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Gaussian maps on curves and algebraic surfaces

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Contents

Introduction

The work contained in this thesis aims to present some new results in the study of Gaussian maps on smooth complex projective curves and smooth projective surfaces. To better describe the results and how they are organized in the thesis we will first recall the definition of Gaussian maps and the classical geometric motivations to study them. Most of the results obtained in the thesis are contained in [30] and [31].

Let X be a smooth complex projective variety and $k \geq 0$ an integer. Consider two line bundles L and M on X. Moreover let $p_i : X \times X \to X$, $i = 1, 2$ be the two projections and denote by $L \boxtimes M$ the line bundle $p_1^* L \otimes p_2^* M$ on $X \times X$. Then consider the inclusion of the $(k+1)$th power of the ideal of the diagonal in the kth power and tensor it by $L \boxtimes M$

$$0 \to I_\Delta^{k+1} \otimes L \boxtimes M \to I_\Delta^k \otimes L \boxtimes M \to I_\Delta^k / I_\Delta^{k+1} \otimes L \boxtimes M \to 0.$$

The kth Gaussian map on X associated with L and M is the map induced at the level of global sections

$$\Phi_{L,M} : H^0(X \times X, I_\Delta^k \otimes L \boxtimes M) \to H^0(X, S^k \Omega_X^1 \otimes L \otimes M).$$

In the following when $k = 1$ or $L = M$, will write $\Phi_{L,M}$ and Φ_L respectively.

The most meaningful case is when X is a smooth complex projective curve, which we denote by C. In this case, the study of Gaussian maps associated with some particular choice of line bundles L and M on C is related to understanding if the curve C can be embedded on a particular class of smooth complex projective surfaces. We make it explicit with some important classical examples.

A fundamental case is when $L = M = \omega_C$, where the latter is the canonical bundle of the curve and $k = 1$. In this case it has been shown by Wahl ([67]) that if C lies on a $K3$ surface then Φ_{ω_C} is not surjective. The significance of this result becomes more evident when it is compared to a theorem of Ciliberto, Harris and Miranda ([20]) which asserts that Φ_{ω_C} is surjective for the general curve in the moduli space of curves of genus g when $g \geq 10$, $g \neq 11$.

Hence the surjectivity of the Gaussian map Φ_{ω_C} (also called Wahl map or Wahl-Gaussian map) gives an obstruction for a curve C to lie on a $K3$ surface. Also the converse is true: given a general enough curve in the moduli space of curves with nonsurjective Wahl map then it lies on a surface which is the limit of a $K3$ surfaces, see [1] and Theorem 1.3.12.

Another situation where it is evident a similar behavior is the case of curves on abelian surfaces. Indeed Colombo, Frediani and Pareschi ([22]) showed that if C lies on an abelian surface then $\Phi^2_{\omega_C}$ is not surjective, whereas Calabri, Ciliberto and Miranda ([12]) proved the surjectivity for the general element in M_g with $g \geq 18$. Observe that in this case the obstruction is given by the surjectivity of a higher-order Gaussian map (i.e. $k = 2$).

A third interesting case is the one of curves on Enriques surfaces. In this setting the obstruction is given by the "mixed" Gaussian map $\Phi_{\omega_C, \omega_C \otimes \alpha}$ where α is the restriction of the canonical bundle of the surface to the curve. In particular, it is a 2-torsion line bundle and the pair (C, α) is called Prym curve. It is an immediate consequence of a more general result of L'Vovsky ([54]) that, if C is a scheme theoretically hyperplane section of an Enriques surface, then $\Phi_{\omega_C, \omega_C \otimes \alpha}$ is not surjective. On the contrary, it has been proven by Ciliberto and Verra ([19]) that it is surjective for the general Prym curve in the moduli space of Prym curves R_g (when $g \geq 12$ and $g \neq 13, 19$). Hence the surjectivity of $\Phi_{\omega_C, \omega_C \otimes \alpha}$ gives an obstruction for the Prym curve (C, α) to lie on an Enriques surface S and α is the restriction of ω_S to C.

Now consider again a Prym curve (C, α). Another natural Gaussian map one can study for this pair is the one associated with the (same) line bundle $L = M = \omega_C \otimes \alpha$. The behavior of this map for $k = 1, 2$ ($\Phi_{\omega_C \otimes \alpha}$ and $\Phi^2_{\omega_C \otimes \alpha}$), is studied for the general element (C, α) in the moduli space of Prym curves R_g by Barchielli and Frediani, and Colombo and Frediani respectively ([6],[24]). Again, it is shown a surjectivity result for the general element as soon as it is possible, namely for $g \geq 12$ and $g \geq 20$ respectively. These results are our motivation to understand what happens when the Prym curve (C, α) lies on an Enriques surface S and $\alpha = \omega_{S_{|C}}$: is $\Phi_{\omega_C \otimes \alpha}$ (and/or $\Phi^2_{\omega_C \otimes \alpha}$) not surjective and hence this map gives an obstruction for a Prym curve to lie on an Enriques surface? We give a negative answer to this question (see Theorem 3.1.3).

Theorem 1. *Let C be a smooth hyperplane section of an unnodal Enriques surface (S, H) with $\phi(H) > 4(k+2)$. Then the kth Gaussian-Prym map $\Phi^k_{\omega_C \otimes \alpha}$ is surjective. In case $k = 1$ it is sufficient to ask $\phi(H) > 6$.*

Here ϕ is a measure of the positivity of the line bundle H on S and it is connected to the notion of $k-$very ampleness. An unnodal Enriques is an Enriques surface that does not contain any rational curve: the general Enriques surface is unnodal. We observe that the statement holds for almost every isomorphism class of line bundles. Indeed if c is a positive integer, then there are finitely many isomorphism classes of line bundles H such that $\phi(H) \leq c$.

In doing so we prove that Gaussian maps associated with a sufficiently positive polarization on an Enriques surface are surjective (see Theorem 2.2.12)

Theorem 2. *Let S be an unnodal Enriques surface, H be a line bundle on S with $\varphi(H) > 2k+4$ and $C \in |H|$. The kth Gaussian map Φ^k_H is surjective.*

These statements are similar to the ones obtained in [58] for $K3$ surfaces. By a theorem of Rios Ortiz ([58] (see Theorem 1.3.24), a sufficient condition for having the surjectivity of Φ^k_H in Theorem 2, is the vanishing of a cohomology group of a certain line bundle on the Hilbert Scheme of 2-points on the surface $\mathrm{S}^{[2]}$. Hence one of the main technical tool is the following ampleness result for line bundles on $\mathrm{S}^{[2]}$(see Proposition 2.2.11).

Proposition 1. *Let L be a line bundle on S such that $\phi(L) = k, k > 4$. Then*

$$\tilde{L} - \left(\frac{k}{2} - 1 - r\right)B \quad \textit{is ample for } \; 1 \leq r < \frac{k}{2} - 1. \tag{0.0.1}$$

on $\mathrm{S}^{[2]}$.

The Picard group of line bundles on $\mathrm{S}^{[2]}$ decomposes as $\mathrm{Pic}(S) \oplus \mathbb{Z}B$, where $2B$ is the exceptional divisor of the Hilbert-Chow morphism. In the statement of Proposition 1, $\tilde{L}$ denotes the line bundle corresponding to L via this identification (see section 1.2). The proof of Theorem 1 also involves showing the vanishing of some cohomology groups of vector bundles on Enriques surfaces. We mention the following result which can interest independently (see Proposition 3.1.4).

Proposition 2. *Let S be an unnodal Enriques surface and $H \in \mathrm{Pic}(S)$ such that $\phi(H) > 4(k+2)$. Then*

$$H^1(S, \mathrm{Sym}^k \Omega^1_S(C)) = 0 \;\; \textit{for all } k \geq 1.$$

Let K_S be the canonical divisor of S. When $k = 1$, the cohomology group $H^1(S, \Omega^1_S(C + K_S)) \simeq H^1(S, T_S(-C))$ has a nice geometric interpretation. Indeed let $\mathcal{E}C_{g,\phi}$ be the moduli space which parametrizes isomorphism classes of 3-tuples (S, H, C) with $\phi(H) = \phi$ and C smooth irreducible curve in $|H|$ of genus g. Then, if $[(S, H, C)]$ is a general point of an irreducible component of $\mathcal{E}C_{g,\phi}$, the dimension of $H^1(S, \Omega^1_S(C+K_S))$ is the dimension of the general fiber of the morphism

$$c_{g,\phi} : \mathcal{E}C_{g,\phi} \to M_g \tag{0.0.2}$$

which sends $[(S, H, C)]$ to $[C]$. Then we obtain the following (see Proposition 3.1.8).

Proposition 3. *For every $\phi > 12$, the moduli map $c_{g,\phi} : \mathcal{E}\mathcal{C}_{g,\phi} \to \mathcal{M}_g$ is generically finite over its image.*

This recovers a result already proven in ([15]) (with different methods), where a complete description of the dimension of the fibers of the moduli map was given for any g, ϕ. Moreover, in ([15]) it was shown that in our hypothesis it is generically injective on any irreducible component.

In the context of giving obstructions for a curve to lie on a particular class of surfaces via Gaussian maps, one can consider also singular curves. One possible approach to do that is to work directly with singular curves ([5]). Another is to study Gaussian maps on the normalization. The last one is the approach of Kemeny ([46]) and Fontanari and Sernesi ([36]). Before presenting our contributions to this problem we will briefly describe their results since they inspired our work.

In [46] it is considered the stack $\mathcal{V}^n_{g,k}$ parametrizing morphisms $[(f : C \to X, L)]$ where (X, L) is a polarized $K3$ surface with $L^2 = 2g-2$, C is a smooth connected curve of arithmetic genus $p(g, k) - n$ with $p(g, k) := k^2(g - 1) + 1$, f is birational onto its image and $f_*C \in |kL|$ is nodal. It is shown - under some assumptions on g and n - that if $[(f : C \to X, L)]$ is a general element of some irreducible component of $\mathcal{V}^n_{g,k}$ and if T is the divisor of the points in C which are mapped to the nodes, then the "marked Wahl map" $\Phi_{\omega_C - T}$ is not surjective (see Theorem 4.1.3). A similar result, proved with very different methods, is given in [36] where it is shown that if X is a general $K3$ surface and $C \to X$ is the normalization of a curve with a node or a cusp, then $\Phi_{\omega_C - T}$ is not surjective (see Theorem 4.1.4).

On the other hand in [46] it is shown that for a general pointed (with unordered points) curve of genus g, i.e. pairs (C, T_d) where T_d is a divisor of degree d, the same map is surjective for infinitely many integers g and d (see Theorem 4.1.2).

In this thesis we also deal with similar questions for singular curves on Enriques surfaces. We consider a polarized Enriques surface (S, H) and a curve C having a morphism $f : C \to S$ birational onto its image and such that $f(C) \in |H|$ has exactly one ordinary singular point of multiplicity d. We set $\alpha = f^*K_S$ and we denote by $(p_1, ..., p_d)$ the d-distinct points that are mapped to the singular point. Then $(C, \alpha, p_1, ..., p_d)$ is called a d-pointed Prym curve. Denoted by T_d the divisor $p_1 + ... + p_d$, we study the mixed Gaussian-Prym maps $\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$ and $\Phi_{\omega_C, \omega_C - T_d + \alpha}$ on the normalization, proving the following (see Theorem 4.1.5).

Theorem 3. *Let (S, H) be a polarized unnodal Enriques surface with $H^2 = 2g - 2$ and let $d \geq 2$. Suppose that $\phi(L) \geq l + 4$, if $1 \leq l \leq 14$, or $\phi(L) \geq \frac{2\sqrt{3}}{3} l + \sqrt{3}$, if $l \geq 15$. Set $g' = g - \binom{d}{2}$ and let C be a smooth curve of genus g' having a birational morphism $f : C \to S$ onto its image such that $f(C) \in |H|$, and $f(C)$ has exactly one ordinary singular point of multiplicity d. Set $\alpha = f^*\omega_{S_{|C}}$ and let $T_d = p_1 + ... + p_d$ be the divisor over the singular point. Then the Gaussian maps $\Phi_{\omega_C, \omega_C - T_d + \alpha}$ and $\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$ are not surjective.*

We recall that an Enriques surface is called unnodal if it does not contain any rational curve. The general Enriques surface is unnodal.

On other hand we prove that if $[(C, \alpha, p_1, ..., p_d)]$ is the class of a general $d-$pointed curve (in the moduli space parametrizing isomorphism classes of $d-$pointed curves $R_{g,d}$), then for any $d \geq 2$ the Gaussian maps are surjective for infinitely many values of g (see Theorem 4.1.6 and Example 4.3.1).

Theorem 4. *Fix an integer $d \geq 2$. Then there exist infinitely many integers g, such that the Gaussian maps $\Phi_{\omega_C, \omega_C - T_d + \alpha}$ and $\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$ are surjective for the general Prym pointed curve $[(C, \alpha, p_1, ..., p_d)]$ in $R_{g,d}$. In case $d = 2, 3$ or $d = 4$ we obtain all the genera $g \geq 76$.*

The proof of Theorem 3 is similar to the proof of Theorem 4.1.4 of Fontanari and Sernesi and uses a result of L'vovsky ([54]). A fundamental point is showing that some line bundles on the blow up at a point of an Enriques surface are very ample.

The study of very ample line bundles on the blowup at a point of an Enriques surface gives in turn the following interesting consequence (see Corollary 4.2.9).

Theorem 5. *Let $l \geq 2$ be an integer, and let (S, H) be polarized unnodal Enriques surface. Suppose that $\phi(L) \geq l + 4$, if $2 \leq l \leq 14$, or $\phi(L) \geq \frac{2\sqrt{3}}{3}l + \sqrt{3}$, if $l \geq 15$. Then there exists a curve C in the linear system $|H|$ with an ordinary singular point of multiplicity l.*

We observe that a similar statement in the case of curves on $K3$ surfaces having an ordinary singular point can be found in [36]. We also mention that in [40] is proven the existence of curves on $K3$ surfaces having singularities of the type A_k.

The proof of Theorem 4 is more involved. A first step consists in showing the surjectivity of $\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$ and $\Phi_{\omega_C, \omega_C - T_d + \alpha}$ on a pointed Prym curve $(C, \alpha, p_1, ..., p_d)$ constructed as curve in a product $C' \times \mathbb{P}^1$ where C' is curve, with some special choices of α and $(p_1, ..., p_d)$. To prove that the surjectivity for the special point implies - by semicontinuity - the surjectivity for the general pointed prym curve in $R_{g,d}$, we need to control the dimension of the spaces of global sections of the line bundles $\omega_C - T_d + \alpha$ and $\omega_C - T_d$. In order to control this spaces we prove a result concerning the gonality of curves lying in a product $C' \times \mathbb{P}^1$ (see Proposition 4.3.6).

Proposition 4. *Let $X \in |p_1^*(D_1) + d_2 C_0|$ be a curve in $C' \times \mathbb{P}^1$ where C_0 is the class of a fiber over $\mathbb{P}^1$. Then*

- *if C' is hyperelliptic,*

$$\mathrm{gon}(X) \geq min(d_1, 2d_2).$$

- *If C' is any curve, $g(X) > 0$ and $d_2 \geq \frac{d_1}{4} + 1 + \frac{1}{d_1}$*

$$\mathrm{gon}(X) \geq min(d_1, d_2 \, \mathrm{gon}(C')).$$

In the process of proving Theorem 4 we also prove a result which gives the surjectivity of "mixed" Gaussian maps on surfaces of the type $C_1 \times C_2$ (see Proposition 2.1.7), generalizing results of Wahl (see [68], Lemma 4.12 and Theorem 4.11)).

Proposition 5. *Let $X = C_1 \times C_2$. Let D_i, $i = 1, 2$ be effective divisors on C_i. Let $p_i : X = C_1 \times C_2 \to C_i$, $i = 1, 2$ be the projections. Let L_i and M_i be line bundles on C_i, $i = 1, 2$, such that $deg(L_i), deg(M_i) \geq 2g_i + 2$ and $deg(L_i) + deg(M_i) \geq 6g_i + 3$, for $i = 1, 2$. Set $L = p_1^* L_1 \otimes p_2^* L_2$ and $M = p_1^* M_1 \otimes p_2^* M_2$. Then $\Phi_{X,L,M}$ is surjective.*

We also give sufficient conditions for the surjectivity of Gaussian maps on curves which lie in a product $C_1 \times C_2$ (see Proposition 4.3.1):

Proposition 6. *With the same hypothesis and notations of Proposition* 5*, let C be a smooth curve in the linear system $|p_1^* D_1 + p_2^* D_2|$. Denote by l_i and m_i the degree of L_i and M_i respectively. Moreover suppose that*

1. $l_i, m_i \geq 2g_i + 2$ *and* $l_i + m_i \geq 6g_i + 3$;
2. $l_i + m_i > 2g_i - 2 + d_i$ *for* $i = 1, 2$
3. $d_2(l_1 + m_1 - (2g_1 - 2)) + d_1(l_2 + m_2 - (2g_2 - 2)) - 4d_1d_2 > 0.$

Then

$$\Phi_{C, L_{|C} \otimes M_{|C}}$$

is surjective.

Now we come back to the Gaussian-Wahl maps $\Phi^k_{\omega_C}$, $k \geq 1$. We have said that the surjectivity of $\Phi^k_{\omega_C}$ represents, when $k = 1$, an obstruction for the curve C to lie on a $K3$ surface and, when $k = 2$ an obstruction for C to lie on an abelian surface. Moreover we have said that Φ_{ω_C} and $\Phi^2_{\omega_C}$ are surjective for the general curve in the moduli space of curves. A natural problem is to understand (for the general curve) the behavior of $\Phi^k_{\omega_C}$ when $k \geq 3$. By specializing on curves that lie in a product $C_1 \times C_2$, we prove the following (see Corollary 3.2.5).

Theorem 6. *Let $k \geq 2$ and let g_i, $i = 1, 2$ and d_i, $i = 1, 2$ be integers satisfying one of the following conditions*

1. $g_1 \geq 2, g_2 \geq 1$, *and* $d_i \geq kg_i + k + 3$ *for* $i = 1, 2$ *or,*
2. $g_1 = 0$, $g_2 \geq 2$, $d_1 > 2(k+1)$, $d_1 > \frac{kd_2}{g_2 - 1}$, $d_2 \geq kg_2 + k + 3$.

Then the general curve of genus

$$g = 1 + (g_2 - 1)d_1 + (g_1 - 1)d_2 + d_1d_2, \tag{0.0.3}$$

has surjective kth Gaussian-Wahl map.

Moreover, for any k, the lowest genus is $6k^2 + 17k + 13$ (see Remark 3.2.4). While proving it we also obtain a statement about the surjectivity of higher Gaussian maps on surfaces of the form $C_1 \times C_2$. This is Proposition 2.1.9. Theorem 6 partially recovers a result of Rios Ortiz ([58], see Theorem 1.3.23) which, by specializing on curves on $K3$ surfaces, gives the surjectivity of $\Phi^k_{\omega_C}$

for the general curve of genus $g > 4(k+2)^2+2$.

The last problem we deal with in the thesis is the computation of the rank of the higher Gaussian maps $\Phi^k_{\omega_C}$ when C is a hyperelliptic curve. The problem of understanding the rank of map on special classes of curves dates back to results of Wahl and Ciliberto and Miranda for example (see [68] and [18]) and it was classically motivated by the interest in stratifying the moduli space of curves M_g by the (co)rank of the Wahl map. In recent years the problem appears even more interesting because it has been shown that the corank of the Wahl map is linked to the property of the curve of being coker(Φ_{ω_C})-extendable (see Theorem [16]). Furthermore, higher Wahl maps appear naturally in the study of the local geometry of the Torelli embedding of M_g in A_g, the moduli space of principally polarized abelian varieties and the problem of understanding their behavior on special loci of M_g is then important (see the introduction of Chapter 5 and section 1.3.4). Coming back to our result, we prove the following (see Theorem 5.2.1).

Theorem 7. *Let C be a hyperelliptic curve of genus $g \geq 3$. Then for every $2 \leq k \leq \frac{g-1}{2}$*

$$\mathrm{rank}(\Phi^{2k}_{\omega_C}) = 2g - (4k+1), \tag{0.0.4}$$

and is zero for every $k > \frac{g-1}{2}$.

Structure of the thesis

The thesis is composed of five chapters. Here we briefly mention the main results of each chapter. We refer to the introduction of each chapter for a more detailed overview.

Chapter 1 contains the necessary background on Gaussian maps, curves on Enriques surfaces, and the Hilbert scheme of points of a smooth projective complex surface.

Chapter 2 contains the results which give sufficient conditions for the surjectivity of (higher) Gaussian maps associated with line bundles on surfaces of the type $C_1 \times C_2$ - where C_i, $i = 1, 2$ is a smooth complex projective curve - and on Enriques surfaces. More precisely in the chapter it is proven Theorem 2 and Proposition 5, together with Proposition 1.

Chapter 3 contains the proof the surjectivity of (higher) Gaussian-Prym maps on curves on Enriques surfaces i.e Theorem 1. Moreover it contains the proofs of Proposition 2 and Proposition 3 and Theorem 6,

Chapter 4 is the chapter devoted to the study of Gaussian maps on normalizations of singular curves on Enriques surfaces. Here Theorem 3, Theorem 4 and Theorem 5 are proved.

Chapter 5 contains the computation of the rank of the higher Wahl maps on any hyperelliptic curve, that is Theorem 7.

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Chapter 1

Preliminaries

This introductory chapter is divided into three parts.

In section 1.1 we present some basic properties of Enriques surfaces, and general classical results on smooth complex projective algebraic surfaces. Moreover we present the definition of an important "positivity" measure of line bundles on Enriques surfaces: the ϕ-function, and its relation with elliptic fibrations on Enriques surfaces.

In section 1.2, after a quick overview of the Hilbert scheme as a functor, we present some background on the Hilbert scheme of two points on a smooth complex projective surface S.

In section 1.3 we give the definition of Gaussian maps on a smooth complex variety X associated with two line bundles on X and present (some of) the main results related to the study of Gaussian maps.

1.1 Curves on Enriques surfaces

In this section we present some background material on Enriques surfaces and curves on them. The main reference for this section is [28], but we also refer to [7] and [44]. We start with the definition of an Enriques surface.

Definition 1.1.1. *An Enriques surface is a smooth complex projective surface such that*

$$2\omega_S \simeq \mathcal{O}_S, \quad \omega_S \not\simeq \mathcal{O}_S \tag{1.1.1}$$

and

$$H^1(S, \mathcal{O}_S) = 0, \tag{1.1.2}$$

where ω_S denotes the canonical bundle.

If $Div(S)$ denotes that group of divisors on S recall that there is an intersection form

$$Div(S) \times Div(S) \to \mathbb{Z}$$

which behaves well with respect to the linear equivalence of divisors and then gives an intersection form on

$$Pic(S) \times Pic(S) \to \mathbb{Z}. \tag{1.1.3}$$

We recall that a line bundle L is numerically trivial if $L \cdot L' = 0$ for every other line bundle L'. From the definition 1.1.1 immediately follows that $\omega_S \equiv 0$, i.e. ω_S is numerically trivial. An important tool we will use in some parts of the thesis is a consequence of the so-called Hodge index Theorem (see 1.1.4 below). Let $Num(S)$ be the group of numerically trivial divisors on S. Recall that the Néron-Severi group of S is defined as the group of numerical classes of divisors

$$NS(S) := Div(S)/Num(S). \tag{1.1.4}$$

$NS(S)$ is a finitely generated abelian group and in the case of Enriques surfaces has rank $\rho(S) = 10$. The intersection form 1.1.3 is well-defined on $NS(S)$ and gives a symmetric bilinear form on $NS(S)_{\mathbb{R}} := NS(S) \otimes \mathbb{R}$. Now we recall the Hodge index theorem.

Theorem 1.1.2 (Hodge index Theorem)**.** *Let H be an ample divisor on a smooth complex projective surface X, and suppose that D is a divisor satisfying $D \not\equiv 0$ and $D \cdot H = 0$. Then $D^2 < 0$.*

A consequence of Theorem 1.1.2 is that the signature of the product form on $NS(S)_{\mathbb{R}}$ is of the form $(1, \rho - 1)$, which gives the following immediate corollary.

Corollary 1.1.3. *Let H be a divisor on a smooth complex projective surface X such that $H^2 > 0$, and suppose that D' is a divisor satisfying $D' \not\equiv 0$ and $D' \cdot H = 0$. Then $D'^2 < 0$.*

From this last result one gets the following very useful inequality.

Corollary 1.1.4. *Let X be a smooth complex projective surface and let H be a divisor such that $H^2 > 0$. Let D be another divisor. Then*

$$H^2 \cdot D^2 \leq (H \cdot D)^2. \tag{1.1.5}$$

Proof. This is a consequence of Corollary 1.1.3 applied with $D' := D - \frac{HD}{H^2}H$. □

Now we recall the definitions of big and nef divisors on a surface X.

Definition 1.1.5. *Let L be a line bundle on a projective complex surface X. Then*

L is nef if $L \cdot C \geq 0$ for every irreducible curve C or equivalently for every effective divisor D.

L is big if $max_{m \geq 1}\{\dim(\phi_{L^{\otimes m}}(X))\} = 2$, where $\phi_{L^{\otimes m}}$ is the rational map associated with $L^{\otimes m}$.

On a surface X it is very useful the following characterization of big and nef line bundles (see [52], Theorem 2.2.16).

Proposition 1.1.6. *A nef line bundle L on a projective complex surface X is big if and only if $L^2 > 0$.*

Now we recall some standard vanishing results of cohomology of line bundles on an Enriques surface S. We follow [28], Chapter 2, section 2.1.

First we observe that the Riemann-Roch theorem for an Enriques surfaces S is the following.

