A variational inequality approach to a class of network games with local complementarities and global congestion

Mauro Passacantando and Fabio Raciti

Abstract We investigate a class of network games with strategic complements and congestion effects, by using the variational inequality approach. Our contribution is twofold. We first express the boundary components of the Nash equilibrium by means of the Katz-Bonacich centrality measure. Then, we propose a new ranking of the network nodes based on the social welfare at equilibrium and compare this solution-based ranking with some classical topological ranking methods.

Key words: Network games; Nash equilibrium; network centrality measures; social welfare

1 Introduction

The topic of Network Games is relatively recent and was formulated in a general framework in the influential paper by Ballester et al. [1], where the authors proposed to model the social and economic interactions among individuals with the help of a graph where each individual (player) is identified with the node of a graph and can interact only with his/her neighbors in the graph, while congestion effects are due to all the players in the network. The solution concept considered is the Nash equilibrium of the game and is related to the so called Katz-Bonacich centrality measure, in the case of interior solution. Although the topic has grown at a high pace in the last fifteen years (see, e.g., [4]), only recently some authors have proposed to use the variational inequality approach to investigate this kind of games. In

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particular, in [9], the authors make an in-depth analysis of uniqueness and sensitivity of equilibrium, with particular emphasis on its connection with the spectral properties of the adjacency matrix of the graph, while in [8] a generalized Nash equilibrium problem is proposed within the framework of variational inequalities.

In this note, we consider a game where the generic player is influenced by his/her neighbors through a local strategic complement term and experiments a global congestion effect. We first derive a new representation formula for the Nash equilibrium, in the case where some of its components reach their upper bound, which makes use of the Katz-Bonacich vector. Then, we focus on the problem of assessing the importance of the players and propose a new centrality measure based on the social welfare computed at the Nash equilibrium.

More specifically, the paper is structured as follows. In Section 2 we first introduce the notation and the basic definitions, as well as the essential tools from variational inequality theory needed for our investigation. We then introduce the utility functions which describe a quadratic model with local complementarities and global congestion. Moreover, we recall the classical Katz-Bonacich formula for the interior solution case, where the strategy set of each player is $\mathbb{R}_+$. In Section 3, we assume that the strategy sets are bounded also from above and derive a representation formula for the solution in the case where some of its components lie on the boundary, which is based on the Katz-Bonacich centrality. In Section 4, we propose to assess the importance of a player by measuring the variation of the social welfare at equilibrium when the player is removed from the network. Moreover, we compare the ranking thus obtained with that one obtained using some classical topological measures in the literature. A small concluding section ends the paper.

2 Network games

2.1 Game Formulation and variational inequality approach

In Network Games players are represented by the nodes of a graph (V, E), where V is the sets of nodes and E is the set of edges formed by pairs of nodes (v, w). Here, we only consider undirected simple graphs. Two nodes v and w are said to be adjacent if they are connected by an edge, i.e., if (v, w) is an edge. The information about the adjacency of nodes can be stored in the adjacency matrix G whose elements g_{ij} are equal to 1 if (v_i, v_j) is an edge, 0 otherwise. G is thus a symmetric and zero-diagonal matrix. Given a node v, the nodes connected to v with an edge are called the *neighbors* of v, and are grouped in the set $N_v(g)$. The number of elements of $N_v(g)$ is the *degree* of v and will be denote by $deg_v(g)$. A *walk* in the graph g is a finite sequence of the form $v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{j_k}$, which consists of alternating nodes and edges of the graph, such that $v_{i_{t-1}}$ and v_{i_t} are end nodes of e_{j_t}. The *length* of a walk is the number of its edges. Let us remark that it is allowed to visit a node or go through an edge more than once. The indirect connections between any two nodes in the graph are described by means of the powers of the adjacency matrix G. Indeed, it can be proved that the element $g_{ij}^{[k]}$ of G^k gives the number of walks of length k between v_i and v_j. We now define some common topological measures used to assess the importance of a node i in a network:

- *degree centrality*: $DC_i = deg_i$;
- *closeness centrality*: $CC_i = \frac{1}{\sum_{j \neq i} d_{ij}}$, where d_{ij} is the shortest path length between i and j;
- *betweennes centrality*: $BC_i = \sum_{s,t \neq i} \frac{n_{st}(i)}{n_{st}}$, where n_{st} is the number of shortest paths between s and t and $n_{st}(i)$ is the number of such paths that pass through node i.

