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Riesz capacities of a set due to Dobiński

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Abstract. We study the Riesz (a, p)-capacity of the so called Dobiński set. We characterize the values of the parameters a and p for which the (a, p)-Riesz capacity of the Dobiński set is positive. In particular we show that the Dobiński set has positive logarithmic capacity, thus answering a question of Dayan, Fernandéz and González. We approach the problem by considering the dyadic analogues of the Riesz (a, p)-capacities which seem to be better adapted to the problem.

Keywords. Riesz capacity, Logarithmic capacity, Dobiński set, Dyadic capacity, Non-linear capacity, Diophantine approxmation.

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1. Introduction and main results

In a series of two papers [11, 12] Dobiński claims that the following identity is true

$$\prod_{n\geq 0} (\tan 2^n \pi x)^{2^{-n}} = (2\sin \pi x)^2,$$

for all real numbers $x \in [0,1]$ which are not dyadic rationals. As it has been already noted in [2] and explained in detail in a recent paper [10] the situation is not quite so simple. In fact, if we consider the same identity with absolute values,

$$\prod_{n\geq 0} |\tan 2^n \pi x|^{2^{-n}} = (2\sin \pi x)^2,$$

so as to avoid issues of defining the powers of negative numbers, in [2] the authors prove that the identity holds if and only if x does not belong to the so called *Dobiński set* $\mathscr{D}$. To define $\mathscr{D}$ let $x \in [0,1]$ be a real number with dyadic expansion $x = (0.a_1 a_2 \dots)_2$ and for $n \geq 1$ let $s_n(x) = \max\{r \in \mathbb{N} : a_n = a_{n+1} = \cdots = a_{n+r}\}$ [1]. Then,

$$\mathscr{D} := \left\{ x \in [0,1] : \limsup_{n\to\infty} \frac{s_n(x)}{2^n} > 0 \right\}.$$

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[1] Let $s_n(x) = \infty$ if x is a dyadic rational

So the Dobiński set comprises real numbers which can be approximated "exceedingly well" by dyadic rationals on every scale. Related problems of diophantine approximation by dyadic rationals have been considered in [14].

In a recent work [10] Dayan, Fernandéz and González prove, among other results, that $\mathscr{D}$ has Hausdorff dimension 0 and *logarithmic* Hausdorff dimension 1. Their techniques are primarily based on the mass transference principle of Beresnevich and Velani [6] which allows one to transfer measure theoretic statements for $\limsup$ subsets of $\mathbb{R}^n$ to statements about Hausdorff measure. The Hausdorff dimension is a precise way to talk about the size of a subset of $\mathbb{R}^n$. Another way to measure the size of subsets of $\mathbb{R}^n$ is by some kind of *capacity*. From now on it will be convenient to consider $\mathscr{D}$ as a subset of the unit circle $\mathbb{T}$ in $\mathbb{R}^2$ via the usual correspondence $x \mapsto e^{2\pi i x}$. Let $0 < a < 1$ and f a positive measurable function on $\mathbb{T}$ the *a-Riesz potential of f* is defined as

$$\mathscr{I}_a f(x) := \int_{\mathbb{T}} \frac{f(y)\mathrm{d}y}{|x-y|^{1-a}}, \quad x \in \mathbb{T},$$

where $\mathrm{d}y$ is the normalized Lebesgue measure on $\mathbb{T}$. Finally let $1 < p < \infty$ and a as before. The *(a,p)-Riesz capacity* of a Borel subset E of $\mathbb{T}$ is defined as

$$R_{a,p}(E) := \inf\left\{\int_{\mathbb{T}} f(y)^p \mathrm{d}y : f \geq 0 \text{ and } \mathscr{I}_a f \geq 1 \text{ on } E\right\}.$$