Theorem 1.1.7 (Riemann-Roch). *Let S be an Enriques surface and L a line bundle on S. Let $\chi(L) := h^0(L) - h^1(L) + h^2(L)$. Then*

$$\chi(L) = 1 + \frac{L^2}{2}. \tag{1.1.6}$$

If L is a line bundle with $h^0(L) \neq 0$, we have that $h^2(L) = 0$ ([28], Lemma 2.2.1). Then 1.1.6 becomes

$$h^0(L) = h^1(L) + 1 + \frac{L^2}{2}. \tag{1.1.7}$$

Moreover if L is a big and nef line bundle also h^1 vanishes ([28], Theorem 2.1.15).

Theorem 1.1.8. *Let L be a big and nef line bundle on an Enriques surface S. Then*

$$h^1(S, L) = h^1(S, L^\vee) = 0. \tag{1.1.8}$$

In this case we have

Proposition 1.1.9. *Let L be a big and nef line bundle on an Enriques surface S. Then*

$$h^0(L) = 1 + \frac{L^2}{2}. \tag{1.1.9}$$

The formula for the arithmetic genus of an integral (irreducible and reduced) curve C on an Enriques surface S is given by

$$p_a(C) = 1 + \frac{C^2}{2}. \tag{1.1.10}$$

Now we recall Reider's Theorem. It is a result useful to prove that a given line bundle on a projective surface X is base point-free or ample. The following statement is [28], Theorem 2.4.5.

Theorem 1.1.10. *Let X be a smooth and proper surface. Let L be a big, nef and effective invertible sheaf.*

1. *Suppose that $L^2 \geq 5$ and $|L \otimes \omega_X|$ has a base point $x \in X$. Then, there exists an effective divisor E that contains x, such that either*

   a. *$E^2 = 0$ and $L \cdot E = 1$, or*

   b. *$E^2 = -1$ and $L \cdot E = 0$.*

2. *Suppose that $L^2 \geq 9$ and $|L \otimes \omega_X|$ does not separate two points x and $y \in X$ (possibly infinitely near). Then, there exists an effective divisor E that contains x and y, such that*

   a. *$E^2 = 0$ and $L \cdot E \leq 2$, or*

   b. *$E^2 = -1$ and $L \cdot E \leq 1$, or*

   c. *$E^2 = -2$ and $L \cdot E = 0$, or*

   d. *$L^2 = 9$, $E^2 = 1$ and $L \equiv 3E$ in $NS(X)$.*

Elliptic pencils play a central role in the geometry of Enriques surfaces. We recall that an elliptic pencil on an Enriques surface S is a surjective morphism

$$f : S \to \mathbb{P}^1 \tag{1.1.11}$$

with connected fibers and such that the general fiber is a smooth curve of genus 1. Among the singular fibers, there are exactly two multiple fibers:

$$2F \text{ and } 2F', \tag{1.1.12}$$

Moreover they satisfy $\mathcal{O}_S(F - F') \simeq \omega_S$ (see for Example [7], Lemma 17.1.).

Definition 1.1.11. *F and F' are called half-fibers of the elliptic fibration f.*

Remark 1.1.12. Every Enriques surfaces carries an elliptic fibration (see [7], Theorem 17.5). A half-fiber F satisfies $H^0(S, F) = 1$, it is a nef isotropic divisor (that is $F^2 = 0$) and its numerical class is primitive in $NS(S)$. Vice versa if F is a primitive, isotropic and nef divisor then either F or $-F$ is effective and (if for example F is effective) $|2F|$ is a base point-free linear system of dimension 1. Then it is an elliptic pencil (using the formula for the arithmetic genus). For more details about these last statements see for example [7], Proposition 16.1 and [28], Proposition 2.2.8.

Now we recall the definition of the ϕ-function which was introduced by Cossec. It is a measure of the positivity of line bundles on Enriques surfaces as we will see.

Definition 1.1.13. *Let S be an Enriques surface and L be a big line bundle such that $L^2 > 0$.*

$$\phi(L) := min\{|L \cdot F| : F \in \mathrm{Pic}(S), F^2 = 0, F \not\equiv 0\}.$$

The value $\phi(L)$ is actually computed by half-pencils ([28], Lemma 2.4.10). Hence $\phi(L)$ gives the minimum value of the intersection between L and elliptic pencils on S (divided by 2). Moreover $\phi(L)$ is always bounded from above by the square root of L^2 ([28], Proposition 2.4.11)

$$\phi(L)^2 \leq L^2. \tag{1.1.13}$$

The $\phi-$function measures the "regularity" of the (rational) map associated with a line bundle. This is well-expressed by the following Theorem (see [28], Theorem 2.4.14, Theorem 2.4.18 and 2.4.19).

Theorem 1.1.14. *Let S be an Enriques surface and let L be a big and nef line bundle on S. Then*

- *$|L|$ is base point-free if and only if $\phi(L) \geq 2$.*
- *$|L|$ is very ample if and only if $\phi(L) \geq 3$ and there exists no effective divisor E on S such that $E^2 = -2$.*

Actually Theorem 1.1.14 is more general, as we will see in Theorem 2.2.5.

If R is an integral curve on S with $R^2 = -2$, then R is isomorphic to $\mathbb{P}^1$ ([28] Proposition 2.1.6). Such a curve is called -2 curve or nodal curve. We end this section with an important definition.

Definition 1.1.15. *An Enriques surface is said to be unnodal if it does not contain any -2 curves. Otherwise it is called nodal.*

There is a 10-dimensional smooth and irreducible moduli space parametrizing Enriques surfaces, which we note (as in [15]) by $\mathcal{E}$. The isomorphism classes of nodal Enriques surfaces forms a divisor in $\mathcal{E}$, hence the general Enriques surface is unnodal.

1.2 The Hilbert scheme of points of on a surface

The main references for this section are [33], [34], [42] and [55]. We start presenting the Hilbert scheme as a functor. The hypothesis will be at first general but then we will focus on the specific case we are interested in.

Let X be a projective variety over an algebraically closed field k and let S be a scheme. We recall the definition of family of subschemes of X.

Definition 1.2.1. *A family of subschemes of X parametrized by S is a closed subscheme*

$$T \subset X \times S, \tag{1.2.1}$$

such that restriction of the projection map $T \to S$ is flat.

Denote by $(T \to S)$ a family of closed subschemes (of some projective variety X). Denote by Schemes the category of schemes and by Sets the category of sets and consider the contravariant functor $\mathcal{H}\text{ilb}(\text{X})$ from Schemes to Sets which associates to a scheme S the set of families of closed subschemes parametrized by S.

Definition 1.2.2. *For any $S \in \text{Schemes}$ and for any $\phi : W \to S$ morphism of schemes,*

$$\mathcal{H}\text{ilb}(\text{X})(S) := \{(T \to S) : \ T \textit{ is a closed subscheme of} X \times S, \ \textit{flat over } S\ \};$$

$$\mathcal{H}\text{ilb}(\text{X})(\phi)(T \to S) := (T \times_S W \to W).$$

The family $(T \times_S W \to W)$ is called the pullback family through the morphism ϕ and we denote it by $\phi^*(T \to S)$. Now let L be a very ample invertible sheaf on X, $T \subset X \times S$ a family of subschemes of X and denote by

$$\begin{aligned} &p : T \to X, \quad \text{and} \\ &q : T \to S \end{aligned}$$

the two projections. Moreover for every $s \in S$, denote by T_s the fiber $q^{-1}(s)$.

Definition 1.2.3. *The Hilbert polynomial of T in s is defined as*

$$P_s(Z)(m) := \chi(\mathcal{O}_{T_s} \otimes_{\mathcal{O}_T} p^* L^m).$$

P_s is a polynomial in the variable m with rational coefficients. Moreover, since $p : T \to S$ is flat, if S is connected, $P_s(T)$ is independent of s. Vice versa let $P \in \mathbb{Q}[m]$ a polynomial. Then one can consider the natural subfunctor of $\mathcal{H}\mathrm{ilb(X)}$ of families of subschemes of X with Hilbert polynomial equal to P.

Definition 1.2.4. $\mathcal{H}\mathrm{ilb^P(X)}$ *is the subfunctor of* $\mathcal{H}\mathrm{ilb(X)}$ *defined on objects of* Schemes *as:*

$$\mathcal{H}\mathrm{ilb^P(X)}(S) :=$$
$$\{T \subset X \times S : \ T \textit{ closed subscheme flat over } S, \ P_s(T) = P \ \forall s \in S\}.$$

It is a classical result that dates back to Grothendieck, that $\mathcal{H}\mathrm{ilb(X)}$ and $\mathcal{H}\mathrm{ilb^P(X)}$ are representable by two k-schemes Hilb(X) and $\mathrm{Hilb^P(X)}$, the latter being a projective scheme.

Remark 1.2.5. The fact that $\mathcal{H}\mathrm{ilb^P(X)}$ is representable by a projective k-scheme $\mathrm{Hilb^P(X)}$ means that that there exists a universal family $T_X \subset X \times \mathrm{Hilb^P(X)}$ of closed subschemes of $\mathrm{Hilb^P(X)}$ with Hilbert polynomial equal to P such that for any family $(T \to S)$ there exists a unique $\Phi_T : S \to \mathrm{Hilb^P(X)}$ such that $(T \to S) \simeq \phi_T^*(T_X)$, with the natural notion of isomorphism for families of closed subschemes of X. This in particular gives that the k-valued points of $\mathrm{Hilb^P(X)}$ are in one to one correspondence with closed subschemes of X with Hilbert polynomial P.

Recall that if X is a projective variety over an algebraically closed field k and if $Z \subset X$ is a 0-dimensional subscheme, the length of Z is defined as the dimension of $H^0(Z, \mathcal{O}_Z)$. Let $n \in \mathbb{N}$, $n \geq 1$ be a natural number and consider the constant polynomial $P = n$.

Definition 1.2.6. *The Hilbert scheme of subschemes of X of length n, also called the Hilbert scheme of n points on X, is the projective scheme that represents the functor* $\mathcal{H}\mathrm{ilb^n}(X)$.

The Hilbert schemes of n points on X is usually denoted by

$$\mathrm{X}^{[n]}\,. \tag{1.2.2}$$

From remark 1.2.5 it follows that closed points of $\mathrm{X}^{[n]}$ are in one-to-one correspondence with closed subschemes of X of length n.

Remark 1.2.7. Actually the definition of the (relative) Hilbert functor and the representability results hold in much more general situations (see [42], Definition 1.1.1).

As before let X be a projective scheme over an algebraically closed field k and $n \geq 1$ an integer. Let $X^{(n)}$ be the $n-$fold symmetric power of X, i.e. the quotient of X^n by the action of the symmetric group Σ_n. The two geometric objects $\mathrm{X}^{[n]}$ and $X^{(n)}$ are linked by the so called Hilbert-Chow morphism. This is a fundamental result of Mumford and Fogarty ([35], Theorem 5.4).

Theorem 1.2.8. *There is a canonical morphism*

$$\rho : \mathrm{X}^{[n]}_{red} \to X^{(n)}. \tag{1.2.3}$$

where, if Z is a 0-dimensional closed subscheme whose support consists of distinct points $\{p_1, ..., p_n\}$ then the image of Z is given by $p_1 + ... + p_n \in S^{(n)}$.

For an overview of the construction of the Hilbert-Chow morphism see also [33].

For the rest of the section X will be a smooth projective complex surface, which we will denote by S. We want to recall some fundamental results proved by Fogarty ([33], Theorem 2.4 and Corollary 2.6).

Theorem 1.2.9. *Let S be a smooth complex projective surface. Then $S^{[n]}$ is a smooth variety of dimension $2n$ and the Hilbert-Chow morphism is birational.*

Since $S^{(n)}$ for $n \geq 2$ is never smooth, we get in particular that $\rho : S^{[n]} \to S^{(n)}$ is a resolution of singularities. Moreover, the resolution is crepant, that is

$$K_{S^{[n]}} = \rho^* K_{S^{(n)}}, \tag{1.2.4}$$

where $K_{S^{[n]}}$ and $K_{S^{(n)}}$ are the canonical divisors. Now consider the case $n = 2$. In [34], Lemma 4.4, it is shown that $(\mathrm{S}^{[2]}, \rho)$ is isomorphic to the blow up of $S^{(2)}$ along the diagonal. We denote by E the exceptional divisor of the Hilbert-Chow morphism $\rho : \mathrm{S}^{[2]} \to S^{(2)}$. The divisor parametrizes the locus of non-reduced 0-dimensional closed subschemes of S of length 2. In [34] it is also shown that the blow-up of S^2 along the diagonal, which we denote by $\widetilde{S}^2$, is isomorphic to the fiber product $S^2 \times_{S^{(2)}} \mathrm{S}^{[2]}$. In particular, it is shown that there is a commutative diagram:

$$\begin{array}{ccc} \widetilde{S}^2 & \xrightarrow{\eta} & \mathrm{S}^{[2]} \\ \downarrow & & \downarrow \rho \\ S^2 & \longrightarrow & S^{(2)}. \end{array} \tag{1.2.5}$$

The map η is a degree 2 morphism which is ramified along the exceptional divisor $\widetilde{E} \subset \widetilde{S}^2$ and the restriction

$$\widetilde{E} \xrightarrow{\eta|_{\widetilde{E}}} E, \tag{1.2.6}$$

is an isomorphism.

If the surface S satisfies $H^1(S, \mathcal{O}_S) = 0$, for example, as in the case of K3, or Enriques surfaces, Fogarty proves a fundamental result concerning the structure of the Picard group of $S^{[n]}$. More precisely:

Theorem 1.2.10.

$$Pic(S^{[n]}) \simeq Pic(S) \oplus \mathbb{Z}B. \tag{1.2.7}$$

In the statement, B is a non-effective class in $Pic(S^{[n]})$ with the property that $2B \sim E$.

Notations 1.2.11. In the rest of the thesis we will denote by $2B$ the exceptional divisor E of the Hilbert-Chow morphism.

The embedding of $Pic(S)$ in $Pic(S^{[n]})$ is given through the following procedure. If $\pi_i : S \times S \to S,\ i = 1, 2$ are the two projections, and $L \in \mathrm{Pic}(S)$ is a line bundle on S, then we can consider the line bundle $L \boxtimes L = \pi_1^* L \otimes \pi_2^* L$ in S^2. This is invariant by the action of Σ_n and naturally descends to a line bundle over $S^{(n)}$. Pulling it back through ρ, we obtain a line bundle $\tilde{L}$ over $S^{[n]}$. This procedure gives an injective group homomorphism of $\mathrm{Pic}(S)$ into $\mathrm{Pic}(S^{[n]})$. We have $\omega_{S^{[n]}} = \widetilde{\omega_S}$ for the canonical bundle.

1.3 Gaussian maps

In this section we present the central topic of the thesis: Gaussian maps. We start by recalling the definition, their different interpretations and some of the most important results in which they appear and play a significant role.

1.3.1 Definition

A standard reference for the definition of Gaussian map is [66]. Let X be a smooth projective variety, let $k \geq 0$ be an integer, and let L and M be two line bundles on X. Let $q_i : X \times X \to X,\ i = 1, 2$ be the two projections. Consider the short exact sequence given by the inclusion of the $(k+1)-$power of the ideal of the diagonal Δ in $X \times X$, and tensor it with $q_1^* L \otimes q_2^* M$, which we denote by $L \boxtimes M$.

$$0 \to I_\Delta^{k+1} \otimes L \boxtimes M \to I_\Delta^k \otimes L \boxtimes M \to I_\Delta^k / I_\Delta^{k+1} \otimes L \boxtimes M \to 0. \tag{1.3.1}$$

Definition 1.3.1 (kth Gaussian map)**.** *The kth Gaussian map associated with L and M is defined as the map induced at the level of global sections:*

$$\begin{array}{ccc} \Phi^k_{L,M} : H^0(X \times X, I^k_\Delta \otimes L \boxtimes M) & \longrightarrow & H^0(X \times X, I^k_\Delta / I^{k+1}_\Delta \otimes L \boxtimes M) \\ & & \downarrow \simeq \\ & & H^0(X, S^k\Omega^1_X \otimes L \otimes M). \end{array} \tag{1.3.2}$$

For $k = 0$ the Gaussian map $\Phi^0_{L,M}$ is the restriction to the diagonal

$$\Phi^0_{L,M} : H^0(X \times X, L \boxtimes M) \to H^0(X, L \otimes M),$$

which is naturally identified by Künneth formula with the multiplication map on global sections

$$\Phi^0_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M).$$

Moreover, from the definition, it follows that the domain of the kth Gaussian map, $k \geq 1$ is the kernel of the previous one. We will denote by $R^k(L, M)$ the kernel of $\Phi^k_{L,M}$. In particular, for the first Gaussian map we have:

$$\Phi_{L,M} : R^0(L, M) \to H^0(\Omega^1_X \otimes L \otimes M). \tag{1.3.3}$$

If $\alpha = \sum l_i \otimes m_i \in \ker(\phi_{L,M})$, $l_i = f_i s$, $m_i = g_i t$, where s and t are two local generators of L and M, respectively, it is locally given by $\Phi_{L,M}(\alpha) = \sum(f_i dg_i - g_i df_i) \otimes s \otimes t$ ([67], Lemma 5.3). Now let $L = M$. Notice that

$$R^0(L, L) = I_2(L) \oplus \Lambda^2 H^0(L), \tag{1.3.4}$$

where we have denoted by $I_2(L)$ the kernel of the restriction of Φ^0 to $S^2H^0(L)$ and used the decomposition $H^0(L)^{\otimes 2} \simeq S^2H^0(L) \oplus \Lambda^2 H^0(L)$. From the local description of Φ_L we have that that it is identically zero on $I_2(L)$. One usually identifies Φ_L with its restriction to $\Lambda^2 H^0(L)$, which we will denote by μ_L

$$\mu_L : \Lambda^2 H^0(L) \to H^0(\Omega^1_X \otimes L^{\otimes 2}). \tag{1.3.5}$$

Observe that again, if $\alpha = \sum l_i \wedge m_i \in \Lambda^2 H^0(L)$, $l_i = f_i s$, $m_i = g_i s$, where s is a local generator of L, the local description of μ_L becomes

$$\mu_L(\alpha) = \sum(f_i dg_i - g_i df_i) \otimes s \otimes s. \tag{1.3.6}$$

Another important description of the first Gaussian map associated with two line bundles L and M is obtained when L is a very ample line bundle giving an embedding $\phi_L : X \hookrightarrow \mathbb{P}^r$. Let M_L be defined by

$$0 \to M_L \to H^0(C, L) \otimes \mathcal{O}_X \to L \to 0,$$

then $\phi_L^* \Omega^1_{\mathbb{P}^r}(1) = \Omega^1_{\mathbb{P}^r}(1)_{|X} \simeq M_L$. Consider indeed the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(-1)^{r+1} \to \mathcal{O}_{\mathbb{P}^r} \to 0,$$

and tensor it with $\mathcal{O}_{\mathbb{P}^r}(1)$:

$$0 \to \Omega^1_{\mathbb{P}^r}(1) \to H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \otimes \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(1) \to 0.$$

Pulling it back by ϕ_L we obtain

$$0 \to \phi_L^* \Omega_{\mathbb{P}^r}(1) \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0,$$

and so we conclude.

In particular it follows that $H^0(\Omega_{\mathbb{P}^r}(1)_{|X} \otimes M) \simeq H^0(M_L \otimes M) \simeq R^0(L, M)$. Now consider a twist by $L \otimes M$ of the conormal exact sequence:

$$0 \to N^\vee_{X/\mathbb{P}^r} \otimes L \otimes M \to M_L \otimes M \to \Omega^1_X \otimes L \otimes M \to 0, \qquad (1.3.7)$$

Under the aforementioned identification:

$$\Phi_{L,M} : H^0(X, M_L \otimes M) \to H^0(X, \Omega^1_X \otimes L \otimes M), \qquad (1.3.8)$$

i.e. $\Phi_{L,M}$ is the map induced at the level of global sections in 1.3.7 ([65], (7.5.2), or [66], Proposition 1.1.10).

1.3.2 Curves on surfaces and Gaussian maps

If C is a smooth irreducible curve we denote by ω_C the canonical bundle. Gaussian maps make their appearance in [67] where it is proven the following fundamental theorem([67], Theorem 5.9).

Theorem 1.3.2 (Wahl)**.** *If C is a curve which sits on a $K3$ surface, then Φ_{ω_C} is not surjective.*

The same theorem is also proved with very different methods in [8].

Definition 1.3.3. *The Gaussian map Φ_{ω_C} (or equivalently μ_{ω_C}) is called Wahl map or Gaussian-Wahl map.*

The importance of this result becomes more evident when compared to the following one of Ciliberto, Harris and Miranda ([20], Main Theorem).

Theorem 1.3.4. *If C is a general curve of genus $g \geq 10$ with $g \neq 11$, then the Wahl map Φ_{ω_C} is surjective.*

Remark 1.3.5. The proof is based on a degeneration argument, that is, it is proved that the map is surjective for some nodal curves in the boundary of the Deligne-Mumford compactification $\bar{M}_g$: since the condition of being surjective is Zariski open, they conclude.

From Theorem 1.3.2 and Theorem 1.3.4 it follows that the surjectivity of the Wahl map for a curve C gives a natural obstruction for the curve to lie on a $K3$ surface. When the genus of the curve less than or equal to 9, then the Wahl map cannot be surjective for dimensional reasons, while for $g = 11$ the Wahl map is never surjective since by a result of Mori and Mukai the general curve of genus 11 lies on a $K3$ surface. We also mention that Theorem 1.3.4 was later reproved by Voisin in [64] with very different techniques.

We also present a similar but much weaker result than Theorem 1.3.4. The following Theorem can be found in [68], Theorem 4.11.

Theorem 1.3.6. *Let C_i be a complete smooth curve of genus g_i $(i = 1, 2)$, ω_i the canonical line bundle on C_i, and D_i a divisor on C_i of degree d_i. Suppose*

(a) D_i is very ample on C_i.

(b) $d_i > max(0, 4 - 4g_i)$.

(c) On C_i $\omega_i(D_i)$ is normally generated and $\Phi_{\omega_i(D_i)}$ is surjective.

(d) $g_2 \geq 2$.

Set $X = C_1 \times C_2$. Then the general element of the complete linear system $|p_1^* D_1 \otimes p_2^* D_2|$ is a smooth curve for which Φ_{ω_C} is surjective and

$$2g(C) - 2 = d_1(2g_1 - 2) + d_2(2g_2 - 2) + 2d_1 d_2. \tag{1.3.9}$$

Theorem 1.3.6 is based on relating the Gaussian maps $\Phi_{\omega_{C_i}(D_i)}$ on C_i with the Gaussian maps Φ_{ω_C} for $C \in |p_1^* D_1 \otimes p_2^* D_2|$. See also ([68], Lemma 4.12).

Now let $S \subset \mathbb{P}^r$ be a smooth complex surface in some projective space, and let C be a hyperplane section of S. We can ask if there is some natural Gaussian map that behaves as the Wahl map, i.e. that "see" the fact the curve C lies on S. The answer is yes if the embedding $C \hookrightarrow \mathbb{P}^{r-1}$ is given by a complete linear system. Indeed this the theorem of L'vovsky ([54], Corollary 2).

Theorem 1.3.7 (L'Vovsky). *Let C be a smooth curve and L a very ample line bundle on C. If the map $\Phi_{\omega_C,L}$ is surjective, then C, in the embedding given by $|L|$, is not a hyperplane section of a projective surface other than a cone.*

In particular if C, in the embedding given by L, is the hyperplane section of a surface different than a cone, then $\Phi_{\omega_C,L}$ is not surjective. This result generalizes Theorem 1.3.2 when C is a hyperplane section of a $K3$ surface.

Remark 1.3.8. Theorem 1.3.7 is actually a corollary of a more general theorem of Zak and L'vovsky (see [54], Theorem 0.1 for more details).

Similar generalizations of Wahl result (1.3.2) have been made by other authors. For example Wahl proves the following(see [66], Theorem 3.10).

Theorem 1.3.9. *Let $C \subset \mathbb{P}^r$ be a projectively normal embedding of a curve, with $L = \mathcal{O}_C(1)$. Suppose that $\Phi_{\omega_C,L}$ is surjective. Then if Y is any normal variety, with hyperplane section C for which the normal bundle is L, then Y is isomorphic to the projective cone over C.*

See also [3] and [4] for other similar results.

Now we want to discuss the case of Enriques surfaces. Let S be an Enriques surface and H a very ample line bundle on S. Let C be a smooth curve in $|H|$ and set $\alpha := \omega_{S_{|C}}$ An immediate consequence of Theorem 1.3.7, or of a similar result of Ballico and Ciliberto (see [4], Theorem of 1.1), is the following.

Corollary 1.3.10. *Let S be an Enriques surface and let H be a very ample line bundle on S. Let C be a smooth curve in $|H|$ and set $\alpha := \omega_{S_{|C}}$. Then the Gaussian map on $\Phi_{\omega_C,\omega_C\otimes\alpha}$ is not surjective.*

Proof. Using that for an Enriques surface $H^1(S, \mathcal{O}_S) = 0$, it is immediate to see that the curve C is embedded by the complete linear system $|H_{|C}|$. Since by adjunction, $H_{|C} = \omega_C \otimes \alpha$, one concludes by applying Theorem 1.3.7. □

Now if (C, α) is a pair as in 1.3.10 then we say it is a Prym curve coming from an Enriques surface. Since there is a (coarse) moduli space R_g parametrizing isomorphism classes of Prym curve of genus g, it is natural to ask - as in the case of $K3$ surfaces - how the Gaussian map $\Phi_{\omega_C,\omega_C\otimes\alpha}$ behaves for a general element in R_g. The answer is given by the following Theorem of Verra ([19] Theorem 1.5).

