On L_p-quantiles for the Student t distribution

L_p-quantili per la distribuzione t di Student

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Abstract L_p-quantiles belong to the class of generalised quantiles investigated in the papers by [5] and [4]. In this contribution we study their mathematical properties and provide some recursive equations to compute them. For the special case of the L_2-quantile, also known as *expectile*, [10] showed the equivalence of the quantile and the expectile for a class of distribution functions that includes the t-distribution with 2 degrees of freedom. We extend this result and show that for a general t-distribution with p degrees of freedom (and any affine transformation), the L_p-quantile and the quantile coincide. This result opens up new research directions for the computation of the t-quantile that is generally not available in closed form and difficult to compute.

Abstract Gli L_p-quantili appartengono alla classe dei quantili generalizzati studiati negli articoli di [5] e [4]. In questo articolo studiamo le loro proprietá matematiche e introduciamo delle equazioni ricorsive per il loro calcolo. Nel caso particolare degli L_2-quantili, meglio noti come *espettili*, [10] mostra l' equivalenza del quantile e dell' espettile per una classe di distribuzioni che include la distribuzione t di Student con 2 gradi di libertá. Noi estendiamo questo risultato e mostriamo come per una distribuzione t di Student con p gradi di libertá (ed ogni sua trasformazione affine), l' L_p-quantile ed il quantile coincidono. Questo risultato apre nuove possibili linee di ricerca per il calcolo del quantile di una Student-t che solitamente non disponibile in forma chiusa e di difficile calcolo.

Key words: L_p-quantiles, expectiles, t-distribution, quantiles, risk measures.

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1 Risk measurement

Financial institutions such as banks and insurance companies, are required to hold some safely invested capital in order to be acceptable from a regulatory perspective. The most well known risk measures used to compute such a risk capital are Value-at-Risk (VaR) and Expected Shortfall (ES).

ES is often regarded as a better risk measure than VaR because it is *coherent*, in particular subadditive, see [2] and it considers the extreme left tail of the distribution. The recent literature on risk measurement has highlighted that, from a statistical point of view, ES presents some important deficiencies. In particular [9] pointed out that ES, contrary to VaR, does not satisfy the so-called *elicitability property*. A risk measure ρ is elicitable if it can be defined as the unique minimiser of a given expected loss function L:

$$\rho(Y) = \arg\min_{m\in\mathbb{R}} \mathbb{E}[L(Y-m)]. \tag{1}$$

This property has attracted major attention in the recent literature because the empirical expected loss can be used as a natural statistics to perform the backtesting of the risk measure and to consistently rank different risk measure forecasts. As a result of the debate between VaR and ES, another risk measure has attracted major attention as a valid alternative to VaR and ES, namely the *expectiles*. Expectiles were introduced by [1] and [12]. They are defined as the unique minimiser of an asymmetric squared function:

$$\mu_\tau(Y) = \arg\min_{m\in\mathbb{R}} \mathbb{E}\left[\left|\tau - \mathbb{I}_{\{Y-m<0\}}\right|(Y-m)^2\right], \quad \text{for all } \tau\in(0,1), \tag{2}$$

for $\tau = 1/2$, $\mu_{1/2}(Y) = \mathbb{E}[Y]$. [4] showed that they are coherent risk measures for $\tau \in (0, \frac{1}{2})$ and according to [14], [3] and [6] they are the *unique* elicitable coherent risk measure. Expectiles are becoming very popular also in the econometric literature (see for instance [11]; [7]; [8], and the references therein).

2 L_p-quantiles

In the present contribution we work in the same direction and consider the class of L_p-*quantiles* introduced by [5]. L_p-quantiles belong to the class of generalised quantiles, also investigated in [4] and share many important properties of the quantiles. For a random variable Y with cumulative distribution function F_Y the L_p-*quantile* at level τ is defined as

$$\rho_{\tau,p}(Y) = \arg\min_{m\in\mathbb{R}} E\left[\left|\tau - \mathbb{I}_{\{Y-m\geq 0\}}\right|(Y-m)^p\right] \quad \text{for all } \tau\in(0,1), p\in\mathbb{N}\setminus\{0\}. \tag{3}$$

According to Corollary 3 in [4], they can alternatively be written as the unique solution to the following equation

$$\tau\mathbb{E}\left[((Y-m)_+)^{p-1}\right]=(1-\tau)\mathbb{E}\left[((Y-m)_-)^{p-1}\right] \quad \text{for all } \tau\in(0,1). \tag{4}$$

Clearly the quantiles and the expectiles coincide with the L_p-quantiles for $p=1$ and $p=2$ respectively. L_p-quantiles represent an important class of risk measures; they are monotone, translation invariant, positive homogeneous and law-invariant. They belong to the class of convex (and coherent) risk measures if and only if $p=2$. Further they are, by definition, elicitable and hence easy to backtest. Having all these properties, L_p-quantiles represent a valid alternative to other popular risk measures such as VaR, ES and expectiles and they provide an important tool for quantitative risk management.

We investigate their properties and financial meanings and explain their link with conditional tail moments. We show that computing the L_p-quantiles of a distribution for different values of p provides important information on the distribution. In particular for $p=3$ and $p=4$, the L_p-quantiles are strictly related to the index of skewness and kurtosis of the distribution.

We show that L_p-quantiles can be written as the unique solution of an equation involving all the truncated moments $G_{j,Y}(m)=\int_{-\infty}^{m}y^p\mathrm{d}F_Y(y)$ for all $m\in\mathbb{R}$ and $j=0,1,\ldots,p-1$. In particular, we show that for p odd the L_p-quantiles can be written as the unique solution to the following equation:

$$\sum_{k=0}^{p-1}\frac{(p-1)!}{k!(p-1-k)!}m^k(-1)^k\left(\tau\mathbb{E}[Y^{p-1-k}]-G_{p-1-k,Y}(m)\right)=0. \tag{5}$$

Similarly, for p even, we have:

$$\sum_{k=0}^{p-1}\frac{(p-1)!}{k!(p-1-k)!}m^k(-1)^k\left(\tau\left(\mathbb{E}[Y^{p-1-k}]-2G_{p-1-k,Y}(m)\right)+G_{p-1-k,Y}(m)\right)=0. \tag{6}$$

The above equations can then be used to interpret L_p-quantiles as quantiles of another distribution H_Y and provide an easy way to compute them. We provide the graphs of L_p-quantiles for different distributions and different values of p and show how they behave depending on the tail of the distribution. From a statistical point of view, we can generalise the notion of quantile regression to the more general L_p-regression by means of the loss function $f(x)=|\tau-\mathbb{I}_{\{x\geq 0\}}|x^p$. This allow us to properly estimate these functionals.Further we generalise the approach of [13] to Bayesian quantile regression to perform Bayesian inference to properly estimate L_p-quantiles and provide numerical examples.

2.1 L_p-quantiles for the Student t distribution

[10] proved the equivalence of the quantile and the expectile for a class of distribution functions that includes the t-distribution with 2 degrees of freedom. In the present contribution we extend this result and show that for a general t-distribution with p degrees of freedom (and any affine transformation), the L_p-quantile and the quantile coincide for any $\tau \in (0,1)$. This is a neat result that contributes to both the literature on risk measurement and statistical theory as it characterises a special feature of the Student t distribution.

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