EIGENVALUES OF THE LAPLACIAN WITH MOVING MIXED BOUNDARY CONDITIONS: THE CASE OF DISAPPEARING NEUMANN REGION

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ABSTRACT. We deal with eigenvalue problems for the Laplacian with varying mixed boundary conditions, consisting in homogeneous Neumann conditions on a vanishing portion of the boundary and Dirichlet conditions on the complement. By the study of an Almgren-type frequency function, we derive upper and lower bounds of the eigenvalue variation and sharp estimates in the case of a strictly star-shaped Neumann region.

Keywords. Asymptotics of Laplacian eigenvalues; mixed boundary conditions; monotonicity formula.

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1. INTRODUCTION

The present paper concerns the eigenvalue problem for the Laplacian in a bounded domain, with mixed Dirichlet-Neumann homogeneous boundary conditions prescribed on variable portions of the boundary. More precisely, we focus on a perturbative problem characterized by the disappearance, in some limiting process, of the region where Neumann boundary conditions are imposed. In this situation, the eigenvalues of the mixed problem converge to Dirichlet eigenvalues: we aim to study the rate of this convergence. This paper is the counterpart of [13], where the case of Dirichlet disappearing region is studied.

In the literature there have been several contributions on asymptotic behaviour of eigenvalues of elliptic boundary value problems under singular perturbation of the boundary conditions. Concerning, in particular, the case treated in the present paper, i.e. the perturbation of a Dirichlet problem by imposing a homogeneous Neumann condition on a vanishing portion of the boundary, we mention the results in [16], where a full asymptotic expansion of perturbed eigenvalues is obtained in dimension 2, see also [3]; we mention additionally the paper [8], concerned with the spectral stability of the first eigenvalue. The complementary problem, i.e. a Neumann problem perturbed with a Dirichlet condition on a small part of the boundary, is treated by [17] in dimension 2 and by [13] in any dimension $N \geq 3$. The approach developed in [13] is based on a capacity argument inspired by [9] and [2], where the problem of spectral stability for the Dirichlet Laplacian in domains with small holes is investigated; in particular in [13] the sharp asymptotic behaviour of perturbed eigenvalues is described in terms of the Sobolev capacity of the boundary portion where the Dirichlet condition is imposed. This kind of method does not seem to be effective in the case of a disappearing Neumann region, being the Dirichlet boundary set not small. Therefore, in the present paper we treat this case with a different approach, based on blow-up analysis for scaled eigenfunctions and energy estimates obtained by monotonicity formulas, in the spirit of [3]; we point out that the case of dimension $N \geq 3$ presents several additional difficulties with respect to the 2-dimensional case treated in [3], because of the occurrence of some effects of the geometry of the Neumann region in the monotonicity argument, see Remark 1.3.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set such that $\partial\Omega$ is of class $C^{1,1}$ in a neighbourhood of $0 \in \partial\Omega$, namely there exist $r_0 \in (0,1)$ and $g \in C^{1,1}(\mathbb{R}^{N-1})$ such that

$$B_{r_0} \cap \Omega = \{x \in B_{r_0} \colon x_N > g(x')\} \quad \text{and} \quad B_{r_0} \cap \partial\Omega = \{x \in B_{r_0} \colon x_N = g(x')\}, \tag{1.1}$$

where $B_{r_0} = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : |x| < r_0\}$ is the ball in $\mathbb{R}^N$ centered at the origin with radius r_0 and $x' = (x_1, \dots, x_{N-1})$. Up to a suitable choice of the Cartesian coordinate system, it

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is not restrictive to assume that

$$g(0) = 0 \quad \text{and} \quad \nabla g(0) = 0, \tag{1.2}$$

i.e. that $\partial\Omega$ is tangent to the coordinate hyperplane $\{x_N = 0\}$ in the origin.

Let $\mathcal{V}$ be a bounded open set in $\mathbb{R}^N$ such that

$$0 \in \mathcal{V} \quad \text{and} \quad \operatorname{diam}(\mathcal{V}) = \sup\{|x-y| : x, y \in \mathcal{V}\} < r_0. \tag{1.3}$$

For every $\varepsilon \in (0,1)$, let

$$\Sigma_\varepsilon := (\varepsilon\mathcal{V}) \cap \partial\Omega, \tag{1.4}$$

where $\varepsilon\mathcal{V} = \{\varepsilon x : x \in \mathcal{V}\} \subset B_{r_0}$. Furthermore, we set

$$\Sigma = \mathcal{V} \cap \partial\mathbb{R}^N_+ = \{x = (x_1, x_2, \dots, x_N) \in \mathcal{V} : x_N = 0\}, \tag{1.5}$$

where $\mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$.

We consider the following eigenvalue problem with mixed boundary conditions

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \setminus \Sigma_\varepsilon, \\ \partial_{\boldsymbol{\nu}} u = 0, & \text{on } \Sigma_\varepsilon, \end{cases} \tag{1.6}$$

and we are interested in the asymptotic behaviour of its eigenvalues as $\varepsilon \to 0^+$, that is when the Neumann region Σ_ε is disappearing.

In order to write the weak formulation of (1.6), we first introduce the suitable functional framework. For any open set $\omega \subseteq \mathbb{R}^N$ and for any closed set $\Gamma \subseteq \partial\omega$, we define $H^1_{0,\Gamma}(\omega)$ as the closure in $H^1(\omega)$ of $C^\infty_c(\overline{\omega} \setminus \Gamma)$; we refer to [11] for a more detailed analysis of this kind of space (see also [6]). We note that, if Ω and $\mathcal{V}$ are sufficiently regular (for example Lipschitz), then we have the following characterization

$$H^1_{0,\partial\Omega\setminus\Sigma_\varepsilon}(\Omega) = \{u \in H^1(\Omega) : \operatorname{Tr}(u) = 0 \text{ on } \partial\Omega \setminus \Sigma_\varepsilon\}$$

for every $\varepsilon \in (0,1)$, where Tr denotes the trace operator (see [6]). We note also that, formally, when $\varepsilon = 0$ the space $H^1_{0,\partial\Omega}(\Omega)$ coincides with the usual Sobolev space $H^1_0(\Omega)$.

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (1.6) if there exists $\varphi \in H^1_{0,\partial\Omega\setminus\Sigma_\varepsilon}(\Omega)$, $\varphi \not\equiv 0$, called an *eigenfunction*, such that

$$\int_\Omega \nabla\varphi \cdot \nabla v \, dx = \lambda \int_\Omega \varphi v \, dx \quad \text{for every } v \in H^1_{0,\partial\Omega\setminus\Sigma_\varepsilon}(\Omega). \tag{1.7}$$

From classical spectral theory, (1.7) admits a diverging sequence of positive eigenvalues

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \lambda_3^\varepsilon \le \cdots \le \lambda_i^\varepsilon \le \cdots,$$

where each eigenvalue is repeated according to its multiplicity. Letting $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$, we denote by $\{\varphi_i^\varepsilon\}_{i\in\mathbb{N}_*}$ a corresponding sequence of eigenfunctions satisfying

$$\int_\Omega \varphi_i^\varepsilon \varphi_j^\varepsilon = \delta_i^j, \tag{1.8}$$

with δ_i^j denoting the usual *Kronecker delta.* In the sense clarified in (1.7), the functions φ_i^ε weakly solve

$$\begin{cases} -\Delta\varphi_i^\varepsilon = \lambda_i^\varepsilon \varphi_i^\varepsilon, & \text{in } \Omega, \\ \varphi_i^\varepsilon = 0, & \text{on } \partial\Omega \setminus \Sigma_\varepsilon, \\ \partial_{\boldsymbol{\nu}} \varphi_i^\varepsilon = 0, & \text{on } \Sigma_\varepsilon. \end{cases} \tag{1.9}$$

In the limit $\varepsilon = 0$, once the Neumann region Σ_ε has disappeared, we formally recover the eigenvalue problem for the standard Dirichlet Laplacian

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

which is well known to admit a diverging sequence of positive eigenvalues

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_i \le \cdots .$$

We denote by $\{\varphi_i\}_{i\in\mathbb{N}_*}$ a corresponding sequence of eigenfunctions satisfying

$$\int_\Omega \varphi_i \varphi_j \, \mathrm{d}x = \delta_i^j. \tag{1.11}$$

More precisely, λ_i and φ_i solve (1.10) in the sense that $\varphi_i \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla\varphi_i \cdot \nabla v \, \mathrm{d}x = \lambda_i \int_\Omega \varphi_i v \, \mathrm{d}x \quad \text{for all } v \in H_0^1(\Omega). \tag{1.12}$$

In Section 2 we prove that, for all $i \in \mathbb{N}_*$,

$$\lambda_i^\varepsilon \to \lambda_i \quad \text{as } \varepsilon \to 0.$$

The main goal of the present paper is to detect the sharp rate of the above convergence.

The vanishing rate of the eigenvalue variation $\lambda_i^\varepsilon - \lambda_i$ turns out to be strongly related to the behaviour of the Dirichlet eigenfunctions φ_i, locally near the point $0 \in \partial\Omega$. We can derive from [12] the classification of possible vanishing orders of φ_i at the boundary: for every $i \in \mathbb{N}_*$, there exist $\gamma_i \in \mathbb{N}_*$, $\Psi_i \in H_0^1(\mathbb{S}_+^{N-1})$, with $\Psi_i \not\equiv 0$, such that

$$\frac{\varphi_i(rx)}{r^{\gamma_i}} \to |x|^{\gamma_i} \Psi_i\left(\frac{x}{|x|}\right) \quad \text{in } H^1(B_\rho) \text{ as } r \to 0, \text{ for every } \rho > 0, \tag{1.13}$$

where $\mathbb{S}_+^{N-1} = \{(x_1,\dots,x_N) \in \mathbb{R}^N : |x| = 1, x_N > 0\}$ and φ_i, respectively Ψ_i, are trivially extended outside Ω, respectively $\mathbb{S}_+^{N-1}$. Moreover the function Ψ_i is the restriction to $\mathbb{S}_+^{N-1}$ of a spherical harmonic odd with respect to the equator $x_N = 0$ and the γ_i-homogeneous function ψ_i with angular profile Ψ_i, i.e.

$$\psi_i(x) = |x|^{\gamma_i} \Psi_i\left(\frac{x}{|x|}\right), \tag{1.14}$$

is a harmonic homogeneous polynomial of degree γ_i vanishing on $\partial\mathbb{R}_+^N$. In particular, being ψ_i harmonic, nontrivial and vanishing on $\partial\mathbb{R}_+^N$, we have that $\partial_{x_N}\psi_i \not\equiv 0$ on Σ.

In the following we fix $n_0 \in \mathbb{N}_*$ such that

$$\lambda_{n_0} \text{ is simple} \tag{1.15}$$

as an eigenvalue of the standard Dirichlet Laplacian in Ω. Moreover, hereafter we denote

$$\gamma := \gamma_{n_0} \tag{1.16}$$

and

$$\Psi := \Psi_{n_0}, \quad \psi := \psi_{n_0}. \tag{1.17}$$

Under assumption (1.15) it is possible to choose the eigenfunctions $\varphi_{n_0}^\varepsilon$, solving (1.9) with $i = n_0$, in such a way that

$$\varphi_{n_0}^\varepsilon \to \varphi_{n_0} \quad \text{in } H^1(\Omega) \quad \text{as } \varepsilon \to 0,$$

see Proposition 2.4.

The scaled shape (1.5) of the Neumann disappearing region emerges in the asymptotic expansion of the eigenvalue variation in the guise of a coefficient $\mathcal{C}_{n_0}(\Sigma)$ admitting a variational characterization. We define $\mathcal{C}_{n_0}(\Pi)$ for any bounded open subset $\Pi \subset \partial\mathbb{R}_+^N$ such that $0 \in \Pi$. Letting $\mathcal{D}^{1,2}(\mathbb{R}_+^N \cup \Pi)$ be the completion of $C_c^\infty(\mathbb{R}_+^N \cup \Pi)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}_+^N \cup \Pi)} = \sqrt{\int_{\mathbb{R}_+^N} |\nabla u|^2 \, \mathrm{d}x},$$

we define

$$\mathcal{C}_{n_0}(\Pi) := -2 \min\Big\{ J_\Pi(u) : u \in \mathcal{D}^{1,2}(\mathbb{R}_+^N \cup \Pi) \Big\}, \tag{1.18}$$

where

$$J_\Pi : \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Pi) \to \mathbb{R}, \quad J_\Pi(u) := \frac{1}{2}\int_{\mathbb{R}^N_+} |\nabla u|^2 \,\mathrm{d}x + \int_\Pi u \partial_{\boldsymbol{\nu}} \psi \,\mathrm{d}x', \tag{1.19}$$

ψ is defined in (1.17), and $\boldsymbol{\nu} = (0,0,\dots,0,-1)$ is the vertical downward unit vector. By classical variational methods one can easily prove that the minimum in (1.18) is attained (by the function $w_{0,\Pi}$ defined in (3.3)) and $\mathcal{C}_{n_0}(\Pi) > 0$, see Section 3. We define

$$\mathcal{C}_{n_0} = \mathcal{C}_{n_0}(B'_1) > 0, \tag{1.20}$$

where we are denoting, for all $r > 0$,

$$B'_r := B_r \cap \partial\mathbb{R}^N_+. \tag{1.21}$$

Our first main result provides asymptotic lower and upper bounds of the eigenvalue variation as $\varepsilon \to 0$.

Theorem 1.1. *Let $\mathcal{V} \subset \mathbb{R}^N$ be a bounded open set satisfying (1.3) and $0 < r_{\mathcal{V}} < R_{\mathcal{V}} < r_0$ be such that $B_{r_{\mathcal{V}}} \subset \mathcal{V} \subset B_{R_{\mathcal{V}}}$. Then*

$$\mathcal{C}_{n_0} r_{\mathcal{V}}^{N+2\gamma-2} \le \liminf_{\varepsilon\to 0} \frac{\lambda_{n_0} - \lambda^\varepsilon_{n_0}}{\varepsilon^{N+2\gamma-2}} \le \limsup_{\varepsilon\to 0} \frac{\lambda_{n_0} - \lambda^\varepsilon_{n_0}}{\varepsilon^{N+2\gamma-2}} \le \mathcal{C}_{n_0} R_{\mathcal{V}}^{N+2\gamma-2}$$

with $\mathcal{C}_{n_0}$ as in (1.20) and γ as in (1.16).

The proof of Theorem 1.1 will be obtained by comparison of the eigenvalues $\lambda^\varepsilon_{n_0}$ of problem (1.9) with the eigenvalues of the analogous problems with $\mathcal{V}$ replaced by the balls $B_{R_{\mathcal{V}}}$ and $B_{r_{\mathcal{V}}}$, for which a more precise asymptotic expansion can be derived, exploiting the star-shapedness of the Neumann region.

We now state the sharper asymptotic estimates which we are able to obtain under stronger regularity and geometric assumptions on the set $\mathcal{V}$. Let us assume, in addition to (1.3), that

$$\mathcal{V} \text{ is of class } C^{1,1} \tag{1.22}$$

and $\mathcal{V}$ is strictly star-shaped with respect to the origin, i.e.

$$\text{there exists } \sigma > 0 \text{ such that } x \cdot \boldsymbol{\nu}(x) \ge \sigma \quad \text{for every } x \in \partial\mathcal{V}, \tag{1.23}$$

where $\boldsymbol{\nu}(x)$ is the exterior unit normal vector at $x \in \partial\mathcal{V}$. We observe that the notion of *strict star-shapedness* given in (1.23) is equivalent to the notion of *star-shapedness with respect to a ball* discussed in [21, Section 1.1.8], see [7, Lemma 1] and [20, Lemma 3.2].

Theorem 1.2. *If $\mathcal{V}$ satisfies (1.22) and (1.23) in addition to (1.3), then the following asymptotic expansion holds*

$$\lambda^\varepsilon_{n_0} = \lambda_{n_0} - \mathcal{C}_{n_0}(\Sigma)\varepsilon^{N+2\gamma-2} + o(\varepsilon^{N+2\gamma-2}) \quad \text{as } \varepsilon \to 0,$$

with $\mathcal{C}_{n_0}(\Sigma)$ as in (1.18) and Σ as in (1.5).

The proof of Theorem 1.2 is obtained through sharp estimates from above and below of the Rayleigh quotients for the eigenvalues $\lambda^\varepsilon_{n_0}$ and λ_{n_0}, which in turn require energy bounds, uniform with respect to ε, on eigenfunctions, provided by an Almgren-type monotonicity argument. The last step in the achievement of sharp eigenvalue estimates consists of a blow-up analysis for scaled eigenfunctions.

The Almgren frequency function at a point is given by the ratio of the local energy over the mass near that point, see [5]; we refer to formula (6.2) in Section 6 for the precise definition of the frequency for our problem. Monotonicity of this quotient implies a uniform control of the local energies and, in the classical case of harmonic functions, such a monotonicity easily follows from the positivity of the derivative. For solutions u of elliptic equations of the type

$$-\Delta u = Vu,$$

with V bounded, the frequency is no more monotone because of the presence of the potential. However, the "perturbed frequency"

$$\mathcal{N}_V(r) := \frac{r \displaystyle\int_{|x-x_0|<r} (|\nabla u|^2 - V u^2)\,\mathrm{d}x}{\displaystyle\int_{|x-x_0|=r} u^2\,\mathrm{d}S}$$

still enjoys some monotonicity properties, in the form of an estimate of the type

$$\mathcal{N}_V'(r) \geq -\mathrm{const}\,\mathcal{N}_V(r),$$

which allows proving boundedness of $\mathcal{N}_V$ and then energy estimates for u. In this spirit, here we mean to prove boundedness of the frequency of eigenfunctions φ_i^ε at the origin (which belongs to the boundary), uniformly with respect to the parameter ε, by establishing its perturbed monotonicity through an estimate from below of its derivative. Since, in the case we are considering here, all neighbourhoods of the origin contain portions of the boundary, some additional boundary terms appear in the derivative of the frequency; star-shapedness assumption (1.23) forces these remainder terms to have a sign which is favorable to the desired estimate. On the other hand, the lack of regularity of the eigenfunctions φ_i^ε at Dirichlet-Neumann junctions prevents us from a direct differentiation of the frequency function, which requires a Pohozaev-type identity based on the integration of the Rellich-Necas identity (5.10). For what concerns this last issue, considerable differences appear between the cases $N = 2$ and $N \geq 3$, as explained in the following Remark.

Remark 1.3. With respect to the 2-dimensional case treated in [3], significant new difficulties arise, mainly due to regularity issues for mixed boundary value problems like (1.9), which turn out to be more delicate in dimension $N \geq 3$ because of the positive dimension of the junction set $\partial\Sigma_\varepsilon$ and some role played by the geometry of Σ_ε, in particular in connection with its star-shapedness. Indeed, when $N = 2$ the interface $\partial\Sigma_\varepsilon$ has zero dimension (basically, it consists of a couple of points) and it is possible to perform an approximation just by removing a small neighbourhood of the junction points, thus allowing quite explicit computations in the derivation of Almgren monotonicity formulas (see [3, Lemma C.5]). In higher dimensions, we overcome the difficulties produced by lack of regularity of solutions by constructing a sequence of approximating problems, for which enough regularity is available to derive Pohozaev-type identities, needed, in turn, to obtain Almgren-type monotonicity formulas and consequently to perform blow-up analysis. In particular, the geometry of the boundary manifests in the form of some extra remainder terms appearing in the Pohozaev-type identity for the regularized problem and depending on the mutual orientation of normal and position vectors, whose control motivates here the geometric assumption (1.23), which is, in fact, a star-shapedness condition on the Neumann region, see Proposition 5.1 and, in particular, (5.27).

Under the same assumptions of Theorem 1.2, the blow-up analysis performed in Section 9 allows us to describe the behaviour of perturbed eigenfunctions $\varphi_{n_0}^\varepsilon$ when they are scaled at the origin, i.e. at the point around which Σ_ε is shrinking, thus yielding the following result.

Theorem 1.4. *Let $\mathcal{V}$ satisfy (1.3), (1.22) and (1.23). Let $U = \psi + w_{0,\Sigma}$, where ψ is as in (1.17) and $w_{0,\Sigma}$ (defined in (3.3)) is the unique minimizer of the functional J_Σ introduced in (1.19). Then, for any $R > 0$ sufficiently large,*

$$\varepsilon^{-N-2\gamma} \int_{\Omega\cap B_{R\varepsilon}} \left|\varphi_{n_0}^\varepsilon\right|^2 \mathrm{d}x \to \int_{B_R^+} U^2\,\mathrm{d}x$$
$$\varepsilon^{-N-2\gamma+2} \int_{\Omega\cap B_{R\varepsilon}} \left|\nabla\varphi_{n_0}^\varepsilon\right|^2 \mathrm{d}x \to \int_{B_R^+} |\nabla U|^2\,\mathrm{d}x,$$

as $\varepsilon \to 0$.

The paper is structured as follows. In Section 2 we prove convergence of eigenvalues and eigenfunctions as $\varepsilon \to 0$ to eigenelements of the unperturbed problem. In Section 3 we construct the limit profiles, which will appear in the blow-up analysis, by minimization of the functional introduced in (1.19). In Section 4 we introduce an equivalent auxiliary problem obtained by deforming the boundary of Ω into a straight hyperplane; to this aim we use a particular diffeomorphism, introduced in [4] and made on purpose to ensure that the equation is conserved by reflection through the straightened boundary. Section 5 is devoted to a Pohozaev-type identity for the approximating problems, which is then used in Section 6 to develop a monotonicity argument, from which energy estimates follow. In Sections 7 and 8 we prove sharp upper and lower bounds for the eigenvalue variation, while Section 9 is devoted to a blow-up analysis for scaled eigenfunctions. In Section 10 we combine the lower/upper estimates on the eigenvalue variation and the blow analysis to prove Theorem 1.2, which is then combined with a comparison and scaling argument to prove Theorem 1.1 in Section 11. Finally, in the appendix we recall some Poincaré-type inequalities and an abstract lemma on maxima of quadratic forms.

2. Convergence of eigenelements

In the following we tacitly assume that the hypotheses on Ω set out in the Introduction and assumption (1.3) on $\mathcal{V}$ are satisfied; consequently we let Σ_ε be as in (1.4). In this section we prove that the eigenvalues and eigenfunctions of the perturbed problem converge, as $\varepsilon \to 0$, to the corresponding unperturbed eigenelements.

Lemma 2.1. *For any $\varepsilon \in (0,1)$, let $\lambda_\varepsilon \in \mathbb{R}$ be an eigenvalue of problem (1.7) and $\varphi_\varepsilon \in H^1_{0,\partial\Omega\setminus\Sigma_\varepsilon}(\Omega)$ be an associated eigenfunction such that $\int_\Omega \varphi_\varepsilon^2 \,\mathrm{d}x = 1$. Let us assume that there exists a decreasing sequence $(\varepsilon_n)_n \subseteq (0,1)$ and a real number λ^* such that $\varepsilon_n \to 0$ and $\lambda_{\varepsilon_n} \to \lambda^*$ as $n \to \infty$. Then there exist a subsequence $(\varepsilon_{n_i})_i$ and $\varphi^* \in H^1_0(\Omega)$ such that*

$$\varphi_{\varepsilon_{n_i}} \rightharpoonup \varphi^* \text{ weakly in } H^1(\Omega) \quad \text{and} \quad \varphi_{\varepsilon_{n_i}} \to \varphi^* \text{ strongly in } L^2(\Omega),$$

as $i \to \infty$. Moreover λ^* is an eigenvalue of the Dirichlet Laplacian in Ω with φ^* as an eigenfunction, in the sense of (1.12), and $\int_\Omega |\varphi^*|^2 \,\mathrm{d}x = 1$.

Proof. By hypothesis we have that

$$\int_\Omega \varphi_{\varepsilon_n}^2 \,\mathrm{d}x = 1 \quad \text{and} \quad \int_\Omega |\nabla \varphi_{\varepsilon_n}|^2 \,\mathrm{d}x = \lambda_{\varepsilon_n} = \lambda^* + o(1)$$

as $n \to \infty$. Therefore there exist a subsequence $(\varepsilon_{n_i})_i$ and $\varphi^* \in H^1(\Omega)$ such that

$$\varphi_{\varepsilon_{n_i}} \rightharpoonup \varphi^* \text{ weakly in } H^1(\Omega) \quad \text{and} \quad \varphi_{\varepsilon_{n_i}} \to \varphi^* \text{ strongly in } L^2(\Omega),$$

as $i \to \infty$. We first aim at showing that $\varphi^* \in H^1_0(\Omega)$. In order to do this, let $r_2 > r_1 > 0$ and let $\zeta \in C^\infty_c(\mathbb{R}^N)$ be such that $\operatorname{supp}(\zeta) \subset B_{r_2}$ and $\zeta(x) = 1$ for every $x \in B_{r_1}$. For every $\delta > 0$ and $x \in \mathbb{R}^N$, we define $\zeta_\delta(x) = \zeta(x/\delta)$. First we notice that $(1-\zeta_\delta)\varphi^* \in H^1_0(\Omega)$ for every $\delta > 0$. Indeed, by approximation of $\varphi_{\varepsilon_{n_i}}$ with $C^\infty_c(\Omega \cup \Sigma_{\varepsilon_{n_i}})$-functions, we see that, for every fixed $\delta > 0$ and for i large enough, $(1-\zeta_\delta)\varphi_{\varepsilon_{n_i}} \in H^1_0(\Omega)$; moreover, $(1-\zeta_\delta)\varphi_{\varepsilon_{n_i}} \rightharpoonup (1-\zeta_\delta)\varphi^*$ weakly in $H^1(\Omega)$ as $i \to \infty$, thus implying that $(1-\zeta_\delta)\varphi^* \in H^1_0(\Omega)$. Secondly, we have that, as $\delta \to 0$,

$$\begin{aligned}\|(1-\zeta_\delta)\varphi^* - \varphi^*\|^2_{H^1(\Omega)} &= \|\zeta_\delta\varphi^*\|^2_{H^1(\Omega)} \le \int_\Omega \left(2|\nabla\zeta_\delta|^2(\varphi^*)^2 + 2\zeta_\delta^2|\nabla\varphi^*|^2 + \zeta_\delta^2(\varphi^*)^2\right)\mathrm{d}x \\ &\le 2\delta^{-2}\|\nabla\zeta\|^2_{L^\infty(\mathbb{R}^N)} \int_{B_{\delta r_2}\setminus B_{\delta r_1}} (\varphi^*)^2 \,\mathrm{d}x + o(1) \le C\left(\int_{B_{\delta r_2}\setminus B_{\delta r_1}} (\varphi^*)^{2^*} \,\mathrm{d}x\right)^{2/2^*} + o(1) = o(1),\end{aligned}$$

for some $C > 0$, with $2^* = 2N/(N-2)$. Hence $(1-\zeta_\delta)\varphi^* \in H^1_0(\Omega)$ for every $\delta > 0$ and converges to φ^* in $H^1(\Omega)$ as $\delta \to 0$, so that $\varphi^* \in H^1_0(\Omega)$.

By strong $L^2(\Omega)$-convergence, we have that $\int_\Omega |\varphi^*| \,\mathrm{d}x = 1$. Finally, by hypothesis we have that, for every i,

$$\int_\Omega \nabla\varphi_{\varepsilon_{n_i}} \cdot \nabla\phi \,\mathrm{d}x = \lambda_{\varepsilon_{n_i}} \int_\Omega \varphi_{\varepsilon_{n_i}} \phi \,\mathrm{d}x,$$

for all $\phi \in H^1_{0,\partial\Omega\setminus\Sigma_{\varepsilon_{n_i}}}(\Omega)$ and, in particular, for any $\phi \in H^1_0(\Omega)$. Passing to the limit as $i \to \infty$ in the previous equation for $\phi \in H^1_0(\Omega)$ proves that φ^* and λ^* satisfy (1.12) thus completing the proof. ☐

Remark 2.2. Some basic relations among the families of perturbed eigenvalues and between the perturbed and unperturbed sequences can be easily observed. The eigenvalues λ_i^ε admit the following classical Min-Max variational characterization

$$\lambda_i^\varepsilon = \min\left\{ \max_{u\in F_i} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} : F_i \subset H^1_{0,\partial\Omega\setminus\Sigma_\varepsilon}(\Omega) \ i\text{-dimensional subspace}\right\}. \tag{2.1}$$

By (1.3), for every $\varepsilon_1 > 0$ there exists $0 < \varepsilon_2 < \varepsilon_1$ such that $H^1_{0,\partial\Omega\setminus\Sigma_{\varepsilon_2}}(\Omega) \subset H^1_{0,\partial\Omega\setminus\Sigma_{\varepsilon_1}}(\Omega)$. Then for every sequence $\varepsilon_n \to 0$ there exists a decreasing subsequence $\{\varepsilon_{n_k}\}$ such that, for every $i \in \mathbb{N}_*$,

$$\lambda_i^{\varepsilon_{n_{k+1}}} \geq \lambda_i^{\varepsilon_{n_k}} \quad \text{for every } k.$$

Moreover, since $H^1_0(\Omega) \subset H^1_{0,\partial\Omega\setminus\Sigma_\varepsilon}(\Omega)$ for every $\varepsilon > 0$, we readily get, for every $i \in \mathbb{N}_*$,

$$\lambda_i \geq \lambda_i^\varepsilon \quad \text{for every } \varepsilon > 0. \tag{2.2}$$

Proposition 2.3. *For any $i \in \mathbb{N}_*$, $\lambda_i^\varepsilon \to \lambda_i$ as $\varepsilon \to 0$.*

Proof. By Remark 2.2 and Urysohn's subsequence principle, it is enough to prove the convergence along sequences $\varepsilon_n \to 0$ for which $n \mapsto \lambda^i_{\varepsilon_n}$ is increasing; then, by (2.2), it is not restrictive to assume that $\varepsilon \mapsto \lambda^i_\varepsilon$ is decreasing for any $i \in \mathbb{N}_*$ and admits a limit $\lambda_i^* \leq \lambda_i$ as $\varepsilon \to 0$. We now prove that, for any $i \in \mathbb{N}_*$, $\lambda_i \leq \lambda_i^*$. We argue by induction on i. From Lemma 2.1 we know that λ_1^* is an eigenvalue of the unperturbed problem so that, $\lambda_1^* \geq \lambda_1$. Let us now assume that

$$\lambda_j^* = \lambda_j \qquad \text{for all } j = 1, \dots, i-1. \tag{2.3}$$

Let $\varphi_1^\varepsilon, \dots, \varphi_i^\varepsilon$ be a family of perturbed eigenfunctions as in (1.8). By Lemma 2.1 there exist a sequence $\varepsilon_n \to 0$ as $n \to \infty$ and functions $u_1^*, \dots, u_i^*$, that are eigenfunctions of the unperturbed problem (1.12), such that

$$\varphi_j^{\varepsilon_n} \rightharpoonup u_j^* \text{ weakly in } H^1(\Omega) \quad \text{and} \quad \varphi_j^{\varepsilon_n} \to u_j^* \text{ strongly in } L^2(\Omega), \tag{2.4}$$

as $n \to \infty$, for every $j = 1, \dots, i$.

