Rational expectations (may) lead to complex dynamics in a Muthian cobweb model with heterogeneous agents

Ahmad Naimzada [a*], Marina Pireddu [b†]

[a] Dept. of Economics, Management and Statistics, University of Milano - Bicocca,
U6 Building, Piazza dell'Ateneo Nuovo 1, 20126 Milano, Italy.

[b] Dept. of Mathematics and its Applications, University of Milano - Bicocca,
U5 Building, Via R. Cozzi 55, 20125 Milano, Italy.

Abstract

Starting from a Muthian cobweb model, we extend the profit-based evolutionary setting in Hommes and Wagener (2010) by assuming that, in addition to pessimistic, optimistic and unbiased fundamentalists, the market is populated by rational producers, which correctly anticipate the next period price. Thanks to their introduction, we find that, differently from the framework in Hommes and Wagener (2010), the map governing the dynamics is no more monotonically decreasing. Hence, if on the one hand adding rational agents enlarges the local stability region of the steady state, on the other hand their consideration opens the door to complex dynamic outcomes, characterized by chaotic attractors and rich multistability phenomena, that we investigate along the paper.

Keywords: Muthian cobweb model; evolutionary learning; heterogeneous agents; rational expectations; complex dynamics.

JEL classification: B52, C62, D83, D84, D91

*Tel.: +39 0264485813. E-mail address: ahmad.naimzada@unimib.it

†Corresponding author. Tel.: +39 0264485767. E-mail address: marina.pireddu@unimib.it

1 Introduction

In Brock and Hommes (1997) a Muthian cobweb type demand-supply model was presented, where producers can choose between rational and naive expectations about prices, selecting the strategy on the basis of the recent profits that the two forecasting rules allowed to realize[1]. In particular, an information cost is associated with the use of the more sophisticated forecasting rule. Dealing with the same share updating mechanism adopted in Brock and Hommes (1997) for the case without memory, Hommes and Wagener (2010) consider a Muthian cobweb model framework in which producers can choose among three different fundamentalistic forecasting rules: unbiased fundamentalists predict that prices will always be at their fundamental value, optimists predict that the price of the good will always be above the fundamental price, whereas pessimists always predict prices below the fundamental price. In Hommes and Wagener (2010) all agents face a common zero information cost and the authors focus on the case in which the Muthian model is globally eductively stable in the sense of Guesnerie (2002), that is, on the case in which the model is stable under naive expectations, as the slopes of demand and supply satisfy the familiar "cobweb theorem" by Ezekiel (1938). They show that the unique steady state, which coincides with the fundamental, is always stable and may coexist with a locally stable two-cycle, where the prices fluctuate around the rational expectations price, and where most agents switch between optimistic and pessimistic strategies. The final sentence in Hommes and Wagener (2010) reads as follows: "The

[1]Namely, Brock and Hommes (1997) started a new phase in the study of cobweb models, in which agents heterogeneous in the decisional mechanism and in regard to the linear forecasting rules, such as fundamentalists, contrarians and Sample AutoCorrelation (SAC) learning users, are considered (see e.g. Goeree and Hommes 2000, where the authors deal with nonlinear, but monotonic, demand and supply curves in a heterogeneous expectations cobweb model with rational versus naive expectations, Branch and Evans 2006 for an evolutionary cobweb model with two types of agents, both adopting ordinary least squares learning of a misspecified model, and Branch and McGough 2008, where the setting with rational and naive expectations introduced in Brock and Hommes 1997 is generalized w.r.t. the replicator mechanism). Before the turning point by Brock and Hommes (1997), in the works by Artstein (1983), Day and Hanson (1991), Jensen and Urban (1984), Lichtenberg and Ujihara (1989) non-monotonic supply functions were introduced, while Chiarella (1988) and Hommes (1991, 1994) analyzed the case of monotone nonlinear demand or supply functions assuming adaptive expectations for agents, which are homogeneous in relation to the learning process and the decisional mechanism. See Hommes (2018) for a recent survey on cobweb dynamics.

study of the stability of evolutionary systems with many trader types in various market settings and with more complicated strategies remains an important topic for future work." In this perspective, the present contribution further develops the approach in Hommes and Wagener (2010) by considering a richer set of forecasting rules, and aims at characterizing the resulting dynamic outcomes. In particular, we extend the model in Hommes and Wagener (2010) by introducing rational producers with perfect foresight expectations about prices, which face an information cost like in Brock and Hommes (1997).[2] In Hommes (2013), Paragraph 5.2, a Muthian cobweb model with fundamentalists and naive expectations is considered, but, to the best of our knowledge, in the literature the setting encompassing fundamentalists and rational agents has not been analyzed yet. And, as we shall see below, such match produces unexpected outcomes. We stress that another motivation for our work is given by the ongoing debate about the stabilizing role of rational agents and non-survival of irrational agents originated by Friedman (1953).

In order to better compare our findings with those obtained in Hommes and Wagener (2010), at first we need to complete the analysis performed therein, by taking into account also the case in which the model is not globally eductively stable in the sense of Guesnerie (2002). Indeed, we have to investigate what happens when the eductive stability assumption for the Muthian model is not fulfilled, because the model outcomes do not coincide when the model is globally eductively stable and when it is not, both when rational agents populate the economy and when they are not present.

The results that we obtain may be summarized as follows[3].

[2] We recall that the framework in Hommes and Wagener (2010) has been extended to encompass heterogeneous information costs also for fundamentalistic agents in Naimzada and Pireddu (2020a), where just one couple of groups of symmetrically biased agents was considered, and in Naimzada and Pireddu (2020b), where several coupled groups of symmetrically biased fundamentalists, differing in the strength of their bias, were taken into account. On the other hand, the present contribution aims at showing the dynamic phenomena arising when introducing rational agents in the original setting in Hommes and Wagener (2010). For such reason, we will confine ourselves to the simplest case, in which only rational agents face an information cost and in which the economy is populated by one couple of groups of symmetrically biased agents.

[3] We stress that, since we are mainly interested in the agents' heterogeneity, rather than describing our findings in terms of the effect produced by increasing the intensity of choice parameter of the evolutive mechanism, like it was done in Hommes and Wagener (2010), we will present our results in terms of the effect produced by increasing the bias.

In the setting analyzed in Hommes and Wagener (2010), without rational agents, when the eductive stability assumption for the Muthian model is not fulfilled, the steady state may first lose and then recover stability through (supercritical or subcritical) period-doubling bifurcations of the map governing the dynamics, being stable just for suitably small and for suitably large values of the bias. This counterintuitive finding can be easily explained in terms of profits of the various kinds of agents and by looking at the intensity of the market mechanism, described by the ratio between the demand and supply curve reactivities. We stress however that the economic interpretation of the dynamic scenarios that we shall detect in this work is based on a combination of several factors, having each time a different weight. Namely, in addition to the intensity of the market mechanism, also the reactivity of the share updating rule, the strength of the bias and, in the case of multistability phenomena, the initial condition for prices play a crucial role.
In particular, when the model is unstable under naive expectations, and both the bias and the ratio between the slopes of demand and supply curves are large enough, any initial condition for prices - lying close to or far from the steady state - produces large price variations, which alternately favor optimists or pessimists, leading to the emergence of the globally stable period-two cycle. However, when the bias is excessively large, for prices close to the steady state it becomes again more profitable being fundamentalists, because the forecast error made by biased agents is too big. The consequent increase in the share of unbiased fundamentalists makes prices converge towards the steady state, that recovers its local stability. On the other hand, for high values of the bias and price initial conditions that are distant from the fundamental, having biased expectations about prices allows to make large profits far from the steady state and this accounts for the coexistence between the steady state and the period-two cycle for high values of the bias. Since the map governing the dynamics is monotonically decreasing, no higher-order cycles or complex attractors may emerge.
On the contrary, with the introduction of rational agents the map governing the dynamics admits critical points and a horizontal asymptote, and thus we can observe the emergence of chaotic attractors and of rich multistability phenomena, that we illustrate in the paper. Such surprising result, which

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As we shall prove in Sections 2 and 3, the bias and the intensity of choice parameters have the same effect on the system stability, and thus our choice does not affect the possible dynamic outcomes of the system.

can be rephrased by saying that rational agents may lead to complex dynamics, is partially mitigated by another finding, i.e., that the local stability region of the steady state increases when the economy is also populated by agents with perfect foresight expectations. The latter fact implies that, like in the setting analyzed in Hommes and Wagener (2010), eductive stability implies evolutionary stability. On the other hand, when the eductive stability assumption for the Muthian model is not fulfilled, even in the presence of rational agents we obtain that the unique steady state, which coincides with the fundamental, is stable for suitably small and for suitably large values of the bias. Like done for the setting without rational agents, the just described findings may be interpreted from an economic viewpoint comparing the profits of the various kinds of agents and by looking at the intensity of the market mechanism.

In particular, the loss of stability of the steady state in favor of the period-two cycle occurs when the bias, starting from lower values and increasing, becomes compatible with the strength of the "separating" effect produced by the price adjustment mechanism, which for a sufficiently high ratio between the slopes of the demand and supply curves reacts quite violently to a production variation, determining prices that are distant from the steady state. We stress that in this framework rational agents do not perform well in terms of profits and shares because of the information cost they face, while the profits of unbiased fundamentalists are not high because their price forecasts are not precise.

When the bias increases further, its value becomes excessive with respect to the intensity of the just described "separating" effect produced by the price adjustment mechanism. Since the reaction of the latter to production variations is no more strong enough, the determined prices are not sufficiently distant from the fundamental to be compatible with the bias and the period-two cycle is not stable anymore. Namely, rational agents become favored thanks to their perfect foresight, despite the information cost they face. Then, a chaotic attractor may emerge, which can coexist with the fundamental steady state, whose basin of attraction is in this case unconnected due to the presence of the horizontal asymptote, with its non-immediate components lying outside the basin of attraction of the chaotic attractor. Indeed, when the value of the initial condition is not distant from the bias, orbits visit the chaotic attractor, while orbits converge toward the steady state when the value of the initial condition is too close to the fundamental or when it is excessively large, and optimists' and pessimists' profits are too

low.
Finally, when the bias still increases, the chaotic attractor disappears since the prices determined through the adjustment mechanism are much smaller than the bias, and optimists and pessimists realize very low profits. In fact, the fundamental steady state recovers its global stability due to the even better than before performance of rational agents with respect to biased fundamentalists. Nonetheless, we prove that chaotic invariant sets still exist when the bias is large, but those sets are no more attractive and thus the iterates of almost all the points lying in a neighborhood of them limit towards the fundamental steady state.
We stress that the scenarios characterized by the presence of the chaotic attractor and by the return to the global stability of the fundamental steady state can not arise when rational agents are disregarded. Namely in both those frameworks, without rational agents, we would observe a regular pattern, characterized by the cyclical alternation between a prevailing optimism or pessimism in the market, according to which group of biased fundamentalists performed better.
The remainder of the paper is organized as follows. In Section 2 we recall the setting and the results in Hommes and Wagener (2010), and we investigate which dynamic phenomena arise when the model is unstable under naive expectations. In Section 3 we present and analyze the model enriched by the presence of rational agents, both when the Muthian model is globally eductively stable and when it is not, comparing the findings with those in Section 2. In Section 4 we briefly discuss our results and describe some possible extensions of the model.

