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Commutativity of quantization with conic reduction for torus actions on compact CR manifolds

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Abstract
We define conic reductions X_ν^{red} for torus actions on the boundary X of a strictly pseudo-convex domain and for a given weight ν labeling a unitary irreducible representation. There is a natural residual circle action on X_ν^{red}. We have two natural decompositions of the corresponding Hardy spaces $H(X)$ and $H(X_\nu^{\text{red}})$. The first one is given by the ladder of isotypes $H(X)_{k\nu}$, $k \in \mathbb{Z}$; the second one is given by the k-th Fourier components $H(X_\nu^{\text{red}})_k$ induced by the residual circle action. The aim of this paper is to prove that they are isomorphic for k sufficiently large. The result is given for spaces of $(0, q)$-forms with L^2-coefficient when X is a CR manifold with non-degenerate Levi form.

Contents

Keywords Hardy space · Torus action · CR conic reduction

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1 Introduction

Let X be the boundary of a strictly pseudo-convex domain D in $\mathbb{C}^{n+1}$. Then $(X,\ T^{1,0}X)$ is a contact manifold of dimension $2n+1$, $n \geq 1$, where $T^{1,0}X$ is the sub-bundle of $TX \otimes \mathbb{C}$ defining the CR structure. We denote by $\omega_0 \in \mathcal{C}^\infty(X, T^*X)$ the contact 1-form whose kernel is the horizontal bundle $HX \subset TX$; we refer to Sect. 2.1 for definitions. Associated

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with these data we can define the Hardy space $H(X)$, it is the space of boundary values of holomorphic functions in D which lie in $L^2(X)$, the Hilbert space of square integrable functions on X. Suppose a contact and CR action of a t-dimensional torus $\mathbb{T}$ is given; we denote by $\mu : X \to \mathfrak{t}^*$ the associated CR moment map. Fix a weight $i\,\nu$ in the lattice $i\,\mathbb{Z}^t \subset \mathfrak{t}^*$, if $0 \in \mathfrak{t}^*$ does not lie in the image of the moment map, the isotypes

$$H(X)_{k\nu} = \{f \in H(X) : (e^{i\,\theta} \cdot f)(x) = e^{ik\,\langle \nu,\theta\rangle} f(x),\ \theta \in \mathbb{R}^t\}, \quad k \in \mathbb{Z},$$

are finite dimensional.

Suppose that the ray $i\,\mathbb{R}_+ \cdot \nu \in \mathfrak{t}^*$ is transversal to μ, then $X_\nu := \mu^{-1}(i\,\mathbb{R}_+ \cdot \nu)$ is a sub-manifold of X of codimension $t - 1$. There is a well-defined locally free action of $\mathbb{T}^{t-1}_\nu := \exp_{\mathbb{T}}(i\ \ker \nu)$ on X_ν; the resulting orbifold X^{red}_ν is called *conic reduction* of X with respect to the weight ν. Let φ be an Euclidean product on $\mathfrak{t}$, we shall also use the symbol $\langle\cdot,\ \cdot\rangle$ and denote by $\lambda^\varphi \in \mathfrak{t}$ be uniquely determined by $\lambda = \varphi(\lambda^\varphi,\ \cdot)$ and $\|\lambda\|$ the corresponding norm. By abuse of notation we write λ for λ^φ and we identify $\mathfrak{t} \cong i\mathbb{R}^t$ with its dual. We set

$$\ker \nu = \nu^\perp := \{\lambda \in \mathfrak{t} : \langle \nu,\ \lambda\rangle = 0\}\,.$$

The locus X_ν is $\mathbb{T}$-invariant; we will always assume that the action of $\mathbb{T}$ on X_ν is locally free. After replacing $\mathbb{T}$ with its quotient by a finite subgroup, we may and will assume without loss of generality that the action is generically free. In Sect. 2.1, we show that X^{red}_ν is CR manifold with positive definite Levi form of dimension $2n - 2t + 3$.

Let us define $\mathbb{T}^1_\nu := \exp_{\mathbb{T}}(i\ \nu)$, if ν is coprime, we have a Lie group isomorphism

$$\kappa_\nu : S^1 \to \mathbb{T}^1_\nu, \quad e^{i\theta} \mapsto e^{i\theta\nu}$$

between $\mathbb{T}^1_\nu := \exp_{\mathbb{T}}(i\ \nu)$ and the circle S^1. Let us denote by

$$\overline{\mathbb{T}^1_\nu} := \mathbb{T}/\mathbb{T}^{t-1}_\nu \cong \mathbb{T}^1_\nu/(\mathbb{T}^1_\nu \cap \mathbb{T}^{t-1}_\nu),$$

then the character $\chi_\nu : \mathbb{T} \to S^1$, $\chi_\nu(e^{i\,\theta}) := e^{ik\,\langle \nu,\,\theta\rangle}$, being trivial on $\mathbb{T}^{t-1}_\nu$, descends to a character $\chi'_\nu : \overline{\mathbb{T}^1_\nu} \to S^1$ which is a Lie group isomorphism, see [14, Lemma 10]. Thus, we have a locally free circle action of $\overline{\mathbb{T}^1_\nu}$ on X^{red}_ν, which induces an action on the Hardy space $H(X^{\mathrm{red}}_\nu)$. Suppose that the action of $\overline{\mathbb{T}^1_\nu}$ on X is transversal to the CR structure. We denote by $H(X^{\mathrm{red}}_\nu)_k$ the corresponding k-th Fourier component, and we call the action of $\overline{\mathbb{T}^1_\nu}$ on X^{red}_ν *residual circle action*. The aim of this paper is to prove that $H(X)_{k\nu}$ and $H(X^{\mathrm{red}}_\nu)_k$ are isomorphic for k sufficiently large.

We prove the aforementioned result in the more general setting of CR manifolds for spaces of $(0, q)$-forms when k is large; more precisely we consider $(0, q)$ forms with L^2 coefficients and the corresponding projector $S^{(q)}$ onto the kernel of the Kohn Laplacian $H^q(X)$. Now, we make more precise the assumptions on the CR manifold X and on the group action.

Assumption 1.1 Let $(X,\ T^{1,0}X)$ be a compact connected orientable CR manifold of dimension $2n + 1$, $n \geq 1$, and let ω be the associated contact 1-form. The Levi form L is non-degenerate of constant signature $(n_-,\ n_+)$ on X. That is, the Levi form has exactly n_- negative and n_+ positive eigenvalues at each point of X, where $n_- + n_+ = n$.

Concerning the group action, we always assume

Assumption 1.2 The action of $\mathbb{T}$ preserves the contact form ω_0 and the complex structure J. That is, $g^*\omega_0 = \omega_0$ on X and $g_*J = Jg_*$ on the horizontal bundle HX for every $g \in \mathbb{T}$ where g^* and g_* denote the pull-back map and push-forward map of $\mathbb{T}$, respectively.

Let $X_\nu^{\text{red}} := X_\nu/\mathbb{T}_\nu^{t-1}$, defined in the same way as before, more precisely we shall assume

Assumption 1.3 The moment map μ is transverse to the ray $i\,\mathbb{R}_+ \cdot \nu \in \mathfrak{t}^*$, the action of $\mathbb{T}$ on X_ν is locally free and for every $x \in X_\nu$

$$\text{val}_x(\nu^\perp) \cap \text{val}_x(\nu^\perp)^{\perp_b} = \{0\}$$

where b is a bilinear form on $H_x X$ such that

$$b(\cdot,\ \cdot) = \mathrm{d}\omega_0(\cdot,\ J\cdot) \tag{1}$$

and it is non-degenerate.