In the sequel, the set of players will be denoted by $\{1, 2, \dots, n\}$ instead of $\{v_1, v_2, \dots, v_n\}$. We denote with $A_i \subset \mathbb{R}$ the action space of player i, while $A = A_1 \times \cdots \times A_n$. For each $a = (a_1, \dots, a_n)$, $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ and the notation $a = (a_i, a_{-i})$ will be used when we want to distinguish the action of player i from the action of all the other players. Each player i is endowed with a payoff function $u_i : A \to \mathbb{R}$ that he/she wishes to maximize. The notation $u_i(a, G)$ is often utilized when one wants to emphasize that the utility of player i also depends on the actions taken by her/his neighbors in the graph.

The solution concept that we consider here is the Nash equilibrium of the game, that is, we seek an element $a^* \in A$ such that for each $i \in \{1, \dots, n\}$:

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \qquad \forall\, a_i \in A_i. \tag{1}$$

According to how variations of the actions of player's i neighbors affect his/her marginal utility, two classes of game can be defined. Specifically, the game has the property of *strategic complements* if: $\frac{\partial^2 u_i}{\partial a_j \partial a_i}(a) > 0, \ \forall (i,j) : g_{ij} = 1, \forall\, a \in A$, while it has the property of *strategic substitutes* if: $\frac{\partial^2 u_i}{\partial a_j \partial a_i}(a) < 0, \ \forall (i,j) : g_{ij} = 1, \forall\, a \in A$.

For the subsequent development it is important to recall that if the u_i are continuously differentiable functions on A, and $u_i(\cdot, a_{-i})$ are concave, the Nash equilibrium problem is equivalent to the variational inequality $VI(F, A)$: find $a^* \in A$ such that

$$F(a^*)^\top (a - a^*) \geq 0, \qquad \forall\, a \in A, \tag{2}$$

where

$$[F(a)]^\top := -\left(\frac{\partial u_1}{\partial a_1}(a), \dots, \frac{\partial u_n}{\partial a_n}(a) \right) \tag{3}$$

is also called the *pseudo-gradient* of the game. For an account of variational inequalities the reader can refer to [6, 7]. We recall here some monotonicity properties.

Definition 1 $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on A iff:

$$[T(x) - T(y)]^\top (x - y) \geq 0, \qquad \forall\, x, y \in A.$$

If the equality holds only when $x = y$, T is said to be strictly monotone. T is said to be β-strongly monotone on A iff there exists $\beta > 0$ such that

$$[T(x) - T(y)]^\top (x - y) \geq \beta \|x - y\|^2, \qquad \forall\, x, y \in A.$$

For linear operators on $\mathbb{R}^n$ the two concepts of strict and strong monotonicity coincide and are equivalent to the positive definiteness of the corresponding matrix.

Conditions that ensure the unique solvability of a variational inequality problem are given by the following theorem (see, e.g. [7]).

Theorem 1 *If $K \subset \mathbb{R}^n$ is compact and convex, and $T : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on K, then the variational inequality problem $VI(F, K)$ admits at least one solution. In the case that K is unbounded, existence of a solution may be established if the following coercivity condition holds, for $x \in K$ and some $x_0 \in K$:*

$$\lim_{\|x\| \to +\infty} \frac{[T(x) - T(x_0)]^\top (x - x_0)}{\|x - x_0\|} = +\infty.$$

Furthermore, if T is strictly monotone on K the solution is unique.

2.2 The linear-quadratic model with local complementarities and global congestion

Let $A_i = \mathbb{R}_+$ for any $i \in \{1, \ldots, n\}$, hence $A = \mathbb{R}^n_+$. The payoff of player i is given by:

$$u_i(a, G) = \alpha a_i - \frac{1}{2} a_i^2 + \phi \sum_{j=1}^{n} g_{ij} a_i a_j - \gamma \sum_{j=1}^{n} a_i a_j, \qquad \alpha, \phi, \gamma > 0. \tag{4}$$