2 The setting without rational agents

At first we recall the discrete-time evolutionary cobweb setting in Hommes and Wagener (2010), in which the economy is populated by unbiased fundamentalists, named just fundamentalists, and by two types of biased fundamentalists, i.e., optimists and pessimists. In particular, biased fundamentalists are gathered in coupled groups of optimists and pessimists, that share the same bias, but that respectively overestimate and underestimate the price of the good they produce. For the clarity's sake, we focus on the simplest case

with two coupled groups[4].
In the Muthian farmer model, agents have to choose the quantity q of a certain good to produce in the next period and are expected profit maximizers. Assuming a quadratic cost function

$$\gamma(q) = \frac{q^2}{2s}, \tag{2.1}$$

with $s > 0$, the supply curve is given by

$$S(p^e) = sp^e, \tag{2.2}$$

where p^e is the expected price and s describes its slope. The demand function is supposed to be linearly decreasing in the market price, i.e.,

$$D(p) = A - dp, \tag{2.3}$$

with A and d positive parameters, representing respectively the market size and the slope of the demand function. We stress that the demand is positive for sufficiently large values of A.
In the case of rational expectations, the price at which demand equals supply is the so-called fundamental price p^*, i.e.,

$$p^* = \frac{A}{d+s}\,. \tag{2.4}$$

This is also the expression of the unique model steady state in Hommes and Wagener (2010).
Agents have heterogeneous expectations about the price of the good they have to produce. In particular, fundamentalists predict that prices will always be at their fundamental value, while optimists (pessimists) predict that the price of the good will always be above (below) the fundamental price.

[4]On the basis of a preliminary study and of the analysis performed in Naimzada and Pireddu (2020b), where we dealt with several types of biased fundamentalists in the presence of information costs, without encompassing rational agents, we expect that adding further couples of groups of biased agents would reproduce the results in Section 4 in Hommes and Wagener (2010) about the coexistence of attractors. Probably, in the presence of rational agents, rather than a multiplicity of period-two cycles, we would witness multistability phenomena involving more complex attractors. Nonetheless, the role of the various parameters should not be affected by the number of considered coupled groups of agents. We will deepen such investigation in a future work.

Hence, assuming a symmetric disposition of the beliefs and characterizing the fundamentalists, pessimists and optimists by subscripts 0, 1, 2, respectively, in symbols we have that their expectations at time t are given by

$$p^e_{i,t} = p^* + b_i, \ i \in \{0,1,2\}, \text{ with } b_0 = 0, \ b_1 = -b, \ b_2 = b, \tag{2.5}$$

where $b > 0$ describes the bias degree of pessimists and optimists. In order to avoid a negative expectation for pessimists, we will restrict our attention to the bias values $b \in (0, p^*)$, with p^* as in (2.4).

Denoting by $\omega_{i,t}$ the share of agents choosing the forecasting rule $i \in \{0,1,2\}$ at time t, the total supply is given by $\sum_{i=0}^{2} \omega_{i,t} S(p^e_{i,t})$ and thus the market equilibrium condition at time t reads as

$$A - dp_t = \sum_{i=0}^{2} \omega_{i,t} S(p^e_{i,t}). \tag{2.6}$$

The price which solves the equation obtained when specifying in (2.6) the expectation formation rules for the various kinds of agents is called market equilibrium price.

As concerns the share updating mechanism, Hommes and Wagener (2010) deal with the discrete choice model in Brock and Hommes (1997) for the case without memory, in which only the most recently realized net profits $\pi_{j,t-1}$, $j \in \{0,1,2\}$, are taken into account. In symbols

$$\omega_{i,t} = \frac{\exp(\beta \pi_{i,t-1})}{\sum_{j=0}^{2} \exp(\beta \pi_{j,t-1})}, \ i \in \{0,1,2\}, \tag{2.7}$$

where $\beta > 0$ is the intensity of choice parameter.

In particular, net profits $\pi_{j,t}$, $j \in \{0,1,2\}$, are defined as

$$\pi_{j,t} = p_t S(p^e_{j,t}) - \gamma(S(p^e_{j,t})), \tag{2.8}$$

with γ and S as in (2.1) and (2.2), respectively.

Introducing the variable $x_t = p_t - p^*$, Hommes and Wagener (2010) write their model dynamic equation in deviation from the fundamental as

$$x_t = -\frac{s}{d} \sum_{i=0}^{2} \omega_{i,t} \, b_i$$

with

$$\omega_{i,t} = \frac{\exp\left(-\frac{\beta s}{2}(x_{t-1} - b_i)^2\right)}{\sum_{j=0}^{2} \exp\left(-\frac{\beta s}{2}(x_{t-1} - b_j)^2\right)},$$

or, more explicitly, recalling (2.5), as

$$\begin{aligned} x_t &= \tfrac{sb}{d}(\omega_{1,t} - \omega_{2,t}) \\ &= \tfrac{sb}{d}\frac{\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right) - \exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)}{\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right) + \exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right) + \exp\left(-\frac{\beta s}{2}x_{t-1}^2\right)}. \end{aligned} \tag{2.9}$$

Rewriting (2.9) as

$$x_t = f(x_{t-1}), \tag{2.10}$$

where the one-dimensional map $f : (-p^*, +\infty) \to \mathbb{R}$ is defined as

$$f(x) = \frac{sb}{d}\frac{\exp\left(-\frac{\beta s}{2}(x+b)^2\right) - \exp\left(-\frac{\beta s}{2}(x-b)^2\right)}{\exp\left(-\frac{\beta s}{2}(x+b)^2\right) + \exp\left(-\frac{\beta s}{2}(x-b)^2\right) + \exp\left(-\frac{\beta s}{2}x^2\right)}, \tag{2.11}$$

we have that f is differentiable. Moreover, Hommes and Wagener (2010) prove in their Theorem A that such map, for all values of s and d, is always decreasing and thus it admits a unique fixed point, which is the steady state of (2.10). Since optimists and pessimists are symmetrically biased, the steady state is given by $x = 0$, which corresponds to $p = p^*$ in (2.4). Hence, at the steady state the market equilibrium price coincides with the fundamental value. The monotonicity of f also prevents the emergence of interesting dynamic phenomena and indeed at most period-two cycles can occur. Nonetheless, in regard to the (local) stability of the steady state, different frameworks may be observed. Namely, as shown in Hommes and Wagener (2010), when $s/d < 1$, i.e., when the slopes of demand and supply satisfy the familiar "cobweb theorem" by Ezekiel (1938), so that the Muthian model is globally eductively stable in the sense of Guesnerie (2002), being stable under naive expectations, the model is also evolutionary stable. More precisely, the steady state may either be globally stable for all positive values of β or b, or $x = 0$ can be just locally stable, due to its coexistence with a period-two cycle. We stress that the condition $s/d < 1$ given in Theorem A in Hommes and Wagener (2010) is just sufficient, but not necessary for the unconditional stability[5] of the steady state. For instance, $x = 0$ is (globally or locally) asymptotically stable for $s \in (0, 1.06)$ when $d = 1$, for all

[5]We call a scenario unconditionally stable when the steady state is (globally or locally) stable for every value of the considered parameter.

positive values of β and b (cf. Figures 1 (A) and 2 for global stability of $x = 0$ when $s = 0.5$, and Figures 1 (B) and 3 for local stability of $x = 0$ when $s = 1.04$). However, in agreement with the stability condition that we shall derive (see (2.12) below), when fixing e.g. $d = 1$ and we let s increase, the steady state may lose and then recover stability through (supercritical or subcritical) period-doubling bifurcations[6] of f, so that $x = 0$ is stable only for sufficiently low and for sufficiently high values of the intensity of choice parameter or of the bias. In this case, after recovering stability, the steady state is just locally stable, because the period-two cycle that has emerged through the supercritical flip bifurcation of f, persists when raising β or b (see Figures 1 (C) and 4 for $s = 1.6$).

The initial global stability of the steady state, its loss and recovery of stability, as well as the persistence of the stable period-two cycle can be easily explained in terms of profits of the various kinds of agents and by looking at the ratio between the demand and supply curve reactivities. In particular, the ratio s/d plays a crucial role in the loss of stability of $x = 0$.

Let us start from the interpretation of the global stability framework when the values of β or b are small. If the value of β is low, agents are scarcely reactive. Nonetheless, when the initial condition is close to the steady-state price, the share of unbiased fundamentalists is reinforced due to their more accurate forecast and prices are led toward the fundamental value (see Hommes 2013 for the stabilizing role of unbiased fundamentalists); when the initial condition for prices is far from the steady state, population shares will not differ too much, but optimists perform a bit better if x_0 is e.g. positive. This causes a moderate increase in the production, so that the excess demand is negative and the price falls in a not too violent manner, making the realized price closer to the steady state with respect to the original price. Then, pessimists perform better in terms of profits. The production decreases, the excess demand becomes positive and the price increases, so that the newly realized price is still closer to the steady state with respect to the previous period price. In this manner prices progressively approach the steady state value in an oscillatory damped fashion, so that unbiased fundamentalists are more and more favored, leading to the convergence toward the fundamental.

[6] We stress that it would be possible to verify both the occurrence and the nature of all the bifurcations mentioned or illustrated along the manuscript by checking that the corresponding conditions reported in Wiggins (2003) are satisfied, similarly to what done, in a different context, e.g. in Proposition 3.1 in Naimzada and Pireddu (2019). For brevity's sake, we omit those proofs.