Theorem 1.3.11. *If $g \geq 12$, $g \neq 13, 19$, then for a general point (C, α) in R_g the map $\Phi_{\omega_C,\omega_C\otimes\alpha}$ is surjective.*

Therefore, as in the case of $K3$ surfaces one has that the surjectivity of the mixed Gaussian-Prym map $\Phi_{\omega_C,\omega_C\otimes\alpha}$ for a Prym curve gives an obstruction for the curve to come from an Enriques surface.
Until this point, we have seen only how the surjectivity of some Gaussian maps on a curve C tells that the curve cannot lie on some specific class of surfaces. In recent years there have been important results also in the other direction and in relation to higher extendability. Indeed Arbarello, Bruno and Sernesi have proved the following ([1]), Theorem 1.1.).

Theorem 1.3.12. *Let C be a Brill-Noether-Petri curve of genus $g \geq 12$. Then C lies on a polarised $K3$ surface, or on a limit thereof, if and only if its Wahl map is not surjective.*

A first (fundamental) step to prove Theorem 1.3.12 is the following ([1], Corollary 1.4).

Theorem 1.3.13. *Let C be a canonical curve of genus $g \geq 11$ with $Cliff(C) \geq 3$. Then C is extendable if and only if Φ_{ω_C} is not surjective.*

Here by extendable, they mean that there exists a projective surface $S \subset \mathbb{P}^g$ having C as a hyperplane section. This is of course a converse of Wahl's Theorem. Equivalently, this last theorem says that if the corank of the Wahl map on (sufficiently general curve C) is strictly greater than 0, then the curve C is extendable. This result has been generalized to higher dimensional extendability by Ciliberto, Dedieu and Sernesi ([16], Theorem 2.1).

Theorem 1.3.14. *Let C be a smooth genus g curve with Clifford index $Cliff(C) \geq 3$, and let r be a non-negative integer. We consider the following two propositions:*

(i) $\mathrm{cork}(\Phi_{\omega_C}) \geq r + 1$.

(ii) There exixts an arithmetically Gorenstein normal variety Y in $\mathbb{P}^{g+r}$, not a cone, with $\dim(Y) = r + 2$, $\omega_Y = \mathcal{O}_Y(-r)$, which has a canonical image of C as a section with a $(g-1)-$dimensional linear subspace of $\mathbb{P}^{g+r}$ (in particular, the curve $C \subset \mathbb{P}^{g-1}$ is $(r+1)-$extendable).

If $g \geq 11$, then (i) implies (ii). Conversely, if $g \geq 22$ and the canonical image of C is a hyperplane section of some smooth $K3$ surface in $\mathbb{P}^g$, then (ii) implies (i).

In the statement cork stands for corank. Actually they prove that if $r := \mathrm{cork}(\Phi_{\omega_C}) - 1$ the extension given by the Theorem is "universal", which informally means that it is an extension of all the possible surfaces which have

C as a hyperplane section. We refer to [16] for many other results connected with these questions. We also refer to [53] for an overview of extendability questions and the connection with Gaussian maps.

Remark 1.3.15. More generally, if C is a smooth irreducible curve, and L is a very ample line bundle on C, it is an important and interesting question to study the extensions (if any) of the curve C in the embedding given by L. In [14] are described the possible extensions of $\Phi_L(C)$ (for the degree of L in some interesting range), and the relation is explored between the elements of $\mathbb{P}(\mathrm{coker}(\Phi_{\omega_C,L}))$ and the possibile extensions of the embedded curve. Moreover in some cases it is proven the existence of a universal extension.

1.3.3 Higher order Gaussian maps

In this section we present some results concerning higher Gaussian maps. We start with a Theorem of Colombo,Frediani and Pareschi ([22], Theorem A).

Theorem 1.3.16. *Let C be a curve contained in an abelian surface S. Then the corank of $\Phi^2_{\omega_C}$ is at least 2.*

Remark 1.3.17. Actually the statement of 1.3.16 is more refined. Indeed they prove that the image of $\Phi^2_{\omega_C}$ is contained in the image of $S^2H^0(\Omega_S) \otimes H^0(\omega_C^2)$ inside $H^0(\omega_C^{\otimes 4})$, via the natural multiplication map (see [22] for more details). Moreover they also prove that for curves lying in a sufficiently positive linear the (first) Wahl map is surjective (Theorem B).

As in the case of the first Wahl map Φ_{ω_C} for $K3$ surfaces and the first mixed Gaussian-Prym map $\Phi_{\omega_C,\omega_C\otimes\alpha}$ for Enriques surfaces, this result has to be compared with another result giving the surjectivity for the general curve in the moduli space of curves. This is indeed the main result of Calabri, Ciliberto, and Miranda in [12].

Theorem 1.3.18. *The second Gaussian map $\Phi^2_{\omega_C}$ for C a general curve is surjective for $g \geq 18$.*

This is the analogous result of Theorem 1.3.4 for $\Phi^2_{\omega_C}$, and together with Theorem 1.3.16, it says that the surjectivity of the second Wahl-Gaussian map gives an obstruction for a curve to lie on an abelian surface. The proof is again based on a degeneration argument.

Remark 1.3.19. Actually in [12] it is proven more than the statement in Theorem 1.3.18. To better explain it, let us introduce the map $\mu^2_{\omega_C}$.

$\Phi^2_{\omega_C}$ is defined on $\ker(\Phi_{\omega_C})$ and the latter decomposes as $I_2(\omega_C) \oplus \ker(\mu_{\omega_C})$, where $I_2(\omega_C)$ is the kernel of the restriction of $\Phi^0_{\omega_C}$ to the symmetric tensors $S^2H^0(\omega_C)$. Since $\Phi^2_{\omega_C}$ identically vanishes on antisymmetric tensors, one usually identifies $\Phi^2_{\omega_C}$ with its restriction to $I_2(\omega_C)$, which is usually denoted by $\mu^2_{\omega_C}$ (we refer to section 5.1, Chapter 5 for more details). Ciliberto, Harris and Miranda prove in [12] that $\mu^2_{\omega_C}$ has maximal rank for the general curve of g as soon as it is possible, i.e. it is injective when $g \leq 17$ and surjective when $g \geq 18$.

While the first Gaussian-Wahl map is not surjective for a curve on a $K3$ surface, this does not happen for the second Wahl map $\Phi^2_{\omega_C}$. This a result of Colombo and Frediani Theorem 3.1, [23]).

Theorem 1.3.20. *If X is a general polarized $K3$ surface of degree $2g-2$ with $g > 280$, and if C is a general hyperplane section of X, then $\Phi^2_{\omega_C}$ is surjective.*

Remark 1.3.21. We mention that the the surjectivity result of Theorem 1.3.18 was first proved in [23] for $g > 280$. Indeed it is an immediate consequence of Theorem 1.3.20. Before that it was known for infinitely many genera $g \geq 71$ by considering curves in the product of two curves. See Theorem 3.2.2.

For higher Wahl maps $\Phi^k_{\omega_C}$ with $k \geq 3$, an answer is given by Rios Ortiz in [58]. Indeed he proves the surjectivity of higher Wahl maps for curves on $K3$ surfaces ([58], Theorem 4.7)

Theorem 1.3.22. *Let (S, L) be a polarized $K3$ surface of degree $2d$ and $k > 1$ an integer. If $d \geq 4(k+2)^2 + \frac{5}{4}$ and C is a smooth hyperplane section of S, then $\Phi^k_{\omega_C}$ is surjective.*

Thus obtaining a result for the general curve ([58], Theorem D)

Theorem 1.3.23. *Let $k > 1$ be an integer. Then for a general curve of genus $g > 4(k+2)2 + 2$ the k-th higher Gaussian map is surjective.*

The proof of Theorem 1.3.22 relies on studying the surjectivity of the Gaussian maps on the $K3$ surface (together with some other restriction maps) as in [23]. In order to prove the surjectivity of some Gaussian maps on the surface, Rios Ortiz proves the following interesting result ([58], Theorem A).

Theorem 1.3.24. *Let S be a projective surface with $H^1(S, \mathcal{O}_S) = 0$ and let L be a line bundle on S. If $H^1(\mathrm{S}^{[2]}, \tilde{L} - (k+2)B) = 0$, then Φ^k_L is surjective,*

where we refer for notations to the end of section 1.2.

1.3.4 Gaussian maps and second fundamental form

In this subsection we briefly present the relation between Gaussian maps and the study of the (local) geometry of the moduli space of curves M_g inside the moduli space of principally polarized abelian varieties A_g.

Let M_g denote the coarse moduli space of smooth complex projective genus g curves and A_g the coarse moduli space of principally polarized Abelian varieties of dimension g over $\mathbb{C}$, and consider the Torelli map

$$j : M_g \to A_g \tag{1.3.10}$$

that associates to the class of a curve $[C]$ the class $[J(C)] \in A_g$ of its Jacobian, which is a principally polarized abelian variety, with polarization given by the theta divisor. By Torelli Theorem, the map j is injective and it is a natural problem to study the geometry of $j(M_g)$ in A_g. Moreover the map is ramified on the hyperelliptic locus and is an immersion outside. Hence, if $x = [C] \in M_g$ is the class of a non-hyperelliptic curve, we have an exact sequence

$$0 \to T_x M_g \stackrel{dj_x}{\to} T_{j(x)} A_g \stackrel{\pi}{\to} N_x \to 0 \tag{1.3.11}$$

where N denotes normal bundle of $j(M_g) \subset A_g$. Since the tangent spaces of the moduli spaces at the point x are given by $H^1(C, T_C)$ and $S^2 H^0(C, \omega_C)^*$, respectively, we get:

$$0 \to H^1(C, T_C) \stackrel{dj_x}{\to} S^2 H^0(C, \omega_C)^* \stackrel{\pi}{\to} N_x \to 0, \tag{1.3.12}$$

where dj_x is the dual of the multiplication map $\mu^0_{\omega_C} : S^2 H^0(\omega_C) \to H^0(\omega_C^{\otimes 2})$. When we think the moduli space A_g as a complex analytic orbifold it comes equipped with a natural symmetric orbifold metric. We denote by ∇ its associated Levi-Civita connection. The map

$$\mathrm{II} : S^2 T M_g \to N, \quad \mathrm{II}(X \odot Y) = \pi(\nabla_X(Y))$$

is the second fundamental form of the Torelli map with respect to the natural metric on A_g.
Dualizing 1.3.12 we have:

$$0 \to I_2(\omega_C) \to S^2 H^0(C, \omega_C) \stackrel{\mu^0_{\omega_C}}{\to} H^0(C, \omega_C^{\otimes 2}) \to 0.$$

It follows that $N_x^* = I_2(K_C)$ and one can study II_x via its dual map

$$\rho_x : I_2(\omega_C) \to S^2 H^0(C, \omega_C^{\otimes 2}). \tag{1.3.13}$$

The study of ρ_x is not easy and it is complicated by the fact that ρ_x does not vary holomorphically in families. A crucial result in the study of the second fundamental form is the connection between ρ_x and the restriction of the second Wahl map

$$\Phi^2_{\omega_C} : I_2(\omega_C) \oplus \Lambda^2 H^0(\omega_C) \to H^0(C, \omega_C^{\otimes 4}), \tag{1.3.14}$$

to $I_2(\omega_x)$, i.e. $\mu^2_{\omega_C}$ (recall Remark 1.3.19).

Indeed it has been proven in [27] that the composition of ρ_x with the multiplication map $S^2H^0(C, \omega_C^{\otimes 2}) \to H^0(C, \omega_C^{\otimes 4})$ is precisely $\mu^2_{\omega_C}$.

Then after composing with the multiplication map we obtain a map (i.e. $\mu^2_{\omega_C}$) which varies holomorphically and therefore it is more suitable to be studied in the context of algebraic geometry. $\mu^2_{\omega_C}$ provides a useful tool to understand the second fundamental form of $j(M_g)$ in A_g, and hence of its local geometry. More precisely, this fact has been useful to approach the Coleman-Oort conjecture, which for high genus, predicts the non-existence of special subvarieties Z of A_g, that are generically contained in the Torelli image (i.e. $Z \subset \overline{j(M_g)}$ and $Z \cap j(M_g) \neq \emptyset$). For the many omitted details in this section and for more others we refer to [25], [21] and [38].

Chapter 2

Higher Gaussian maps on some special surfaces

This chapter is devoted to the study of (higher) Gaussian maps on some class of smooth projective surfaces. It is divided in two parts.

In section 2.1 it is considered the case of a surface X which is the product of two smooth projective curves $C_1 \times C_2$. In loc. cit. there are two main results: Proposition 2.1.7 which is a statement about the surjectivity of "mixed" Gaussian maps $\Phi^1_{L,M}$ where L and M are line bundles on $X = C_1 \times C_2$ of the form $L = p_1^*L_1 \otimes p_2^*L_2$ and $M = p_1^*M_1 \otimes p_2^*M_2$ where L_i and M_i are line bundles on C_i, $i = 1, 2$. This result can be also found in [30], Proposition 3.1. The other result is Proposition 2.1.9. This is a statement regarding the surjectivity of Gaussian maps of the form Φ^k_L, $k \geq 2$, where L is a line bundle on X of the form $L = p_1^*L_1 \otimes p_2^*L_2$. In order to prove this statement we prove Lemma 2.1.1, Lemma 2.1.3 and Lemma 2.1.4. These allow us to relate the Gaussian maps we want to study on the surface $C_1 \times C_2$ to the Gaussian maps on the curves C_1 and C_2.

Proposition 2.1.7 and Proposition 2.1.9 will be applied in section 3.2 and section 4.3.1 to study Gaussian map on curves on the product surface X.

In section 2.2 it is considered the case of an Enriques surface S. The results contained in this section can be found in [31]. The main result of this section is Theorem 2.2.12 which is a surjectivity statement about (higher) Gaussian maps on an unnodal Enriques surfaces. The proof relies on Theorem 1.3.24 of Rios Ortiz and follows a similar approach as in [58]. More specifically it relies on proving a positivity statement (Proposition 2.2.9) for line bundles on the Hilbert scheme of 2-points on an Enriques surface. Theorem 2.2.12 will be then used, together with other results, in section 3.1 to prove Theorem 3.1.3 which also appears in [30].

2.1 Gaussian maps on product of curves

As explained in the introduction of the chapter, in this section we give sufficient conditions for the surjectivity of mixed Gaussian maps on surfaces that are the product of two curves. The results will then be used in section 3.2 and section 4.3.1 to study Gaussian maps on curves on the product surface X.

The central idea to study Gaussian maps on X is to relate them with Gaussian maps on the curves. This idea was already first used by Wahl ([67], Lemma 4.12) for the first Wahl map and then by Colombo and Frediani in [26], Theorem 3.1 for the second.

We start proving some lemmas we will use in Proposition 2.1.7. They are probably well known, but we were not able to find a reference. Even if we will use the lemmas in a very specific situation, i.e. when X_1 and X_2 (below) are smooth complex projective curves, we will prove them under more general hypothesis.

Let k be an algebraically closed field of characteristic different than 2, let X_i, $i = 1, 2$ be two separated k-schemes and let $X := X_1 \times X_2$ be the fiber product over Spec(k). Let $\phi_1 : (X_1 \times X_2) \times (X_1 \times X_2) \to (X_1 \times X_1)$ and $\phi_2 : (X_1 \times X_2) \times (X_1 \times X_2) \to (X_2 \times X_2)$ be the projections. The separatedness hypothesis guarantees the diagonals are closed subschemes. The first lemma relates the ideal of the diagonal in $X \times X$ with the ideals of the diagonals in $X_i \times X_i$, $i = 1, 2$.

Lemma 2.1.1. *Let I be the ideal of the diagonal Δ in $X \times X$, I_1 be the ideal of the diagonal Δ_1 in $X_1 \times X_1$ and let I_2 be the ideal of the diagonal Δ_2 in $X_2 \times X_2$. For $i = 1, 2$, denote by $\phi_i^* I_i$ the pull-back of the ideal sheaves. Then*

$$I \simeq \phi_1^* I_1 + \phi_2^* I_2,$$

where we use the identification of the sheaves $\phi_i^* I_i$ with $\phi^{-1} I_i \cdot \mathcal{O}_{X \times X}$, granted by the flatness of ϕ_i.

Proof. $(X_1 \times X_2) \times (X_1 \times X_2)$ is locally isomorphic to Spec$(A \otimes B \otimes A \otimes B)$, where Spec$(A) \subset X_1$ and Spec$(B) \subset X_2$ are two open affine subspace. The sections of I over this affine open set are given by the ideal generated by $\langle a\otimes b\otimes 1\otimes 1-1\otimes 1\otimes a\otimes b : a\in A, b\in B\rangle$ as an $A\otimes B\otimes A\otimes B$ -mod. The sections of $\phi_1^*I_1$ (on this open set) are given by the image of $\langle a\otimes 1-1\otimes a : a\in A\rangle\otimes_{A\otimes A}(A\otimes B\otimes A\otimes B)$ under the isomorphism $A\otimes A\otimes_{A\otimes A}(A\otimes B\otimes A\otimes B)\simeq (A\otimes B\otimes A\otimes B)$, i.e. $\langle a\otimes 1\otimes 1\otimes 1-1\otimes 1\otimes a\otimes 1 : a\in A\rangle$. Analogously the ones of $\phi_2^*I_2$ are given by $\langle 1\otimes b\otimes 1\otimes 1-1\otimes 1\otimes 1\otimes b : b\in B\rangle$. Now observe that for every $a\in A$ and for every $b\in B$,

$$1\otimes b\otimes 1\otimes 1(a\otimes 1\otimes 1\otimes 1-1\otimes 1\otimes a\otimes 1)+ \tag{2.1.1}$$
$$1\otimes 1\otimes 1\otimes b(a\otimes 1\otimes 1\otimes 1-1\otimes 1\otimes a\otimes 1)+ \tag{2.1.2}$$
$$a\otimes 1\otimes 1\otimes 1(1\otimes b\otimes 1\otimes 1-1\otimes 1\otimes 1\otimes b)+ \tag{2.1.3}$$
$$1\otimes 1\otimes a\otimes 1(1\otimes b\otimes 1\otimes 1-1\otimes 1\otimes 1\otimes b)= \tag{2.1.4}$$
$$=2(a\otimes b\otimes 1\otimes 1-1\otimes 1\otimes a\otimes b). \tag{2.1.5}$$

The conclusion then follows. □

Remark 2.1.2. Using the result of the previous lemma 2.1.1 and reasoning locally one easily gets

$$\mathcal{O}_\Delta \simeq \phi_1^*\mathcal{O}_{\Delta_1}\otimes_{\mathcal{O}_{X\times X}}\phi_2^*\mathcal{O}_{\Delta_2}. \tag{2.1.6}$$

Now suppose X_i, $i=1,2$ are smooth varieties over an algebraically closed field of k such that $char(k)\neq 2$. We use the same notations as before and let $p_i : X=X_1\times X_2\to X_i$, $i=1,2$ be the (other) two projections. Moreover, for $i=1,2$ we denote by I_i^e the sheaf $\phi_i^*I_i$ when thought as a sheaf of ideals in $X\times X$. We have just seen in Lemma 2.1.1 that $I=I_1^e+I_2^e$.

The following lemma reinterprets the decomposition $\Omega_X^1 \simeq p_1^*\Omega_{X_1}\oplus p_2^*\Omega_{X_2}$ under the isomorphism (of $\mathcal{O}_{X\times X}$ and $\mathcal{O}_X$- sheaves of modules) $\Omega_X^1\simeq I_\Delta\otimes_{\mathcal{O}_{X\times X}}\mathcal{O}_\Delta$.

Lemma 2.1.3.

$$I\otimes_{\mathcal{O}_{X\times X}}\mathcal{O}_\Delta\simeq(\phi_1^*I_1\otimes_{\mathcal{O}_{X\times X}}\mathcal{O}_\Delta)\oplus(\phi_2^*I_2\otimes_{\mathcal{O}_{X\times X}}\mathcal{O}_\Delta). \tag{2.1.7}$$

Proof. $X\times X=(X_1\times X_2)\times(X_1\times X_2)$ is locally isomorphic to Spec$(A\otimes B\otimes A\otimes B)$, where Spec$(A)\subset X_1$ and Spec$(B)\subset X_2$ are two open affine subspace. The sections of $(\phi_1^*I_1\otimes_{\mathcal{O}_{X\times X}}\mathcal{O}_\Delta)$ (as presheaf) over this affine open set are given by the ideal generated by

$$(I_1\otimes_{A\otimes A}A\otimes B\otimes A\otimes B)\otimes_{A\otimes B\otimes A\otimes B}(A\otimes B\otimes A\otimes B/(I_1^e,I_2^e)) \tag{2.1.8}$$
$$\simeq(I_1\otimes_{A\otimes A}(A\otimes B\otimes A\otimes B/(I_1^e,I_2^e)) \tag{2.1.9}$$

Using the isomorphisms $X \simeq \Delta$ and $X_1 \simeq \Delta_1$, we can locally describe the sections of $p_1^*\Omega_{X_1}$ (as preshaf) over $\mathrm{Spec}(A \otimes B) \simeq \mathrm{Spec}(A \otimes B \otimes A \otimes B/(I_1^e, I_2^e))$ as

$$(I_1 \otimes_{A\otimes A} (A \otimes A/I_1)) \otimes_{(A\otimes A)/I_1} (A \otimes B \otimes A \otimes B/(I_1^e, I_2^e)) \qquad (2.1.10)$$
$$\simeq (I_1 \otimes_{A\otimes A} (A \otimes B \otimes A \otimes B/(I_1^e, I_2^e)). \qquad (2.1.11)$$

This gives natural isomorphism of presheaves between $(\phi_1^* I_{\Delta_1} \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_\Delta)$ and $p_1^*\Omega_{X_1}$, and hence isomorphism of sheaves. The same holds for $i = 2$. □

Now suppose X_1 and X_2 are smooth projective algebraic curves and let $k \geq 1$ be an integer. We want to give a decomposition of $I_\Delta^k \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_\Delta$ for any $k \geq 1$ similar to the one of lemma 2.1.3 for $k = 1$. On some affine open set U of $X \times X$ the ideals $\phi_i^* I_{\Delta_i}$ correspond to some ideals I_i^e with the property that

$$I_1^e I_2^e = I_1^e \cap I_2^e \qquad (2.1.12)$$

in the ring $\mathcal{O}_{X\times X}(U)$, which we denote by R. Since $\mathrm{Tor}_1^R(R/I_1^e, R/I_2^e) = (I_1^e \cap I_2^e)/(I_1^e I_2^e)$ (see [29], page 48), to see that 2.1.12 holds, it is equivalent to show that $\mathrm{Tor}_1^R(R/I_1^e, R/I_2^e) = 0$. Now observe the closed subschemes corresponding to $\phi_i^* I_{\Delta_i}$ are smooth varieties (respectively isomorphic to $\Delta_1 \times X_2 \times X_2$ and $\Delta_2 \times X_1 \times X_1$) which intersect properly in Δ. Moreover by [29], Theorem 1.26 $-$ (c), the intersection multiplicity is 1. This gives the vanishing of $\mathrm{Tor}_1^R(R/I_1^e, R/I_2^e)$. See for example the final part ot the proof of Lemma 14.3, here.

Lemma 2.1.4.

$$I_\Delta^k \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_\Delta \simeq (\phi_1^* I_{\Delta_1} \oplus \phi_2^* I_{\Delta_2})^{\otimes k} \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_\Delta. \qquad (2.1.13)$$

Proof. On some affine open set U consider the short exact sequence of $\mathcal{O}_{X\times X}$ - modules:

$$0 \to K \to (I_1^e \oplus I_2^e)^{\otimes k} \to (I_1^e + I_2^e)^k \to 0 \qquad (2.1.14)$$
$$(i_1, j_1) \otimes ... \otimes (i_k, j_k) \to (i_1 - j_1)...(i_k - j_k), \qquad (2.1.15)$$

where we have denoted by K the kernel. Using that in this case $\phi_i^* I_{\Delta_i}$, $i = 1, 2$ are invertible sheaves, and 2.1.12, it is easy to see that the kernel K is given by $(I_1^e I_2^e) \otimes (I_1^e \oplus I_2^e)^{\otimes(k-1)} \oplus ... \oplus (I_1^e \oplus I_2^e)^{\otimes(k-1)} \otimes (I_1^e I_2^e)$ inside $(I_1^e \oplus I_2^e)^{\otimes k}$. Tensoring with $A \otimes B \otimes A \otimes B/(I_1^e, I_2^e)$, we conclude. □

Remark 2.1.5. The previous Lemma 2.1.4, for $k = 1$, gives another proof of lemma 2.1.3 at least when X_1 and X_2 are smooth curves.

Now consider two smooth complex projective algebraic curves C_i of genus g_i, $i = 1, 2$. In the following proposition we prove the surjectivity of Gaussian maps on the surface $C_1 \times C_2$, associated with line bundles that are pull-back of line bundles on the curves. This result can be seen as a generalization to mixed Gaussian maps of a result of Wahl ([68], Lemma 4.12).

In order to prove Proposition 2.1.7 below it is crucial the following result of Bertram, Ein and Lazarsfeld ([10], Theorem 1 and Theorem 1.7)).

Theorem 2.1.6. *Let C be a smooth curve of genus g and let L and M be line bundles of degree d and m respectively. Let $k \geq 1$ be an integer and assume that $d, e \geq (k+1)(g+1)$.*

i If $d + e \geq (k+1)(2g+2) + 2g - 1$, then $\Phi^k_{L,M}$ is surjective.

ii If C is not hyperelliptic $d + e \geq (k+1)(2g+2) + 2g - 2$, then $\Phi^k_{L,M}$ is surjective.

Now we prove our result.