On the one hand, by passing to the limit as $n \to \infty$ in (1.8), we obtain

$$\int_\Omega u_j^* u_l^* \, dx = \delta_j^l. \tag{2.5}$$

for all $j, l \in \{1, \dots, i\}$. On the other hand, by (2.3) and (2.4), for every $j = 1, \dots, i-1$, u_j^* is a $L^2(\Omega)$-normalized eigenfunction corresponding to the eigenvalue λ_j . Therefore, in view also of (2.5), $u_i^* \in \operatorname{span}\{u_1^*, \dots, u_{i-1}^*\}^\perp$. From the iterative variational characterization of the eigenvalues (see e.g. [19, Section 11.5]) we have that

$$\lambda_i^* = \int_\Omega |\nabla u_i^*|^2 \, dx \geq \min_{u\in \operatorname{span}\{u_1^*,\dots,u_{i-1}^*\}^\perp} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} = \lambda_i$$

and this concludes the proof. ☐

Proposition 2.4. *Let λ_i be a simple eigenvalue of (1.12) and let φ_i be a $L^2(\Omega)$-normalized associated eigenfunction. For any $\varepsilon \in (0,1)$ let λ_i^ε be the i-th eigenvalue of (1.9) and let φ_i^ε be a $L^2(\Omega)$-normalized associated eigenfunction such that*

$$\int_\Omega \varphi_i^\varepsilon \varphi_i \, dx \geq 0. \tag{2.6}$$

Then $\varphi_i^\varepsilon \to \varphi_i$ in $H^1(\Omega)$ as $\varepsilon \to 0$.

Proof. From Lemma 2.1 and Proposition 2.3 we infer that there exist a sequence $\varepsilon_n \to 0$ as $n \to \infty$ and $\varphi^* \in H^1_0(\Omega)$, eigenfunction associated to λ_i, such that

$$\varphi_i^{\varepsilon_n} \rightharpoonup \varphi^* \text{ weakly in } H^1(\Omega) \quad \text{and} \quad \varphi_i^{\varepsilon_n} \to \varphi^* \text{ strongly in } L^2(\Omega)$$

as $n \to \infty$. In particular, by strong $L^2(\Omega)$-convergence, φ^* is $L^2(\Omega)$-normalized. Being λ_i simple, we have that either $\varphi^* = \varphi_i$ or $\varphi^* = -\varphi_i$. The assumption (2.6) rules out the second possibility, allowing us to conclude that $\varphi^* = \varphi_i$.

Finally, in view of Proposition 2.3,

$$\|\varphi_i^{\varepsilon_n}\|^2_{H^1(\Omega)} = \lambda_i^{\varepsilon_n} + 1 \to \lambda_i + 1 = \|\varphi_i\|^2_{H^1(\Omega)}$$

as $n \to \infty$. Hence $\varphi_i^{\varepsilon_n} \to \varphi_i$ strongly in $H^1(\Omega)$ as $n \to \infty$. By Urysohn's subsequence principle the convergence holds as $\varepsilon \to 0$, thus concluding the proof. □

3. Limit profiles

We now introduce the functions that appear as limit profiles in the blow-up analysis. From here on, for any $R > 0$, we denote by η_R a cut-off function such that

$$\eta_R \in C^\infty(\overline{\mathbb{R}^N_+}), \quad 0 \le \eta_R \le 1, \quad |\nabla \eta_R| \le \frac{4}{R}, \quad \eta_R(x) := \begin{cases} 1, & \text{if } |x| \ge R, \\ 0, & \text{if } |x| \le R/2. \end{cases} \tag{3.1}$$

Lemma 3.1. *Let Π be a bounded open subset of $\partial\mathbb{R}^N_+$ such that $0 \in \Pi$ and let $f \in L^2(\Pi)$. Then there exists a unique function $w = w(f, \Pi) \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Pi)$ such that*

$$\begin{cases} -\Delta w = 0, & \text{in } \mathbb{R}^N_+, \\ w = 0, & \text{on } \partial\mathbb{R}^N_+ \setminus \Pi, \\ \partial_{\boldsymbol{\nu}} w = f, & \text{on } \Pi, \end{cases}$$

where $\boldsymbol{\nu} = (0, \dots, 0, -1)$, in a weak sense, that is

$$\int_{\mathbb{R}^N_+} \nabla w \cdot \nabla v \, \mathrm{d}x = \int_\Pi f v \, \mathrm{d}x' \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Pi). \tag{3.2}$$

Proof. The result is a direct consequence of the Lax-Milgram Theorem. □

Given $\psi \in C^\infty(\overline{\mathbb{R}^N_+})$ as in (1.17), we define

$$w_{0,\Pi} := w(-\partial_{\boldsymbol{\nu}}\psi, \Pi), \tag{3.3}$$

where $w(\cdot,\cdot)$ is defined in Lemma 3.1. We point out that the function $w_{0,\Pi} \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Pi)$ is the unique minimizer, among all the possible $u \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Pi)$, of the functional J_Π defined in (1.19). We denote

$$m_{n_0}(\Pi) := J(w_{0,\Pi}) = \min_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Pi)} J_\Pi(u). \tag{3.4}$$

Since, for any bounded open set $\Pi \subset \partial\mathbb{R}^N_+$, $\partial_{x_N}\psi \not\equiv 0$ on Π, we have that $w_{0,\Pi} \not\equiv 0$ and hence, choosing $v = w_{0,\Pi}$ in (3.2), we obtain that

$$m_{n_0}(\Pi) = \frac{1}{2}\int_\Pi w_{0,\Pi}\partial_{\boldsymbol{\nu}}\psi \, \mathrm{d}x' = -\frac{1}{2}\int_{\mathbb{R}^N_+} |\nabla w_{0,\Pi}|^2 \, \mathrm{d}x < 0. \tag{3.5}$$

The following lemma is a consequence of the homogeneity of the function ψ.

Lemma 3.2. *For every $r > 0$ we have that $m_{n_0}(B'_r) = r^{N+2\gamma-2} m_{n_0}(B'_1)$, with γ as in (1.16).*

Proof. Since $\partial_{\boldsymbol{\nu}}\psi(rx') = r^{\gamma-1}\partial_{\boldsymbol{\nu}}\psi(x')$, a scaling argument easily yields that

$$w_{0,B'_r}(x) = r^\gamma w_{0,B'_1}\Big(\frac{x}{r}\Big).$$

Hence, by a change of variable, we obtain that

$$m_{n_0}(B'_r) = -\frac{1}{2}\int_{\mathbb{R}^N_+} \left|\nabla w_{0,B'_r}(x)\right|^2 \, \mathrm{d}x = -\frac{1}{2} r^{2(\gamma-1)} \int_{\mathbb{R}^N_+} \left|\nabla w_{0,B'_1}(x/r)\right|^2 \, \mathrm{d}x$$
$$= -\frac{1}{2} r^{N+2\gamma-2} \int_{\mathbb{R}^N_+} \left|\nabla w_{0,B'_1}(y)\right|^2 \, \mathrm{d}y = r^{N+2\gamma-2} m_{n_0}(B'_1),$$

thus concluding that proof. □

Hereafter in this section, we let Σ be as in (1.5), for some $\mathcal{V}$ satisfying (1.3), (1.22) and (1.23), and define

$$w_0 := w_{0,\Sigma}$$

and

$$U = w_0 + \psi. \tag{3.6}$$

One can see that $U \in \psi + \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Sigma)$ weakly satisfies the following boundary value problem

$$\begin{cases} -\Delta U = 0, & \text{in } \mathbb{R}^N_+, \\ U = 0, & \text{on } \partial\mathbb{R}^N_+ \setminus \Sigma, \\ \partial_{\boldsymbol{\nu}} U = 0, & \text{on } \Sigma, \end{cases} \tag{3.7}$$

i.e. $\int_{\mathbb{R}^N_+} \nabla U \cdot \nabla v \, \mathrm{d}x = 0$ for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Sigma)$.

The two following existence results Lemma 3.3 and Lemma 3.4 can be easily proved by standard minimization methods. Henceforth we denote, for all $R > 0$,

$$B^+_R := B_R \cap \mathbb{R}^N_+, \quad \text{and} \quad S^+_R := \partial B_R \cap \mathbb{R}^N_+. \tag{3.8}$$

Lemma 3.3. *Let $\Sigma \subseteq \partial\mathbb{R}^N_+$ be as in (1.5). For every $R > 1$ there exists a unique function $U_R \in \psi + H^1_{0,\partial B^+_R \setminus \Sigma}(B^+_R)$ achieving*

$$\min\left\{ \|\nabla u\|^2_{L^2(B^+_R)} : u \in \psi + H^1_{0,\partial B^+_R \setminus \Sigma}(B^+_R) \right\}.$$

Moreover, U_R weakly solves

$$\begin{cases} -\Delta U_R = 0, & \text{in } B^+_R, \\ U_R = \psi, & \text{on } \partial B^+_R \setminus \Sigma, \\ \partial_{\boldsymbol{\nu}} U_R = 0, & \text{on } \Sigma, \end{cases} \tag{3.9}$$

that is

$$\begin{cases} U_R - \psi \in H^1_{0,\partial B^+_R \setminus \Sigma}(B^+_R), \\ \int_{B^+_R} \nabla U_R \cdot \nabla \phi \, \mathrm{d}x = 0 \quad \text{for all } \phi \in H^1_{0,\partial B^+_R \setminus \Sigma}(B^+_R). \end{cases} \tag{3.10}$$

Lemma 3.4. *Let $\Sigma \subseteq \partial\mathbb{R}^N_+$ be as in (1.5), $R > 2$, η_R as in (3.1), and U as in (3.6). Then there exists a unique $Z_R \in \eta_R U + H^1_0(B^+_R)$ achieving*

$$\min\left\{ \|\nabla u\|^2_{L^2(B^+_R)} : u \in \eta_R U + H^1_0(B^+_R) \right\}.$$

Moreover, Z_R weakly solves

$$\begin{cases} -\Delta Z_R = 0, & \text{in } B^+_R, \\ Z_R = U, & \text{on } S^+_R, \\ Z_R = 0, & \text{on } B'_R, \end{cases} \tag{3.11}$$

that is

$$\begin{cases} Z_R \in \eta_R U + H^1_0(B^+_R), \\ \int_{B^+_R} \nabla Z_R \cdot \nabla \phi \, \mathrm{d}x = 0 \quad \text{for all } \phi \in H^1_0(B^+_R). \end{cases}$$

The function U_R naturally appears as a limit profile of a scaled convenient competitor in the estimate of the eigenvalue variation $\lambda_{n_0} - \lambda_{n_0}^\varepsilon$ from below. Indeed, to estimate $\lambda_{n_0}^\varepsilon$ from above with the most precise approximation of λ_{n_0}, we are led to test the Rayleigh quotient for $\lambda_{n_0}^\varepsilon$ with test functions obtained by modifying the limit eigenfunction φ_{n_0} (inside balls $B_{R\varepsilon}$) into a solution of the mixed boundary problem, in the less expensive way from the energetic point of view. By the Dirichlet principle, among all functions satisfying some prescribed boundary conditions, the energy is minimized by harmonic ones, so that the above defined harmonic functions U_R turn out to be the blown-up limit profile of best competitors. In a similar way the function Z_R is the limit profile of scaled best competitors for the estimate of the eigenvalue variation $\lambda_{n_0} - \lambda_{n_0}^\varepsilon$ from above.

The function U_R, introduced in Lemma 3.3, is locally a good approximation of the limit profile U, defined in (3.6), for large values of R.

Lemma 3.5. *For any $r > 2$, $U_R \to U$ in $H^1(B_r^+)$ as $R \to +\infty$.*

Proof. Let $R > r$ and η_R as in (3.1). The function $U_R - U \in H^1(B_r^+)$ weakly solves

$$\begin{cases} -\Delta(U_R - U) = 0, & \text{in } B_R^+, \\ U_R - U = 0, & \text{on } B_R' \setminus \Sigma, \\ \partial_{\boldsymbol{\nu}}(U_R - U) = 0, & \text{on } \Sigma, \\ U_R - U = \psi - U, & \text{on } S_R^+. \end{cases}$$

By the Dirichlet principle, we observe that $\eta_R(\psi - U) \in H^1(B_R^+)$ has higher energy than $U_R - U$ in B_R^+. Hence

$$\begin{aligned} \int_{B_r^+} |\nabla(U_R - U)|^2 \, dx &\le \int_{B_R^+} |\nabla(U_R - U)|^2 \, dx \le \int_{B_R^+} |\nabla(\eta_R(\psi - U))|^2 \, dx \\ &\le 2\int_{B_R^+} |\nabla \eta_R|^2 |\psi - U|^2 \, dx + 2\int_{B_R^+} \eta_R^2 |\nabla(\psi - U)|^2 \, dx \\ &\le \frac{32}{R^2} \int_{B_R^+ \setminus B_{R/2}^+} |\psi - U|^2 \, dx + 2\int_{B_R^+ \setminus B_{R/2}^+} |\nabla(\psi - U)|^2 \, dx \\ &\le 32 \int_{B_R^+ \setminus B_{R/2}^+} \frac{|\psi - U|^2}{|x|^2} \, dx + 2\int_{B_R^+ \setminus B_{R/2}^+} |\nabla(\psi - U)|^2 \, dx. \end{aligned}$$

The last two terms vanish as $R \to +\infty$ thanks to the validity of the Hardy inequality and the fact that $\psi - U \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Sigma)$. Since $B_r^+ \setminus \Sigma$ has positive $(N-1)$-dimensional measure, by Poincaré inequality we may conclude the proof. □

4. An equivalent problem on a domain with a flat crack

Sections 4 to 10 are devoted to the development of the monotonicity formula and the consequent energy and eigenvalue estimates needed to prove Theorem 1.2 and requiring regularity and star-shapedness assumptions on the open set $\mathcal{V}$. Then throughout Sections 4 to 10 we tacitly assume hypotheses (1.3), (1.22), and (1.23) on $\mathcal{V}$, besides the assumptions on Ω set out in the Introduction; consequently we let Σ_ε be as in (1.4) and Σ as in (1.5).

In the present section, we first introduce an equivalent problem, obtained by straightening the boundary of Ω locally around the origin. Then, we prove that the star-shapedness of the Neumann region is preserved by such a transformation, see Section 4.2.

4.1. **Flattening the boundary of the domain.** In this section, we consider a particular diffeomorphism straightening the boundary of Ω near 0, first introduced in [4], see also [13]. Let $g \in C^{1,1}(\mathbb{R}^{N-1})$ be as in (1.1). Let $\zeta \in C_c^\infty(\mathbb{R}^{N-1})$ be such that $\operatorname{supp}\zeta \subset B_1'$, $\zeta \ge 0$ in $\mathbb{R}^{N-1}$, $\int_{\mathbb{R}^{N-1}} \zeta(y')\, dy' = 1$ (see (1.21) for the notation B_1'). For every $\delta > 0$ we consider the mollifier

$$\zeta_\delta(y') = \delta^{-N+1} \zeta\left(\frac{y'}{\delta}\right).$$

For all $j = 1, \dots, N-1$, we define

$$G_j(y', y_N) = \begin{cases} \left(\zeta_{y_N} \star \dfrac{\partial g}{\partial y_j}\right)(y'), & \text{if } y' \in \mathbb{R}^{N-1},\ y_N > 0, \\ \dfrac{\partial g}{\partial y_j}(y'), & \text{if } y' \in \mathbb{R}^{N-1},\ y_N = 0, \end{cases}$$

where $\star$ denotes the convolution product. We observe that, for all $j = 1, \dots, N-1$, $G_j \in C^\infty(\mathbb{R}^N_+)$, G_j is Lipschitz continuous in $\overline{\mathbb{R}^N_+}$, and $\frac{\partial G_j}{\partial y_i} \in L^\infty(\mathbb{R}^N_+)$ for every $i \in \{1, \dots, N\}$. Furthermore we have that, for all $j = 1, \dots, N-1$ and $i = 1, \dots, N$,

$$y_N \frac{\partial G_j}{\partial y_i} \quad \text{is Lipschitz continuous in } \overline{\mathbb{R}^N_+}.$$

Let, for every $j = 1, \dots, N-1$,

$$\widetilde{G}_j : \mathbb{R}^N \to \mathbb{R}, \quad \widetilde{G}_j(y', y_N) := G_j(y', |y_N|)$$

and

$$F_j : \mathbb{R}^N \to \mathbb{R}, \quad F_j(y', y_N) = y_j - y_N \widetilde{G}_j(y', y_N).$$

It follows that $\widetilde{G}_j$ is Lipschitz continuous in $\mathbb{R}^N$ and F_j belongs to $C^{1,1}(\mathbb{R}^N)$ (i.e. it is continuously differentiable with Lipschitz gradient) for all $j = 1, \dots, N-1$.

In particular, defining $\widetilde{G} : \mathbb{R}^N \to \mathbb{R}^{N-1}$ as

$$\widetilde{G}(y', y_N) = (\widetilde{G}_1(y', y_N), \widetilde{G}_2(y', y_N), \dots, \widetilde{G}_{N-1}(y', y_N)),$$

we have that

$$J_{\widetilde{G}} \in L^\infty(\mathbb{R}^N, \mathbb{R}^{N(N-1)}),$$

where $J_{\widetilde{G}}(y', y_N)$ is the Jacobian matrix of $\widetilde{G}$ at (y', y_N), and

$$|\widetilde{G}(y', y_N) - \nabla g(y')| \le C\,|y_N| \quad \text{for all } (y', y_N) \in \mathbb{R}^N, \tag{4.1}$$

for some constant $C > 0$ independent of (y', y_N).

Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be defined as follows

$$F(y', y_N) := (F_1(y', y_N), \dots, F_{N-1}(y', y_N), y_N + g(y')). \tag{4.2}$$

Using the above function F, we are going to construct a diffeomorphism which straightens the boundary. To prove Lemma 5.2, which will be crucial in the monotonicity argument, we will need a quite precise quantification of the behaviour of all entries of the Jacobian matrix of F. Hence, by direct computations and (4.1) we have that

$$J_F(y', y_N) = \begin{pmatrix} 1 - y_N \frac{\partial \widetilde{G}_1}{\partial y_1} & -y_N \frac{\partial \widetilde{G}_1}{\partial y_2} & \cdots & -y_N \frac{\partial \widetilde{G}_1}{\partial y_{N-1}} & -\widetilde{G}_1 - y_N \frac{\partial \widetilde{G}_1}{\partial y_N} \\ -y_N \frac{\partial \widetilde{G}_2}{\partial y_1} & 1 - y_N \frac{\partial \widetilde{G}_2}{\partial y_2} & \cdots & -y_N \frac{\partial \widetilde{G}_2}{\partial y_{N-1}} & -\widetilde{G}_2 - y_N \frac{\partial \widetilde{G}_2}{\partial y_N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -y_N \frac{\partial \widetilde{G}_{N-1}}{\partial y_1} & -y_N \frac{\partial \widetilde{G}_{N-1}}{\partial y_2} & \cdots & 1 - y_N \frac{\partial \widetilde{G}_{N-1}}{\partial y_{N-1}} & -\widetilde{G}_{N-1} - y_N \frac{\partial \widetilde{G}_{N-1}}{\partial y_N} \\ \frac{\partial g}{\partial y_1}(y') & \frac{\partial g}{\partial y_2}(y') & \cdots & \frac{\partial g}{\partial y_{N-1}}(y') & 1 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} I_{N-1} - y_N J_{\widetilde{G}} & -\nabla g(y') + O(y_N) \\ \hline (\nabla g(y'))^T & 1 \end{array}\right),$$

where I_{N-1} is the $(N-1) \times (N-1)$ identity matrix, $\nabla g(y')$ is a column vector and $(\nabla g(y'))^T$ denotes its transpose; henceforth, the notation $O(y_N)$ will be used to denote blocks of matrices with all entries being $O(y_N)$ as $y_N \to 0$ uniformly with respect to y'.

From (1.2) and the assumption that $g \in C^{1,1}(\mathbb{R}^{N-1})$ we deduce that $\nabla g(y') = O(|y'|)$ as $|y'| \to 0$, so that

$$\det J_F(y', y_N) = 1 + |\nabla g(y')|^2 + O(y_N) = 1 + O(|y'|^2) + O(y_N) \tag{4.3}$$

as $y_N \to 0$ and $|y'| \to 0$.

In particular, $\det J_F(0) = 1 \neq 0$. Hence, by the Inverse Function Theorem, F is invertible in a neighbourhood of the origin, i.e. there exists $r_1 \in (0, r_0)$ such that F is a diffeomorphism of class $C^{1,1}$ from B_{r_1} to $\mathcal{U} = F(B_{r_1})$ for some $\mathcal{U}$ open neighbourhood of 0. Moreover, it is possible to choose r_1 sufficiently small so that

$$F^{-1}(\mathcal{U} \cap \Omega) = \mathbb{R}^N_+ \cap B_{r_1} = B^+_{r_1} \quad \text{and} \quad F^{-1}(\mathcal{U} \cap \partial\Omega) = \partial\mathbb{R}^N_+ \cap B_{r_1} = B'_{r_1}, \tag{4.4}$$

i.e. near the origin the image of Ω through F^{-1} has flat boundary (lying on $\partial\mathbb{R}^N_+$). Let

$$\Phi : \mathcal{U} \to B_{r_1}, \quad \Phi := F^{-1}. \tag{4.5}$$

From the fact that

$$\Phi \in C^{1,1}(\mathcal{U}, B_{r_1}), \quad \Phi^{-1} \in C^{1,1}(B_{r_1}, \mathcal{U}), \quad \Phi(0) = \Phi^{-1}(0) = 0, \quad J_\Phi(0) = J_{\Phi^{-1}}(0) = I_N,$$

it follows that

$$J_\Phi(x) = I_N + O(|x|) \quad \text{and} \quad \Phi(x) = x + O(|x|^2) \quad \text{as } |x| \to 0, \tag{4.6}$$

$$J_{\Phi^{-1}}(x) = I_N + O(|x|) \quad \text{and} \quad \Phi^{-1}(x) = x + O(|x|^2) \quad \text{as } |x| \to 0. \tag{4.7}$$

Let $i \in \mathbb{N}_*$ and $\bar\varepsilon \in (0,1)$ be such that $\varepsilon\mathcal{V} \subset \mathcal{U}$ for all $\varepsilon \in (0, \bar\varepsilon)$ (so that $\Phi(\varepsilon\mathcal{V}) \subset B_{r_1}$ for all $\varepsilon \in (0, \bar\varepsilon)$). For $y \in \Phi(\mathcal{U} \cap \Omega) = B^+_{r_1}$, we define

$$u_i^\varepsilon(y) := \varphi_i^\varepsilon(\Phi^{-1}(y)) = \varphi_i^\varepsilon(F(y)).$$

From (1.9) we deduce that

$$\int_{B^+_{r_1}} A(y)\nabla u_i^\varepsilon(y) \cdot \nabla\phi(y)\,\mathrm{d}y = \lambda_i^\varepsilon \int_{B^+_{r_1}} p(y) u_i^\varepsilon(y)\phi(y)\,\mathrm{d}y \tag{4.8}$$

for all $\phi \in H^1_{0,\partial B^+_{r_1} \setminus \widetilde\Sigma_\varepsilon}(B^+_{r_1})$, where

$$\widetilde\Sigma_\varepsilon = \Phi(\Sigma_\varepsilon) = \Phi((\varepsilon\mathcal{V}) \cap \partial\Omega) = \Phi(\varepsilon\mathcal{V}) \cap \partial\mathbb{R}^N_+, \quad p(y) = |\det J_F(y)|, \tag{4.9}$$

and

$$A(y) = (J_F(y))^{-1}((J_F(y))^{-1})^T |\det J_F(y)|. \tag{4.10}$$

Notice that $u_i^\varepsilon \in H^1_{0,B'_{r_1} \setminus \widetilde\Sigma_\varepsilon}(B^+_{r_1})$ for every $\varepsilon \in (0, \bar\varepsilon)$. We observe that (4.8) is the weak formulation of the problem

$$\begin{cases} -\operatorname{div}(A(y)\nabla u_i^\varepsilon(y)) = \lambda_i^\varepsilon\, p(y) u_i^\varepsilon(y), & \text{in } B^+_{r_1}, \\ u_i^\varepsilon = 0, & \text{on } \widetilde\Gamma_{\varepsilon,r_1}, \\ A(y)\nabla u_i^\varepsilon(y) \cdot \boldsymbol{\nu} = 0, & \text{on } \widetilde\Sigma_\varepsilon, \end{cases}$$

where

$$\widetilde\Gamma_{\varepsilon,r_1} = \overline{B'_{r_1}} \setminus \widetilde\Sigma_\varepsilon$$

and $\boldsymbol{\nu}$ is the exterior unit normal vector on $\widetilde\Sigma_\varepsilon$, which is equal to $-\boldsymbol{e}_N$ with $\boldsymbol{e}_N = (0, 0, \ldots, 1)$ since $\widetilde\Sigma_\varepsilon \subset \partial\mathbb{R}^N_+$.

From (4.10) it follows directly that A is symmetric. Moreover, by direct computations we have that

$$A(y) = \alpha(y) B(y) (B(y))^T$$

where, taking into account (4.3),

$$\alpha(y) = \frac{1}{|\det J_F(y)|} = 1 + O(|y'|^2) + O(y_N) \tag{4.11}$$

as $y_N \to 0$ and $|y'| \to 0$, and

$$B = \left(\begin{array}{cccc|c} 1+\sum\limits_{j\neq 1}|\frac{\partial g}{\partial y_j}|^2+O(y_N) & -\frac{\partial g}{\partial y_1}\frac{\partial g}{\partial y_2}+O(y_N) & \cdots & -\frac{\partial g}{\partial y_1}\frac{\partial g}{\partial y_{N-1}}+O(y_N) & \\ -\frac{\partial g}{\partial y_2}\frac{\partial g}{\partial y_1}+O(y_N) & 1+\sum\limits_{j\neq 2}|\frac{\partial g}{\partial y_j}|^2+O(y_N) & \cdots & -\frac{\partial g}{\partial y_2}\frac{\partial g}{\partial y_{N-1}}+O(y_N) & \nabla g+O(y_N) \\ \vdots & \vdots & \ddots & \vdots & \\ -\frac{\partial g}{\partial y_{N-1}}\frac{\partial g}{\partial y_1}+O(y_N) & -\frac{\partial g}{\partial y_{N-1}}\frac{\partial g}{\partial y_2}+O(y_N) & \cdots & 1+\sum\limits_{j\neq N-1}|\frac{\partial g}{\partial y_j}|^2+O(y_N) & \\ \hline \multicolumn{4}{c|}{-(\nabla g)^T+O(y_N)} & 1+O(y_N) \end{array}\right).$$

Hence we have that

$$A(y', y_N) = \alpha(y', y_N)\left(\begin{array}{c|c} I_{N-1}+O(|y'|^2)+O(y_N) & O(y_N) \\ \hline O(y_N) & 1+O(|y'|^2)+O(y_N) \end{array}\right) \tag{4.12}$$

where here $O(y_N)$, respectively $O(|y'|^2)$, denotes blocks of matrices with all entries being $O(y_N)$ as $y_N \to 0$, respectively $O(|y'|^2)$ as $|y'| \to 0$.

From (4.11) and (4.12) it follows that, if r_1 is chosen sufficiently small, $\alpha(y) > 0$ and A is uniformly elliptic in $B_{r_1}^+$. Moreover, by (4.10) and the fact that $F \in C^{1,1}(B_{r_1}, \mathbb{R}^N)$, we have that, if we denote $A(y) = (a_{i,j}(y))_{i,j=1,\dots,N}$, then

$$a_{i,j} \in C^{0,1}(B_{r_1}^+ \cup B'_{r_1}), \tag{4.13}$$

while (4.12) ensures that

$$a_{i,N}(y', 0) = a_{N,i}(y', 0) = 0 \quad \text{for all } i = 1, \dots, N-1. \tag{4.14}$$

The even reflection through the hyperplane $\{y_N = 0\}$ of u_i^ε

$$v_i^\varepsilon(y', y_N) = u_i^\varepsilon(y', |y_N|), \quad (y', y_N) \in B_{r_1}, \tag{4.15}$$

belongs to $H^1_{0,\widetilde{\Gamma}_{\varepsilon,r_1}}(B_{r_1})$ and

$$\int_{B_{r_1}} \widetilde{A}(y)\nabla v_i^\varepsilon(y) \cdot \nabla \phi(y)\,\mathrm{d}y = \lambda_i^\varepsilon \int_{B_{r_1}} \widetilde{p}(y) v_i^\varepsilon(y)\phi(y)\,\mathrm{d}y \tag{4.16}$$

for all $\phi \in H^1_{0,\partial B_{r_1}\cup\widetilde{\Gamma}_{\varepsilon,r_1}}(B_{r_1})$, i.e. v_i^ε weakly solves

$$\begin{cases} -\operatorname{div}(\widetilde{A}(y)\nabla v_i^\varepsilon(y)) = \lambda_i^\varepsilon \widetilde{p}(y) v_i^\varepsilon(y), & \text{in } B_{r_1} \setminus \widetilde{\Gamma}_{\varepsilon,r_1}, \\ v_i^\varepsilon = 0, & \text{on } \widetilde{\Gamma}_{\varepsilon,r_1}, \end{cases} \tag{4.17}$$

where

$$\widetilde{p}(y', y_N) = \begin{cases} p(y', y_N), & \text{if } y_N \geq 0, \\ p(y', -y_N), & \text{if } y_N < 0, \end{cases} \quad \text{and} \quad \widetilde{A}(y', y_N) := \begin{cases} A(y', y_N), & \text{if } y_N > 0, \\ QA(y', -y_N)Q, & \text{if } y_N < 0, \end{cases} \tag{4.18}$$

with

$$Q := \left(\begin{array}{ccc|c} & & & 0 \\ & I_{N-1} & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & -1 \end{array}\right).$$

We observe that (4.13) and (4.14) ensure that the coefficients of the matrix $\widetilde{A}$ are Lipschitz continuous in B_{r_1}.