The global stability of the steady state for low values of the bias can be explained in a similar manner. Indeed, if b is small and the initial condition is close to the steady state, all kinds of agents perform well in terms of profits and the chosen production levels keep prices close to the fundamental. This reinforces the share of unbiased fundamentalists, who lead prices toward the steady state. If the initial condition is far from the steady state, since the bias is small, all agents make big forecasting errors, but if x_0 is e.g. positive optimists perform better. This causes a moderate increase in the production, so that the excess demand is negative and the price falls, but in a not too violent manner. Then, pessimists perform better in terms of profits and such alternation makes prices progressively approach the steady state in an oscillatory fashion.
Since the destabilizing role of the intensity of choice parameter has been often witnessed and commented on in the existing literature (see e.g. Brock and Hommes 1997 and Hommes 2013), we now focus on the effect produced by the bias on the system stability when, starting from lower values of b, we increase it. If the model is unstable under naive expectations, we observe in Figure 1 two different scenarios according to the value of s/d. If the latter ratio is moderate, in (B) the steady state is always stable but raising the value of the bias it starts coexisting with a locally stable period-two cycle, along which agents switch between optimism and pessimism. When instead the ratio s/d is larger, for increasing values of b we find in Figure 1 (C) that $x = 0$ suddenly loses stability in favor of a globally stable period-two cycle. This dissimilarity comes from the fact that, in the scenario considered in Figure 1 (B), the ratio between the demand and supply curve reactivities is not excessive, while it becomes more pronounced in the framework illustrated in Figure 1 (C). Namely, in the scenario depicted in Figure 1 (B) an initial condition for prices close to the steady state produces a small price variation due to the functioning of the price adjustment mechanism, which allows for a balance between demand and supply. This favors unbiased fundamentalists, whose production choices lead prices toward the fundamental. On the other hand, an initial condition for prices lying far from the steady state produces larger price variations, which alternately favor optimists or pessimists, leading to the emergence of the locally stable period-two cycle. When moving to the scenario considered in Figure 1 (C), since the ratio s/d is larger, even for an initial condition for prices close to the steady state, the adjustment mechanism will determine a price far from the fundamental, so that optimists or pessimists are favored. Hence, $x = 0$ is not locally stable

anymore and the period-two cycle becomes globally stable.
We can then conclude that b and β start playing a role, leading to the emergence of a (locally or globally) stable period-two cycle, just when s/d is large enough. Namely, when the ratio between the slopes of demand and supply curves is small, the price adjustment mechanism determines a price for the next period which is not too far from the steady state, so that unbiased fundamentalists are favored, leading prices toward the fundamental. This does not occur when s/d increases, as in such case the next period price will be distant from the fundamental and one kind of biased agents (optimists or pessimists) will have more accurate predictions, performing better from an evolutive viewpoint, so that prices will not approach the steady state anymore.
However, when the bias is excessively large, for prices close to the steady state it becomes again more profitable being fundamentalists, because the forecast error made by biased agents is too big. The consequent increase in the share of unbiased fundamentalists makes prices converge towards the steady state, that recovers its local stability. On the other hand, for high values of the bias and price initial conditions that are distant from the fundamental, having biased expectations about prices allows to make large profits far from the steady state and this accounts for the persistence of the stable period-two cycle when raising b, so that in Figure 1 (C) we observe the coexistence between the steady state and the period-two cycle for high values of the bias. A similar argument allows to explain why we witness the same phenomenon when β is large enough. Namely, if the intensity of choice level is high, profits are taken into big account, so that close to the steady state unbiased fundamentalists perform much better from an evolutive viewpoint, leading prices toward the fundamental, while far from the steady state the share of biased fundamentalists is high, and their production choices make the period-two cycle persist.
Since the map f in (2.11) is decreasing, no further frameworks may arise. We stress that the occurrence of one or the other of the possible outcomes is also influenced by the values of β and b, which have a destabilizing effect when they are too large. Thus, considering lower values of those parameters, the system will be unconditionally stable even for larger values of $s/d > 1$.

After this preliminary discussion, we derive in the next result the stability condition for $x = 0$ with respect to the intensity of choice parameter, which is the bifurcation parameter considered in Hommes and Wagener (2010):

Proposition 2.1 *Equation* (2.10) *admits* $x = 0$ *as unique steady state. The equilibrium* $x = 0$ *is locally asymptotically stable for map* f *in* (2.11) *if*

$$\beta < \frac{d\left(2 + \exp\left(\frac{\beta b^2 s}{2}\right)\right)}{2b^2 s^2}. \tag{2.12}$$

Hence, according to the considered parameter configuration, $x = 0$ *is stable for any* $\beta > 0$ *or there exist* $0 < \beta' \leq \beta''$ *such that* $x = 0$ *is stable for each* $\beta \in (0, \beta') \cup (\beta'', +\infty)$.

Proof. It is immediate to check that $x = 0$ solves the fixed-point equation $f(x) = x$, with f as in (2.11).
In order to show that $x = 0$ is the unique steady state it suffices to recall that, according to Theorem A in Hommes and Wagener (2010), the map f is decreasing.
The stability condition follows by imposing that $f'(0) \in (-1, 1)$. By direct computations, we have

$$f'(0) = \frac{-2b^2 \beta s^2 \exp\left(\frac{-\beta b^2 s}{2}\right)}{d\left(2\exp\left(\frac{-\beta b^2 s}{2}\right) + 1\right)}.$$

Since $f'(0)$ is always negative, the stability of $x = 0$ is guaranteed when $f'(0) > -1$, which is equivalent to (2.12). In particular, setting $\phi_1(\beta) = \beta$ and $\phi_2(\beta) = \left(d\left(2 + \exp\left(\frac{\beta b^2 s}{2}\right)\right)\right) / (2b^2 s^2)$, we notice that for $\beta \geq 0$ both ϕ_1 and ϕ_2 are increasing, convex maps with $\phi_1(0) < \phi_2(0)$. Since ϕ_2 tends to $+\infty$ faster than ϕ_1 for $\beta \to +\infty$ due to the presence of the exponential function, the graphs of ϕ_1 and ϕ_2 intersect never or twice according to the considered parameter configuration. We can have just one intersection between the graphs of ϕ_1 and ϕ_2 only when their are tangent at some point.
This concludes the proof. □

We stress that (2.12) implies that d has a stabilizing effect on $x = 0$, as d enlarges the stability condition, while s plays an ambiguous role on the stability of the steady state. Namely, considering ϕ_2 in the proof of Proposition 2.1 as a function of s, it holds that $\lim_{s \to 0} \phi_2(s) = \lim_{s \to +\infty} \phi_2(s) = +\infty$ and thus, for suitably low (high) positive values of s, such parameter has a destabilizing (stabilizing) effect on $x = 0$. However, for the parameter configuration we shall consider below, and whose dynamic outcomes are reported in Figures

1–4, d does not vary and we witness just the destabilizing effect of s. Indeed, s describes the slope of the supply function. When its value is moderate and it increases, assuming that the initial price is e.g. high, optimists produce even more, making the supply offer increase a lot. The price adjustment mechanism, which allows for a balance between demand and supply, makes then prices heavily fall, operating in a more violent manner. Hence, pessimists' forecast is now more accurate and, obtaining higher profits, their share increases. Pessimists produce less, making the supply offer decrease, so that the price raises again, and this process gives rise to wide oscillations, which do not converge towards the steady state, thus leading to a reduction of the steady state local stability region.

The stabilizing effect of d can be explained in a similar way. Namely, assuming that the initial price is e.g. high, optimists produce a lot, making the supply offer increase. However, if d raises, the demand function is more reactive and this weakens the price oscillations needed to reach a balance between demand and supply. In such manner, prices are less distant from the fundamental value, so that unbiased fundamentalists' forecast is precise enough. This makes their share increase, so that orbits approach the steady state, with a consequent enlargement in the steady state local stability region.

In regard to s and d, as explained above, a sufficiently high ratio between the slopes of demand and supply allows for the loss of stability of the steady state for increasing values of β or b.

We also remark that, rather than dealing with the intensity of choice parameter as done in Hommes and Wagener (2010), in what follows we will consider the bias as bifurcation parameter, measuring the influence of agents' heterogeneity through the parameter describing the degree of optimism and pessimism. Such choice is motivated by our interest in studying the agents' asymptotic heterogeneity and it has no consequences on the observed model dynamic outcomes since, as we shall prove in Corollary 2.1, the bias has the same effect on the system stability as the intensity of choice parameter.

In fact, rewriting the stability conditions in Proposition 2.1 in terms of the bias, we obtain the next result:

Corollary 2.1 *The equilibrium $x = 0$ is locally asymptotically stable for* (2.10) *if*

$$b^2 < \frac{d\left(2 + \exp\left(\frac{\beta b^2 s}{2}\right)\right)}{2\beta s^2}. \tag{2.13}$$

Hence, depending on the considered parameter configuration, $x = 0$ is stable for any $b > 0$ or there exist $0 < b' \leq b''$ such that $x = 0$ is stable for each $b \in (0, b') \cup (b'', +\infty)$.

Thus, according to Proposition 2.1 and Corollary 2.1, recalling also Theorem A in Hommes and Wagener (2010), when the eductive stability assumption for the Muthian model is not fulfilled, there are up to two possible stability thresholds for $x = 0$ with respect to β and b, and $x = 0$ may be locally stable just for sufficiently low and for sufficiently high values of the intensity of choice parameter and of the bias. In particular, this means that, while an intermediate beliefs' heterogeneity may have a destabilizing effect on the steady state, sufficiently strong biases can be stabilizing, as we justified above looking at profits.

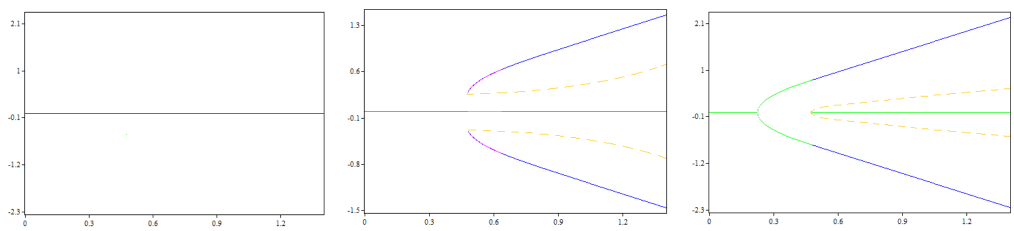
We now report in Figure 1 the three scenarios compatible with Corollary 2.1 for increasing values of the bias. In particular, we fix the other parameters as follows: $A = 18$, $\beta = 15$, $d = 1$, considering $s = 0.5$ in (A), $s = 1.04$ in (B) and $s = 1.6$ in (C). As initial conditions in (A) we have $x_0 = 1$; in (B) we have $x_0 = 0.01$ for the green points, $x_0 = 0.3$ for the magenta points and $x_0 = 1$ for the blue points; in (C) we have $x_0 = 0.01$ for the green points and $x_0 = 1$ for the blue points.

