On some generalizations of Hadamard's inversion theorem beyond differentiability

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ABSTRACT
A recognized trend of research investigates generalizations of the Hadamard's inversion theorem to functions that may fail to be differentiable. In this vein, the present paper explores some consequences of a recent result about the existence of global Lipschitz continuous inverse by translating its assumptions of metric nature in terms of nonsmooth analysis constructions. This exploration focuses on continuous, but possibly not locally Lipschitz mappings, acting in finite-dimensional Euclidean spaces. As a result, sufficient conditions for global invertibility are formulated by means of strict estimators, $\stackrel{*}{-}$-difference of convex compacta and regular/basic coderivatives. These conditions qualify the global inverse as a Lipschitz continuous mapping and provide quantitative estimates of its Lipschitz constant in terms of the above constructions.

KEYWORDS
Global inversion; linear openness; metric injectivity; strong regularity; strict μ-estimator; difference of convex compacta; coderivatives.

1. Introduction

The celebrated Hadamard's inversion theorem establishes a connection between the global invertibility of a mapping and certain differential properties of it. Global as well as local invertibility is a key property that mappings may have, widely employed in various contexts often in synergy with more structured properties (see, for instance, [5]). Such a property can be considered in highly abstract settings, because in its essence it is free from references to specific structures, such as the topological, the metric and the differential ones. This freedom however may result in a poor interplay with the latter ones, provided that they are at disposal, as illustrated by the example below.

Example 1.1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = x \cdot \chi_{\mathbb{Q}}(x) - x \cdot \chi_{\mathbb{R}\setminus\mathbb{Q}}(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ -x, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

If $\mathbb{R}$ is equipped with its usual Euclidean space structure, it is clear that f is nowhere

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differentiable, while is continuous only at 0, yet it is globally invertible, namely there exists $f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ (which is given, in the present case, by f itself).

That said, it must be recognized that the study of invertibility conditions took benefits from tools of differential calculus and linear algebra. It is well known indeed that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuously differentiable in a neighbourhood of a point $\bar{x} \in \mathbb{R}^n$ and its (Fréchet) derivative $\mathrm{D}f(\bar{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ at $\bar{x}$ is invertible (the Jacobian matrix $\mathrm{J}f(\bar{x})$ representing the linear mapping $\mathrm{D}f(\bar{x})$ is nonsingular), then f admits a local inverse, which is continuously differentiable in a neighbourhood of $f(\bar{x})$. This classic result was subsequently generalized to contexts of possible lack of differentiability in several ways (see, among the others, [2], [4, Theorem 1E.3], [11, Theorem 3.3.1]).

Since derivatives (or their surrogates) provide only a local approximation of a mapping, the above result is expected to hold only locally, as it actually does. The great advance made with the Hadamard's theorem consists in providing a sufficient condition for the global invertibility of a mapping in terms of derivatives. Its statement is recalled below.

Theorem 1.2 (Hadamard's inversion theorem)**.** *Given a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, suppose that:*

(i) $f \in \mathrm{C}^1(\mathbb{R}^n)$*;*
(ii) for every $x \in \mathbb{R}^n$ *there exists* $\mathrm{D}f(x)^{-1}$*;*
(iii) there exists $\kappa > 0$ *such that* $\sup_{x\in\mathbb{R}^n} \|\mathrm{D}f(x)^{-1}\|_{\mathcal{L}} \leq \kappa$*.*

Then f is a global diffeomorphism, namely:

(t) $\exists f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$*;*
(tt) f^{-1} *is (continuously) differentiable on* $\mathbb{R}^n$*.*

A further step in providing more general sufficient conditions for the global invertibility of a mapping has been done in the early 80s with the appearance of [17], when the unnecessary smoothness assumption on f was dropped out. Its role was replaced by Lipschitz continuity and a proper condition on the Clarke subdifferential of the mapping. Recall that a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be Lipschitz continuous with constant $\ell > 0$ in a subset $U \subseteq \mathbb{R}^n$ if

$$\|f(x_1) - f(x_2)\| \leq \ell\|x_1 - x_2\|, \quad \forall x_1, x_2 \in U. \tag{1}$$

Whenever inequality (1) holds in a neighbourhood of a given point $\bar{x}$, f is said to be locally Lipschitz around $\bar{x}$. In such an event, the infimum over all ℓ, for which there exists a neighbourhood U of $\bar{x} \in \mathbb{R}^n$ such that (1) holds, will be denoted throughout the paper by $\mathrm{lip}(f; \bar{x})$. According to the Rademacher's theorem, a mapping f which is locally Lipschitz around $\bar{x}$ turns out to be Fréchet differentiable in a (Lebesgue) full measure subset of a neighbourhood of $\bar{x}$. Thus, it is possible to define

$$\partial^\circ f(\bar{x}) = \mathrm{conv}\,\{M \in \mathcal{M}_{m\times n}(\mathbb{R}) \mid M = \lim_{k\to\infty} \mathrm{J}f(x_k), \text{ for some } \{x_k\}_{k\in\mathbb{N}},\ x_k \notin \Omega_f\},$$

where Ω_f denotes the null measure subset of a proper neighbourhood of $\bar{x}$ containing all those points at which f is not Fréchet differentiable and $\mathrm{conv}\, S$ denotes the convex closure of the set S.

Theorem 1.3 (Pourciau's inversion theorem). *Given a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, suppose that:*

(i) f *is locally Lipschitz around each point* $x \in \mathbb{R}^n$*;*
(ii) for every $x \in \mathbb{R}^n$ *and for every* $M \in \partial^\circ f(x)$*, there exists* M^{-1}*;*
(iii) there exists $\kappa > 0$ *such that* $\sup_{x\in\mathbb{R}^n} \sup_{M\in\partial^\circ f(x)} \|M^{-1}\| \leq \kappa$*.*

Then f *is a global homeomorphism, namely:*

(t) $\exists f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$*;*
(tt) f^{-1} *is Lipschitz continuous on* $\mathbb{R}^n$ *with constant* κ*.*

Since then, the study of conditions for global invertibility remained in the agenda of several groups of researchers. A recent trend of investigation is exploring the potential of concepts emerged in modern variational analysis properly combined with nonsmooth analysis constructions in deriving new generalizations of Hadamard's theorem (see, among the others, [1,7,8,10]).

The purpose of the present paper is to carry on such a trend, sheding light on some possible combinations, which seem not to have been gleaned yet. More precisely, following existent approaches, this is done by analyzing local invertibility through metric regularity, metric injectivity and strong regularity and their stability properties under perturbation. Such an approach leads to replace generalized derivatives with looser approximation concepts (strict μ-estimators or coderivatives), which can be exploited in formulating general conditions for global invertibility.

The contents of the paper are arranged as follows. In Section 2 a general scheme for deriving conditions for global inversion of possibly nonsmooth mappings is presented. It stems from a recent global inversion result relying on variational analysis concepts of purely metric nature. This approach leads to formulate two generalizations of Hadamard's theorem, where derivatives are replaced by strict estimators. In Section 3 a result obtained in the preceding section is specialized to the mappings admitting strict estimators, which are dually representable by means of pairs of convex compacta. These representations are known to enjoy a rich calculus. Section 4 comes back to a more general setting, providing a global invertibility condition, where the role of derivatives is played by regular/basic coderivatives. Conclusions are gathered in Section 5.

The notations in use throughout the paper are standard. In an Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, $\| \cdot \|$ stands for the Euclidean norm, while $\mathrm{B}[x; r]$ denotes the closed ball, with center at x and radius r, while $\mathrm{B}(x; r)$ stands for its open counterpart. The unit sphere centered at the null vector $\mathbf{0} \in \mathbb{R}^n$ is indicated by $\mathbb{S}$. The metric enlargement with radius r of a set $S \subseteq \mathbb{R}^n$ is denoted by $\mathrm{B}(S; r) = \{x \in \mathbb{R}^n \mid \mathrm{dist}\,(x, S) < r\}$, where $\mathrm{dist}\,(x, S) = \inf_{z\in S} \|z - x\|$ is the distance of x from S. $\mathcal{S}(\mathbb{R}^n)$ denotes the cone of all sublinear (i.e. p.h. and convex) functions defined on $\mathbb{R}^n$, whereas $\mathcal{DS}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) - \mathcal{S}(\mathbb{R}^n)$. $\mathcal{K}(\mathbb{R}^n)$ is the class of all nonempty convex compact subsets of $\mathbb{R}^n$. By $\varsigma\,(\cdot; S)$ the support function associated to the subset $S \subseteq \mathbb{R}^n$ is denoted. If $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ is a convex function, $\partial\varphi(x)$ stands for its subdifferential at $x \in \mathrm{dom}\,\varphi = \varphi^{-1}(\mathbb{R})$ in the sense of convex analysis, whereas if φ is concave $\partial^+\varphi(x)$ is the superdifferential of φ at x. If $l : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear mapping, $\ker l$, $\mathrm{im}\, l$ and $\|l\|_{\mathcal{L}}$ indicate the kernel, the range and the operator norm of l, respectively. If $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued mapping, $\mathrm{gph}\, F$ denotes the graph of F. The acronyms p.h. and u.s.c. stand for positively homogeneous and upper semicontinuous, respectively. Further special notations will be introduced subsequently, contextually to their use.