Remark 4.1. From (4.6)–(4.7) we easily deduce that, as $\varepsilon \to 0$, $\mathbb{R}^N \setminus \big(\partial\mathbb{R}^N_+ \setminus \big(\frac{1}{\varepsilon}\widetilde{\Sigma}_\varepsilon\big)\big)$ converges in the sense of Mosco (see [10, 22]) to the set $\mathbb{R}^N \setminus \big(\partial\mathbb{R}^N_+ \setminus \Sigma\big)$, where Σ is defined in (1.5). In particular, for every $R > 0$, the weak limit points in $H^1(B_R^+)$ as $\varepsilon \to 0$ of a family of functions $\{w_\varepsilon\}_\varepsilon$ with $w_\varepsilon \in H^1_{0,\partial B_R^+\setminus(\frac{1}{\varepsilon}\widetilde{\Sigma}_\varepsilon)}(B_R^+)$ belong to $H^1_{0,\partial B_R^+\setminus\Sigma}(B_R^+)$.

4.2. **Star-shapedness of the transformed domains.** We are interested in proving that the star-shapedness of the set Σ_ε is preserved under the transformation Φ introduced in (4.5). To this aim, we first observe that $\Phi(\varepsilon\mathcal{V})$ is strictly star-shaped provided ε is sufficiently small.

Lemma 4.2. *Let $\mathcal{V}$ be a bounded open set in $\mathbb{R}^N$ satisfying (1.3), (1.22), and (1.23). Then*

(i) *there exists a function $\rho : \mathbb{S}^{N-1} \to \mathbb{R}$ of class $C^{1,1}$ such that $\rho \geq 0$,*

$$\mathcal{V} = \{r\theta : \theta \in \mathbb{S}^{N-1} \text{ and } 0 \leq r < \rho(\theta)\}, \quad \text{and} \quad \partial\mathcal{V} = \{\rho(\theta)\theta : \theta \in \mathbb{S}^{N-1}\}; \tag{4.19}$$

(ii) *letting Φ be as in (4.5), $\Phi(\varepsilon\mathcal{V})$ is strictly star-shaped with respect to 0 provided ε is sufficiently small.*

Proof. (i) Let us define, for all $\theta \in \mathbb{S}^{N-1}$, $\rho(\theta) = \sup\{r > 0 : r\theta \in \mathcal{V}\}$. Assumption (1.23) implies that (4.19) is satisfied. To show that the function ρ is of class $C^{1,1}$ we use the Implicit Function Theorem. To this aim, let us fix $P_0 \in \partial\mathcal{V}$, so that $P_0 = \rho(\theta_0)\theta_0$ for some $\theta_0 \in \mathbb{S}^{N-1}$. Up to a rotation, it is not restrictive to assume that $\theta_0 = \boldsymbol{e_N} = (0, 0, \dots, 1)$. Let $\widetilde{\rho} : B_1' \to \mathbb{R}$, $\widetilde{\rho}(x') = \rho(x', \sqrt{1 - |x'|^2})$. We observe that $\widetilde{\rho}(0) = \rho(\theta_0)$ and that $\widetilde{\rho}$ is the composition of ρ with a smooth local parametrization of $\mathbb{S}^{N-1}$ near 0; hence ρ is of class $C^{1,1}$ in a neighbourhood θ_0 if and only if $\widetilde{\rho}$ is of class $C^{1,1}$ in a neighbourhood 0. Since $\mathcal{V}$ is of class $C^{1,1}$, there exist $\delta > 0$ and a $C^{1,1}$–function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that

$$\mathcal{V} \cap B_\delta(P_0) = \{(x', x_N) \in B(P_0, \delta) : x_N < \varphi(x')\},$$
$$\partial\mathcal{V} \cap B_\delta(P_0) = \{(x', x_N) \in B_\delta(P_0) : x_N = \varphi(x')\}.$$

Let $H : (0, +\infty) \times B_1' \to \mathbb{R}$, $H(r, x') = \varphi(rx') - r\sqrt{1 - |x'|^2}$. We have that H is of class $C^{1,1}$ in a neighbourhood $(\rho_0, 0)$, $H(\rho_0, 0) = 0$ and $\frac{\partial H}{\partial r}(\rho_0, 0) = -1 \neq 0$; furthermore

$$H(\widetilde{\rho}(x'), x') = 0 \quad \text{for } |x'| \text{ small}.$$

By the Implicit Function Theorem we can then conclude that $\widetilde{\rho}$ is of class $C^{1,1}$ in a neighbourhood 0.

(ii) Since $\mathcal{V}$ is of class $C^{1,1}$, there exists a function $G \in C^{1,1}(\mathbb{R}^N)$ such that

$$\mathcal{V} = \{x \in \mathbb{R}^N : G(x) < 0\}, \quad \partial\mathcal{V} = \{x \in \mathbb{R}^N : G(x) = 0\} \quad \text{and} \quad \nabla G(x) \neq 0 \text{ for all } x \in \partial\mathcal{V}.$$

In particular we have that $\boldsymbol{\nu}(x) = \frac{\nabla G(x)}{\|\nabla G(x)\|}$ for all $x \in \partial\mathcal{V}$, hence assumption (1.23) can be reformulated as follows:

$$\text{there exists } \tilde{\sigma} > 0 \text{ such that } x \cdot \nabla G(x) \geq \tilde{\sigma} \quad \text{for every } x \in \partial\mathcal{V}. \tag{4.20}$$

We observe that

$$\partial\left(\Phi(\varepsilon\mathcal{V})\right) = \{x \in \mathbb{R}^N : G_\varepsilon(x) = 0\}, \tag{4.21}$$

where

$$G_\varepsilon(x) = G\left(\frac{\Phi^{-1}(x)}{\varepsilon}\right).$$

From (4.7) we deduce that (by choosing r_1 smaller if necessary) there exists some positive constant $C > 0$ independent of ε such that

$$|x| \leq C\varepsilon \quad \text{for all } x \in \partial\left(\Phi(\varepsilon\mathcal{V})\right). \tag{4.22}$$

Since the exterior unit normal at $x \in \partial\left(\Phi(\varepsilon\mathcal{V})\right)$ has the same direction as $\nabla G_\varepsilon(x)$, to prove assertion (ii) we have to show that, if ε is sufficiently small,

$$\inf_{x \in \partial(\Phi(\varepsilon\mathcal{V}))} x \cdot \nabla G_\varepsilon(x) > 0. \tag{4.23}$$

From (4.7), (4.20), and (4.22) it follows that, for all $x \in \partial(\Phi(\varepsilon\mathcal{V}))$,

$$\begin{aligned} x \cdot \nabla G_\varepsilon(x) &= \frac{x}{\varepsilon} \cdot \nabla G\Big(\frac{\Phi^{-1}(x)}{\varepsilon}\Big) J_{\Phi^{-1}}(x) \\ &= \Big(\frac{\Phi^{-1}(x)}{\varepsilon} + \frac{x - \Phi^{-1}(x)}{\varepsilon}\Big) \cdot \nabla G\Big(\frac{\Phi^{-1}(x)}{\varepsilon}\Big)(I_N + O(|x|)) \\ &= \frac{\Phi^{-1}(x)}{\varepsilon} \cdot \nabla G\Big(\frac{\Phi^{-1}(x)}{\varepsilon}\Big) + \frac{1}{\varepsilon} O(|x|^2) \geq \tilde{\sigma} - O(\varepsilon) > \frac{\tilde{\sigma}}{2} \end{aligned}$$

provided ε is sufficiently small, thus proving claim (4.23). $\square$

We now prove that sections of strictly star-shaped sets are strictly star-shaped.

Lemma 4.3. *Let ω be a $C^{1,1}$ bounded open set in $\mathbb{R}^N$ such that $0 \in \omega$ and ω is strictly star-shaped with respect to the origin (i.e. ω satisfies (1.23)). Then the set $\omega' = \{x' \in \mathbb{R}^{N-1} : (x', 0) \in \omega\}$ is a $C^{1,1}$ strictly star-shaped open subset of $\mathbb{R}^{N-1}$.*

Proof. Since ω is of class $C^{1,1}$, there exists $\tilde{G} \in C^{1,1}(\mathbb{R}^N)$ such that $\omega = \{x \in \mathbb{R}^N : \tilde{G}(x) < 0\}$ and $\partial\omega = \{x \in \mathbb{R}^N : \tilde{G}(x) = 0\}$. Then

$$\omega' = \{x' \in \mathbb{R}^{N-1} : \tilde{g}(x') < 0\}$$

where $\tilde{g}(x') = \tilde{G}(x', 0)$, $\tilde{g} \in C^{1,1}(\mathbb{R}^{N-1})$. We claim that

$$\partial\omega' = \{x' \in \mathbb{R}^{N-1} : (x', 0) \in \partial\omega\}. \tag{4.24}$$

It is easy to verify that, if $x' \in \partial\omega'$, then $(x', 0) \in \partial\omega$. Thus, to prove claim (4.24) it is enough to show that, if $(x', 0) \in \partial\omega$, then $x' \in \partial\omega'$. To this aim, the assumption of strict star-shapedness of ω plays a crucial role. Let $(x', 0) \in \partial\omega$. Then $\tilde{G}(x', 0) = 0$; hence for every $n \in \mathbb{N}_*$ there exists $\xi_n \in \big[0, \frac{1}{n}\big]$ such that

$$\tilde{G}\big((1 - \tfrac{1}{n})x', 0\big) = \nabla\tilde{G}((1 - \xi_n)x', 0) \cdot (-\tfrac{x'}{n}, 0) = -\tfrac{1}{n}\Big(\nabla\tilde{G}(x', 0) \cdot (x', 0) + o(1)\Big)$$

as $n \to +\infty$. The assumption that ω is strictly star-shaped with respect to the origin yields that $\nabla\tilde{G}(x', 0) \cdot (x', 0) > 0$, hence we conclude that $\tilde{G}\big((1 - \frac{1}{n})x', 0\big) < 0$ for n sufficiently large, so that $\big(1 - \frac{1}{n}\big)x' \in \omega'$ and $\big(1 - \frac{1}{n}\big)x' \to x'$ as $n \to +\infty$. In a similar way, we can prove that $\big(1 + \frac{1}{n}\big)x' \notin \omega'$ and $\big(1 + \frac{1}{n}\big)x' \to x'$ as $n \to +\infty$. Hence we conclude that $x' \in \partial\omega'$, thus proving claim (4.24).

From (4.24) it follows that

$$\partial\omega' = \{x' \in \mathbb{R}^{N-1} : \tilde{g}(x') = 0\}.$$

We observe that, for $x' \in \partial\omega'$,

$$\nabla\tilde{g}(x') \cdot x' = \nabla\tilde{G}(x', 0) \cdot (x', 0) > 0 \tag{4.25}$$

by strict star-shapedness of ω. In particular $\nabla\tilde{g}(x') \neq 0$ for all $x' \in \partial\omega'$, hence by the Implicit Function Theorem the boundary of ω' can be locally parametrized as the graph of a $C^{1,1}$-function, i.e. ω' is of class $C^{1,1}$. The strict star-shapedness of ω' directly follows from (4.25). $\square$

From Lemmas 4.2 and 4.3 we may directly conclude the following result.

Corollary 4.4. *The set $\widetilde{\Sigma}_\varepsilon$ defined in (4.9) is of class $C^{1,1}$ and strictly star-shaped in $\mathbb{R}^{N-1}$ with respect to 0 for ε sufficiently small.*

Corollary 4.4 achieves our aim of proving preservation of star-shapedness after the action of the diffeomorphism Φ. The following Lemma 4.5 provides a quantitative estimate of the size of the transformed Neumann region, proving that its size remains of order ε.

From Lemma 4.2 (ii) we have that there exists $\varepsilon_0 \in (0, 1)$ such that $\Phi(\varepsilon\mathcal{V})$ is strictly star-shaped with respect to 0 for all $\varepsilon \in (0, \varepsilon_0)$. Then, applying Lemma 4.2 (i) to $\Phi(\varepsilon\mathcal{V})$, for all $\varepsilon \in (0, \varepsilon_0)$ there exists a function $\rho_\varepsilon : \mathbb{S}^{N-1} \to \mathbb{R}$ of class $C^{1,1}$ such that $\rho_\varepsilon \geq 0$,

$$\Phi(\varepsilon\mathcal{V}) = \{r\theta : \theta \in \mathbb{S}^{N-1} \text{ and } 0 \leq r < \rho_\varepsilon(\theta)\}, \quad \text{and} \quad \partial(\Phi(\varepsilon\mathcal{V})) = \{\rho_\varepsilon(\theta)\theta : \theta \in \mathbb{S}^{N-1}\}. \tag{4.26}$$

Lemma 4.5. *For every $\varepsilon \in (0, \varepsilon_0)$, let ρ_ε be as in (4.26). Then there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $\kappa > 1$ (independent of ε) such that, for all $\varepsilon \in (0, \varepsilon_1)$,*

$$\frac{\varepsilon}{\kappa} \le \rho_\varepsilon(\theta) \le \kappa\varepsilon \quad \text{for all } \theta \in \mathbb{S}^{N-1} \tag{4.27}$$

and

$$|\nabla_{\mathbb{S}^{N-1}} \rho_\varepsilon(\theta)| \le \kappa\varepsilon \quad \text{for all } \theta \in \mathbb{S}^{N-1}. \tag{4.28}$$

Proof. Estimate (4.27) follows from (4.6) and the fact that, being $\mathcal{V}$ a bounded open set containing 0, $0 < \inf_{x\in\partial\mathcal{V}} |x| \le \sup_{x\in\partial\mathcal{V}} |x| < +\infty$.

To prove (4.28), we observe that, by (4.26) and (4.21),

$$\partial(\Phi(\varepsilon\mathcal{V})) = \left\{ x \in \mathbb{R}^N \setminus \{0\} : |x| - \rho_\varepsilon\left(\frac{x}{|x|}\right) = 0 \right\} = \left\{ x \in \mathbb{R}^N \setminus \{0\} : G\left(\frac{\Phi^{-1}(x)}{\varepsilon}\right) = 0 \right\},$$

with G being as in the proof of Lemma 4.2. Therefore, for every $\varepsilon \in (0, \varepsilon_0)$ and $x \in \partial(\Phi(\varepsilon\mathcal{V}))$, there exists $c_\varepsilon(x) \in \mathbb{R}$ such that

$$\frac{1}{\varepsilon} \nabla G\left(\frac{\Phi^{-1}(x)}{\varepsilon}\right) J_{\Phi^{-1}}(x) = c_\varepsilon(x) \left(\frac{x}{|x|} - \frac{1}{|x|} \nabla_{\mathbb{S}^{N-1}} \rho_\varepsilon\left(\frac{x}{|x|}\right) \right). \tag{4.29}$$

Hence, from (4.29), (4.20), (4.7), and (4.27) we deduce

$$\begin{aligned} c_\varepsilon(x)|x| &= \nabla G\left(\frac{\Phi^{-1}(x)}{\varepsilon}\right) J_{\Phi^{-1}}(x) \cdot \frac{x}{\varepsilon} \\ &= \nabla G\left(\frac{\Phi^{-1}(x)}{\varepsilon}\right) \cdot \frac{\Phi^{-1}(x)}{\varepsilon} + \frac{1}{\varepsilon} \nabla G\left(\frac{\Phi^{-1}(x)}{\varepsilon}\right) \cdot \left(J_{\Phi^{-1}}(x)x - \Phi^{-1}(x)\right) \\ &\ge \tilde\sigma + \frac{1}{\varepsilon} O(|x|^2) = \tilde\sigma + O(\varepsilon) > \frac{\tilde\sigma}{2} \quad \text{for all } x \in \partial(\Phi(\varepsilon\mathcal{V})) \end{aligned} \tag{4.30}$$

if ε is sufficiently small. On the other hand, multiplying both sides of (4.29) by $\nabla_{\mathbb{S}^{N-1}} \rho_\varepsilon\left(\frac{x}{|x|}\right)$ we obtain that

$$\frac{1}{\varepsilon} \nabla G\left(\frac{\Phi^{-1}(x)}{\varepsilon}\right) J_{\Phi^{-1}}(x) \cdot \nabla_{\mathbb{S}^{N-1}} \rho_\varepsilon\left(\frac{x}{|x|}\right) = -\frac{c_\varepsilon(x)}{|x|} \left| \nabla_{\mathbb{S}^{N-1}} \rho_\varepsilon\left(\frac{x}{|x|}\right) \right|^2$$

and hence, in view of (4.30),

$$\frac{\tilde\sigma}{2|x|^2} \left| \nabla_{\mathbb{S}^{N-1}} \rho_\varepsilon\left(\frac{x}{|x|}\right) \right|^2 \le \frac{|c_\varepsilon(x)|}{|x|} \left| \nabla_{\mathbb{S}^{N-1}} \rho_\varepsilon\left(\frac{x}{|x|}\right) \right|^2 \le \text{const} \, \frac{1}{\varepsilon} \left| \nabla_{\mathbb{S}^{N-1}} \rho_\varepsilon\left(\frac{x}{|x|}\right) \right|$$

for all $x \in \partial(\Phi(\varepsilon\mathcal{V}))$ and for some const > 0 independent of x and ε. Therefore, taking into account (4.27), we have that, for all $x \in \partial(\Phi(\varepsilon\mathcal{V}))$,

$$\left| \nabla_{\mathbb{S}^{N-1}} \rho_\varepsilon\left(\frac{x}{|x|}\right) \right| \le \text{const} \, \frac{2|x|^2}{\tilde\sigma\, \varepsilon} \le \text{const}\, \varepsilon,$$

thus proving (4.28). □

Remark 4.6. Lemma 4.5 implies that $\widetilde\Sigma_\varepsilon \subset B'_{\kappa\varepsilon}$ for all $\varepsilon \in (0, \varepsilon_1)$.

We conclude this section with the following refined Poincaré-type inequality for function vanishing on the crack $\widetilde\Gamma_{\varepsilon, r_1}$.

Lemma 4.7. *For all $\tau \in (0,1)$ there exists $M_\tau > 1$ such that, for all $r > 0$ and $\varepsilon < \min\{\varepsilon_1, \frac{r}{\kappa M_\tau}\}$,*

$$\frac{1-\tau}{r} \int_{\partial B_r} u^2 \, \mathrm{d}S \le \int_{B_r} |\nabla u|^2 \, \mathrm{d}x \quad \text{for all } u \in H^1_{0,\widetilde\Gamma_{\varepsilon,r}}(B_r)$$

and

$$\frac{N-1}{r^2} \int_{B_r} u^2 \, \mathrm{d}x \le \left(1 + \frac{1}{1-\tau}\right) \int_{B_r} |\nabla u|^2 \, \mathrm{d}x \quad \text{for all } u \in H^1_{0,\widetilde\Gamma_{\varepsilon,r}}(B_r).$$

Proof. It follows directly from [14, Lemma 4.2, Corollaries 4.3 and 4.4], recalling that Lemma 4.5 implies $B'_r \setminus B'_{\kappa\varepsilon} \subseteq \widetilde\Gamma_{\varepsilon,r}$ for $\varepsilon < \min\{\varepsilon_1, \frac{r}{\kappa}\}$, as also observed in Remark 4.6. □

5. A Pohozaev-type inequality

Pohozaev-type identites play a pivotal role in the differentiation of the frequency function; indeed, by the coarea formula,

$$\frac{\mathrm{d}}{\mathrm{d}r}\int_{B_r}\widetilde{A}\nabla v\cdot\nabla v\,\mathrm{d}x=\int_{\partial B_r}\widetilde{A}\nabla v\cdot\nabla v\,\mathrm{d}x$$

and the Pohozaev identity allows to rewrite the latter boundary integral in terms of volume integrals and boundary integrals of normal derivatives. Therefore, this section is devoted to the proof of a Pohozaev-type inequality for solutions to problem (4.17).

Let $\widetilde{A}$ be the matrix-valued function introduced in (4.10)–(4.18). From (4.12) and (4.13) it follows easily that

$$\widetilde{A}(0)=I_N,\quad \widetilde{A}(y)=I_N+O(|y|)\quad\text{as }|y|\to 0. \tag{5.1}$$

Recalling that r_1 was defined in (4.4), we define, for all $y\in B_{r_1}$,

$$\mu(y)=\frac{\widetilde{A}(y)y\cdot y}{|y|^2}\quad\text{and}\quad \boldsymbol{b}(y)=\frac{1}{\mu(y)}\widetilde{A}(y)y. \tag{5.2}$$

From (4.13) and (5.1) we have that

$$\mu\in C^{0,1}(B_{r_1}),\quad \mu(y)=1+O(|y|),\quad \nabla\mu(y)=O(1)\quad\text{as }|y|\to 0, \tag{5.3}$$

and

$$\boldsymbol{b}(y)=y+O(|y|^2),\quad J_{\boldsymbol{b}}(y)=I_N+O(|y|),\quad \operatorname{div}\boldsymbol{b}(y)=N+O(|y|)\quad\text{as }|y|\to 0. \tag{5.4}$$

Furthermore we have that, possibly choosing r_1 smaller, for every $y\in B_{r_1}$

$$\frac{1}{2}|\xi|^2\le\widetilde{A}(y)\xi\cdot\xi\le\frac{3}{2}|\xi|^2\text{ for all }\xi\in\mathbb{R}^N, \tag{5.5}$$

$$\frac{1}{2}\le\mu(y)\le\frac{3}{2}, \tag{5.6}$$

$$\frac{1}{2}\le\widetilde{p}(y)\le\frac{3}{2}. \tag{5.7}$$

Finally we have that

$$\boldsymbol{b}(y)\cdot\frac{y}{|y|}=|y|\quad\text{for all }y\in B_{r_1}. \tag{5.8}$$

Proposition 5.1. *Let n_0 be as in (1.15) and let ε_1,κ be as in Lemma 4.5. For $i=1,\dots,n_0$ and $\varepsilon\in(0,\varepsilon_1)$, let v_i^ε solve (4.17). There exists $\tilde r\in(0,r_1)$ such that, for all $\varepsilon\in(0,\min\{\varepsilon_1,\tilde r/\kappa\})$ and a.e. $r\in(\kappa\varepsilon,\tilde r)$,*

$$\begin{aligned}r\int_{\partial B_r}\widetilde{A}\nabla v_i^\varepsilon\cdot\nabla v_i^\varepsilon\,\mathrm{d}S-\int_{B_r}\Big((\operatorname{div}\boldsymbol{b})\widetilde{A}\nabla v_i^\varepsilon\cdot\nabla v_i^\varepsilon-2J_{\boldsymbol{b}}(\widetilde{A}\nabla v_i^\varepsilon)\cdot\nabla v_i^\varepsilon+(d\widetilde{A}\nabla v_i^\varepsilon\nabla v_i^\varepsilon)\cdot\boldsymbol{b}\Big)\,\mathrm{d}y\\ \ge 2r\int_{\partial B_r}\frac{1}{\mu}|\widetilde{A}\nabla v_i^\varepsilon\cdot\boldsymbol{\nu}|^2\,\mathrm{d}S+2\lambda_i^\varepsilon\int_{B_r}\widetilde{p}(\boldsymbol{b}\cdot\nabla v_i^\varepsilon)v_i^\varepsilon\,\mathrm{d}y.\end{aligned} \tag{5.9}$$

The proof of a Pohozaev-type identity for an equation of type (4.17) is classically based on the integration by Divergence Theorem of the following Rellich-Nečas identity

$$\begin{aligned}\operatorname{div}\big((\widetilde{A}\nabla w\cdot\nabla w)\boldsymbol{b}-2(\boldsymbol{b}\cdot\nabla w)\widetilde{A}\nabla w\big)=(\operatorname{div}\boldsymbol{b})\widetilde{A}\nabla w\cdot\nabla w-2(J_{\boldsymbol{b}}\widetilde{A}\nabla w)\cdot\nabla w\\ +(d\widetilde{A}\nabla w\nabla w)\cdot\boldsymbol{b}-2(\boldsymbol{b}\cdot\nabla w)\operatorname{div}(\widetilde{A}\nabla w),\end{aligned} \tag{5.10}$$

which holds in a distributional sense for any H^2-function w. Nevertheless, the highly non-smoothness of the cracked domain $B_{r_1}\setminus\widetilde{\Gamma}_{\varepsilon,r_1}$, on which equation (4.17) is satisfied, prevents us from the direct use of the Divergence Theorem, because of lack of regularity of the solution v_i^ε, which could indeed fail to be H^2.

In order to overcome this regularity issue, we perform an approximation process.

5.1. **Regular approximation of the cracked domain.** The first step of our regularization procedure relies in the construction of a family of sets which approximate the cracked domain, being of class $C^{1,1}$ and star-shaped with respect to the origin.

To this aim, let us first consider the sequence of functions

$$h_n : \mathbb{R} \to \mathbb{R}, \quad h_n(t) = \left(n^2 t^2 + \frac{1}{n^2}\right)^{1/8}.$$

By direct computations it is easy to verify that

$$h_n(t) \geq 4t h_n'(t) \quad \text{for all } t \in \mathbb{R} \tag{5.11}$$

and

$$\text{if } (y', y_N) \in B_r \text{ and } |y'| \geq h_n(y_N), \quad \text{then } |y_N| \leq \frac{r^4}{n}. \tag{5.12}$$

Recall the definition of ρ_ε in (4.26). For every $\varepsilon \in (0, \min\{\varepsilon_1, r_1/\kappa\})$, $r \in (\kappa\varepsilon, r_1]$, and $n \in \mathbb{N}_*$ we define

$$D^n_{\varepsilon,r} = \left\{x = (x', x_N) \in B_r : |x'| < \rho_\varepsilon\left(\tfrac{x'}{|x'|}\right) + h_n(x_N)\right\},$$
$$\Gamma^n_{\varepsilon,r} = \left\{x = (x', x_N) \in B_r : |x'| = \rho_\varepsilon\left(\tfrac{x'}{|x'|}\right) + h_n(x_N)\right\} \subset \partial D^n_{\varepsilon,r},$$
$$S^n_{\varepsilon,r} = \partial D^n_{\varepsilon,r} \setminus \Gamma^n_{\varepsilon,r}.$$

We note that, for $\varepsilon \in (0, \min\{\varepsilon_1, r_1/\kappa\})$ and $r \in (\kappa\varepsilon, r_1]$ fixed, $D^n_{\varepsilon,r} \neq B_r$ and $\Gamma^n_{\varepsilon,r}$ is not empty provided n is sufficiently large.

Lemma 5.2. *There exists $r_2 \in (0, r_1)$ such that, for every $\varepsilon \in (0, \min\{\varepsilon_1, r_2/\kappa\})$ (with κ as in Lemma 4.5), there exists $n_\varepsilon \in \mathbb{N}_*$ such that $A(y)y \cdot \boldsymbol{\nu}(y) > 0$ on $\Gamma^n_{\varepsilon,r_2}$ for all $n \geq n_\varepsilon$, where $\boldsymbol{\nu}(y)$ is the exterior unit normal at $y \in \partial D^n_{\varepsilon,r_2}$.*

Proof. Because of the definition of $\Gamma^n_{\varepsilon,r}$, we have that, if $y \in \Gamma^n_{\varepsilon,r}$,

$$\boldsymbol{\nu}(y) = \frac{\nabla G^n_\varepsilon(y)}{\|\nabla G^n_\varepsilon(y)\|},$$

where $G^n_\varepsilon(y', y_N) = |y'| - h_n(y_N) - \rho_\varepsilon\left(\frac{y'}{|y'|}\right)$. Then, to prove the lemma it is enough to show that, for some $r_2 \in (0, r_1)$ sufficiently small and all $\varepsilon \in (0, \min\{\varepsilon_1, r_2/\kappa\})$, $A(y)y \cdot \nabla G^n_\varepsilon(y) > 0$ for all $y \in \Gamma^n_{\varepsilon,r_2}$, provided n is sufficiently large.

We observe that

$$\nabla G^n_\varepsilon(y', y_N) = \left(\frac{y'}{|y'|} - \frac{1}{|y'|}\nabla_{\mathbb{S}^{N-1}}\rho_\varepsilon\left(\tfrac{y'}{|y'|}\right), -h_n'(y_N)\right).$$

From (4.12) and (4.11) it follows that, as $|y'| \to 0$, $|y_N| \to 0$,

$$A(y', y_N)(y', y_N) = \Big(y' + O(|y'|^3) + |y'|O(|y_N|) + O(y_N^2),\, y_N + |y'|O(|y_N|) + O(y_N^2)\Big),$$

so that

$$\begin{aligned} A(y', y_N)(y', y_N) \cdot \nabla G^n_\varepsilon(y', y_N) &= |y'| + O(|y'|^3) + O(|y'|^2)\left|\nabla_{\mathbb{S}^{N-1}}\rho_\varepsilon\left(\tfrac{y'}{|y'|}\right)\right| + |y'|O(|y_N|) \\ &\quad + O(|y_N|)\left|\nabla_{\mathbb{S}^{N-1}}\rho_\varepsilon\left(\tfrac{y'}{|y'|}\right)\right| + O(y_N^2) \\ &\quad + \frac{O(y_N^2)}{|y'|}\left|\nabla_{\mathbb{S}^{N-1}}\rho_\varepsilon\left(\tfrac{y'}{|y'|}\right)\right| - y_N h_n'(y_N)\big(1 + O(|y'|) + O(y_N)\big), \end{aligned}$$

uniformly with respect to $\varepsilon \in (0, \varepsilon_1)$. Therefore, in view of (4.28),

$$\begin{aligned} A(y', y_N)(y', y_N) \cdot \nabla G^n_\varepsilon(y', y_N) &= F_1(y', y_N, \varepsilon) + |y'|O(|y_N|) \\ &\quad + \varepsilon\, O(|y_N|) + O(y_N^2) + \varepsilon\,\frac{O(y_N^2)}{|y'|} - y_N h_n'(y_N) F_2(y', y_N), \end{aligned} \tag{5.13}$$

where

$$F_1(y', y_N, \varepsilon) = |y'| + O(|y'|^3) + \varepsilon\, O(|y'|^2) \quad \text{and} \quad F_2(y', y_N) = 1 + O(|y'|) + O(y_N)$$

as $|y'| \to 0$ and $|y_N| \to 0$ uniformly with respect to $\varepsilon \in (0, \varepsilon_1)$.

