ASIDE:

SPATIAL ANTIDERIVATIVES OF DISTRIBUTIONS

Let $R \in L^2(T; H^{-1}(D))$ — a distribution

then $\{ R^{[-1]} \} \subset L^2(T; L^1_{loc}(D))$ s.t., — the antiderivative

$$\langle R^{[-1]} \mid w \rangle = - \langle R \mid w_\varepsilon \rangle + \langle c_1 \mid w \rangle \, , \forall \, w \in \mathbb{W}$$

where

$$\mathbb{W} := \{ w \mid w \in L^2(T; H^1_0(D)); \, w_t \in L^2(T; H^{-1}(D)); \, w(\,.,\, t_1\,) = 0 \, \}$$

the space of test functions

$$w_\varepsilon(.,t) := \int_{x_0}^{x} [w(\sigma, t) - \omega_\varepsilon(\sigma - \sigma_0) \int_D w(\psi, t) d\psi \,] d\sigma$$

$$w \in \mathbb{W} \, , \; \omega_\varepsilon(x) = c_\varepsilon \exp[\frac{\varepsilon^2}{x^2 - \varepsilon^2}]$$

a *bell*–function

Rem.: antidifferentiation, $(.)^{[-1]}$, is *set valued*

Notation

$\langle h \rangle$:= spatial average of $h \in C^0(\bar{T}; L^1(D))$,

($\Rightarrow$ $\langle h \rangle \in C^0(\bar{T})$)

$R_0^{[-1]}$:= antiderivative, which vanishes @ x_0

(requires $\exists \lim_{x \to x_0^+} R^{[-1]}$)

UNIQUENESS CONDITIONS:
2 – *REGULAR* CAUCHY PBM.

Notation:

$r := -V_t + (aV_x)_x$ (the *defect*)

$R := \frac{r}{v_x}$ (provided ...)

Def. $\mathbb{B}_{ad} := \{ B \mid B \in L^{\infty}(D); \exists \lim_{x \to x_0^+} B \wedge \lim_{x \to x_0^+} B = 0 \}$

Prop. (*uniqueness*)

$f \in C^0(\bar{T}; H^{-1}(D))$; $u, v \in \mathbb{X}$

$\exists\ \tau \in \bar{T} \cdot \ni \cdot v_x \neq 0, \forall x \in \bar{D}$

$\exists\ \hat{a}(u, f) \in \mathbb{A}_{ad}$; $a, b \cdot \ni \cdot B \in \mathbb{B}_{ad}$

$\exists \lim_{x \to x_0^+} R^{[-1]}(\tau)$

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IF $\{ V(\ .\ ,\tau) = 0\ \forall x \in \bar{D} \} \wedge \{ V_t(\ .\ ,\tau) =_{\text{a.e.}} 0 \text{ in } D \}$

THEN $B =_{\text{a.e.}} 0$ in D .

Applications:

Regular Cauchy data from
- $a(x_0)$ known
- $\{u, f\}, \{v, g\}$ independent
- $a'(x_0) = 0$ (*zoning*)

UNIQUENESS CONDITIONS:
3 – *SINGULAR* CAUCHY PBM.

Def. (set of points where u stationary = critical points)

$$E_u(t) := \mathrm{clos}\,\{\ \xi(t) \mid \xi(t) \in \overline{D}\ ,$$

$$u_x(\xi(t), t) = 0 \ \vee \lim_{x \to y^{\pm}} u_x(\xi(t), t) = 0\ \}$$

$$\exists\, \tilde{a} \in A_{ad} \cdot\!\ni\!\cdot\ u(\hat{a}, f) =_{\text{a.e.}} u(\tilde{a}, f) \text{ in } Q.$$

Prop. (*uniqueness*)

$$f \in L^2(T\,;\, H^{-1}(D))$$

$$\exists\, \hat{a} \in \mathbb{A}_{ad} \cdot\!\ni\!\cdot u(\hat{a}, f) \in \mathbb{U}_{ad}$$

$$\exists\, \tilde{a} \in \mathbb{A}_{ad} \cdot\!\ni\!\cdot u(\tilde{a}, f) \in \mathbb{U}_{ad}$$

IF EITHER

$$\exists\, \tau \in \overline{T} \cdot\!\ni\!\cdot \{E_u(\tau) \neq \emptyset \wedge \mathrm{meas}[E_u(\tau)] = 0\ \} \quad (*)$$

OR $$E_u(t) \neq \emptyset,\ \forall\, t \in T \wedge \mathrm{meas}[\ \bigcap_t E_u(t)\ ] = 0\ \} \quad (**)$$

THEN $$\tilde{a} =_{\text{a.e.}} \hat{a} \text{ in } D\ .$$

Rem.:

Uniqueness relies on set E_u of critical points.

Kitamura & Nakagiri' s (1977) uniqueness conditions apply to more regular data.