Proposition 2.1.7. *Let $X = C_1 \times C_2$. Let $p_i : X = C_1 \times C_2 \to C_i$, $i = 1, 2$ be the projections. Let L_i and M_i be line bundles on C_i, $i = 1, 2$, such that $deg(L_i), deg(M_i) \geq 2g_i + 2$ and $deg(L_i) + deg(M_i) \geq 6g_i + 3$, for $i = 1, 2$. Set $L = p_1^* L_1 \otimes p_2^* L_2$ and $M = p_1^* M_1 \otimes p_2^* M_2$. Then $\Phi_{X,L,M}$ is surjective.*

Proof. We want to relate the Gaussian map $\Phi_{X,L,M}$ with Gaussian maps on C_i, $i = 1, 2$. Let $q_i : X \times X \to X$, $i = 1, 2$ the two projections. Denote by Δ, Δ_i, $i = 1, 2$, respectively the diagonal in $X \times X$ and $C_i \times C_i$, $i = 1, 2$. Recall that $\Phi_{X,L,M}$ is given by:

$$\Phi_{X,L,M} : H^0(X \times X, I_\Delta \otimes q_1^* L \otimes q_2^* M) \to H^0(X \times X, I_\Delta / I_\Delta^2 \otimes q_1^* L \otimes q_2^* M)$$

Let $q_{i,1} : C_1 \times C_1 \to C_1$ for $i = 1, 2$ be the projections and analogously $q_{i,2} : C_2 \times C_2 \to C_2$. Let (ϕ_1, ϕ_2) be the isomorphism which exchange factors:

$$X \times X = (C_1 \times C_2) \times (C_1 \times C_2) \xrightarrow{(\phi_1,\phi_2)} (C_1 \times C_1) \times (C_2 \times C_2),$$

i.e. $\phi_i((x_1, x_2), (y_1, y_2) = (x_i, y_i)$. Observe that by Lemma 2.1.1, $I_\Delta \simeq \phi_1^* I_{\Delta_i} + \phi_2^* I_{\Delta_2}$, where $\phi_i^* I_{\Delta_i}$, $i = 1, 2$, are the inverse image ideal sheaves or equivalently the pullbacks sheaves (because projections are flat). Now consider the isomorphism of $\mathcal{O}_X$-modules:

$$\Omega^1_X \simeq I_\Delta \otimes_{\mathcal{O}_{X \times X}} \mathcal{O}_\Delta.$$

By Lemma 2.1.3, the decomposition

$$\Omega^1_X \simeq p_1^* \Omega^1_{C_1} \oplus p_2^* \Omega^1_{C_2}$$

can be read as

$$I_\Delta \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_\Delta \simeq (\phi_1^* I_{\Delta_1} \oplus \phi_2^* I_{\Delta_2}) \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_\Delta.$$

So we obtain the following commutative diagram:

$$\begin{array}{ccc} (\phi_1^* I_{\Delta_1} \oplus \phi_2^* I_{\Delta_2}) \otimes q_1^* L \otimes q_2^* M & \longrightarrow & (\phi_1^* I_{\Delta_1} \oplus \phi_2^* I_{\Delta_2}) \otimes q_1^* L \otimes q_2^* M \otimes \mathcal{O}_\Delta \\ \downarrow & & \downarrow \simeq \\ I_\Delta \otimes q_1^* L \otimes q_2^* M & \longrightarrow & I_\Delta / I_\Delta^2 \otimes q_1^* L \otimes q_2^* M \end{array}$$

Taking global sections we obtain

$$\begin{array}{ccc} H^0((\phi_1^* I_{\Delta_1} \oplus \phi_2^* I_{\Delta_2}) \otimes q_1^* L \otimes q_2^* M) & \xrightarrow{\psi} & H^0((\phi_1^* I_{\Delta_1} \oplus \phi_2^* I_{\Delta_2}) \otimes q_1^* L \otimes q_2^* M \otimes \mathcal{O}_\Delta) \\ \downarrow & & \downarrow \simeq \\ H^0(I_\Delta \otimes q_1^* L \otimes q_2^* M) & \xrightarrow{\Phi_{X,L,M}} & H^0(I_\Delta / I_\Delta^2 \otimes q_1^* L \otimes q_2^* M) \end{array}$$

In order to show that $\Phi_{X,L,M}$ is surjective we will show the surjectivity of ψ. Cleary ψ is surjective if each of the direct sum map is surjective:

$$\psi_1 : H^0(\phi_1^* I_{\Delta_1} \otimes q_1^* L \otimes q_2^* M) \to H^0((\phi_1^* I_{\Delta_1} \otimes q_1^* L \otimes q_2^* M) \otimes \mathcal{O}_\Delta)$$

and

$$\psi_2 : H^0(\phi_2^* I_{\Delta_2} \otimes q_1^* L \otimes q_2^* M) \to H^0((\phi_2^* I_{\Delta_2} \otimes q_1^* L \otimes q_2^* M) \otimes \mathcal{O}_\Delta)$$

Let us deal with the first map. The same argument will apply also to the second one. Observe that

$$p_j \circ q_i = q_{i,j} \circ \phi_j.$$

Then we can write

$$\begin{aligned} q_1^* L \otimes q_2^* M &= q_1^*(p_1^* L_1 \otimes p_2^* L_2) \otimes q_2^*(p_1^* M_1 \otimes p_2^* M_2) & (2.1.16) \\ &= \phi_1^*(q_{1,1}^* L_1 \otimes q_{2,1}^* M_1) \otimes \phi_2^*(q_{1,2}^* L_2 \otimes q_{2,2}^* M_2). & (2.1.17) \end{aligned}$$

And so we obtain

$$\phi_1^* I_{\Delta_1} \otimes q_1^* L \otimes q_2^* M \simeq \phi_1^*(I_{\Delta_1} \otimes (q_{1,1}^* L_1 \otimes q_{2,1}^* M_1)) \otimes \phi_2^*((q_{1,2}^* L_2 \otimes q_{2,2}^* M_2))$$

Using $\mathcal{O}_\Delta \simeq \phi_1^* \mathcal{O}_{\Delta_1} \otimes \phi_2^* \mathcal{O}_{\Delta_2}$(Remark 2.1.2) we also obtain

$$\phi_1^* I_{\Delta_1} \otimes q_1^* L \otimes q_2^* M \otimes \mathcal{O}_\Delta \simeq$$

$$\simeq \phi_1^*(I_{\Delta_1} \otimes (q_{1,1}^* L_1 \otimes q_{2,1}^* M_1) \otimes \mathcal{O}_{\Delta_1}) \otimes \phi_2^*((q_{1,2}^* L_2 \otimes q_{2,2}^* M_2) \otimes \mathcal{O}_{\Delta_2})$$

So ψ_1 becomes a map:

$$H^0(\phi_1^*(I_{\Delta_1}\otimes(q_{1,1}^*L_1\otimes q_{2,1}^*M_1))\otimes\phi_2^*((q_{1,2}^*L_2\otimes q_{2,2}^*M_2)))$$
$$\downarrow$$
$$H^0(\phi_1^*(I_{\Delta_1}\otimes(q_{1,1}^*L_1\otimes q_{2,1}^*M_1)\otimes\mathcal{O}_{\Delta_1})\otimes\phi_2^*((q_{1,2}^*L_2\otimes q_{2,2}^*M_2)\otimes\mathcal{O}_{\Delta_2}))$$

Now using that $X\times X\stackrel{\simeq}{\to}(C_1\times C_1)\times(C_2\times C_2)$ and Künneth formula we get:

$$\begin{aligned}&H^0(X\times X,\phi_1^*(I_{\Delta_1}\otimes(q_{1,1}^*L_1\otimes q_{2,1}^*M_1))\otimes\phi_2^*((q_{1,2}^*L_2\otimes q_{2,2}^*M_2)))\\ \simeq&H^0(C_1\times C_1,I_{\Delta_1}\otimes(q_{1,1}^*L_1\otimes q_{2,1}^*M_1))\otimes H^0(C_2\times C_2,(q_{1,2}^*L_2\otimes q_{2,2}^*M_2)),\end{aligned}$$

and

$$\begin{aligned}&H^0(X\times X,\phi_1^*(I_{\Delta_1}\otimes(q_{1,1}^*L_1\otimes q_{2,1}^*M_1)\otimes\mathcal{O}_{\Delta_1})\otimes\phi_2^*((q_{1,2}^*L_2\otimes q_{2,2}^*M_2))\otimes\mathcal{O}_{\Delta_2})\\ \simeq&H^0(C_1\times C_1,I_{\Delta_1}\otimes(q_{1,1}^*L_1\otimes q_{2,1}^*M_1)\otimes\mathcal{O}_{\Delta_1}))\otimes H^0(C_2\times C_2,(q_{1,2}^*L_2\otimes q_{2,2}^*M_2)\otimes\mathcal{O}_{\Delta_2})).\end{aligned}$$

Under these identifications ψ_1 becomes:

$$H^0(I_{\Delta_1}\otimes q_{1,1}^*L_1\otimes q_{2,1}^*M_1)\otimes H^0(q_{1,2}^*L_2\otimes q_{2,2}^*M_2)$$
$$\downarrow\psi_1$$
$$H^0(I_{\Delta_1}\otimes q_{1,1}^*L_1\otimes q_{2,1}^*M_1\otimes\mathcal{O}_{\Delta_1})\otimes H^0(q_{1,2}^*L_2\otimes q_{2,2}^*M_2\otimes\mathcal{O}_{\Delta_2})$$

and it is given by the tensor product $\Phi_{C_1,L_1,M_1}\otimes\Phi^0_{C_2,L_2,M_2}$, where

$$\Phi_{C_1,L_1,M_1}:H^0(I_{\Delta_1}\otimes q_{1,1}^*L_1\otimes q_{2,1}^*M_1)\to H^0(I_{\Delta_1}\otimes q_{1,1}^*L_1\otimes q_{2,1}^*M_1\otimes\mathcal{O}_{\Delta_1})$$

and

$$\Phi^0_{C_2,L_2,M_2}:H^0(q_{1,2}^*L_2\otimes q_{2,2}^*M_2)\to H^0(q_{1,2}^*L_2\otimes q_{2,2}^*M_2\otimes\mathcal{O}_{\Delta_2}).$$

Analogously one can show that $\psi_2=\Phi^0_{C_1,L_1,M_1}\otimes\Phi_{C_2L_2,M_2}$. Therefore we obtain

$$\psi=\Phi_{C_1,L_1,M_1}\otimes\Phi^0_{C_2,L_2,M_2}\oplus\Phi^0_{C_1,L_1,M_1}\otimes\Phi_{C_2L_2,M_2}. \tag{2.1.18}$$

Now observe that if $\deg(L_i)$, $\deg(M_i)\geq 2g_i+2$ for $i=1,2$, then by Theorem 2.1.6, each Gaussian map is surjective, and by a classical result of Mumford also the multiplication maps are (since $deg(L_i),deg(M_i)\geq 2g_i+1$). Then ψ is. □

Remark 2.1.8. Let X_1 and X_2 be two smooth varieties of any dimension. Let L_1, M_1 and L_2, M_2 be two line bundles on X_1 and X_2 respectively. Denote by $L=L_1\boxtimes L_2$ and $M=M_1\boxtimes M_2$. We observe that a similar proof gives a lifting of $\Phi_{L,M}$ by $\Phi_{X_1,L_1,M_1}\otimes\Phi^0_{X_2,L_2,M_2}\oplus\Phi^0_{X_1,L_1,M_1}\otimes\Phi_{X_2,L_2,M_2}$.

Now prove an analogous statement for higher Gaussian maps. We use the same notations as above and since the proof is very similar to the one of Proposition 2.1.7 many details are not repeated.

Proposition 2.1.9. *Let $k \geq 1$ and let L_1 be a line bundle on C_1 of degree l_1 and L_2 be a line bundle on C_2 of degree l_2. Suppose that g_i and l_i satisfy the hypothesis of Theorem 2.1.6 with $d = e = l_1$ and $d = e = l_2$ and denote by $L = p_1^*L_1 \otimes p_2^*L_2$. Then the higher Gaussian map Φ_L^k is surjective.*

Proof. With the same notations as before, recall that Φ_L^k is given by:

$$\Phi_L^k : L^0(X \times X, I_\Delta^k \otimes q_1^*L \otimes q_2^*L) \to L^0(X, I_\Delta^k \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_\Delta \otimes q_1^*L \otimes q_2^*L).$$

Using 2.1.4 we have the following commutative diagram:

$$\begin{array}{ccc} H^0(\phi_1^*I_{\Delta_i} \oplus \phi_2^*I_{\Delta_2})^{\otimes k} \otimes H^{\boxtimes^2}) & \xrightarrow{\psi} & H^0((\phi_1^*I_{\Delta_1} \oplus \phi_2^*I_{\Delta_2})^{\otimes k} \otimes H^{\boxtimes^2} \otimes \mathcal{O}_\Delta) \\ \downarrow & & \downarrow \simeq \\ H^0(I_\Delta^k \otimes L^{\boxtimes^2}) & \xrightarrow{\Phi_L^k} & H^0(I_\Delta^k / I_\Delta^{k+1} \otimes L^{\boxtimes^2}) \end{array}$$

where we have denoted by $L^{\boxtimes^2} = L \boxtimes L$ the tensor product $q_1^*L \otimes q_2^*L$. Again it is sufficient to show that ψ is surjective. Now observe that ψ decompses as a direct sum of maps

$$\begin{array}{c} H^0((\phi_1^*I_{\Delta_i})^{\otimes i} \otimes (\phi_2^*I_{\Delta_2})^{\otimes j} \otimes L^{\boxtimes^2})) \\ \downarrow \psi_{i,j} \\ H^0((\phi_1^*I_{\Delta_i})^{\otimes i} \otimes (\phi_2^*I_{\Delta_2})^{\otimes j} \otimes L^{\boxtimes^2} \otimes \mathcal{O}_\Delta), \end{array}$$

where i, j varies among the pairs of non negative integers such that $i + j = k$. As in the proof of Proposition 2.1.7, using Künneth formula and the fact $L = p_1^*L_1 \otimes p_1^*L_2$ and $\mathcal{O}_\Delta \simeq \phi_1^*\mathcal{O}_{\Delta_1} \otimes \phi_2^*\mathcal{O}_{\Delta_2}$, $\psi_{i,j}$ becomes the tensor product $\phi_{1,L_1}^i \otimes \phi_{2,L_2}^j$, where ϕ_{1,L_i}^r is the rth Gaussian map on C_i associated with the line bundle L_i, $i = 1, 2$. Since we are assuming that g_i and $deg(L_i)$ satisfy the hypothesis of Theorem 2.1.6), each of the Gaussian map is surjective and so each of the $\psi_{i,j}$ is. Hence we conclude that ψ is surjective. □

Remark 2.1.10. More generally one can prove an analogous surjectivity statement for mixed higher Gaussian maps on $X = C_1 \times C_2$, i.e. higher Gaussian maps associated with line bundles $L = p_1^*L_1 \otimes p_2^*L_2$ and $M = p_1^*M_1 \otimes p_2^*M_2$ as in Proposition 2.1.7.

For future convenience we state the following immediate corollary.

Corollary 2.1.11. *Let $X = C_1 \times C_2$ and let $C \in |p_1^*(D_1) \otimes p_2^*(D_2)|$, where $p_i : X := C_1 \times C_2 \to C_i$ are the usual projections, and let H be the line bundle $\omega_X(C)$ on X. If*

$$g_i \geq 0, d_i \geq kg_i + k + 3, \ i = 1, 2. \tag{2.1.19}$$

Then the higher Gaussian map Φ_H^k is surjective.

Proof. Apply Proposition 2.1.9 with $L_i = \omega_i(D_i)$ where ω_i is the canonical bundle on C_i. □

2.2 Gaussian maps on Enriques surfaces

In this section we prove a statement about surjectivity of higher Gaussian maps on an unnodal Enriques surfaces S, i.e. Theorem 2.2.12 below. The result will then be applied in section 3.1 to prove Theorem 3.1.3. We refer to that section for motivations about studying this problem.

As already mentioned in the introduction of the chapter, the proof of Theorem 2.2.12 relies on the following Theorem of Rios Ortiz (Theorem 1.3.24).

Theorem 2.2.1. *Let S be a smooth projective surface with $H^1(S, \mathcal{O}_S) = 0$ and let L be a line bundle on S. If $H^1(\mathrm{S}^{[2]}, \tilde{L} - (k+2)B) = 0$, then Φ_L^k is surjective.*

Here $\mathrm{S}^{[2]}$ denotes the Hilbert scheme of two points on an Enriques surface (we refer to section 1.2 for definitions and properties). In order to apply the Theorem in our situation we prove some results about ample line bundles on Enriques surface. This is the content of subsection 2.2.1.

The material in this section appears in [31].

2.2.1 Ample line bundles on the Hilbert scheme of points of an Enriques surface

As we have already seen in section 1.1, the geometry of curves on Enriques surfaces is strongly related to the ϕ-function, that we recall.

Definition 2.2.2. *Let H be a line bundle on an Enriques surface S such that $H^2 > 0$.*

$$\phi(H) := min\{|H \cdot F| : F \in \mathrm{Pic}(S), F^2 = 0, F \not\equiv 0\}.$$

In section 1.1 we have described the relation between $\phi(H)$ and the linear system$|H|$ being base-point-free or very ample. In what follows we need to deal with a generalization of the notion of very ampleness, which requires the map associated with the linear system $|H|$ to be even more regular. This is the notion of k-very ampleness. We start by recalling it in the case of a smooth complex surface S.

Definition 2.2.3. *Let S be a smooth complex connected surface over the complex numbers and let $k \geq 0$ be an integer. A line bundle H is said to be k-very ample if for any 0-dimensional subscheme $(Z, \mathcal{O}_Z)$ of length $k+1$ the restriction map $H^0(H) \to H^0(H \otimes \mathcal{O}_Z)$ is surjective.*

Remark 2.2.4. Observe that a line bundle is 0-very ample if and only if it is globally generated and 1-very ample if it is very ample.

Knutsen and Szemberg proved independently the following:

Theorem 2.2.5 ([62], [48])**.** *Let S be an Enriques surface. Then H is k-very ample if and only if $\phi(H) \geq k+2$ and there exists no effective divisor E such that $E^2 = -2$ and $H \cdot E \leq k+1$.*

Thus, one immediately gets the following

Corollary 2.2.6. *Let S be an unnodal Enriques surfaces. The line bundle H is k-very ample if and only if $\phi(H) \geq k+2$.*

The notion of k-very ampleness is useful to construct (very) ample line bundles on Hilbert scheme of points. Indeed, let H be a line bundle on a smooth projective surface S and let Z be a 0-dimensional subscheme of S of length $k+1$. Let

$$0 \to H \otimes \mathcal{I}_Z \to H \to H \otimes \mathcal{O}_Z \to 0.$$

be the exact sequence defining Z as a subscheme, tensored by H. If H is k-very ample, $H^0(S, H \otimes \mathcal{I}_Z)$ is a codimension $k+1$ linear subspace of $H^0(S, H)$. Thus, we have a map:

$$\begin{aligned} \psi_H : S^{[k+1]} &\to Gr((k+1), h^0(S, H)) \\ (Z, \mathcal{I}_Z) &\to H^0(S, H \otimes \mathcal{I}_Z). \end{aligned} \tag{2.2.1}$$

In [13], Catanese and Göttsche showed that (2.2.1) is an embedding if and only if H is $(k+1)$-very ample. Now recall that if H is a line bundle on S we denote by $\widetilde{H}$ the corresponding line bundle on $\mathrm{S}^{[2]}$. In the appendix of [9] it is shown by Göttsche that $\tilde{H} - B$ is the pull-back of the very ample line bundle $\mathcal{O}(1)$ on the Grassmanian. Therefore, if H is $(k+1)$-very ample then $\tilde{H} - B$ is very ample. For our convenience, we state the following

Proposition 2.2.7. *Let S be an unnodal Enriques surface and H a line bundle such that $\phi(H) \geq 4$. Then $\tilde{H} - B$ is a very ample line bundle on* $\mathrm{S}^{[2]}$.

Proof. From Corollary 2.2.6, we have that H is 2-very ample, Then the statement follows since ψ_H is an embedding. □

Before coming to the central result of the section, we need the description of nef divisor classes in the Hilbert scheme of a Enriques surface, due to Nuer([56]).

Theorem 2.2.8 ([56])**.** *Let S be an unnodal Enriques surface and $k \geq 2$. Then $\tilde{L} - aB \in \mathrm{Nef}(S^{[k]})$ if and only if $L \in \mathrm{Nef}(S)$ and $0 \leq a \leq \phi(L)/k$.*

Now we are ready to prove the following.

Proposition 2.2.9. *Let S be an unnodal Enriques surface. If L is a line bundle on S such that $\phi(L) > 2k+4$, then $\widetilde{L} - (k+2)B$ is ample on* $\mathrm{S}^{[2]}$.

Proof. Since S is unnodal, L is nef. Therefore, using Nuer's description of $\mathrm{Nef}(\mathrm{S}^{[2]})$ (see Theorem 2.2.8), we conclude that

$$\widetilde{L} - (k+1)B \qquad \text{and} \qquad \widetilde{L} - \frac{\phi(L)}{2}B$$

both belong to $\mathrm{Nef}(\mathrm{S}^{[2]})$. Furthermore, for the same reasons, $\widetilde{L} - (k+2)B$ is nef.

Assume now, by contradiction, that there exists $D \in \overline{NE}(\mathrm{S}^{[2]})$ violating Kleiman's criterion for ampleness, namely such that $(\widetilde{L} - (k+2)B) \cdot D = 0$. Thus we have $\widetilde{L} \cdot D = (k+2)(B \cdot D)$. Since

$$(\widetilde{L} - (k+1)B) \cdot D \geq 0 \qquad \text{and} \qquad (\widetilde{L} - \frac{\phi(L)}{2}B) \cdot D \geq 0,$$

we obtain

$$B \cdot D = \tilde{L} \cdot D = 0.$$

Indeed, $(\widetilde{L}-(k+1)B)\cdot D \geq 0$ yields $(k+2)(B\cdot D) = \widetilde{L}\cdot D \geq (k+1)(B\cdot D)$, thus $B\cdot D \geq 0$. On the other hand, $(\widetilde{L}-\frac{\phi(L)}{2}B)\cdot D \geq 0$ yields $(\frac{\phi(L)}{2}-k-2)(B\cdot D) \leq 0$, thus $B \cdot D \leq 0$.

By Proposition 2.2.7, $\tilde{L} - B$ is very ample. Thus, the condition

$$(\tilde{L} - B) \cdot D > 0 \text{ for all } D \in \overline{NE}(\mathrm{S}^{[2]})$$

yields a contradiction. □

Corollary 2.2.10. *Let $H \in \mathrm{Pic}(S)$ be such that $\phi(H) > 2k+4$. Then $\widetilde{H - K_S} - (k+2)B$ is ample.*

Proof. Being K_S numerically trivial, we have $\phi(H - K_S) = \phi(H)$. Therefore, we just need to apply Proposition 2.2.9. $\square$

More generally, the strategy of Proposition 2.2.9 leads to the following.

Proposition 2.2.11. *Let L be a line bundle on S such that $\phi(L) = k, k > 4$. Then*

$$\tilde{L} - \left(\frac{k}{2} - 1 - r\right)B \quad \textit{is ample for } 1 \leq r < \frac{k}{2} - 1. \tag{2.2.2}$$

Proof. By Theorem 2.2.8, we have that $\tilde{L} - \frac{k}{2}B$ and $\tilde{L} - (\frac{k}{2} - 2)B$ belong to $\mathrm{Nef}(\mathrm{S}^{[2]})$. Arguing as before by contradiction, we obtain $\tilde{L} - (\frac{k}{2} - 1)B$ ample. This implies 2.2.2. Indeed, if there exists $D \in \overline{NE}(\mathrm{S}^{[2]})$ such that $(\tilde{L} - (\frac{k}{2} - 1 - r)B) \cdot D = 0$, then

$$\left(\tilde{L} - \left(\frac{k}{2} - 1\right)B\right) \cdot D = -r(B \cdot D)$$

and so, being the left hand side strictly positive, we would get $B \cdot D < 0$. Now, since $\tilde{L}$ is nef, we would have

$$0 \leq \tilde{L} \cdot D = \left(\frac{k}{2} - 1 - r\right)(B \cdot D) < 0$$

when $1 \leq r < \frac{k}{2} - 1$. Thus we have a contradiction.

$\square$

2.2.2 Surjectivity of higher Gaussian maps on Enriques surfaces

Now we are ready to prove the main result of the section. It will be applied to the study of (higher) Gaussian maps for curves on Enriques surfaces. The motivation for studying this problem can be found at the beginning of subsection 3.1.1.

Theorem 2.2.12. *Let S be an unnodal Enriques surface, H be a line bundle on S with $\varphi(H) > 2k+4$ and $C \in |H|$. The kth Gaussian map Φ_H^k is surjective.*

Proof. Take $H \in \mathrm{Pic}(S)$ such that $\phi(H) > 2(k+2)$. By Theorem 2.2.1, to show the surjectivity of Φ_H^k, we just need to show that $H^1(\mathrm{S}^{[2]}, \tilde{H}-(k+2)B) = 0$. Using that $K_{S^{[n]}} \simeq \widetilde{K}_S$, we have

$$H^1(\mathrm{S}^{[2]}, \tilde{H}-(k+2)B) = H^1(\mathrm{S}^{[2]}, \widetilde{H-K_S} - (k+2)B + K_{S^{[n]}}).$$

Since by Corollary 2.2.10 $\widetilde{H-K_S} - (k+2)B$ is is ample we conclude by Kodaira vanishing. □

Chapter 3

Gaussian maps for curves on surfaces

This chapter is devoted to the study of (higher) Gaussian maps on some class of smooth projective surfaces. It is divided in two parts.

In section 3.1 we consider the problem of studying the Gaussian-Prym map $\Phi^k_{\omega_C \otimes \alpha}$ when C is a smooth curve lying in a sufficiently positive linear system $|H|$ on an unnodal Enriques surface S and α is the restriction $\omega_{S_{|C}}$. In subsection 3.1.1 the motivation for studying this problem is given. In subsection 3.1.2 the relation is described between line bundles on the projectivized cotangent bundle $\mathbb{P}(\Omega^1_S)$ and on the the exceptional divisor of the Hilbert-Chow morphism. In subsection 3.1.3 it is proved the main Theorem. The proof follows similar steps as in [58] (Theorem 1.3.22). Moreover it uses the main result of section 2.2.