Since in Figure 1 (A) it holds that $s/d < 1$, and thus the Muthian model is globally eductively stable, the steady state is always stable. In particular, it is here globally asymptotically stable, since there are no other attractors, as we can observe by looking at Figure 2, where for $s = 0.5$ we report the graph of the second iterate of f for $b = 1$ in (A), $b = 2$ in (B) and $b = 3$ in (C).

In Figure 1 (B), although $s/d > 1$ and thus the eductive stability assumption for the Muthian model does not hold true anymore, the steady state is still stable for all values of the bias. However, while $x = 0$ is globally stable for $b \in (0, 0.48)$, for $b = 0.48$ a stable period-two cycle emerges (represented in Figure 1 (B) with a solid line), together with an unstable period-two cycle (represented in orange, dashed line), through a double fold bifurcation of f^2 and coexists with $x = 0$ for increasing values of b, so that for $b > 0.48$ the steady state is just locally stable. We report in Figure 3 the graph of the second iterate of f for $s = 1.04$ and $b = 0.3$ in (A), $b = 0.48$ in (B) and $b = 0.6$ in (C), in order to illustrate the double fold bifurcation of f^2.

Finally, for a still larger value of $s/d > 1$, in Figure 1 (C) we find that $x = 0$ is not stable for intermediate values of the bias. Indeed, according to Corollary 2.1, $x = 0$ is stable for $b \in (0, b') \cup (b'', +\infty)$, with $b' = 0.223$

and $b'' = 0.478$, and unstable otherwise. In particular, $x = 0$ is globally stable for $b \in (0, 0.223)$, while for $b = 0.223$ a supercritical flip bifurcation of f occurs, at which $x = 0$ loses stability in favor of a stable period-two cycle (represented in Figure 1 (C) with a solid line), that persists for larger values of the bias. However, $x = 0$ recovers its (local) stability for $b = 0.478$ through a subcritical flip bifurcation of f, at which an unstable period-two cycle emerges (represented in orange, dashed line). Since supercritical and subcritical flip bifurcations of f correspond to pitchfork bifurcations of f^2, in order to illustrate the main steps that we observe in Figure 1 (C) when the bias increases, we report in Figure 4 the graph of the second iterate of f for $s = 1.6$ and $b = 0.2$ in (A), $b = 0.223$ in (B), $b = 0.3$ in (C), $b = 0.478$ in (D) and $b = 0.6$ in (E).

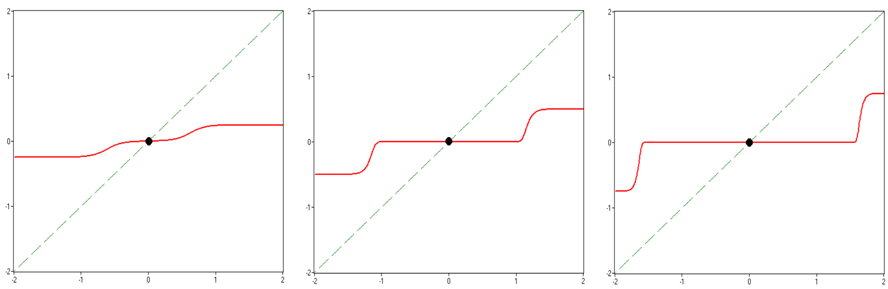


(A) (B) (C)

Figure 1: The bifurcation diagram of f for $b \in (0, 1.4)$ and different initial conditions, for $A = 18$, $\beta = 15$, $d = 1$, and $s = 0.5$ in (A), $s = 1.04$ in (B) and $s = 1.6$ in (C).

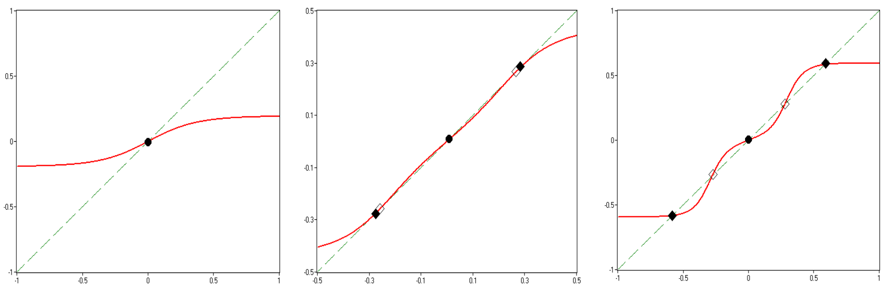
3 The model with rational agents

We now introduce rational agents in the economy and we investigate the effects that they produce both when the model is globally eductively stable and when it is not. In particular, we enrich the set of expectation rules in (2.5) by assuming that agents may also be rational and thus, being endowed with perfect foresight, they correctly predict the next period price. Moreover, when their share is strongly prevailing in the population, in choosing the production level which allows them to maximize profits, they determine an aggregate production level that, together with the demand, generates a market equilibrium price close to the fundamental value. Characterizing



(A) (B) (C)

Figure 2: The graph of the second iterate of f for $s = 0.5$, and $b = 1$ in (A), $b = 2$ in (B) and $b = 3$ in (C).



(A) (B) (C)

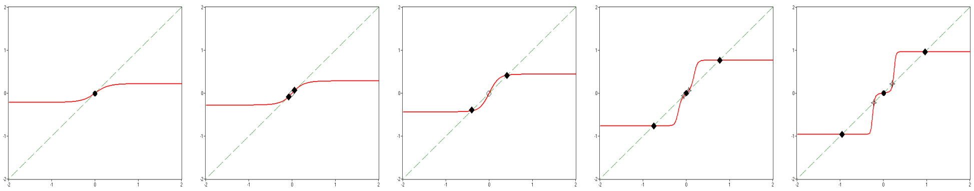
Figure 3: The graph of the second iterate of f for $s = 1.04$, and $b = 0.3$ in (A), $b = 0.48$ in (B) and $b = 0.6$ in (C).

rational agents by the subscript -1, in symbols we have

$$p^e_{-1,t} = p_t. \tag{3.1}$$

The cost function and the demand function are supposed to be described by (2.1) and (2.3), respectively[7]. At the fundamental price $p = p^*$ it still holds that demand equals supply. In fact, $p = p^*$ in (2.4) is again the only steady

[7]In this respect, we stress that we investigated the just described setting in which, in addition to the introduction of rational agents, we took into account more general technologies. In particular, we considered the cost function $\tilde{\gamma}(q) = 1/s_0 + q/s_1 + q^2/(2s_2)$, with s_0, s_1, $s_2 > 0$, which coincides with $c(q)$ in (2.1) when letting s_0, $s_1 \to +\infty$ and identifying s_2 with s. Although the position of the unique steady state, which corresponds to the fundamental price $\tilde{p} = (A + \frac{s_2}{s_1})/(d + s_2)$, is influenced by the value of s_1, when



(A) (B) (C) (D) (E)

Figure 4: The graph of the second iterate of f for $s = 1.6$, and $b = 0.2$ in (A), $b = 0.223$ in (B), $b = 0.3$ in (C), $b = 0.478$ in (D) and $b = 0.6$ in (E).

state (cf. Proposition 3.1 below).
Like in Brock and Hommes (1997), we assume that rational agents face an information cost $C > 0$, so that their net profits are described by

$$\pi_{-1,t} = p_t S(p^e_{-1,t}) - \gamma(S(p^e_{-1,t})) - C = p_t S(p_t) - \gamma(S(p_t)) - C. \qquad (3.2)$$

Consequently, denoting by $\omega_{i,t}$ the share of agents choosing the forecasting rule $i \in \{-1, 0, 1, 2\}$ at time t, the evolutive mechanism in (2.7), based on the most recently realized net profits $\pi_{j,t-1}$, $j \in \{-1, 0, 1, 2\}$, becomes

$$\omega_{i,t} = \frac{\exp(\beta \pi_{i,t-1})}{\sum_{j=-1}^{2} \exp(\beta \pi_{j,t-1})}, \; i \in \{-1, 0, 1, 2\}. \qquad (3.3)$$

Introducing the variable $x_t = p_t - p^*$, simple computations allow to write our model dynamic equation in deviation from the fundamental as

$$x_t = -\frac{s}{d + \omega_{-1,t}\, s} \sum_{i=0}^{2} \omega_{i,t} b_i$$

---

expressing the model in deviation from the fundamental neither s_0 nor s_1 enter the model equation. This happens because of the formulation of the share updating rule, as s_0 and s_1 affect net profits through multiplicative terms which can be simplified between numerator and denominator in (3.4) and in (3.5). Due to such considerations, for sake of simplicity, we decided to present our model as an extension of the setting in Hommes and Wagener (2010) only in regard to the set of considered forecasting rules, rather than introducing the more general cost function $\tilde{\gamma}(q)$. Such choice allows also for neater conclusions about the role of rational agents on the model stability, as their effect need not be disentangled from that of other modifications with respect to the original framework in Hommes and Wagener (2010).

with

$$\omega_{i,t} = \frac{\exp\left(-\frac{\beta s}{2}(x_{t-1}-b_i)^2\right)}{\left(\sum_{j=0}^{2}\exp\left(-\frac{\beta s}{2}(x_{t-1}-b_j)^2\right)\right)+\exp(-\beta C)} \tag{3.4}$$

for $i \in \{0,1,2\}$ and with

$$\omega_{-1,t} = \frac{\exp(-\beta C)}{\left(\sum_{j=0}^{2}\exp\left(-\frac{\beta s}{2}(x_{t-1}-b_j)^2\right)\right)+\exp(-\beta C)}. \tag{3.5}$$

More explicitly, recalling (2.5), we obtain

$$\begin{aligned} x_t &= \tfrac{bs}{d+\omega_{-1,t}s}(\omega_{1,t}-\omega_{2,t}) \\ &= \tfrac{bs\left(\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)-\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)\right)}{d\left(\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)+\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)+\exp\left(-\frac{\beta s}{2}x_{t-1}^2\right)\right)+(d+s)\exp(-\beta C)}. \end{aligned} \tag{3.6}$$

Expressing the model in terms of x_t, according to Proposition 3.1 the unique steady state is still given by $x^* = 0$.