2. A general scheme for global inversion without derivatives

The analysis here exposed starts with a recent global invertibility result established in [1, Theorem 2.2], which relies on a suitable metric behaviour of mappings, disregarding their differential properties. Such a metric behaviour deals with a covering property formalized as follows.

Definition 2.1. Given $\alpha > 0$ and $x \in \mathbb{R}^n$, a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be *α-covering at x* if for every $\epsilon > 0$ there exists $r \in (0, \epsilon]$ such that

$$\mathrm{B}[f(x); \alpha r] \subseteq f(\mathrm{B}[x; r]). \tag{2}$$

The value

$$\mathrm{cov}(f; x) = \sup\{\alpha > 0 \mid \text{ for which (2) holds}\}$$

is called *covering bound* of f at x. If inclusion (2) holds true for all $x \in \mathbb{R}^n$ and $r \geq 0$, with the same value of α, then f will be called *α-covering*.

In what follows, a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be locally injective provided that every $x \in \mathbb{R}^n$ admits a neighbourhood in which f is injective.

Theorem 2.2 ([1]). *Given a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, suppose that:*

(i) f is continuous on $\mathbb{R}^n$;
(ii) f is locally injective;
(iii) f is α-covering at each $x \in \mathbb{R}^n$.

Then

(t) $\exists f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$;
(tt) f^{-1} is Lipschitz continuous on $\mathbb{R}^n$ with constant α^{-1}.

It is worth noticing that Theorem 2.2 avoids any reference to (traditional or generalized) differential properties of mappings, while focusing on basic, yet more than set-theoretic (topological and metric) properties, namely continuity, injectivity and metric covering. A related feature is that its thesis is not only qualitative (f is a homeomorphism), but it provides quantitative estimates for the Lipschitz constant of f^{-1}. In the above theorem, a key role is played by hypothesis (iii), which is indeed a metric surjection assumption. The property of being α-covering at a given point is a weaker (point-based) variant of the linear openness (a.k.a. openness at a linear rate), which is well-known and largely employed in variational analysis. Recall that a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be linearly open around a point $\bar{x} \in \mathbb{R}^n$ if there exists a constant $\alpha > 0$ together with neighbourhoods U of $\bar{x}$ and V of $f(\bar{x})$ such that

$$\mathrm{B}(f(x); \alpha r) \cap V \subseteq f\left(\mathrm{B}(x; r)\right), \quad \forall x \in \ U, \forall r > 0. \tag{3}$$

The quantity

$$\mathrm{lop}(f; \bar{x}) = \sup\{\alpha > 0 \mid \exists U,\ V \text{ for which (3) holds}\}$$

is called (exact) linear openness bound of f around $\bar{x}$ (see [4,13,14]).

Remark 1. If a continuous mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linearly open around $\bar{x}$, then for any $\alpha < \mathrm{lop}(f;\bar{x})$ it is also α-covering at $\bar{x}$ in the sense of Definition 2.1, and it is true that $\mathrm{lop}(f;\bar{x}) \le \mathrm{cov}(f;\bar{x})$. On the other hand, if the property of being α-covering holds true at each point of $\mathbb{R}^n$ with the same $\alpha > 0$, then it implies linear openness around each point x of $\mathbb{R}^n$, with $\mathrm{lop}(f;x) \ge \alpha$. Indeed, as done at the beginning of the proof of [1, Theorem 2.2] by means of a Caristi-like condition (see [1, Lemma 4.2]), it is possible to prove that, for a continuous mapping, α-covering at each point of $\mathbb{R}^n$ actually implies α-covering.

The concept of linear openness turned out to be highly fruitful for the development of several topics in variational and nonlinear analysis, in particular under its equivalent reformulation in terms of metric regularity. It is indeed known that f is linearly open around $\bar{x}$ iff there exists $\kappa > 0$ together with a neighbourhood U of $\bar{x}$ and V of $f(\bar{x})$ such that

$$\mathrm{dist}\left(x, f^{-1}(y)\right) \le \kappa \|y - f(x)\|, \quad \forall (x,y) \in U \times V.$$

The infimum over all κ for which there exist U and V satisfying the above inequality is called (exact) metric regularity bound of f around $\bar{x}$ and will be denoted throughout the paper by $\mathrm{reg}(f;\bar{x})$. In other terms, both linear openness and metric regularity are seemingly different manifestations of the same metric behaviour of a mapping. The linear openness bound and the metric regularity bound are known to be intertwined by the sharp relation

$$\mathrm{lop}(f;\bar{x}) \cdot \mathrm{reg}(f;\bar{x}) = 1, \tag{4}$$

with the conventions that $0 \cdot \infty = 1$, while $\mathrm{lop}(f;\bar{x}) = 0$ and $\mathrm{reg}(f;\bar{x}) = \infty$ indicate the lack of the related property (see [4, Theorem 3E.6], [14, Theorem 3.2]).

In its essence, differentiability of a mapping amounts to be suitable for being locally approximated by linear mappings. Nonsmooth analysis went beyond this property, by admitting relaxed forms of approximation (via directional restrictions or metric estimates) and by removing the linearity of the approximating terms. Among several possible ways to accomplish this (see [3,11,13,15]) in [4, Chapter 1.5] this approach resulted in the following notion.

Definition 2.3 (Strict μ-estimator)**.** Consider a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, a point $\bar{x} \in \mathbb{R}^n$ and $\mu \ge 0$. A function $h_{\bar{x}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be a *strict μ-estimator* of f at $\bar{x}$ if

(i) $h_{\bar{x}}(\bar{x}) = f(\bar{x})$;
(ii) $\mathrm{lip}(f - h_{\bar{x}};\bar{x}) \le \mu$.

More than generalized derivatives, strict μ-estimators provide a scheme for building various kind of controlled metric perturbations of a given mapping. In the example below, some specific implementations of this concept are presented.

Example 2.4 (Strict first-order approximation)**.** Following [4, Chapter 1.5], given a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a point $\bar{x} \in \mathbb{R}^n$, a mapping $\mathrm{A}f(\bar{x};\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be a strict first-order approximation of f at $\bar{x}$ if $\mathrm{A}f(\bar{x};\bar{x}) = f(\bar{x})$ and

$$\mathrm{lip}(f - \mathrm{A}f(\bar{x};\cdot);\bar{x}) = 0. \tag{5}$$

These conditions obviously imply that any strict first-order approximation of f at $\bar{x}$ is, in particular, a strict μ-estimator of f at $\bar{x}$, for every $\mu \geq 0$. It is worth observing that the axiomatic notion of strict first-order approximation can cover several classic and generalized (less classic) situations of differentiability. For instance, if $\mathrm{A}f(\bar{x};\cdot)$ can be taken in the class of all affine mappings, then, whenever f is strictly differentiable at $\bar{x}$, by setting

$$\mathrm{A}f(\bar{x};x) = f(\bar{x}) + \mathrm{D}f(\bar{x})[x - \bar{x}]$$

one sees that its first-order expansion at $\bar{x}$, expressed by its Fréchet derivative, falls in the general scheme of strict first-order approximation.

Another specific instance of strict first-order approximation is given by the notion of strong Bouligand derivative. According to [6, Definition 3.1.2], $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is strongly Bouligand differentiable at $\bar{x}$ if it is locally Lipschitz around $\bar{x}$ and condition (5) holds with

$$\mathrm{A}f(\bar{x};x) = f(\bar{x}) + f'(\bar{x};x - \bar{x}),$$

where $f'(\bar{x};v)$ denotes the directional derivative of f at $\bar{x}$, in the direction $v \in \mathbb{R}^n$.

Let us further mention that strict first-order approximations appeared in a more general setting (normed linear spaces) already in [18] under the name of strong approximations.

Remark 2. (i) From Definition 2.3(ii) it follows that if $h_{\bar{x}}$ (resp. f) is locally Lipschitz around $\bar{x}$, then f (resp. $h_{\bar{x}}$) inherits the same property around $\bar{x}$. Nevertheless, it may happen that mappings failing to be locally Lipschitz around a reference point do admit estimators at that point, of course affected by the same lack of Lipschitz continuity (see Example 2.17).