We shall refer to the capacities $R_{a,p}$ as *linear* if $p = 2$ and as *non-linear* if $p \neq 2$. We shall also assume that $ap \leq 1$, otherwise singletons have positive capacity. There exists a remarkable relation between Hausdorff dimension, which we will denote by dim, and linear Riesz capacities for a Borel set $E \subseteq \mathbb{T}$ established by Frostman [1, Corollary 5.1.14]

$$\dim E = \sup\{1 - ap : R_{a,p}(E) > 0\}. \tag{1}$$

This fact, together with the standard comparison results between capacities [1, Section 5.5] and the fact that the Dobiński set has vanishing Hausdorff dimension implies that $R_{a,p}(\mathscr{D}) = 0$ when $ap < 1$. In the same work where the Hausdorff dimension of $\mathscr{D}$ was studied the authors ask whether also $R_{\frac{1}{2},2}(\mathscr{D}) = 0$ or not [10, Section 5]. In fact they formulate the question in terms of *logarithmic capacity* in the complex plane but it is well known that for subsets of $\mathbb{T}$, logarithmic capacity is bounded below and above by the Riesz $(\frac{1}{2},2)$-capacity [7, Lemma 2.2] and [8, Corollary 2.6].

We have been able to answer the above question for all Riesz $(\frac{1}{p},p)$-capacities.

Theorem 1. *Let $\mathscr{D}$ be the Dobiński set and $p > 1$. Then,*

$$R_{\frac{1}{p},p}(\mathscr{D}) = \begin{cases} R_{\frac{1}{p},p}(\mathbb{T}) & \textit{if } 1 < p \leq 2 \\ 0 & \textit{if } p > 2. \end{cases}$$

Somewhat surprisingly the capacity of $\mathscr{D}$ exhibits a jump from full to 0 at the critical value $p = 2$. It should be mentioned that this statement implies, via (1), that the Hausdorff dimension of $\mathscr{D}$ is 0 and that the logarithmic Hausdorff dimension is 1 by [1, Corollary 5.1.14]. The proof of the above theorem is presented in Section 3 and it applies to a more general class of Dobiński type sets and all (a,p) Riesz capacities (see Theorem 5). The proof rests on two ideas. One is the use of a discrete/dyadic version of the Riesz capacity. Discrete type capacities have appeared in potential theory in the past (see for example [5, 15]) and their "combinatorial" nature suites very well the dyadic structure of $\mathscr{D}$. In concrete terms one can show using a *recursive formula* (Lemma 3) that $\mathscr{D}$ has positive discrete capacity and this, through a comparison theorem for discrete and Riesz capacities, [4, Theorem 1] allows one to deduce that $\mathscr{D}$ has positive Riesz $(\frac{1}{p},p)$-capacity for $1 < p \leq 2$ and zero capacity when $p > 2$. Finally we prove a "Kolmogorov $0-1$" type lemma (Lemma 4) from which we can deduce that in fact $\mathscr{D}$ is of *full* capacity, i.e. $R_{\frac{1}{p},p}(\mathscr{D}) = R_{\frac{1}{p},p}(\mathbb{T})$

when $1 < p \le 2$. The general type Dobiński sets exhibit a similar behavior also with respect to their Hausdorff h-measure (associated to sufficiently regular gauge function h). This in fact is the main result of Dayan et al. [10, Theorem 1].

It is worth also noticing that the same transition phenomenon appears in the study of logarithmic capacity of uniform G_δ-sets [13, Theorem 1.2].

2. Trees, dyadic capacity and the recursion formula

Let $T := \{0,1\}^*$ the free monoid generated by the language $\{0,1\}$ with neutral element e. In this context we shall call T the *dyadic tree*. The length of a word x is denoted $|x|$. For two words $x, y \in T$ we denote the largest common prefix of x and y by $x \wedge y$. If $x \wedge y = x$ we write $x \le y$. Finally we use the notation $x_- := x0$, $x_+ := x1$. The (Poisson) boundary ∂T of T can be identified with the metric space $\{0,1\}^{\mathbb{N}}$ equipped with the metric

$$d(x, y) := 2^{-|x\wedge y|}.$$

We will write $\overline{T} := \partial T \cup T$. There exists a natural mapping from ∂T to $[0,1]$,

$$\Lambda : \partial T \mapsto [0,1], \quad \Lambda(w_1 w_2 \cdots) := (0.w_1 w_2 \cdots)_2 \tag{2}$$

which is onto and Lipschitz continuous. Moreover, every $x \in [0,1]$ which is not a dyadic rational has a unique pre-image. Dyadic rationals have two pre-images under Λ.