Finally in subsection 3.1.4 it is described the connection between a vanishing result proved in subsection 3.1.3 (Proposition 3.1.4) and the problem of studying the moduli of curves on Enriques surfaces. The material in this section is partially contained in [31].

In section 3.2 is is proved that for any $k \geq 3$ the Gaussian map $\Phi^k_{\omega_C}$ is surjective for infinitely many genera. This is Corollary 3.2.5. The proof relies on results of section 2.1 and considers curves on the product of two curves. A more powerful result has already been proved in [58] (Theorem 1.3.23) by considering curves on $K3$ surfaces. Nevertheless, we decided to present our proof because it is different.

3.1 Gaussian-Prym maps and curves on Enriques surfaces

3.1.1 Motivation

Let (C, α) be a Prym curve of genus g, that is C is a smooth projective curve of genus g and α is a 2-torsion line bundle. In section 1.3 we have seen that the surjectivity of the "mixed" Gaussian-Prym map $\Phi_{\omega_C, \omega_C \otimes \alpha}$ gives an obstruction for a curve to lie on an Enriques surface (Theorem 1.3.10 and Theorem 1.3.11).

Given a Prym curve (C, α) one can consider another natural Gaussian map, i.e. the Gaussian-Prym map $\Phi_{\omega_C \otimes \alpha, \omega_C \otimes \alpha}$. Denoted by R_g the coarse moduli space parametrizing isomorphism classes of Prym curves, Barchielli and Frediani have proved the surjectivity of the Gaussian- Prym map $\Phi_{\omega_C \otimes \alpha}$ for a general Prym curve in R_g ([6] Theorem 4.1).

Theorem 3.1.1. *Let $[(C, \alpha)]$ be a general Prym curve of genus $g \geq 12$. Then $\Phi_{\omega_C \otimes \alpha}$ is surjective.*

A similar result was then proved for the second Gaussian-Prym map by Colombo and Frediani ([24], Theorem 5.1).

Theorem 3.1.2. *Let $[(C, \alpha)]$ be a general Prym curve of genus $g \geq 20$. Then $\Phi^2_{\omega_C \otimes \alpha}$ is surjective.*

These two results were our starting motivation to understand how the Gaussian-Prym map behaves for Prym pairs (C, α) coming from an Enriques surface, that is when C lies on an Enrique surface and $\alpha \simeq \omega_{|_C}$. In Theorem 3.1.3 we prove that if C lives in a sufficiently positive linear system $|H|$ and S is unnodal (Definition 1.1.15), then both Gaussian maps are surjective. Hence they do not give any obstructions for a Prym curve to come from an Enriques surface. We observe that (still in the unnodal hypothesis) there remain only a finite number of isomorphism classes of polarization for which the surjectivity is unknown (see Remark 3.1.6).

3.1.2 Line bundles on the exceptional divisor of the Hilbert-Chow morphism

In this section we are going to complement the material in section 1.2 presenting some interesting properties of the exceptional divisor of the Hilbert scheme of 2-points of an Enriques surface.

More precisely we are going to describe the identification between line bundles on the projectivized cotangent bundle of the surface with line bundles on the exceptional divisor of $\mathrm{S}^{[2]}$. We refer to section 1.2 for the background material on the Hilbert scheme, whereas the reference for the material in this section is [43], section 2. In [43] the authors deal with the case of a $K3$ surface, but the identifications hold of course in much more generality. Since in [43] these natural identifications are not explained in detail we decided to give some more details.

In the following S is an Enriques surface. Recall ([44]) that the projectivized cotangent bundle

$$\pi : \mathbb{P}(\Omega^1_S) \to S, \tag{3.1.1}$$

is defined as the **Proj** of the sheaf of the symmetric algebra of $\mathcal{O}_S$-modules

$$\mathcal{S} = \oplus_{d>0} S^d(\Omega^1_S) \simeq \oplus_{d>0} S^d(I_\Delta/I_\Delta^2), \tag{3.1.2}$$

where I_Δ is the diagonal in $S \times S$. Moreover recall that if

$$f : \widetilde{X} \to X \tag{3.1.3}$$

is the blow up of a smooth variety Y contained in a smooth variety X and if I is the ideal sheaf of Y, then the exceptional divisor is isomorphic to $\mathbb{P}(I/I^2)$. From these considerations it immediately follows that we have an isomorphism

$$\mathbb{P}(\Omega^1_S) \simeq \mathbb{P}(I_\Delta/I_\Delta^2), \tag{3.1.4}$$

where $\mathbb{P}(I_\Delta/I_\Delta^2)$ is the exceptional divisor of the blow up of the diagonal Δ in $S \times S$. Now recall from section 1.2 that we have a commutative diagram

$$\begin{array}{ccc} \widetilde{S}^2 \simeq S^2 \times_{S^{(2)}} \mathrm{S}^{[2]} & \xrightarrow{\eta} & \mathrm{S}^{[2]} \\ \big\downarrow \pi & & \big\downarrow \rho \\ S^2 & \longrightarrow & S^{(2)}, \end{array} \tag{3.1.5}$$

where $\widetilde{S}^2$ is the blow up of the diagonal in $S \times S$, ρ is the Hilbert-Chow morphism and η is a degree 2 morphism which is ramified along the exceptional divisor of $E = \mathbb{P}(I_\Delta/I_\Delta^2) \subset \widetilde{S}^2$ and whose restriction on E gives an isomorphism with $2B \subset \mathrm{S}^{[2]}$. Then we have that the projectivized cotangent bundle of the Enriques surface embeds in $\mathrm{S}^{[2]}$ as the exceptional divisor.

$$\eta : \mathbb{P}(\Omega^1_S) \xrightarrow{\simeq} 2B \subset \mathrm{S}^{[2]}\,. \tag{3.1.6}$$

Observe that we have denoted by π both the blow up morphism in 3.1.5 and its restriction to $\mathbb{P}(\Omega^1_S)$, i.e. the bundle morphism in 3.1.1. From the commutativity of the diagram it immediately follows that for any $H \in \mathrm{Pic}(S)$

$$2\pi^* H = \tilde{H}_{|2B}, \tag{3.1.7}$$

via the identification 3.1.6. Now let ξ the tautological line bundle $\mathcal{O}_{\mathbb{P}(\Omega^1_S)}(1)$. This is the dual of the normal bundle of $\mathbb{P}(\Omega^1_S)$ in $\widetilde{S}$

$$-\xi = \mathcal{O}_{\mathbb{P}(\Omega^1_S)}(\mathbb{P}(\Omega^1_S)). \tag{3.1.8}$$

Now, since η is a degree 2 cover branched along the divisor $2B$, from the theory of cyclic coverings one has:

$$\eta^* \mathcal{O}_{\mathrm{S}^{[2]}}(B) = \mathcal{O}_{\widetilde{S}}(\mathbb{P}(\Omega^1_S)). \tag{3.1.9}$$

Then

$$-\xi = \eta^*(B_{|2B}). \tag{3.1.10}$$

3.1.3 Proof of the main theorem

Let S be an Enriques surface and let $H \in \mathrm{Pic}(S)$ be a very ample line bundle. Take $C \in |H|$ and denote by α the restriction $\omega_{S_{|C}}$, as usual. In this section we prove that if H is sufficiently positive then the (higher) Gaussian-Prym maps $\Phi^k_{\omega_C \otimes \alpha}$ are surjective. More precisely we prove:

Theorem 3.1.3. *Let C be a smooth hyperplane section of an unnodal Enriques surface (S, H) with $\phi(H) > 4(k+2)$. Then the kth Gaussian-Prym map $\Phi^k_{\omega_C \otimes \alpha}$ is surjective. In case $k = 1$ it is sufficient to ask $\phi(H) > 6$.*

When one wants to study the surjectivity of Gaussian maps for curves lying on some surfaces, a standard argument is to consider a diagram like the one that follows.

$$\begin{array}{ccc}
H^0(S \times S, \mathcal{I}^k_{\Delta_S}(H \boxtimes H)) & \xrightarrow{\Phi^k_H} & H^0(S, \mathrm{Sym}^k \Omega^1_S(2C)) \\
 & & \quad\searrow^{p_1} \\
\downarrow & & \qquad H^0(C, \mathrm{Sym}^k \Omega^1_S(2C)|_C) \\
 & & \quad\swarrow_{p_2} \\
H^0(C \times C, \mathcal{I}^k_{\Delta_C}((\omega_C \otimes \alpha) \boxtimes (\omega_C \otimes \alpha))) & \xrightarrow{\Phi^k_{\omega_C \otimes \alpha}} & H^0(C, \omega_C^{\otimes k+2}).
\end{array} \tag{3.1.11}$$

Here Φ^k_H is the k-th Gaussian map (on S) associated with the line bundle $H = \mathcal{O}_S(C)$. The vertical arrow and p_1 are restriction maps. Finally, p_2 comes from the k-th symmetric power of the conormal bundle sequence

$$0 \to \mathrm{Sym}^{k-1}\Omega^1_{S_{|C}}(-C) \to \mathrm{Sym}^k\Omega^1_{S_{|C}} \to \omega_C^{\otimes k} \to 0 \tag{3.1.12}$$

tensored by $\mathcal{O}_C(2C)$.

We prove that $\Phi^k_{\mathcal{O}_S(C)}, p_1,$ and p_2 are surjective. From this we obtain the surjectivity of $\Phi_{\omega_C \otimes \alpha}$.

Surjectivity of $\Phi^k_{\mathcal{O}_S(C)}$. Take S, $H \in \mathrm{Pic}(S)$ such that $\phi(H) > 2(k+2)$ and C as above (notice that the assumption on $\phi(H)$ is weaker for this step). The map is surjective by Theorem 2.2.12.

Surjectivity of p_1. Let S, $H \in \mathrm{Pic}(S)$ be such that $\phi(H) > 4(k+2)$ and $C \in |H|$. Let us consider the following short exact sequence:

$$0 \to \mathrm{Sym}^k\Omega^1_S(C) \to \mathrm{Sym}^k\Omega^1_S(2C) \to \mathrm{Sym}^k\Omega^1_{S|_C}(2C) \to 0.$$

In order to prove the surjectivity of p_1, it is enough to prove the following lemma.

Proposition 3.1.4. *Let S be an unnodal Enriques surface and $H \in \mathrm{Pic}(S)$ such that $\phi(H) > 4(k+2)$. Then*

$$H^1(S, \mathrm{Sym}^k\Omega^1_S(C)) = 0 \quad \textit{for all } k \geq 0.$$

Proof. The case $k = 0$ follows from the exact sequence

$$0 \to \omega_S(-C) \to \omega_S \to \omega_{S|_C} \to 0.$$

using Serre duality and the fact that S is an Enriques surface. For $k \geq 1$ we proceed in the same way as in [58]. Let $\pi : \mathbb{P}(\Omega^1_S) \to S$ be the projectivisation of Ω^1_S and let ξ be the class of the tautological line bundle $\mathcal{O}_{\mathbb{P}(\Omega^1_S)}(1)$ on $\mathbb{P}(\Omega^1_S)$. Then the following properties hold:

$$\begin{aligned} \pi_*(\mathcal{O}_{\mathbb{P}(\Omega^1_S)}(k\xi)) &= \mathrm{Sym}^k\Omega^1_S; \\ R^i\pi_*(\mathcal{O}_{\mathbb{P}(\Omega^1_S)}(k\xi)) &= 0 \quad \forall i > 0. \end{aligned} \tag{3.1.13}$$

By the projection formula, it then follows that

$$R^i\pi_*(\mathcal{O}_{\mathbb{P}(\Omega^1_S)}(k\xi + \pi^*H)) = R^i\pi_*(\mathcal{O}_{\mathbb{P}(\Omega^1_S)}(k\xi)) \otimes \mathcal{O}_S(C) = 0 \quad \forall i > 0,$$

and so, by degeneration of the Leray spectral sequence, one gets

$$\begin{aligned} H^1(\mathbb{P}(\Omega^1_S), \mathcal{O}_{\mathbb{P}(\Omega^1_S)}(k\xi + \pi^* H)) &\simeq H^1(S, \pi_*(\mathcal{O}_{\mathbb{P}(\Omega^1_S)}(k\xi + \pi^* H))) \\ &\simeq H^1(S, \mathrm{Sym}^k \Omega^1_S(C)). \end{aligned}$$

As we have seen in the previous section the bundle $\mathbb{P}(\Omega^1_S)$ embeds in $\mathrm{S}^{[2]}$ and under this identification we have

$$\tilde{H}_{|2B} = 2\pi^* H \qquad \text{and} \qquad \xi = -B_{|2B}. \tag{3.1.14}$$

To show that $H^1(\mathbb{P}(\Omega^1_S), k\xi + \pi^* H) = 0$, we apply Kodaira vanishing theorem. We prove that $k\xi + \pi^* H - K_{\mathbb{P}(\Omega^1_S)}$ is ample. Using (3.1.14), the fact that $K_S^{[2]} = \widetilde{K}_S$, and the adjunction formula for the divisor $\mathbb{P}(\Omega^1_S) \simeq 2B$, we get

$$2(k\xi + \pi^* H - K_{\mathbb{P}(\Omega^1_S)}) = (-2kB + \widetilde{H} - 2\tilde{K}_S - 4B)_{|2B} = (\widetilde{H} - 2(k+2)B)_{|2B}.$$

The assumption $\phi(H) > 4(k+2)$ allows to conclude. Indeed, by Proposition 2.2.9, the latter is the restriction of an ample line bundle. Hence it is ample. □

Surjectivity of p_2. Assume $H \in \mathrm{Pic}(S)$ with $\phi(H) > 4(k+1)$. By 3.1.12 twisted by $\mathcal{O}_C(2C)$, it is enough to show that $H^1(C, \mathrm{Sym}^{k-1}\Omega^1_{S_{|C}}(C)) = 0$, equivalently

$$H^0(C, \mathrm{Sym}^{k-1}\mathcal{T}_{S_{|C}}(\alpha)) = 0, \tag{3.1.15}$$

by Serre duality. By

$$0 \to \mathrm{Sym}^{k-1}\mathcal{T}_S(-C + K_S) \to \mathrm{Sym}^{k-1}\mathcal{T}_S(K_S) \to \mathrm{Sym}^{k-1}\mathcal{T}_{S_{|C}}(\alpha) \to 0,$$

it is enough to prove that

$$H^0(S, \mathrm{Sym}^{k-1}\mathcal{T}_S(K_S)) = 0 \qquad \text{and} \qquad H^1(S, \mathrm{Sym}^{k-1}\mathcal{T}_S(-C + K_S)) = 0. \tag{3.1.16}$$

The right hand vanishing follows from Serre duality and Lemma 3.1.4, along with our assumptions on $\phi(H)$. To prove the left hand vanishing of (3.1.16), let $Y \xrightarrow{\pi} S$ be the $K3$ double cover of S, namely the degree 2 (cyclic) covering associated with the pair (S, ω_S). As π is unramified, we have $\mathrm{Sym}^{k-1}\mathcal{T}_Y = \pi^*\mathrm{Sym}^{k-1}\mathcal{T}_S$. Using the projection formula we obtain

$$H^0(Y, \mathrm{Sym}^{k-1}\mathcal{T}_Y) = H^0(S, \mathrm{Sym}^{k-1}\mathcal{T}_S) \oplus H^0(S, \mathrm{Sym}^{k-1}\mathcal{T}_S(K_S)).$$

A result of Kobayashi ([50]) asserts that $H^0(Y, \mathrm{Sym}^{k-1}\mathcal{T}_Y) = 0$, and so we get $H^0(\mathrm{Sym}^{k-1}\mathcal{T}_S(K_S)) = 0$, as desired. This ends the proof of the Main Theorem.

Remark 3.1.5. In case $k = 1$ it is sufficient to require $\phi(H) > 6$ (instead of $\phi(H) > 12$). Indeed, by Theorem 2.2.12, if $\phi(H) > 6$, the map $\Phi^1_{\mathcal{O}_S(C)}$ is surjective. As for p_1, when $k = 1$, condition (3.1.4) becomes

$$H^1(S, \Omega^1_S(C)) = 0, \tag{3.1.17}$$

which by Serre duality is equivalent to

$$H^1(S, \mathcal{T}_S \otimes \otimes\omega_S(-C)) = 0. \tag{3.1.18}$$

Let $Y \xrightarrow{\pi} S$ be the $K3$ double cover of S as above. Again, using the projection formula, we get

$$\begin{aligned} H^1(Y, \mathcal{T}_Y(-(\pi^*H)) &\simeq H^1(Y, \pi^*(\mathcal{T}_S(-H))) \\ &\simeq H^1(S, \mathcal{T}_S(-H)) \oplus H^1(S, \mathcal{T}_S \otimes \omega_S(-H)). \end{aligned} \tag{3.1.19}$$

By [15], Lemma 6.3, if $\phi(H) \geq 5$ then $H^1(Y, \mathcal{T}_Y(-\pi^*H)) = 0$, thus, using (3.1.19), we get (3.1.18). This yields the surjectivity of p_1.

The surjectivity of p_2 follows as (3.1.15) for $k = 1$ reads $H^0(C, \alpha) = 0$, which is satisfied since α is a non-trivial 2-torsion line bundle on C.

Remark 3.1.6. If S is an Enriques surface and $n \geq 0$ is an integer, there are at most a finite number of isomorphism classes of line bundles H such that $\phi(H) < n$. Then Theorem 3.1.3 holds for all but a finitely many isomorphism classes of line bundles on an unnodal Enriques surface S.

3.1.4 A remark on the moduli of curves on Enriques surfaces

In this subsection we say a few words about the vanishing 3.1.17 and its relation with the moduli of curves on Enriques surfaces. We borrow from [15] the necessary background.

Let $\mathcal{E}_{g,\phi}$ be the moduli space which parametrizes pairs (S, H) where S is an Enriques surface, H is an ample line bundle of degree $H^2 = 2g - 2$ and $\phi(H) = \phi$. The moduli space $\mathcal{E}_{g,\phi}$ is not in general irreducible. For any values of g and ϕ its irreducible components have been recently characterized in terms of a generalized ϕ-vector (see [49], Theorem 1.4 for more details). Let $\mathcal{EC}_{g,\phi}$ be the moduli space which parametrizes triples (S, H, C) where S is an Enriques surface, H is an ample line bundle of degree $H^2 = 2g - 2$, $\phi(H) = \phi$ and $C \in |H|$. $\mathcal{EC}_{g,\phi}$ has as many irreducible components as $\mathcal{E}_{g,\phi}$.

There are natural forgetful morphisms

$$\chi_{g,\phi} : \mathcal{EC}_{g,\phi} \to R_g \tag{3.1.20}$$

which associate to a isomorphism class of 3-tuples $[(S, H, C)]$ the isomorphism class of the Prym pair $[(C, \omega_{S_{|C}})]$, and

$$c_{g,\phi} : \mathcal{EC}_{g,\phi} \to M_g. \tag{3.1.21}$$

which gives just the isomorphism class of C. In [15] - for any values of g and ϕ - it is computed the dimension of the general fiber of the restriction of $\chi_{g,\phi}$ to any irreducible component of $\mathcal{EC}_{g,\phi}$. In particular in case $\phi \geq 3$ it is proven the following ([15], Theorem 1).

Theorem 3.1.7. *Assume that $\phi \geq 3$ (whence $g \geq 6$). The map $\chi_{g,\phi}$ is generically injective on any irreducible component of $\mathcal{EC}_{g,\phi}$ not appearing in the list below (see list below [15], Theorem 1).*

Of course this says that $c_{g,\phi}$ is generically injective on the same irreducible components.
The kernel of the differential of $c_{g,\phi}$ at (S, H, C) is given by $H^1(T_S(-C))$. (see [15], Lemma 3.1). If $[(S, H, C)]$ is a general element of an irreducible component of $\mathcal{EC}_{g,\phi}$, then the dimension of the general fiber of the restriction to the irreducible component of the morphism $c_{g,\phi}$ is given by $H^1(T_S(-C))$. In [15] it is done a careful analysis of the cases when $H^1(T_S(-C)) = 0$ and the vanishing is linked to the structure of the simple isotropic decomposition of H. See [15], Corollary 3.3, Lemma 6.2 and Lemma 6.3 for more details. Here we want just to observe that Proposition 3.1.4, for $k = 1$, gives another proof, with very different methods, of the following.

Proposition 3.1.8. *For every $\phi > 12$ the moduli map $c_{g,\phi} : \mathcal{EC}_{g,\phi} \to \mathcal{M}_g$ is generically finite over its image.*

Indeed this follows immediately from Proposition 3.1.4 using that

$$H^1(S, T_S(-C)) \simeq H^1(S, \Omega^1_S(C + K_S))$$

together with the fact that $\phi(\mathcal{O}_S(C)) = \phi(\mathcal{O}_S(C+K_S))$. Of course we remark that Theorem 3.1.7 is a far much better and complete result.

3.2 Higher Gaussian-Wahl maps

Let C be a general curve in the moduli space of curves M_g. Ciliberto, Harris and Miranda have proved that for the general curve of genus $g \geq 10$ and $g \neq 11$ the Gaussian-Wahl map Φ_{ω_C} is surjective (Theorem 1.3.4). Then Colombo and Frediani have proved in [23] an analogous statement for the second Wahl map $\Phi^2_{\omega_C}$ for $g > 280$ (Remark 1.3.21), which was later refined by Calabri, Ciliberto and Miranda obtaining the surjectivity for $g \geq 18$ (Theorem 1.3.18). In the same article the authors ask what happens for higher Gaussian maps. Rios Ortiz then gives in [58] an "asymptotically optimal" answer proving Theorem 1.3.23, which we recall here.

Theorem 3.2.1. *Let $k > 1$ an integer. Then for a general curve of genus*

$$g > 4(k+2)^2 + 2 \tag{3.2.1}$$

the k-th higher Gaussian map is surjective.

In this section we provide another proof of a weaker version of Theorem 3.2.1 using curves on the product of two curves and relying on results of section 2.1. The idea of using curves lying in the product of some other curves (to prove some surjectivity statement) dates back to Wahl (Theorem 1.3.6) for the first Wahl map, and was also used by Colombo and Frediani in [26] to prove an analogous statement for the second Wahl map. We recall here the theorem.

Theorem 3.2.2. *Let C_i, $i = 1, 2$ be smooth projective algebraic curves of genus g_i and let D_i be effective divisors on C_i of degree d_i. Suppose that*

1. *$g_1 \geq 1, g_2 \geq 2$ or $g_1 \geq 2, g_2 \geq 1$ and $d_i \geq 2g_i + 5$ or,*
2. *$g_1 \geq 2$, $g_2 = 0$, $d_1 \geq 2g_1 + 5$, $d_2 \geq 7$ and $d_2(g_1 - 1) > 2d_1 \geq 4g_1 + 10$.*

Then $\Phi^2_{\omega_C}$ is surjective for any smooth curve $C \in |p_1^* D_1 \otimes p_2^* D_2|$. Therefore, under these assumptions and for the general curve of genus

$$g = 1 + (g_2 - 1)d_1 + (g_1 - 1)d_2 + d_1 d_2,$$

the second Gaussian map $\Phi^2_{\omega_C}$ is surjective.

The proof of 3.2.2 is based on linking the second Gaussian map $\Phi^2_{\omega_C}$ with the maps $\phi^k_{\omega_i(D_i)}$ for $0 \leq k \leq 2$ and relies on Theorem 2.1.6 of Ein and Lazarsfeld. Now we come the main result of this section.

Proposition 3.2.3. *Let $k \geq 2$ be an integer. Let C_i, $i = 1, 2$ be smooth projective algebraic curves of genus g_i, let D_i be an effective divisor on C_i of degree d_i. Denote by $p_i : X := C_1 \times C_2 \to C_i$ the projections and suppose that*

1. *$g_1 \geq 2, g_2 \geq 1$ or $g_1 \geq 1, g_2 \geq 2$, and $d_i \geq kg_i + k + 3$ for $i = 1, 2$ or,*
2. *$g_1 = 0$, $g_2 \geq 2$, $d_1 > 2(k+1)$, $d_1 > \frac{kd_2}{g_2 - 1}$, $d_2 \geq kg_2 + k + 3$.*

Then for any irreducible smooth curve C in the linear system $|p_1^*(D_1) \otimes p_2^*(D_2)|$, $\Phi^k_{\omega_C}$ is surjective.

Proof. Consider as in section 3.1.3 the following commutative diagram

$$\begin{array}{ccccc}
H^0(X \times X, \mathcal{I}^k_{\Delta_X}(H \boxtimes H)) & \overset{\Phi^k_H}{\to} & H^0(X, \mathrm{Sym}^k\Omega^1_X \otimes H^{\otimes 2}) & \xrightarrow{p_1} & H^0(C, (\mathrm{Sym}^k\Omega^1_X \otimes H^{\otimes 2})_{|C}) \\
\downarrow & & & & \\
H^0(C \times C, \mathcal{I}^k_{\Delta_C}(\omega_C \boxtimes \omega_C)) & \xrightarrow{\Phi^k_{\omega_C \otimes \alpha}} & H^0(C, \omega_C^{\otimes k+2}) & \xleftarrow{p_2} &
\end{array} \tag{3.2.2}$$

where H denote the line bundle $\omega_X(C)$ and I_{Δ_X} is the ideal of the diagonal in $X \times X$. As in subsection 3.1.3, we have that the vertical arrow and p_1 are restriction maps and p_2 comes from the k-th symmetric power of the conormal bundle sequence

$$0 \to \mathrm{Sym}^{k-1}\Omega^1_{S_{|C}}(-C) \to \mathrm{Sym}^k\Omega^1_{S_{|C}} \to \omega_C^{\otimes k} \to 0 \tag{3.2.3}$$

tensored by $\omega_C^{\otimes 2}$, i.e.

$$0 \to \mathrm{Sym}^{k-1}\Omega^1_{S_{|C}} \otimes \omega_C^{\otimes 2}(-C) \to \mathrm{Sym}^k\Omega^1_{S_{|C}} \otimes \omega_C^{\otimes 2} \to \omega_C^{\otimes k+2} \to 0 \tag{3.2.4}$$

The surjectivity of $\Phi^k_{\omega_C}$ will follow from the surjectivity of Φ^k_H, p_1 and p_2.