(ii) For the subject under investigation, the question of existence of μ-estimators for a given mapping at a given point becomes a relevant issue. Example 2.4 and the point (i) of the present remark should offer perspectives for addressing this question. Obviously, whenever f is locally Lipschitz around $\bar{x}$, it admits $h_{\bar{x}} \equiv 0$ as a μ-estimator at $\bar{x}$, for every $\mu < \mathrm{lip}(f;\bar{x})$. On the other hand, in any case f admits $h_{\bar{x}} = f$ as a μ-estimator at $\bar{x}$, for every $\mu \geq 0$ and $\bar{x} \in \mathbb{R}^n$. The contexts of employment should determine the convenience of choices to be made, through specific requirements on the class $\{h_x : \ x \in \mathbb{R}^n\}$.

The next proposition justifies the employment of μ-estimators in the current approach by asserting the stability behaviour of linear openness/metric regularity in the presence of additive Lipschitz perturbations, which was independently observed by A.A. Milyutin and S.M. Robinson. Actually, it captures the quantitative stability phenomenon behind many recent extensions of the celebrated Lyusternik-Graves theorem. As presented below, it is a reformulation of [4, Theorem 3F.1], adapted in the light of the relation (4).

Proposition 2.5 (Linear openness via strict estimators)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a given mapping, let $\bar{x} \in \mathbb{R}^n$, and $\mu \geq 0$. Suppose that:*

(i) f admits a strict μ-estimator $h_{\bar{x}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at $\bar{x}$;
(ii) $\mathrm{lop}(h_{\bar{x}};\bar{x}) > \mu$.

Then, f is linearly open around $\bar{x}$, with $\mathrm{lop}(f;\bar{x}) \geq \mathrm{lop}(h_{\bar{x}};\bar{x}) - \mu$. *In particular, f*

is α-covering at $\bar{x}$ with any $\alpha < \mathrm{lop}(h_{\bar{x}}; \bar{x}) - \mu$.

The impact of Proposition 2.5 becomes clear when studying linear openness of $h_{\bar{x}}$ is easier than studying the same property of f, by virtue of the specific features of $h_{\bar{x}}$ (e.g. linearity, positive homogeneity, some form of convexity property for vector-valued mappings, and so on).

Together with surjection properties, an enhanced (again, metric) form of injectivity will be employed in the sequel. To the best of the author's knowledge, such a property was first formalized as follows in [7], wherefrom the terminology is borrowed.

Definition 2.6. A mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be *metrically injective* around $\bar{x} \in \mathbb{R}^n$ if there exist constants β, $\delta > 0$ such that

$$\|f(x_1) - f(x_2)\| \geq \beta \|x_1 - x_2\|, \quad \forall x_1,\, x_2 \in \mathrm{B}[\bar{x}; \delta]. \tag{6}$$

The value

$$\mathrm{inj}(f; \bar{x}) = \sup\{\beta > 0 \mid \exists \delta > 0 \text{ for which (6) holds}\}$$

is called (exact) *metric injection bound* of f around $\bar{x}$.

Remark 3. (i) It is plain to see that, whenever a mapping f is metrically injective around a point, it must be locally injective around that point, whereas the converse implication fails to be true, in general (see Example 2.10).

(ii) Notice that the metric injection bound of a mapping f around $\bar{x}$ can be expressed in terms of displacement rate of the values taken by f as follows

$$\mathrm{inj}(f; \bar{x}) = \liminf_{\substack{x_1,\, x_2 \to \bar{x} \\ x_1 \neq x_2}} \frac{\|f(x_1) - f(x_2)\|}{\|x_1 - x_2\|}.$$

Example 2.7 (Characterization of metric injectivity for linear mappings)**.** Let $l : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear mapping. As a preliminary remark about metric injectivity of linear mappings, notice that, because this property implies mere injectivity and hence the relation $\dim \ker l + \dim \mathrm{im}\, l = 0 + \dim \mathrm{im}\, l = n$ must be true, then a necessary dimensional condition for metric injectivity of linear mapping to hold is that $m \geq n$.

Then, the following are equivalent:

(i) l is metrically injective all over $\mathbb{R}^n$ (i.e. inequality (6) holds with $\delta = +\infty$);
(ii) l is metrically injective around $\mathbf{0}$;
(iii) $\mathrm{dist}\,(\mathbf{0}, l[\mathbb{S}]) > 0$.

Implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii): Since, in particular, for some β, $\delta > 0$, it is $\|l[x]\| \geq \beta\|x\|$ for every $x \in \mathrm{B}[\mathbf{0}; \delta]$, one obtains by linearity

$$\|l[u]\| \geq \beta, \quad \forall u \in \mathbb{S}, \tag{7}$$

whence, passing to the infimum over $\mathbb{S}$, it immediately follows

$$\mathrm{dist}\,(\mathbf{0}, l[\mathbb{S}]) \geq \beta > 0.$$

(iii) ⇒ (i): Set $\beta = \text{dist}\,(\mathbf{0}, l[\mathbb{S}])$. Then inequality (7) must be true. Thus, by taking an arbitrary pair $x_1,\, x_2 \in \mathbb{R}^n$, with $x_1 \neq x_2$, one can write

$$\left\| l\left[\frac{x_1 - x_2}{\|x_1 - x_2\|}\right]\right\| \geq \beta.$$

From the last inequality, one obtains by linearity

$$\|l[x_1] - l[x_2]\| \geq \beta\|x_1 - x_2\|, \quad \forall x_1,\, x_2 \in \mathbb{R}^n. \tag{8}$$

The reader should notice that, by exploiting the above inequalities, one can deduce also that it holds

$$\text{inj}(l; x) = \text{inj}(l) = \sup\{\beta > 0 \mid (8) \text{ holds }\} = \text{dist}\,(\mathbf{0}, l[\mathbb{S}])\,,\ \forall x \in \mathbb{R}^n.$$

In operator theory, the quantity $\text{dist}\,(\mathbf{0}, l[\mathbb{S}])$ is connected with the Banach constant of a bounded linear operator between Banach spaces. More precisely, if referred to the adjoint operator to l, here denoted by l^*, the Banach constant quantifies the (global) linear openness of an onto operator, i.e. $\text{lop}(l;\mathbf{0}) = \text{lop}(l;x) = \text{cov}(l;\mathbf{0}) = \text{dist}\,(\mathbf{0}, l^*[\mathbb{S}])$ (see, for instance, [13, Corollary 1.58]). This notion appears in [10] under the name of surjectivity index. The link between injectivity of l and openness of l^* is well known in operator theory and it is summarized by the condition on the annihilator of the kernel of a regular operator: $(\ker l)^\perp = \text{im}\, l^*$, where $S^\perp$ denotes the annihilator of a subspace S of a Banach space.

Example 2.8 (Metric injectivity of p.h. scalar functions)**.** Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a p.h. continuous function, taking the form

$$h(x) = \begin{cases} \theta_+ x, & \text{if } x \geq 0, \\ \theta_- x, & \text{if } x < 0, \end{cases}$$

for proper $\theta_+,\, \theta_- \in \mathbb{R}$. Then h is metrically injective around 0 iff it is locally injective around the same point. This happens iff h is injective all over $\mathbb{R}$, namely iff it is strictly monotone (increasing or decreasing), or, equivalently, iff

$$\theta_- \cdot \theta_+ > 0. \tag{9}$$

Upon condition (9), if $0 < \theta_- \leq \theta_+$, h turns out to be sublinear and $0 \notin \partial h(0) = [\theta_-, \theta_+]$. Upon condition (9), if $0 < \theta_+ \leq \theta_-$, h turns out to be superlinear and $0 \notin \partial^+ h(0) = [\theta_+, \theta_-]$. Again under condition (9), if $\theta_- \leq \theta_+ < 0$, h is sublinear and $0 \notin \partial h(0) = [\theta_-, \theta_+]$, whereas if $\theta_+ \leq \theta_- < 0$, h is superlinear and $0 \notin \partial^+ h(0) = [\theta_+, \theta_-]$. In any case,

$$\text{inj}(h; 0) = \min\{|\theta_+|, |\theta_-|\} = \begin{cases} \text{dist}\,(0, \partial h(0))\,, & \text{if } h \text{ is sublinear,} \\ \text{dist}\,(0, \partial^+ h(0))\,, & \text{if } h \text{ is superlinear.} \end{cases}$$

The next proposition establishes a useful stability behaviour of metric injectivity under additive Lipschitz perturbations.

Proposition 2.9. *Given two mappings $g,\, h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, suppose that:*

(i) g is metrically injective around $\bar{x} \in \mathbb{R}^n$;
(ii) h is locally Lipschitz around $\bar{x}$, with $\mathrm{lip}(h; \bar{x}) < \mathrm{inj}(g; \bar{x})$.