We note that $b(U,\ V) = 2\,L(U,\ V)$ for every $U,\ V \in HX$. By assumptions above, we will show that X_ν^{red} is a CR manifold with natural CR structure induced by $T^{1,0}X$ of dimension $2n - 2(t-1) + 1$. Let $L_{X_\nu^{\text{red}}}$ be the Levi form on X_ν^{red} induced naturally from the Levi form L on X. For a given subspace $\mathfrak{s}$ of $\mathfrak{t}$, we denote $\mathfrak{s}_X$ the subspace of infinitesimal vector fields on X. Let us consider

$$B = \ker \nu_X \oplus J \ker \nu_X.$$

Hence, b has constant signature on $B \times B$, suppose b has r negative eigenvalues on $B \times B$ where $r \leq n_-$ since L and b have the same number of negative eigenvalues on HX. Fix $q = n_-$; hence, by Lemma 2.1, $L_{X_\nu^{\text{red}}}$ has $q - r$ negative eigenvalues at each point of X_ν^{red}. We refer to Sect. 2.1 for definitions; we have:

Theorem 1.1 *Suppose that $\Box_b^q$ has L^2 closed range. Fix a maximal coprime weight $\nu \neq 0$ in the lattice inside $\mathfrak{t}^*$ and assume that the circle action $\overline{\mathbb{T}_\nu^1}$ is a transversal CR action. Fix $q = n_-$, under the assumptions above, X_ν^{red} is a compact CR manifold with non-degenerate Levi form having $q - r$-negative eigenvalues. There is a natural isomorphism of vector spaces $\sigma_k : H^q(X)_{k\nu} \to H^{q-r}(X_\nu^{\text{red}})_k$ for k sufficiently large.*

For strictly pseudoconvex domain, we have $q = n_- = r = 0$ and thus we have quantization commutes with reduction for spaces of functions for k large. We give a proof in Sect. 3, which is inspired from [7] (see [8] for the full extension of [7]), and it is a consequence of the microlocal properties of the projector $S_{k\nu}^{(q)}$ described in Sect. 2.2 and calculus of Fourier integral operators of complex type, see [11]. Furthermore, we recall that the way to establish the isometry from kernel expansion for k large comes from [10].

The conic reductions defined above appear naturally in geometric quantization. In fact, given a Hamiltonian and holomorphic action with moment map Φ of a compact Lie group G on a Hodge manifold (M, ω) with quantizing circle bundle $\pi : X \to M$, one can always define an infinitesimal action of the Lie algebra $\mathfrak{g}$ on X. If it can be integrated to an action of the whole group G, then one has a representation of G on $H(X)$. In [5] it was observed that associated to group actions one can define reductions M_{red}^{Θ} by pulling back a G-invariant and proper sub-manifold Θ of $\mathfrak{g}^*$ via the moment map Φ. When Θ is chosen to be a cone through a co-adjoint orbit $C(\mathcal{O}_\nu)$, one has associated reduction whose Hardy space of its quantization is

$$H(X_{\text{red}}^{C(\mathcal{O}_\nu)}) = \bigoplus_{k \in \mathbb{Z}} H(X_{\text{red}}^{C(\mathcal{O}_\nu)})_k,$$

where k labels an irreducible representation of the residual circle action described above. Now, it is natural to ask if this spaces are related to the decomposition induced by G on

$H(X)$. When $G = \mathbb{T}$ Theorem 1.1 states that they are isomorphic to $H(X)_{k\nu}$; the canonical decomposition of the Hardy space $H(X)$ of the quantitation by the built-in circle action does not play any role. The semi-classical parameter k is the one induced by the ladder $k\,\nu$, $k = 0, 1, 2, \ldots$, labeling unitary irreducible representations.

Further geometrical motivations for this theorem are explained in paper [14], where it is proved that δ_k is an isomorphism for k large enough in the setting when X is the circle bundle of a polarized Hodge manifold whose Grauert tube is D. Thus, Theorem 1.1 generalizes the main theorem in [14] to compact "quantizable" pseudo-Kähler manifolds. We also refer to [12] for examples and the explicit expression of the leading term of the asymptotic expansion of $\dim H(X)_{k\nu}$ as k goes to infinity. Along this line of research in [13] Toeplitz operators were studied for circle action; in [1] the study of asymptotics of compositions of Toeplitz operators with quantomopomorphism is addressed for torus actions.

2 Preliminaries

2.1 Geometric setting

We recall some notations concerning CR and contact geometry. Let $(X, T^{1,0}X)$ be a compact, connected and orientable CR manifold of dimension $2n+1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of X. There is a unique sub-bundle HX of TX such that $HX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$. Let $J : HX \to HX$ be the complex structure map given by $J(u + \overline{u}) = iu - i\overline{u}$, for every $u \in T^{1,0}X$. By complex linear extension of J to $TX \otimes \mathbb{C}$, the i-eigenspace of J is $T^{1,0}X$. We shall also write (X, HX, J) to denote a CR manifold.

Since X is orientable, there always exists a real non-vanishing 1-form $\omega_0 \in \mathcal{C}^\infty(X, T^*X)$ so that $\langle\, \omega_0(x),\ u \,\rangle = 0$, for every $u \in H_xX$, for every $x \in X$; ω_0 is called contact form and it naturally defines a volume form on X. For each $x \in X$, we define a quadratic form on HX by

$$L_x(U, V) = \frac{1}{2}\mathrm{d}\omega_0(JU, V), \qquad \forall\, U, V \in H_xX.$$

Then, we extend L to $HX \otimes \mathbb{C}$ by complex linear extension; for $U, V \in T_x^{1,0}X$,

$$L_x(U, \overline{V}) = \frac{1}{2}\,\mathrm{d}\omega_0(JU, \overline{V}) = -\frac{1}{2i}\,\mathrm{d}\omega_0(U, \overline{V}).$$

The Hermitian quadratic form L_x on $T_x^{1,0}X$ is called Levi form at x. In the case when X is the circle bundle of an Hodge manifold (M, ω), the positivity of ω implies that the number of negative eigenvalues of the Levi form is equal to n. The Reeb vector field $R \in \mathcal{C}^\infty(X, TX)$ is defined to be the non-vanishing vector field determined by

$$\omega_0(R) \equiv 1, \quad \mathrm{d}\omega_0(R, \cdot) \equiv 0 \quad \text{on } TX.$$

Fix a smooth Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TX$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X$, $\langle\, u \mid v \,\rangle$ is real if u, v are real tangent vectors, $\langle\, R \mid R \,\rangle = 1$ and R is orthogonal to $T^{1,0}X \oplus T^{0,1}X$. For $u \in \mathbb{C}TX$, we write $|u|^2 := \langle\, u \mid u \,\rangle$. Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. They can be identified with sub-bundles of the complexified cotangent bundle $\mathbb{C}T^*X$.

Assume that X admits an action of t-dimensional torus $\mathbb{T}$. In this work, we assume that the $\mathbb{T}$-action preserves ω_0 and J; that is, $t^*\omega_0 = \omega_0$ on X and $t_*J = Jt_*$ on HX. Let t denote the Lie algebra of T, we identify $\mathfrak{t}$ with its dual $\mathfrak{t}^*$ by means of the scalar product $\langle\cdot,\cdot\rangle$. For any $\xi \in \mathfrak{t}$, we write ξ_X to denote the vector field on X induced by ξ. The moment map associated to the form ω_0 is the map $\mu : X \to \mathfrak{t}^*$ such that, for all $x \in X$ and $\xi \in \mathfrak{t}$, we have

$$\langle \mu(x), \xi\rangle = \omega_0(\xi_X(x)). \tag{2}$$

Fix a maximal weight $\nu \neq 0$ in the lattice inside $\mathfrak{t}^*$. Suppose that $i\mathbb{R}_+ \cdot \nu$ is transversal to μ, so we have

$$\mathfrak{t}^* = i\mathbb{R}_+ \cdot \nu \oplus \mathrm{d}_p\mu(T_pX) \tag{3}$$

and X_ν is a sub-manifold of X of codimension $2n+2-t$. We claim that the action of $\mathbb{T}_\nu^{t-1}$ on X_ν is locally free. In fact, by the contrary suppose that there exists $\xi \in \ker\nu$ such that $\xi_X(x) = 0$ on T_xX_ν, $x \in X_\nu$. For each $v \in T_xX$, we have

$$(\mathrm{d}_x\mu(v))(\xi) = \mathrm{d}_x\omega_0(\xi_X, v) = 0$$

which contradicts (3).