Surjectivity of Φ^k_H.
This follows immediately from Proposition 2.1.9.

Surjectivity of p_1.
In order to show the surjectivity of p_1 it is enough to prove that $H^1(\mathrm{Sym}^k\Omega^1_X \otimes \omega_X^{\otimes 2}(C)) = 0$. Denoted by ω_i the canonical bundle of C_i for $i = 1, 2$ observe that

$$\omega_X^2(C) = p_1^*(\omega_1^2) \otimes p_2^*(\omega_2^2)(C) \tag{3.2.5}$$

$$S^k\Omega^1_X \simeq S^k(p_1^*\omega_1 \oplus p_2^*\omega_2) \simeq \bigoplus_{\substack{i+j=k\\ i,j\geq 0}} p_1^*(\omega_1^i)\otimes p_2^*(\omega_2^j). \tag{3.2.6}$$

Hence

$$S^k\Omega^1_X \otimes \omega^2_X(C) = \bigoplus_{\substack{i+j=k\\ i,j\geq 0}} (p_1^*(\omega_1^{i+2}(D_1)) \otimes p_2^*(\omega_2^{j+2}(D_2)) \tag{3.2.7}$$

Then by Küneth formula we have that

$$H^1(S^k\Omega^1_X \otimes \omega^2_X(C)) = H^1(\bigoplus_{\substack{i+j\simeq k\\ i,j\geq 0}} (p_1^*(\omega_1^{i+2}(D_1)) \otimes p_2^*(\omega_2^{j+2}(D_2))).$$

$$\simeq \bigoplus_{\substack{i+j=k\\ i,j\geq 0}} H^1(\omega_1^{i+2}(D_1))\otimes H^0(\omega_2^{j+2}(D_2)) \oplus \bigoplus_{\substack{i+j=k\\ i,j\geq 0}} H^0(\omega_1^{i+2}(D_1))\otimes H^1(\omega_2^{j+2}(D_2))$$

$$\simeq \bigoplus_{\substack{i+j=k\\ i,j\geq 0}} H^1(\omega_1^{i+2}(D_1))\otimes H^0(\omega_2^{j+2}(D_2)) \oplus \bigoplus_{\substack{i+j=k\\ i,j\geq 0}} H^0(\omega_1^{i+2}(D_1))\otimes H^1(\omega_2^{j+2}(D_2)).$$

Now observe that since by Serre duality

$$H^1(\omega_1^{i+2}(D_1)) \simeq H^0(\omega_1^{-i-1}(-D_1)) \text{ and } H^1(\omega_1^{j+2}(D_2)) \simeq H^0(\omega_2^{-j-1}(-D_2)),$$

to have the desired vanishing it is sufficient that for any $0 \leq i, j \leq k$

$$d_1 > -(i+1)(2g_1-2);$$
$$d_2 > -(j+1)(2g_2-2).$$

If $g_1, g_2 > 0$, these of course occur for every $d_1, d_2 \geq 1$. If $g_1 = 0$ ($g_2 = 0$) the conditions become

$$d_1 > 2(k+1) \quad (d_2 > 2(k+1)) \tag{3.2.8}$$

Surjectivity of p_2
In order to show to the surjectivity of p_2 by 3.2.4 it is enough to prove that $H^1(S^{k-1}\Omega^1_{X_{|C}} \otimes \omega^2_X(C)) = 0$. Observe that

$$S^{k-1}\Omega^1_{X_{|C}} \otimes \omega^2_C(-C) = S^{k-1}[(p_1^*\omega_{C_1} \oplus p_2^*\omega_{C_2})_{|C}] \otimes \omega^2_C(-C), \tag{3.2.9}$$

which is isomorphic to

$$= \bigoplus_{\substack{i+j=k-1\\ i,j\geq 0}} p_1^*\omega_{C_1}^{i+2}(D_1)_{|C} \otimes p_2^*\omega_{C_2}^{j+2}(D_2)_{|C}. \tag{3.2.10}$$

Then we get

$$H^1(S^{k-1}\Omega^1_{X_{|C}}\otimes\omega_C^2(-C))=\bigoplus_{\substack{i+j=k-1\\ i,j\geq 0}}H^1(p_1^*\omega_{C_1}^{i+2}(D_1)_{|C}\otimes p_2^*\omega_{C_2}^{j+2}(D_2)_{|C}).$$

Again, by Serre duality, in order to have the vanishing of H^1, it is enough that

$$2g(C)-2<\deg(p_1^*\omega_{C_1}^{i+2}(D_1)_{|C}\otimes p_2^*\omega_{C_2}^{j+2}(D_2)_{|C}),$$

i.e.

$$(2g_1-2)d_2+(2g_2-2)d_1+2d_1d_2<(i+2)(2g_1-2)d_2+(j+2)(2g_2-2)d_1+2d_1d_2$$

for every i,j. If $g_1\geq 2$ and $g_2\geq 1$ or vice versa then the conditions hold for every $d_1,d_2\geq 1$. If $g_1=0$, $g_2\geq 1$ the conditions become

$$-2d_2+(2g_2-2)d_1<-(i+2)2d_2+(j+2)(2g_2-2)d_1 \tag{3.2.11}$$

for every $i,j\geq 0, i+j=k-1$, which is equivalent to the condition relative to $i=k-1$, $j=0$. That is

$$(g_2-1)d_1>d_2k.$$

Then if $g_1=0$ we need $g_2\geq 2$ and $d_1>\frac{kd_2}{(g_2-1)}$. Analogously if $g_2=0$ we need $g_1\geq 2$ and $d_2\frac{kd_1}{(g_1-1)}$. Comparing all the conditions we conclude. □

Remark 3.2.4. If D_1 and D_2 in Proposition 3.2.3 are general divisors on C_1 and C_2 then the linear system $|p_1^*(D_1)\otimes p_2^*(D_2)|$ is base-point-free, then there actually exists a smooth curve $C\in|p_1^*(D_1)\otimes p_2^*(D_2)|$. This follows from the fact that under the hypothesis of Proposition 3.2.3, we always have $d_i\geq g_i+1$ (see Remark 4.3.3). The general curve $C\in|p_1^*(D_1)\otimes p_2^*(D_2)|$ is a smooth curve of genus

$$g(C)=1+(g_2-1)d_1+(g_1-1)d_2+d_1d_2.$$

In particular the lowest genus (depending on k) is obtained by choosing $g_1=0$ and $g_2=2$, $d_1=3k+3$ and $d_2=2k+3$. This is

$$6k^2+17k+13.$$

Unfortunately this is higher than the bound in Theorem 3.2.1.

A consequence of the previous Proposition is the following.

Corollary 3.2.5. *Let $k\geq 2$. For all g_i and d_i satisfying the hypothesis of Proposition 3.2.3 the general curve of genus*

$$g=1+(g_2-1)d_1+(g_1-1)d_2+d_1d_2, \tag{3.2.12}$$

has surjective kth Gaussian-Wahl map.

Chapter 4

Gaussian maps for singular curves on Enriques surfaces

The content of this chapter is contained in [30].

4.1 Motivation and overview of the chapter

In section 1.3 we have given the definition of Gaussian maps (Definition 1.3.1) for smooth varieties. Actually, the same definition can be given for non-smooth varieties. Moreover, there are results similar to L'vovsky Theorem (Theorem 1.3.7), which hold also in the non-smooth case. For example we have the following theorem by Ballico and Fontanari ([5], Theorem 1).

Theorem 4.1.1. *Le $n \geq 3$ and let $C \subset \mathbb{P}^n$ be an integral non-degenerate and locally complete intersection curve of genus $g > 0$ embedded by a very ample line bundle L. Let $S \subset \mathbb{P}^{n+1}$ be an integral surface such that C is scheme-theoretically a hyperplane section of S. If the Gaussian map $\Phi_{\omega_C,L}$ is surjective then S is a cone over C.*

Another way to deal with Gaussian maps on singular curves is to study "natural" Gaussian maps on their normalization. This is the approach of Kemeny in [46]. In the article, in addition to study the moduli of nodal curves on $K3$ surfaces, the author gives an obstruction in terms of a suitable Gaussian map for a curve to have a nodal model lying on a $K3$ surface. The Gaussian map is the one associated with the canonical bundle twisted by the divisor of the points that are mapped to the nodes. More precisely, if

$$f : C \to f(C) \subset S \tag{4.1.1}$$

is the normalization of a nodal curve with l nodes on a $K3$ surface, and T denotes the divisor of degree $2l$ of the points which is mapped to the nodes, Kemeny studies the Gaussian map $\Phi_{\omega_C(-T)}$.

Following the author notations, denote by $\bar{\mathcal{M}}_{h,2l}$ the moduli stack of smooth curves of genus h with $2l$ marked points and by $\widetilde{\mathcal{M}}_{h,2l} = \bar{\mathcal{M}}_{h,2l}/\Sigma_{2n}$ the stack of curves with unordered marking. If h, l are two positive integers and $[(C,T)] \in \widetilde{\mathcal{M}}_{h,2l}$, the author calls $\Phi_{\omega_C - T}$: marked Wahl map, and proves the following result([46], Theorem 1.6).

Theorem 4.1.2 ([46])**.** *Fix any integer integer $l \in \mathbb{Z}$. Then there exist infinitely many integers $h(l)$, such that the general element $[(C,T)] \in \widetilde{\mathcal{M}}_{h(l),2l}$ has surjective marked Wahl map.*

The author then proves that the same maps are not surjective when C is the normalization of a curve with nodes and T is the divisor given by the points over the nodes: denote by $\mathcal{V}^n_{g,k}$ the stack parametrizing morphisms $[(f : C \to X, L)]$ where (X, L) is a polarized $K3$ surface with $L^2 = 2g-2$, C is a smooth connected curve of genus $p(g,k)-n$ with $p(g,k) := k^2(g-1)+1$, f is birational onto its image and $f(C) \in |kL|$ is nodal, we have the following ([46], Theorem 1.7).

Theorem 4.1.3. *Assume $g - n \geq 13$ for $k = 1$ or $g \geq 8$ for $k > 1$, and let $n \leq \frac{p(g,k)-2}{5}$. Then there is an irreducible component $I^0 \subseteq \mathcal{V}^n_{g,k}$ such that for a general $[(f : C \to X, L)] \in I^0$ the marked Wahl map $\Phi_{\omega_C - T}$ is non-surjective, where $T \subseteq C$ is the divisor over the nodes of $f(C)$.*

The same marked Wahl maps have been studied by Fontanari and Sernesi in [36], where they proved, using very different methods from [46], the following theorem.

Theorem 4.1.4 ([36])**.** *Fix an integer $g \geq 9$ Let (S, H) be a polarized $K3$ surface with $Pic(S) = \mathbb{Z}H$ and $H^2 = 2g-2$. Let C be a smooth curve of genus $g-1$ endowed with a morphism $f : C \to S$ birational onto its image and such that $f(C) \in |H|$. If $T = P + Q \subseteq C$ is the divisor over the singular point of $f(C)$, then the Gaussian map $\Phi_{\omega_C - T}$ is not surjective.*

In this chapter, we want to study similar problems for singular curves on Enriques surfaces.

In our case we consider a polarized Enriques surface (S, H) and a curve C having a morphism $f : C \to S$, birational onto its image and such that $f(C) \in |H|$ has exactly one ordinary singular point of multiplicity d. We set $\alpha = f^*K_S$ and we denote by $(p_1, ..., p_d)$ the d distinct points that are mapped to the singular point. Then $(C, \alpha, p_1, ..., p_d)$ is a d-pointed Prym curve. Denoted by T_d the divisor $p_1 + ... + p_d$, we study the mixed Gaussian-Prym maps $\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$ and $\Phi_{\omega_C, \omega_C - T_d + \alpha}$ on the normalization proving that if (S, H) is a sufficiently positive polarized Enriques surface the maps are not surjective. More precisely we prove the following.

Theorem 4.1.5. *Let (S, H) be a polarized unnodal Enriques surface with $H^2 = 2g - 2$ and let $d \geq 2$. Suppose that $\phi(L) \geq l + 4$, if $1 \leq l \leq 14$, or $\phi(L) \geq \frac{2\sqrt{3}}{3} l + \sqrt{3}$, if $l \geq 15$. Set $g' = g - \binom{d}{2}$ and let C be a smooth curve of genus g' having a birational morphism $f : C \to S$ onto its image such that $f(C) \in |H|$ and $f(C)$ has exactly one ordinary singular point of multiplicity d. Set $\alpha = f^* K_{S_{|_C}}$ and let $T_d = p_1 + ... + p_d$ be the divisor over the singular point. Then the Gaussian maps $\Phi_{\omega_C, \omega_C - T_d + \alpha}$ and $\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$ are not surjective.*

Moreover in the process of doing so we prove that under the same hypothesis there actually exists a curve in $|H|$ having only one ordinary singular point of multiplicity d (Corollary 4.2.9). This is the content of section 4.2. The strategy of the proof of the theorem is similar to the one of Theorem 4.1.4.

In section 4.3 we consider the coarse moduli space parametrizing smooth d-pointed Prym-curves, which we denote by $R_{g,d}$ and we prove that for infinitely many values of d and g the Gaussian maps are surjective. More precisely, let S be the following set:

$$S := \{(g_1, d_1, d_2) : g_1 \geq 3, d_2 \geq 4, d_2(g_1 - 2) > d_1 \geq g_1 + 5, d_1 > d_2\}, \quad (4.1.2)$$

and denote by $R_{g,d}$ the moduli space of $d-$pointed Prym curves. We prove the following.

Theorem 4.1.6. *Let (g_1, d_1, d_2) be in S (4.1.2), and $g = (g_1 - 1)d_2 + d_1(d_2 - 1) + 1$. Let d be an integer such that $2 \leq d \leq d_2$. If $[(C, \alpha, (p_1, ..., p_d))]$ is a general element in $R_{g,d}$, then the Gaussian maps*

$$\Phi_{C, \omega_C - T_d, \omega_C - T_d + \alpha}$$

and

$$\Phi_{C, \omega_C, \omega_C - T_d + \alpha}$$

are surjective.

In case $d = 2, 3$ or $d = 4$ (see example 4.3.1) we obtain the surjectivity for all genera greater than or equal to 76. More generally, for every $d \geq 2$, we obtain infinitely many genera for which the marked Gaussian maps (we are considering) are surjective. However we expect our result far from being sharp (see remark 4.3.17).

4.2 Non surjectivity

In this section we are going to prove Theorem 4.1.5.

4.2.1 Cokernels of Wahl maps

Recall from section 1.3(1.3.8), that if L is a very ample line bundle on a smooth projective curve C, the Gaussian map $\Phi_{L,M}$ can be thought as the map

$$\Phi_{L,M} : H^0(C, M_L \otimes M) \to H^0(C, \omega_C \otimes L \otimes M)$$

coming from

$$0 \to N^{\vee}_{C/\mathbb{P}^r} \otimes L \otimes M \to M_L \otimes M \to \omega_C \otimes L \otimes M \to 0, \tag{4.2.1}$$

if $\phi_{|L|} \hookrightarrow \mathbb{P}^r$ is the embedding given by L. Moreover we recall the following useful construction of Lazarsfeld.

Proposition 4.2.1 (Lemma 1.4.1, [51])). *Let $p_1, ..., p_n \in C$ be distinct points such that $L(-\sum_{i=1}^n p_i)$ is generated by global sections, and assume $h^1(L(-\sum_{i=1}^n p_i)) = h^1(L)$. Then one has an exact sequence:*

$$0 \to M_{L(-\sum_{i=1}^n p_i)} \to M_L \to \bigoplus_{i=1}^n \mathcal{O}_C(-p_i) \to 0. \tag{4.2.2}$$

We now observe that a slight modification of [36],Theorem 8, gives the following result which relates cokernels of gaussian maps in different embeddings. In the following $X = C$ will be a smooth complex algebraic curve.

Proposition 4.2.2. *Let C be a smooth complex projective algebraic curve. Let $T_n = p_1 + ... + p_n$ be an effective divisor of degree n on C with $p_i \neq p_j$ for $i \neq j$. Let L and M be two very ample line bundles such that $L - T_n$ is very ample and $h^1(L) = h^1(L - T_n)$. Then there exists a surjection between the cokernels of the Gaussian maps:*

$$\mathrm{coker}(\Phi_{L-T_n,M}) \to \mathrm{coker}(\Phi_{L,M}).$$

The proof follows the same steps of [36], Theorem 8. We present it for completeness.

Proof. The core of the proof is showing the existence of the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r-n}} \otimes L(-T_n) \otimes M & \longrightarrow & M_{L(-T_n)} \otimes M & \longrightarrow & \omega_C \otimes L(-T) \otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r}} \otimes L \otimes M & \longrightarrow & M_L \otimes M & \longrightarrow & \omega_C \otimes L \otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{i=1}^{n} M(-2p_i) & \xrightarrow{g} & \bigoplus_{i=1}^{n} M(-p_i) & \longrightarrow & \bigoplus_{i=1}^{n} M_{|p_i}(-p_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.2.3}$$

where the first two rows are 4.2.1 for the line bundles L and $L - T_n$, the second column is (4.2.2) twisted by M, and the third column is just the restriction modulo the identification $\omega_C \otimes L \otimes \mathcal{O}_T \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{p_i}(-p_i)$, and then tensored by M. In the diagram above all rows and columns are exact. We now explain its construction.

First observe that by hypothesis, $L(-T_n)$ e L are two very ample line bundles and $H^0(L(-T_n)) \subset H^0(L)$. This gives the following commutative diagram:

$$\begin{array}{ccc}
C & \xrightarrow{\phi_{|L|}} & \mathbb{P}^r = \mathbb{P}(H^0(L)^{\vee}) \\
 & \searrow^{\phi_{|L(-T_n)|}} & \downarrow \\
 & & \mathbb{P}^{r-n} = \mathbb{P}(H^0(L(-T_n))^{\vee})
\end{array}$$

where $\phi_{|L(-T_n)|}$ and $\phi_{|L|}$ are the embeddings associated with the linear systems $|L(-T_n)|$ and $|L|$, and $\mathbb{P}^r \to \mathbb{P}^{r-n}$ is a projection. From this, we obtain the following commutative diagram (with exact rows and columns) involving the two conormal exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_C & \longrightarrow & T_{\mathbb{P}^r_{|C}} & \longrightarrow & N_{C/\mathbb{P}^r} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & T_C & \longrightarrow & T_{\mathbb{P}^{r-n}_{|C}} & \longrightarrow & N_{C/\mathbb{P}^{r-n}} & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Considering the duals we then get

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r-n}} & \longrightarrow & \Omega_{\mathbb{P}^{r-n}_{|C}} & \longrightarrow & \omega_C & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r}} & \longrightarrow & \Omega_{\mathbb{P}^{r}_{|C}} & \longrightarrow & \omega_C & \longrightarrow & 0,
\end{array}
$$

where the morphism

$$\Omega_{\mathbb{P}^{r-n}_{|C}} \to \Omega_{\mathbb{P}^{r}_{|C}} \tag{4.2.4}$$

is the dual of the differential of the restriction to C of the projection morphism $\mathbb{P}^r \to \mathbb{P}^{r-n}$. Tensoring with the inclusion $0 \to L(-T_n) \to L$, we then get

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r-n}} \otimes L(-T_n) & \longrightarrow & \Omega_{\mathbb{P}^{r-n}_{|C}} \otimes L(-T_n) & \longrightarrow & \omega_C \otimes L(-T_n) & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r}} \otimes L & \longrightarrow & \Omega_{\mathbb{P}^{r}_{|C}} \otimes L & \longrightarrow & \omega_C \otimes L & \longrightarrow & 0
\end{array}
\tag{4.2.5}
$$

Observe that in the above diagram, all rows and columns are exact. Now let M_L and $M_{L(-T_n)}$ (as usual) be the kernels of the evaluation maps of sections associated with L and $L - T_n$, respectively. That is:

$$0 \to M_L \to H^0(C, L) \otimes \mathcal{O}_C \to L \to 0, \tag{4.2.6}$$

and

$$0 \to M_{L(-T_n)} \to H^0(C, L(-T_n)) \otimes \mathcal{O}_C \to L(-T_n) \to 0. \tag{4.2.7}$$

From the commutativity of the diagram

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ H^0(C, L(-T_n)) \otimes \mathcal{O}_C & \longrightarrow & L(-T_n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ H^0(C, L) \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0, \end{array}$$

it follows that there is an injective morphism between the kernels: $M_{L(-T_n)} \to M_L$. This is the morphism that appears in Proposition 4.2.1. Since L and $L(-T_n)$ are very ample, we have that $\Omega_{\mathbb{P}^r_{|C}} \otimes L \simeq M_L$ and analogously $\Omega_{\mathbb{P}^{r-n}_{|C}} \otimes L(-T_n) \simeq M_{L(-T_n)}$ (see subsection 1.3.1). In these identifications the morphism

$$\phi : \Omega_{\mathbb{P}^{r-n}_{|C}} \otimes L(-T_n) \to \Omega_{\mathbb{P}^r_{|C}} \otimes L$$

in the diagram 4.2.5, is identified with the morphism $M_{L(-T_n)} \to M_L$. Therefore diagram 4.2.5 becomes:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N^\vee_{C/\mathbb{P}^{r-n}} \otimes L(-T_n) & \longrightarrow & M_{L(-T_n)} & \longrightarrow & \omega_C \otimes L(-T_n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N^\vee_{C/\mathbb{P}^r} \otimes L & \longrightarrow & M_L & \longrightarrow & \omega_C \otimes L & \longrightarrow & 0 \end{array}$$

Since by hypothesis $h^1(L(-T_n)) = h^1(L)$, and $L(-T_n)$ is globally generated (actually it is very ample), applying Proposition 4.2.1 we have that the cokernel of the morphism $M_{L(-T_n)} \to M_L$ is isomorphic to $\bigoplus_{i=1}^n \mathcal{O}_C(-p_i)$. Then we obtain:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r-n}} \otimes L(-T_n) & \longrightarrow & M_{L(-T_n)} & \longrightarrow & \omega_C \otimes L(-T_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r}} \otimes L & \longrightarrow & M_L & \longrightarrow & \omega_C \otimes L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \bigoplus_{i=1}^n \mathcal{O}_C(-p_i) & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_{p_i}(-p_i) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where we have used the commutativity of the diagram, together with the fact that cokernel of the morphism $\omega_C \otimes L(-T_n) \to \omega_C \otimes L$ is isomorphic to $\bigoplus_{i=1}^n \mathcal{O}_{p_i} \simeq \bigoplus_{i=1}^n \mathcal{O}_{p_i}(-p_i)$. Now observe that the kernel of the morphism:

$$\bigoplus_{i=1}^{n} \mathcal{O}_C(-p_i) \to \bigoplus_{i=1}^{n} \mathcal{O}_{p_i}(-p_i)$$

is isomorphic to $\bigoplus_{i=1}^n \mathcal{O}_C(-2p_i)$. Then we obtain:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r-n}} \otimes L(-T_n) & \longrightarrow & M_{L(-T_n)} & \longrightarrow & \omega_C \otimes L(-T_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C/\mathbb{P}^{r}} \otimes L & \longrightarrow & M_L & \longrightarrow & \omega_C \otimes L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_C(-2p_i) & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_C(-p_i) & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_{p_i}(-p_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}
$$

where all rows and columns are exact. Tensoring by M we then obtain (4.2.3).

Passing to cohomology in diagram (4.2.3) we get

$$\begin{array}{ccccccccc}
0 \to & \operatorname{coker}(\Phi_{L(-T_n)}, M) & \to & H^1(N^\vee_{C/\mathbb{P}^{r-n}} \otimes L \otimes (-T_n) \otimes M) & \to & H^1(M_{L(-T_n)}) \otimes M) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \operatorname{coker}(\Phi_{L,M}) & \longrightarrow & H^1(N^\vee_{C/\mathbb{P}^r} \otimes L \otimes M) & \longrightarrow & H^1(M_L \otimes M) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \ker(H^1(g)) & \longrightarrow & H^1(\bigoplus M(-2p_i)) & \xrightarrow{H^1(g)} & H^1(\bigoplus M(-p_i)) & \longrightarrow 0
\end{array}$$

Being M very ample, from Riemann-Roch it follows that $h^1(\bigoplus M(-2p_i)) = h^1(\bigoplus M(-p_i))$. Then $\ker(H^1(g)) = 0$. □

4.2.2 Positivity result on blow up

In this section we are going to prove Theorem 4.1.5. We proceed in a similar way as in [36]: we will obtain the non-surjectivity result applying Theorem 1.3.7 together with a result about very ampleness of line bundles on the blow up at a point of an Enriques surface.

Let S be an Enriques surface. Recall that the ϕ function of a big and nef line bundle is defined as

$$\phi(H) := \min\{|H \cdot F| : F \in \operatorname{Pic}(S), F^2 = 0, F \not\equiv 0\},$$

In section 1.1 (and also in section 2.2, Theorem 2.2.5) we have seen that the bigger is the value of the ϕ function, the more regular is the map associated to the line bundle. There is another measure of the positivity of line bundles on an Enriques surface: the Seshadri constant.

Definition 4.2.3. *Let H be a big and nef line bundle on S and $x \in S$ a point. Set*

$$\epsilon(H, x) := \inf_{x \in C} \frac{H \cdot C}{mult_x C}$$

where the infimum is taken over all curves C passing through x. The Seshadri constant $\epsilon(H)$ of H is defined as

$$\epsilon(H) := \inf_{x \in X} \epsilon(H, x).$$

The value $\epsilon(H)$ is strictly bigger than 0 if and only if H is ample. Moreover we have the following inequalities

$$0 \le \epsilon(H)^2 \le \phi(H)^2 \le H^2. \tag{4.2.8}$$

See for example the introduction of [41] and [28], Theorem 2.4.21. Now let $\sigma : S' \to S$ be the blow up at a point p. We will now give, in terms of ϕ, sufficient conditions for a line bundle of the form $\sigma^*H - lE$ to be big and nef. More precisely we will prove the following.