Then, the mapping $g + h$ is metrically injective, with

$$\mathrm{inj}(g + h; \bar{x}) \geq \mathrm{inj}(g; \bar{x}) - \mathrm{lip}(h; \bar{x}). \tag{10}$$

Proof. It is clear that, by the triangle inequality, it holds

$$\begin{aligned} \|g(x_1) - g(x_2)\| &\leq \|g(x_1) + h(x_1) - g(x_2) - h(x_2)\| \\ &+ \|h(x_1) - h(x_2)\|, \quad \forall x_1,\, x_2 \in \mathbb{R}^n, \end{aligned}$$

whence it follows

$$\|(g+h)(x_1) - (g+h)(x_2)\| \geq \|g(x_1) - g(x_2)\| - \|h(x_1) - h(x_2)\|, \quad \forall x_1,\, x_2 \in \mathbb{R}^n. \tag{11}$$

By hypothesis (i), taken an arbitrary $\beta \in (0, \mathrm{inj}(g; \bar{x}))$ there exists $\delta_\beta > 0$ such that

$$\|g(x_1) - g(x_2)\| \geq \beta\|x_1 - x_2\|, \quad \forall x_1,\, x_2 \in \mathrm{B}[\bar{x}; \delta_\beta]. \tag{12}$$

By hypothesis (ii), taken an arbitrary $\ell > \mathrm{lip}(h; \bar{x})$ there exists $\delta_\ell > 0$ such that

$$\|h(x_1) - h(x_2)\| \leq \ell\|x_1 - x_2\|, \quad \forall x_1,\, x_2 \in \mathrm{B}[\bar{x}; \delta_\ell]. \tag{13}$$

Thus, by taking $\delta = \min\{\delta_\beta,\, \delta_\ell\}$ and combining inequalities (11), (12), and (13), one obtains

$$\|(g + h)(x_1) - (g + h)(x_2)\| \geq (\beta - \ell)\|x_1 - x_2\|, \quad \forall x_1,\, x_2 \in \mathrm{B}[\bar{x}; \delta],$$

which shows that $g + h$ is metrically injective around $\bar{x}$ and

$$\mathrm{inj}(g + h; \bar{x}) \geq \beta - \ell.$$

Since the last inequality is true for every $\beta < \mathrm{inj}(g; \bar{x})$ and $\ell > \mathrm{lip}(h; \bar{x})$, then by passing to the supremum over β and the infimum over ℓ, one gets inequality (10), thereby completing the proof. □

In order for demonstrating that an analogous stability behaviour does not hold for mere injectivity, one can consider the following example.

Example 2.10. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $f(x) = x^3$. This function is globally injective. Therefore, fixing $\bar{x} = 0$, f is injective in particular in a neighbourhood of 0, whereas it fails to be metrically injective at the same point. As a perturbation terms let us take the linear (and hence Lipschitz continuous) functions $h_\zeta : \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$h_\zeta(x) = -\zeta^2 x, \qquad \zeta \in \mathbb{R}.$$

For every $\delta > 0$ one can find a function h_ζ, with $\zeta \in (0, \delta)$, such that the sum $f + h_\zeta$, namely the function

$$x \mapsto f(x) + h_\zeta(x) = x(x - \zeta)(x + \zeta),$$

is clearly not injective in $[-\delta, \delta]$. Notice that $\mathrm{lip}(h_\zeta; 0) = \zeta^2$ can be chosen arbitrarily small.

The stability behaviour stated in Proposition 2.9 enables one to employ estimators for detecting metric injectivity, in the same way as done for linear openness.

Corollary 2.11 (Metric injectivity via strict estimators)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a mapping and let $\bar{x} \in \mathbb{R}^n$. If f admits a strict η-estimator $h_{\bar{x}}$ at $\bar{x}$ and $\mathrm{inj}(h_{\bar{x}}; \bar{x}) > \eta$, then f inherits from $h_{\bar{x}}$ the metric injectivity around $\bar{x}$, with bound*

$$\mathrm{inj}(f; \bar{x}) \geq \mathrm{inj}(h_{\bar{x}}; \bar{x}) - \eta.$$

Proof. Since by hypothesis it is $\mathrm{lip}(f - h_{\bar{x}}; \bar{x}) \leq \eta$, it suffices to apply Proposition 2.9 with $g = h_{\bar{x}}$ and $h = f - h_{\bar{x}}$. □

On the base of the above constructions, the following condition for global inversion can be established.

Proposition 2.12. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a mapping and let $\mu \geq 0$. Suppose that:*

(i) f is continuous on $\mathbb{R}^n$;
(ii) for every $x \in \mathbb{R}^n$ f admits a strict η_x-estimator g_x such that $\mathrm{inj}(g_x; x) > \eta_x$;
(iii) for every $x \in \mathbb{R}^n$ f admits a strict μ-estimator h_x and

$$\sigma_f = \inf_{x \in \mathbb{R}^n} \mathrm{lop}(h_x; x) > \mu.$$

Then,

(t) $\exists f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$;
(tt) f^{-1} is Lipschitz continuous on $\mathbb{R}^n$ with constant $(\sigma_f - \mu)^{-1}$.

Proof. By hypothesis (ii) and Corollary 2.11, f is metrically injective at each point of $\mathbb{R}^n$ and hence locally injective. Since for any $x \in \mathbb{R}^n$ f admits a strict μ-estimator h_x, with $\mathrm{lop}(h_x; x) > \mu$, according to Proposition 2.5 f is α-covering at x, with any $\alpha < \sigma_f - \mu$. As a consequence, by recalling what observed in Remark 1, f is α-covering with any $\alpha < \sigma_f - \mu$. These facts mean that all the hypotheses of Theorem 2.2 happen to be fulfilled. Accordingly, there exists $f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, which is Lipschitz continuous on $\mathbb{R}^n$ with constant

$$\alpha^{-1} > \frac{1}{\sigma_f - \mu}.$$

By arbitrariness of $\alpha < \sigma_f - \mu$ also assertion (tt) in the thesis follows. This completes the proof. □

The aim of Proposition 2.12 is to define an interplay between metric properties and approximation tools, able to rule viable methods for deriving conditions for global invertibility, which are formulated in terms of more specific nonsmooth analysis constructions. In order to implement the above scheme, the next property may be of help.

Definition 2.13 (Strong regularity)**.** A mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be *strongly metrically regular* (henceforth, for short, *strongly regular*) around a point $\bar{x} \in \mathbb{R}^n$ if it has both the following properties:

(i) f is metrically regular (equivalently, linearly open) around $\bar{x}$;
(ii) the set-valued mapping $f^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ has a graphical localization around $(f(\bar{x}), \bar{x})$, which is nowhere multi-valued, i.e. there exist neighbourhoods V of $f(\bar{x})$ and U of $\bar{x}$ such that

$$f^{-1}(y) \cap U = \{x\}, \quad \forall y \in V.$$

The above enhanced form of metric regularity can be equivalently reformulated by postulating that the multifunction f^{-1} admits a single-valued localization $f^\sharp : V \longrightarrow \mathbb{R}^n$ around $(f(\bar{x}), \bar{x})$, which is locally Lipschitz around $f(\bar{x})$, with

$$\mathrm{lip}(f^\sharp; f(\bar{x})) = \mathrm{reg}(f; \bar{x}) = \mathrm{lop}(f; \bar{x})^{-1}. \tag{14}$$

(see [4, Proposition 3G.1]).

For the purposes of the present paper, it is convenient to provide the following characterization of strong regularity for continuous mapping, which involves the notion of metric injection. To the best of the author's knowledge, it has remained still unnoticed within the variational analysis literature.

Proposition 2.14. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a mapping continuous at $\bar{x} \in \mathbb{R}^n$. f is strongly regular around $\bar{x}$ iff it is metrically regular and metrically injective around $\bar{x}$. In such an event, it holds*

$$\mathrm{inj}(f; \bar{x}) \geq \mathrm{lop}(f; \bar{x}). \tag{15}$$

Proof. Assume first that f is metrically regular and metrically injective around $\bar{x}$. Since f is continuous at $\bar{x}$ and linearly open around $\bar{x}$, one has in particular

$$\mathrm{B}[f(\bar{x}); \alpha r] \subseteq f(\mathrm{B}[\bar{x}; r]), \quad \forall r \in (0, r_*], \tag{16}$$

for $\alpha < \mathrm{lop}(f; \bar{x})$ and for a proper $r_* > 0$. By the local injectivity of f around $\bar{x}$, taken an arbitrary $0 < \beta < \mathrm{inj}(f; \bar{x})$ there exists $\delta_\beta > 0$ such that

$$\|f(x_1) - f(x_2)\| \geq \beta \|x_1 - x_2\|, \quad \forall x_1,\, x_2 \in \mathrm{B}[\bar{x}; \delta_\beta]. \tag{17}$$

Thus, if taking $r_0 = \min\{r_*, \delta_\beta\}$, from inclusion (16) it follows that

$$\forall y \in \mathrm{B}[f(\bar{x}); \alpha r_0] \quad \exists x \in \mathrm{B}[\bar{x}; r_0] : \ y = f(x). \tag{18}$$