The action of $\mathbb{T}$ restricts to an action of $\mathbb{T}^{t-1}$ whose moment map is given by

$$\mu_{|\mathbb{T}^{t-1}} = p_\nu \circ \mu, \quad \text{where } p_\nu : \mathfrak{t}^* \to \mathfrak{t}_\nu^*$$

is the canonical projection onto the $t-1$-dimensional subspace $\nu^\perp$ in $\mathfrak{t}^*$. The transversality condition in 1.2 implies that $0 \in \mathfrak{t}_\nu^{t-1}$ is a regular value for the moment $\mu_{|\mathbb{T}^{t-1}}$. Thus, we have

$$X_\nu/\mathbb{T}_\nu^{t-1} = \mu^{-1}_{|\mathbb{T}^{t-1}}(0)/\mathbb{T}_\nu^{t-1}$$

and by assumptions 1.2, 1.3 and [7, Section 2.5] (we shall also refer to [3] for definitions concerning CR structures on orbifolds) we have

Lemma 2.1 *The space of orbits $X_\nu/\mathbb{T}_\nu^{t-1}$ is a CR orbifold. Let us denote by $\pi : X_\nu \to X_\nu^{\mathrm{red}}$ and $\iota : X_\nu \hookrightarrow X$ the natural projection and inclusion, respectively, then there is a unique induced contact form ω_0^{red} on $X_\nu/\mathbb{T}_\nu^{t-1}$ such that*

$$\pi^*\omega_0^{\mathrm{red}} = \iota^*\omega_0.$$

In particular, set $HX_\nu = TX_\nu \cap HX$, we have

$$HX = HX_\nu \oplus J\, i\nu_X^\perp \quad \text{and} \quad HX_\nu = i\nu_X^\perp \oplus \mathrm{d}\pi^* HX_\nu^{\mathrm{red}}.$$

We will also assume that $\overline{\mathbb{T}_\nu^1}$-action is transversal CR, that is, the infinitesimal vector field

$$(\nu_X u)(x) = \frac{\partial}{\partial t}\left(u(\exp(it\,\nu)\circ x)\right)|_{t=0}, \quad \text{for any } u \in C^\infty(X),$$

preserves the CR structure $T^{1,0}X$, so that ν_X and $T^{1,0}X \oplus T^{0,1}X$ generate the complex tangent bundle to X,

$$\mathbb{C}T_xX = \mathbb{C}\nu_X(x) \oplus \mathbb{C}T_x^{1,0}X \oplus \mathbb{C}T_x^{0,1}X \qquad (x \in X).$$

We define local coordinates that will be useful later. Recall that X admits a CR and transversal $\overline{\mathbb{T}_\nu^1}$-action which is locally free on X_ν, $T \in \mathcal{C}^\infty(X, TX)$ denotes the global real vector field given by this infinitesimal circle action. We will take T to be our Reeb vector field R. In a similar way as in Theorem 3.6 in [7], there exist local coordinates $v = (v_1, \ldots, v_{t-1})$ of $\mathbb{T}_\nu^{t-1}$ in a small neighborhood V_0 of the identity e with $v(e) = (0, \ldots, 0)$, local coordinates

$x = (x_1 \dots, x_{2n+1})$ defined in a neighborhood $U_1 \times U_2$ of $p \in X_\nu$, where $U_1 \subseteq \mathbb{R}^{t-1}$ (resp. $U_2 \subseteq \mathbb{R}^{2n+2-t}$) is an open set of $0 \in \mathbb{R}^{t-1}$ (resp. $0 \in \mathbb{R}^{2n+2-t}$) and $p \equiv 0 \in \mathbb{R}^{2n+1}$, and a smooth function $\gamma = (\gamma_1, \dots, \gamma_{t-1}) \in \mathcal{C}^\infty(U_2, U_1)$ with $\gamma(0) = 0$ such that

$$\begin{aligned}&(v_1, \dots, v_{t-1}) \circ (\gamma(x_t, \dots, x_{2n+1}), x_t, \dots, x_{2n+1})\\&= (v_1 + \gamma_1(x_t, \dots, x_{2n+1}), \dots, v_{t-1} + \gamma_d(x_t, \dots, x_{2n+1}), x_t, \dots, x_{2n+1})\end{aligned}$$

for each $(v_1, \dots, v_{t-1}) \in V_0$ and $(x_t, \dots, x_{2n+1}) \in U_2$. Furthermore, we have

$$\mathfrak{t} = \operatorname{span}\left\{\partial_{x_j}\right\}_{j=1,\dots,t-1}, \quad \mu^{-1}(i\,\mathbb{R}_\nu \cdot \nu) \cap U = \{x_{2d-t+1} = \dots = x_{2d} = 0\},$$

on $\mu^{-1}(i\,\mathbb{R}_\nu \cdot \nu) \cap U$ there exist smooth functions a_j's with $a_j(0) = 0$ for every $0 \le j \le t-1$ and independent on $x_1, \dots, x_{2(t-1)}, x_{2n+1}$ such that

$$J\left(\partial_{x_j}\right) = \partial_{x_{t-1+j}} + a_j(x)\partial_{x_{2n+1}} \qquad j = 1, \dots, t-1,$$

the Levi form L_p, the Hermitian metric $\langle \cdot | \cdot \rangle$ and the 1-form ω_0 can be written

$$L_p(Z_j, \overline{Z}_k) = \mu_j\, \delta_{j,k}, \qquad \langle Z_j | \overline{Z}_k \rangle = \delta_{j,k} \qquad (1 \le j, k \le n),$$

and

$$\begin{aligned}\omega_0(x) =&(1 + O(|x|))\mathrm{d}x_{2n+1} + \sum_{j=1}^{t-1} 4\mu_j x_{t-1+j}\mathrm{d}x_j + \sum_{j=t}^{n} 2\mu_j x_{2j}\mathrm{d}x_{2j-1}\\&- \sum_{j=t}^{n} 2\mu_j x_{2j-1}\mathrm{d}x_{2j} + \sum_{j=r}^{2n} b_j x_{2n+1}\mathrm{d}x_j + O(|x|^2)\end{aligned}$$

where $b_r, \dots, b_{2n} \in \mathbb{R}$,

$$T_p^{1,0}X = \operatorname{span}\{Z_1, \dots, Z_n\}$$

and

$$\begin{aligned}Z_j &= \frac{1}{2}(\partial_{x_j} - i\,\partial_{x_{t-1+j}})(p) \qquad (j = 1, \dots, t-1),\\Z_j &= \frac{1}{2}(\partial_{x_{2j-1}} - i\,\partial_{x_{2j}})(p) \qquad (j = t, \dots, n).\end{aligned}$$

We need to define in local coordinates we just introduced the phase function of the $\mathbb{T}_\nu^{t-1}$-invariant Szegő kernel $\Phi_-(x, y) \in \mathcal{C}^\infty(U \times U)$ which is independent of $(x_1, \dots, x_{t-1})$ and $(y_1, \dots, y_{t-1})$. Hence, we write $\Phi_-(x, y) = \Phi_-((0, x''), (0, y'')) := \Phi_-(x'', y'')$ and $\mathring{x}'' := (x_t, \dots, x_{2n})$, $\mathring{y}'' := (y_t, \dots, y_{2n})$. Moreover, there is a constant $c > 0$ such that

$$\operatorname{Im}\Phi_-(x'', y'') \ge c\left(|\mathring{x}''|^2 + |\mathring{y}''|^2 + |\mathring{x}'' - \mathring{y}''|^2\right), \quad \text{for all } ((0, x''), (0, y'')) \in U \times U. \tag{4}$$