Our next goal is to develop a potential theory on ∂T which parallels the one we have already seen in $\mathbb{T}$. A more detailed exposition of the potential theory on the boundary of the tree can be found in [3, 4, 9]. Here we shall present only the elements that are essential for our problem.

Let φ a non negative function defined on T. The potential of φ is given by

$$I\varphi(x) := \sum_{y\le x} \varphi(y), \quad x \in \overline{T}.$$

Let π be a positive weight function defined on T. Then for a set $E \subseteq \partial T$ we define its π (discrete) capacity as follows

$$\operatorname{cap}_\pi(E) := \inf\left\{\sum_{x\in T} \varphi(x)^p \pi(x) : \varphi \ge 0 \text{ and } I\varphi \ge 1 \text{ on } E\right\}.$$

When $\pi(x) = 2^{-|x|(1-ap)}$ we shall refer to the capacity $\operatorname{cap}_\pi =: \operatorname{cap}_{a,p}$ as discrete (a, p) capacity. The relation between the Riesz and discrete capacities can be made explicit. This has been first noted in [5] for $a = 1/2, p = 2$ and generalized further in [4]. In particular, [4, Theorem 1] specialized to the case where the metric space is the interval $[0,1]$ with the Euclidean metric and as "tree decomposition" we consider the set of dyadic intervals in $[0,1]$ we get the following;

Theorem 2. *Let $p > 1, 0 < a \le 1/p$. There exists a constant $c = c(a, p) > 0$ such that for any compact set $K \subseteq \partial T$*

$$c^{-1} \operatorname{cap}_{a,p}(K) \le R_{a,p}(\Lambda(K)) \le c \operatorname{cap}_{a,p}(K).$$

In fact the restriction that K should be a compact set can be relaxed considerably. By Choquet's capacitability theorem [1, Theorem 2.3.11], Theorem 2 holds for all Suslin sets, in particular for all Borel sets.

Discrete capacities satisfy a recursive formula, which is of fundamental importance for our computations. It relates the capacity of a set to the capacities of the parts of its dyadic decomposition. Let $x \in T$ and $E \subseteq \partial T$. Let also $E_x := \{w \in \partial T : xw \in E\}$ and $\pi_x(w) = \pi(xw)$. Then we define

$$\operatorname{cap}_\pi(E, x) := \operatorname{cap}_{\pi_x}(E_x).$$

Informally, $\operatorname{cap}_\pi(E, x)$ is the capacity of the portion of E that stays below x "viewed" from the root x.

Theorem 3 ([4, Theorem 30]). *Let $E \subseteq \partial T$ a Borel set. For every $x \in T$ the following equality holds*

$$\operatorname{cap}_\pi(E,x) = \frac{\operatorname{cap}_\pi(E,x_-)+\operatorname{cap}_\pi(E,x_+)}{\left(1+\left[\frac{\operatorname{cap}_\pi(E,x_-)+\operatorname{cap}_\pi(E,x_+)}{\pi(x)}\right]^{p'-1}\right)^{p-1}}. \tag{3}$$

Finally let us introduce a more general class of Dobiński type sets on the boundary of the tree. This is a rather natural generalization of the set $\mathscr{D}$. Suppose that κ_n is a sequence of positive integers. Let

$$D(n,\kappa_n) := \left\{w \in \partial T : w_{n+1} = w_{n+2} = \dots w_{n+\kappa_n} = 0\right\}.$$

We define the *Dobiński* type set associated to κ_n as

$$D := \limsup_{n\to\infty} D(n,\kappa_n).$$

Notice that if we consider the set $\Lambda(\cup_m D_m)$ where D_m is the Dobiński type set corresponding to the sequence $\kappa_n = 2^n m^{-1}$, we obtain "one half" of the Dobiński set $\mathscr{D}$. The other half is obtained by considering the same construction, where instead of "strings of 0's" in the definition of $D(n,\kappa_n)$ we consider strings of 1's. These two sets cover the Dobiński set but they have non empty intersection.