In view of the analysis we shall perform below, it is expedient to rewrite (3.6) as

$$x_t = g(x_{t-1}), \tag{3.7}$$

where the one-dimensional map $g : (-p^*, +\infty) \to \mathbb{R}$ is defined as

$$g(x) = \frac{bs\left(\exp\left(-\frac{\beta s}{2}(x+b)^2\right)-\exp\left(-\frac{\beta s}{2}(x-b)^2\right)\right)}{d\left(\exp\left(-\frac{\beta s}{2}(x+b)^2\right)+\exp\left(-\frac{\beta s}{2}(x-b)^2\right)+\exp\left(-\frac{\beta s}{2}x^2\right)\right)+(d+s)\exp(-\beta C)}. \tag{3.8}$$

We stress that, like f in (2.11), also g is differentiable. Moreover, recalling the expression of p^* in (2.4), the domain of g is enlarged by considering increasing values of A. When extending its domain to $\mathbb{R}$, the map is odd[8]. Namely, replacing x with $-x$ leaves the denominator unchanged, while the two terms on the numerator of g are interchanged, so that $g(-x) = -g(x)$ for every $x \in \mathbb{R}$. We also observe that the extension of g to $\mathbb{R}$ admits the x-axis as horizontal asymptote for $x \to \pm\infty$. Hence, unlike f in (2.11), g is not monotone and indeed, as we shall see below, the introduction of rational agents may lead to complex dynamics. We notice however that chaotic phenomena can occur only when the eductive stability assumption

[8]We remark that analogous properties hold for f in (2.11) as well, even if we did not need to use them in Section 2.

for the Muthian model is not fulfilled. Indeed, according to Proposition 3.1, the local stability region is enlarged by the introduction of rational agents and thus, like in Hommes and Wagener (2010), when $s/d < 1$ the steady state $x = 0$ is always stable. More precisely, since a stable period-two cycle emerges through a supercritical flip bifurcation of g, rather than through a fold bifurcation of g^2, and since for $s/d < 1$ the equilibrium is always stable, it cannot coexist neither with the period-two cycle, nor with any chaotic or periodic attractor following the period-two cycle, when the model is stable under naive expectations. Hence, as we shall notice below, a scenario analogous to Figure 1 (B) cannot occur when rational agents are taken into account. On the other hand, for $s/d > 1$ we witness interesting dynamic outcomes of the model. Before illustrating the possible scenarios in Figures 5–9, we derive the stability conditions for the steady state $x = 0$ with respect to the intensity of choice parameter in Proposition 3.1 and with respect to the bias in Corollary 3.1. The proof of Proposition 3.1, where we also show that $x = 0$ is the unique steady state for (3.7), is based on the same argument that we used to derive the stability region in Proposition 2.1. Nonetheless, we report all the details for the sake of completeness.

Proposition 3.1 *Equation* (3.7) *admits* $x = 0$ *as unique steady state. The equilibrium* $x = 0$ *is locally asymptotically stable for map* g *in* (3.8) *if*

$$\beta < \frac{2d + \exp\left(\frac{\beta b^2 s}{2}\right)(d + (d+s)\exp(-\beta C))}{2b^2 s^2}. \tag{3.9}$$

Hence, according to the considered parameter configuration, $x = 0$ *is stable for any* $\beta > 0$ *or there exist* $0 < \beta'_R \leq \beta''_R$ *such that* $x = 0$ *is stable for each* $\beta \in (0, \beta'_R) \cup (\beta''_R, +\infty)$.

Proof. A straightforward check ensures that $x = 0$ solves the fixed-point equation $g(x) = x$, with g as in (3.8).
In order to show that $x = 0$ is the unique steady state it suffices to observe that g is positive if and only if x is negative.
The stability condition follows by imposing that $g'(0) \in (-1, 1)$. By direct computations, we have

$$g'(0) = \frac{-2b^2 \beta s^2 \exp\left(\frac{-\beta b^2 s}{2}\right)}{d\left(2\exp\left(\frac{-\beta b^2 s}{2}\right) + 1\right) + (d+s)\exp(-\beta C)}.$$

Since $g'(0)$ is always negative, the stability of $x = 0$ is guaranteed when $g'(0) > -1$, which is equivalent to (3.9). In particular, setting $\varphi_1(\beta) = \beta$ and $\varphi_2(\beta) = (2d + \exp(\frac{\beta b^2 s}{2})(d + (d+s)\exp(-\beta C)))/(2b^2 s^2)$, we notice that for $\beta \geq 0$ both φ_1 and φ_2 are convex maps with $0 = \varphi_1(0) < \varphi_2(0)$. Since φ_2 tends to $+\infty$ faster than φ_1 for $\beta \to +\infty$ due to the presence of the exponential function, the graphs of φ_1 and φ_2 intersect never or twice according to the considered parameter configuration. We can have just one intersection between the graphs of φ_1 and φ_2 only when their are tangent at some point. This concludes the proof. □

Comments analogous to those made after Proposition 2.1 about the stabilizing role of d and the ambiguous role of s hold in relation to (3.9) as well. Again, for the parameter configuration we shall consider below and whose dynamic outcomes are reported in Figures 5–9, we witness just the destabilizing effect of s, while d does not vary.
Moreover, rewriting (3.9) as

$$\beta < \frac{d\left(2 + \exp\left(\frac{\beta b^2 s}{2}\right)\right) + (d+s)\exp(\beta(\frac{b^2 s}{2} - C))}{2b^2 s^2} \tag{3.10}$$

and comparing such expression with (2.12), we notice that the right-hand side in (3.10) is larger than that in (2.12) for any $C > 0$. Hence, as expected, we can conclude that rational agents have a stabilizing effect on the system stability, no matter what is the information cost C they face. Nonetheless, as the right-hand side in (3.10) is decreasing in C, raising the latter parameter has a destabilizing effect on the steady state. Indeed, the choices of rational agents lead prices towards the fundamental value. Raising their information cost makes the corresponding share decrease, due to their resulting lower fitness in terms of profits, not only for prices far from the steady state, but also in a neighborhood of it, and this may lead to a destabilization of the steady state. In particular, in the limit $C \to +\infty$ the term on the right-hand side in (3.10) coincides with that on the right-hand side in (2.12), meaning that the stabilizing effect of rational agents tends to disappear when C is excessively large. Namely, in correspondence to $x = 0$ for the profits and the share of rational agents we find $\pi^*_{-1} = \frac{s}{2}(p^*)^2 - C$ and $\omega^*_{-1} = \exp(-\beta C) \,/\, \left(2\exp\left(-\frac{\beta s}{2}b^2\right) + 1 + \exp(-\beta C)\right)$, respectively, from which it follows that, when C is too high, their profits become negative and their share tends to vanish. On the other hand, in addition to the information cost faced by rational agents, also the bias degree plays an important role in determining which forecasting strategy is more profitable at the steady state. Indeed, in correspondence to $x = 0$ the profits of biased fundamentalists are given by $\pi_1^* = \pi_2^* = \frac{s}{2}\left((p^*)^2 - b^2\right)$, and thus it holds that $\pi_{-1}^* > \pi_1^* = \pi_2^*$ if and only if $C < \frac{s}{2}b^2$. We can then conclude that, when the bias is very large, due to the high inaccuracy degree of the price forecasts by optimists and pessimists, it is still more convenient being rational at the steady state, despite the information cost faced by rational agents.

As done in Section 2, also in the framework encompassing rational agents we will consider as bifurcation parameter the bias, which again has the same effect on the system stability as the intensity of choice parameter. Namely, rewriting the stability conditions in Proposition 3.1 in terms of the bias, we obtain the next result:

Corollary 3.1 *The equilibrium $x = 0$ is locally asymptotically stable for (3.7) if*

$$b^2 < \frac{2d + \exp\left(\frac{\beta b^2 s}{2}\right)(d + (d+s)\exp(-\beta C))}{2\beta s^2}. \tag{3.11}$$

Hence, according to the considered parameter configuration, $x = 0$ is stable for any $b > 0$ or there exist $0 < b'_R \leq b''_R$ such that $x = 0$ is stable for each $b \in (0, b'_R) \cup (b''_R, +\infty)$.

Thus, we found up to two stability thresholds for $x = 0$ with respect to b as well, and the steady state is locally stable for sufficiently low and for sufficiently high values of the bias. We are going to illustrate in Figures 5–9 the latter double effect of b, as well as the interesting dynamic phenomena which may arise when the economy is populated by rational agents, too.
In particular, we report in Figure 5 the bifurcation diagrams corresponding to the three main scenarios[9] compatible with Corollary 3.1 for increasing values of the bias. In particular, like in Section 2 we fix the other parameters

[9] We stress that for the parameter configuration considered in Figure 5 (B), where in particular $s = 1.6$, the stable and unstable period-two cycles persist for all $b \geq 0.282$. Nonetheless, for lower values of s the period-two cycles may disappear for sufficiently high values of the bias, because the maximum and minimum values of g^2 are not pronounced enough. Since increasing values of b produce an horizontal translation of those extrema, if they are not sufficiently high in absolute value, they do not exceed the 45-degree line when b is too large. We observe the just described phenomenon e.g. with $s = 1.5$, in correspondence to which, drawing in Figure 8 the bifurcation diagram with respect to b, like in Figure 5 (B) we still find a double stability threshold, but the period-two cycles