Such x must be unique in $\mathrm{B}[\bar{x}; r_0]$ by virtue of inequality (17). Therefore, one can define as a single-valued graphical localization of f^{-1} the mapping $f^\sharp : \mathrm{B}[f(\bar{x}); \alpha r_0] \longrightarrow \mathbb{R}^n$ given by $f^\sharp(y) = x$, where x is as in (18). Furthermore, one can readily see that $f^\sharp$ is

Lipschitz continuous in $\mathrm{B}[f(\bar{x}); \alpha r_0]$, because, on account of inequality (17), it holds

$$\begin{aligned}\|f^\sharp(y_1) - f^\sharp(y_2)\| &= \|x_1 - x_2\| \leq \beta^{-1}\|f(x_1) - f(x_2)\| \\ &= \beta^{-1}\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathrm{B}[f(\bar{x}); \alpha r_0].\end{aligned}$$

Vice versa, assume now that f is strongly regular around $\bar{x}$. According to the aforementioned equivalent reformulation, fixed an arbitrary $\kappa > \mathrm{reg}(f; \bar{x})$ there exist neighbourhoods U_κ of $\bar{x}$ and V_κ of $f(\bar{x})$ and a graphical single-valued localization $f^\sharp : V_\kappa \longrightarrow \mathbb{R}^n$ such that

$$\|f^\sharp(y_1) - f^\sharp(y_2)\| \leq \kappa\|y_1 - y_2\|, \quad \forall y_1, y_2 \in V_\kappa. \tag{19}$$

Since f is continuous at $\bar{x}$, fixing $\alpha < \mathrm{lop}(f; \bar{x})$, it is possible to pick $r > 0$ in such a way that

$$\mathrm{B}[\bar{x}; r] \subseteq U_\kappa \quad \text{and} \quad f(\mathrm{B}[\bar{x}; \alpha r]) \subseteq V_\kappa. \tag{20}$$

Take arbitrary $x_1, x_2 \in \mathrm{B}[\bar{x}; r]$. By virtue of the second inclusion in (20), it must be $y_i = f(x_i) \in V_\kappa$, for $i = 1, 2$. On the other hand, by virtue of the first inclusion in (20), one has

$$f^\sharp(y_i) = x_i = f^{-1}(y_i) \cap U_\kappa, \quad i = 1, 2.$$

Therefore, by taking into account inequality (19), one obtains

$$\|x_1 - x_2\| \leq \kappa\|y_1 - y_2\| = \kappa\|f(x_1) - f(x_2)\|,$$

which shows that f is metrically injective around $\bar{x}$ with $\mathrm{inj}(f; \bar{x}) \geq \kappa^{-1}$. By passing to the infimum over all $\kappa > \mathrm{reg}(f; \bar{x}) = \mathrm{lop}(f; \bar{x})^{-1}$, one achieves the inequality in (15). This completes the proof. □

One is now in a position to formulate the main result of the current section.

Theorem 2.15 (Global invertibility via strict estimators)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous mapping and let $\mu \geq 0$. Suppose that for every $x \in \mathbb{R}^n$ f admits a strict μ-estimator h_x, which is strongly regular around x, and such that*

$$\sigma_f = \inf_{x \in \mathbb{R}^n} \mathrm{lop}(h_x; x) > \mu. \tag{21}$$

Then,

(t) $\exists f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$;
(tt) f^{-1} is Lipschitz continuous on $\mathbb{R}^n$ with constant $(\sigma_f - \mu)^{-1}$.

Proof. For every $x \in \mathbb{R}^n$, in the light of Proposition 2.14, h_x is metrically injective and, by virtue of condition (21), taking into account inequality (15) one has $\mathrm{inj}(h_x; x) \geq \mathrm{lop}(h_x; x) > \mu$. So h_x satisfies hypothesis (ii) of Proposition 2.12 for every $x \in \mathbb{R}^n$. The same mappings play the role of a strict μ-estimator of f at x, satisfying hypothesis (iii) of Proposition 2.12. These facts make it possible to invoke that result, wherefrom all the assertions in the thesis follow. □

As a first comment about Theorem 2.15, let us observe that it can be regarded as a kind of 'globalization' of the local invertibility condition expressed in terms of estimators by [4, Theorem 1E.3]. As its local counterpart, the quantitative condition tying the parameter μ and the bound of linear openness plays a crucial role. Roughly speaking, this condition says that as higher is σ_f as looser can be the estimate of f provided by the family of mappings $\{h_x : x \in \mathbb{R}^n\}$. Furthermore, the difference $\sigma_f - \mu$ takes a transparent part in the estimate of the Lipschitz constant of f^{-1}.

Another point deals with the scope of Theorem 2.15. It offers a starting point for establishing more specific global inversion results beyond differentiability. An example is provided in considering the following corollary, which is derived by replacing C^1 smoothness with existence of strict first-order approximations. Its proof follows at once on account of Example 2.4.

Corollary 2.16. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous mapping. If for every $x \in \mathbb{R}^n$ f admits a strict first-order approximation $\mathrm{A}f(x;\cdot)$ at x, which is strongly regular around x, with*

$$\alpha_f = \inf_{x\in\mathbb{R}^n} \mathrm{lop}(\mathrm{A}f(x;\cdot);x) > 0,$$

then

(t) $\exists f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$*;*
(tt) f^{-1} *is Lipschitz continuous on* $\mathbb{R}^n$ *with constant* α_f^{-1}*.*

Example 2.17. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$f(x) = \begin{cases} \mathrm{sgn}\,(x)\sqrt{|x|}, & \text{if } |x| \le 1, \\ x, & \text{if } |x| > 1, \end{cases} \qquad \text{where} \qquad \mathrm{sgn}\,(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Such a function is continuous in $\mathbb{R}$, but fails to be locally Lipschitz in $\mathbb{R}$ because, as one readily sees, inequality (1) does not hold in any neighbourhood of 0. Moreover, f lacks of differentiability at $x = 0$ and at $x = \pm 1$. These pathological features make f to fall out from the scope of application of Theorem 1.2 and Theorem 1.3. Nevertheless, by direct inspection one can check that f is globally invertible, with $f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ being given by

$$f^{-1}(y) = \begin{cases} \mathrm{sgn}\,(y)y^2, & \text{if } |y| \le 1, \\ y, & \text{if } |y| > 1. \end{cases}$$

Besides, since it holds

$$|f^{-1}(y_1) - f^{-1}(y_2)| \le 2|y_1 - y_2|, \quad \forall y_1, y_2 \in \mathbb{R},$$

then, in contrast to f, function f^{-1} turns out to be Lipschitz continuous in $\mathbb{R}$ with constant $\ell = 2$. Let us illustrate how such an instance can be put in the framework of global invertibility via strict estimators, where Theorem 2.15 comes into play. To start with, observe that f admits strict 0-estimators at each point of $\mathbb{R}$. More precisely, for all those points $x \in \mathbb{R}$ at which f is strictly differentiable one can choose as strict 0-estimator the affine functions $h_x = f(x) + \mathrm{D}f(x)(\cdot - x)$. Thus, it results in

$$h_x(t) = \begin{cases} f(x) + \dfrac{t-x}{2\sqrt{|x|}}, & \text{if } 0 < |x| < 1, \\ \\ t, & \text{if } |x| > 1. \end{cases}$$

For the point $x = 1$, one can choose $h_1 : \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$h_1(t) = 1 + \max\left\{\frac{t-1}{2},\, t-1\right\}.$$

Indeed, fixed an arbitrary $\epsilon \in (0,1)$, by setting $\delta_\epsilon = \frac{\epsilon^2+2\epsilon}{(\epsilon+1)^2}$, one finds

$$|(f-h_1)(x_1) - (f-h_1)(x_2)| \leq \epsilon|x_1 - x_2|, \quad \forall x_1,\, x_2 \in (1-\delta_\epsilon, 1+\delta_\epsilon),$$

so $\mathrm{lip}(f - h_1; 1) = 0$. The above inequality is obviously true if $x_1,\, x_2 \in [1, 1+\delta_\epsilon)$. In the case $x_1,\, x_2 \in (1-\delta_\epsilon, 1]$, since the function $x \mapsto \sqrt{x} - [1 + \frac{1}{2}(x-1)]$ is C^1 in an open set containing $[1-\delta_\epsilon, 1]$, by the mean-value theorem one can write

$$\begin{aligned} |(f-h_1)(x_1) - (f-h_1)(x_2)| &\leq \sup_{x\in[1-\delta_\epsilon,1]} \left|\frac{1}{2\sqrt{x}} - \frac{1}{2}\right| |x_1 - x_2| \\ &= \frac{1}{2}\left|\frac{1}{\sqrt{1-\delta_\epsilon}} - 1\right| |x_1 - x_2| \\ &= \frac{\epsilon}{2}|x_1 - x_2|, \quad \forall x_1,\, x_2 \in (1-\delta_\epsilon, 1]. \end{aligned}$$