Let us choose $r_2 \in (0, r_1)$ such that

$$F_1(y', y_N, \varepsilon) \geq \frac{1}{2}|y'| \quad \text{and} \quad F_2(y', y_N) \leq 2 \quad \text{for all } (y', y_N) \in B_{r_2} \text{ and } \varepsilon \in (0, \varepsilon_1). \tag{5.14}$$

We note that, if $\varepsilon \in (0, \min\{\varepsilon_1, r_2/\kappa\})$, then $\widetilde{\Sigma}_\varepsilon \subset B'_{r_2}$ and $\Gamma^n_{\varepsilon,r_2}$ is not empty for n sufficiently large. Moreover by (4.27) we have that, if $(y', y_N) \in \Gamma^n_{\varepsilon,r_2}$, then $|y'| \geq \rho_\varepsilon\big(\frac{y'}{|y'|}\big) \geq \frac{\varepsilon}{\kappa}$, so that

$$\frac{\varepsilon}{|y'|} \leq \kappa \quad \text{for all } \varepsilon \in (0, \min\{\varepsilon_1, r_2/\kappa\}) \text{ and } (y', y_N) \in \Gamma^n_{\varepsilon,r_2}. \tag{5.15}$$

From (5.12), (5.13), (5.14), (5.15), the definition of $\Gamma^n_{\varepsilon,r_2}$, (5.11), and (4.27), we conclude that, for all $\varepsilon \in (0, \min\{\varepsilon_1, \frac{r_2}{\kappa}\})$ and $(y', y_N) \in \Gamma^n_{\varepsilon,r_2}$,

$$\begin{aligned}
A(y', y_N)(y', y_N)\cdot\nabla G^n_\varepsilon(y', y_N) &\geq \frac{1}{2}|y'| - 2y_N h'_n(y_N) + O\left(\frac{1}{n}\right) \\
&= \frac{1}{2}\left(h_n(y_N) + \rho_\varepsilon\big(\tfrac{y'}{|y'|}\big)\right) - 2y_N h'_n(y_N) + O\left(\frac{1}{n}\right) \\
&\geq \frac{1}{2}\left(h_n(y_N) - 4y_N h'_n(y_N)\right) + \frac{\varepsilon}{2\kappa} + O\left(\frac{1}{n}\right) \geq \frac{\varepsilon}{2\kappa} + O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.
\end{aligned}$$

From the above estimate it follows that, choosing r_2 as above, for every $\varepsilon \in (0, \min\{\varepsilon_1, \frac{r_2}{\kappa}\})$ there exists $n_\varepsilon \in \mathbb{N}_*$ such that $A(y)y \cdot \nabla G^n_\varepsilon(y) > 0$ for all $y \in \Gamma^n_{\varepsilon,r_2}$ with $n \geq n_\varepsilon$, thus completing the proof. □

Let $\tilde{\alpha} \in (0, 1)$ and μ as in (5.2). In view of (4.11), (4.12), (4.9), (4.3), (4.18), (5.3) and Lemma A.1, there exists

$$\tilde{r} \in \left(0, \min\left\{r_2, \frac{\tilde{\alpha}}{2\|\mu\|_{L^\infty(B_{r_2})}}\right\}\right) \tag{5.16}$$

such that

$$\widetilde{A}(y)\xi \cdot \xi \geq \tilde{\alpha}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{ and } y \in B_{\tilde{r}} \tag{5.17}$$

and, for all $r \in (0, \tilde{r}]$,

$$\int_{B_r} \left(\widetilde{A}\nabla w \cdot \nabla w - \lambda_{n_0}\widetilde{p}w^2\right) dx + \int_{\partial B_r} \mu w^2 \, dS \geq \tilde{\alpha} \int_{B_r} |\nabla w|^2 \, dx \quad \text{for all } w \in H^1(B_r). \tag{5.18}$$

In particular, from (5.18) it follows that, for all $r \in (0, \tilde{r}]$

$$\int_{B_r} \widetilde{A}\nabla w \cdot \nabla w \, dx - \lambda_{n_0} \int_{B_r} \widetilde{p}w^2 \, dx \geq \tilde{\alpha} \int_{B_r} |\nabla w|^2 \, dx \quad \text{for all } w \in H^1_0(B_r). \tag{5.19}$$

We denote

$$D^n_\varepsilon = D^n_{\varepsilon,\tilde{r}}, \quad \Gamma^n_\varepsilon = \Gamma^n_{\varepsilon,\tilde{r}}, \quad S^n_\varepsilon = S^n_{\varepsilon,\tilde{r}}, \quad \widetilde{\Gamma}_\varepsilon = \overline{B'_{\tilde{r}}} \setminus \widetilde{\Sigma}_\varepsilon.$$

5.2. **Regular approximation of v_i^ε.** Let v_i^ε be the functions defined in (4.15). Let us fix

$$i \in \{1, \dots, n_0\} \quad \text{and} \quad \varepsilon \in (0, \min\{\varepsilon_1, \tilde{r}/\kappa\}).$$

Since $v_i^\varepsilon \in H^1_{0,\widetilde{\Gamma}_\varepsilon}(B_{\tilde{r}})$, there exists a sequence of functions $v_n = v^\varepsilon_{i,n}$ such that

$$v_n \in C^\infty_c(\overline{B_{\tilde{r}}} \setminus \widetilde{\Gamma}_\varepsilon) \quad \text{and} \quad v_n \to v_i^\varepsilon \text{ in } H^1(B_{\tilde{r}}) \text{ as } n \to \infty. \tag{5.20}$$

The functions v_n can be chosen in such a way that

$$v_n(x', x_N) = 0 \quad \text{if } x' \in \widetilde{\Gamma}_\varepsilon \text{ and } |x_N| \leq \frac{\tilde{r}^4}{n}. \tag{5.21}$$

Remark 5.3. We observe that $v_n \equiv 0$ in $B_{\tilde r} \setminus D^n_\varepsilon$ (in particular v_n has null trace on Γ^n_ε). Indeed, let $x = (x', x_N) \in B_{\tilde r} \setminus D^n_\varepsilon$. Then

$$|x'| \geq h_n(x_N) + \rho_\varepsilon\big(\tfrac{x'}{|x'|}\big) > \rho_\varepsilon\big(\tfrac{x'}{|x'|}\big),$$

so that $x' \in \widetilde\Gamma_\varepsilon$; moreover

$$\tilde r \geq |x'| \geq h_n(x_N) \geq n^{1/4}|x_N|^{1/4},$$

so that $|x_N| \leq \tilde r^4/n$. Then $v_n(x) = 0$ in view of (5.21).

We are going to construct a sequence of approximated solutions $\{w_n\}_{n\in\mathbb{N}}$ on the sets D^n_ε. Let $\tilde n_\varepsilon \geq n_\varepsilon$ be such that Γ^n_ε is not empty for all $n \geq \tilde n_\varepsilon$.

Lemma 5.4. *For every $n \geq \tilde n_\varepsilon$, there exists a unique weak solution $w_n \in H^1(D^n_\varepsilon)$ to the problem*

$$\begin{cases} -\operatorname{div}(\widetilde A(y)\nabla w_n(y)) = \lambda_i^\varepsilon \widetilde p(y) w_n(y), & \text{in } D^n_\varepsilon,\\ w_n = v_n, & \text{on } \partial D^n_\varepsilon. \end{cases} \tag{5.22}$$

Furthermore, extending w_n trivially to zero in $B_{\tilde r} \setminus D^n_\varepsilon$, we have that

$$w_n \to v_i^\varepsilon \quad \text{in} \quad H^1(B_{\tilde r}) \quad \text{as } n \to +\infty. \tag{5.23}$$

Proof. Letting $W_n := w_n - v_n$, we observe that w_n is a weak solution to (5.22) if and only if $W_n \in H^1_0(D^n_\varepsilon)$ weakly solves

$$\begin{cases} -\operatorname{div}(\widetilde A\nabla W_n) - \lambda_i^\varepsilon\, \widetilde p\, W_n = \lambda_i^\varepsilon \widetilde p\, v_n + \operatorname{div}(\widetilde A\nabla v_n) & \text{in } D^n_\varepsilon,\\ W_n = 0 & \text{on } \partial D^n_\varepsilon. \end{cases} \tag{5.24}$$

i.e.

$$a_n(W_n, \phi) = \langle F_n, \phi\rangle \quad \text{for all } \phi \in H^1_0(D^n_\varepsilon) \tag{5.25}$$

where

$$a_n : H^1_0(D^n_\varepsilon) \times H^1_0(D^n_\varepsilon) \to \mathbb{R}, \quad a_n(\phi_1,\phi_2) = \int_{D^n_\varepsilon} \widetilde A\nabla\phi_1\cdot\nabla\phi_2\,dy - \lambda_i^\varepsilon \int_{D^n_\varepsilon} \widetilde p\,\phi_1\,\phi_2\,dy,$$

$$F_n \in H^{-1}(D^n_\varepsilon), \quad {}_{H^{-1}(D^n_\varepsilon)}\langle F_n, \phi\rangle_{H^1_0(D^n_\varepsilon)} = \int_{D^n_\varepsilon} \Big(\lambda_i^\varepsilon \widetilde p\, v_n\,\phi - \widetilde A\nabla v_n\cdot\nabla\phi\Big)\,dy.$$

Since $\widetilde p \in L^\infty(D^n_\varepsilon)$, by the Poincaré inequality the bilinear form a_n is continuous, whereas estimate (5.19) implies that a_n is coercive on $H^1_0(D^n_\varepsilon)$. The Lax-Milgram Theorem ensures the existence of a unique weak solution W_n to (5.24) and, consequently, of a unique weak solution $w_n = W_n + v_n$ to (5.22), for all $n \geq \tilde n_\varepsilon$.

From the Poincaré inequality and boundedness of $\{v_n\}$ in $H^1(B_{\tilde r})$ we can easily deduce that

$$|\langle F_n, \phi\rangle| \leq c\,\|\phi\|_{H^1_0(D^n_\varepsilon)} \quad \text{for all } \phi \in H^1_0(D^n_\varepsilon),$$

for some constant $c > 0$ which may depend on $i, \varepsilon, N, \tilde r$ but is independent of n. Therefore, choosing $\phi = W_n$ in (5.25) and using estimates (5.19) and (2.2), we obtain that

$$\tilde\alpha \|W_n\|^2_{H^1_0(D^n_\varepsilon)} \leq a_n(W_n, W_n) = \langle F_n, W_n\rangle \leq c\,\|W_n\|_{H^1_0(D^n_\varepsilon)},$$

so that

$$\|W_n\|_{H^1_0(B_{\tilde r})} \leq \frac{c}{\tilde\alpha} \quad \text{for all } n \geq \tilde n_\varepsilon,$$

where W_n is extended trivially to zero in $B_{\tilde r} \setminus D^n_\varepsilon$. Therefore there exist $W \in H^1_0(B_{\tilde r})$ and a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ such that

$$W_{n_k} \rightharpoonup W \quad \text{weakly in } H^1_0(B_{\tilde r}). \tag{5.26}$$

We observe that, for any $\delta > 0$ small, the set $\Lambda^\varepsilon_\delta = \Big\{(x', 0) \in B_{\tilde r} : |x'| \geq \delta + \rho_\varepsilon\big(\tfrac{x'}{|x'|}\big)\Big\}$ is contained in $B_{\tilde r} \setminus \overline{D^\varepsilon_n}$ for n sufficiently large (how large it should be depends on δ); hence, for any $\delta > 0$ small fixed, $W_n \in H^1_{0,\Lambda^\varepsilon_\delta\cup\partial B_{\tilde r}}(B_{\tilde r})$ for n sufficiently large. Since $H^1_{0,\Lambda^\varepsilon_\delta\cup\partial B_{\tilde r}}(B_{\tilde r})$ is weakly closed in $H^1(B_{\tilde r})$, we deduce that $W \in H^1_{0,\Lambda^\varepsilon_\delta\cup\partial B_{\tilde r}}(B_{\tilde r})$ for all $\delta > 0$ small; therefore we conclude that $W \in H^1_{0,\partial B_{\tilde r}\cup\widetilde\Gamma_\varepsilon}(B_{\tilde r})$.

Then, from (4.16), (5.20), (5.26), and (5.25) we deduce that

$$\begin{aligned}0 &= -\int_{B_{\tilde r}}\Big(\widetilde{A}\nabla v_i^\varepsilon\cdot\nabla W\,\mathrm{d}y - \lambda_i^\varepsilon\widetilde{p}\,v_i^\varepsilon\,W\Big)\,\mathrm{d}y\\ &= -\lim_{k\to+\infty}\int_{B_{\tilde r}}\Big(\widetilde{A}\nabla v_{n_k}\cdot\nabla W_{n_k}\,\mathrm{d}y - \lambda_i^\varepsilon\widetilde{p}\,v_{n_k}\,W_{n_k}\Big)\,\mathrm{d}y\\ &= \lim_{k\to+\infty}\langle F_{n_k},W_{n_k}\rangle = \lim_{k\to+\infty} a_{n_k}(W_{n_k},W_{n_k})\end{aligned}$$

thus concluding that $\|W_{n_k}\|_{H_0^1(B_{\tilde r})}\to 0$ as $k\to+\infty$ in view of (5.19). Hence $W_{n_k}\to 0$ in $H_0^1(B_{\tilde r})$ and $w_{n_k}=W_{n_k}+v_{n_k}\to v_i^\varepsilon$ in $H^1(B_{\tilde r})$ as $k\to+\infty$ thanks to (5.20). By Urysohn's subsequence principle, we finally conclude that $w_n\to v_i^\varepsilon$ in $H^1(B_{\tilde r})$ as $n\to+\infty$. □

5.3. **Proof of Proposition 5.1.** Let us fix $i=1,\dots,n_0$ and $\varepsilon\in(0,\min\{\varepsilon_1,\tilde r/\kappa\})$ with $\tilde r$ being as in section 5.1 (see (5.16)–(5.19)). Let v_i^ε solve (4.17) and, for all $n\geq\tilde n_\varepsilon$, let $w_n\in H^1(D_\varepsilon^n)$ be as in Lemma 5.4.

Let $r\in(\kappa\varepsilon,\tilde r)$. By classical elliptic regularity theory (see e.g. [18, Theorem 2.2.2.3]) we have that $w_n\in H^2(D^n_{\varepsilon,r})$. Then

$$(\widetilde{A}\nabla w_n\cdot\nabla w_n)\boldsymbol{b} - 2(\boldsymbol{b}\cdot\nabla w_n)\widetilde{A}\nabla w_n\in W^{1,1}(D^n_{\varepsilon,r})$$

so that we can use the integration by parts formula for Sobolev functions on the Lipschitz domain $D^n_{\varepsilon,r}$ and obtain, in view of (5.10), (5.22), and (5.8),

$$\begin{aligned}r\int_{S^n_{\varepsilon,r}}\widetilde{A}\nabla w_n\cdot\nabla w_n\,\mathrm{d}S - \int_{D^n_{\varepsilon,r}}\Big((\operatorname{div}\boldsymbol{b})\widetilde{A}\nabla w_n\cdot\nabla w_n - 2(J_{\boldsymbol{b}}\widetilde{A}\nabla w_n)\cdot\nabla w_n + (d\widetilde{A}\nabla w_n\nabla w_n)\cdot\boldsymbol{b}\Big)\,\mathrm{d}y,\\ = 2r\int_{S^n_{\varepsilon,r}}\frac{1}{\mu}|\widetilde{A}\nabla w_n\cdot\boldsymbol{\nu}|^2\,\mathrm{d}S + 2\lambda_i^\varepsilon\int_{D^n_{\varepsilon,r}}(\boldsymbol{b}\cdot\nabla w_n)\widetilde{p}\,w_n\,\mathrm{d}y\\ + \int_{\Gamma^n_{\varepsilon,r}}\big(-(\widetilde{A}\nabla w_n\cdot\nabla w_n)\boldsymbol{b}\cdot\boldsymbol{\nu} + 2(\boldsymbol{b}\cdot\nabla w_n)\widetilde{A}\nabla w_n\cdot\boldsymbol{\nu}\big)\,\mathrm{d}S\end{aligned}$$

where $\boldsymbol{\nu}=\boldsymbol{\nu}(y)$ is the exterior unit normal at $y\in\partial D^n_{\varepsilon,r}=\Gamma^n_{\varepsilon,r}\cup S^n_{\varepsilon,r}$. On $\Gamma^n_{\varepsilon,r}$ we have that $w_n=0$ so that $\nabla w_n=\frac{\partial w_n}{\partial\boldsymbol{\nu}}\boldsymbol{\nu}$; hence

$$-(\widetilde{A}\nabla w_n\cdot\nabla w_n)\boldsymbol{b}\cdot\boldsymbol{\nu} + 2(\boldsymbol{b}\cdot\nabla w_n)\widetilde{A}\nabla w_n\cdot\boldsymbol{\nu} = \frac{1}{\mu}\left|\frac{\partial w_n}{\partial\boldsymbol{\nu}}\right|^2(\widetilde{A}y\cdot\boldsymbol{\nu})(\widetilde{A}\boldsymbol{\nu}\cdot\boldsymbol{\nu})\geq 0\quad\text{on }\Gamma^n_{\varepsilon,r}\tag{5.27}$$

thanks to Lemma 5.2 and (5.5). Then we obtain the following inequality

$$\begin{aligned}r\int_{\partial B_r}\widetilde{A}\nabla w_n\cdot\nabla w_n\,\mathrm{d}S\\ -\int_{B_r}\Big((\operatorname{div}\boldsymbol{b})\widetilde{A}\nabla w_n\cdot\nabla w_n - 2(J_{\boldsymbol{b}}\widetilde{A}\nabla w_n)\cdot\nabla w_n + (d\widetilde{A}\nabla w_n\nabla w_n)\cdot\boldsymbol{b}\Big)\,\mathrm{d}y,\\ \geq 2r\int_{\partial B_r}\frac{1}{\mu}|\widetilde{A}\nabla w_n\cdot\boldsymbol{\nu}|^2\,\mathrm{d}S + 2\lambda_i^\varepsilon\int_{B_r}(\boldsymbol{b}\cdot\nabla w_n)\widetilde{p}\,w_n\,\mathrm{d}y\end{aligned}\tag{5.28}$$

for all $n\geq\tilde n_\varepsilon$, where w_n is extended trivially to zero in $B_r\setminus D_\varepsilon^n$.

For i and ε fixed as above, we now intend to pass to the limit in (5.28) as $n\to+\infty$ for every $r\in(\kappa\varepsilon,\tilde r)$. The strong H^1-convergence of w_n to v_i^ε stated in (5.23) directly implies that

$$\begin{aligned}\lim_{n\to+\infty}\int_{B_r}\Big((\operatorname{div}\boldsymbol{b})\widetilde{A}\nabla w_n\cdot\nabla w_n - 2(J_{\boldsymbol{b}}\widetilde{A}\nabla w_n)\cdot\nabla w_n + (d\widetilde{A}\nabla w_n\nabla w_n)\cdot\boldsymbol{b}\Big)\,\mathrm{d}y\\ = \int_{B_r}\Big((\operatorname{div}\boldsymbol{b})\widetilde{A}\nabla v_i^\varepsilon\cdot\nabla v_i^\varepsilon - 2(J_{\boldsymbol{b}}\widetilde{A}\nabla v_i^\varepsilon)\cdot\nabla v_i^\varepsilon + (d\widetilde{A}\nabla v_i^\varepsilon\nabla v_i^\varepsilon)\cdot\boldsymbol{b}\Big)\,\mathrm{d}y\end{aligned}$$

and

$$\lim_{n\to+\infty}\int_{B_r}(\boldsymbol{b}\cdot\nabla w_n)\widetilde{p}\,w_n\,\mathrm{d}y = \int_{B_r}(\boldsymbol{b}\cdot\nabla v_i^\varepsilon)\widetilde{p}\,v_i^\varepsilon\,\mathrm{d}y,$$

for all $r \in (\kappa\varepsilon, \tilde r)$. In order to deal with the boundary integrals in (5.28), we observe that, by the strong H^1-convergence (5.23) of w_n to v_i^ε,

$$\lim_{n\to+\infty}\int_0^{\tilde r}\left(\int_{\partial B_r}|\nabla(w_n-v_i^\varepsilon)|^2\,\mathrm{d}S\right)\mathrm{d}r=0,$$

i.e., letting $F_n(r)=\int_{\partial B_r}|\nabla(w_n-v_i^\varepsilon)|^2\,\mathrm{d}S$, $F_n\to 0$ in $L^1(0,\tilde r)$. Then there exists a subsequence F_{n_k} such that $F_{n_k}(r)\to 0$ for a.e. $r\in(0,\tilde r)$, hence

$$\nabla w_{n_k}\to\nabla v_i^\varepsilon \quad \text{in } L^2(\partial B_r) \quad \text{as } k\to+\infty \quad \text{for a.e. } r\in(0,\tilde r).$$

Therefore, for a.e. $r\in(0,\tilde r)$,

$$\lim_{k\to+\infty}\int_{\partial B_r}\widetilde A\nabla w_{n_k}\cdot\nabla w_{n_k}\,\mathrm{d}S=\int_{\partial B_r}\widetilde A\nabla v_i^\varepsilon\cdot\nabla v_i^\varepsilon\,\mathrm{d}S,$$
$$\lim_{k\to+\infty}\int_{\partial B_r}\frac{1}{\mu}|\widetilde A\nabla w_n\cdot\boldsymbol{\nu}|^2\,\mathrm{d}S=\int_{\partial B_r}\frac{1}{\mu}|\widetilde A\nabla v_i^\varepsilon\cdot\boldsymbol{\nu}|^2\,\mathrm{d}S.$$

Hence we can pass to the limit in (5.28) as $n\to+\infty$ for a.e. $r\in(\kappa\varepsilon,\tilde r)$, thus obtaining (5.9). ☐

6. Energy estimates via an Almgren-type frequency function

6.1. **Monotonicity formula.**

For every $\lambda\in\mathbb{R}$, $r>0$, and $v\in H^1(B_r)$ we define

$$E(v,r,\lambda)=r^{2-N}\int_{B_r}\left(\widetilde A\nabla v\cdot\nabla v-\lambda\widetilde p v^2\right)\mathrm{d}y,$$

where $\widetilde A$ and $\widetilde p$ have been introduced in (4.18), and

$$H(v,r)=r^{1-N}\int_{\partial B_r}\mu\, v^2\,\mathrm{d}S,$$

where μ has been introduced in (5.2). We observe that from (5.18) it follows that

$$E(v,r,\lambda)+rH(v,r)\geq 0 \quad \text{for all } r\in(0,\tilde r],\ \lambda\leq\lambda_{n_0} \text{ and } v\in H^1(B_r). \tag{6.1}$$

We also define the Almgren-type frequency function as

$$N(v,r,\lambda):=\frac{E(v,r,\lambda)}{H(v,r)}. \tag{6.2}$$

Lemma 6.1. *Let v_i^ε be as in (4.15).*

- (i) *$H(v_i^\varepsilon,r)>0$ for all $\varepsilon\in(0,\min\{\varepsilon_1,\tilde r/\kappa\})$, $r\in[\kappa\varepsilon,\tilde r]$, and $1\leq i\leq n_0$.*
- (ii) *For every $r\in(0,\tilde r]$, there exist $C_r>0$ and $\alpha_r\in(0,r/\kappa)$ such that $H(v_i^\varepsilon,r)\geq C_r$ for all $0<\varepsilon<\min\{\alpha_r,\varepsilon_1\}$ and $1\leq i\leq n_0$.*

Proof. To prove (i) we argue by contradiction and assume that there exist $\varepsilon\in(0,\min\{\varepsilon_1,\tilde r/\kappa\})$, $r\in[\kappa\varepsilon,\tilde r]$, and $1\leq i\leq n_0$ such that $H(v_i^\varepsilon,r)=0$, i.e. $v_i^\varepsilon=0$ on ∂B_r. Testing (4.17) with v_i^ε, integrating over B_r, and using estimates (5.19) and (2.2), we obtain that

$$0=\int_{B_r}\widetilde A\nabla v_i^\varepsilon\cdot\nabla v_i^\varepsilon\,\mathrm{d}y-\lambda_i^\varepsilon\int_{B_r}\widetilde p|v_i^\varepsilon|^2\,\mathrm{d}y\geq\tilde\alpha\int_{B_r}|\nabla v_i^\varepsilon|^2\,\mathrm{d}y$$

and hence $v_i^\varepsilon\equiv 0$ in B_r. It follows that $u_i^\varepsilon\equiv 0$ in B_r^+, i.e. $\varphi_i^\varepsilon\equiv 0$ in $F(B_r^+)$, with F as in (4.2), so that from the classical unique continuation principle for elliptic equations we may conclude that $\varphi_i^\varepsilon\equiv 0$ in Ω, a contradiction.

In order to prove (ii), suppose by contradiction that there exist $0<r\leq\tilde r$, $\varepsilon_\ell\to 0$, and $i_\ell\in\{1,\dots,n_0\}$ such that

$$\lim_{\ell\to+\infty}H(v_{i_\ell}^{\varepsilon_\ell},r)=0.$$

From (1.9), (1.8), and (2.2) we have that $\{\varphi_{i_\ell}^{\varepsilon_\ell}\}_\ell$ is bounded in $H^1(\Omega)$. Then there exist $\lambda\in[0,\lambda_{n_0}]$ and $\varphi\in H^1(\Omega)$ such that, along a subsequence, $\lambda_{i_\ell}^{\varepsilon_\ell}\to\lambda$ and $\varphi_{i_\ell}^{\varepsilon_\ell}\to\varphi$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. From (1.8) it follows that $\int_\Omega\varphi^2\,\mathrm{d}x=1$ and then $\varphi\not\equiv 0$ in Ω. Moreover φ weakly satisfies $-\Delta\varphi=\lambda\varphi$ in Ω.

The weak convergence $\varphi_{i_\ell}^{\varepsilon_\ell} \rightharpoonup \varphi$ in $H^1(\Omega)$ implies tha $v_{i_\ell}^{\varepsilon_\ell} \rightharpoonup v$ weakly in $H^1(B_{r_1})$, where v is the even reflection through the hyperplane $\{y_N = 0\}$ of $u = \varphi \circ F$. Since $v_{i_\ell}^{\varepsilon_\ell} \in H^1_{0,\widetilde{\Gamma}_{\varepsilon_\ell, r_1}}(B_{r_1})$, we have that $v \in H^1_{0,\overline{B'_{r_1}}}(\Omega)$. Moreover, v weakly solves

$$\begin{cases} -\operatorname{div}(\widetilde{A}\nabla v) = \lambda \widetilde{p}\, v, & \text{in } B_{r_1} \setminus B'_{r_1}, \\ v = 0, & \text{on } B'_{r_1}. \end{cases} \tag{6.3}$$

By compactness of the trace embedding $H^1(B_r) \hookrightarrow L^2(\partial B_r)$, we also have that

$$0 = \lim_{\ell\to\infty} H(v_{i_\ell}^{\varepsilon_\ell}, r) = \lim_{\ell\to\infty} r^{1-N} \int_{\partial B_r} \mu\, |v_{i_\ell}^{\varepsilon_\ell}|^2 \,\mathrm{d}S = r^{1-N} \int_{\partial B_r} \mu\, |v|^2 \,\mathrm{d}S,$$

which implies that $v = 0$ on ∂B_r. Testing (6.3) by v in B_r, from (5.19) we deduce that

$$0 = \int_{B_r} \big(\widetilde{A}\nabla v \cdot \nabla v - \lambda \widetilde{p} v^2\big) \,\mathrm{d}x \geq \tilde{\alpha} \int_{B_r} |\nabla v|^2 \,\mathrm{d}x.$$

Then $v \equiv 0$ in B_r and, consequently, $\varphi \equiv 0$ in $F(B_r^+)$, so that from the classical unique continuation principle for elliptic equations we may conclude that $\varphi \equiv 0$ in Ω, a contradiction. $\square$

As a consequence of Lemma 6.1 the function $r \mapsto N(v_i^\varepsilon, r, \lambda_i^\varepsilon)$ is well defined in the interval $[\kappa\varepsilon, \tilde{r}]$ for all $\varepsilon \in (0, \min\{\varepsilon_1, \tilde{r}/\kappa\})$ and $1 \leq i \leq n_0$. Furthermore, estimate (6.1) implies that

$$N(v_i^\varepsilon, r, \lambda_i^\varepsilon) + 1 \geq 1 - \tilde{r} > 0 \quad \text{for all } r \in [\kappa\varepsilon, \tilde{r}].$$

In order to differentiate the function N, we need to differentiate both E and H. We start here by deriving a formula for the derivative of H, which turns out to be expressible in terms of the function E, see (6.6).

Lemma 6.2. *For all $\varepsilon \in (0, \min\{\varepsilon_1, \tilde{r}/\kappa\})$ and $1 \leq i \leq n_0$, $H(v_i^\varepsilon, \cdot) \in W^{1,1}(\kappa\varepsilon, \tilde{r})$,*

$$\frac{d}{dr} H(v_i^\varepsilon, r) = 2r^{1-N} \int_{\partial B_r} \mu v_i^\varepsilon \frac{\partial v_i^\varepsilon}{\partial \boldsymbol{\nu}} \,\mathrm{d}S + O(1) H(v_i^\varepsilon, r) \quad \text{as } r \to 0, \tag{6.4}$$

$$\frac{d}{dr} H(v_i^\varepsilon, r) = 2r^{1-N} \int_{\partial B_r} (\widetilde{A}\nabla v_i^\varepsilon \cdot \boldsymbol{\nu}) v_i^\varepsilon \,\mathrm{d}S + O(1) H(v_i^\varepsilon, r) \quad \text{as } r \to 0, \tag{6.5}$$

and

$$\frac{d}{dr} H(v_i^\varepsilon, r) = \frac{2}{r} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) + O(1) H(v_i^\varepsilon, r) \quad \text{as } r \to 0, \tag{6.6}$$

where the derivative is meant in a distributional sense and a.e. in $(\kappa\varepsilon, \tilde{r})$, $\boldsymbol{\nu} = \boldsymbol{\nu}(y) = \frac{y}{|y|}$ is the unit outer normal vector to ∂B_r, and $O(1)$ denotes terms which are bounded for r in a neighbourhood of 0 uniformly with respect to ε.

Proof. By direct calculations we have that $H(v_i^\varepsilon, \cdot) \in W^{1,1}(\kappa\varepsilon, \tilde{r})$ and

$$\frac{d}{dr} H(v_i^\varepsilon, r) = 2r^{1-N} \int_{\partial B_r} \mu v_i^\varepsilon \frac{\partial v_i^\varepsilon}{\partial \boldsymbol{\nu}} \,\mathrm{d}S + r^{1-N} \int_{\partial B_r} |v_i^\varepsilon|^2 \frac{\partial \mu}{\partial \boldsymbol{\nu}} \,\mathrm{d}S$$

in a weak sense, from which (6.4) follows in view of (5.3).