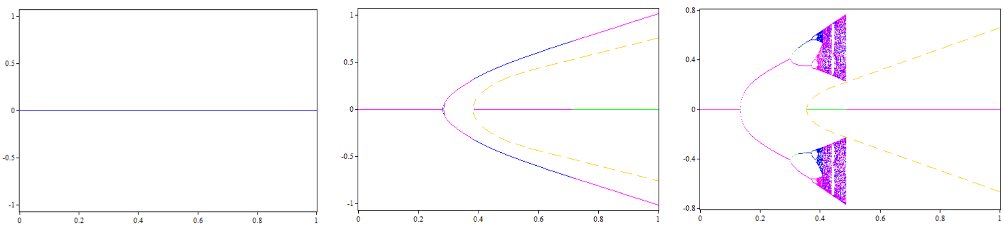
as follows: $A = 18$, $\beta = 15$, $d = 1$, considering $C = 0.1$ and $s = 1.2$ in (A), $s = 1.6$ in (B) and $s = 3$ in (C). As initial conditions in (A) we have $x_0 = 1$; in (B) we have $x_0 = 0.01$ for the green points, $x_0 = 0.5$ for the magenta points and $x_0 = 1$ for the blue points; in (C) we have $x_0 = 0.01$ for the green points, $x_0 = 0.35$ for the magenta points and $x_0 = 0.55$ for the blue points.
Although in Figure 5 (A) it holds that $s/d > 1$, and thus the eductive stability assumption for the Muthian model is not fulfilled, the steady state is always globally asymptotically stable, since there are no other attractors, as we can observe by looking at Figure 6, where we report the graph of the second iterate of g for $b = 0.4$ in (A), $b = 0.9$ in (B) and $b = 1.4$ in (C). Namely, also in the setting with rational agents, the condition $s/d < 1$ is just sufficient, but not necessary for the unconditional stability of the steady state. This phenomenon is strengthened by the stabilizing effect produced by the introduction of rational agents, whose presence enlarges the local stability region, as discussed just after Proposition 3.1. Indeed, for $A = 18$, $\beta = 15$, $d = 1$, in the setting considered in Section 2 it holds that $x = 0$ is globally asymptotically stable for all values of the bias when $s \in (0, 0.768)$, while in the framework with rational agents facing an information cost $C = 0.1$ we observe the unconditional stability of $x = 0$ for a larger interval of values for s, i.e., for $s \in (0, 1.422)$. On the other hand, due to the destabilizing effect produced by an increase in the information cost faced by rational agents, when C raises it holds that $x = 0$ is globally asymptotically stable for all values of the bias when s varies in a smaller interval. For instance, with $C = 0.2$ the unconditional stability of $x = 0$ holds just for $s \in (0, 1.099)$, which is anyway larger than the interval $(0, 0.768)$ found in regard to the setting without rational agents.
In Figure 5 (B), we have again $s/d > 1$, but this time we find a scenario similar to Figure 1 (C), in which $x = 0$ is unstable for intermediate values

disappear for $b = 1.307$ through a double reverse fold bifurcation of g^2, at which they coincide, and after that $x = 0$ is globally asymptotically stable. Although the latter framework is new with respect to the scenarios portrayed in Section 2, we chose not to deal with it in details because for $s = 1.5$ the map g^2 almost coincides with the 45-degree line in a neighborhood of $x = 0$ and thus the bifurcations illustrated in Figure 7 would not be clearly visible.
We also remark that, for values of s larger than those used in Figure 5 (C), the chaotic attractor could not coexist with the steady state and that, for still higher values of s, the chaotic attractor in two pieces could become a chaotic attractor in one piece before disappearing. All such effects are in agreement with the destabilizing role of s, highlighted and interpreted in Section 2.

of the bias. Indeed, according to Corollary 3.1, for the considered parameter configuration $x = 0$ is stable just for $b \in (0, b'_R) \cup (b''_R, +\infty)$, with $b'_R = 0.282$ and $b''_R = 0.386$. In particular, $x = 0$ is globally stable for $b \in (0, 0.282)$, while for $b = 0.282$ a supercritical flip bifurcation of g occurs, at which $x = 0$ loses stability in favor of a stable period-two cycle (represented with a solid line), which persists for larger values of the bias. However, $x = 0$ recovers its (local) stability for $b = 0.386$ through a subcritical flip bifurcation of g, at which an unstable period-two cycle emerges (represented in orange, dashed line). Recalling that flip bifurcations of g correspond to pitchfork bifurcations of g^2, in order to illustrate the main steps that we observe in Figure 5 (B) as the bias increases, we report in Figure 7 the graph of g^2 for $b = 0.2$ in (A), $b = 0.282$ in (B), $b = 0.36$ in (C), $b = 0.386$ in (D) and $b = 0.5$ in (E).
Before discussing the framework in Figure 5 (C), we stress that in the presence of rational agents it is not possible to observe a scenario analogous to Figure 1 (B). Namely in that case $x = 0$ was locally stable even after the emergence of the stable period-two cycle through a double fold bifurcation of f^2. On the contrary, with rational agents the stable period-two cycle arises through a supercritical flip bifurcation of g and thus the steady state can not be stable for all values of the bias. However, like in Figure 1 (C), also with rational agents $x = 0$ recovers stability through a subcritical flip bifurcation of g. Another difference between the frameworks with and without rational agents lies in the fact that, as remarked in Section 2, with the map f in (2.11) once a period-two cycle has emerged, it persists for increasing values of the bias. Vice versa, as explained in Footnote 9, when dealing with g in (3.8), for suitable values of s such as $s = 1.5$, the stable and unstable period-two cycles, born respectively through a supercritical and a subcritical flip bifurcation of g at $x = 0$, may disappear for sufficiently high values of the bias through a double reverse fold bifurcation of g^2 and after that $x = 0$ is globally asymptotically stable. We show the corresponding bifurcation diagram for g in Figure 8, in which $s = 1.5$ and b varies in $(0, 1.4)$, and where we represent with a solid line the stable period-two cycle born through a supercritical flip bifurcation of g at $x = 0$ for $b = 0.319$ and with an orange dashed line the unstable period-two cycle born through a subcritical flip bifurcation of g at $x = 0$ for $b = 0.371$. The stable and unstable period-two cycles coincide and disappear through a double reverse fold bifurcation of g^2 occurring for $b = 1.307$. Due to unconnectedness of the basin of attraction of $x = 0$, we need different initial conditions to represent the whole stable period-two cycle. Indeed, in Figure 8 as initial conditions we have $x_0 = 0.01$

for the green points, $x_0 = 0.5$ for the magenta points, $x_0 = 1$ for the blue points and $x_0 = 1.2$ for the red points.
Passing now to Figure 5 (C), for a still larger value of $s/d > 1$ we finally observe the presence of chaotic dynamics, which could not arise in the framework without rational agents, where the map f governing the dynamics is monotonically decreasing. On the contrary, quite surprisingly, when the economy is populated by agents endowed with perfect foresight, too, we may witness complex behaviors and variegated multistability phenomena. Namely, according again to Corollary 3.1, for the considered parameter configuration $x = 0$ is stable just for $b \in (0, b'_R) \cup (b''_R, +\infty)$, with $b'_R = 0.134$ and $b''_R = 0.354$. In particular, $x = 0$ is globally stable for $b \in (0, 0.134)$, while it becomes unstable and recovers its local stability through flip bifurcations of g, occurring at $b = b'_R$ and $b = b''_R$, respectively. More precisely, the flip bifurcation occurring at $b = b'_R$ is supercritical and in correspondence to it $x = 0$ loses stability in favor of a stable period-two cycle, which undergoes a cascade of flip bifurcations leading to chaos. We notice that the external - first periodic, and then chaotic - attractor coexists with the locally stable steady state for $b \in (0.354, 0.487)$, while for $b = 0.487$ the chaotic attractor suddenly disappears due to a contact bifurcation with the unstable period-two cycle, represented with an orange dashed line in Figure 5 (C), born through the subcritical flip bifurcation occurring at $x = 0$ for $b = b''_R$. Indeed, for $b = 0.487$ the forward iterates of the extrema of g enter the basin of attraction of $x = 0$ exceeding the unstable period-two cycle, which separates the basin of attraction of $x = 0$ from that of the chaotic attractor. After the disappearance of the chaotic attractor, $x = 0$ is again globally stable.
In order to better describe the dynamic phenomena related to Figure 5 (C), we chose to report in Figure 9 the graph of g rather than that of g^2. This allows us to represent the periodic and chaotic trajectories of the system, at the cost of not showing the bifurcations occurring as the bias value raises. We stress however that the two pitchfork bifurcations of g^2, corresponding to the supercritical and subcritical flip bifurcations of g through which the steady state loses and recovers its stability, are analogous to those depicted in Figure 7. In more detail, in Figure 9 we draw the graph of g for various increasing values of the bias, in order to illustrate how $x = 0$, after losing stability for $b = 0.134$, recovers stability for $b = 0.354$, at first locally and then globally. In particular, in (A), for $b = 0.1$, the steady state $x = 0$ is globally asymptotically stable. In (B), for $b = 0.2$, $x = 0$ has become unstable and we represent the stable period-two cycle that has arisen through the

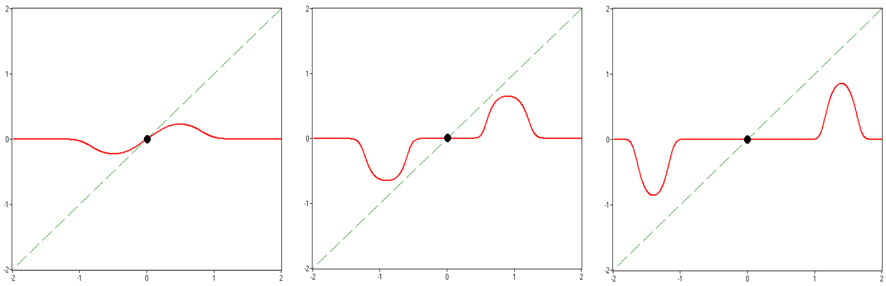
supercritical flip bifurcation of g occurring for $b = 0.134$. After recovering its local stability through the subcritical flip bifurcation of g, the steady state coexists with two stable period-two cycles, born through a flip bifurcation of the period-two cycle occurring for $b = 0.295$. Due the oddness of the map g, the two-cycles are symmetric and for e.g. $b = 0.36$ they are composed by the periodic points $\{\hat{x}_1, \hat{x}_2\} = \{-0.351, 0.553\}$ and $\{\bar{x}_1, \bar{x}_2\} = \{-0.553, 0.351\}$, respectively. Raising b in Figure 9 to 0.46, $x = 0$ is still locally stable and it is surrounded by a one-piece chaotic attractor. In particular, in (C) we show an orbit visiting the chaotic attractor, while in (D), still for $b = 0.46$, we illustrate an orbit which directly hits the steady state, since the initial condition belongs to a non-immediate component of its basin of attraction, contained in the x-axis. We remark that the basin of attraction of $x = 0$ is unconnected due to presence of rational agents, since now the map generating the dynamics is no more monotone. Finally, in Figure 9 (E), for $b = 0.5$, $x = 0$ is again globally stable because of the disappearance of the chaotic attractor due to the above described contact bifurcation with the unstable period-two cycle born through the subcritical flip bifurcation of g at $x = 0$.