Furthermore,

$$\begin{aligned}\Phi_-(x'', y'') = &-x_{2n+1} + y_{2n+1} + 2i\sum_{j=1}^{t-1}|\mu_j|\,y_{t-1+j}^2 + 2i\sum_{j=1}^{t-1}|\mu_j|\,x_{t-1+j}^2 + i\sum_{j=t}^{n}|\mu_j|\,|z_j - w_j|^2\\&+ \sum_{j=t}^{n} i\mu_j(\overline{z}_j w_j - z_j\overline{w}_j) + \sum_{j=1}^{d}(-b_{t-1+j}x_{d+j}x_{2n+1} + b_{t-1+j}y_{t-1+j}y_{2n+1})\end{aligned}$$

$$+\sum_{j=t}^{n}\frac{1}{2}(b_{2j-1}-ib_{2j})(-z_jx_{2n+1}+w_jy_{2n+1})+\sum_{j=t}^{n}\frac{1}{2}(b_{2j-1}+ib_{2j})(-\overline{z}_jx_{2n+1}+\overline{w}_jy_{2n+1})$$
$$+(x_{2n+1}-y_{2n+1})f(x,y)+O(|(x,y)|^3), \tag{5}$$

where $z_j=x_{2j-1}+ix_{2j}$, $w_j=y_{2j-1}+iy_{2j}$, $j=t,\dots,n$, μ_j, $j=1,\dots,n$, f is smooth and satisfies $f(0,0)=0$, $f(x,y)=\overline{f}(y,x)$.

We now consider $\mathbb{T}^1_\nu$ circle action on X. Let $p\in\mu^{-1}(i\mathbb{R}_+\cdot\nu)$, there exist local coordinates $v=(v_1,\dots,v_{t-1})$ of $\mathbb{T}^{t-1}$ in a small neighborhood V_0 of e with $v(e)=(0,\dots,0)$, local coordinates $x=(x_1\dots,x_{2n+1})$ defined in a neighborhood $U_1\times U_2$ of p, where $U_1\subseteq\mathbb{R}^{t-1}$ (resp. $U_{t-1}\subseteq\mathbb{R}^{2n+t}$) is an open set of $0\in\mathbb{R}^{t-1}$ (resp. $0\in\mathbb{R}^{2n+t}$) and $p\equiv 0\in\mathbb{R}^{2n+1}$, and a smooth function $\gamma=(\gamma_1,\dots,\gamma_t)\in\mathcal{C}^\infty(U_2,U_1)$ with $\gamma(0)=0$ such that $T=-\frac{\partial}{\partial x_{2n+1}}$ and all the properties for the local coordinates defined before hold. The phase function Ψ satisfies $\Psi(x,y)=-x_{2n+1}+y_{2n+1}+\hat{\Psi}(\mathring{x}'',\mathring{y}'')$, where $\hat{\Psi}(\mathring{x}'',\mathring{y}'')\in\mathcal{C}^\infty(U\times U)$ and Ψ satisfies (5).

2.2 Hardy spaces

We denote by $L^2_{(0,q)}(X)$, $q=0,1,\dots,n$, the completion of $\Omega^{0,q}(X)$ with respect to $(\cdot\,|\,\cdot)$. We extend $(\cdot\,|\,\cdot)$ to $L^2_{(0,q)}(X)$ in the standard way. We extend $\overline{\partial}_b$ to $L^2_{(0,q)}(X)$ by

$$\overline{\partial}_b:\mathrm{Dom}\,\overline{\partial}_b\subset L^2_{(0,q)}(X)\to L^2_{(0,q+1)}(X),$$

where $\mathrm{Dom}\,\overline{\partial}_b:=\{u\in L^2_{(0,q)}(X);\ \overline{\partial}_bu\in L^2_{(0,q+1)}(X)\}$ and, for any $u\in L^2_{(0,q)}(X)$, $\overline{\partial}_bu$ is defined in the sense of distributions. We also write

$$\overline{\partial}^*_b:\mathrm{Dom}\,\overline{\partial}^*_b\subset L^2_{(0,q+1)}(X)\to L^2_{(0,q)}(X)$$

to denote the Hilbert space adjoint of $\overline{\partial}_b$ in the L^2 space with respect to $(\cdot\,|\,\cdot)$. There is a well-defined orthogonal projection

$$S^{(q)}:L^2_{(0,q)}(X)\to\mathrm{Ker}\,\Box^q_b \tag{6}$$

with respect to the L^2 inner product $(\cdot\,|\,\cdot)$ and let

$$S^{(q)}(x,y)\in\mathcal{D}'(X\times X,T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$$

denote the distribution kernel of $S^{(q)}$. We will write $H^q(X)$ for $\ker\Box^q_b$, and for functions, we simply write $H(X)$ to mean $H^0(X)$. The distributional kernel $S^{(q)}(x,y)$ was studied in [6], before recalling an explicit description describing the oscillatory integral defining the distribution $S^{(q)}(x,y)$ we need to fix some notation. To begin, let us review the concept of the Hörmander symbol space. Let $D\subset X$ be a local coordinate patch with local coordinates $x=(x_1,\dots,x_{2n+1})$.

Definition 2.1 For every $m\in\mathbb{R}$, we denote with

$$S^m_{1,0}(D\times D\times\mathbb{R}_+,T^{*0,q}X\boxtimes(T^{*0,q}X)^*)\subseteq\mathcal{C}^\infty(D\times D\times\mathbb{R}_+,T^{*0,q}X\boxtimes(T^{*0,q}X)^*) \tag{7}$$

the space of all a such that, for all compact $K\Subset D\times D$ and all $\alpha,\beta\in\mathbb{N}_0^{2n+1}$, $\gamma\in\mathbb{N}_0$, there is a constant $C_{\alpha,\beta,\gamma}>0$ such that

$$\left|\partial^\alpha_x\partial^\beta_y\partial^\gamma_t a(x,y,t)\right|\le C_{\alpha,\beta,\gamma}(1+|t|)^{m-\gamma},\quad\text{for every }(x,y,t)\in K\times\mathbb{R}_+,\ t\ge 1.$$

For simplicity we denote with $S^m_{1,0}$ the spaces defined in (7); furthermore, we write

$$S^{-\infty}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*) := \bigcap_{m\in\mathbb{R}} S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*).$$

Let $a_j \in S^{m_j}_{1,0}$, $j = 0, 1, 2, \ldots$ with $m_j \to -\infty$, as $j \to \infty$. Then there exists $a \in S^{m_0}_{1,0}$ unique modulo $S^{-\infty}$, such that

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\big)$$

for $k = 0, 1, 2, \ldots$. If a and a_j have the properties above, we write $a \sim \sum_{j=0}^{\infty} a_j$ in $S^{m_0}_{1,0}$.

It is known that the characteristic set of $\Box^q_b$ is given by

$$\Sigma = \Sigma^- \cup \Sigma^+, \quad \Sigma^- = \left\{(x, \lambda\omega_0(x)) \in T^*X;\ \lambda < 0\right\},$$

and Σ^+ is defined similarly for $\lambda > 0$. We recall the following theorem (see [6, Theorem 1.2]).

Theorem 2.1 *Suppose that the Levi form is non-degenerate and $\Box^q_b$ has L^2 closed range. Then, there exist continuous operators $S_-, S_+ : L^2_{(0,q)}(X) \to \operatorname{Ker}\Box^q_b$ such that*

$$S^{(q)} = S_- + S_+, \quad S_+ \equiv 0 \quad \textit{if}\ q \neq n_+$$

and

$$\mathrm{WF}'(S_-) = \mathrm{diag}\,(\Sigma^- \times \Sigma^-), \quad \mathrm{WF}'(S_+) = \mathrm{diag}\,(\Sigma^+ \times \Sigma^+) \quad \textit{if}\ q = n_- = n_+,$$

where $\mathrm{WF}'(S_-) = \{(x, \xi, y, \eta) \in T^*X \times T^*X;\ (x, \xi, y, -\eta) \in \mathrm{WF}(S_-)\}$, $\mathrm{WF}(S_-)$ is the wave front set of S_- in the sense of Hörmander.