Data @ t = τ only.

UNIQUENESS CONDITIONS:
4 – *SELF–IDENTIFIABILITY*

Def. (admissible data pairs with least possible regularity)

$$\mathbb{P}_{ad} := \{ \{ u, f \} \mid u \in \mathbb{U}_{ad} , f \in C^0(\bar{T}; H^{-1}(D)) ;$$

$$(u_t + f)^{[-1]} := g^{[-1]} \subset C^0(\bar{D} \setminus S_g(t)) \cap L^2(D) \ \forall t \in \bar{T} \}$$

where

$$S_g(t) \cap \{ x_0 ; x_1 \} = \emptyset \ \forall \ t \in \bar{T}$$

Thm. (*self–identifiability*)

IF $\{ u, f \} \in \mathbb{P}_{ad}$; $\text{meas}[E_u(\tau)] = 0$

THEN the following are equivalent

i) $\exists \ \hat{a} \in \mathbb{A}_{ad}$, $\exists \ \tau \in \bar{T} \cdot\ni\cdot <\hat{a}\, u_x>(\tau) = 0$

ii) $\exists! \ \hat{a} \wedge \hat{a} = \left(\frac{g_0^{[-1]} - \langle g_0^{[-1]} \rangle}{u_x} \right)(\tau)$

Rem.

non–local condition

$\exists$ link with singular Cauchy if $g_0^{[-1]} \in C^0(\bar{D})$

counterexample

UNIQUENESS CONDITIONS:
5 – COUNTEREXAMPLE

Features:

$t = \tau$, fixed

self–identifiable solution s.t., singular Cauchy uniqueness Hp. do not apply.

Let

$$D = (-1+\varepsilon\,,\,1-\varepsilon)\,;\; u_x = \begin{cases} -1-x\,, & -1+\varepsilon \le x < 0 \\ \frac{1}{2}-\frac{x}{2}\,, & 0 < x \le 1-\varepsilon \end{cases}$$

$\hat{a} = 1 + \theta(x)$ (unit step function)

Note

$$<\hat{a}\,u_x> = 0\;,\;\exists \lim_{x \to x_0^+} \hat{a}\,u_x \;(= -\varepsilon)$$

$$g_0^{[-1]} = 2\,\theta(x) - 1 - x + \varepsilon\;(\notin C^0(\bar{D}))\;;\; <g_0^{[-1]}> = \varepsilon$$

hence

$$\hat{a} = \frac{g_0^{[-1]} - \left\langle g_0^{[-1]} \right\rangle}{u_x}$$

Since both $(\frac{1}{u_x})(\tau)$ and $(g_0^{[-1]})(\tau) \in L^4(D)$, then the stability estimate applies

$$\|\, B \,\|_1 \le \text{const.}\;[\, \| v - u \|_{1,4} + \| (v_t - u_t)_0^{[-1]} \|_{0,4}](\tau)$$

STABILITY ESTIMATES:
1 – *REGULAR* CAUCHY – UNIQUE SOLN.

(Regularization conditions denoted by ☜)

Thm.

IF $u, v \in \mathbb{X}$

$\exists\, \tau \in \bar{T} \cdot\ni\cdot \left|\frac{1}{u_x}\right|(\tau)\,,\ \left|\frac{1}{v_x}\right|(\tau) \leq c_v$ ☜

$||v_{xx}||_{0,2}(\tau) \leq c_s$ ☜

$\exists\, \hat{a} \in \mathbb{A}_C := \{\, a \mid a \in \mathbb{A}_{ad}\,;$ ☜

$$a' = \sum_{i=1}^{\infty} c_i\, \delta\,(x - \xi_i) + \sum_{i=1}^{\infty} \psi_i\, \chi_{(\xi_i, \xi_{i+1})}$$

$$y_i \neq x_0\,,\ x_1\,;\ \xi_i < \xi_{i+1}\,,\ \forall\, i\,;\ \sum_{i=1}^{\infty} |c_i| < \infty$$

$$\psi_i \in L^1((\xi_i, \xi_{i+1}))\,\}$$

$B \in \mathbb{B}_{ad}\,;\ \exists \lim_{x \to x_0^+} R^{[-1]}(\tau)$

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THEN

$$||B||_{0,\infty} \leq c_v[1 + ||\hat{a}\,||_{0,\infty} + ||\ |\hat{a}_x|_0^{[-1]}||_{0,\infty}\,].$$

$$.||V||_{\mathbb{X}_\tau} \exp[c_v c_s \sqrt{\text{meas}[D]}\,]$$

where

$$||V||_{\mathbb{X}_\tau} := \max_{\bar{D}} \{\ |V|(\tau),\ |V_t|(\tau)\ \} +$$

$$+ \sqrt{\text{meas}[D]}\ \ [\ ||V_t||_{0,2} +\ ||V_{xx}||_{0,2}](\tau)$$