3. Proof of the main result

The general Dobiński set as defined above enjoy a natural invariance under rotation by dyadic rational angles. This follows from the fact that for a point $e^{2\pi i x}$, $x = (0.a_1a_2\dots)_2$ belonging or not to a Dobiński type set D is a property which depends only on the eventual behaviour of the sequence $(a_1a_2\dots)$. Since adding a dyadic rational number to x only changes a finite number of the binary digits of x, Dobiński sets are invariant under such rotations. This observation motivates the following lemma.

Lemma 4. *Let $p > 1$ and $a > 0$ such that $ap \le 1$ and $E \subseteq \mathbb{T}$ a Borel set which is invariant under rotations by angles θ, where θ is a dyadic rational number. Then, either $R_{a,p}(E) = R_{a,p}(\mathbb{T})$ or $R_{a,p}(E) = 0$.*

Proof. We recall the fact that the Riesz (a,p) capacity of the circle is finite. More precisely an elementary argument shows that

$$R_{a,p}(\mathbb{T})^{-\frac{1}{p}} = \int_{\mathbb{T}} \frac{\mathrm{d}y}{|1-y|^{1-a}} < +\infty.$$

Assume now that $R_{a,p}(E) \neq 0$. By Theorem [1, Theorem 2.3.10], since E has finite capacity, there exists a unique non negative function $f_E \in L^p(\mathbb{T})$ such that

$$\mathscr{I}_a(f_E) \ge 1, \quad R_{a,p}\text{-q.e. on } E, \tag{4}$$

and

$$\int_{\mathbb{T}} f_E^p(x)\mathrm{d}x = R_{a,p}(E). \tag{5}$$

Let θ a dyadic rational and define $\rho_\theta f(x) := f(e^{2\pi i\theta}x)$. Then it is clear that $\mathscr{I}_a(\rho_\theta f_E) = \rho_\theta(\mathscr{I}_a(f_E)) \ge 1$ on $e^{2\pi i\theta}E = E$. Therefore $\rho_\theta f_E$ satisfies equations (4) and (5) and by uniqueness $\rho_\theta f_E = f_E$. Since θ is dense in $[0,1]$ a calculation with the Fourier coefficients of f_E shows that $f_E = c$ Lebesgue a.e. on $\mathbb{T}$ for some positive constant c.

By equation (5) and the fact that $0 < R_{a,p}(E) \le R_{a,p}(\mathbb{T})$ we get $0 < c^p \le R_{a,p}(\mathbb{T})$. Finally let $y_0 \in E$, such that (4) holds;

$$1 \le \mathscr{I}_a(f_E)(y_0) = c\int_{\mathbb{T}} \frac{\mathrm{d}x}{|y_0-x|^{1-a}} = cR_{a,p}(\mathbb{T})^{-\frac{1}{p}}. \qquad \square$$

We now turn to the main theorem. The calculation can be carried out for general (a,p),

Theorem 5. *Let D a Dobiński set associated to a sequence κ_n and $a>0, 1<p<\infty$. In the case $ap=1$ we have*

(i) *If* $\limsup_{n\to\infty}\kappa_n^{-(p-1)}2^n>0$, *then* $\operatorname{cap}_{\frac{1}{p},p}(D)>0$.

(ii) *If* $\sum_{n=1}^{\infty}\kappa_n^{-(p-1)}2^n<\infty$, *then* $\operatorname{cap}_{\frac{1}{p},p}(D)=0$.

While in the case $ap<1$ we have

(a) *If* $\limsup_{n\to\infty}(apn-(1-ap)\kappa_n)>-\infty$ *then* $\operatorname{cap}_{a,p}(D)>0$.,

(b) $\sum_{n=0}^{\infty}2^{apn-(1-ap)\kappa_n}<+\infty$ *then* $\operatorname{cap}_{a,p}(D)=0$.