In this model α and ϕ take on the same value for all players, which then only differ according to their position in the network. The third term describes the interaction between neighbors and since $\phi > 0$ this interaction falls in the class of strategic complements. On the other term, since $\gamma > 0$ the last term falls in the class of strategic substitutes and models the overall congestion effects in the network. Thus, without further hypotheses, this model does not belong to any of the above mentioned class. The pseudo-gradient's components of this game are easily computed as: $F_i(a) = (1 + \gamma) a_i - \alpha - \phi \sum_{j=1}^{n} g_{ij} a_j + \gamma \sum_{j=1}^{n} a_j,\ \ i \in \{1, \ldots, n\}$, which can be written in compact form as $F(a) = [(1 + \gamma) I - \phi G + \gamma U] a - \alpha \mathbf{1}$, where $U_{ij} = 1$ for any $i, j = 1, \ldots, n$ and $\mathbf{1} = (1, \ldots, 1)^\top \in \mathbb{R}^n$. We will seek Nash equilibrium points by solving the variational inequality:

$$F(a^*)^\top (a - a^*) \geq 0, \qquad \forall\, a \in \mathbb{R}^n_+. \tag{5}$$

Since the constraint set is unbounded, to ensure solvability we require that F be strongly monotone, which also guarantees the uniqueness of the solution. In the next lemma we recall a well known result about matrices.

Lemma 1 *Let T be a symmetric matrix and $\rho(T)$ the spectral radius of T. If $\rho(T) < 1$, then the matrix $I - T$ is positive definite and $(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$.*

We now give an important definition for our analysis.

Definition 2 For any weight $w \in \mathbb{R}_+^n$, and $\phi > 0$ the weighted vector of Katz-Bonacich [2] for the network G is given by:

$$b_w(G, \phi) = M(G, \phi) = (I - \phi G)^{-1} w = \sum_{p=0}^{\infty} \phi^p G^p w. \tag{6}$$

In the case where $w = \mathbf{1}$, the (non weighted) centrality measure of Katz-Bonacich of node i can be interpreted as the total number of walks in the graph, which start at node i, exponentially damped by ϕ. The connection between the Katz-Bonacich vector and the Nash equilibrium is given in the following theorem.

Theorem 2 (see Theorem 1 in [1])

If $\phi\rho(G) < 1 + \gamma$, then the unique Nash equilibrium of the game with utility functions (4) *and $A = \mathbb{R}_+^n$ is interior and given by:*

$$a^* = \frac{\alpha}{1 + \gamma + \gamma \sum_{i=1}^{n} \left(b_{\mathbf{1}} \left(G, \frac{\phi}{1+\gamma} \right) \right)_i} b_{\mathbf{1}} \left(G, \frac{\phi}{1+\gamma} \right). \tag{7}$$

For the subsequent developments we also need to define the social welfare:

$$W(a, G) = \sum_{i=1}^{n} u_i(a, G). \tag{8}$$

3 A Katz-Bonacich representation formula

We now assume that the strategies of each player have an upper bound and derive a Katz-Bonacich type representation of the solution, in the case where exactly k components take on their maximum value.

Theorem 3 *Let u_i be defined as in* (4)*, $a_i \in [0, L_i]$ for any $i \in \{1, \ldots, n\}$, and $\phi\rho(G) < 1 + \gamma$. Then there exists a unique Nash equilibrium a^* of the game and $a_i^* > 0$ holds for any $i \in \{1, \ldots, n\}$. Moreover, assume that exactly k components of a^* take on their maximum value: $a_{i_1}^* = L_{i_1}, \ldots, a_{i_k}^* = L_{i_k}$, and denote with $\tilde{a}^* = (\tilde{a}_{i_{k+1}}^*, \ldots, \tilde{a}_{i_n}^*)$ the subvector of the nonboundary components of a^*. We then get:*

$$\tilde{a}^* = \left(\frac{1}{1+\gamma} \right) b_{\mathbf{w}} \left(G_1, \frac{\phi}{1+\gamma} \right)$$

$$- \left(\frac{\gamma}{1+\gamma}\right) \frac{\sum\limits_{m=k+1}^{n} \left(b_w\left(G_1, \frac{\phi}{1+\gamma}\right)\right)_{i_m}}{1+\gamma+\gamma \sum\limits_{m=k+1}^{n} \left(b_{\mathbf{1}_{n-k}}\left(G_1, \frac{\phi}{1+\gamma}\right)\right)_{i_m}} \, b_{\mathbf{1}_{n-k}}\left(G_1, \frac{\phi}{1+\gamma}\right), \qquad (9)$$

where G_1, G_2, U_1, U_2 are submatrices of G and U defined as follows:

$$G = \begin{array}{c} \\ i_1 \\ \vdots \\ i_k \\ i_{k+1} \\ \vdots \\ i_n \end{array} \begin{array}{c} i_1 \; \dots \; i_k \mid i_{k+1} \; \dots \; i_n \\ \left(\begin{array}{c|c} * & * \\ \hline G_2 & G_1 \end{array}\right) \end{array}, \qquad U = \begin{array}{c} \\ i_1 \\ \vdots \\ i_k \\ i_{k+1} \\ \vdots \\ i_n \end{array} \begin{array}{c} i_1 \; \dots \; i_k \mid i_{k+1} \; \dots \; i_n \\ \left(\begin{array}{c|c} * & * \\ \hline U_2 & U_1 \end{array}\right) \end{array},$$

$w = [\alpha \mathbf{1}_{n-k} + (\phi G_2 - \gamma U_2)\, L]/(1+\gamma)$ *and* $L = (L_{i_1}, \dots, L_{i_k})^\top$.

Proof Let us notice that the matrix γU is positive semidefinite and $(1+\gamma)I - \phi G$ is positive definite by Lemma 1, hence the map F is strongly monotone on $\mathbb{R}^n$ and the game has a unique Nash equilibrium a^*, which solves the variational inequality

$$F(a^*)^\top (a - a^*) \geq 0, \qquad \forall\, a \in K, \qquad (10)$$

where $K = [0, L_1] \times \ldots \times [0, L_n]$. Let us notice that $a^* \neq 0$, otherwise we have $0 \leq -\alpha \mathbf{1}^\top a$ holds for any $a \in K$, that is impossible. Define the set $S \subseteq \{1, \dots, n\}$ such that $a_i^* > 0,\ \forall i \in S$, $a_i^* = 0,\ \forall i \notin S$. We then get that a^* solves the KKT system associated with $VI(F; K)$:

$$\begin{array}{lll} [(1+\gamma)I - \phi G + \gamma U]a^* - \alpha \mathbf{1} - \lambda + \mu = 0, & & \\ \lambda_i a_i^* = 0, & \lambda_i \geq 0, & i = 1, \dots, n, \\ \mu_i^*(a_i^* - L_i) = 0, & \mu_i^* \geq 0, & i = 1, \dots, n, \end{array}$$

which implies:

$$(1+\gamma)a_i^* - \phi \sum_{j \in S} g_{ij} a_i^* + \gamma \sum_{j \in S} a_i^* - \alpha + \mu_i = 0, \qquad \forall\, i \in S, \qquad (11)$$

$$- \phi \sum_{j \in S} g_{ij} a_i^* - \alpha + \gamma \sum_{j \in S} a_i^* - \lambda_i = 0, \qquad \forall\, i \notin S. \qquad (12)$$

We then get: $((1+\gamma)I_S - \phi G_S)a_S^* = (\alpha - \gamma \sum_{j=1}^n a_j^*)\mathbf{1}_S - \mu_S$, and because $\phi\rho(G_S) \leq \phi\rho(G) < 1+\gamma$, we also have $((1+\gamma)I_S - \phi G_S)^{-1} \geq 0$, hence:

$$0 < a_S^* = \left(\alpha - \gamma \sum_{j=1}^n a_j^*\right)((1+\gamma)I_S - \phi G_S)^{-1}\mathbf{1}_S - ((1+\gamma)I_S - \phi G_S)^{-1}\mu_s$$

$$\leq \left(\alpha - \gamma \sum_{j=1}^{n} a_j^* \right) ((1+\gamma) I_S - \phi G_S)^{-1} \mathbf{1}_S,$$

which implies $\alpha - \gamma \sum_{j=1}^n a_j^* > 0$. If there exists an index $i \notin S$, then from (12) we get the contradiction: $0 < \alpha - \gamma \sum_{j=1}^n a_j^* = -\phi \sum_{j \in S} g_{ij} a_i^* - \lambda_i \leq 0$. Therefore, $a_i^* > 0$ holds for any $i \in \{1, \dots, n\}$.