(A) (B) (C)

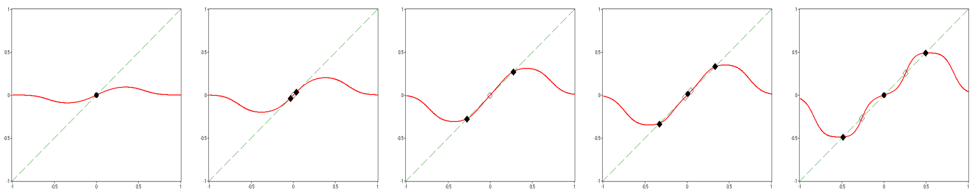
Figure 5: The bifurcation diagram of g for $b \in (0, 1)$ and different initial conditions, for $C = 0.1$, $A = 18$, $\beta = 15$, $d = 1$, and $s = 1.2$ in (A), $s = 1.6$ in (B) and $s = 3$ in (C).

We stress that, although the chaotic attractor has disappeared, two unstable invariant chaotic sets still exist for the parameter values considered in Figure 9 (E). We do not see them in the bifurcation diagram in Figure 5 (C) because the forward iterates of almost all the points in a neighborhood of those sets limit towards $x = 0$. In order to show their existence, we rely on the method of the strictly turbulent maps in Block and Coppel (1992), which guarantees the existence of invariant, chaotic sets, but in general not their attractiveness. Such technique requires to find for a given continuous map $\psi : J \to J$,



(A) (B) (C)

Figure 6: The graph of the second iterate of g for $C = 0.1$, $s = 1.2$, and $b = 0.4$ in (A), $b = 0.9$ in (B) and $b = 1.4$ in (C).



(A) (B) (C) (D) (E)

Figure 7: The graph of the second iterate of g for $C = 0.1$, $s = 1.6$, and $b = 0.2$ in (A), $b = 0.282$ in (B), $b = 0.36$ in (C), $b = 0.386$ in (D) and $b = 0.5$ in (E).

where $\emptyset \neq J \subset \mathbb{R}$ is a compact interval, two nonempty compact disjoint sub-intervals J_0 and J_1 of J such that

$$J_0 \cup J_1 \subseteq f(J_0) \cap f(J_1). \tag{3.12}$$

If the latter property is fulfilled, then the map ψ is called strictly turbulent in Block and Coppel (1992) and it is therein shown to display some of the typical features associated with the concept of chaos like, e.g., existence of periodic points of each period (cf. Lemma 3 on page 26), semi-conjugacy to the Bernoulli shift of ψ restricted to a suitable compact invariant set (see Proposition 15 on page 35), and thus positive topological entropy, as defined in Adler et al. (1965).
We will use the just described methodology in the proof of Proposition 3.2. In particular, since we need an invariant compact interval J in order to apply

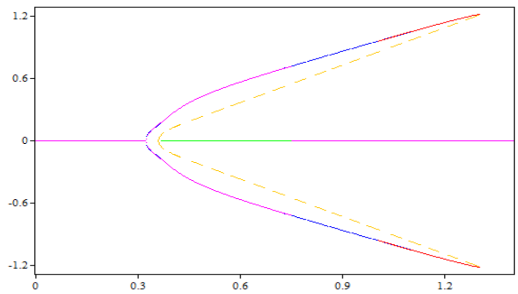
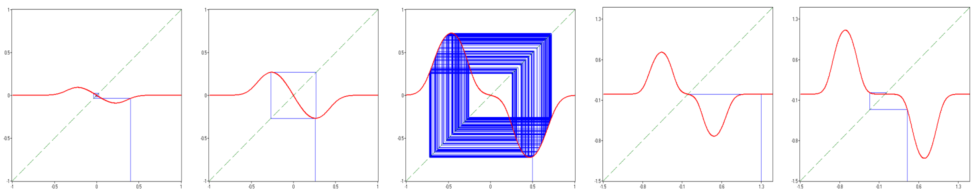


Figure 8: The bifurcation diagram of g for $b \in (0, 1.4)$ and different initial conditions, for $C = 0.1$, $A = 18$, $\beta = 15$, $d = 1$ and $s = 1.5$.



(A) (B) (C) (D) (E)

Figure 9: The graph of map g for $C = 0.1$, $s = 3$, and $b = 0.1$, $x_0 = 0.4$ in (A), $b = 0.2$, $x_0 = 0.26$ in (B), $b = 0.46$, $x_0 = 0.5$ in (C), $b = 0.46$, $x_0 = 1.3$ in (D) and $b = 0.7$, $x_0 = 0.4$ in (E).

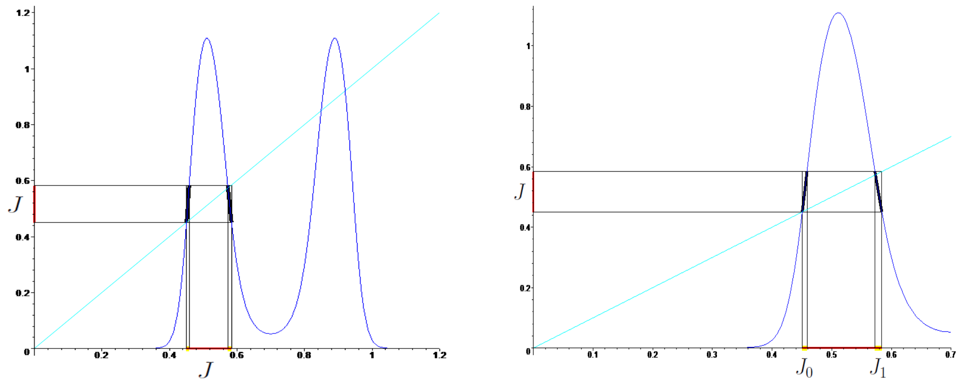
the method of the strictly turbulent maps and since the map g is odd and positive just when x is negative, we have to deal with the second iterate of g, so that for us $\psi = g^2$. For brevity's sake, we will focus on what happens on the positive horizontal axis only, but a completely symmetric picture occurs for g^2 for negative values of x, as well.

Proposition 3.2 *Let us consider the map g in* (3.8) *for the parameter configuration used in Figure 9 (E), i.e., $C = 0.1$, $A = 18$, $\beta = 15$, $d = 1$, $s = 3$ and $b = 0.7$. Setting $J = [0.451, 0.584]$, $J_0 = [0.451, 0.458]$ and $J_1 = [0.572, 0.584]$, it holds that*

$$g^2(J_i) = J, \ i = 0, 1. \tag{3.13}$$

Hence, the map g^2 is strictly turbulent and, in particular, it follows that $h_{\text{top}}(g) \geq \log(\sqrt{2})$, where we denote by $h_{\text{top}}(g)$ the topological entropy of g.

Proof. The map g^2 is strictly turbulent because it is continuous on $\mathbb{R}$, and thus on J, and J_0, J_1 are nonempty compact disjoint sub-intervals of J for which (3.13) is satisfied. In fact, the left endpoint of J_0 is a fixed point for g^2, that we call x^* and whose value is given by $x = 0.451$, and the right endpoint of J_1, which coincides with the right endpoint of J, is the smallest solution on the right of x^* to the equation $g^2(x) = x^*$. Notice that x^* is also the left endpoint of J. Moreover, the right endpoint of J_0 and the left endpoint of J_1 are the solutions to the equation $g^2(x) = 0.584$, where 0.584 is the right endpoint of J. Thus, by construction, it holds that $g^2(J_i) = J$, $i = 0, 1$, and g^2 is strictly turbulent. By Corollary 15, page 200, in Block and Coppel (1992), it follows that $h_{\text{top}}(g^2) \geq \log(2)$ and, by Theorem 2 in Adler et al. (1965), we can conclude[10] that $h_{\text{top}}(g) \geq \log(\sqrt{2})$. □



(A) (B)

Figure 10: The graph of g^2 for positive values of x in (A) and an enlargement in (B), together with a pictorial illustration of Proposition 3.2. We draw with a ticker line the interval J, its sub-intervals J_0 and J_1, as well as their images through g^2.

[10]Actually, by the shape of the graph of g^2 it is easy to show that taking $J = [0.451, 0.950]$, where $x = 0.950$ is the largest solution on the right of x^* (visible in Figure 10 (A)) to the equation $g^2(x) = x^* = 0.451$, then there exist four pairwise disjoint compact sub-intervals J_i of J, with $i \in \{0, 1, 2, 3\}$, for which it holds that $g^2(J_i) = J$. Indeed, by a geometrical construction analogous to the one described in the proof of Proposition 3.2, we have $J_0 = [0.451, 0.483]$, $J_1 = [0.542, 0.584]$, $J_2 = [0.816, 0.859]$, $J_3 = [0.918, 0.950]$. Then, by Proposition 8, page 196, in Block and Coppel (1992), it follows that $h_{\text{top}}(g^2) \geq \log(4)$ and, by Theorem 2 in Adler et al. (1965), we obtain the better estimate $h_{\text{top}}(g) \geq 2$.

See Figure 10 for a graphical illustration of the result. In particular, in (B) we do not use the same scaling on the horizontal and vertical axes, in order to better show the interval J and its sub-intervals J_0 and J_1.
We stress that, although in the statement of Proposition 3.2 we have considered the parameter configuration used in Figure 9 (E), the same conclusions hold for several different sets of parameter values, as well.
We also remark that, once that a result analogous to Proposition 3.2 is proven for a certain parameter configuration, by continuity, the same conclusions still hold, suitably modifying the intervals J, J_0 and J_1, for small variations in those parameters. Hence, for instance, Proposition 3.2 actually allows to infer the existence of complex dynamics for the map g^2 when b lies in a neighborhood of 0.7, or when s lies in a neighborhood of 3, and for some suitable values of the other parameters.