In the case $x_1 \in (1-\delta_\epsilon, 1)$ and $x_2 \in [1, 1+\delta_\epsilon)$, by exploiting again the above inequality for the pair $x_1,\ 1 \in (1-\delta_\epsilon, 1]$, one has

$$\begin{aligned} |(f-h_1)(x_1) - (f-h_1)(x_2)| &\leq |(f-h_1)(x_1) - (f-h_1)(1)| \\ &+ |(f-h_1)(1) - (f-h_1)(x_2)| \\ &= |(f-h_1)(x_1) - (f-h_1)(1)| \leq \frac{\epsilon}{2}|x_1 - 1| \\ &\leq \frac{\epsilon}{2}|x_1 - x_2|. \end{aligned}$$

Similarly, one sees that for the point $x = -1$ it is possible to take $h_{-1} : \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$h_{-1}(t) = -1 + \min\left\{\frac{t+1}{2},\, t+1\right\}.$$

Finally, for $x = 0$ one can choose the strict 0-estimator $h_0 : \mathbb{R} \longrightarrow \mathbb{R}$

$$h_0(t) = \mathrm{sgn}\,(t)\sqrt{|t|}.$$

One has now to show that, by virtue of the above choices, each function h_x is strongly regular around x, then to provide an estimate of each $\mathrm{lop}(h_x; x)$, and finally to check that condition (21) is fulfilled, namely $\sigma_f > 0$. To this purpose, notice that, in the case $0 < |x| < 1$, as h_x are affine and invertible, they are strongly regular around every point. In particular, if setting $l_x(t) = \frac{t}{2\sqrt{|x|}}$, one has

$$\mathrm{lop}(h_x; x) = \mathrm{lop}(l_x; 0) = \mathrm{reg}(l_x; 0)^{-1} = \frac{1}{2\sqrt{|x|}} \geq \frac{1}{2}, \quad \forall x \in \mathbb{R} : \ 0 < |x| < 1. \tag{22}$$

Analogously, in the case $|x| > 1$, as a linear invertible function h_x are strongly regular around each point, with

$$\mathrm{lop}(h_x; x) = 1 \geq \frac{1}{2}, \quad \forall x \in \mathbb{R} : \ |x| > 1 \tag{23}$$

(remember Example 2.10). In the case $x = 1$, as h_1 is globally invertible, having inverse which is Lipschitz continuous in $\mathbb{R}^n$ with $\mathrm{lip}(h_1^{-1}; 1) = 2$, it is strongly regular around $x = 1$, and in the light of (14) one has

$$\mathrm{lop}(h_1; 1) = \frac{1}{2}. \tag{24}$$

The case $x = -1$ can be treated in a similar manner, after noticing that $\mathrm{gph}\, f$ is centrally symmetric (f being an odd function). To prove that h_0 is strongly regular around 0 it suffices to observe that h_0^{-1} coincides with its graphical single-valued localization $h_0^\sharp$ (actually defined all over $\mathbb{R}$), given by

$$h_0^\sharp(y) = \mathrm{sgn}\,(y) \cdot y^2,$$

which is locally Lipschitz around 0. To estimate $\mathrm{lop}(h_0; 0)$ it is useful to notice that, since $h_0^\sharp$ is C^1 around 0, with $\mathrm{D}h_0^\sharp(y) = |2y|$, a straightforward application of the mean-value theorem gives

$$\mathrm{lip}(h_0^\sharp; 0) \leq 2.$$

Consequently, one obtains

$$\mathrm{lop}(h_0; 0) \geq \frac{1}{2}. \tag{25}$$

Inequalities (22), (23), (24) and (25) allow one to deduce that $\sigma_f \geq 1/2$. Thus, it is possible to apply Theorem 2.15. Its thesis leads to conclusions that are consistent with what one can directly check.

3. Invertibility via pairs of convex compacta

In order to illustrate how Theorem 2.15 may interact with constructive elements of nonsmooth analysis, let us recall the following notions from quasidifferential calculus.

Definition 3.1. According to [21, Definition 2.1], a mapping $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be *scalarly quasidifferentiable* (for short, scalarly q.d.) at $\bar{x} \in \mathbb{R}^n$ if for every $w \in \mathbb{R}^m$

the scalar function $\langle w, g\rangle : \mathbb{R}^n \longrightarrow \mathbb{R}$, i.e. $x \mapsto \langle w, g(x)\rangle = w^\top g(x)$, is quasidifferentiable at $\bar{x}$ in the sense of Demyanov-Rubinov.

Here $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product of $\mathbb{R}^m$ and $w^\top$ the transpose vector of w.

On the base of [3,15], Definition 3.1 amounts to say that, for every $w \in \mathbb{R}^m$, $\langle w, g\rangle$ is directionally differentiable at $\bar{x}$, in any direction $v \in \mathbb{R}^n$, and there exist two elements g_w^+, $g_w^- \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\langle w, g\rangle'(\bar{x}; v) = \langle w, g'(\bar{x}; v)\rangle = g_w^+(v) - g_w^-(v), \quad \forall v \in \mathbb{R}^n. \tag{26}$$

According to the Minkowski duality $\varsigma : \mathcal{K}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, the functions g_w^+ and g_w^- can be dually represented by means of elements of the semigroup $(\mathcal{K}(\mathbb{R}^n), +)$ through their support functions, namely

$$g_w^+(v) = \varsigma\left(v; \partial g_w^+(\mathbf{0})\right), \qquad g_w^-(v) = \varsigma\left(v; \partial g_w^-(\mathbf{0})\right), \qquad v \in \mathbb{R}^n.$$

(see, for instance, [16]). It is well known that the representation in (26) can not be unique and hence so does the dual pair $(\partial g_w^+(\mathbf{0}), \partial g_w^-(\mathbf{0})) \in \mathcal{K}^2(\mathbb{R}^n) = \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n)$. To restore the uniqueness, and with that the preliminary condition for a well defined and effective calculus, the following equivalence relation $\sim\, \subseteq \mathcal{K}^2(\mathbb{R}^n)$ is employed, which was already introduced by L. Hörmander:

$$(A, B) \sim (C, D) \qquad \text{if} \qquad A + D = B + C.$$

Let us denote by $[\partial g_w^+(\mathbf{0}), \partial g_w^-(\mathbf{0})]_\sim$ the equivalence class containing the pair $(\partial g_w^+(\mathbf{0}), \partial g_w^-(\mathbf{0}))$. Thus, whenever g is scalarly q.d. at $\bar{x}$ one can consider the mapping $\widetilde{\mathrm{D}}^*g(\bar{x}) : \mathbb{R}^m \longrightarrow \mathcal{K}(\mathbb{R}^n)^2/_\sim$, which is well defined by

$$\widetilde{\mathrm{D}}^*g(\bar{x})(w) = \left[\partial g_w^+(\mathbf{0}), \partial g_w^-(\mathbf{0})\right]_\sim.$$

The above generalized differentiation concept inherits from $\mathcal{K}(\mathbb{R}^n)^2/_\sim$ a rich calculus.

Remark 4. In view of next nonsmooth analysis constructions, it is convenient to notice that, since for every $\lambda > 0$ it holds

$$\begin{aligned}\langle \lambda w, g\rangle'(\bar{x}; v) &= \lambda\langle w, g\rangle'(\bar{x}; v) = \lambda g_w^+(v) - \lambda g_w^-(v) \\ &= \varsigma\left(v; \lambda\partial g_w^+(\mathbf{0})\right) - \varsigma\left(v; \lambda\partial g_w^-(\mathbf{0})\right), \quad \forall v \in \mathbb{R}^n,\end{aligned}$$

and, by known calculus rules in $\mathcal{K}(\mathbb{R}^n)^2/_\sim$, it is

$$\left[\lambda\partial g_w^+(\mathbf{0}), \lambda\partial g_w^-(\mathbf{0})\right]_\sim = \lambda\left[\partial g_w^+(\mathbf{0}), \partial g_w^-(\mathbf{0})\right]_\sim,$$

then it results in

$$\widetilde{\mathrm{D}}^*g(\bar{x})(\lambda w) = \lambda\widetilde{\mathrm{D}}^*g(\bar{x})(w), \quad \forall\lambda > 0,\ \forall w \in \mathbb{R}^m.$$

To carry out the further construction needed for the present analysis, recall that, given two subsets A, $B \in \mathcal{K}_0(\mathbb{R}^n) = \mathcal{K}(\mathbb{R}^n) \cup \{\varnothing\}$, the operation $\overset{*}{-} : \mathcal{K}_0(\mathbb{R}^n) \times$

$\mathcal{K}_0(\mathbb{R}^n) \longrightarrow \mathcal{K}_0(\mathbb{R}^n)$ defined by

$$A \overset{*}{-} B = \{x \in \mathbb{R}^n \mid x + B \subseteq A\}$$

is known as $\overset{*}{-}$ (or *Pontryagin*) *-difference* of compact convex sets (see, for instance, [20]). As other difference operations employed in nonsmooth analysis (e.g. the Demyanov's difference [20]), actually it can be regarded as a particular instance of a more general approach to defining algebraic operations over elements of $\mathcal{K}(\mathbb{R}^n)$ (see, for more details, [16,19,20]). One is now in a position to introduce the tool to be used in formulating a global invertibility condition for nonsmooth mappings.