To prove (6.5) we define

$$\boldsymbol{a}(y) = \frac{\mu(y)(\boldsymbol{b}(y) - y)}{|y|}$$

and observe that

$$\int_{\partial B_r} (\widetilde{A}\nabla v_i^\varepsilon \cdot \boldsymbol{\nu}) v_i^\varepsilon \,\mathrm{d}S = \int_{\partial B_r} \mu v_i^\varepsilon \frac{\partial v_i^\varepsilon}{\partial \boldsymbol{\nu}} \,\mathrm{d}S + \frac{1}{2} \int_{\partial B_r} \boldsymbol{a} \cdot \nabla((v_i^\varepsilon)^2) \,\mathrm{d}S. \tag{6.7}$$

By (5.8) we have that

$$\boldsymbol{a}(y) \cdot y = 0. \tag{6.8}$$

From (5.3), (5.4), and (6.8) we deduce that

$$\operatorname{div} \boldsymbol{a} = \frac{\nabla\mu}{|y|} \cdot (\boldsymbol{b} - y) + \frac{\mu}{|y|}(\operatorname{div} \boldsymbol{b} - N) = O(1) \quad \text{as } |y| \to 0. \tag{6.9}$$

From (6.7), (6.8), and (6.9) it follows that

$$\begin{aligned} \int_{\partial B_r} (\widetilde{A}\nabla v_i^\varepsilon \cdot \boldsymbol{\nu}) v_i^\varepsilon \, \mathrm{d}S &= \int_{\partial B_r} \mu v_i^\varepsilon \frac{\partial v_i^\varepsilon}{\partial \boldsymbol{\nu}} \, \mathrm{d}S - \frac{1}{2} \int_{\partial B_r} (\operatorname{div} \boldsymbol{a}) |v_i^\varepsilon|^2 \, \mathrm{d}S \\ &= \int_{\partial B_r} \mu v_i^\varepsilon \frac{\partial v_i^\varepsilon}{\partial \boldsymbol{\nu}} \, \mathrm{d}S + O(1) r^{N-1} H(v_i^\varepsilon, r) \end{aligned} \tag{6.10}$$

as $r \to 0$ (uniformly in ε). Combining (6.4) and (6.10) we obtain estimate (6.5).

To prove (6.6) we test (4.17) with v_i^ε and integrate over B_r thus obtaining

$$r^{N-2} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) = \int_{B_r} \left(\widetilde{A}\nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon - \lambda_i^\varepsilon \widetilde{p} |v_i^\varepsilon|^2 \right) \mathrm{d}y = \int_{\partial B_r} (\widetilde{A}\nabla v_i^\varepsilon \cdot \boldsymbol{\nu}) v_i^\varepsilon \, \mathrm{d}S,$$

whose combination with (6.5) immediately yields (6.6). □

Remark 6.3. We observe that (6.6) implies that there exist $\bar{r}_0 \in (0, \tilde{r})$ and $C_0 > 0$ (independent of ε) such that, for all $\varepsilon \in (0, \bar{r}_0/\kappa)$,

$$\frac{2}{r} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) - \frac{1}{C_0} H(v_i^\varepsilon, r) \le \frac{d}{dr} H(v_i^\varepsilon, r) \le \frac{2}{r} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) + C_0 H(v_i^\varepsilon, r) \quad \text{a.e. } r \in (\varepsilon\kappa, \bar{r}_0).$$

Lemma 6.4. *For all $\varepsilon \in (0, \min\{\varepsilon_1, \tilde{r}/\kappa\})$ and $1 \le i \le n_0$, $E(v_i^\varepsilon, \cdot, \lambda_i^\varepsilon) \in W^{1,1}(\kappa\varepsilon, \tilde{r})$ and*

$$\frac{d}{dr} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) \ge 2r^{2-N} \int_{\partial B_r} \frac{1}{\mu} |\widetilde{A}\nabla v_i^\varepsilon \cdot \boldsymbol{\nu}|^2 \, \mathrm{d}S + O(1) E(v_i^\varepsilon, r, \lambda_i^\varepsilon) + r O(1) H(v_i^\varepsilon, r)$$

as $r \to 0$, where the derivative is meant in a distributional sense and a.e. in $(\kappa\varepsilon, \tilde{r})$ and $O(1)$ denotes terms which are bounded for r in a neighbourhood of 0 uniformly with respect to ε.

Proof. By direct calculations we have that $E(v_i^\varepsilon, \cdot, \lambda_i^\varepsilon) \in W^{1,1}(\kappa\varepsilon, \tilde{r})$ and

$$\begin{aligned} \frac{d}{dr} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) = (2-N) r^{1-N} &\int_{B_r} \left(\widetilde{A}\nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon - \lambda_i^\varepsilon \widetilde{p} |v_i^\varepsilon|^2 \right) \mathrm{d}y \\ &+ r^{2-N} \int_{\partial B_r} \left(\widetilde{A}\nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon - \lambda_i^\varepsilon \widetilde{p} |v_i^\varepsilon|^2 \right) \mathrm{d}S. \end{aligned}$$

Hence, in view of Proposition 5.1,

$$\begin{aligned} &\frac{d}{dr} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) \\ &\ge 2r^{2-N} \int_{\partial B_r} \frac{1}{\mu} |\widetilde{A}\nabla v_i^\varepsilon \cdot \boldsymbol{\nu}|^2 \, \mathrm{d}S - \lambda_i^\varepsilon r^{2-N} \int_{\partial B_r} \widetilde{p} |v_i^\varepsilon|^2 \, \mathrm{d}S \\ &\quad + r^{1-N} \int_{B_r} \Big((2-N) \widetilde{A}\nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon + (\operatorname{div} \boldsymbol{b}) \widetilde{A}\nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon - 2 J_{\boldsymbol{b}} (\widetilde{A}\nabla v_i^\varepsilon) \cdot \nabla v_i^\varepsilon + (d\widetilde{A}\nabla v_i^\varepsilon \nabla v_i^\varepsilon) \cdot \boldsymbol{b} \Big) \mathrm{d}y \\ &\quad + \lambda_i^\varepsilon r^{1-N} \int_{B_r} \left(2\widetilde{p} (\boldsymbol{b} \cdot \nabla v_i^\varepsilon) v_i^\varepsilon + (N-2) \widetilde{p} |v_i^\varepsilon|^2 \right) \mathrm{d}y. \end{aligned}$$

Therefore, in view of (4.13), (5.1), (5.4), (5.3), Lemma A.1, and (5.18)

$$\begin{aligned} \frac{d}{dr} E(v_i^\varepsilon, \cdot, \lambda_i^\varepsilon) &\ge 2r^{2-N} \int_{\partial B_r} \frac{1}{\mu} |\widetilde{A}\nabla v_i^\varepsilon \cdot \boldsymbol{\nu}|^2 \, \mathrm{d}S \\ &\quad + O(1) r^{2-N} \int_{B_r} |\nabla v_i^\varepsilon|^2 \, \mathrm{d}y + O(1) r^{2-N} \int_{\partial B_r} \mu |v_i^\varepsilon|^2 \, \mathrm{d}S \\ &= 2r^{2-N} \int_{\partial B_r} \frac{1}{\mu} |\widetilde{A}\nabla v_i^\varepsilon \cdot \boldsymbol{\nu}|^2 \, \mathrm{d}S + O(1) E(v_i^\varepsilon, r, \lambda_i^\varepsilon) + r O(1) H(v_i^\varepsilon, r) \end{aligned}$$

thus proving the lemma. □

Lemma 6.5. *There exist $\bar r \in (0, \bar r_0)$ and $C > 0$ such that, for all $\varepsilon \in (0, \min\{\varepsilon_1, \bar r/\kappa\})$ and $1 \le i \le n_0$, $N(v_i^\varepsilon, \cdot, \lambda_i^\varepsilon) \in W^{1,1}(\kappa\varepsilon, \bar r)$ and*

$$\frac{d}{dr} N(v_i^\varepsilon, r, \lambda_i^\varepsilon) \ge -C(N(v_i^\varepsilon, r, \lambda_i^\varepsilon) + 1) \quad \text{for a.e. } r \in (\kappa\varepsilon, \bar r). \tag{6.11}$$

Furthermore, for all $\varepsilon \in (0, \min\{\varepsilon_1, \bar r/\kappa\})$, $1 \le i \le n_0$, and $\kappa\varepsilon \le r < R \le \bar r$

$$N(v_i^\varepsilon, r, \lambda_i^\varepsilon) + 1 \le e^{C(R-r)}(N(v_i^\varepsilon, R, \lambda_i^\varepsilon) + 1). \tag{6.12}$$

Proof. The fact that $N(v_i^\varepsilon, \cdot, \lambda_i^\varepsilon) \in W^{1,1}(\kappa\varepsilon, \bar r)$ follows directly from the fact that $H(v_i^\varepsilon, \cdot)$ and $E(v_i^\varepsilon, \cdot, \lambda_i^\varepsilon)$ belong to $W^{1,1}(\kappa\varepsilon, \bar r)$ and Lemma 6.1.

From (6.6) it follows that, as $r \to 0$,

$$E(v_i^\varepsilon, r, \lambda_i^\varepsilon) = \frac{r}{2}\frac{d}{dr} H(v_i^\varepsilon, r) + rO(1)H(v_i^\varepsilon, r),$$

which, together with (6.5) and again (6.6), yields

$$\begin{aligned}
&E(v_i^\varepsilon, r, \lambda_i^\varepsilon)\frac{d}{dr}H(v_i^\varepsilon, r)\\
&= \left(r^{2-N}\int_{\partial B_r}(\widetilde A\nabla v_i^\varepsilon\cdot\boldsymbol\nu)v_i^\varepsilon\,\mathrm dS + rO(1)H(v_i^\varepsilon, r)\right)\left(2r^{1-N}\int_{\partial B_r}(\widetilde A\nabla v_i^\varepsilon\cdot\boldsymbol\nu)v_i^\varepsilon\,\mathrm dS + O(1)H(v_i^\varepsilon, r)\right)\\
&= 2r^{3-2N}\left(\int_{\partial B_r}(\widetilde A\nabla v_i^\varepsilon\cdot\boldsymbol\nu)v_i^\varepsilon\,\mathrm dS\right)^2 + rO(1)(H(v_i^\varepsilon, r))^2\\
&\qquad + r^{2-N}O(1)H(v_i^\varepsilon, r)\left(\int_{\partial B_r}(\widetilde A\nabla v_i^\varepsilon\cdot\boldsymbol\nu)v_i^\varepsilon\,\mathrm dS\right)\\
&= 2r^{3-2N}\left(\int_{\partial B_r}(\widetilde A\nabla v_i^\varepsilon\cdot\boldsymbol\nu)v_i^\varepsilon\,\mathrm dS\right)^2 + rO(1)(H(v_i^\varepsilon, r))^2\\
&\qquad + rO(1)H(v_i^\varepsilon, r)\left(\frac12\frac{d}{dr}H(v_i^\varepsilon, r) + O(1)H(v_i^\varepsilon, r)\right)\\
&= 2r^{3-2N}\left(\int_{\partial B_r}(\widetilde A\nabla v_i^\varepsilon\cdot\boldsymbol\nu)v_i^\varepsilon\,\mathrm dS\right)^2 + rO(1)(H(v_i^\varepsilon, r))^2\\
&\qquad + rO(1)H(v_i^\varepsilon, r)\left(\frac1r E(v_i^\varepsilon, r, \lambda_i^\varepsilon) + O(1)H(v_i^\varepsilon, r)\right).
\end{aligned}$$

The above estimate, Lemma 6.4, the Cauchy-Schwarz inequality, and (6.1) imply that

$$\begin{aligned}
\frac{d}{dr}N(v_i^\varepsilon, r, \lambda_i^\varepsilon) &= \frac{H(v_i^\varepsilon, r)\frac{d}{dr}E(v_i^\varepsilon, r, \lambda_i^\varepsilon) - E(v_i^\varepsilon, r, \lambda_i^\varepsilon)\frac{d}{dr}H(v_i^\varepsilon, r)}{(H(v_i^\varepsilon, r))^2}\\
&\ge 2r^{3-2N}\frac{\left(\int_{\partial B_r}\frac1\mu|\widetilde A\nabla v_i^\varepsilon\cdot\boldsymbol\nu|^2\,\mathrm dS\right)\left(\int_{\partial B_r}\mu|v_i^\varepsilon|^2\,\mathrm dS\right) - \left(\int_{\partial B_r}(\widetilde A\nabla v_i^\varepsilon\cdot\boldsymbol\nu)v_i^\varepsilon\,\mathrm dS\right)^2}{(H(v_i^\varepsilon, r))^2}\\
&\quad + O(1)N(v_i^\varepsilon, r, \lambda_i^\varepsilon) + r\,O(1)\\
&\ge -C\left(N(v_i^\varepsilon, r, \lambda_i^\varepsilon) + 1\right)
\end{aligned}$$

a.e. in $(\varepsilon\kappa, \bar r)$, for some $\bar r \in (0, \bar r_0)$ and $C > 0$ independent of ε.

Finally, estimate (6.12) follows by integration of (6.11) over the interval $[r, R]$. $\square$

Lemma 6.6. *For $\tau \in \left(0, \frac12\right)$, let M_τ be as in Lemma 4.7. Let $i \in \{1, \dots, n_0\}$. If $\varepsilon < \min\{\varepsilon_1, \frac{\bar r}{\kappa M_\tau}\}$ and $\kappa M_\tau\varepsilon \le s_1 < s_2 \le \bar r$, then*

$$\frac{H(v_i^\varepsilon, s_2)}{H(v_i^\varepsilon, s_1)} \ge e^{-(4+C_0^{-1})\bar r}\left(\frac{s_2}{s_1}\right)^{\frac{2\tilde\alpha(1-\tau)}{\|\mu\|_{L^\infty(B_{\bar r})}}}.$$

Proof. Let $\tilde\alpha \in (0,1)$ be as in (5.17). By (5.18), (2.2) and Lemma 4.7, for every $\varepsilon < \min\{\varepsilon_1, \frac{\bar r}{\kappa M_\tau}\}$ and $r \in [\kappa M_\tau \varepsilon, , \bar r]$

$$\begin{aligned}\tilde\alpha \int_{B_r} |\nabla v_i^\varepsilon|^2 \,\mathrm{d}x &\le \int_{B_r} \left(\widetilde A \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon - \lambda_i^\varepsilon \widetilde p |v_i^\varepsilon|^2\right) \mathrm{d}x + \|\mu\|_{L^\infty(B_{\bar r})} \int_{\partial B_r} |v_i^\varepsilon|^2 \,\mathrm{d}S \\ &\le \int_{B_r} \left(\widetilde A \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon - \lambda_{n_0} \widetilde p |v_i^\varepsilon|^2\right) \mathrm{d}x + \|\mu\|_{L^\infty(B_{\bar r})} \int_{\partial B_r} |v_i^\varepsilon|^2 \,\mathrm{d}S \\ &\le r^{N-2} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) + \frac{\|\mu\|_{L^\infty(B_{\bar r})} r}{1-\tau} \int_{B_r} |\nabla v_i^\varepsilon|^2 \,\mathrm{d}x\end{aligned}$$

so that, using again Lemma 4.7 and recalling that $\tilde\alpha - 2\|\mu\|_{L^\infty(B_{\bar r})} r > 0$ in view of (5.16), we obtain that

$$\begin{aligned}r^{N-2} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) &\ge \left(\tilde\alpha - \frac{\|\mu\|_{L^\infty(B_{\bar r})} r}{1-\tau}\right) \int_{B_r} |\nabla v_i^\varepsilon|^2 \,\mathrm{d}x \ge \left(\tilde\alpha - 2\|\mu\|_{L^\infty(B_{\bar r})} r\right) \int_{B_r} |\nabla v_i^\varepsilon|^2 \,\mathrm{d}x \\ &\ge \left(\tilde\alpha - 2\|\mu\|_{L^\infty(B_{\bar r})} r\right) \frac{1-\tau}{r} \int_{\partial B_r} |v_i^\varepsilon|^2 \,\mathrm{d}S \\ &\ge \left(\tilde\alpha - 2\|\mu\|_{L^\infty(B_{\bar r})} r\right) \frac{1-\tau}{\|\mu\|_{L^\infty(B_{\bar r})}} r^{N-2} H(v_i^\varepsilon, r) \\ &\ge \frac{\tilde\alpha(1-\tau)}{\|\mu\|_{L^\infty(B_{\bar r})}} r^{N-2} H(v_i^\varepsilon, r) - 2r^{N-1} H(v_i^\varepsilon, r).\end{aligned} \tag{6.13}$$

From Remark 6.3 and (6.13) it follows that

$$\frac{d}{dr} H(v_i^\varepsilon, r) \ge \frac{2}{r} E(v_i^\varepsilon, r, \lambda_i^\varepsilon) - \frac{1}{C_0} H(v_i^\varepsilon, r) \ge \frac{2\tilde\alpha(1-\tau)}{\|\mu\|_{L^\infty(B_{\bar r})}} \frac{H(v_i^\varepsilon, r)}{r} - (4 + C_0^{-1}) H(v_i^\varepsilon, r)$$

in $[\kappa M_\tau \varepsilon, , \bar r]$. The conclusion then follows by integration between s_1 and s_2. ☐

6.2. **Energy estimates.** By a combination of Lemma A.1 with estimates (5.5)–(5.7) it is possible to prove the following perturbed Poincaré-type inequality.

Lemma 6.7. *For any $r \le r_1$ and for any $u \in H^1(B_r)$ there holds*

$$\frac{N-1}{r^2} \int_{B_r} \widetilde p u^2 \,\mathrm{d}y \le 3 \left(\int_{B_r} \widetilde A \nabla u \cdot \nabla u \,\mathrm{d}y + \frac{1}{r} \int_{\partial B_r} \mu u^2 \,\mathrm{d}S \right).$$

Proposition 6.8. *For any R, K such that $R \ge K \ge \kappa$ there holds*

$$\int_{B_{R\varepsilon}} \widetilde A \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon \,\mathrm{d}y = O(\varepsilon^{N-2} H(v_i^\varepsilon, K\varepsilon)) \quad \text{as } \varepsilon \to 0, \tag{6.14}$$

$$\int_{B_{R\varepsilon}} \widetilde p \, |v_i^\varepsilon|^2 \,\mathrm{d}y = O(\varepsilon^N H(v_i^\varepsilon, K\varepsilon)) \quad \text{as } \varepsilon \to 0, \tag{6.15}$$

$$\int_{\partial B_{R\varepsilon}} \mu \, |v_i^\varepsilon|^2 \,\mathrm{d}S = O(\varepsilon^{N-1} H(v_i^\varepsilon, K\varepsilon)) \quad \text{as } \varepsilon \to 0, \tag{6.16}$$

for all $i \in \{1, \dots, n_0\}$.

Proof. First of all, we prove that

$$\mathcal N(v_i^\varepsilon, \bar r, \lambda_i^\varepsilon) = O(1) \quad \text{as } \varepsilon \to 0. \tag{6.17}$$

We notice that

$$\begin{aligned}E(v_i^\varepsilon, \bar r, \lambda_i^\varepsilon) &\le \bar r^{2-N} \int_{B_{\bar r}} \widetilde A \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon \,\mathrm{d}y \\ &= 2\bar r^{2-N} \int_{\Phi^{-1}(B_{\bar r}^+)} |\nabla \varphi_i^\varepsilon|^2 \,\mathrm{d}x \le 2\bar r^{2-N} \int_\Omega |\nabla \varphi_i^\varepsilon| = 2\bar r^{2-N} \lambda_i^\varepsilon.\end{aligned}$$

Since $\lambda_i^\varepsilon \le \lambda_{n_0}$ for all $\varepsilon \in (0,1)$ and all $1 \le i \le n_0$, we have that $E(v_i^\varepsilon, \bar r, \lambda_i^\varepsilon)$ is bounded for $\varepsilon \in (0, \min\{\varepsilon_1, \bar r/K\})$. From Lemma 6.1 (ii) we know that there exists $C_{\bar r} > 0$ and $\alpha_{\bar r} \in (0, \bar r/K)$

such that $H(v_i^\varepsilon, \bar r) \geq C_{\bar r}$ for all $\varepsilon \in (0, \min\{\alpha_{\bar r}, \varepsilon_1\})$. Therefore (6.17) is proved. Hence from estimate (6.12) we deduce that there exists $c_0 > 0$ such that

$$\mathcal{N}(v_i^\varepsilon, r, \lambda_i^\varepsilon) \leq c_0 \tag{6.18}$$

for all $\varepsilon \in (0, \min\{\varepsilon_1, \bar r/K, \alpha_{\bar r}\})$ and all $K\varepsilon \leq r \leq \bar r$.

By Lemma 6.2 and (6.18) there exist $R_1 \in (0, \bar r)$, $c_1 > 0$ and $\bar\varepsilon \in (0, \min\{\varepsilon_1, R_1/R\})$ such that, for any $\varepsilon \in (0, \bar\varepsilon)$ and for any $K\varepsilon \leq r \leq R_1$

$$\frac{\frac{d}{dr} H(v_i^\varepsilon, r)}{H(v_i^\varepsilon, r)} \leq c_1 \left(\frac{1}{r} + 1\right). \tag{6.19}$$

By integration of (6.19) in $(K\varepsilon, R\varepsilon)$ we obtain that

$$\frac{H(v_i^\varepsilon, R\varepsilon)}{H(v_i^\varepsilon, K\varepsilon)} \leq \left(\frac{R}{K}\right)^{c_1} e^{c_1 \varepsilon (R-K)}$$

which, in turn, implies (6.16).

From (5.19) and (5.5) we have that there exists $R_2 \in (0, \tilde r)$ such that

$$\int_{B_{R\varepsilon}} (\widetilde{A} \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon - \lambda_{n_0} \widetilde{p} |v_i^\varepsilon|^2)\, dy \geq \frac{\tilde\alpha}{2} \int_{B_{R\varepsilon}} \widetilde{A} \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon \, dy$$

for all $\varepsilon \in (0, \min\{\varepsilon_1, R_2/R\})$. Therefore, since $\lambda_i^\varepsilon \leq \lambda_{n_0}$, there exists $c_2 > 0$ such that

$$\int_{B_{R\varepsilon}} \widetilde{A} \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon \, dy \leq c_2 \varepsilon^{N-2} E(v_i^\varepsilon, R\varepsilon, \lambda_i^\varepsilon)$$

for all $\varepsilon \in (0, \min\{\varepsilon_1, R_2/R\})$. Then (6.18) (with $r = R\varepsilon$) yields

$$\int_{B_{R\varepsilon}} \widetilde{A} \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon \, dy \leq c_0 c_2 \varepsilon^{N-2} H(v_i^\varepsilon, R\varepsilon), \tag{6.20}$$

for all $\varepsilon \in (0, \min\{\varepsilon_1, R_2/R, \alpha_{\bar r}\})$. This fact, together with (6.16), proves (6.14). Applying Lemma 6.7 with $r = R\varepsilon$, for ε sufficiently small, and $u = v_i^\varepsilon$, in view of (6.14) and (6.16) we obtain (6.15), thus concluding the proof. □

Hereafter, we denote

$$\beta := 2\tilde\alpha / \|\mu\|_{L^\infty(B_{\bar r})}. \tag{6.21}$$

Proposition 6.9. *Let $\tau \in (0, 1/2)$, $M_\tau > 1$ as in Lemma 4.7 and β as in (6.21). Then, for any $R \geq M_\tau \kappa$, there holds*

$$\int_{B_{R\varepsilon}} \widetilde{A} \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon \, dy = O(\varepsilon^{N-2+\beta(1-\tau)}) \quad \text{as } \varepsilon \to 0, \tag{6.22}$$

$$\int_{B_{R\varepsilon}} \widetilde{p} |v_i^\varepsilon|^2 \, dy = O(\varepsilon^{N+\beta(1-\tau)}) \quad \text{as } \varepsilon \to 0, \tag{6.23}$$

$$\int_{\partial B_{R\varepsilon}} \mu |v_i^\varepsilon|^2 \, dS = O(\varepsilon^{N-1+\beta(1-\tau)}) \quad \text{as } \varepsilon \to 0, \tag{6.24}$$

for all $i \in \{1, \dots, n_0\}$.

Proof. From Lemma 6.6 we know that there exists a constant $C > 0$ such that

$$H(v_i^\varepsilon, R\varepsilon) \leq C \varepsilon^{\beta(1-\tau)} H(v_i^\varepsilon, \bar r) \quad \text{for all } \varepsilon \in (0, \bar r/R). \tag{6.25}$$

Combining estimates (5.5) and (5.6) with Lemma 4.7, we obtain that

$$H(v_i^\varepsilon, \bar r) \leq \frac{3\bar r^{2-N}}{(1-\tau)} \int_{B_{\bar r}} \widetilde{A} \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon \, dy. \tag{6.26}$$

By definition of $\widetilde{A}$ and monotonicity of eigenvalues, we have that

$$\int_{B_{\bar r}} \widetilde{A} \nabla v_i^\varepsilon \cdot \nabla v_i^\varepsilon \, dy = 2 \int_{\Phi^{-1}(B_{\bar r}^+)} |\nabla \varphi_i^\varepsilon|^2 \, dx \leq 2\lambda_i^\varepsilon \leq 2\lambda_{n_0}.$$

This, together with (6.26) and (6.25), implies (6.24). Moreover, (6.22) follows from (6.20) and (6.24), while (6.23) comes as a consequence of Lemma 6.7, (6.22) and (6.24). □

The following result is a straightforward consequence of the previous two propositions.

Corollary 6.10. *Let $\tau \in (0, 1/2)$. Then for any $K \geq \kappa$ there exist $\bar{C}, q, \tilde{\varepsilon} > 0$ such that*

$$H(v_{n_0}^\varepsilon, K\varepsilon) \geq \bar{C}\varepsilon^q \quad \text{for all } \varepsilon \in (0, \tilde{\varepsilon}). \tag{6.27}$$

Moreover, letting M_τ be as in Lemma 4.7, β as in (6.21) and $K \geq M_\tau \kappa$, we have that, for all $i \in \{1, \dots, n_0\}$,

$$H(v_i^\varepsilon, K\varepsilon) = O(\varepsilon^{\beta(1-\tau)}) \quad \text{as } \varepsilon \to 0. \tag{6.28}$$

Proof. If we integrate (6.19) between $K\varepsilon$ and R_1 we obtain that

$$\frac{H(v_{n_0}^\varepsilon, R_1)}{H(v_{n_0}^\varepsilon, K\varepsilon)} \leq \left(\frac{R_1 e^{R_1}}{K}\right)^{c_1} \varepsilon^{-c_1}.$$

Then, in view of Lemma 6.1 point (ii), (6.27) follows with

$$\bar{C} := C_{R_1} \left(\frac{K}{R_1 e^{R_1}}\right)^{c_1} \quad \text{and} \quad q := c_1.$$

Finally (6.28) directly comes from Proposition 6.9. □

7. Upper bound on $\lambda_{n_0} - \lambda_{n_0}^\varepsilon$

Hereafter we fix $\tau \in (0, 1/2)$ and

$$K_\tau > 2\kappa M_\tau$$

with κ as in Lemma 4.5 and M_τ as in Lemma 4.7. For convenience in the exposition, hereafter we denote

$$\Theta_r := \Phi^{-1}(B_r^+) \tag{7.1}$$

for any $r \in (0, r_1)$, with Φ as in (4.5).

For every $i \in \{1, \dots, n_0\}$, $R \geq K_\tau$ and $\varepsilon \in (0, \min\{\varepsilon_1, \tilde{r}/R\})$ we consider the following minimization problem

$$\min\left\{\int_{\Theta_{R\varepsilon}} |\nabla u|^2 \, dx \colon u \in H^1(\Theta_{R\varepsilon}),\ u - (\eta_{R\varepsilon} \circ \Phi)\varphi_i^\varepsilon \in H_0^1(\Theta_{R\varepsilon})\right\}, \tag{7.2}$$

where $\eta_{R\varepsilon}(x) = \eta_R(x/\varepsilon)$ and η_R is as in (3.1). By standard variational methods, it is easy to prove that this problem has a unique solution $\xi_{i,R,\varepsilon}^{\rm int}$, which weakly satisfies

$$\begin{cases} -\Delta \xi_{i,R,\varepsilon}^{\rm int} = 0, & \text{in } \Theta_{R\varepsilon}, \\ \xi_{i,R,\varepsilon}^{\rm int} = \varphi_i^\varepsilon, & \text{on } (\partial\Theta_{R\varepsilon})^+, \\ \xi_{i,R,\varepsilon}^{\rm int} = 0, & \text{on } (\partial\Theta_{R\varepsilon})^0, \end{cases}$$

where

$$(\partial\Theta_{R\varepsilon})^+ := \partial\Theta_{R\varepsilon} \cap \Omega \quad \text{and} \quad (\partial\Theta_{R\varepsilon})^0 := \partial\Theta_{R\varepsilon} \cap \partial\Omega.$$

Lemma 7.1. *For any $R \geq K_\tau$ the following estimates hold as $\varepsilon \to 0$*

$$\int_{\Theta_{R\varepsilon}} \left|\nabla \xi_{i,R,\varepsilon}^{\rm int}\right|^2 dx = O(\varepsilon^{N-2} H(v_i^\varepsilon, K_\tau\varepsilon)), \tag{7.3}$$

$$\int_{\Theta_{R\varepsilon}} \left|\xi_{i,R,\varepsilon}^{\rm int}\right|^2 dx = O(\varepsilon^{N} H(v_i^\varepsilon, K_\tau\varepsilon)), \tag{7.4}$$

$$\int_{(\partial\Theta_{R\varepsilon})^+} \left|\xi_{i,R,\varepsilon}^{\rm int}\right|^2 dS = O(\varepsilon^{N-1} H(v_i^\varepsilon, K_\tau\varepsilon)), \tag{7.5}$$

together with

$$\int_{\Theta_{R\varepsilon}} \left|\nabla \xi_{i,R,\varepsilon}^{\mathrm{int}}\right|^2 \,\mathrm{d}x = O(\varepsilon^{N-2+\beta(1-\tau)}), \tag{7.6}$$

$$\int_{\Theta_{R\varepsilon}} \left|\xi_{i,R,\varepsilon}^{\mathrm{int}}\right|^2 \,\mathrm{d}x = O(\varepsilon^{N+\beta(1-\tau)}), \tag{7.7}$$

$$\int_{(\partial\Theta_{R\varepsilon})^+} \left|\xi_{i,R,\varepsilon}^{\mathrm{int}}\right|^2 \,\mathrm{d}S = O(\varepsilon^{N-1+\beta(1-\tau)}), \tag{7.8}$$

for all $i \in \{1, \ldots, n_0\}$, where β is defined in (6.21).