Moreover, consider any small local coordinate patch $D \subset X$ with local coordinates $x = (x_1, \ldots, x_{2n+1})$, then $S_-(x, y)$, $S_+(x, y)$ satisfy

$$S_\mp(x, y) \equiv \int_0^\infty e^{i\varphi_\mp(x,y)t} s_\mp(x, y, t)dt \quad \textit{on}\ D,$$

with

$$s_\mp(x, y, t) \sim \sum_{j=0}^{\infty} s^j_\mp(x, y)t^{n-j} \ \textit{in}\ S^n_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),$$

$s_+(x, y, t) = 0$ if $q \neq n_+$ and $s^0_-(x, x) \neq 0$ for all $x \in D$. The phase functions φ_-, φ_+ satisfy

$$\varphi_+, \varphi_- \in \mathcal{C}^\infty(D \times D), \ \ \mathrm{Im}\,\varphi_\mp(x, y) \geq 0,\ \varphi_-(x, x) = 0,\ \ \varphi_-(x, y) \neq 0 \ \textit{if}\ x \neq y,$$

and

$$d_x\varphi_-(x, y)\big|_{x=y} = -\omega_0(x),\ \ d_y\varphi_-(x, y)\big|_{x=y} = \omega_0(x),\ -\overline{\varphi}_+(x, y) = \varphi_-(x, y).$$

Remark 2.1 Kohn [9] proved that if $q = n_- = n_+$ or $|n_- - n_+| > 1$ then $\Box^q_b$ has L^2 closed range. For a description of the phase function in local coordinates see chapter 8 of part I in [6].

Now we focus on the decomposition induced by the group action on $H^q(X)$. Fix $t \in \mathbb{T}$ and let $t^* : \Lambda^r_x(\mathbb{C}T^*X) \to \Lambda^r_{t^{-1}\circ x}(\mathbb{C}T^*X)$ be the pull-back map. Since $\mathbb{T}$ preserves J, we have $t^* : T^{*0,q}_x X \to T^{*0,q}_{g^{-1}\circ x} X$, for all $x \in X$. Thus, for $u \in \Omega^{0,q}(X)$, we have $t^* u \in \Omega^{0,q}(X)$. Put

$$\Omega^{0,q}(X)_{k\nu} := \left\{ u \in \Omega^{0,q}(X);\ (e^{i\theta})^* u = e^{i\,k\langle \nu,\,\theta\rangle} u,\ \ \forall \theta \in \mathbb{R}^r \right\}.$$

Since the Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ is $\mathbb{T}$-invariant, the L^2 inner product $(\cdot | \cdot)$ on $\Omega^{0,q}(X)$ induced by $\langle \cdot | \cdot \rangle$ is $\mathbb{T}$-invariant. Let $u \in L^2_{(0,q)}(X)$ and $t \in \mathbb{T}$, we can also define $t^* u$ in the standard way. We introduce the following notation

$$L^2_{(0,q)}(X)_{k\nu} := \left\{ u \in L^2_{(0,q)}(X);\ (e^{i\theta})^* u = e^{i\,k\langle \nu,\,\theta\rangle} u,\ \ \forall \theta \in \mathbb{R}^r \right\},$$

and put

$$(\mathrm{Ker}\, \Box^q_b)_{k\nu} := \mathrm{Ker}\, \Box^q_b \cap L^2_{(0,q)}(X)_{k\nu}.$$

The equivariant Szegő projection is the orthogonal projection

$$S^{(q)}_{k\nu} : L^2_{(0,q)}(X) \to (\mathrm{Ker}\, \Box^q_b)_{k\nu}$$

with respect to $(\cdot | \cdot)$. Let $S^{(q)}_{k\nu}(x, y) \in \mathcal{D}'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be the distribution kernel of $S^{(q)}_{k\nu}$. The asymptotic expansion for the distributional kernel of the projector $S^{(0)}_{k\nu}$ was studied in [12] when X is the quantizing circle bundle of a given Hodge manifold in Heisenberg local coordinates. Using similar ideas as in [7] one can generalize the results for $(0, q)$-forms in the following theorem, we give a sketch of the proof, which is similar to the proof of [7, Theorem 1.8], here the group G in [7] is $\mathbb{T}^{t-1}_\nu$, and the circle action in [7] is given by the action of $\overline{\mathbb{T}^1_\nu}$.

Since the action is locally free we need to recall some notations from [2]. Since the action is locally free $\mu^{-1}(0)/\overline{\mathbb{T}^1_\nu}$ is an orbifold, let us denote with $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/\overline{\mathbb{T}^1_\nu}$ the projection. Furthermore the action of $\overline{\mathbb{T}^1_\nu}$ commutes with the one of $\mathbb{T}^{t-1}_\nu$; then, we have a smooth locally free action of $\mathbb{T}^{t-1}_\nu$ on $\mu^{-1}(0)/\overline{\mathbb{T}^1_\nu}$. Given $y \in X$ and g in the stabilizer $(\mathbb{T}^{t-1}_\nu)_{\pi(y)}$, there exist $|(\overline{\mathbb{T}^1_\nu})^1_y|$ elements $e^{i\,\theta_{g,j}} \in \overline{\mathbb{T}^1_\nu}$ $(j = 1, \dots, |(\overline{\mathbb{T}^1_\nu})^1_y|)$ such that

$$g \circ y = e^{-i\,\theta_{g,j}} \circ y.$$

Thus, all the elements $(e^{i\,\theta},\ g) \in \overline{\mathbb{T}^1_\nu} \times \mathbb{T}^{t-1}_\nu = \mathbb{T}$ satisfying

$$e^{i\,\theta} \cdot g \circ y = y.$$

are of the form $(e^{i\theta_{g,j}}, g)$ for each $g \in (\mathbb{T}^{t-1}_\nu)_{\pi(y)}$.

Theorem 2.2 *Suppose that $\Box^q_b$ has L^2 closed range. Then, there exist continuous operators $S^-_{k\nu}, S^+_{k\nu} : L^2_{(0,q)}(X) \to (\mathrm{Ker}\, \Box^q_b)_{k\nu}$ such that*

$$S^{(q)}_{k\nu} = S^-_{k\nu} + S^+_{k\nu}, \quad S_+ \equiv 0 \quad \text{if } q \neq n_+$$

Suppose for simplicity that $q \neq n_+$. If $q \neq n_-$, then $S^{(q)}_{k\nu} \equiv O(k^{-\infty})$ on X.

Suppose $q = n_-$ and let D be an open set in X such that the intersection $\mu^{-1}(i\,\mathbb{R}_+ \cdot \nu) \cap D = \emptyset$. Then $S^{(q)}_{k\nu} \equiv O(k^{-\infty})$ on D.

Let $p \in \mu^{-1}(i\,\mathbb{R}_+ \cdot \nu)$ and let U a local neighborhood of p with local coordinates $(x_1, \dots x_{2n+1})$. Then, if $q = n_-$, for every fix $y \in U$, we consider $S^{(q)}_{k\nu}(x, y)$ as a k-dependent smooth function in x, then

$$S^{(q)}_{k\nu}(x,\ y) = \sum_{h\in G_{\pi(y)}} \sum_{j=1}^{|S^1_x|} e^{ik\,\theta_{h,j}}\, e^{ik\|\nu\|\,\Psi(x,\,y)}\, b(x,\ y,\ k\|\nu\|) + O(k^{-\infty}).$$

for every $x \in U_y$, where U_y is a small open neighborhood of y. The phase function $\Psi(x,\ y)$ is defined in local coordinates in the end of Sect. 2.1; the symbol satisfies

$$b(x,\ y,\ k\|\nu\|) \in S^{n+(1-t)/2}_{\mathrm{loc}}(1,\ U \times U,\ T^{*\,(0,q)}X \boxtimes (T^{*\,(0,q)})^*)$$

and the leading term of $b(x, x, \|\nu\|)$ is nonzero.