Let us pause for a second to show how one can derive Theorem 1 from Theorem 5.

Proof of Theorem 1. Let $p>1$. Consider the Dobiński type set D_m corrisponding to the sequence $\kappa_n=2^nm^{-1}$ for some fixed m. As noted before $\Lambda(D_m)\subseteq\mathscr{D}$. For $1<p\le 2$, by Theorem 5 (i), we have that $\operatorname{cap}_{\frac{1}{p},p}(D_m)>0$. Therefore the comparison principle, Theorem 2, gives $R_{\frac{1}{p},p}(\mathscr{D})\ge R_{\frac{1}{p},p}(\Lambda(D_m))\ge c^{-1}\operatorname{cap}_{\frac{1}{p},p}(D_m)>0$. Since $\mathscr{D}$ is invariant under rotations by dyadic rationals, Lemma 4 implies that $\mathscr{D}$ has full capacity.

While when $p>2$, by Theorem 5 (ii), $\operatorname{cap}_{\frac{1}{p},p}(D_m)=0$ for all $m\in\mathbb{N}$. By countable subadditivity and the comparison theorem, $\operatorname{cap}_{\frac{1}{p},p}(\cup_m D_m)=R_{\frac{1}{p},p}(\Lambda(\cup_m D_m))=0$. The "other half" of Dobiński's set has zero capacity for the same reason, which concludes the proof. □

Proof of Theorem 5. Let $D(n,\kappa_n)$ as before. We start with deriving an exact formula for the discrete (a,p) capacity of the set $D(n,\kappa_n)$ using the recursive formula (equation (3)). For a positive parameter $r>0$ define the function

$$\Phi_r(x):=\frac{x}{(1+rx^{p'-1})^{p-1}},\ x>0.$$

An elementary computations shows that the following semigroup law is satisfied

$$\Phi_r\circ\Phi_s(x)=\Phi_{r+s}(x),\ \forall\ r,s,x>0.$$

Next we apply $n+\kappa_n$ times the recursive formula (3) for the set $D(n,\kappa_n)$. In the following $c:=\operatorname{cap}_{a,p}(\partial T)$. If we use the symbol $\prod$ for repeated composition of functions we have

$$\operatorname{cap}_{a,p}(D(n,\kappa_n))=2^n\prod_{m=1}^{n}\Phi_{2^{(p'-1)[(n+1-m)+(m-1)(1-ap)]}}\circ\prod_{m=n+1}^{n+\kappa_n}\Phi_{2^{(p'-1)(m-1)(1-ap)}}(c)=2^n\Phi_\sigma(c)\qquad(6)$$

where we have used the fact that $\Phi_r(2x)=2\Phi_{2^{p'-1}r}(x)$ and σ is given by

$$\sigma=\begin{cases}\displaystyle\sum_{m=1}^{n}2^{(p'-1)[(n+1-m)+(m-1)(1-ap)]}+\sum_{m=n+1}^{n+\kappa_n}2^{(p'-1)(m-1)(1-ap)}\\[2ex] \displaystyle\qquad\frac{2^{n(p'-1)}-2^{n(1-ap)(p'-1)}}{1-2^{-ap(p'-1)}}+\frac{2^{(n+\kappa_n)(1-ap)(1-p')}-2^{n(1-ap)(p'-1)}}{2^{(1-ap)(p'-1)}-1} & \text{when } ap<1,\\[2ex] \displaystyle\frac{2^{n(p'-1)}-1}{1-2^{1-p'}}+\kappa_n, & \text{when } ap=1.\end{cases}$$

Consequently if $ap=1$,

$$\begin{aligned}\operatorname{cap}_{\frac{1}{p},p}(D(n,\kappa_n))&=\frac{2^nc}{\left(1+\kappa_nc^{p'-1}+\dfrac{2^{n(p'-1)}-1}{1-2^{1-p'}}c^{p'-1}\right)^{p-1}}\\&=\frac{c}{\left(2^{n(1-p')}+\kappa_n2^{n(1-p')}c^{p'-1}+\dfrac{1-2^{n(1-p')}}{1-2^{1-p'}}c^{p'-1}\right)^{p-1}}.\end{aligned}$$