Proof. By the change of variable induced by the diffeomorphism Φ, problem (7.2) is equivalent to

$$\min\left\{\int_{B_{R\varepsilon}^+} A\nabla u\cdot\nabla u \,\mathrm{d}y \colon u \in H^1(B_{R\varepsilon}^+),\ u - \eta_{R\varepsilon}u_i^\varepsilon \in H_0^1(B_{R\varepsilon}^+)\right\},$$

with A as in (4.10), and the minimum is attained by $\xi_{i,R,\varepsilon}^{\mathrm{int}}\circ\Phi^{-1}$. If one tests the problem above with $u = \eta_{R\varepsilon}u_i^\varepsilon$, the following is obtained, in view also of (5.5), (5.7) and (3.1),

$$\begin{aligned}\int_{\Theta_{R\varepsilon}} \left|\nabla \xi_{i,R,\varepsilon}^{\mathrm{int}}\right|^2 \,\mathrm{d}x &= \int_{B_{R\varepsilon}^+} A\nabla(\xi_{i,R,\varepsilon}^{\mathrm{int}}\circ\Phi^{-1})\cdot\nabla(\xi_{i,R,\varepsilon}^{\mathrm{int}}\circ\Phi^{-1}) \,\mathrm{d}y\\ &\le \frac{3}{2}\int_{B_{R\varepsilon}^+} \left|\eta_{R\varepsilon}\nabla u_i^\varepsilon + u_i^\varepsilon\nabla\eta_{R\varepsilon}\right|^2 \,\mathrm{d}y\\ &\le 96\int_{B_{R\varepsilon}^+}\left(A\nabla u_i^\varepsilon\cdot\nabla u_i^\varepsilon + \frac{1}{(R\varepsilon)^2}\,p\,|u_i^\varepsilon|^2\right)\mathrm{d}y.\end{aligned}$$

Combining this estimate with (6.14) and (6.15) proves (7.3), while combining it with (6.22) and (6.23) proves (7.6). Since $\xi_{i,R,\varepsilon}^{\mathrm{int}} = \varphi_i^\varepsilon$ on $(\partial\Theta_{R\varepsilon})^+$, estimates (7.5) and (7.8) are trivial in view of (6.16) and (6.24). Finally, (7.4) and (7.7) come from the other estimates, Lemma 6.7, and the change of variable induced by the diffeomorphism Φ. □

Using the functions $\xi_{i,R,\varepsilon}^{\mathrm{int}}$ that solve (7.2), we construct a family of competitors (see (7.20)) to test the Rayleigh quotient for λ_{n_0} and obtain a sharp estimate from above of the eigenvalue variation $\lambda_{n_0} - \lambda_{n_0}^\varepsilon$. To this aim, we also provide suitable energy estimates for such competitors, see (7.21)–(7.26). For all $i \in \{1, \ldots, n_0\}$, $R \ge K_\tau$ and $\varepsilon \in (0, \min\{\varepsilon_1, \tilde{r}/R\})$ we define

$$\xi_{i,R,\varepsilon}(x) := \begin{cases}\varphi_i^\varepsilon(x), & \text{if } x \in \Omega\setminus\Theta_{R\varepsilon},\\ \xi_{i,R,\varepsilon}^{\mathrm{int}}(x), & \text{if } x \in \Theta_{R\varepsilon}.\end{cases} \tag{7.9}$$

We observe that $\xi_{i,R,\varepsilon} \in H_0^1(\Omega)$ thanks to the fact that $R \ge K_\tau > 2\kappa$, which guarantees that $\varepsilon\mathcal{V} \subset \Theta_{\frac{R\varepsilon}{2}}$. Moreover it is easy to verify that the family $\{\xi_{1,R,\varepsilon}, \ldots, \xi_{n_0,R,\varepsilon}\}$ is linearly independent in $H_0^1(\Omega)$, for ε sufficiently small. We also define

$$Z_R^\varepsilon(x) := \frac{(\xi_{n_0,R,\varepsilon}^{\mathrm{int}}\circ\Phi^{-1})(\varepsilon x)}{\sqrt{H(\varepsilon)}}, \quad \Upsilon^\varepsilon(x) := \frac{u_{n_0}^\varepsilon(\varepsilon x)}{\sqrt{H(\varepsilon)}}, \tag{7.10}$$

where we denote

$$H(\varepsilon) := \frac{1}{2}H(v_{n_0}^\varepsilon, K_\tau\varepsilon) = \frac{1}{(K_\tau\varepsilon)^{N-1}}\int_{S_{K_\tau\varepsilon}^+} \mu|u_{n_0}^\varepsilon|^2 \,\mathrm{d}S. \tag{7.11}$$

As a consequence of the estimates given in Propositions 6.8 and 6.9 and Lemma 7.1, we are able to prove the following result.

Lemma 7.2. *For any $R \geq K_\tau$ we have that, as $\varepsilon \to 0$,*

$$\int_\Omega |\nabla \xi_{n_0,R,\varepsilon}|^2 \,\mathrm{d}x = \lambda_{n_0}^\varepsilon + \varepsilon^{N-2} H(\varepsilon) \left(\int_{B_R^+} A(\varepsilon y) \nabla Z_R^\varepsilon \cdot \nabla Z_R^\varepsilon \mathrm{d}y - \int_{B_R^+} A(\varepsilon y) \nabla \Upsilon^\varepsilon \cdot \nabla \Upsilon^\varepsilon \mathrm{d}y \right), \tag{7.12}$$

$$\int_\Omega |\nabla \xi_{i,R,\varepsilon}|^2 \,\mathrm{d}x = \lambda_i^\varepsilon + O(\varepsilon^{N-2+\beta(1-\tau)}), \quad \textit{for all } i = 1, \dots, n_0, \tag{7.13}$$

$$\int_\Omega \nabla \xi_{i,R,\varepsilon} \cdot \nabla \xi_{n_0,R,\varepsilon} \,\mathrm{d}x = O\big(\varepsilon^{N-2+\frac{\beta}{2}(1-\tau)} \sqrt{H(\varepsilon)}\big), \quad \textit{for all } i = 1, \dots, n_0 - 1, \tag{7.14}$$

$$\int_\Omega \nabla \xi_{i,R,\varepsilon} \cdot \nabla \xi_{j,R,\varepsilon} \,\mathrm{d}x = O(\varepsilon^{N-2+\beta(1-\tau)}), \quad \textit{for all } i, j = 1, \dots, n_0, \ i \neq j, \tag{7.15}$$

$$\int_\Omega |\xi_{n_0,R,\varepsilon}|^2 \,\mathrm{d}x = 1 + O(\varepsilon^N H(\varepsilon)), \tag{7.16}$$

$$\int_\Omega |\xi_{i,R,\varepsilon}|^2 \,\mathrm{d}x = 1 + O(\varepsilon^{N+\beta(1-\tau)}), \quad \textit{for all } i = 1, \dots, n_0, \tag{7.17}$$

$$\int_\Omega \xi_{i,R,\varepsilon}\, \xi_{n_0,R,\varepsilon} \,\mathrm{d}x = O(\varepsilon^{N+\frac{\beta}{2}(1-\tau)} \sqrt{H(\varepsilon)}), \quad \textit{for all } i = 1, \dots, n_0 - 1, , \tag{7.18}$$

$$\int_\Omega \xi_{i,R,\varepsilon}\, \xi_{j,R,\varepsilon} \,\mathrm{d}x = O(\varepsilon^{N+\beta(1-\tau)}), \quad \textit{for all } i, j = 1, \dots, n_0, \ i \neq j. \tag{7.19}$$

Proof. By definition of $\xi_{n_0,R,\varepsilon}$ we have

$$\int_\Omega |\nabla \xi_{n_0,R,\varepsilon}|^2 \,\mathrm{d}x = \int_\Omega \left|\nabla \varphi_{n_0}^\varepsilon\right|^2 \,\mathrm{d}x - \int_{\Theta_{R\varepsilon}} \left|\nabla \varphi_{n_0}^\varepsilon\right|^2 \,\mathrm{d}x + \int_{\Theta_{R\varepsilon}} \left|\nabla \xi_{n_0,R,\varepsilon}^{\mathrm{int}}\right|^2 \,\mathrm{d}x.$$

Since, by (1.8)– (1.9), $\int_\Omega \left|\nabla \varphi_{n_0}^\varepsilon\right|^2 \,\mathrm{d}x = \lambda_{n_0}^\varepsilon$, by the change of variable $y = \Phi(x)$ and the definition of Z_R^ε and Υ^ε given in (7.10), we obtain (7.12). Similarly, for any $i = 1, \dots, n_0$,

$$\int_\Omega |\nabla \xi_{i,R,\varepsilon}|^2 \,\mathrm{d}x = \lambda_i^\varepsilon - \int_{\Theta_{R\varepsilon}} |\nabla \varphi_i^\varepsilon|^2 \,\mathrm{d}x + \int_{\Theta_{R\varepsilon}} \left|\nabla \xi_{i,R,\varepsilon}^{\mathrm{int}}\right|^2 \,\mathrm{d}x.$$

Then (7.13) follows from (6.22) and (7.6).

For all $i = 1, \dots, n_0 - 1$ we have that

$$\int_\Omega \nabla \xi_{i,R,\varepsilon} \cdot \nabla \xi_{n_0,R,\varepsilon} \,\mathrm{d}x = - \int_{\Theta_{R\varepsilon}} \nabla \varphi_i^\varepsilon \cdot \nabla \varphi_{n_0}^\varepsilon \,\mathrm{d}x + \int_{\Theta_{R\varepsilon}} \nabla \xi_{i,R,\varepsilon}^{\mathrm{int}} \cdot \nabla \xi_{n_0,R,\varepsilon}^{\mathrm{int}} \,\mathrm{d}x,$$

since the perturbed eigenfunctions are orthogonal, therefore (7.14) follows from Cauchy-Schwartz inequality and estimates (6.14), (6.22), (7.3) and (7.6). Finally, again by orthogonality, for any $i, j = 1, \dots, n_0$, $i \neq j$, we have that

$$\int_\Omega \nabla \xi_{i,R,\varepsilon} \cdot \nabla \xi_{j,R,\varepsilon} \,\mathrm{d}x = - \int_{\Theta_{R\varepsilon}} \nabla \varphi_i^\varepsilon \cdot \nabla \varphi_j^\varepsilon \,\mathrm{d}x + \int_{\Theta_{R\varepsilon}} \nabla \xi_{i,R,\varepsilon}^{\mathrm{int}} \cdot \nabla \xi_{j,R,\varepsilon}^{\mathrm{int}} \,\mathrm{d}x$$

and so (7.15) easily follows from Cauchy-Schwartz inequality and estimates (6.22) and (7.6). The proof of (7.16)–(7.19) is completely analogous and it is therefore omitted. □

We now construct an orthogonal basis $\{\hat{\xi}_{1,R,\varepsilon}, \dots, \hat{\xi}_{n_0,R,\varepsilon}\}$ of the space span $\{\xi_{1,R,\varepsilon}, \dots, \xi_{n_0,R,\varepsilon}\}$. To this aim we recursively define

$$\hat{\xi}_{n_0,R,\varepsilon} := \xi_{n_0,R,\varepsilon} \quad \text{and} \quad \hat{\xi}_{i,R,\varepsilon} := \xi_{i,R,\varepsilon} - \sum_{j=i+1}^{n_0} d_{i,j}^{R,\varepsilon} \hat{\xi}_{j,R,\varepsilon} \quad \text{for } i = 1, \dots, n_0 - 1, \tag{7.20}$$

where

$$d_{i,j}^{R,\varepsilon} := \frac{\int_\Omega \xi_{i,R,\varepsilon} \hat{\xi}_{j,R,\varepsilon} \,\mathrm{d}x}{\int_\Omega |\hat{\xi}_{j,R,\varepsilon}|^2 \,\mathrm{d}x}.$$

The functions $\{\hat{\xi}_{1,R,\varepsilon},\dots,\hat{\xi}_{n_0,R,\varepsilon}\}$ are orthogonal in $L^2(\Omega)$. Moreover they satisfy the following estimates:

$$\int_\Omega |\nabla\hat{\xi}_{n_0,R,\varepsilon}|^2 \mathrm{d}x = \lambda^\varepsilon_{n_0} + \varepsilon^{N-2}H(\varepsilon)\left(\int_{B_R^+} A(\varepsilon y)\nabla Z_R^\varepsilon\cdot\nabla Z_R^\varepsilon \,\mathrm{d}y - \int_{B_R^+} A(\varepsilon y)\nabla\Upsilon^\varepsilon\cdot\nabla\Upsilon^\varepsilon \,\mathrm{d}y\right), \tag{7.21}$$

$$\int_\Omega |\nabla\hat{\xi}_{i,R,\varepsilon}|^2 \,\mathrm{d}x = \lambda^\varepsilon_i + O(\varepsilon^{N-2+\beta(1-\tau)}), \quad \text{for all } i=1,\dots,n_0, \tag{7.22}$$

$$\int_\Omega \nabla\hat{\xi}_{i,R,\varepsilon}\cdot\nabla\hat{\xi}_{n_0,R,\varepsilon} \,\mathrm{d}x = O(\varepsilon^{N-2+\frac{\beta}{2}(1-\tau)}\sqrt{H(\varepsilon)}), \quad \text{for all } i=1,\dots,n_0-1, \tag{7.23}$$

$$\int_\Omega \nabla\hat{\xi}_{i,R,\varepsilon}\cdot\nabla\hat{\xi}_{j,R,\varepsilon} \,\mathrm{d}x = O(\varepsilon^{N-2+\beta(1-\tau)}), \quad \text{for all } i,j=1,\dots,n_0,\ i\neq j, \tag{7.24}$$

$$\int_\Omega |\hat{\xi}_{n_0,R,\varepsilon}|^2 \,\mathrm{d}x = 1 + O(\varepsilon^N H(\varepsilon)), \tag{7.25}$$

$$\int_\Omega |\hat{\xi}_{i,R,\varepsilon}|^2 \,\mathrm{d}x = 1 + O(\varepsilon^{N+\beta(1-\tau)}), \quad \text{for all } i=1,\dots,n_0. \tag{7.26}$$

The proof of estimates (7.21)–(7.26) consists in direct computations and comes from Lemma 7.2 and the following estimates on the coefficients $d^{R,\varepsilon}_{i,j}$

$$d^{R,\varepsilon}_{j,n_0} = O(\varepsilon^{N+\frac{\beta}{2}(1-\tau)}\sqrt{H(\varepsilon)}) \quad \text{for all } j=1,\dots,n_0-1,$$
$$d^{R,\varepsilon}_{j,k} = O(\varepsilon^{N+\beta(1-\tau)}) \quad \text{for all } k=2,\dots,n_0,\ j<k.$$

We are now ready to prove an upper bound of the eigenvalue variation $\lambda_{n_0}-\lambda^\varepsilon_{n_0}$.

Proposition 7.3. *For any $R\geq K_\tau$ we have that*

$$\lambda_{n_0}-\lambda^\varepsilon_{n_0} \leq \varepsilon^{N-2}H(\varepsilon)(f_R(\varepsilon)+o(1)) \quad \text{as } \varepsilon\to 0, \tag{7.27}$$

where

$$f_R(\varepsilon) = \int_{B_R^+} A(\varepsilon y)\nabla Z_R^\varepsilon\cdot\nabla Z_R^\varepsilon \,\mathrm{d}y - \int_{B_R^+} A(\varepsilon y)\nabla\Upsilon^\varepsilon\cdot\nabla\Upsilon^\varepsilon \,\mathrm{d}y. \tag{7.28}$$

Moreover

$$f_R(\varepsilon) = O(1) \quad \text{as } \varepsilon\to 0. \tag{7.29}$$

Proof. By the Courant-Fischer Min-Max variational characterization of the eigenvalues, see (2.1), we have that

$$\lambda_{n_0} = \min\left\{ \max_{\substack{a_1,\dots,a_{n_0}\in\mathbb{R}\\ \sum_{i=1}^{n_0} a_i^2=1}} \frac{\|\nabla(\sum_{i=1}^{n_0} a_i u_i)\|^2_{L^2(\Omega)}}{\|\sum_{i=1}^{n_0} a_i u_i\|^2_{L^2(\Omega)}} : \begin{array}{l}\{u_1\dots,u_{n_0}\}\subset H^1_0(\Omega)\\ \text{linearly independent}\end{array}\right\}.$$

We test the above minimization problem with the orthonormal family

$$\left\{ u_i := \frac{\hat{\xi}_{i,R,\varepsilon}}{\|\hat{\xi}_{i,R,\varepsilon}\|_{L^2(\Omega)}} \right\}_{i=1,\dots,n_0},$$

where $\hat{\xi}_{i,R,\varepsilon}$ is defined in (7.20); we thus obtain

$$\lambda_{n_0}-\lambda^\varepsilon_{n_0} \leq \max_{\substack{a_1,\dots,a_{n_0}\in\mathbb{R}\\ \sum_{i=1}^{n_0} a_i^2=1}} \int_\Omega \left|\nabla\left(\sum_{i=1}^{n_0} a_i \frac{\hat{\xi}_{i,R,\varepsilon}}{\|\hat{\xi}_{i,R,\varepsilon}\|_{L^2(\Omega)}}\right)\right|^2 \mathrm{d}x - \lambda^\varepsilon_{n_0} = \max_{\substack{a_1,\dots,a_{n_0}\in\mathbb{R}\\ \sum_{i=1}^{n_0} a_i^2=1}} \sum_{i,j=1}^{n_0} M^\varepsilon_{i,j}a_i a_j,$$

with

$$M^\varepsilon_{i,j} = \frac{\int_\Omega \nabla\hat{\xi}_{i,R,\varepsilon}\cdot\nabla\hat{\xi}_{j,R,\varepsilon}\,\mathrm{d}x}{\|\hat{\xi}_{i,R,\varepsilon}\|_{L^2(\Omega)}\|\hat{\xi}_{j,R,\varepsilon}\|_{L^2(\Omega)}} - \lambda^\varepsilon_{n_0}\delta^j_i.$$

From estimates (7.21)–(7.26) we deduce the behaviour of the coefficients $M^\varepsilon_{i,j}$'s as $\varepsilon \to 0$, that is

$$\begin{aligned}
&M^\varepsilon_{n_0,n_0} = \varepsilon^{N-2} H(\varepsilon)(f_R(\varepsilon) + o(1)),\\
&M^\varepsilon_{i,i} = \lambda^\varepsilon_i - \lambda^\varepsilon_{n_0} + o(1), \quad \text{for all } i = 1, \dots, n_0 - 1,\\
&M^\varepsilon_{i,n_0} = O(\varepsilon^{N-2+\frac{\beta}{2}(1-\tau)}\sqrt{H(\varepsilon)}), \quad \text{for all } i = 1, \dots, n_0 - 1,\\
&M^\varepsilon_{i,j} = O(\varepsilon^{N-2+\beta(1-\tau)}), \quad \text{for all } i, j = 1, \dots, n_0 - 1,\ i \neq j.
\end{aligned}$$

Moreover, (6.14) and (7.3) yield that $f_R(\varepsilon) = O(1)$ as $\varepsilon \to 0$, while from Corollary 6.10 we have that, for all $\varepsilon \in (0, \tilde\varepsilon)$, $H(\varepsilon) \geq \bar C \varepsilon^q$ for some $\bar C, q, \tilde\varepsilon > 0$. Therefore the assumptions of Lemma A.2 are fulfilled with

$$\begin{aligned}
&\sigma(\varepsilon) = \varepsilon^{N-2} H(\varepsilon), \quad \mu(\varepsilon) = f_R(\varepsilon) + o(1), \quad \text{as } \varepsilon \to 0,\\
&a = \frac{1}{2}(N - 2 + \beta(1-\tau)) > 0, \quad M > \frac{N-2+q}{a} - 2.
\end{aligned}$$

Hence

$$\max_{\substack{a_1,\dots,a_{n_0}\in\mathbb{R}\\ \sum_{i=1}^{n_0} a_i^2 = 1}} \sum_{i,j=1}^{n_0} M^\varepsilon_{i,j} a_i a_j = \varepsilon^{N-2} H(\varepsilon)(f_R(\varepsilon) + o(1))$$

as $\varepsilon \to 0$ and the proof of (7.27) is complete. As already observed, (7.29) is a consequence of (6.14) and (7.3). □

8. Lower bound on $\lambda_{n_0} - \lambda^\varepsilon_{n_0}$

In this section we provide a lower bound for the eigenvalue variation $\lambda_{n_0} - \lambda^\varepsilon_{n_0}$. In order to do this, we first construct a family of competitors for the Rayleigh quotient of $\lambda^\varepsilon_{n_0}$; then, exploiting the local energy estimates stated in Lemma 8.2, we prove a blow-up result for their scaling, see Lemma 8.3.

Recalling the definition of ψ_i given in (1.14), from (1.13) and (4.6)–(4.7) we easily deduce that

$$\frac{(\varphi_i \circ \Phi^{-1})(rx)}{r^{\gamma_i}} \to \psi_i \quad \text{in } H^1(B_R) \text{ as } r \to 0, \text{ for every } R > 0. \tag{8.1}$$

From (8.1) and Lemma A.1 we deduce the following estimates for $\varphi_i \circ \Phi^{-1}$, $i = 1, \dots, n_0$.

Lemma 8.1. *There exists $\tilde C > 0$ such that, for all $i \in \{1, \dots, n_0 - 1\}$, for all $R > 1$ and $\varepsilon \in (0, \frac{r_1}{R})$,*

$$\begin{aligned}
&\|\nabla(\varphi_i \circ \Phi^{-1})\|^2_{L^2(B^+_{R\varepsilon})} \leq \tilde C (R\varepsilon)^N,\\
&\|\varphi_i \circ \Phi^{-1}\|^2_{L^2(B^+_{R\varepsilon})} \leq \tilde C (R\varepsilon)^{N+2},\\
&\|\varphi_i \circ \Phi^{-1}\|^2_{L^2(S^+_{R\varepsilon})} \leq \tilde C (R\varepsilon)^{N+1},
\end{aligned}$$

where, for $r > 0$, B^+_r and S^+_r are defined in (3.8), and

$$\begin{aligned}
&\|\nabla(\varphi_{n_0} \circ \Phi^{-1})\|^2_{L^2(B^+_{R\varepsilon})} \leq \tilde C (R\varepsilon)^{N+2\gamma-2},\\
&\|\varphi_{n_0} \circ \Phi^{-1}\|^2_{L^2(B^+_{R\varepsilon})} \leq \tilde C (R\varepsilon)^{N+2\gamma},\\
&\|\varphi_{n_0} \circ \Phi^{-1}\|^2_{L^2(S^+_{R\varepsilon})} \leq \tilde C (R\varepsilon)^{N-1+2\gamma},
\end{aligned}$$

where γ is defined in (1.16).

For every $R > \kappa$ and $0 < \varepsilon < \frac{r_1}{R}$, we define

$$w_{n_0,R,\varepsilon}(x) = \begin{cases} \varphi_{n_0}(x), & \text{if } x \in \Omega \setminus \Theta_{R\varepsilon},\\ w^{\text{int}}_{n_0,R,\varepsilon}(x), & \text{if } x \in \Theta_{R\varepsilon}, \end{cases} \tag{8.2}$$

where the notation $\Theta_{R\varepsilon}$ has been introduced in (7.1),

$$w^{\text{int}}_{n_0,R,\varepsilon}(x) = w_{R,\varepsilon}(\Phi(x))$$

and $w_{R,\varepsilon}$ is the unique $H^1(B^+_{R\varepsilon})$-function satisfying $w_{R,\varepsilon} - (\varphi_{n_0} \circ \Phi^{-1}) \in H^1_{0,\partial B^+_{R\varepsilon} \setminus \widetilde{\Sigma}_\varepsilon}(B^+_{R\varepsilon})$ and achieving

$$\min \left\{ \|\nabla w\|^2_{L^2(B^+_{R\varepsilon})} : w \in H^1(B^+_{R\varepsilon}),\ w - (\varphi_{n_0} \circ \Phi^{-1}) \in H^1_{0,\partial B^+_{R\varepsilon} \setminus \widetilde{\Sigma}_\varepsilon}(B^+_{R\varepsilon}) \right\}.$$

By classical variational methods, $w_{R,\varepsilon}$ exists and it is the unique solution to

$$\begin{cases} w_{R,\varepsilon} - (\varphi_{n_0} \circ \Phi^{-1}) \in H^1_{0,\partial B^+_{R\varepsilon} \setminus \widetilde{\Sigma}_\varepsilon}(B^+_{R\varepsilon}), \\ \int_{B^+_{R\varepsilon}} \nabla w_{R,\varepsilon} \cdot \nabla \phi \, \mathrm{d}x = 0 \text{ for every } \phi \in H^1_{0,\partial B^+_{R\varepsilon} \setminus \widetilde{\Sigma}_\varepsilon}(B^+_{R\varepsilon}). \end{cases} \tag{8.3}$$

As a consequence of Lemma 8.1 and Lemma A.1 we obtain the following estimates.

Lemma 8.2. *Letting $\tilde{C} > 0$ be as in Lemma 8.1, we have that, for all $R > \kappa$ and $\varepsilon \in (0, \frac{r_1}{R})$,*

$$\|\nabla w_{R,\varepsilon}\|^2_{L^2(B^+_{R\varepsilon})} \leq \tilde{C}(R\varepsilon)^{N+2\gamma-2},$$
$$\|w_{R,\varepsilon}\|^2_{L^2(B^+_{R\varepsilon})} \leq \tilde{C}(R\varepsilon)^{N+2\gamma},$$
$$\|w_{R,\varepsilon}\|^2_{L^2(S^+_{R\varepsilon})} \leq \tilde{C}(R\varepsilon)^{N+2\gamma-1}.$$

For every $R > \kappa$, $0 < \varepsilon < \frac{r_1}{R}$, and $x \in B^+_R$, we let

$$U_{R,\varepsilon}(x) = \frac{w_{R,\varepsilon}(\varepsilon x)}{\varepsilon^\gamma}. \tag{8.4}$$

Lemma 8.3. *For all $R > \kappa$, $\lim_{\varepsilon \to 0} \|U_{R,\varepsilon} - U_R\|_{H^1(B^+_R)} = 0$, where U_R is as in Lemma 3.3.*

Proof. Let $R > \kappa$. From a change of variables and Lemma 8.2 we have

$$\|\nabla U_{R,\varepsilon}\|^2_{L^2(B^+_R)} = \varepsilon^{-N-2\gamma+2} \|\nabla w_{R,\varepsilon}\|^2_{L^2(B^+_{R\varepsilon})} \leq \tilde{C} R^{N+2\gamma-2}$$

and

$$\int_{S^+_R} U^2_{R,\varepsilon} \, \mathrm{d}S = \varepsilon^{1-N-2\gamma} \int_{S^+_{R\varepsilon}} w^2_{R,\varepsilon} \, \mathrm{d}S \leq \tilde{C} R^{N+2\gamma-1}$$

for all $\varepsilon \in (0, r_1/R)$, so that the family $\{U_{R,\varepsilon}\}_{\varepsilon \in (0,r_1/R)}$ is bounded in $H^1(B^+_R)$ in view of Lemma A.1.

We deduce that there exist $W \in H^1(B^+_R)$ and a sequence $\varepsilon_n \to 0$ such that $U_{R,\varepsilon_n} \rightharpoonup W$ weakly in $H^1(B^+_R)$ as $n \to \infty$. Letting

$$V_\varepsilon(x) = \frac{(\varphi_{n_0} \circ \Phi^{-1})(\varepsilon x)}{\varepsilon^\gamma}, \tag{8.5}$$

from (8.1) we have that

$$V_\varepsilon \to \psi \quad \text{in } H^1(B^+_R) \text{ as } \varepsilon \to 0. \tag{8.6}$$

Hence

$$U_{R,\varepsilon_n} - V_{\varepsilon_n} \rightharpoonup W - \psi \quad \text{as } n \to \infty \text{ in } H^1(B^+_R).$$

Since (8.3) yields that $U_{R,\varepsilon} - V_\varepsilon \in H^1_{0,\partial B^+_R \setminus (\frac{1}{\varepsilon}\widetilde{\Sigma}_\varepsilon)}(B^+_R)$, the above convergence and Remark 4.1 imply that

$$W - \psi \in H^1_{0,\partial B^+_R \setminus \Sigma}(B^+_R).$$

We observe that the equation satisfied by $U_{R,\varepsilon}$ is

$$\int_{B^+_R} \nabla U_{R,\varepsilon} \cdot \nabla \phi \, \mathrm{d}x = 0 \quad \text{for every } \phi \in H^1_{0,\partial B^+_R \setminus (\frac{1}{\varepsilon}\widetilde{\Sigma}_\varepsilon)}(B^+_R). \tag{8.7}$$

Let $\phi \in C^\infty_c(B^+_R \cup \Sigma)$. From (4.6)–(4.7) we easily deduce that $\phi \in C^\infty_c(B^+_R \cup (\frac{1}{\varepsilon}\widetilde{\Sigma}_\varepsilon))$ for ε sufficiently small, hence $\phi \in H^1_{0,\partial B^+_R \setminus (\frac{1}{\varepsilon}\widetilde{\Sigma}_\varepsilon)}(B^+_R)$ and (8.7) is satisfied for $\varepsilon = \varepsilon_n$ and large n. Therefore we can pass to the limit to infer that $\int_{B^+_R} \nabla W \cdot \nabla \phi \, \mathrm{d}x = 0$ for every $\phi \in C^\infty_c(B^+_R \cup \Sigma)$ and then, by density, for all $\phi \in H^1_{0,\partial B^+_R \setminus \Sigma}(B^+_R)$. By uniqueness of the solution to (3.10), we conclude that

$W = U_R$. Since the limit U_R is the same along every subsequence, the Urysohn's Subsequence Principle implies that the whole family $U_{R,\varepsilon}$ weakly converges to U_R in $H^1(B_R^+)$ as $\varepsilon \to 0$.