Let $\tilde{K}$ denote the face of K obtained intersecting K with the hyperplanes: $a_{i_1} = L_{i_1}, \dots, a_{i_k} = L_{i_k}$. Moreover, let $\tilde{a} = (a_{i_{k+1}}, \dots, a_{i_n})$, $\tilde{a}^* = (\tilde{a}^*_{i_{k+1}}, \dots, \tilde{a}^*_{i_n})$ and define $\tilde{F} : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ such that $\tilde{F}_{i_l}(\tilde{a})$ is obtained by fixing $a_{i_1} = L_{i_1}, \dots, a_{i_k} = L_{i_k}$ in $F_{i_l}(a)$. We consider now the restriction of (10) to $\tilde{K}$, which reads:

$$\sum_{l=k+1}^{n} \tilde{F}_{i_l}(\tilde{a}^*)(\tilde{a}_{i_l} - \tilde{a}^*_{i_l}) \geq 0, \qquad \forall\, \tilde{a} \in \tilde{K}.$$

Since we are assuming that exactly k components of the solution a^* reach their upper bounds, it follows that $\tilde{a}^*$ lies in the interior of $\tilde{K}$, hence $\tilde{F}(\tilde{a}^*) = 0$, that is equivalent to

$$(1+\gamma) a^*_{i_l} - \phi \sum_{m=k+1}^{n} g_{i_l i_m} a^*_{i_m} + \gamma \sum_{m=k+1}^{n} a^*_{i_m} = \alpha + \phi \sum_{m=1}^{k} g_{i_l i_m} L_{i_m} - \gamma \sum_{m=1}^{k} L_{i_m},$$

for any $l = k+1, \dots, n$, which yields $[(1+\gamma) I_{n-k} - \phi G_1 + \gamma U_1] \tilde{a}^* = \alpha \mathbf{1}_{n-k} + \phi G_2 L - \gamma U_2\, L$, which can be written as

$$\left[I_{n-k} - \frac{\phi}{1+\gamma} G_1 + \frac{\gamma}{1+\gamma} U_1 \right] \tilde{a}^* = \frac{1}{1+\gamma} w,$$

with $w = \alpha \mathbf{1}_{n-k} + (\phi G_2 - \gamma U_2) L$. To derive $\tilde{a}^*$ let us first notice that $U_1 \tilde{a}^* = \left(\sum_{m=k+1}^{n} \tilde{a}^*_{i_m} \right) \mathbf{1}_{n-k}$ and the matrix $(I_{n-k} - \frac{\phi}{1+\gamma} G_1)$ is not singular because $\frac{\phi}{1+\gamma} \rho(G_1) < 1$, thus we get:

$$\tilde{a}^* = \frac{1}{1+\gamma} \left[I_{n-k} - \frac{\phi}{1+\gamma} G_1 \right]^{-1} w - \frac{\gamma}{1+\gamma} \left(\sum_{m=k+1}^{n} \tilde{a}^*_{i_m} \right) \left[I_{n-k} - \frac{\phi}{1+\gamma} G_1 \right]^{-1} \mathbf{1}_{n-k}, \tag{13}$$

which, from the definition of the Katz-Bonachich vector (6), yields:

$$(1+\gamma)\, \tilde{a}^* + \gamma \left(\sum_{m=k+1}^{n} \tilde{a}^*_{i_m} \right) b_{\mathbf{1}_{n-k}} \left(G_1, \frac{\phi}{1+\gamma} \right) = b_w \left(G_1, \frac{\phi}{1+\gamma} \right), \tag{14}$$

which can be exploited to derive $\sum_{m=k+1}^{n} \tilde{a}^*_{i_m}$. Indeed, summing up on the components of both the left and right handside of (14) we get:

$$\sum_{m=k+1}^{n} \tilde{a}^*_{i_m} = \frac{\sum\limits_{m=k+1}^{n} (b_w(G_1, \frac{\phi}{1+\gamma}))_{i_m}}{1+\gamma+\gamma \sum\limits_{m=k+1}^{n} (b_{\mathbf{1}_{n-k}}(G_1, \frac{\phi}{1+\gamma}))_{i_m}}. \tag{15}$$

Inserting (15) into (13), we finally obtain (9) and the proof is completed. □

4 A social welfare centrality measure

In this section we propose a new centrality measure of nodes (players) of a network game based on the social welfare computed at the Nash equilibrium. Specifically, for any node $i \in \{1, \dots, n\}$, we measure the importance of i as the percentage variation of the social welfare computed at the Nash equilibrium after i is removed from the network, that is

$$SWC(i) = 100 \cdot \frac{W(NE(G)) - W(NE(G \setminus \{i\}))}{W(NE(G))}, \tag{16}$$

where $NE(G)$ is the Nash equilibrium in the network G, $NE(G \setminus \{i\})$ is the Nash equilibrium in the network G where node i has been removed, and W is the social welfare function (8). Note that $SWC(i)$ can be negative if the total social welfare of the network increases after removing the node i (as Example 1 below shows). This situation can be compared to the well-known Braess paradox [3], where the efficiency of a network improves due to the removal of a link.