Definition 3.2. Given a mapping $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, suppose that g is scalarly q.d. at $\bar{x} \in \mathbb{R}^n$. The set-valued mapping $\widetilde{\mathrm{D}}^{\overset{*}{-}} g(\bar{x}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined as being

$$\widetilde{\mathrm{D}}^{\overset{*}{-}} g(\bar{x})(w) = \partial g_w^+(\mathbf{0}) \overset{*}{-} \partial g_w^-(\mathbf{0}), \tag{27}$$

is called $\overset{*}{-}$*-coquasiderivative* of g at $\bar{x}$.

As a first comment to Definition 3.2 let us remark that the equality in (27) singles out a uniquely defined derivative object (in $\mathcal{K}_0(\mathbb{R}^n)$), which is independent of the representation of the scalarized coquasiderivative. This happens because the $\overset{*}{-}$-difference is an operation which turns out to be invariant with respect to the equivalence relation $\sim$, namely

$$(A, B) \sim (C, D) \qquad \text{implies} \qquad A \overset{*}{-} B = C \overset{*}{-} D$$

(see, for instance, [20, Chapter 5]). Another feature to be pointed out is that, as a set-valued mapping, $\widetilde{\mathrm{D}}^{\overset{*}{-}} g(\bar{x})$ turns out to be p.h..

Condition ($\mathfrak{C}$) A mapping $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is scalarly q.d. in a neighbourhood U of $\bar{x}$, is said to satisfy *condition* ($\mathfrak{C}$) around $\bar{x}$ provided that the set-valued mapping $G_{\mathbb{S}} : U \rightrightarrows \mathbb{R}^n$, given by

$$G_{\mathbb{S}}(x) = \widetilde{\mathrm{D}}^{\overset{*}{-}} g(x)(\mathbb{S}) = \bigcup_{v \in \mathbb{S}} \widetilde{\mathrm{D}}^{\overset{*}{-}} g(x)(v),$$

is u.s.c. at $\bar{x}$, i.e. for every open set $O \supseteq \widetilde{\mathrm{D}}^{\overset{*}{-}} g(\bar{x})(\mathbb{S})$ there exists $\delta_O > 0$ such that

$$\widetilde{\mathrm{D}}^{\overset{*}{-}} g(x)(\mathbb{S}) \subseteq O, \quad \forall x \in \mathrm{B}[\bar{x}; \delta_O].$$

The above elements enable one to establish the following sufficient condition for the linear openness around $\mathbf{0}$ of the class of those continuous p.h. mappings with scalarizations in $\mathcal{DS}(\mathbb{R}^n)$ (for short, scalarly d.s.), which may be of independent interest.

Proposition 3.3 (Linear openness of scalarly d.s. mappings)**.** *Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a p.h. mapping. Suppose that:*

(i) for every $w \in \mathbb{R}^m$, $\langle w, h \rangle \in \mathcal{DS}(\mathbb{R}^n)$;
(ii) h satisfies condition ($\mathfrak{C}$) around $\mathbf{0}$;

(iii) $\flat_0^{\overset{*}{-}}(h) = \mathrm{dist}\left(\mathbf{0}, \widetilde{\mathrm{D}}^{\overset{*}{-}} h(\mathbf{0})(\mathbb{S})\right) > 0.$

Then, h is linearly open around $\mathbf{0}$ and

$$\mathrm{lop}(h; \mathbf{0}) \geq \flat_0^{\overset{*}{-}}(h). \tag{28}$$

Proof. A short way of proving this proposition is to apply [21, Theorem 4.1], which provides sufficient conditions for the metric regularity of scalarly q.d. mappings in terms of $\overset{*}{-}$-coquasiderivative. To see how, let us start with observing that, as h acts between finite-dimensional Euclidean spaces, the assumption on the Fréchet smooth renorming of the range space as well as the assumption on the trustwortiness of the domain space are automatically fulfilled. Then, notice that, by virtue of hypothesis (i), h is in particular scalarly q.d. at each point of $\mathbb{R}^n$. Moreover, the former property entails that h is continuous in $\mathbb{R}^n$ (in particular, each component of h is an element of $\mathcal{DS}(\mathbb{R}^n)$).

Now, by virtue of condition $(\mathfrak{C})$, fixed an arbitrary $\epsilon \in (0, \flat_0^{\overset{*}{-}}(h))$, for a proper $\delta_\epsilon > 0$ one has

$$\widetilde{\mathrm{D}}^{\overset{*}{-}} h(x)(\mathbb{S}) \subseteq \mathrm{B}(\widetilde{\mathrm{D}}^{\overset{*}{-}} h(\bar{x})(\mathbb{S}); \epsilon), \quad \forall x \in \mathrm{B}[\bar{x}; \delta_\epsilon].$$

As a consequence, one obtains

$$\inf\left\{\mathrm{dist}\left(\mathbf{0}, \widetilde{\mathrm{D}}^{\overset{*}{-}} h(x)(\mathbb{S})\right) \mid x \in \mathrm{B}[\bar{x}; \delta_\epsilon]\right\} \geq \flat_0^{\overset{*}{-}}(h) - \epsilon > 0.$$

The last inequality ensures that also hypothesis (1) in Theorem 4.1 is fulfilled. Thus h is metrically regular/linearly open around $\mathbf{0}$. Furthermore, a perusal of the proof of [21, Theorem 3.2], on which Theorem 4.1 relies, reveals that it holds

$$\mathrm{lop}(h; \mathbf{0}) \geq \flat_0^{\overset{*}{-}}(h) - \epsilon.$$

The arbitrariness of ϵ enables one to achieve the estimate in the thesis. □

By combining Proposition 3.3 and Theorem 2.15 it is possible to achieve the following sufficient condition for the global invertibility of a special class of nonsmooth mappings.

Theorem 3.4 (Global invertibility via $\overset{*}{-}$-difference of convex compacta)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous mapping and $\mu \geq 0$. Suppose that:*

(i) for every $x \in \mathbb{R}^n$, f admits a strict μ-estimator of the form $\widetilde{h}_x = f(x) + h_x(\cdot - x)$;
(ii) for every $w \in \mathbb{R}^n$, $\langle w, h_x\rangle \in \mathcal{DS}(\mathbb{R}^n)$;
(iii) for every $x \in \mathbb{R}^n$, h_x satisfies condition $(\mathfrak{C})$ around $\mathbf{0}$;
(iv) $\flat_f^{\overset{*}{-}} = \inf_{x\in\mathbb{R}^n} \mathrm{dist}\left(\mathbf{0}, \widetilde{\mathrm{D}}^{\overset{*}{-}} h_x(\mathbf{0})(\mathbb{S})\right) > \mu$;
(v) for every $x \in \mathbb{R}^n$, $\mathrm{inj}(h_x; \mathbf{0}) > \mu$.

Then,

(t) $\exists f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$;
(tt) f^{-1} is Lipschitz continuous on $\mathbb{R}^n$ with constant $(\flat_f^{\overset{*}{-}} - \mu)^{-1}$.