Like done for the setting without rational agents, the scenarios illustrated in Figure 9 for increasing values of the bias may be interpreted from an economic viewpoint comparing the profits of the various kinds of agents and by looking at the ratio between the demand and supply curve reactivities.
Let us start from the global asymptotic stability framework in (A), whose explanation bears a strong resemblance to that presented for Figure 1 (A), being based on the functioning of the price adjustment mechanism and on the stabilizing role of unbiased fundamentalists, which is now strengthened by the presence of rational agents. In more detail, when the bias is small and the initial condition for prices is close to the steady state, recalling the information cost faced by rational agents, unbiased fundamentalists realize higher profits due to their more accurate price forecasts with respect to biased fundamentalists, which however do not perform badly since the bias is small. Then the share of unbiased fundamentalists increases more and, due to the positive effect they have on the system stability (see Hommes 2013), prices converge towards the fundamental. When the initial condition is far from the steady state, since the bias is small, biased fundamentalists make big forecast errors and thus they are not favored by the share updating mechanism. More precisely, if like in Figure 9 (A) the initial condition is positive, rational agents and, in a much reduced manner, optimists perform well. Hence, the share of rational agents is that which increases more and the total supply approaches its stationary value. Close to the steady state, the forecasting errors made by both biased and unbiased fundamentalists are small, being the bias low. In particular, unbiased fundamentalists' forecast

is now the most precise among fundamentalists, while the profits realized by rational agents are low due to the information cost they face. Nonetheless, the convergence towards the steady state occurs like in the case in which the initial condition for prices is close to the steady state.
Also the loss of stability of the steady state in favor of the period-two cycle that we observe in Figure 9 (B) can be interpreted similarly to what done with Figure 1 (C). Indeed, since b has increased with respect to (A), its value has become compatible with the strength of the "separating" effect produced by the price adjustment mechanism, which determines prices that are quite distant from the steady state. Namely, due to the high value of s, the price adjustment mechanism reacts violently to a production variation due to the strong difference in the slopes of the demand and supply curves. Such compatibility between the bias and the prices allows to explain the global stability of the period-two cycle, characterized by an alternation of optimism and pessimism. We stress that in this framework rational agents do not perform as well as biased fundamentalists in terms of profits and shares due to the information cost they face, while the profits of unbiased fundamentalists are not high because their price forecasts are not precise.
When b increases further moving to Figures 9 (C) and (D), its value becomes excessive with respect to the intensity of the just described "separating" effect produced by the price adjustment mechanism. Since the reaction of the latter to production variations is no more strong enough, the determined prices are not sufficiently distant from the fundamental to be compatible with the bias and the period-two cycle is not stable anymore. Namely, rational agents become favored thanks to their perfect foresight, despite the information cost they face. According to the chosen initial condition for prices, orbits visit the chaotic attractor, when like in (C) the value of the initial condition is not distant from the bias, or they converge toward the steady state, when the value of the initial condition is too close to the fundamental or when like in (D) it is excessively large, and optimists' and pessimists' profits are too low. In particular, if the initial condition is close to the fundamental, unbiased fundamentalists perform better than rational agents, while if the initial condition is much higher than the bias then rational agents perform well and this immediately leads prices toward the steady state, where unbiased fundamentalists obtain higher profits. The latter scenario, reported in Figure 9 (D), is made possible by the unconnectedness of the basin of attraction of the steady state. Indeed, the initial condition $x_0 = 1.3$ belongs to a non-immediate component of the basin of attraction, contained in the

horizontal asymptote, located on the x-axis. We recall that without rational agents the map governing the dynamics would be decreasing, and thus the basin of attraction of the steady state would be connected.
Finally, when like in Figure 9 (E) the bias still increases, the chaotic attractor disappears since the prices determined through the adjustment mechanism are much smaller than the bias, and optimists and pessimists realize very low profits. In fact, the fundamental steady state recovers its global stability due to the even better than before performance of rational agents and of unbiased agents with respect to biased fundamentalists. In particular, in Figure 9 (E) we depict a situation in which the bias, being very large, exceeds the initial condition. Hence, rational agents obtain the highest profits, due to their more accurate forecast, and this makes prices approach the steady state, so that unbiased agents start performing well. We stress that if in (E) we started from an initial condition larger than the bias, then the convergence towards the steady state could be explained again thanks to the prevailing role played by rational agents in the beginning, witnessing however less oscillations with respect to the case depicted in (E), since the initial condition would probably belong to a non-immediate component of the basin of attraction like in (D).
We remark that the phenomena portrayed in Figure 9 (C)–(E) can not occur when rational agents are not taken into account. Indeed in framework (C), rather than irregular oscillations, without rational agents we would observe a regular pattern, like that found in (B), characterized by the alternation between a prevailing optimism or pessimism in the market, according to which group of biased fundamentalists performed better. Similarly in (D) and (E), if rational agents were not present, rather than the convergence toward the steady state, we would observe again a periodic behavior of the same kind.
Making a comparison with the role played by rational agents in the framework considered in Brock and Hommes (1997), where producers can choose between costly rational and costless naive expectations about prices, some similarities emerge, but also a few crucial differences. Namely, in that setting the emergence of chaos was explained by the existence of homoclinic orbits, which start close to the unstable steady state, move away from its neighborhood due to the destabilizing effect played by naive agents, and then return close to the steady state thanks to the stabilizing role of rational forecast. Indeed, with prices close to the steady state, naive agents perform better than rational agents in terms of realized profits because they can free ride on the rational expectations, without facing any information cost. When the intensity of choice parameter is high enough, most agents will then switch

to the cheap predictor, and this leads prices far from the equilibrium in a fluctuating manner. In the unstable phase however the profits realized by using naive expectations are low and thus, when the intensity of choice is high, most agents are willing to switch to rational expectations, despite the related information cost. In such manner prices return close to the steady state, where naive agents are favored again and the process repeats, with the alternation between the prevalence of a centrifugal force generated by the simple predictor and the prevalence of the centripetal force induced by the sophisticated predictor. Our setting shares with the Brock and Hommes (1997) framework the centripetal force induced by the rational predictor when initial conditions for prices are very far from the steady state. Close to the steady state, we witness a better performance by unbiased fundamentalists and by biased fundamentalists, too, when the bias is low. In this case, the steady state is locally or globally stable. We observe the centrifugal force produced by biased agents when their bias is moderately high. More precisely, if it is compatible with the "separating" effect produced by the price adjustment mechanism, a globally stable period-two cycle emerges, characterized by an alternation of optimism and pessimism, and rational agents do not play any role. When instead the value of the bias becomes too large with respect to the strength of the price adjustment mechanism, rational agents become favored, despite the information cost they face, and break the symmetry of the former configuration, leading to the emergence of complex dynamics. In particular, if the steady state is still locally stable, orbits visit the chaotic attractor when the initial condition is not distant from the bias value, while orbits converge toward the steady state when the initial condition is close to the fundamental or when it is excessively large. Indeed, differently from the Brock and Hommes (1997) framework in which chaos can only occur when the unique equilibrium is an unstable saddle point, we can witness the coexistence between the complex attractor and the locally stable fundamental steady state. The chaotic attractor for us persists as long as the bias value is too large to be compatible with the strength of the price adjustment mechanism, but without being excessively high, since otherwise optimists and pessimists would realize very low profits with respect to rational agents, whose prevalence would lead prices towards the fundamental, thanks to the stabilizing centripetal effect that they produce.

Hence, in order to understand when complex dynamics can arise in our setting we have to jointly take into account the intensity of the market mechanism, defined as the ratio between the demand and supply curve reactivities, the

value of the bias, and also the initial condition for prices in case of coexistence between the chaotic attractor and the locally stable steady state.
Summarizing, we have shown that, despite being locally stabilizing, globally the introduction of rational agents opens the door to complex dynamic outcomes, characterized not only by chaotic attractors, but also by rich multistability phenomena. Namely, differently from the context considered in Hommes and Wagener (2010), the basin of attraction of the steady state may be unconnected due to the presence of the horizontal asymptote, with its non-immediate components lying outside the basin of attraction of the chaotic attractor, when they coexist.

4 Conclusion

In the present contribution, following the final suggestion by Hommes and Wagener (2010) according to which "The study of the stability of evolutionary systems with many trader types in various market settings and with more complicated strategies remains an important topic for future work", we enriched the set of forecasting rules considered in that paper by assuming that the economy is populated by rational agents, too. We found that, on the one hand, their presence enlarges the steady state local stability region, but, on the other hand, they allow for the emergence of chaotic attractors and variegated multistability phenomena, since the map governing the dynamics is no more monotonically decreasing. Hence, we can say that, quite unexpectedly, rational agents may lead to complex dynamics.
We deem that the proposed setting can be the starting point for other interesting investigations.
A first natural extension of the present framework consists in considering several couples of groups of symmetrically biased fundamentalists, which differ in the strength of the bias, in view of investigating the effect of their presence on the system stability and on the possible model dynamic outcomes. We recall that we dealt with several types of biased fundamentalists also in Naimzada and Pireddu (2020b), where we introduced heterogeneous information costs in the original framework in Hommes and Wagener (2010), without encompassing rational agents.
In regard to information costs, we have here considered the simplest setting in which, like in Brock and Hommes (1997), only rational agents face a nonzero cost. However, as done in Naimzada and Pireddu (2020a), where we

dealt with just one couple of groups of symmetrically biased agents, and in Naimzada and Pireddu (2020b), with several types of biased fundamentalists, the present framework could be extended so as to encompass for all groups of agents heterogeneous information costs, that are directly proportional to their rationality degree. Indeed, in the present contribution we found that rational agents have a stabilizing effect on the steady state, while in Naimzada and Pireddu (2020a, 2020b) we discovered that introducing heterogeneous information costs for fundamentalists may have a destabilizing effect. Hence, it would be interesting to analyze the setting including both rational agents and heterogeneous information costs, in order to investigate whether the stabilizing effect of the former element or the destabilizing effect of the latter factor prevails. Such study will be the topic of a future work.
A further research question concerning rational agents would consist in the investigation of their effect in the macroeconomic setting considered in Anufriev et al. (2013), where, dealing just with biased and unbiased fundamentalists, the map linking the variable which describes the dynamical system to the expectations about it formulated by the various groups of agents is increasing, rather than decreasing like in the framework analyzed in Hommes and Wagener (2010).
Looking from a formal viewpoint at the model here proposed, two weak points emerge, that partially pertain also to the setting in Hommes and Wagener (2010). The first limit concerns the assumed symmetry in the bias of optimists and pessimists, without which the steady state would not necessarily coincide with the fundamental value. The second issue regards the formulation of the considered evolutionary mechanism, that does encompass the extinction of any group of agents neither at the steady state, nor along orbits. It would then be worth investigating how the results obtained in the present work change when dealing with one or both those improvements. In particular, we stress that, even in the presence of asymmetric biases, one of the steady states could coincide with the fundamental value if the share updating rule allowed for the extinction of some kinds of agents.
A different modification of the model, which would also lead to an increase in the number of dynamic equations describing the system, would consist in introducing into the original framework in Hommes and Wagener (2010) a group of agents endowed with naive expectations, in addition to rational agents. We recall indeed that a Muthian cobweb model with fundamentalists and naive expectations has been considered in Hommes (2013). However, to the best of our knowledge, in the literature the setting encompassing fun-

damentalists, rational agents and naive expectations has not been analyzed yet.
Another variant of the proposed setting, which would raise the number of dynamic equations describing the system, too, would consist in the introduction of memory in the share updating mechanism, so that agents, in choosing the heuristics to adopt, rather than taking into account just the most recently realized profit, would consider the performance of the various forecasting rules in terms of realized profits in the recent past.

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