Proposition 4.2.4. *Let S be an unnodal Enriques surface and $l \geq 1$ be an integer. Let H be a big and nef line bundle on S and suppose one of the following holds:*

i) $\phi(H) = l$ and H is not of the type $H \equiv \frac{l}{2}(E_1 + E_2)$ with E_i, $i = 1, 2$, effective isotropic vectors such that $E_1 \cdot E_2 = 2$, or

ii) $\phi(H) \geq l + 1$.

. Then $\sigma^*H - lE$ is big and nef.

We recall that a nodal Enriques surface is one that contains -2 curves. In the moduli space of Enriques surface these correspond to a divisor. An Enriques surface not containing a -2 curve is usually called unnodal.

To prove Proposition 4.2.4 we will need the following result of Galati and Knutsen ([41], Corollary 4.6).

Theorem 4.2.5. *Let S be an Enriques surface and H a big and nef line bundle on S. Except possibly for countably many x lying on $(-2)-$curves, we have $\epsilon(H, x) \geq \phi(H)$.*

which gives, together with 4.2.8, the following immediate corollary.

Corollary 4.2.6. *Let S be an unnodal Enriques surface and H a big and nef line bundle on S. Then $\epsilon(H) = \phi(H)$.*

Now we come to the proof of 4.2.4 .

Proof. First we show that $\sigma^*H - lE$ is nef. From [52], Proposition 5.1.5., it follows that $\sigma^*(H) - lE$ is nef if and only if $\epsilon(H) \geq l$. Since S is unnodal, we conclude by Corollary 4.2.6.

From the inequality 4.2.8 and the hypothesis $l \geq 1$ in case (ii) we get $H^2 \geq \phi(H)^2 > l^2$ and hence $\sigma^*H - lE$ is also big. Consider now case (i) and suppose that $\sigma^*H - lE$ is not big, i.e. $H^2 = l^2$. Then, again by 4.2.8, we have $H^2 = \phi(H)^2 = l^2$. By [47], Proposition 1.4, we must have $H \equiv l(E_1 + E_2)$, where E_i, $i = 1, 2$ are isotropic effective divisor such that $E_1 \cdot E_2 = 2$. □

The proof of the next result is a direct application of Reider's Theorem (Theorem 1.1.10).

Proposition 4.2.7. *Let $l \geq 1$ be an integer and let (S, H) be a polarized unnodal Enriques surface. Suppose that $\phi(L) \geq l + 4$, if $1 \leq l \leq 14$, or $\phi(L) \geq \frac{2\sqrt{3}}{3}l + \sqrt{3}$, if $l \geq 15$. Let $\sigma : S' \to S$ be the blow up at a point and E be the exceptional divisor. Then $\sigma^*H - lE$ is a very ample line bundle on S'.*

Proof. First observe that $\sigma^*H - lE = \sigma^*(H + K_S) - (l + 1)E + K_{S'}$. Set $H' = H + K_S$. By Proposition 4.2.4 $\sigma^*H' - (l + 1)E$ is big and nef. Indeed $\phi(H') = \phi(H) \geq l + 2$. Observe that it is also effective. Indeed suppose by contradiction it is not. Then, by Riemann-Roch and Serre duality, $K_{S'} \otimes (\sigma^*H' - (l+1)E)^\vee = -(\sigma^*H - (l+2)E)$ is effective. Now take a nef effective divisor L in S. Since σ^*L is also nef we obtain $0 \leq \sigma^*L \cdot (-(\sigma^*H - (l + 2)E)) = -L \cdot H < 0$, where the latter is just the fact that H is ample and L effective. Then we conclude that $\sigma^*H' - (l + 1)E$ is effective. Now suppose by contradiction that $\sigma^*H - lE$ is not very ample. Since $\sigma^*H' - (l + 1)E$ is an effective, big and nef divisor and $H^2 \geq \phi^2(H) \geq 9 + (l+1)^2$, we can apply Reider's theorem (Theorem 1.1.10). Then there exists a non trivial effective divisor D in S' such that either one of the following holds:

(a) $D^2 = 0$ and $(\sigma^*H' - (l + 1)E)D \leq 2$;

(b) $D^2 = -1$ and $(\sigma^*H' - (l + 1)E)D \leq 1$;

(c) $D^2 = -2$ and $(\sigma^*H' - (l + 1)E)D = 0$;

(d) $(\sigma^*H' - (l + 1)E)^2 = 9$, $D^2 = 1$ and $(\sigma^*H' - (l + 1)E) \equiv 3D$ in $Num(S')$.

Now we show that none of these can happen.

Let $D \sim \sigma^*L - aE$, for some $L \in \text{Pic}(S)$ and $a \in \mathbb{Z}$. Suppose we are in case (a). Then we have $H'L \leq (l + 1)a + 2$ and $L^2 = a^2$ and so we obtain the following inequalities:

$$\phi(H')^2a^2 \leq H'^2a^2 = H'^2L^2 \leq (H' \cdot L)^2 \leq ((l + 1)a + 2)^2, \tag{4.2.9}$$

where the second inequality follows by Hodge index theorem (Corollary 1.1.4). If $|a| \geq 2$ we obtain

$$\phi(H') \leq |\frac{(l + 1)a + 2}{a}| \leq (l + 1) + |\frac{2}{a}| \leq (l + 1) + 1,$$

which contradicts the hypothesis. If $|a| = 1$ from 4.2.9 we get $\phi(H) = \phi(H') \leq (l + 1) + 2$ which again is not possible. If $a = 0$ we get $D = \sigma^*L$ with L effective, not numerically trivial and such that $L^2 = 0$ and $H'L \leq 2$. This gives $\phi(H) \leq 2$ and we conclude.

Suppose now we are in case (b). As before we have $L^2 = a^2 - 1$, $H'L \leq a(l+1)+1$. Therefore we obtain

$$\phi(H')^2(a^2-1) \leq H'^2(a^2-1) = H'^2L^2 \leq (H' \cdot L)^2 \leq (a(l+1)+1)^2.$$

If $|a| \geq 2$ we find $\phi(H') < \sqrt{2}(l+2)$. If $a = 1$ then L is an effective divisor such that $L^2 = 0$ and $H'L \leq l+2$. Moreover observe that L is not numerically trivial since otherwise $D \equiv -E$, which is not possible because D is an effective nontrivial divisor. Therefore we obtain $\phi(H') \leq l+2$. The case $a = -1$ cannot happen if $l \geq 1$ because H' is nef and L is effective and $L \cdot H' = -l$. If $a = 0$ then $L^2 = -1$. This is not possible for Enriques surfaces.

Suppose now we are in case (c). Then $H'L = a(l+1)$ and $L^2 = a^2 - 2$. Then, as before,

$$\phi(H')^2(a^2-2) \leq H'^2(a^2-2) = H'^2L^2 \leq (H' \cdot L)^2 \leq a^2(l+1)^2.$$

Observe that if $|a| \geq 2$ this gives $\phi(H') \leq \sqrt{2}(l+1)$ and hence we conclude. Observe that $|a| = 1$ cannot happen because otherwise $L^2 = -1$ and this, again, is not possible. If $a = 0$ then L is an effective divisor such that $L^2 = -2$ and $H'L = 0$. This cannot happen because $H' \cdot L = (H + K_S) \cdot L$, H is ample and L is effective.

Suppose we are now in case (d). Then $H'^2 = 9 + (l+1)^2$ which is not possible since $H'^2 \geq \phi(H)^2 > 9 + (l+1)^2$ by hypothesis. □

Corollary 4.2.8. *With the same hypothesis of the previous result, we have* $\sigma^*H - lE + \sigma^*K_S = \sigma^*(H + K_S) - lE$ *is very ample on* S'.

Proof. Apply proposition 4.2.7 with $H + K_S$ instead of H. □

We observe that Proposition 4.2.7 has the following immediate corollary.

Corollary 4.2.9. *Let* $l \geq 2$ *be an integer, and let* (S, H) *be an unnodal polarized Enriques surface. Suppose that* $\phi(L) \geq l+4$, *if* $2 \leq l \leq 14$, *or* $\phi(L) \geq \frac{2\sqrt{3}}{3}l + \sqrt{3}$, *if* $l \geq 15$. *Then there exists a curve* C *in the linear system* $|H|$ *with an ordinary singular point of multiplicity* l.

Proof. Let $\sigma : S' \to S$ be the blow up at a point $p \in S$. Under our hypothesis - applying Proposition 4.2.7 - we have that the line bundle $\sigma^*H - lE$ is very ample on S'. Then the general curve C in the linear system $|\sigma^*H - lE|$ meets the exceptional divisor E in l distinct points. Then $\sigma(C)$ is a curve in S with an ordinary singular point of multiplicity l. □

Now we conclude with the proof of Theorem 4.1.5.

4.2.3 Proof of the main theorem

Proof. 4.1.5] Let $p \in f(C)$ be an ordinary singular point of multiplicity d. Let $\sigma : S' \to S$ be the blow up at p and E the exceptional divisor. By the universal property of normalization we can suppose $C \in |\sigma^* H - dE|$ and $\alpha = \sigma^* K_{S_{|C}}$. From Proposition 4.2.7 it follows that $\mathcal{O}_C(C) = \omega_C - T_d + \alpha$ is very ample. Observe that $h^0(C, \mathcal{O}_C(C)) = h^0(S', \mathcal{O}_{S'}(C)) - 1$. Applying Theorem 1.3.7 we obtain that $\Phi_{\omega_C, \omega_C - T_d + \alpha}$ is not surjective.

Now we want to prove that also $\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$ is not surjective using Proposition 4.2.2 with $L = \omega_C$, $M = \omega_C - T_d + \alpha$ and $n = d$. Observe that since $\mathcal{O}_{S'}(C + K_{S'}) \simeq \mathcal{O}_{S'}(\sigma^*(H + K_S) - (d-1)E)$, $\omega_C = \mathcal{O}_C(C + K_{S'})$ is very ample by Corollary 4.2.8. Analogously $\mathcal{O}_C(C + \sigma^* K_S) = \omega_C - T_d$ is very ample. It remains to show that $h^1(\omega_C) = h^1(\omega_C - T_d)$ or equivalently that $h^0(\omega_C - T_d) = h^0(\omega_C) - d$. Consider then the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{S'}(K_S' - E) & \longrightarrow & \mathcal{O}_{S'}(K_{S'}) & \longrightarrow & \mathcal{O}_E(K_{S'}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{S'}(C + K_{S'} - E) & \longrightarrow & \mathcal{O}_{S'}(C + K_{S'}) & \longrightarrow & \mathcal{O}_E(C + K_{S'}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C(\omega_C - T_d) & \longrightarrow & \mathcal{O}_C(\omega_C) & \longrightarrow & \bigoplus_{i=1}^d \mathcal{O}_{p_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

and the one induced on global sections:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\mathcal{O}_{S'}(C + K_{S'} - E)) & \longrightarrow & H^0(\mathcal{O}_{S'}(C + K_{S'})) & \longrightarrow & \mathbb{C}^d & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\mathcal{O}_C(\omega_C - T_d)) & \longrightarrow & H^0(\mathcal{O}_C(\omega_C)) & \longrightarrow & \mathbb{C}^d & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where we are using that $H^0(\mathcal{O}_{S'}(K_{S'})) \simeq H^0(\mathcal{O}_S(K_S)) = 0$, $E \simeq \mathbb{P}^1$ and $\mathcal{O}_E(K_{S'})$ is a divisor of degree -1 in $E \simeq \mathbb{P}^1$, $h^1(\mathcal{O}_{S'}(C + K_{S'} - E)) = 0$ by Kawamata vanishing theorem since $\mathcal{O}_{S'}(C - E) = \mathcal{O}_{S'}(\sigma^* H - (d+1)E)$ is big and nef. We have $H^1(K_{S'}) \simeq H^1(\mathcal{O}_{S'}) = 0$ because S is an Enriques surface and S' is a blow up. Hence we conclude that $h^0(\omega_C(-T_d)) = h^0(\omega_C) - d$. $\square$

4.3 Surjectivity

In this section we are going to prove Theorem 4.1.6. We present a brief overview of the section.

In subsection 4.3.1 we prove that the marked Gaussian-Prym maps we want to study are surjective when we consider a particular construction of curves in the product of two curves. This is Construction 4.3.2. The main results of the subsection are Corollary 4.3.5 (i.e. the mentioned surjectivity result) and Proposition 4.3.4, which is a surjectivity result for the multiplication maps on the same curves.

In subsection 4.3.2 we prove a result giving a bound on the gonality of curves lying on surfaces of the form $C \times \mathbb{P}^1$. This is Proposition 4.3.6. Moreover we prove an easy lemma giving sufficient conditions for a line bundle on a curve C of the form $\omega_C - T_m + \alpha$ to be base-point-free and very ample. This is Lemma 4.3.9. These results will be used in the final steps of the proof of the main theorem.

In subsection 4.3.3 we informally present the coarse moduli space parametrizing isomorphism classes of smooth $d-$pointed Prym curves, i.e. objects of the form $(C, \alpha, p_1, ..., p_d)$ where $p_1, ..., p_d$ are pairwise distinct points and α is a $2-$ torsion line bundle on C. Finally in section 4.3.4 we prove Theorem 4.1.6.

4.3.1 Surjectivity for special curves

We are now going to prove a surjectivity result for mixed gaussian maps on curves lying on the product of two curves.

Proposition 4.3.1. *Let $X = C_1 \times C_2$. Let D_i, $i = 1, 2$ be effective divisors on C_i. Let $p_i : X = C_1 \times C_2 \to C_i$, $i = 1, 2$ be the projections. Let L_i and M_i be line bundles on C_i, $i = 1, 2$. Let $L = p_1^* L_1 \otimes p_2^* L_2$ and $M = p_1^* M_1 \otimes p_2^* M_2$. Denote by l_i, m_i and g_i, the degree of L_i and M_i, and the genus of C_i respectively. Moreover, suppose that*

1. $l_i, m_i \geq 2g_i + 2$ *and* $l_i + m_i \geq 6g_i + 3$*;*

2. $l_i + m_i > 2g_i - 2 + d_i$ *for* $i = 1, 2$*;*

3. $d_2(l_1 + m_1 - (2g_1 - 2)) + d_1(l_2 + m_2 - (2g_2 - 2)) - 4d_1d_2 > 0.$

Let C *be a smooth irreducible curve in the linear system* $|p_1^*D_1 + p_2^*D_2|$*. Then*

$$\Phi_{C, L_{|C}, M_{|C}}$$

is surjective.

Proof. Consider as usual the following commutative diagram

$$\begin{array}{ccccc}
H^0(X \times X, \mathcal{I}_{\Delta_X} \otimes L \boxtimes M) & \xrightarrow{\Phi_{X,L,M}} & H^0(X, \Omega^1_X \otimes L \otimes M) & & \\
\Big\downarrow & & & \searrow^{\pi_1} & \\
 & & & & H^0(C, \Omega^1_X \otimes L \otimes M|_C) \\
 & & & \swarrow_{\pi_2} & \\
H^0(C \times C, \mathcal{I}_{\Delta_C} \otimes L_{|C} \boxtimes M_{|C}) & \xrightarrow{\Phi_{L_{|C}, M_{|C}}} & H^0(C, \omega_C \otimes L_{|C} \otimes M_{|C}). & &
\end{array} \tag{4.3.1}$$

Recall that the vertical arrow and π_1 are restriction maps, whereas π_2 comes from the conormal bundle sequence

$$0 \to \mathcal{O}_C(-C) \to \Omega^1_{X_{|C}} \to \omega_C \to 0$$

tensored by $\mathcal{O}_C(L + M)$. We prove that $\Phi_{X,L,M}, \pi_1,$ and π_2 are surjective. From this we obtain the desired surjectivity result. The surjectivity of $\Phi_{X,L,M}$ is just Proposition 2.1.7. The surjectivity of π_1 will follow from the vanishing of $H^1(X, \Omega_X \otimes L \otimes M(-C)) \simeq H^1(X, p_1^*\omega_{C_1} \otimes L \otimes M(-C)) \oplus H^1(X, p_2^*\omega_{C_2} \otimes L \otimes M(-C))$. Consider the first piece. Observe that

$$H^1(X, p_1^*\omega_{C_1} \otimes L \otimes M(-C)) \simeq H^1(X, p_1^*(\omega_{C_1} \otimes L_1 \otimes M_1(-D_1)) \otimes p_2^*(L_2 \otimes M_2(-D_2))).$$

By Künneth this is just

$$\begin{array}{c}
H^0(C_1, \omega_{C_1} \otimes L_1 \otimes M_1(-D_1)) \otimes H^1(C_2, L_2 \otimes M_2(-D_2)). \\
\oplus \\
H^1(C_1, \omega_{C_1} \otimes L_1 \otimes M_1(-D_1)) \otimes H^0(C_2, L_2 \otimes M_2(-D_2)).
\end{array}$$

Now observe that $h^1(C_2, L_2 \otimes M_2(-D_2)) = 0$ and $h^1(C_1, \omega_{C_1} \otimes L_1 \otimes M_1(-D_1)) = 0$ by Serre duality and hypothesis 2. Analogously $H^1(X, p_2^* \omega_{C_2} \otimes L \otimes M(-C))$ decomposes as

$$H^0(C_1, L_1 \otimes M_1(-D_1)) \otimes H^1(C_2, \omega_{C_2} \otimes L_2 \otimes M_2(-D_2)).$$
$$\oplus$$
$$H^1(C_1, L_1 \otimes M_1(-D_1)) \otimes H^0(C_2, \omega_{C_2} \otimes L_2 \otimes M_2(-D_2)).$$

Again, $h^1(C_2, \omega_{C_2} \otimes L_2 \otimes M_2(-D_2))$ and $h^1(C_1, L_1 \otimes M_1(-D_1))$ are zero by Serre duality and the hypothesis 2. The surjectivity of π_2 will follow from the vanishing of $H^1(C, (L_{|C} + M_{|C} - C_{|C})$. By Serre duality it will be enough to show that

$$\deg(K_C + C_{|C} - L_{|C} - M_{|C}) < 0.$$

This is just hypothesis 3. □

Main construction 4.3.2. In this remark we consider a construction we will use in the following corollary. First observe that if S is a smooth surface, H is an ample divisor on S and $C \in |H|$ is a smooth curve, then the restriction map

$$\mathrm{Pic}^0_S \to \mathrm{Pic}^0_C$$

is injective by Lefschetz hyperplane Theorem (see for example [39], Theorem C).

Now let C_1 and C_2 be two curves. Let X be the product $C_1 \times C_2$. Let $\mathrm{pr}_i : X \to C_i$, $i = 1, 2$ be the two projections and let D_i be effective divisors of degree d_i such that $|\mathrm{pr}_1^* D_1 + \mathrm{pr}_2^* D_2|$ is base-point-free. Let C be a smooth irreducible curve in the linear system $|\mathrm{pr}_1^* D_1 + \mathrm{pr}_2^* D_2|$. Let $\alpha' \in \mathrm{Pic}^0(C_1)$ a non trivial 2-torsion element (in particular $g(C_1) \geq 1$). Then $\alpha_1 := \mathrm{pr}_1^* \alpha'$ is a non trivial 2-torsion element in $\mathrm{Pic}(X)$ and $\alpha := {\alpha_1}_{|C}$ is a non trivial 2-torsion element in $\mathrm{Pic}(C)$. Assume $\mathrm{supp}(D_1) = \{p_{1,1}, ..., p_{1,d_1}\}$ and denote by $p_1, ..., p_{d_2}$ the d_2 distinct points of intersection between the fiber $\mathrm{pr}_1^{-1}(p_{1,1})$ and C (since $|\mathrm{pr}_1^* D_1 + \mathrm{pr}_2^* D_2|$ is base-point-free we can suppose that the intersection multiplicity at each point of intersection is 1). Moreover denote by T_{d_2} the associated divisor.

Remark 4.3.3. We observe that a sufficient condition for $\mathcal{O}_X(\mathrm{pr}_1^* D_1 + \mathrm{pr}_2^* D_2)$ to be base-point free is that both $\mathcal{O}_{C_1}(D_1)$ and $\mathcal{O}_{C_2}(D_2)$ are. Observe that if C is any curve of genus $g \geq 1$, a general effective divisor D of degree $d \geq g + 1$ is basepoint-free. This follows from classical results but we recall it.

Since every divisor of degree $2g$ is base-point-free, we can restrict to the case $g \geq 2$ and $g+1 \leq d \leq 2g-1$. Consider first the case $d = 2g-1$. Let D' be a general divisor of degree $2g-2$ and $p \in C$ be a point. Then, by Riemann-Roch, it immediately follows that $D'+p$ is a base-point free divisor of degree $2g-1$. Now suppose $g+1 \leq d \leq 2g-2$ and consider the Brill-Noether variety W_d^r parametrizing isomorphism classes of line bundles of degree d such that the dimension of the space of global sections is greater than or equal to $r+1$. Since d is greater than $g+1$, by Riemann-Roch, $Pic^d(C) = W_d^{d-g}$. Hence we have to show that a general element of W_d^{d-g}, with $g+1 \leq d \leq 2g-2$, is base-point-free. Line bundles with base points are given, inside W_d^{d-g}, by the image of the natural map

$$W_{d-1}^{d-g} \times W_1^0 \to W_d^{d-g}. \tag{4.3.2}$$

Consider the isomorphism $W_{d-1}^{d-g} \simeq W_{2g-1-d}^0$ given by $L \to \omega_C \otimes L^\vee$. Since $0 \leq 2g-1-d \leq g$, the last one is birational to $\mathrm{Sym}^{2g-1-d} C$ and hence has dimension $2g-1-d$. Then the image of 4.3.2 has dimension $2g-d$. On the other hand W_d^{d-g} has dimension greater than or equal to $\rho(g, d-g, d) = g$. We conclude that if $d \geq g+1$ the image of 4.3.2 is a proper subvariety and hence the general element is base-point-free.

Corollary 4.3.4. *Using Construction 4.3.2 suppose that one of the following holds:*

1. $g_i \geq 2$ *i=1,2*, $d_1 \geq 5$, $d_2 \geq 4$, $d_1 \geq g_1+5$, $d_2 \geq g_2+4$;
2. $g_1 = 1$, $g_2 \geq 2$, $d_1 \geq 6$, $d_2 \geq 4$, $d_2 \geq g_2+4$, $d_1 > \frac{d_2}{g_2-1}$;
3. $g_1 \geq 3$, $g_2 = 1$, $d_1 \geq 5$, $d_2 \geq 5$, $d_1 \geq g_1+5$;
4. $C_1 = \mathbb{P}^1$, $g_2 \geq 2$, $d_1 \geq 5$, $d_2 \geq 4$, $d_2 \geq g_2+4$, $d_1(g_2-1) > 2d_2$;
5. $g_1 \geq 3$, $C_2 = \mathbb{P}^1$, $d_1 \geq 5$, $d_2 \geq 4$, $d_2(g_1-2) > d_1 \geq g_1+5$.

Then

$$\Phi_{C,\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha},$$

and

$$\Phi_{C,\omega_C, \omega_C - T_{d_2} + \alpha}$$

are surjective.

Proof. Set $L_1 = \omega_{C_1} + D_1 - p_{1,1}$, $L_2 = \omega_{C_2} + D_2$, $M_1 = \omega_{C_1} + D_1 - p_{1,1} + \alpha'$, $M_2 = \omega_{C_2} + D_2$ and $L_1' = \omega_{C_1} + D_1$, $L_2' = \omega_{C_2} + D_2$, $M_1' = \omega_{C_1} + D_1 - p_{1,1} + \alpha'$ and $M_2' = \omega_{C_2} + D_2$. Denote by l_i, m_i, $i = 1,2$ and l_i', m_i', $i = 1,2$ their degrees. To prove the surjectivity of the Gaussian maps we want to apply Proposition 4.3.1 with $L_i, M_i,\ i = 1, 2$ in the first case, and $L_i', M_i',\ i = 1, 2$, in the second. Since $l_i' \geq l_i,\ i = 1, 2,\ m_i' \geq m_i, i = 1, 2$, it is enough to verify the hypothesis of Proposition 4.3.1 in the first situation. It is easy to see that the conditions become: $d_1 \geq 5,\ d_2 \geq 4,\ d_1 \geq g_1 + 5,\ d_2 \geq g_2 + 4$ and $d_2(g_1 - 2) + d_1(g_2 - 1) > 0$. Then we conclude as in the statement. □

We end this section with a surjectivity result for the associated multiplication maps.

Proposition 4.3.5. *Using Construction 4.3.2 suppose that $d_2 \geq 3$ and $d_1 \geq 4$, $g_1 \geq 1$, or $g_1 = 1$ and $d_2 \geq 3$. Then*

$$\Phi^0_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha} \tag{4.3.3}$$

and

$$\Phi^0_{\omega_C, \omega_C - T_{d_2} + \alpha} \tag{4.3.4}$$

are surjective.