Example 1. We consider the network game on the small network shown in Fig. 1 with three nodes and two links. We assume that the game parameters are $\alpha = 1$, $\phi = 2$, $\gamma = 3$ and $a_i \in [0, 1]$. Note that $\rho(G) = \sqrt{2}$ so that Theorem 3 guarantees the existence and uniqueness of the Nash equilibrium in the original network and the sub-networks obtained by removing one node at a time. The Nash equilibrium in the network G is $(2/17, 3/34, 3/34)$ and the corresponding social welfare is equal to $7/68$. When the node 1 is removed, the network becomes disconnected and the total social welfare at equilibrium decreases to 7/100. On the other hand, when node 2 or 3 is removed, the social welfare at equilibrium increases to 7/64, hence the social welfare centrality of nodes 2 and 3 takes on negative values. More precisely, we have $SWC = (32, -25/4, -25/4)$.

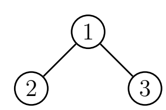

Fig. 1 Network topology of Example 1.

Example 2. We now compare the social welfare centrality with three well-known topological centrality measures: degree, closeness and betweenness. We consider a network game on a graph with 10 nodes, where the adjacency matrix G has been randomly generated (see Fig. 2), and the game parameters are $\alpha = 1$, $\gamma = 1$, $\phi = 0.9(1+\gamma)/\rho(G)$ and $a_i \in [0, 1]$ for any $i = 1, \ldots, 10$. Table 1 shows the ranking of nodes according to the social welfare centrality measure and the three considered topological centrality measures. It is interesting noting that the ranking defined by the new measure is quite different from that provided by the other measures. Moreover, Fig. 2 gives a graphical representation of the values associated to the network nodes for each considered centrality measure.

Table 1 Ranking of nodes according to the new social welfare centrality measure and the well known degree, closeness and betweenness centrality measures.

	Centrality measures			
Rank	Social Welfare	Degree	Closeness	Betweenness
1	2	7	7	7
2	9	2	2	2
3	4	9	9	3
4	7	3	3	9
5	6	1	4	1
6	10	4	1	8
7	3	5	5	4
8	5	6	6	5
9	1	8	8	6
10	8	10	10	10

5 Conclusions and further research perspectives

In future work, we aim to apply our results to some specific social or economic problems. An interesting account of these applications can be found in the survey [4]. Also the theory of stochastic variational inequalities [5] could be used to cope with uncertain parameters in the model.

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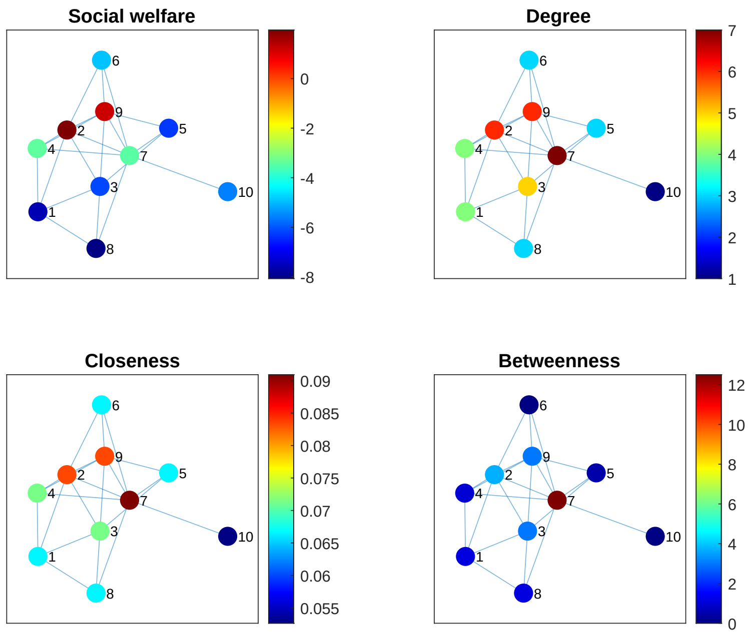

Fig. 2 Comparison between the social welfare centrality measure and degree, closeness and betweenness centrality measures.

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