It remains to show the strong H^1-convergence. To this aim, we choose in (8.7) $\phi = U_{R,\varepsilon} - V_\varepsilon$ for all $\varepsilon \in (0, r_1/R)$. We obtain

$$\int_{B_R^+} |\nabla U_{R,\varepsilon}|^2 \,\mathrm{d}x = \int_{B_R^+} \nabla U_{R,\varepsilon} \cdot \nabla V_\varepsilon \,\mathrm{d}x \to \int_{B_R^+} \nabla U_R \cdot \nabla \psi \,\mathrm{d}x = \int_{B_R^+} |\nabla U_R|^2 \,\mathrm{d}x,$$

as $\varepsilon \to 0$, where the convergence above is justified by the fact that $U_{R,\varepsilon_n} \rightharpoonup U_R$ weakly in $H^1(B_R^+)$ and $V_\varepsilon \to \psi$ strongly in $H^1(B_R^+)$, and the last equality follows from (3.10). □

From (8.2), (8.4), and (8.5) it follows that, for all $R > \kappa$ fixed,

$$\begin{aligned}\int_\Omega |\nabla w_{n_0,R,\varepsilon}(x)|^2 \,\mathrm{d}x &= \lambda_{n_0} - \int_{\Theta_{R\varepsilon}} |\nabla \varphi_{n_0}(x)|^2 \,\mathrm{d}x + \int_{\Theta_{R\varepsilon}} |\nabla w^{\mathrm{int}}_{n_0,R,\varepsilon}(x)|^2 \,\mathrm{d}x \\ &= \lambda_{n_0} - \varepsilon^{N-2+2\gamma} \int_{B_R^+} \Big(A(\varepsilon y)\nabla V_\varepsilon(y) \cdot \nabla V_\varepsilon(y) - A(\varepsilon y)\nabla U_{R,\varepsilon}(y) \cdot \nabla U_{R,\varepsilon}(y)\Big)\, dy, \\ &= \lambda_{n_0} - \varepsilon^{N-2+2\gamma} \bigg(\int_{B_R^+} \Big(|\nabla V_\varepsilon(y)|^2 - |\nabla U_{R,\varepsilon}(y)|^2\Big)\, dy + o(1)\bigg),\end{aligned} \tag{8.8}$$

as $\varepsilon \to 0$, where the last estimate follows from (5.1) and boundedness of $\{V_\varepsilon\}_\varepsilon$ and $\{U_{R,\varepsilon}\}_\varepsilon$ in $H^1(B_R^+)$ (see Lemma 8.3).

The main goal of this section is to prove the following result.

Proposition 8.4. *For all $R > \kappa$, we have that*

$$\lambda^\varepsilon_{n_0} - \lambda_{n_0} \le \varepsilon^{N+2\gamma-2}(g_R(\varepsilon) + o(1)) \quad \text{as } \varepsilon \to 0, \tag{8.9}$$

where

$$g_R(\varepsilon) = \int_{B_R^+} \Big(|\nabla U_{R,\varepsilon}(y)|^2 - |\nabla V_\varepsilon(y)|^2\Big)\, dy \tag{8.10}$$

with V_ε and $U_{R,\varepsilon}$ defined in (8.5) and (8.4) respectively. Furthermore $\lim_{\varepsilon\to 0} g_R(\varepsilon) = g_R$, where

$$g_R = \|\nabla U_R\|^2_{L^2(B_R^+)} - \|\nabla \psi\|^2_{L^2(B_R^+)}. \tag{8.11}$$

Proof. We use the Min-Max characterization of the eigenvalue $\lambda^\varepsilon_{n_0}$ recalled in (2.1), that we rewrite as follows

$$\lambda^\varepsilon_{n_0} = \min \left\{ \max_{\substack{a_1,\dots,a_{n_0}\in\mathbb{R} \\ \sum_{i=1}^{n_0} a_i^2 = 1}} \frac{\|\nabla(\sum_{i=1}^{n_0} a_i u_i)\|^2_{L^2(\Omega)}}{\|\sum_{i=1}^{n_0} a_i u_i\|^2_{L^2(\Omega)}} : \begin{array}{l}\{u_1 \dots, u_{n_0}\} \subset H^1_{0,\partial\Omega\setminus\Sigma_\varepsilon}(\Omega) \\ \text{linearly independent}\end{array} \right\}. \tag{8.12}$$

Let us fix $R > \kappa$. We define

$$\hat{w}_{i,R,\varepsilon} = \varphi_i \quad \text{for all } i = 1, 2, \dots, n_0 - 1,$$

and

$$\hat{w}_{n_0,R,\varepsilon} = w_{n_0,R,\varepsilon} - \sum_{i=1}^{n_0-1} c_i^\varepsilon \varphi_i,$$

where

$$c_i^\varepsilon = \int_\Omega w_{n_0,R,\varepsilon}\varphi_i \,\mathrm{d}x.$$

Let

$$\tilde{w}_{i,R,\varepsilon} = \frac{\hat{w}_{i,R,\varepsilon}}{\|\hat{w}_{i,R,\varepsilon}\|_{L^2(\Omega)}}, \quad i = 1, \dots, n_0.$$

We note that the family $\{\tilde{w}_{i,R,\varepsilon}\}_{i=1,\dots,n_0}$ is orthonormal in $L^2(\Omega)$ and linearly independent in $H^1_{0,\partial\Omega\setminus\Sigma_\varepsilon}(\Omega)$.

Lemmas 8.1 and 8.2, (8.2) and (4.7) imply that, for all $i \in \{1, 2, \dots, n_0 - 1\}$,

$$c_i^\varepsilon = O(\varepsilon^{N+1+\gamma}) \quad \text{and} \quad \int_\Omega \nabla w_{n_0,R,\varepsilon} \cdot \nabla \varphi_i \, dx = O(\varepsilon^{N+\gamma-1}) \quad \text{as } \varepsilon \to 0. \tag{8.13}$$

Then, from (8.8) we deduce that

$$\begin{aligned}\int_\Omega |\nabla \hat{w}_{n_0,R,\varepsilon}(x)|^2 \, dx \\ &= \lambda_{n_0} - \varepsilon^{N-2+2\gamma}\left(\int_{B_R^+} \left(|\nabla V_\varepsilon(y)|^2 - |\nabla U_{R,\varepsilon}(y)|^2\right) dy + o(1)\right) + O(\varepsilon^{2(N+\gamma)}) \\ &= \lambda_{n_0} - \varepsilon^{N-2+2\gamma}\left(\int_{B_R^+} \left(|\nabla V_\varepsilon(y)|^2 - |\nabla U_{R,\varepsilon}(y)|^2\right) dy + o(1)\right)\end{aligned} \tag{8.14}$$

as $\varepsilon \to 0$. Furthermore (8.13) implies that, for all $i = 1, \dots, n_0 - 1$,

$$\int_\Omega \nabla \hat{w}_{n_0,R,\varepsilon}(x) \cdot \nabla \hat{w}_{i,R,\varepsilon}(x) \, dx = O(\varepsilon^{N+\gamma-1}) \quad \text{as } \varepsilon \to 0, \tag{8.15}$$

while (8.2), Lemma 8.1, Lemma 8.2, and (8.13) yield

$$\int_\Omega |\hat{w}_{n_0,R,\varepsilon}(x)|^2 \, dx = 1 + O(\varepsilon^{N+2\gamma}), \quad \text{as } \varepsilon \to 0. \tag{8.16}$$

Choosing as test functions in (8.12) $u_i = \tilde{w}_{i,R,\varepsilon}$ we obtain the following estimate

$$\lambda_{n_0}^\varepsilon - \lambda_{n_0} \le \max_{\substack{a_1,\dots,a_{n_0} \in \mathbb{R} \\ \sum_{i=1}^{n_0} |a_i|^2 = 1}} \int_\Omega \left|\nabla\left(\sum_{i=1}^{n_0} a_i \tilde{w}_{i,R,\varepsilon}\right)\right|^2 dx - \lambda_{n_0} = \max_{\substack{a_1,\dots,a_{n_0} \in \mathbb{R} \\ \sum_{i=1}^{n_0} |a_i|^2 = 1}} \sum_{i,j=1}^{n_0} L_{i,j}^\varepsilon a_i a_j,$$

where

$$L_{i,j}^\varepsilon = \frac{\int_\Omega \nabla \hat{w}_{i,R,\varepsilon} \cdot \nabla \hat{w}_{j,R,\varepsilon} \, dx}{\|\hat{w}_{i,R,\varepsilon}\|_{L^2(\Omega)} \|\hat{w}_{j,R,\varepsilon}\|_{L^2(\Omega)}} - \lambda_{n_0} \delta_i^j.$$

From estimates (8.14), (8.15), and (8.16) it follows that

$$\begin{gathered}L_{n_0,n_0}^\varepsilon = \varepsilon^{N+2\gamma-2}(g_R(\varepsilon) + o(1)), \quad L_{i,n_0}^\varepsilon = L_{n_0,i}^\varepsilon = O(\varepsilon^{N+\gamma-1}) \quad \text{for all } i < n_0, \\ L_{i,i}^\varepsilon = \lambda_i - \lambda_{n_0} < 0 \quad \text{for all } i < n_0, \quad L_{i,j}^\varepsilon = 0 \quad \text{for all } i, j < n_0, \quad i \ne j,\end{gathered}$$

as $\varepsilon \to 0$. We observe that $g_R(\varepsilon) = O(1)$ as $\varepsilon \to 0$ by (8.6) and Lemma 8.3. Therefore estimate (8.9) follows from Lemma A.2. Finally the limit $\lim_{\varepsilon \to 0} g_R(\varepsilon) = g_R$ is a direct consequence of (8.10), Lemma 8.3, and (8.6). □

With the purpose of deducing from (8.9) a more precise estimate from above of the eigenvalue variation $\lambda_{n_0}^\varepsilon - \lambda_{n_0}$ and, in particular, of recognizing a sign in the right-hand side of (8.9), we are now going to compute the limit of the function g_R, as R diverges. To this aim we define

$$\chi_R(r) := \int_{S_1^+} U_R(r\theta)\Psi(\theta) \, dS, \quad \text{for } R > 2 \text{ and } 1 \le r \le R, \tag{8.17}$$

$$\chi(r) := \int_{S_1^+} U(r\theta)\Psi(\theta) \, dS, \quad \text{for } r \ge 1. \tag{8.18}$$

In addition, hereafter we denote

$$\pi_0 = \pi_0(n_0) := \int_{S_1^+} \Psi^2 \, dS. \tag{8.19}$$

We first establish the following preliminary result.

Lemma 8.5. *Let $m_{n_0}(\Sigma)$ be the constant defined in (3.4), let χ_R be as in (8.17), χ as in (8.18) and π_0 as in (8.19). Then*

$$\chi(1) = \lim_{R \to +\infty} \chi_R(1) = \pi_0 - \frac{2m_{n_0}(\Sigma)}{N + 2\gamma - 2}. \tag{8.20}$$

Proof. The fact that $\chi_R(1) \to \chi(1)$ as $R \to +\infty$ is a consequence of Lemma 3.5 and continuity of the trace map from $H^1(B_R^+)$ to $L^2(S_1^+)$. We now claim that

$$r^{-\gamma}\chi(r) \to \pi_0 \quad \text{as } r \to +\infty. \tag{8.21}$$

To prove (8.21), we first observe that

$$\chi(r) = \pi_0 r^\gamma + \int_{S_1^+} w_0(r\theta)\Psi(\theta)\,\mathrm{d}S, \tag{8.22}$$

since $U(x) = w_0(x) + |x|^\gamma\,\Psi(x/|x|)$ by (3.6). By considering the Kelvin transform of the restriction of w_0 on $\mathbb{R}^N_+ \setminus B_1^+$ and observing that it must vanish at 0 at least with vanishing order 1 (see [12]), we deduce that

$$|w_0(x)| = O(|x|^{-N+1}) \quad \text{as } |x| \to +\infty.$$

Combining the above estimate with (8.22) we obtain claim (8.21), being $\gamma \geq 1$.

In view of the definition of χ given in (8.18), the equation satisfied by U and the fact that Ψ is a spherical harmonic of degree γ, it's easy to prove that $\chi(r)$ solves the following differential equation

$$(r^{N+2\gamma-1}(r^{-\gamma}\chi(r))')' = 0 \quad \text{in } [1, +\infty).$$

Integration of the the above equation yields that

$$r^{-\gamma}\chi(r) = \chi(1) + C\frac{1 - r^{-N-2\gamma+2}}{N + 2\gamma - 2}, \quad \text{for all } r \geq 1 \tag{8.23}$$

and for some $C \in \mathbb{R}$. Taking into account (8.21), we obtain the exact value of the constant C, i.e.

$$C = (N + 2\gamma - 2)(\pi_0 - \chi(1)).$$

Then (8.23) can be rewritten as

$$\chi(r) = \pi_0 r^\gamma - (\pi_0 - \chi(1))r^{-N-\gamma+2}, \tag{8.24}$$

whose derivative is

$$\begin{aligned}\chi'(r) &= \pi_0\gamma r^{\gamma-1} - (N + \gamma - 2)(\chi(1) - \pi_0)r^{-N-\gamma+1} \\ &= (N + 2\gamma - 2)\pi_0 r^{\gamma-1} - \frac{N + \gamma - 2}{r}\chi(r), \quad \text{for all } r \geq 1.\end{aligned} \tag{8.25}$$

Then, by computing the derivative in (8.18) as well, and evaluating it at $r = 1$, we have that

$$\int_{S_1^+} \Psi\partial_{\boldsymbol{\nu}} U\,\mathrm{d}S = (N + 2\gamma - 2)\pi_0 - (N + \gamma - 2)\chi(1). \tag{8.26}$$

Thanks to the harmonicity of the function ψ, the definition of U given in (3.6) and (3.5), one can see that

$$\int_{B_1^+} \nabla\psi \cdot \nabla U\,\mathrm{d}x = \int_{S_1^+} U\partial_{\boldsymbol{\nu}}\psi + 2m_{n_0}(\Sigma).$$

On the other hand, being ψ a γ-homogeneous polynomial $\partial_{\boldsymbol{\nu}}\psi = \gamma\psi$ on S_1^+ and so

$$\int_{B_1^+} \nabla\psi \cdot \nabla U\,\mathrm{d}x = \gamma\chi(1) + 2m_{n_0}(\Sigma). \tag{8.27}$$

Moreover, by (3.7) and integration by parts we have that

$$\int_{B_1^+} \nabla\psi \cdot \nabla U\,\mathrm{d}x = \int_{S_1^+} \Psi\partial_{\boldsymbol{\nu}} U\,\mathrm{d}S.$$

Combining the identity above with (8.27) and (8.26) and rearranging the terms, we finally obtain (8.20) and complete the proof. □

As a byproduct of the proof of the previous Lemma, we obtain the following result, which is needed in the Section 10.

Corollary 8.6. *For all $R > 1$ there holds*

$$\int_{S_R^+} \psi \partial_{\boldsymbol{\nu}} U \, \mathrm{d}S = \pi_0 \gamma R^{N+2\gamma-2} + \frac{2(N+\gamma-2)}{N+2\gamma-2} m_{n_0}(\Sigma) \tag{8.28}$$

as well as

$$\chi(R) = \pi_0 R^\gamma - \frac{2m_{n_0}(\Sigma)}{N+2\gamma-2} R^{-N-\gamma+2}. \tag{8.29}$$

Proof. By definition (8.18) we have that

$$\int_{S_R^+} \psi \partial_{\boldsymbol{\nu}} U \, \mathrm{d}S = R^{N+\gamma-1} \chi'(R).$$

Plugging (8.25), (8.24) and (8.20) into the previous identity, one can deduce (8.28). On the other hand (8.29) can be easily proved by plugging (8.20) into (8.24). □

We are now able to compute the limit of g_R as R diverges.

Lemma 8.7. *Let g_R be as in (8.11) and $m_{n_0}(\Sigma)$ as in (3.4). Then $\lim_{R\to\infty} g_R = 2m_{n_0}(\Sigma)$.*

Proof. From (3.9), harmonicity of ψ and the fact that $\psi = 0$ on $\partial \mathbb{R}^N_+$ it follows that

$$g_R = \int_{S_R^+} \psi \partial_{\boldsymbol{\nu}} U_R \, \mathrm{d}S - \int_{S_R^+} \psi \partial_{\boldsymbol{\nu}} \psi \, \mathrm{d}S. \tag{8.30}$$

Let us compute the two terms at the right hand side of (8.30). If χ_R is the function defined in (8.17), then

$$\chi_R'(R) = \int_{S_1^+} \Psi(\theta) \partial_{\boldsymbol{\nu}} U_R(R\theta) \, \mathrm{d}S = R^{-N-\gamma+1} \int_{S_R^+} \psi \partial_{\boldsymbol{\nu}} U_R \, \mathrm{d}S. \tag{8.31}$$

On the other hand, one can easily prove that the function χ_R solves the following ODE

$$(r^{N+2\gamma-1}(r^{-\gamma}\chi_R(r))')' = 0 \quad \text{in } [1, R],$$

so that, by integration, there exists $C \in \mathbb{R}$ such that

$$r^{-\gamma}\chi_R(r) = \chi_R(1) + C\frac{1 - r^{-N-2\gamma+2}}{N+2\gamma-2} \quad \text{for all } r \in [1, R].$$

Since $U_R = \psi = R^\gamma \Psi$ on S_R^+, then, by (8.17) and (8.19), $\chi_R(R) = \pi_0 R^\gamma$. Therefore the constant C above is explicitly given by

$$C = \frac{(N+2\gamma-2)(\pi_0 - \chi_R(1))}{1 - R^{-N-2\gamma+2}}.$$

Hence, in view of (8.31), we can rewrite the first term in (8.30) as

$$\int_{S_R^+} \psi \partial_{\boldsymbol{\nu}} U_R \, \mathrm{d}S = R^{N+\gamma-1} \chi_R'(R) = \frac{\pi_0(N+\gamma-2) - \chi_R(1)(N+2\gamma-2) + \pi_0 \gamma R^{N+2\gamma-2}}{1 - R^{-N-2\gamma+2}}. \tag{8.32}$$

Concerning the second term in (8.30), from (8.19) and the fact that

$$\psi \partial_{\boldsymbol{\nu}} \psi = \gamma R^{2\gamma-1} \Psi^2 \quad \text{on } S_R^+,$$

we may easily deduce that

$$\int_{S_R^+} \psi \partial_{\boldsymbol{\nu}} \psi \, \mathrm{d}S = \pi_0 \gamma R^{N+2\gamma-2}.$$

Plugging the previous identity and (8.32) into (8.30) we obtain that

$$g_R = \frac{(\pi_0 - \chi_R(1))(N+2\gamma-2)}{1 - R^{-N-2\gamma+2}}.$$

In view of Lemma 8.5, passing to the limit as $R \to \infty$ in the previous identity, we draw the conclusion. □

Combining Proposition 8.4 and Lemma 8.7 we directly obtain the following lower bound for the eigenvalue variation $\lambda_{n_0} - \lambda_{n_0}^\varepsilon$.

Corollary 8.8. *We have that*

$$\liminf_{\varepsilon\to 0}\frac{\lambda_{n_0}-\lambda_{n_0}^\varepsilon}{\varepsilon^{N+2\gamma-2}}\geq -2m_{n_0}(\Sigma)>0.$$

Combining Proposition 7.3 and Corollary 8.8 we finally obtain the following result.

Corollary 8.9. *For any $R\geq K_\tau$ fixed we have that*

$$-2m_{n_0}(\Sigma)+o(1)\leq\frac{\lambda_{n_0}-\lambda_{n_0}^\varepsilon}{\varepsilon^{N+2\gamma-2}}\leq\frac{H(\varepsilon)}{\varepsilon^{2\gamma}}(f_R(\varepsilon)+o(1))\quad \text{as } \varepsilon\to 0,$$

where $f_R(\varepsilon)$ and $H(\varepsilon)$ are defined in (7.28) and (7.11) respectively. In particular

$$\frac{\varepsilon^{2\gamma}}{H(\varepsilon)}=O(1)\quad \text{as } \varepsilon\to 0. \tag{8.33}$$

9. Blow-up analysis

The analysis performed in the previous sections led, in Corollary 8.9, to an estimate of the eigenvalue variation in terms of the normalization factor $H(\varepsilon)$. In order to detect the sharp asymptotic behaviour of $H(\varepsilon)$ as $\varepsilon\to 0$, in the present section we perform a blow-up analysis for scaled eigenfunctions. The identification of the limit profile of blown-up eigenfunctions will be possible thanks to the energy estimate in Proposition 9.3 below, which is based on the invertibility of the Fréchet derivative of the operator T, defined as

$$\begin{aligned} T\colon H^1_0(\Omega)\times\mathbb{R}&\longrightarrow H^{-1}(\Omega)\times\mathbb{R} \\ (\varphi,\lambda)&\longmapsto T(\varphi,\lambda):=\left(-\Delta\varphi-\lambda\varphi,\int_\Omega|\nabla\varphi|^2\,\mathrm{d}x-\lambda_{n_0}\right),\end{aligned}\tag{9.1}$$

where

$${}_{H^{-1}(\Omega)}\langle-\Delta\varphi-\lambda\varphi,v\rangle_{H^1_0(\Omega)}:=\int_\Omega(\nabla\varphi\cdot\nabla v-\lambda\varphi v)\,\mathrm{d}x.$$

From the normalization (1.11) it easily follows that

$$T(\varphi_{n_0},\lambda_{n_0})=(0,0).$$

Additionally, as a consequence of the simplicity assumption (1.15) and the Fredholm Alternative, it is easy to prove the following invertibility result for the Fréchet derivative of T at $(\varphi_{n_0},\lambda_{n_0})$. One can see [1, Lemma 7.1] for the proof in a similar framework.

Lemma 9.1. *The functional T defined in (9.1) is Fréchet-differentiable at $(\varphi_{n_0},\lambda_{n_0})$ and its Fréchet derivative*

$$\begin{aligned}&\mathrm{d}T(\varphi_{n_0},\lambda_{n_0})\colon H^1_0(\Omega)\times\mathbb{R}\longrightarrow H^{-1}(\Omega)\times\mathbb{R},\\ &\mathrm{d}T(\varphi_{n_0},\lambda_{n_0})(\varphi,\lambda)=\left(-\Delta\varphi-\lambda\varphi_{n_0}-\lambda_{n_0}\varphi,\,2\int_\Omega\nabla\varphi_{n_0}\cdot\nabla\varphi\,\mathrm{d}x\right),\end{aligned}$$

is invertible.

The following Lemma states that the function $\xi_{n_0,R,\varepsilon}$, defined in (7.9), is a good approximation of the limit eigenfunction φ_{n_0} for small values of ε.

Lemma 9.2. *Let $R\geq K_\tau$ and let $\xi_{n_0,R,\varepsilon}$ be as in (7.9). Then*

$$\xi_{n_0,R,\varepsilon}\to\varphi_{n_0}\quad \text{in } H^1_0(\Omega),\quad \text{as } \varepsilon\to 0.$$

Proof. We first observe that, by definition,

$$\begin{aligned}\int_\Omega|\nabla(\xi_{n_0,R,\varepsilon}-\varphi_{n_0})|^2\,\mathrm{d}x&=\int_\Omega\left|\nabla(\varphi_{n_0}^\varepsilon-\varphi_{n_0})\right|^2\,\mathrm{d}x\\ &\quad-\int_{\Theta_{R\varepsilon}}\left|\nabla(\varphi_{n_0}^\varepsilon-\varphi_{n_0})\right|^2\,\mathrm{d}x+\int_{\Theta_{R\varepsilon}}\left|\nabla(\xi^{\mathrm{int}}_{n_0,R,\varepsilon}-\varphi_{n_0})\right|^2\,\mathrm{d}x.\end{aligned}\tag{9.2}$$

In view of Proposition 2.4, estimates (6.22), (7.6) and Lemma 8.1, we can estimate the right hand side of (9.2), thus obtaining

$$\int_\Omega |\nabla(\xi_{n_0,R,\varepsilon}-\varphi_{n_0})|^2\,dx \le o(1)+O(\varepsilon^{N-2+\beta(1-\tau)})+O(\varepsilon^{N+2\gamma-2})=o(1)\quad\text{as }\varepsilon\to 0.$$

The proof is thereby complete. □

We now state a crucial energy estimate that quantifies the rate of convergence in Lemma 9.2.

Proposition 9.3. *Let $R\ge K_\tau$. Then*

$$\int_\Omega |\nabla(\xi_{n_0,R,\varepsilon}-\varphi_{n_0})|^2\,dx = O(\varepsilon^{N-2}H(\varepsilon)) \tag{9.3}$$

and

$$\int_{\Omega\setminus\Theta_{R\varepsilon}} \left|\nabla(\varphi^\varepsilon_{n_0}-\varphi_{n_0})\right|^2 dx = O(\varepsilon^{N-2}H(\varepsilon)),\quad \text{as }\varepsilon\to 0. \tag{9.4}$$

Proof. Let T be as in (9.1). Being T differentiable at $(\varphi_{n_0},\lambda_{n_0})$, in view of Lemma 9.2 and Proposition 2.3 there holds

$$\begin{aligned} T(\xi_{n_0,R,\varepsilon},\lambda^\varepsilon_{n_0}) &= dT(\varphi_{n_0},\lambda_{n_0})(\xi_{n_0,R,\varepsilon}-\varphi_{n_0},\lambda^\varepsilon_{n_0}-\lambda_{n_0})\\ &\qquad + o(\|\xi_{n_0,R,\varepsilon}-\varphi_{n_0}\|_{H^1_0(\Omega)}+\left|\lambda^\varepsilon_{n_0}-\lambda_{n_0}\right|) \end{aligned}\tag{9.5}$$

as $\varepsilon\to 0$, where

$$\|v\|_{H^1_0(\Omega)} := \left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2}\quad \text{for all } v\in H^1_0(\Omega).$$

Applying $(dT(\varphi_{n_0},\lambda_{n_0}))^{-1}$ to both sides in (9.5) and taking the norms, we obtain that

$$\|\xi_{n_0,R,\varepsilon}-\varphi_{n_0}\|_{H^1_0(\Omega)}+\left|\lambda^\varepsilon_{n_0}-\lambda_{n_0}\right| \le \left\|dT(\varphi_{n_0},\lambda_{n_0})^{-1}\right\|\left\|T(\xi_{n_0,R,\varepsilon},\lambda^\varepsilon_{n_0})\right\|_{H^{-1}(\Omega)\times\mathbb{R}}(1+o(1)),$$

as $\varepsilon\to 0$, where the norm of $dT(\varphi_{n_0},\lambda_{n_0})^{-1}$ is intended in the space of linear bounded operators from $H^{-1}(\Omega)\times\mathbb{R}$ to $H^1_0(\Omega)\times\mathbb{R}$ and it is a constant independent of R and ε. Therefore

$$\begin{aligned} &\|\xi_{n_0,R,\varepsilon}-\varphi_{n_0}\|_{H^1_0(\Omega)}+\left|\lambda^\varepsilon_{n_0}-\lambda_{n_0}\right|\\ &\qquad \le C\left(\left\|-\Delta\xi_{n_0,R,\varepsilon}-\lambda^\varepsilon_{n_0}\xi_{n_0,R,\varepsilon}\right\|_{H^{-1}(\Omega)}+\left|\|\xi_{n_0,R,\varepsilon}\|^2_{H^1_0(\Omega)}-\lambda_{n_0}\right|\right)(1+o(1)), \end{aligned}\tag{9.6}$$

as $\varepsilon\to 0$. We first observe that the definition of $\xi_{n_0,R,\varepsilon}$ given in (7.9), (7.3), (6.14), and Proposition 7.3 imply that

$$\left|\|\xi_{n_0,R,\varepsilon}\|^2_{H^1_0(\Omega)}-\lambda_{n_0}\right| \le \left|\int_\Omega |\nabla\xi_{n_0,R,\varepsilon}|^2\,dx-\lambda^\varepsilon_{n_0}\right|+\left|\lambda^\varepsilon_{n_0}-\lambda_{n_0}\right| = O(\varepsilon^{N-2}H(\varepsilon)), \tag{9.7}$$

as $\varepsilon\to 0$. Let us now study the other term at the right hand side of (9.6). For any $v\in H^1_0(\Omega)$ we have that, by definition of $\xi_{n_0,R,\varepsilon}$ as in (7.9),

$$\begin{aligned} {}_{H^{-1}(\Omega)}\langle -\Delta\xi_{n_0,R,\varepsilon}-\lambda^\varepsilon_{n_0}\xi_{n_0,R,\varepsilon},v\rangle_{H^1_0(\Omega)} &= \int_\Omega(\nabla\xi_{n_0,R,\varepsilon}\cdot\nabla v-\lambda^\varepsilon_{n_0}\xi_{n_0,R,\varepsilon}v)\,dx\\ &= \int_{\Theta_{R\varepsilon}}(\nabla\xi^{\rm int}_{n_0,R,\varepsilon}\cdot\nabla v-\lambda^\varepsilon_{n_0}\xi^{\rm int}_{n_0,R,\varepsilon}v)\,dx-\int_{\Theta_{R\varepsilon}}(\nabla\varphi^\varepsilon_{n_0}\cdot\nabla v-\lambda^\varepsilon_{n_0}\varphi^\varepsilon_{n_0}v)\,dx. \end{aligned}$$

Now, thanks to the boundedness with respect to ε of $\{\lambda^\varepsilon_{n_0}\}$, Cauchy-Schwartz inequality and by virtue of estimates (6.14), (6.15), (7.3), (7.4) and Poincaré inequality, we have that

$$\begin{aligned} {}_{H^{-1}(\Omega)}\langle -\Delta\xi_{n_0,R,\varepsilon}-\lambda^\varepsilon_{n_0}\xi_{n_0,R,\varepsilon},v\rangle_{H^1_0(\Omega)} &= O\left(\varepsilon^{\frac{N-2}{2}}\sqrt{H(\varepsilon)}\right)\|v\|_{H^1_0(\Omega)}+O\left(\varepsilon^{\frac{N}{2}}\sqrt{H(\varepsilon)}\right)\|v\|_{L^2(\Omega)}\\ &= O\left(\varepsilon^{\frac{N-2}{2}}\sqrt{H(\varepsilon)}\right)\|v\|_{H^1_0(\Omega)}, \end{aligned}$$

as $\varepsilon\to 0$ and this readily implies that

$$\left\|-\Delta\xi_{n_0,R,\varepsilon}-\lambda^\varepsilon_{n_0}\xi_{n_0,R,\varepsilon}\right\|_{H^{-1}(\Omega)} = O\left(\varepsilon^{\frac{N-2}{2}}\sqrt{H(\varepsilon)}\right),\quad \text{as }\varepsilon\to 0.$$

Statement (9.3) follows by plugging the previous estimate and (9.7) into (9.6).

Since by (7.9) we have that

$$\int_\Omega |\nabla(\xi_{n_0,R,\varepsilon} - \varphi_{n_0})|^2 \, dx \geq \int_{\Omega\setminus\Theta_{R\varepsilon}} \left|\nabla(\varphi_{n_0}^\varepsilon - \varphi_{n_0})\right|^2 \, dx,$$

estimate (9.4) directly follows from (9.3). □

We are now ready to perform a blow-up analysis for scaled eigenfunctions.