Proof Suppose $q = n_-$, on small local neighborhood D of a point $p \in X_\nu$ we have

$$S^{(q)}_{k\nu}(x,\ y) = \frac{1}{(2\,\pi)^t}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} e^{ik\,\langle\nu,\,\theta\rangle}\, S^{(q)}(x,\ e^{i\,\theta}\cdot y)\,\mathrm{d}\theta$$

where $\theta = (\theta_1, \dots, \theta_t)$. It is easy to prove that the oscillatory integral has a rapidly decreasing asymptotic as $k \to +\infty$ far away from a local neighborhood of those elements $(e^{i\,\theta},\ g) \in \overline{\mathbb{T}^1_\nu} \times \mathbb{T}^{t-1}_\nu = \mathbb{T}$ such that

$$e^{i\,\theta}\cdot g \circ y = y.$$

We shall consider the case of a local neighborhood of the identity in $\mathbb{T}$; we set θ_t the variable for circle action of $\overline{\mathbb{T}^1_\nu}$ and $(\theta_1, \dots, \theta_{t-1})$ the variables for $\mathbb{T}^{t-1}_\nu$. We can then use local coordinates defined in 2.1; we have

$$S^{(q)}_{k\nu}(x,\ y) = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} e^{ik\,\|\nu\|\theta_t - i\,kx_{2n+1} + i\,ky_{2n+1}}\, S^{(q)}_{\mathbb{T}^{t-1}}(\mathring{x},\ e^{i\,\theta_t}\cdot\mathring{y})\,\mathrm{d}\theta_t \tag{8}$$

where

$$S^{(q)}_{\mathbb{T}^{t-1}}(x,\ y) = \frac{1}{(2\,\pi)^{t-1}}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} S^{(q)}(x,\ e^{i\,\mathrm{diag}(\theta_1,\dots,\theta_{t-1})}\cdot y)\,\mathrm{d}\theta_1 \dots \mathrm{d}\theta_{t-1}, \tag{9}$$

and $\mathring{x} = (x_1, \dots, x_n, 0)$ and $\mathring{y} = (y_1, \dots, y_n, 0)$. The action of $\mathbb{T}$ restricts to an action of $\mathbb{T}^{t-1}$ whose moment map is given by

$$\mu_{|\mathbb{T}^{t-1}} = p_\nu \circ \mu, \quad \text{where } p_\nu : \mathfrak{t}^* \to \mathfrak{t}^*_\nu$$

is the canonical projection onto the $t-1$-dimensional subspace $\nu^\perp$ in $\mathfrak{t}^*$. Since we are assuming $X_\nu \neq \emptyset$, we have that $0 \in \mathfrak{t}^*_{t-1}$ does lie in the image of the moment map $\mu_{|\mathbb{T}^{t-1}}$. The proof follows in a similar way as in Theorem 1.8 in [7] where here we have the action of $G = \mathbb{T}^{t-1}$ with moment map $\mu_{\mathbb{T}^{t-1}}$ and the $k\nu$ Fourier components are the ones induced by the transversal $\overline{\mathbb{T}^1_\nu}$-action; notice that [7, Assumption 1.7] is satisfied. We shall also refer to [2, Theorem 5.4] for the locally free action case. □

For ease of notation we introduce a definition to say that an operator behaves microlocally as equivariant Szegő projector we just studied. For simplicity we assume $q = n_-$.

Definition 2.2 (Equivariant Szegő-type operator) Suppose that $q = n_-$ and consider $H : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ be a continuous operator with distribution kernel

$$H(x, y) \in \mathcal{D}'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*).$$

We say that H is a complex Fourier integral operator of equivariant Szegő type of order $n+(1-t)/2 \in \mathbb{Z}$ if H is smoothing away $\mu^{-1}(i\,\mathbb{R}_+ \cdot \nu)$ and for given $p \in \mu^{-1}(i\,\mathbb{R}_+ \cdot \nu)$ let D a local neighborhood of p with local coordinates $(x_1, \dots x_{2n+1})$. Then, the distributional kernel of H satisfies

$$H_k(x, y) = e^{ik\,\Psi(x,y)}\,a(x,\ y,\ k) + O(k^{-\infty}) \quad \text{on } D$$

where $a \in S^{n+(1-t)/2}_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, and Ψ is as in the end of Sect. 2.1.

For $k = 0$, $H^{(q)}(X)_0$ is the space of $\mathbb{T}$-fixed vectors, by Theorem 1.5 in [7] we have $\dim H^{(q)}_0(X) < +\infty$ since when $0 \notin \mu(X)$ the projector $S^{(q)}_{\mathbb{T}}$ is smoothing. Now, given $k_1, k_2 \in \mathbb{Z}$ with $k_1, k_2 \neq 0$, consider $f_1 \in H^{(q)}(X)_{k_1\nu}$ and $f_2 \in H^{(q)}(X)_{k_2\nu}$, we have $f_1 \cdot f_2 \in H^{(q)}(X)_{(k_1+k_2)\nu}$; we can study $\dim H(X)_{k\nu}$ for k large; we have

$$\dim H^{(q)}(X)_{k\nu} = \int_X S^{(q)}_{k\nu}(x,\ x)\,\mathrm{dV}_X(x).$$

These dimensions can be studied as $k \to +\infty$ by using the microlocal properties of the Szegő kernel and Stationary Phase Lemma in a similar way as in Corollary 1.3 in [12] we have $\dim H^{(q)}(X)_{k\nu} = O(k^{d+1-t})$. So, we have the following generalization of Theorem A.3 in [4], where it is proved for spaces of functions $H(X)_{k\nu}$.

Lemma 2.2 *If $0 \notin \mu(X)$, then $H^q(X)_{k\nu}$ are finite dimensional.*

3 Proof of Theorem 1.1

In this section we shall explain how to prove asymptotic commutativity for quantization and reduction for spaces of $(0, q)$-forms. We recall that X^{red}_ν is a CR orbifold whose Levi form is non-degenerate, with $n_- - r$ negative eigenvalues; let us denote with $S^{(q)}_{\mathrm{red}}$ the corresponding Szegő kernel for $(0, q)$-forms whose k-th Fourier components are the one induced by the $\overline{\mathbb{T}^1_\nu}$-action on X^{red}_ν. We shall recall briefly its microlocal expression by [3, Theorem 1.2]. Now, let us denote by $e^{i\theta}\cdot$ the transversal and CR locally free $\overline{\mathbb{T}^1_\nu}$-action and we take the Reeb vector field R to be the vector field on X induced by it.

Let $q = n_- - r$, and consider an open set $U \subset X$, $p \in U$, and an orbifold chart $(\widetilde{U}, G_U) \to U$; we denote by $\widetilde{x}$ the coordinates on $\widetilde{U}$. For every $\ell \in \mathbb{N}$, put

$$X_\ell := \{x \in X;\ e^{i\theta}x \neq x, \theta \in [0, 2\pi/\ell[, e^{i\,2\pi/\ell}x = x\}.$$

With the assumptions and notations used above, assume that $p \in X_\ell$, for some $\ell \in \mathbb{N}$. We have as $k \to +\infty$,

$$S^{(q)}_{\mathrm{red},k}(x, y) = \sum_{j=0}^{\ell-1} \sum_{g \in G_U} e^{\frac{2\pi kj}{\ell}}\, e^{ik\,\Psi(\widetilde{x}, e^{i\frac{2\pi j}{\ell}}\cdot g\cdot\widetilde{y})} b(\widetilde{x},\ e^{i\frac{2\pi j}{\ell}}\cdot g\cdot\widetilde{y}, k) + O(k^{-\infty}) \qquad (10)$$

where the phase function

$$\Psi \in C^\infty(\widetilde{U} \times \widetilde{U}), \qquad \Psi(\widetilde{x}, \widetilde{x}) = 0, \quad \text{for all } \widetilde{x} \in \widetilde{U},$$
$$\inf_{e^{i\theta} \in S^1} \left\{ \mathrm{dist}^2(\widetilde{x}, e^{i\theta}\, \widetilde{y}) \right\} / C \le \mathrm{Im}\, \Psi(\widetilde{x}, \widetilde{y}) \le C \inf_{e^{i\theta} \in S^1} \left\{ \mathrm{dist}^2(\widetilde{x}, e^{i\theta}\, \widetilde{y}) \right\}$$

for each $(\widetilde{x}, \widetilde{y}) \in \widetilde{U} \times \widetilde{U}$, $C > 1$ is a constant, and the symbol satisfies

$$b(\widetilde{x}, \widetilde{y}, k) \sim \sum_{j=0}^{+\infty} b_j(\widetilde{x}, \widetilde{y})\, k^{n-j} \quad \text{in} \quad S^{n-(t-1)}(1; \widetilde{U} \times \widetilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

and $b_0(\widetilde{x}, \widetilde{x})$ is nonzero.

