *Each eigenvalue with index n has multiplicity $|n|$.*

The last effort is to twist the coefficients. However, we do not have finite representations of $\pi_1(H_{\mathrm{cpt}}) = \Gamma$ that may twist the construction along the fibre of H_{cpt}. We proceed as follows. For each $0 < \alpha < 1$, we consider the families $S_\alpha^{(q)} = \{s^q_{\alpha;m,n}\}_{n,m}$:

$$\begin{aligned}S_\alpha^{(0)} = S_\alpha^{(3)} =& \{(2m+1)|n+\alpha| + (n+\alpha)^2\}_{m=0,n=-\infty}^{\infty}, \\ S_\alpha^{(1)} = S_\alpha^{(2)} =& \{(n+\alpha)^2, (|n+\alpha|+1)^2\} \cup \{(2m+1)|n+\alpha| + (n+\alpha)^2\}_{m=0,n=-\infty}^{\infty} \\ & \cup \left\{ \left(\sqrt{|n+\alpha|(2m+1) + (n+\alpha)^2 + \frac{1}{4}} \pm \frac{1}{2} \right)^2 \right\}_{m=0,n=-\infty}^{\infty}.\end{aligned}$$

Then, we define a spectral invariant on H_{cpt} precisely as we do with analytic torsion assuming the family above to be the spectrum of the Hodge Laplace operator with twisted coefficients, namely we set

$$\zeta(s, S_\alpha^{(q)}) = \sum_{m=0,n=-\infty}^{\infty} |n|(s^q_{\alpha;m,n})^{-s},$$

$$z(s; H_{\rm cpt}, \alpha) = \sum_{q=0}^{3} (-1)^q \zeta(s, S_\alpha^{(q)}),$$

and

$$Z(H_{\rm cpt}, \alpha) = z'(0; H_{\rm cpt}, \alpha).$$

5.11. The relative analytic torsion II

We may use the results in the last two sections to deal with the terms in formula (5.4) as in the abelian case. First, rewrite the last equation as

$$\begin{aligned} \mathfrak{t}_\Gamma(s; H) &= \int_0^1 \sum_{n\in\mathbb{Z}} \mathfrak{t}(s; H, n+\alpha)|n| d\alpha - 2\int_0^1 \mathfrak{t}(s; H, \alpha) d\alpha \\ &+ \int_0^1 \sum_{n=0}^{\infty} \mathfrak{t}(s; H, n+\alpha)\alpha d\alpha - \int_0^1 \sum_{n=-\infty}^{-1} \mathfrak{t}(s; H, n+\alpha)\alpha d\alpha. \end{aligned}$$

Next, from one side observe that we have the equivalence

$$\sum_{n\in\mathbb{Z}} \mathfrak{t}(s; H, n+\alpha)|n| = z(s; H_{\rm cpt}, \alpha),$$

where $z(s; H_{\rm cpt}, \alpha)$ was defined at the end of Section 5.10, and from the other that

$$\sum_{n=0}^{\infty} \mathfrak{t}(s; H, n+\alpha) - \sum_{n=-\infty}^{-1} \mathfrak{t}(s; H, n+\alpha) = e(s; H_{\rm red}, \alpha),$$

where $e(s; H_{\rm red}, \alpha)$ is the invariant introduced at the end of Section 5.9.

Whence,

$$\mathfrak{t}_\Gamma(s; H) = \int_0^1 z(s; H_{\rm cpt}, \alpha) d\alpha - 2\int_0^1 \mathfrak{t}(s; H, \alpha)\alpha d\alpha + \int_0^1 e(s; H_{\rm red}, \alpha)\alpha d\alpha.$$

Proceeding as in Section 4, and according to equation (5.2), we have the following result.

Proposition 5.11.1.

$$\mathfrak{T}_\Gamma(H) = \int_0^1 Z(H_{\text{cpt}}, \alpha)\alpha d\alpha + \int_0^1 E(H_{\text{red}}, \alpha) d\alpha.$$

Data availability

No data was used for the research described in the article.

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