Theorem 9.4 (Blow-up)**.** *Let U be as in (3.6) and Υ^ε be as in (7.10). Then, for all $R \geq K_\tau$,*

$$\Upsilon^\varepsilon \to \frac{1}{\sqrt{\Lambda_\tau}} U \quad \text{in } H^1(B_R^+) \quad \text{as } \varepsilon \to 0 \tag{9.8}$$

and

$$\frac{H(\varepsilon)}{\varepsilon^{2\gamma}} \to \Lambda_\tau \quad \text{as } \varepsilon \to 0, \tag{9.9}$$

where

$$\Lambda_\tau := \frac{1}{K_\tau^{N-1}} \int_{S_{K_\tau}^+} U^2 \, dS > 0. \tag{9.10}$$

In particular, for all $R \geq K_\tau$, we have that

$$\frac{u_{n_0}^\varepsilon(\varepsilon x)}{\varepsilon^\gamma} \to U(x) \quad \text{in } H^1(B_R^+) \quad \text{as } \varepsilon \to 0. \tag{9.11}$$

Proof. Let $\varepsilon_n \to 0$ as $n \to \infty$. Firstly, from (8.33) we deduce that there exists $c \in \mathbb{R}$ such that $c \geq 0$ and, up to a subsequence,

$$q(\varepsilon_n) := \frac{\varepsilon_n^\gamma}{\sqrt{H(\varepsilon_n)}} \to c \quad \text{as } n \to \infty. \tag{9.12}$$

Secondly, thanks to Proposition 6.8, we have that, for any $R \geq K_\tau$,

$$\int_{B_R^+} A(\varepsilon_n x)\nabla\Upsilon^{\varepsilon_n}(x)\cdot\nabla\Upsilon^{\varepsilon_n}(x)\, dx = O(1) \quad \text{and} \quad \int_{B_R^+} p(\varepsilon_n x)\,|\Upsilon^{\varepsilon_n}|^2 \, dx = O(1)$$

as $n \to \infty$. Therefore, by a diagonal process there exists $\tilde U \in H^1_{\rm loc}(\overline{\mathbb{R}^N_+})$ such that, up to a subsequence,

$$\Upsilon^{\varepsilon_n} \rightharpoonup \tilde U \quad \text{weakly in } H^1(B_R^+), \quad \Upsilon^{\varepsilon_n} \to \tilde U \quad \text{strongly in } L^2(B_R^+), \tag{9.13}$$

and

$$\Upsilon^{\varepsilon_n} \to \tilde U \quad \text{strongly in } L^2(S_R^+), \tag{9.14}$$

as $n \to \infty$ and for all $R \geq K_\tau$. Since, by definition,

$$\int_{S_{K_\tau}^+} \mu(\varepsilon_n x)\,|\Upsilon^{\varepsilon_n}|^2 \, dS = K_\tau^{N-1},$$

thanks to (5.3) and (9.14) we can pass to the limit and infer that

$$\int_{S_{K_\tau}^+} \tilde U^2 \, dS = K_\tau^{N-1}, \tag{9.15}$$

which implies that $\tilde U \not\equiv 0$ in $\mathbb{R}^N_+$. From the convergence, as $\varepsilon \to 0$, in the sense of Mosco of $\mathbb{R}^N \setminus \big(\partial\mathbb{R}^N_+ \setminus \big(\frac{1}{\varepsilon}\widetilde\Sigma_\varepsilon\big)\big)$ to the set $\mathbb{R}^N \setminus \big(\partial\mathbb{R}^N_+ \setminus \Sigma\big)$, observed in Remark 4.1, we derive that $\tilde U \in H^1_{0,B_R'\setminus\Sigma}(B_R^+)$ for all $R \geq K_\tau$. In addition, $\tilde U$ weakly solves

$$\begin{cases} -\Delta \tilde U = 0, & \text{in } \mathbb{R}^N_+, \\ \tilde U = 0, & \text{on } \partial\mathbb{R}^N_+ \setminus \Sigma, \\ \partial_\nu \tilde U = 0, & \text{on } \Sigma. \end{cases} \tag{9.16}$$

In particular

$$\int_{B_R^+} |\nabla \tilde{U}|^2 \, \mathrm{d}x = \int_{S_R^+} \tilde{U} \partial_{\boldsymbol{\nu}} \tilde{U} \, \mathrm{d}S, \quad \text{for all } R \geq K_\tau. \tag{9.17}$$

We now aim at proving that

$$\Upsilon^{\varepsilon_n} \to \tilde{U} \quad \text{strongly in } H^1(B_R^+), \quad \text{as } n \to \infty, \tag{9.18}$$

for all $R \geq K_\tau$. For every $R \geq K_\tau$, we have that, for n sufficiently large, Υ^{ε_n} weakly solves

$$\begin{cases} -\operatorname{div}(A(\varepsilon_n x)\nabla \Upsilon^{\varepsilon_n})(x) = \varepsilon_n^2 \lambda_{n_0}^{\varepsilon_n} p(\varepsilon_n x) \Upsilon^{\varepsilon_n}(x), & \text{in } B_R^+, \\ \Upsilon^{\varepsilon_n}(x) = 0, & \text{on } B_R' \setminus \frac{1}{\varepsilon_n}\widetilde{\Sigma}_{\varepsilon_n}, \\ A(\varepsilon_n x)\nabla \Upsilon^{\varepsilon_n}(x) \cdot \boldsymbol{\nu}(x) = 0, & \text{on } \frac{1}{\varepsilon_n}\widetilde{\Sigma}_{\varepsilon_n}, \\ \Upsilon^{\varepsilon_n}(x) = \dfrac{u_{n_0}^{\varepsilon_n}(\varepsilon_n x)}{\sqrt{H(\varepsilon_n)}}, & \text{on } S_R^+. \end{cases}$$

For $R \geq K_\tau$, if we consider the restriction of Υ^{ε_n} to $B_R^+ \setminus B_{R/2}^+$ and we oddly reflect it through the hyperplane $\{x_N = 0\}$, given the equation this function satisfies, from classical elliptic regularity theory (see e.g. [18, Theorem 2.3.3.2]) we know that $\{\Upsilon^{\varepsilon_n}\}_n$ is bounded in $H^2(B_R \setminus B_{R/2})$. Therefore, up to a subsequence (still denoted by ε_n), we have that

$$\partial_{\boldsymbol{\nu}} \Upsilon^{\varepsilon_n} \to \partial_{\boldsymbol{\nu}} \tilde{U} \quad \text{in } L^2(S_R^+), \quad \text{as } n \to \infty. \tag{9.19}$$

Furthermore, from the equation satisfied by Υ^{ε_n}, (5.1), (5.7), (9.14) and (9.19) we have, as $n \to \infty$,

$$\begin{aligned} \int_{B_R^+} |\nabla \Upsilon^{\varepsilon_n}|^2 \, \mathrm{d}x &= (1 + o(1)) \int_{B_R^+} A(\varepsilon_n x) \nabla \Upsilon^{\varepsilon_n}(x) \cdot \nabla \Upsilon^{\varepsilon_n}(x) \, \mathrm{d}x \\ &= (1 + o(1)) \left(O(1) \varepsilon_n^2 \lambda_{n_0}^{\varepsilon_n} \int_{B_R^+} |\Upsilon^{\varepsilon_n}|^2 \, \mathrm{d}x + \int_{S_R^+} \tilde{U} \partial_{\boldsymbol{\nu}} \tilde{U} \, \mathrm{d}S + o(1) \right). \end{aligned}$$

Therefore, thanks to (9.13) and (9.17), we conclude that

$$\int_{B_R^+} |\nabla \Upsilon^{\varepsilon_n}|^2 \, \mathrm{d}x \to \int_{B_R^+} |\nabla \tilde{U}|^2 \, \mathrm{d}x,$$

which, together with (9.13), proves (9.18).

Now let us fix $R \geq K_\tau$. From (9.4), we know that there exist $C_R > 0$ and $n_R \in \mathbb{N}$ such that

$$\int_{\Theta_{\tilde{R}\varepsilon} \setminus \Theta_{R\varepsilon}} \left| \nabla (\varphi_{n_0}^{\varepsilon_n} - \varphi_{n_0}) \right|^2 \, \mathrm{d}x \leq C_R \varepsilon_n^{N-2} H(\varepsilon_n),$$

for all $\tilde{R} > R$ and $n > n_R$. In fact, up to a change of variable, this is equivalent to

$$\int_{B_{\tilde{R}}^+ \setminus B_R^+} A(\varepsilon_n x) \nabla \left(\Upsilon^{\varepsilon_n} - q(\varepsilon_n) V_{\varepsilon_n} \right)(x) \cdot \nabla \left(\Upsilon^{\varepsilon_n} - q(\varepsilon_n) V_{\varepsilon_n} \right)(x) \, \mathrm{d}x \leq C_R$$

for all $\tilde{R} > R$ and $n > n_R$, where $q(\varepsilon_n)$ is defined in (9.12) and V_{ε_n} in (8.5). Passing to the limit as $n \to \infty$ in the above estimate and taking into account (9.12), (9.18), (5.1) and (8.6), we obtain that

$$\int_{B_{\tilde{R}}^+ \setminus B_R^+} |\nabla \tilde{U} - c \nabla \psi|^2 \, \mathrm{d}x \leq C_R$$

for all $\tilde{R} > R$ and this readily implies that

$$\int_{\mathbb{R}^N_+} |\nabla \tilde{U} - c \nabla \psi|^2 \, \mathrm{d}x < \infty. \tag{9.20}$$

We now claim that $c > 0$. Indeed, if this were not the case, then $\int_{\mathbb{R}^N_+} |\nabla \tilde{U}|^2 \, \mathrm{d}x < \infty$. Then, since $U = 0$ on $\partial \mathbb{R}^N_+ \setminus \Sigma$, in view of [15, Lemma 2.3] we would have $\tilde{U} \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Sigma)$; since $\tilde{U}$ weakly solves (9.16), this would imply that $\tilde{U} \equiv 0$, thus raising a contradiction.

From (9.20) and [15, Lemma 2.3] it follows that $c^{-1}\tilde{U}-\psi \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Sigma)$; hence, by uniqueness of the limit profile constructed in Lemma 3.1, we conclude that $c^{-1}\tilde{U}-\psi = w_0$. Hence, by the definition of U in (3.6), we have that

$$\tilde{U} = cU.$$

Moreover, in view of (9.15), we conclude that

$$c = \frac{1}{\sqrt{\Lambda_\tau}},$$

with Λ_τ as in (9.10). Since the limit of Υ^{ε_n} is independent of the choice of the sequence $\{\varepsilon_n\}_n$ and of the extracted subsequence, by Uryshon Subsequence Principle we may conclude that the convergence holds as $\varepsilon \to 0$. Finally, (9.11) is a direct consequence of (9.8) and (9.9). □

As a consequence of the Blow-up Theorem 9.4, we are able to prove the strong convergence as $\varepsilon \to 0$ of the family $\{Z_R^\varepsilon\}_\varepsilon$ defined in (7.10).

Corollary 9.5. *For any $R > K_\tau$, there holds*

$$Z_R^\varepsilon \to \frac{1}{\sqrt{\Lambda_\tau}} Z_R \quad \text{in } H^1(B_R^+) \quad \text{as } \varepsilon \to 0,$$

where Z_R is defined in Lemma 3.4.

Proof. Let us fix $R > K_\tau$. We observe that Z_R^ε weakly solves

$$\begin{cases} -\operatorname{div}(A(\varepsilon x)\nabla Z_R^\varepsilon) = 0, & \text{in } B_R^+, \\ Z_R^\varepsilon = \Upsilon^\varepsilon, & \text{on } S_R^+, \\ Z_R^\varepsilon = 0, & \text{on } B_R', \end{cases}$$

hence the function

$$W_R^\varepsilon := Z_R^\varepsilon - \Lambda_\tau^{-1/2} Z_R - \eta_R(\Upsilon^\varepsilon - \Lambda_\tau^{-1/2} U),$$

with η_R being as in (3.1), weakly solves

$$\begin{cases} -\operatorname{div}(A(\varepsilon x)\nabla W_R^\varepsilon) = \operatorname{div}\left(\dfrac{A(\varepsilon x) - I_N}{\sqrt{\Lambda_\tau}}\nabla Z_R + A(\varepsilon x)\nabla\Big(\eta_R\Big(\Upsilon^\varepsilon - \tfrac{U}{\sqrt{\Lambda_\tau}}\Big)\Big)\right), & \text{in } B_R^+, \\ W_R^\varepsilon = 0, & \text{on } \partial B_R^+, \end{cases}$$

i.e.

$$\begin{aligned} \int_{B_R^+} A(\varepsilon x)\nabla W_R^\varepsilon \cdot \nabla\phi \, dx = &-\frac{1}{\sqrt{\Lambda_\tau}}\int_{B_R^+} (A(\varepsilon x) - I_N)\nabla Z_R \cdot \nabla \phi \, dx \\ &- \int_{B_R^+} A(\varepsilon x)\nabla\Big(\eta_R\Big(\Upsilon^\varepsilon - \frac{U}{\sqrt{\Lambda_\tau}}\Big)\Big)\cdot\nabla\phi\, dx \quad \text{for every } \phi \in H_0^1(B_R^+). \end{aligned}$$

Testing the above equation with $\phi = W_R^\varepsilon$ and using (5.1), we then obtain that

$$\int_{B_R^+} A(\varepsilon x)\nabla W_R^\varepsilon \cdot \nabla W_R^\varepsilon \, dx \le \text{const}\, \|\nabla W_R^\varepsilon\|_{L^2(B_R^+)}\left(\varepsilon + \left\|\eta_R\Big(\Upsilon^\varepsilon - \frac{U}{\sqrt{\Lambda_\tau}}\Big)\right\|_{H^1(B_R^+)}\right),$$

which implies that

$$W_R^\varepsilon \to 0 \quad \text{in } H_0^1(B_R^+) \quad \text{as } \varepsilon \to 0, \tag{9.21}$$

thanks to (5.1) and Theorem 9.4. Since $Z_R^\varepsilon - \Lambda_\tau^{-1/2}Z_R = W_R^\varepsilon + \eta_R(\Upsilon^\varepsilon - \Lambda_\tau^{-1/2}U)$, from (9.21) and Theorem 9.4 we deduce that $Z_R^\varepsilon - \Lambda_\tau^{-1/2}Z_R \to 0$ in $H^1(B_R^+)$. □

We conclude this section with the proof of Theorem 1.4.

Proof of Theorem 1.4. It can be easily derived from the change of variable $x = \Phi(y)$, (9.11) and Dominated Convergence Theorem. □

10. Proof of Theorem 1.2

From Theorem 9.4 and Corollary 9.5 it follows that, letting $f_R(\varepsilon)$ be as in (7.28),

$$\lim_{\varepsilon\to 0} f_R(\varepsilon) = \frac{1}{\Lambda_\tau}\left(\int_{B_R^+} |\nabla Z_R|^2 \,\mathrm{d}x - \int_{B_R^+} |\nabla U|^2 \,\mathrm{d}x\right) \tag{10.1}$$

for all $R > K_\tau$, in view also of (5.1).

Combining (10.1) with (9.9) and Corollary 8.9, at this point we know that

$$-2m_{n_0}(\Sigma) \le \liminf_{\varepsilon\to 0} \frac{\lambda_{n_0} - \lambda_{n_0}^\varepsilon}{\varepsilon^{N+2\gamma-2}} \le \limsup_{\varepsilon\to 0} \frac{\lambda_{n_0} - \lambda_{n_0}^\varepsilon}{\varepsilon^{N+2\gamma-2}} \le \int_{B_R^+} |\nabla Z_R|^2 \,\mathrm{d}x - \int_{B_R^+} |\nabla U|^2 \,\mathrm{d}x,$$

or all $R > K_\tau$. Therefore the proof of Theorem 1.2 amounts to the proof of the following Lemma, which the rest of the Section is devoted to.

Lemma 10.1. *There holds*

$$\lim_{R\to+\infty}\left(\int_{B_R^+} |\nabla Z_R|^2 \,\mathrm{d}x - \int_{B_R^+} |\nabla U|^2 \,\mathrm{d}x\right) = -2m_{n_0}(\Sigma). \tag{10.2}$$

A first step in this direction is given by the following lemma.

Lemma 10.2. *There holds*

$$\lim_{R\to+\infty}\left(\int_{B_R^+} |\nabla Z_R|^2 \,\mathrm{d}x - \int_{B_R^+} |\nabla U|^2 \,\mathrm{d}x - \int_{S_R^+} \psi \partial_{\boldsymbol{\nu}}(Z_R - U)\,\mathrm{d}S\right) = 0.$$

Proof. Integration by parts and equations (3.7) and (3.11) imply that

$$\begin{aligned}\int_{B_R^+} |\nabla Z_R|^2 \,\mathrm{d}x &- \int_{B_R^+} |\nabla U|^2 \,\mathrm{d}x - \int_{S_R^+} \psi \partial_{\boldsymbol{\nu}}(Z_R - U)\,\mathrm{d}S \\ &= \int_{S_R^+} (U-\psi)\partial_{\boldsymbol{\nu}}(\psi - U)\,\mathrm{d}S + \int_{S_R^+} (U-\psi)\partial_{\boldsymbol{\nu}}(Z_R - \psi)\,\mathrm{d}S.\end{aligned}$$

Therefore the conclusion follows if we prove that

$$\lim_{R\to+\infty} \int_{S_R^+} (U-\psi)\partial_{\boldsymbol{\nu}}(U-\psi)\,\mathrm{d}S = 0, \tag{10.3}$$

$$\lim_{R\to+\infty} \int_{S_R^+} (U-\psi)\partial_{\boldsymbol{\nu}}(Z_R-\psi)\,\mathrm{d}S = 0. \tag{10.4}$$

First, we observe that integration by parts and the fact that $U - \psi \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Sigma)$ is harmonic in $\mathbb{R}^N_+$ imply that

$$\int_{S_R^+} (U-\psi)\partial_{\boldsymbol{\nu}}(U-\psi)\,\mathrm{d}S = \int_{\mathbb{R}^N_+\setminus B_R^+} |\nabla(U-\psi)|^2 \,\mathrm{d}x.$$

Since $U - \psi \in \mathcal{D}^{1,2}(\mathbb{R}^N_+ \cup \Sigma)$, the right hand side vanishes as $R \to +\infty$, thus implying (10.3).

In order to prove (10.4), we let $R > 2$ and consider the equation satisfied by $Z_R - \psi \in H^1(B_R^+)$ in B_R^+, i.e.

$$\begin{cases} -\Delta(Z_R-\psi) = 0, & \text{in } B_R^+, \\ Z_R - \psi = 0, & \text{on } B_R', \\ Z_R - \psi = U - \psi, & \text{on } S_R^+. \end{cases}$$

If we multiply both sides of the above equation by $\eta_R(U-\psi)$, where η_R is as in (3.1), and integrate by parts, we obtain that

$$\int_{S_R^+} (U-\psi)\partial_{\boldsymbol{\nu}}(Z_R-\psi)\,\mathrm{d}S = \int_{B_R^+} \nabla(Z_R-\psi)\cdot\nabla(\eta_R(U-\psi))\,\mathrm{d}x.$$

Therefore, from the Cauchy-Schwartz inequality and the Dirichlet principle it follows that

$$\left|\int_{S_R^+}(U-\psi)\partial_{\boldsymbol{\nu}}(Z_R-\psi)\,\mathrm{d}S\right| \le \int_{B_R^+}|\nabla(\eta_R(U-\psi))|^2\,\mathrm{d}x. \tag{10.5}$$

Thanks to (3.1), we have that

$$\int_{B_R^+}|\nabla(\eta_R(U-\psi))|^2\,\mathrm{d}x \le 32\left(\int_{B_R^+\setminus B_{R/2}^+}\frac{|U-\psi|^2}{|x|^2}\,\mathrm{d}x+\int_{B_R^+\setminus B_{R/2}^+}|\nabla(U-\psi)|^2\,\mathrm{d}x\right). \tag{10.6}$$

Now, since $U-\psi\in\mathcal{D}^{1,2}(\mathbb{R}^N_+\cup\Sigma)$ and since the Hardy inequality holds in this space, we have that

$$\int_{B_R^+\setminus B_{R/2}^+}\frac{|U-\psi|^2}{|x|^2}\,\mathrm{d}x+\int_{B_R^+\setminus B_{R/2}^+}|\nabla(U-\psi)|^2\,\mathrm{d}x\to 0 \quad \text{as } R\to+\infty.$$

Combining this fact with (10.6) and (10.5) we obtain (10.4), thus concluding the proof. □

We are now ready to prove Lemma 10.1.

Proof of Lemma 10.1. By virtue of Lemma 10.2, to prove (10.2) it is enough to show that

$$\lim_{R\to+\infty}\int_{S_R^+}\psi\partial_{\boldsymbol{\nu}}(Z_R-U)\,\mathrm{d}S=-2m_{n_0}(\Sigma). \tag{10.7}$$

For $R>2$ we let

$$\Gamma_R(r):=\int_{S_1^+}Z_R(r\theta)\Psi(\theta)\,\mathrm{d}S \quad \text{for any } 0<r\le R.$$

From (3.11) and the fact that Ψ is a spherical harmonic of degree γ it easily follows that

$$(r^{N+2\gamma-1}(r^{-\gamma}\Gamma_R(r))')'=0 \quad \text{in } (0,R).$$

Integrating this ODE in (r,R) we obtain that

$$r^{-\gamma}\Gamma_R(r)=R^{-\gamma}\Gamma_R(R)+\frac{C}{N+2\gamma-2}\left[R^{-N-2\gamma+2}-r^{-N-2\gamma+2}\right]$$

for some constant $C\in\mathbb{R}$ and for all $r\in(0,R)$. Multiplying both sides by $r^{N+2\gamma-2}$ leads to

$$r^{N+\gamma-2}\Gamma_R(r)=R^{-\gamma}r^{N+2\gamma-2}\Gamma_R(R)+\frac{C}{N+2\gamma-2}\left[R^{-N-2\gamma+2}r^{N+2\gamma-2}-1\right].$$

Tanking into account that, by regularity of Z_R, $\lim_{r\to0}\Gamma_R(r)$ is finite, thanks to the previous identity, we may conclude that $C=0$, thus implying that

$$\Gamma_R(r)=\left(\frac{r}{R}\right)^\gamma\Gamma_R(R).$$

Moreover, since $Z_R=U$ on S_R^+, we have that $\Gamma_R(R)=\chi(R)$ and then

$$\Gamma_R(r)=\left(\frac{r}{R}\right)^\gamma\chi(R). \tag{10.8}$$

By definition of Γ_R, we know that

$$\int_{S_R^+}\psi\partial_{\boldsymbol{\nu}}Z_R\,\mathrm{d}S=R^{N+\gamma-1}\Gamma_R'(R)$$

which, in view of (10.8), becomes

$$\int_{S_R^+}\psi\partial_{\boldsymbol{\nu}}Z_R\,\mathrm{d}S=\gamma R^{N+\gamma-2}\chi(R).$$

Then, taking into account (8.29), we have that

$$\int_{S_R^+}\psi\partial_{\boldsymbol{\nu}}Z_R\,\mathrm{d}S=\pi_0\gamma R^{N+2\gamma-2}-\frac{2\gamma m_{n_0}(\Sigma)}{N+2\gamma-2}.$$

Combining this identity with (8.28) yields

$$\int_{S_R^+} \psi \partial_{\boldsymbol{\nu}}(Z_R - U)\,\mathrm{d}S = -\frac{2\gamma m_{n_0}(\Sigma)}{N+2\gamma-2} - \frac{2(N+\gamma-2)}{N+2\gamma-2} m_{n_0}(\Sigma) = -2m_{n_0}(\Sigma),$$

which implies (10.7). The proof is thereby complete. □

11. Proof of Theorem 1.1

In this section, we drop assumptions (1.22)–(1.23) on the set $\mathcal{V}$ and prove Theorem 1.1 under the sole assumption (1.3) on $\mathcal{V}$. Let $0 < r_{\mathcal{V}} < R_{\mathcal{V}} < r_0$ be such that $B_{r_{\mathcal{V}}} \subset \mathcal{V} \subset B_{R_{\mathcal{V}}}$ (such $r_{\mathcal{V}}, R_{\mathcal{V}}$ exist because $\mathcal{V}$ is an open bounded set containing 0). For every $\omega \subset \mathbb{R}^N$ bounded open set, we denote as $\lambda_{n_0}^{\varepsilon}(\omega)$ the n_0-th eigenvalue of problem (1.9) with Σ_ε given by $(\varepsilon\omega) \cap \partial\Omega$ (i.e. with $\mathcal{V}$ replaced by ω). Then, from (2.1) and the fact that $\varepsilon B_{r_{\mathcal{V}}} \subset \varepsilon\mathcal{V} \subset \varepsilon B_{R_{\mathcal{V}}}$ it follows that

$$\lambda_{n_0}^{\varepsilon}(B_{R_{\mathcal{V}}}) \leq \lambda_{n_0}^{\varepsilon}(\mathcal{V}) \leq \lambda_{n_0}^{\varepsilon}(B_{r_{\mathcal{V}}}). \tag{11.1}$$

Since $B_{r_{\mathcal{V}}}$ and $B_{R_{\mathcal{V}}}$ satisfy assumptions (1.22) and (1.23), Theorem 1.2 and Lemma 3.2 yield the following asymptotic expansions for $\lambda_{n_0}^{\varepsilon}(B_{R_{\mathcal{V}}}), \lambda_{n_0}^{\varepsilon}(B_{r_{\mathcal{V}}})$:

$$\lambda_{n_0}^{\varepsilon}(B_{R_{\mathcal{V}}}) = \lambda_{n_0} - R_{\mathcal{V}}^{N+2\gamma-2} \mathcal{C}_{n_0} \varepsilon^{N+2\gamma-2} + o(\varepsilon^{N+2\gamma-2}),$$
$$\lambda_{n_0}^{\varepsilon}(B_{r_{\mathcal{V}}}) = \lambda_{n_0} - r_{\mathcal{V}}^{N+2\gamma-2} \mathcal{C}_{n_0} \varepsilon^{N+2\gamma-2} + o(\varepsilon^{N+2\gamma-2}),$$

as $\varepsilon \to 0$, so that, in view of (11.1),

$$\begin{aligned} r_{\mathcal{V}}^{N+2\gamma-2}\mathcal{C}_{n_0} + o(1) = \frac{\lambda_{n_0} - \lambda_{n_0}^{\varepsilon}(B_{r_{\mathcal{V}}})}{\varepsilon^{N+2\gamma-2}} &\leq \frac{\lambda_{n_0} - \lambda_{n_0}^{\varepsilon}(\mathcal{V})}{\varepsilon^{N+2\gamma-2}} \\ &\leq \frac{\lambda_{n_0} - \lambda_{n_0}^{\varepsilon}(B_{R_{\mathcal{V}}})}{\varepsilon^{N+2\gamma-2}} = R_{\mathcal{V}}^{N+2\gamma-2}\mathcal{C}_{n_0} + o(1) \end{aligned}$$

as $\varepsilon \to 0$. The above chain of inequalities directly proves Theorem 1.1.

Appendix A.

We recall from [14, Lemma 4.1] the following Poincaré-type inequality on balls and half-balls.

Lemma A.1. *Let $r > 0$. Then*

$$\frac{N-1}{r^2} \int_{B_r^+} u^2\,\mathrm{d}x \leq \int_{B_r^+} |\nabla u|^2\,\mathrm{d}x + \frac{1}{r}\int_{S_r^+} u^2\,\mathrm{d}S \quad \text{for every } u \in H^1(B_r^+),$$

and

$$\frac{N-1}{r^2} \int_{B_r} u^2\,\mathrm{d}x \leq \int_{B_r} |\nabla u|^2\,\mathrm{d}x + \frac{1}{r}\int_{\partial B_r} u^2\,\mathrm{d}S \quad \text{for every } u \in H^1(B_r).$$

From [1] we recall the following result, regarding the maximum of quadratic forms with coefficients depending on a parameter (see also [14]).

Lemma A.2. *For every $\varepsilon > 0$ let us consider a quadratic form*

$$Q_\varepsilon \colon \mathbb{R}^{n_0} \longrightarrow \mathbb{R},$$
$$Q_\varepsilon(z_1, \dots, z_{n_0}) = \sum_{i,j=1}^{n_0} M_{i,j}(\varepsilon) z_i z_j,$$

with real coefficients $M_{i,j}(\varepsilon)$ such that $M_{i,j}(\varepsilon) = M_{j,i}(\varepsilon)$. Let us assume that there exist $a > 0$, $\varepsilon \mapsto \sigma(\varepsilon) \in \mathbb{R}$ with $\sigma(\varepsilon) \geq 0$ and $\sigma(\varepsilon) = O(\varepsilon^{2a})$ as $\varepsilon \to 0$, and $\varepsilon \mapsto \mu(\varepsilon) \in \mathbb{R}$ with $\mu(\varepsilon) = O(1)$ as

$\varepsilon \to 0$, such that the coefficients $M_{i,j}(\varepsilon)$ satisfy the following conditions:

$$\begin{aligned}
&M_{n_0,n_0}(\varepsilon) = \sigma(\varepsilon)\mu(\varepsilon),\\
&\textit{for all } i < n_0 \ M_{i,i}(\varepsilon) \to M_i < 0, \ \textit{as } \varepsilon \to 0,\\
&\textit{for all } i < n_0 \ M_{i,n_0}(\varepsilon) = O(\varepsilon^a \sqrt{\sigma(\varepsilon)}) \ \textit{as } \varepsilon \to 0,\\
&\textit{for all } i,j < n_0 \textit{ with } i \neq j \ M_{i,j} = O(\varepsilon^{2a}) \ \textit{as } \varepsilon \to 0,\\
&\textit{there exists } M \in \mathbb{N} \textit{ such that } \varepsilon^{(2+M)a} = o(\sigma(\varepsilon)) \ \textit{as } \varepsilon \to 0.
\end{aligned}$$

Then

$$\max_{\substack{z \in \mathbb{R}^n_0 \\ \|z\|=1}} Q_\varepsilon(z) = \sigma(\varepsilon)(\mu(\varepsilon) + o(1)) \quad \textit{as } \varepsilon \to 0,$$

where $\|z\| = \|(z_1, \dots, z_{n_0})\| = \left(\sum_{i=1}^{n_0} z_i^2\right)^{1/2}$.

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