Proof. Fix $x \in \mathbb{R}^n$. Owing to assumptions (ii), (iii) and (iv), the mapping h_x is linearly open around $\mathbf{0}$ with $\mathrm{lop}(h_x; \mathbf{0}) \geq \flat_0^{\overset{*}{-}}(h_x)$. Consequently, as the Euclidean distance in $\mathbb{R}^n$ is invariant under translation, the mapping $\widetilde{h}_x$ is linearly open around x, with $\mathrm{lop}(\widetilde{h}_x; x) = \mathrm{lop}(h_x; \mathbf{0}) \geq \flat_0^{\overset{*}{-}}(h_x)$. For the same reason, $\widetilde{h}_x$ is metrically injective around x, with $\mathrm{inj}(\widetilde{h}_x; x) = \mathrm{inj}(h_x; \mathbf{0}) > \mu$ by hypothesis (v). Thus, according to Proposition 2.14, $\widetilde{h}_x$ turns out to be strongly regular around x. Moreover, by hypothesis (iv), one has

$$\inf_{x \in \mathbb{R}^n} \mathrm{lop}(\widetilde{h}_x; x) \geq \inf_{x \in \mathbb{R}^n} \flat_0^{\overset{*}{-}}(h_x) = \flat_f^{\overset{*}{-}} > \mu,$$

which says that also condition (21) is satisfied. Thus, the thesis follows from Theorem 2.15. □

Remark 5. The reader who remembers Remark 2(i) will observe that, because of hypotheses (i) and (ii), f is implicitly supposed to be locally Lipschitz around each $x \in \mathbb{R}^n$. Therefore Theorem 3.4 is less general than analogous existent results, which are expressed by constructions allowing for a more general approach (for instance, in the case of pseudo-Jacobians, consider [10, Corollary 3.10]). Nonetheless, utmost generality is not the feature aimed at in Theorem 3.4. Instead, it focuses on a special class of possibly nondifferentiable mappings, admitting approximations with a special structure. Such a structure paves the way to the employment of all the benefits given by the calculus in $\mathcal{K}^2(\mathbb{R}^n)/_\sim$, when hypothesis (iv) must be checked. Take into account that $\widetilde{\mathrm{D}}^{\overset{*}{-}} h_x(\mathbf{0})(\mathbb{S}) = \partial(h_x)^+_w(\mathbf{0}) \overset{*}{-} \partial(h_x)^-_w(\mathbf{0})$ is expected to be computed easily enough, as h_x is p.h..

4. Global invertibility via regular coderivatives

Theorem 2.2 enables one to formulate further conditions for global invertibility of nonsmooth mappings, even in the lack of local Lipschitz continuity, without a direct employment of Theorem 2.15. This can be seen by selecting adequate nonsmooth analysis tools. An highly successful approach to generalized differentiability relies on the geometry of normals and graphical differentiation, which was initiated in [12]. It revealed to be effective in providing characterizations for those properties discussed in Section 2 that are fundamental for the approach at the issue. Let us briefly recall the elements needed for the present analysis.

Given a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\bar{x} \in \mathbb{R}^n$, the regular coderivative (a.k.a. prederivative) of f at $\bar{x}$ is the set-valued mapping $\widehat{\mathrm{D}}^* f(\bar{x}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$\widehat{\mathrm{D}}^* f(\bar{x})(v) = \{u \in \mathbb{R}^n \mid (u, -v) \in \widehat{\mathrm{N}}((\bar{x}, f(\bar{x})); \mathrm{gph}\, f)\}, \quad v \in \mathbb{R}^m.$$

Here, given $W \subseteq \mathbb{R}^p$ and $\bar{w} \in W$, the subset

$$\widehat{\mathrm{N}}(\bar{w}; W) = \left\{ v \in \mathbb{R}^p \;\middle|\; \limsup_{\substack{w \in W \\ w \to \bar{w}}} \left\langle v, \frac{w - \bar{w}}{\|w - \bar{w}\|} \right\rangle \leq 0 \right\}$$

stands for the regular normal cone to W at $\bar{w}$. Whenever f is Fréchet differentiable at $\bar{x}$, its regular coderivative becomes single-valued, taking the form $\widehat{\mathrm{D}}^* f(\bar{x})(v) = \{\mathrm{D}f(\bar{x})^*[v]\}$ (see [13, Theorem 1.38]). In the case in which f is locally Lipschitz around $\bar{x}$, the following scalarized representation of the regular coderivative is valid (see [14, Exercise 1.70(ii)]

$$\widehat{\mathrm{D}}^* f(\bar{x})(v) = \widehat{\partial}\langle w, f\rangle(\bar{x}), \quad \forall w \in \mathbb{R}^m,$$

where, given a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\mp\infty\}$ and $\bar{x} \in \operatorname{dom}\varphi$, the set

$$\widehat{\partial}\varphi(\bar{x}) = \left\{ v \in \mathbb{R}^n \;\middle|\; \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0 \right\}$$

is the regular subdifferential of φ at $\bar{x}$ (see [14, Chapter 1.3.4]).

In combination with the above nonsmooth analysis constructions, the following monotonicity property for set-valued mappings plays a crucial role in the present context.

Definition 4.1. A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be *locally hypomonotone* around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there exist a neighbourhood $U \times V$ of $(\bar{x}, \bar{y})$ and a constant $\gamma > 0$ such that

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq -\gamma\|x_1 - x_2\|, \quad \forall (x_1, y_1), (x_2, y_2) \in \operatorname{gph} F \cap (U \times V).$$

As a comment to Definition 4.1, it is worth mentioning that all globally hypomonotone set-valued mappings, i.e. set-valued mappings $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $F + r\,\mathrm{id}$, where id stands for the identity operator, is monotone on $\mathbb{R}^n$ for some $r > 0$, are in particular locally hypomonotone. Moreover, all locally monotone as well as Lipschitz continuous single-valued mappings are locally hypomonotone (see [14, Chapter 5]). Local hypomonotonicity in synergy with a positive-definiteness condition expressed in terms of regular coderivative is known to yield strong metric regularity. As a consequence, the following global invertibility condition can be established via regular coderivatives.

Theorem 4.2 (Global invertibility via regular coderivatives)**.** *Given a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, suppose that*

(i) f is continuous in $\mathbb{R}^n$;
(ii) f is locally hypomonotone around each $x \in \mathbb{R}^n$;
(iii) for every $x \in \mathbb{R}^n$ there exist positive constants $\widehat{\alpha}_x$ and η_x such that

$$\langle u, v\rangle \geq \widehat{\alpha}_x\|v\|^2, \quad \forall u \in \widehat{\mathrm{D}}^* f(z)(v),\ \forall z \in \mathrm{B}[x; \eta_x]; \tag{29}$$

(iv) $\widehat{\alpha} = \inf_{x \in \mathbb{R}^n} \widehat{\alpha}_x > 0$.

Then,

(t) $\exists f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$;
(tt) f^{-1} is Lipschitz continuous on $\mathbb{R}^n$ with constant $\widehat{\alpha}$.

Proof. Observe that the continuity of f at each $x \in \mathbb{R}^n$ (hypothesis (i)) ensures the validity of the inequality in (29) for every $u \in \widehat{\mathrm{D}}^* f(z)(v)$ and every $(z, f(z)) \in \operatorname{gph} f \cap \mathrm{B}[(x, f(x)); \eta_x]$, up to a proper reduction in the value of $\eta_x > 0$. The latter condition, along with the hypothesis of local hypomonotonicity of f around x, is sufficient to guarantee that the mapping f is strongly regular around each x, with $\mathrm{lop}(f;x) \geq \widehat{\alpha}_x \geq \widehat{\alpha}$, as stated in [14, Corollary 5.15]. By virtue of such a property, f turns out to be metrically injective and hence locally injective around each x, as well as $\widehat{\alpha}$-covering at each x. Thus the thesis follows from Theorem 2.2. □

Remark 6. By replacing the regular coderivative with the basic (a.k.a. Mordukhovich) coderivative, in the particular case in which f is locally Lipschitz it is possible to obtain a point-based version of condition (29). Indeed, in such a setting, by applying the scalarized representation valid for basic coderivatives (see [14, Theorem 1.32]), condition (29) can be reformulated as follows: for every $x \in \mathbb{R}^n$ there exists a positive constant α_x such that

$$\langle w, v \rangle \geq \alpha_x, \quad \forall v \in \partial_M \langle w, f \rangle(x), \quad \forall w \in \mathbb{R}^n \backslash \{\mathbf{0}\},$$

where $\partial_M \varphi(\bar{x}) = \operatorname{Limsup}_{x \xrightarrow{\varphi} \bar{x}} \widehat{\partial}\varphi(x)$ denotes the basic subdifferential of φ at $\bar{x} \in \mathrm{dom}\,\varphi$ and $\operatorname{Limsup}_{x \xrightarrow{\varphi} \bar{x}}$ denotes the Painlevé-Kuratowski upper limit of the set-valued mapping $\widehat{\partial}\varphi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as $x \to \bar{x}$ and $\varphi(x) \to \varphi(\bar{x})$ (see [14, Theorem 5.16] and [14, Section 5.4]).

5. Conclusions

The investigations exposed in the present paper stem from a recent result about global inversion of functions in a finite-dimensional Euclidean space setting, which relies on their metric behaviour. In order to develop the potential of this result, its assumptions are interpreted in the light of some specific nonsmooth analysis constructions. The main findings of the paper provide further evidences that generalization of Hadamard's inversion theorem to nonsmooth functions not only are possible (what was already known), but they can be achieved through different paths, which lead to a variety of conditions expanding their scope. The specific nonsmooth analysis constructions here considered play a certain role not only in formulating conditions for global invertibility, but also in providing quantitative estimates for the Lipschitz constant of the inverse, in the spirit of modern variational analysis.

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