Let us recall briefly why we have a local chart $(\widetilde{U},\ G_U) \to U$, for ease of notation let us put $G = \mathbb{T}_\nu^{t-1}$. We recall that, for every $x \in X_\nu$, by the slice theorem a neighborhood of any orbit $\mathbb{T}_\nu^{t-1} \cdot x = x_0$ is equivariantly diffeomorphic to a neighborhood of the zero section of the associated principal bundle

$$G \times_{G_x} N_x,$$

where N_x is the normal space to $G \cdot x$ in X_ν and G_x is the stabilizer of x for the action of G, which is finite. Therefore, for some $\epsilon > 0$ and for an open ball $B_{2e}(\epsilon) \subseteq N_x$, one has a homeomorphism $B_{2e}(\epsilon)/G_x \cong \overline{U}$ onto some neighborhood of x_0 in X_ν^{red}.

Since $\nu_X^\perp := \mathrm{val}(\nu^\perp)$ is orthogonal to

$$H_x X_\nu \cap J_x H_x X_\nu \quad \text{and} \quad H_x X_\nu \cap J_x H_x X_\nu \subset (\nu_X^\perp)^{\perp_b}$$

for every $x \in X_\nu$, we can find a $\mathbb{T}$-invariant orthonormal basis $\{Z_1,\ \ldots,\ Z_n\}$ of $T^{*\,0,1}X$ on X_ν such that for each $j,\ k = 1, \ldots n$,

$$L_x(Z_j(x),\ \overline{Z}_j(x)) = \delta_{jk}\, \lambda_j(x),$$

where

$$Z_j(x) \in (\nu_{X,x}^\perp + i\, J\, \nu_{X,x}^\perp) \text{ for each } j = 1,\ \ldots,\ t-1,$$
$$Z_j(x) \in H_x X_\nu \cap J_x(H_x X_\nu) \text{ for each } j = r,\ \ldots,\ n.$$

Let $\{Z_1^*,\ \ldots,\ Z_n^*\}$ denote the orthonormal basis of $T^{*\,0,1}X$ on X_ν, dual to $Z_1,\ \ldots,\ Z_n$. Fix $s = 0,\ 1,\ 2, \ldots,\ n - r + 1$. For $x \in X_\nu$, put

$$B_x^{*\,0,s}X = \left\{ \sum_{r \le j_1 < \cdots < j_s \le n} a_{j_1,\ldots,j_s} Z_{j_1}^*,\ \ldots,\ Z_{j_s}^*;\ a_{j_1,\ldots,j_s} \in \mathbb{C}, \right\}$$

and let $B^{*\,0,s}X$ be the vector bundle of X_ν with fiber $B_x^{*\,0,s}$, $x \in X_\nu$. Let $C^\infty(X_\nu, B^{*\,0,s}X)^{\mathbb{T}}$ denote the set of all $\mathbb{T}$-invariant sections of X_ν with values in $B_x^{*\,0,s}X$; let

$$\iota_{\mathrm{red}} : C^\infty(X_\nu, B^{*\,0,s}X)^{\mathbb{T}} \to \Omega^{0,s}(X_\nu^{\mathrm{red}})$$

be the natural identification. Before defining the map between $H^q(X)_{k\nu}$ and $H^{q-r}(X_\nu^{\mathrm{red}})_k$ we need one more piece of notation. We can assume that $\lambda_1 < 0, \cdots, \lambda_r < 0$ and also $\lambda_t < 0, \ldots,\ \lambda_{n_- - r + t - 1} < 0$. For $x \in X_\nu$, set

$$\hat{\mathcal{N}}(x, n_-) = \{c\, Z_t^* \wedge \cdots \wedge Z_{n_- - r + t - 1}^*, c \in \mathbb{C}\}$$

and let $\hat{p} : \mathcal{N}(x, n_-) \to \hat{\mathcal{N}}(x, n_-)$ to be

$$\hat{p}(c\, Z_1^* \wedge \cdots \wedge Z_r^* \wedge Z_t^* \wedge \cdots \wedge Z_{n_- - r+t-1}^*) := c\, Z_t^* \wedge \cdots \wedge Z_{n_- - r+t-1}^*$$

where c is a complex number. Put $\iota_\nu : X_\nu \to X$ be the natural inclusion and let $\iota_\nu^* : \Omega^{0,q}(X) \to \Omega^{0,q}(X_\nu)$ be the pull-back of ι_ν.

Let $q = n_-$; now, inspired by [7], we define the map

$$\sigma_{k\nu} : H^q(X)_{k\nu} \to H^{q-r}(X_\nu^{\mathrm{red}})_k$$

given by

$$\sigma_{k\nu}(u) := (k\nu)^{(t-1)/4}\, S_{\mathrm{red},k}^{(q-r)} \circ \iota_{\mathrm{red}} \circ \hat{p} \circ \tau_{x,n_-} \circ e \circ \iota_\nu^* \circ S_{k\nu}^{(q)}(u), \tag{11}$$

here, $e(x)$ is a $\mathbb{T}$-invariant smooth function on X_ν that can be found explicitly. Notice that operators $S_{\mathrm{red},k}^{(q-r)}$ and $S_{k\nu}^{(q)}$ appearing in (11) are known explicitly. $S_{\mathrm{red},k}^{(q-r)}$ is the k-th Fourier component of the standard Szegő kernel for the CR manifold X_ν^{red}; on the other hand, $S_{k\nu}^{(q)}$ is described in Sect. 2.2.

Before stating the next theorem we shall specialize the local coordinates defined in Sect. 2.1. Let us consider $p \in X_\nu$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates in an open neighborhood U of p defined in Sect. 2.1. We may assume that $U = U_1 \times U_2 \times U_3 \times U_4$, where $U_1 \subset \mathbb{R}^{t-1}$, $U_2 \subset \mathbb{R}^{t-1}$ are open sets of $0 \in \mathbb{R}^{t-1}$, $U_3 \subset \mathbb{R}^{2n-2(t-1)}$ is an open set of $0 \in \mathbb{R}^{2n-2(t-1)}$ and U_4 is an open set of $0 \in \mathbb{R}$. From now on, we can identify U_2 with

$$\{(0, \ldots, 0, x_t, \ldots, x_{2(t-1)}, 0, \ldots, 0) \in U : (x_t, \ldots, x_{2(t-1)}) \in U_2\}$$

and U_3 with

$$\{(0, \ldots, 0, x_{2t-1}, \ldots, x_{2n}, 0) \in U : (x_t, \ldots, x_{2n}) \in U_3\}.$$

For a given orbifold chart $(\widetilde{U}, G_U) \to U$ we write $\tilde{U}_i$ for the corresponding open sets in $\tilde{U}$. Eventually we recall that $\widetilde{x}'' := (x_{2t-1}, \ldots, x_{2n+1})$.