From which is easily verified that there exists a constant $A>0$ such that

$$\frac{1}{A}\operatorname{cap}_{\frac{1}{p},p}(D(n,\kappa_n)) \le \kappa_n^{-(p-1)}2^n \le A\operatorname{cap}_{\frac{1}{p},p}(D(n,\kappa_n)), \quad \forall\, n\in\mathbb{N}.$$

Similarly when $ap<1$,

$$\begin{aligned}
&\operatorname{cap}_{\frac{1}{p},p}(D(n,\kappa_n))\\
&\quad= \frac{2^n c}{\left(1+\left[\frac{2^{n(p'-1)}-2^{n(1-ap)(p'-1)}}{1-2^{-ap(p'-1)}}+\frac{2^{(n+\kappa_n)(1-ap)(p'-1)}-2^{n(1-ap)(p'-1)}}{2^{(1-ap)(p'-1)}-1}\right]c^{p'-1}\right)^{p-1}}\\
&\quad= \frac{c}{\left(2^{-n(p'-1)}+\left[\frac{1-2^{-nap(p'-1)}}{1-2^{-ap(p'-1)}}+\frac{2^{-nap(p'-1)+\kappa_n(1-ap)(p'-1)}-2^{-nap(p'-1)}}{2^{(1-ap)(p'-1)}-1}\right]c^{p'-1}\right)^{p-1}}.
\end{aligned}$$

Hence, for some constant $A>0$,

$$\frac{1}{A}\operatorname{cap}_{\frac{1}{p},p}(D(n,\kappa_n)) \le 2^{apn-(1-ap)\kappa_n} \le A\operatorname{cap}_{\frac{1}{p},p}(D(n,\kappa_n)), \quad \forall\, n\in\mathbb{N}.$$

The theorem then follows from the estimates which hold for all $ap\le 1$. We can estimate the capacity of D from above using subadditivity

$$\operatorname{cap}_{a,p}(D)=\operatorname{cap}_{a,p}\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}D(n,\kappa_n)\right)\le\sum_{n=m}^{\infty}\operatorname{cap}_{a,p}D(n,\kappa_n), \quad \forall\, m\in\mathbb{N}.$$

And from below;

$$\operatorname{cap}_{a,p}(D)=\operatorname{cap}_{a,p}\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}D(n,\kappa_n)\right)=\lim_m\left(\operatorname{cap}_{a,p}\bigcup_{n=m}^{\infty}D(n,\kappa_n)\right)\ge\limsup_n\operatorname{cap}_{a,p}D(n,\kappa_n). \qquad \square$$

It is clear that Theorem 5 leaves a gap between the sufficient and the necessary condition for a general Dobiński type set to have positive capacity, although for the Dobiński set considered in [10] this turns out not to be a problem. We conjecture that the series condition in Theorem 1 in fact characterizes the vanishing of the capacity of any Dobiński type sets. This conjecture is supported by the following observation. Consider a $1<p<+\infty$ fixed and define s,a by the relation $s=1-ap$. Then, at least when $s>0$, applying [10, Theorem 1] to the Dobiński type set D associated to the sequence κ_n and the gauge function $h(t)=t^s$ we get that the s-Hausdorff measure of D is zero if and only if the series $\sum_{n\in\mathbb{N}}2^{(1-s)n-s\kappa_n}$ converges. Thanks to equation (1)

$$\sup\{s:R_{a,p}(D)>0\}=\inf\left\{s:\sum_{n\in\mathbb{N}}2^{(1-s)n-s\kappa_n}<+\infty\right\}.$$

Under the hypothesis that the sup in the above equation is a maximum if and only if the inf in the same equation is a minimum, one can conclude that the series characterizes the vanishing of the Riesz (a,p) capacity. Unfortunately we have been unable to prove such a condition.

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