Theorem 3.1 *Under the assumptions above, if $x \notin X_\nu$, then for every sufficiently small open set D of x with $\overline{D} \cap X_\nu = \emptyset$, we have $\sigma_{k\nu} = O(k^{-\infty})$ on $X_\nu^{\mathrm{red}} \times D$.*

Let $\pi : X_\nu \to X_\nu^{\mathrm{red}}$ the projection. If $x, y \in X_\nu$ and $\pi(x) \neq \pi(e^{i\theta} \cdot y)$ for every $e^{i\theta} \in \overline{\mathbb{T}_\nu^1}$, then there exist open set U of $\pi(x)$ in X_ν^{red} and V of y in X such that $\sigma_{k\nu} = O(k^{-\infty})$ on $U \times V$.

Eventually, let $p \in X_\nu \cap X_\ell$, using the local coordinates defined above, we have

$$\sigma_{k\nu}(\tilde{x}'', y'') = \sum_{j=0}^{\ell-1} \sum_{g \in G_U} e^{\frac{2\pi kj}{\ell}} e^{ik\,\Psi(\widetilde{x},\, e^{\frac{2\pi j}{\ell}} \cdot g \cdot \widetilde{y})}\, \alpha(\widetilde{x},\, e^{\frac{2\pi j}{\ell}} \cdot g \cdot \widetilde{y}, k) + O(k^{-\infty}) \quad \text{on } (\tilde{U}_3 \times \tilde{U}_4) \times \tilde{U}$$

where

$$\alpha(\tilde{x}'', y'', k) \in S_{\mathrm{loc}}^{n-\frac{3}{4}(t-1)}(1;\ (\tilde{U}_3 \times \tilde{U}_4) \times \tilde{U},\ T^{*,(0,q-r)} X_\nu^{\mathrm{red}} \boxtimes (T^{*,(0,q)} X)^*)$$

and the leading term α_0 in the expansion of α can be computed explicitly along the diagonal.

Proof Since, by Theorem 2.2, $S_{k\nu}^{(q)}$ has rapidly decreasing asymptotics as k goes to infinity away X_ν, we obtain that $\sigma_k = O(k^{-\infty})$ on $X_\nu^{\mathrm{red}} \times D$.

Now, let us prove the second statement. If $x, y \in X_\nu$ and $\pi(x) \neq \pi(e^{i\theta} \cdot y)$ for every $e^{i\theta} \in \overline{\mathbb{T}_\nu^1}$, since $S_{\mathrm{red},k}^{(q-r)}$ is smoothing away from diagonal, then there exist open set U of $\pi(x)$ in X_ν^{red} and V of y in X such that $\chi\, S_{\mathrm{red},k}^{(q-r)} \eta = O(k^{-\infty})$ on X_ν^{red}, where $\chi \in C_0^\infty(U)$ and

$\eta \in C_0^\infty(V)$. Furthermore by (8), we see that, for sufficiently small neighborhoods of x and y, we can integrate by parts in $d\theta_t$ and see that $S_{k\nu}^{(q)}$ has rapidly decreasing asymptotic. The second statements follow.

Eventually, let $p \in X_\nu$ and consider a small open neighborhood U with coordinates defined as above. Let $\chi \in C_0^\infty(\tilde{U}_3)$ be a G_U invariant bump function and assume $\chi = 1$ on some neighborhood of p, it extends naturally to a function on $\mathbb{T} \cdot \tilde{U}_3$. Let us put $U^\sharp = \{\pi(x) : x \in U\}$. Let $\eta \in C_0^\infty(X_\nu^{\mathrm{red}})$ such that $\eta = 1$ on some neighborhood of $U^\sharp$ and

$$\mathrm{supp}(\eta) \subset \{\pi(x) \in X_\nu^{\mathrm{red}} : x \in X_\nu,\ \chi(x) = 1\}.$$

Thus, we get

$$\eta\, \sigma_{k\nu} \sim (k\nu)^{(t-1)/4}\, \eta\, S_{\mathrm{red},k}^{(q-r)} \circ \iota_{\mathrm{red}} \circ \hat{p} \circ \tau_{x,n_-} \circ e \circ \iota_\nu^* \circ \chi\, S_{k\nu}^{(q)}.$$

Now, we can compose the operators and we can use the complex stationary phase formula of [11]; we get the theorem. □

In fact, $\sigma_{k\nu}$ is an isomorphism for k large if we can prove that

$$\sigma_{k\nu} : H^q(X)_{k\nu} \to H^{q-r}(X_\nu^{\mathrm{red}})_k \quad \text{and} \quad \sigma_{k\nu}^* : H^{q-r}(X_\nu^{\mathrm{red}})_k \to H^q(X)_{k\nu}$$

are injective for k large. Notice that, by Theorem 3.1, we have

$$\sigma_{k\nu}^*(x'', \tilde{y}'') = \sum_{j=0}^{\ell-1} \sum_{g \in G_U} e^{\frac{2\pi k j}{\ell}} e^{-ik\, \overline{\Psi(x'', e^{\frac{2\pi j}{\ell}} \cdot g \cdot \tilde{y}'')}}\, \beta(x'',\, e^{\frac{2\pi j}{\ell}} \cdot g \cdot \tilde{y}'', k) + O(k^{-\infty}) \quad (12)$$

on $\tilde{U} \times (\tilde{U}_3 \times \tilde{U}_4)$ where we can check $\beta_0(\tilde{x}'', \tilde{x}'') = \alpha_0(\tilde{x}'', \tilde{x}'')$.

The injectivity of the map $\sigma_{k\nu}$ is a consequence of the following theorem which is an adaptation of proof of the main theorem in [7].

Theorem 3.2 *There exists a Fourier integral operator R_k of equivariant Szegő type of degree $n - (t-1)/2$ such that*

$$\sigma_{k\nu}^* \sigma_{k\nu} \equiv c_0\, (1 + R_k)\, S_{k\nu}^{(q)} \quad (13)$$

where c_0 is a positive constant and $1 + R_k : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ is an injective Fourier integral operator of equivariant Szegő type.

Proof By Theorem 3.1 and equation (12) we can compose $\sigma_{k\nu}^*$ and $\sigma_{k\nu}$ using the complex stationary phase formula of [11]. We can pick e in the definition of $\sigma_{k\nu}$ so that the leading term of the symbol of $\sigma_{k\nu}^* \sigma_{k\nu}$ agrees with the one of $S_{k\nu}^{(q)}$ along the diagonal. Thus we can find an operator

$$R_k(x,\ y) = \sum_{j=0}^{\ell-1} \sum_{g \in G_U} e^{\frac{2\pi k j}{\ell}} e^{ik\, \Psi(x'', e^{\frac{2\pi j}{\ell}} \cdot g \cdot y'')}\, r(x'',\, e^{\frac{2\pi j}{\ell}} \cdot g \cdot y'', k) + O(k^{-\infty}) \quad \text{on } \tilde{U} \times \tilde{U}$$

such that

$$r \in S^{n-(t-1)/2}(1;\ \tilde{U} \times \tilde{U},\ T^{*(0,q)}X \boxtimes (T^{*(0,q)}X)^*)$$

and

$$|r_0(x,\ y)| \le C|(x, y) - (x_0, x_0)|$$

for all $x_0 \in X_\nu \cap \tilde{U}$, where $C > 0$ is a constant.

The injectivity of the operator $1 + R_k$ follows from direct computations of the leading symbol of R_k which vanishes at $\text{diag}(X_\nu \times X_\nu)$. In fact, as a consequence of this, by Lemma 6.7 and Lemma 6.8 in [7] we have that $\|R_k u\| \leq \epsilon_k \|u\|$ for all $u \in \Omega^{0,q}(X)$, for all $k \in \mathbb{N}$ where ϵ_k is a sequence with $\lim_{k\to+\infty} \epsilon_k = 0$; this in turn implies, if k is large enough, that the map $1 + R_k$ is injective. □

Analogously, the injectivity of the map $\sigma^*_{k\nu}$ follows by studying $\sigma_{k\nu}\sigma^*_{k\nu}$. We do not repeat the proof here since it is an application of the stationary phase formula for Fourier integral operators of complex type, see [11], and it follows in a similar way as above.

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