Proof. Consider first $\Phi^0_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha}$ and denote it by Φ^0. Set $L = K_X + C - \mathrm{pr}_1^*(p_{1,1})$ and $M = K_X + C - \mathrm{pr}_1^*(p_{1,1}) + \alpha_1$. Then $L_{|C} = \omega_C - T_{d_2}$, $M_{|C} = \omega_C - T_{d_2} + \alpha$. Consider the following commutative diagram

$$\begin{array}{ccc} H^0(X, L) \otimes H^0(X, M) & \xrightarrow{\Phi^0_{L,M}} & H^0(X, L \otimes M) \\ \downarrow & & \downarrow p \\ H^0(C, L_{|C}) \otimes H^0(C, M_{|C}) & \xrightarrow{\Phi^0} & H^0(C, L_{|C} \otimes M_{|C}). \end{array} \tag{4.3.5}$$

where p is the restriction map. In order to prove the surjectivity result, again, it is sufficient to prove that $\Phi^0_{L,M}$ and p are surjective. The multiplication map:

$$H^0(X \times X, q_1^* L \otimes q_2^* M) \xrightarrow{\Phi^0_{L,M}} H^0(X \times X, q_1^* L \otimes q_2^* M \otimes \mathcal{O}_{\Delta_{X \times X}})$$

decomposes, using the identifications in 2.1.16 with $L_1 = \omega_{C_1} + D_1 - p_{1,1}$ and $L_2 = \omega_{C_2} + D_2$, $M_1 = \omega_{C_1} + D_1 - p_{1,1} + \alpha_1$, $M_2 = \omega_{C_2} + D_2$, and Künneth Theorem as before, as the tensor product of the multiplication maps on the curves $C_i :\ i = 1, 2$:

$$\Phi^0 = \Phi^0_{L_1, M_1} \otimes \Phi^0_{L_2, M_2}.$$

Since $l_i, m_i \geq 2g_i + 1,\ i = 1, 2$, each of the multiplication maps is surjective by a classical result of Mumford. The surjectivity of p will follow from the vanishing of $H^1(X, L \otimes M - C)$. By Künneth this is isomorphic to

$$\begin{gathered} H^0(C_1, L_1 \otimes M_1 - D_1) \otimes H^1(C_2, L_2 \otimes M_2 - D_2) \\ \oplus \\ H^1(C_1, L_1 \otimes M_1 - D_1) \otimes H^0(C_2, L_2 \otimes M_2 - D_2). \end{gathered}$$

Now observe that $h^1(C_2, L_2 \otimes M_2(-D_2)) = h^1(C_1, L_1 \otimes M_1(-D_1)) = 0$. This is a consequence of Serre duality and the fact that $l_i + m_i > 2g_i - 2 + d_i$. This ends the proof of the surjectivity of 4.3.3. An identical proof, with $L_1 = \omega_{C_1} + D_1$, $L_2 = \omega_{C_2} + D_2$, $M_1 = \omega_{C_1} + D_1 - p_{1,1} + \alpha'$ and $M_2 = \omega_{C_2} + D_2$, gives the surjectivity of 4.3.4. □

4.3.2 Some useful lemmas

In this section we prove some results we will need in the proof of the final theorem of the section. Let C be a curve. We will need an upper bound on the gonality of curves in the surface $C \times \mathbb{P}^1$. The proof is very much inspired by [45] (see Lemma 2.8 and Theorem 6.1).

Let $\mathrm{pr}_1 : C \times \mathbb{P}^1 \to C$, $\mathrm{pr}_2 : C \times \mathbb{P}^1 \to \mathbb{P}^1$ be the two projections. Let C_0 be the class of a fiber of pr_2. Recall that

$$Pic(C \times \mathbb{P}^1) = \mathrm{pr}_1^*(Pic(C)) \oplus \mathbb{Z}C_0,$$

and that the Néron-Severi is generated by C_0 and the class of a fiber of pr_1, which we will call f. We are going to prove the following:

Proposition 4.3.6. *Let $X \in |\,\mathrm{pr}_1^*(D_1) + d_2C_0|$ be a curve in $C \times \mathbb{P}^1$. Then*

- *if C is hyperelliptic,*

$$\mathrm{gon}(X) = \min(d_1, 2d_2).$$

- *If C is any curve, $g(X) > 0$ and $d_2 \geq \frac{d_1}{4} + 1 + \frac{1}{d_1}$*

$$\mathrm{gon}(X) = \min(d_1, d_2 \,\mathrm{gon}(C)).$$

For the proof we will use the following theorem of Serrano (see [60]):

Theorem 4.3.7. *Let X be a smooth curve on a smooth surface S. Let $\phi : X \to \mathbb{P}^1$ be a surjective morphism of degree d. Suppose that either* *(a)* $X^2 > (d+1)^2$*, or*

(b) $X^2 > \frac{1}{2}(d+2)^2$ *and* K_S *is numerically even.*

Then there exists a morphism $\psi : S \to \mathbb{P}^1$ *such that* $\psi_{|X} = \phi$*.*

Recall that a divisor D is called numerically even if $D \cdot E$ is even for any other divisor E. In our case, being $K_{C\times\mathbb{P}^1} \equiv -2C_0 + (2g(C)-2)f$, we have that $K_{C\times\mathbb{P}^1}$ is numerically even. Before giving the proof of Proposition 4.3.6 we will need the folllowing:

Lemma 4.3.8. *Let* $X \in |\operatorname{pr}_1^*(D_1) + d_2C_0|$ *be a curve in* $C \times \mathbb{P}^1$*. Let* $\phi : X \to \mathbb{P}^1$ *be a morphism such that there exists* $\psi : S \to \mathbb{P}^1$ *such thath* $\psi_{|X} = \phi$*. Then deg(*ϕ*)* $\geq$ *min(*$d_2 \operatorname{gon}(C), d_1$*).*

Proof. Let D be a fiber of ψ. Then $D \sim \operatorname{pr}_1^* B + aC_0$, with $a \in \mathbb{Z}$ and B a divisor in C of degree b. Numerically: $D \equiv bf + aC_0$. From $f \cdot D \geq 0$, $C_0 \cdot D \geq 0$, and $D^2 = 0$ one finds $a \geq 0$, $b \geq 0$ and $2ab = 0$. Then we have two cases:

(i) $a = 0$. In this case $D \sim \operatorname{pr}_1^* B$. Then $\deg(\phi) = \deg(\psi_{|X}) = X \cdot D = d_2 b \geq d_2 \operatorname{gon}(C)$, where the latter inequality follows from the observation that the restriction of ψ to a fiber of pr_2 gives a morphism $C \to \mathbb{P}^1$ of degree greater than or equal to $C_0 \cdot D = b$. And so $b \geq \operatorname{gon}(C)$.

(ii) $b = 0$. In this situation $D \sim aC_0$ and then $\deg(\phi) = \deg(\psi_{|X}) = aC_0 \cdot X = ad_1 \geq d_1$.

□

Proof of Proposition 4.3.6. Let $X \in |\operatorname{pr}_1^*(D_1) + d_2C_0|$ be a curve in $C \times \mathbb{P}^1$ as before. First we prove that $\operatorname{gon}(X) \geq \min(d_1, d_2 \operatorname{gon}(C))$. Denote by k the gonality of X and let $\phi : X \to \mathbb{P}^1$ be a morphism of degree k. If ϕ is extendable we conclude using Lemma 4.3.8. Then, assume that ϕ is not extendable. By contradiction suppose $k < \min(d_1, d_2 \operatorname{gon}(C))$. By Serrano's theorem we get $X^2 = 2d_1d_2 \leq \frac{1}{2}(k+2)^2 < \frac{1}{2}(d_1+2)^2$. That cannot happen if $d_2 \geq \frac{d_1}{4} + 1 + \frac{1}{d_1}$. Finally observe that from $k < \min(d_1, d_2 \operatorname{gon}(C))$, we get $(k+1)^2 \leq d_1 d_2 \operatorname{gon}(C)$ and so, if C is hyperellliptic, we get $(k+1)^2 \leq 2d_1d_2 = X^2 \leq \frac{1}{2}(k+2)^2$, so $k = 1$ and $X \simeq \mathbb{P}^1$. Then $\operatorname{gon}(X) \geq \min(d_1, d_2 \operatorname{gon}(C))$. To conclude that we have an equality, observe that the restrictions of the projections give a morphism of degree d_1 from X to $\mathbb{P}^1$, and a morphism of degree d_2 from X to C. □

We end this subsection proving a lemma that gives a criterion for a line bundle of the type $\omega_C - T_m + \alpha$ to be base-point-free/ very ample. We will use it in Proposition 4.3.16 and Theorem 4.1.6.

Since we want this lemma to hold for any effective divisor T_m of degree m, we have to suppose $m \leq g-3$. This condition in fact guarantees that $h^0(C, \omega_c - T_m + \alpha) \geq 2$.

Lemma 4.3.9. *Let C be a curve, T_m an effective divisor of degree $m \leq g-3$ and α a (non trivial) $2-$torsion element.*

(a) Suppose $\omega_C - T_m + \alpha$ is not base-point-free. Then

  (i) $h^0(C, T_m+\alpha) = 0$ and there exists a point p such that $\dim(|2(T_m+p)|) \geq 1$; or

  (ii) $h^0(C, T_m + \alpha) \geq 1$ and there exists a point p such that $\dim(|T_m + \alpha + p)|) \geq 1$.

(b) Suppose $\omega_C - T_m + \alpha$ is not very ample. Then

  (i) there exist points p and q such that $h^0(C, T_m + \alpha + p) = 0$ and $\dim(|2(T_m + p + q)|) \geq 1$; or

  (ii) there exist points p and q such that $h^0(C, T_m + \alpha + p) \geq 1$ and $\dim(|T_m + \alpha + p + q)|) \geq 1$.

Proof.

(a) Suppose $\omega_C - T_m + \alpha$ has a base point p. Then, by Riemann-Roch, $h^0(T_m + p + \alpha) = h^0(T_m + \alpha) + 1$. If $h^0(T_m + \alpha) \geq 1$ we conclude. If $h^0(T_m+\alpha) = 0$, then $h^0(T_m+\alpha+p) = 1$. Then there exists an effective divisor E such that $E \sim T_m + p + \alpha$. This gives $2E \sim 2(T_m + p)$. Now observe that $h^0(2(T_m + p)) \geq 2$ since otherwise $2E = 2(T_m + p)$ and hence $E = T_m + p$ which gives $\alpha = 0$. This cannot be the case because α is not trivial by hypothesis.

(b) Suppose there exist two points p and q such that q is a base point of $\omega_C - T_m + \alpha - p$. Then, by Riemann-Roch, $h^0(T_m+p+q+\alpha) = h^0(T_m+p+\alpha)+1$. If $h^0(T_m + p + \alpha) \geq 1$ we conclude. If $h^0(T_m + p + \alpha) = 0$, then $h^0(T_m + \alpha + p + q) = 1$. Then there exists an effective divisor E such that $E \sim T_m + p + q + \alpha$. Then $2E \sim 2(T_m + p + q)$. As before, it follows $h^0(2(T_m + p + q)) \geq 2$.

□

4.3.3 Moduli space of pointed Prym curves

This subsection aims to present informally some of the technical machinery which is needed to construct the coarse moduli space parametrizing isomorphism classes of objects of the form $(C, \alpha, \eta, p_1, ..., p_n)$ where C is a smooth irreducible projective complex curve of genus g, α is a line bundle on C, $\eta : \alpha^{\otimes 2} \to \mathcal{O}_C$ is an isomorphism, and $p_1, ..., p_n$ are ordered pairwise distinct points on C. There is a natural notion of smooth family of n-pointed Prym curves, which is the natural generalization of family of smooth Prym curves.

Definition 4.3.10. *A family of n-pointed Prym smooth curves of genus g over a scheme S is the the datum of $(F : \mathcal{C} \to S, \sigma, \mathcal{A}, \mathcal{B})$ where*

1. *$F : \mathcal{C} \to S$ is smooth and proper morphism such that every geometric fiber C_s is a smooth irreducible curve of genus g;*
2. *$\sigma = (\sigma_1, .., \sigma_n)$ where $\sigma_i : S \to \mathcal{C}$ are n sections such that $\sigma_i(s) \neq \sigma_j(s)$ if $i \neq j$ for any s (giving the geometric fiber);*
3. *$\mathcal{A}$ is a (non-trivial) invertible sheaf on $\mathcal{C}$ and $\mathcal{B} : \mathcal{A}^{\otimes 2} \to \mathcal{O}_{\mathcal{C}}$ is a homomorphism such that the restriction to every geometric fiber $C_s := F^{-1}(s)$ gives an isomorphism $\eta_s : \alpha_s^{\otimes 2} \to \mathcal{O}_s$.*

Remark 4.3.11. There is a natural notion of morphism between two families of smooth $n-$pointed Prym curves.

In the following we will often just write $(\mathcal{C} \to S)$ for a family of Prym n-pointed curves.

There is a natural functor

$$\widetilde{\mathcal{R}_{g,n}} : \text{Schemes} \to \text{Sets}, \tag{4.3.6}$$

which associates to a scheme S the set of families of $n-$pointed Prym curves over S.

Definition 4.3.12. *The coarse moduli space for the functor $\widetilde{\mathcal{R}_{g,n}}$ is a variety* $\mathrm{R}_{g,n}$ *over $\mathbb{C}$ such that there is a morphism of functors*

$$\widetilde{\mathcal{R}_{g,n}} \to \mathcal{H}om(\ \ , \mathrm{R}_{g,n}). \tag{4.3.7}$$

which induces a bijection between isomorphism classes of $n-$pointed smooth Prym curves and complex-valued points of the coarse moduli space, and such that for any other scheme M and morphism of functors $\widetilde{\mathcal{R}_{g,n}} \to \mathcal{H}om(\ \ , N)$, there is a unique morphism of functors which makes the following diagram commutative.

$$\begin{array}{ccc} \widetilde{\mathcal{R}_{g,n}} & & \\ \downarrow & \searrow & \\ \mathcal{H}om(\ \ ,\mathrm{R}_{g,n}) & \dashrightarrow & \mathcal{H}om(\ \ ,N). \end{array}$$

Here we have decided to present an informal construction for $R_{g,n}$ Many details are omitted, however. The construction relies on powerful results on stacks. For the material and results used in this section we refer to Jarod Alper notes([2]).

The set of families of $n-$pointed Prym smooth curves together with natural morphisms of families form a category that we denote by $\mathcal{R}_{g,n}$. This leads to the natural definition of the prestack of smooth $n-$pointed Prym curves of genus g. We follow the definition and conventions in [2].

Definition 4.3.13. *The prestack of smooth $n-$pointed Prym curves of genus g is the category $\mathcal{R}_{g,n}$ together with the natural functor*

$$p_{\mathcal{R}_{g,n}} : \mathcal{R}_{g,n} \to \text{Schemes} \tag{4.3.8}$$

given by $(\mathcal{C} \to S) \mapsto S$.

For the definition of Prestack see [2], Definition 2.3.1. It is straightforward to show that it indeed satisfies property (a) and (b) of the definition. We observe - as the author says - that this is a non-standard definition (see Caution 2.3.6).

For a prestack over Schemes to be a stack we need first to endow the category Schemes with a Grothendieck topology (Definition 2.1.1, [2]), which is loosely speaking the assignment to every object X in Schemes of a "covering", that is a collection of special morphisms $\{X_i \to X :\ X_i \in \text{Schemes}\ \ i \in I\}$ where I is a set, such that they satisfy some compatibility conditions. For the category Schemes one considers the so called (big) étale topology: in this case the coverings maps will be étale maps. See for details [2], Example 2.1.5. We notes by $\text{Schemes}_{\text{ét}}$ the category of schemes endowed with this addictional structure.

For a prestack over a category with a Grothendieck topology to be a stack, one has to require some other technical conditions (see Definition 2.4.1). Since showing directly that $\mathcal{R}_{g,n}$ is indeed a stack over Schemes_{et} would require introducing a lot of machinery we will follow a more straightforward route relying on the existence of two well-known moduli stacks, i.e. the stack of $n-$pointed smooth curves of genus g:

$$p_{\mathcal{M}_{g,n}} : \mathcal{M}_{g,n} \to \text{Schemes}_{et}, \tag{4.3.9}$$

and the stack of smooth Prym curves of genus g:

$$p_{\mathcal{R}_g} : \mathcal{R}_g \to \text{Schemes}_{et} . \tag{4.3.10}$$

A morphism of stacks over $Schemes_{\text{ét}}$ is a morphism of prestacks, which in turn is a functor that commutes with the morphisms over $Schemes_{\text{ét}}$ ([2], Definition 2.3.16.). We have natural morphisms of (pre)stacks:

$$\begin{array}{ccc} \mathcal{M}_{g,n} & \longrightarrow & \mathcal{M}_g \\ & \searrow \quad \swarrow & \\ & Schemes_{\text{ét}} & \end{array}$$

and

$$\begin{array}{ccc} \mathcal{R}_g & \longrightarrow & \mathcal{M}_g \\ & \searrow \quad \swarrow & \\ & Schemes & \end{array} ,$$

given by forgetting the sections of a family and the Prym line bundle respectively. One can then consider the fiber product of prestacks:

$$\mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{M}_{g,n} \to Schemes. \tag{4.3.11}$$

Here an object of the category $\mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{M}_{g,n}$ is given by 3-uple

$$((\mathcal{C} \to S), (\mathcal{C}' \to S), \Phi), \tag{4.3.12}$$

where $(\mathcal{C} \to S)$ is an object in $\mathcal{R}_g$, $(\mathcal{C}' \to S)$ is an object in $\mathcal{M}_{g,n}$ and Φ is an isomorphism between the families $(\mathcal{C} \to S)$ and $(\mathcal{C}' \to S)$ in the category $\mathcal{M}_g$ ([2], Construction 2.3.31.). Actually the fiber product of stacks (over a stack) is a stack ([2], Exercise 2.4.6). Then one immediately gets that

$$\mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{M}_{g,n} \to Schemes_{\text{ét}} \tag{4.3.13}$$

is a stack. Now to give an isomorphism of (pre)stacks is to give a morphism which is fully faithful and essentially surjective ([2], Exercise 2.3.18). Then it is easy to see that $\mathcal{R}_{g,n}$ (as we have defined it) is isomorphic to $\mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{M}_{g,n}$ as prestacks. Then $\mathcal{R}_{g,n}$ has naturally a structure of stack.

Actually $\mathcal{R}_g$, $\mathcal{M}_{g,n}$, and $\mathcal{M}_g$ are more than stacks: they are Deligne Mumford stacks ([2], Definition 3.1.4). These properties are well behaved under fiber product ([2], Exercise 3.1.9) and hence $\mathcal{R}_{g,n}$ is a Deligne Mumford stack.

A coarse moduli space for the Deligne-Mumford stack $\mathcal{R}_{g,n}$ is an algebraic space (that is a again a "nicer" space than a Deligne-Mumford algebraic stack, see [2], Definition 3.1.2) $R_{g,n}$ such that there exists a morphism of stacks

$$\mathcal{R}_{g,n} \xrightarrow{\pi} R_{g,n} \tag{4.3.14}$$

such that

- for every algebraically closed field k, the induced map

  $$\mathcal{R}_{g,n}(k)/\sim \;\rightarrow R_{g,n}(k) \tag{4.3.15}$$

  from the set of isomorphism classes of objects of $\mathcal{R}_{g,n}(k)$ over k is bijective and

- π is universal for maps to algebraic spaces, i.e. every map $\mathcal{R}_{g,n} \rightarrow Y$, where Y is an algebraic space, uniquely factors as

$$\begin{array}{ccc} \mathcal{R}_{g,n} & & \\ \downarrow{\scriptstyle \pi} & \searrow & \\ R_{g,n} & \dashrightarrow & Y. \end{array}$$

See also [2], Definition 4.3.1. Now we point out that every scheme X is itself a stack over $Schemes_{\text{ét}}$ when one thinks of X as the category of $\{Hom(S, X) : S \in Schemes\}$. The functor $X \rightarrow Schemes_{\text{ét}}$ is then the obvious one, i.e. the one which sends $Hom(S, X)$ to S. Then it is immediate that if $\mathcal{R}_{g,n}$ admits a coarse moduli space in the sense of 4.3.14 which is a scheme (in this latter sense), then one gets a coarse moduli space in the sense of Definition 4.3.12.

The fact that $\mathcal{R}_{g,n}$ admits a coarse moduli space is a consequence of the important Keel-Mori Theorem ([2], Theorem 4.3.12.).

Theorem 4.3.14. *Let $\mathcal{X}$ be a Deligne Mumford stack separated and of finite type over a Noetherian algebraic space S. Then there exists a coarse moduli space $\pi : \mathcal{X} \rightarrow X$ with $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$ such that*

1 X is separated and of finite type over S,

2 π is a proper universal homeomorphism, and

3 for every flat morphism $X' \to X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.

Without entering in technical details, we observe that $\mathcal{R}_{g,n}$ is indeed separated and of finite type over Spec($\mathbb{Z}$) because $\mathcal{M}_{g,n} \to$ Spec($\mathbb{Z}$) is separated and of finite type and Proposition 10.1.6., [57]. This powerful Theorem gives the existence of the coarse moduli space $R_{g,n}$ as a separated algebraic space over Spec($\mathbb{Z}$). $R_{g,n}$ is actually a scheme because it comes equipped with a (locally quasi-) finite morphism to the coarse moduli space of $\mathcal{M}_{g,n}$

$$R_{g,n} \to M_{g,n}, \tag{4.3.16}$$

which is a scheme (See [2], Corollary 4.4.7). Clearly, it is also finite type over Spec($\mathbb{C}$) (because $M_{g,n}$ is for example) and hence a variety. For the irreducibility of $R_{g,n}$ one can argue in this way. Consider

$$\begin{array}{ccc} R_{g,n} & & \\ \downarrow f & \searrow & \\ R_g & \longrightarrow & M_g. \end{array}$$

The morphism $R_{g,n} \to M_g$ is proper since it is the composition of the finite morphism: $R_{g,n} \to M_{g,n}$ with the proper morphism $M_{g,n} \to M_g$ (see for example [59], Lemma 2.1 for the last statement). The morphism $R_g \to M_g$ is finite (hence proper). Then it follows that $R_{g,n} \to R_g$ is proper by [61, Tag 01W0], Lemma 29.41.7. Now the fiber of f over a point $[C, \alpha]$ of R_g is isomorphic to $C^n \setminus \Delta$ (where Δ is the big diagonal). Hence every fiber is irreducible and of the same dimension. Then one concludes using that R_g is irreducible (see for example [63], Exercise 12.4.D.)

4.3.4 Surjectivity for general Prym pointed curves

Let $R_{g,d}$ be the coarse moduli space of smooth Prym curves of genus g with $d-$marked ordered points. If $[(C, \alpha, \eta, p_1, ..., p_d)]$ is a point in $R_{g,d}$, we will often just write $[(C, \alpha, p_1, ..., p_d)]$. Given such a point in $R_{g,d}$ we denote by T_d the divisor of points $p_1 + ... + p_d$. We want to show that for the general point in $R_{g,d}$, under some assumptions on g, d, the Gaussian maps

$$\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}, \text{ and } \Phi_{\omega_C, \omega_C - T_d + \alpha} \tag{4.3.17}$$

are surjective.

More precisely let us introduce the following set:

$$S := \{(g_1, d_1, d_2) : g_1 \geq 3, d_2 \geq 4, d_2(g_1-2) > d_1 \geq g_1+5, d_1 > d_2\}. \quad (4.3.18)$$

Fix $(g_1, d_1, d_2) \in S$ and set $g = (g_1 - 1)d_2 + d_1(d_2 - 1) + 1$. We are going to prove that for all $0 \leq d \leq d_2$, if $[(C, \alpha, p_1, ..., p_d)] \in R_{g,d}$, the Gaussian maps are surjective.

Notations 4.3.15. In the following we will denote by $[(C^*, \alpha^*, p_1^*, ..., p_{d_2}^*)]$ the class of a point in R_{g,d_2} constructed as in Construction 4.3.2 with D_1 general and taking $C_2 = \mathbb{P}^1$, C_1 hyperelliptic, (g_1, d_1, d_2) belonging to S and $g = (g_1 - 1)d_2 + d_1(d_2 - 1) + 1$. We observe that the conditions $g_1 \geq 3, d_2 \geq 4, d_2(g_1 - 2) > d_1 \geq g_1 + 5$ guarantee that C^* does exist - by Remark 4.3.3 - and the surjectivity of the aforementioned Gaussian maps for the special point (see Corollary 4.3.4). We require C_1 to be hyperelliptic and $d_1 > d_2$ because in the proof of Proposition 4.3.16 we will need $h^0(C^*, T_{d_2}^*) = 1$: this will follow applying Proposition 4.3.6.

Proposition 4.3.16. *Let (g_1, d_1, d_2) be in S (4.3.18), and $g = (g_1 - 1)d_2 + d_1(d_2 - 1) + 1$. Then the Gaussian maps*

$$\Phi_{\omega_C, \omega_C - T_{d_2} + \alpha} : R(\omega_C, \omega_C - T_{d_2} + \alpha) \to H^0(C, 3\omega_C - T_{d_2} + \alpha)$$

and

$$\Phi_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha} : R(\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha) \to H^0(C, 3\omega_C - 2T_{d_2} + \alpha)$$

are surjective for the general d_2-pointed Prym curve in R_{g,d_2} .

Proof. We will prove the result for $\Phi_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha}$. An identical proof gives the surjectivity of $\Phi_{\omega_C, \omega_C - T_{d_2} + \alpha}$. For the rest of the proof we denote $\Phi^i_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha}$, $i = 0, 1$ just by Φ^i, $i = 0, 1$.

Let X be the product $C_1 \times \mathbb{P}^1$ with $g(C_1) = g_1 \geq 3$ and C_1 hyperelliptic. Let $(C^*, \alpha^*, p_1^*, ..., p_{d_2}^*)$ be a d_2-pointed Prym curve as in 4.3.15. First, we show that $h^0(C, \omega_C - T_{d_2})$, $h^0(C, \omega_C - T_{d_2} + \alpha)$ and $h^0(C, 2\omega_C - 2T_{d_2} + \alpha)$ are locally constant in a neighborhood of $[(C^*, \alpha^*, p_1^*, ..., p_{d_2}^*)]$. For the latter, it follows immediately from Riemann-Roch. So let's focus on the other two. By Riemann-Roch it is equivalent to show that $h^0(C, T_{d_2} + \alpha)$ and $h^0(C, T_{d_2})$ are locally constant in a neighborhood of $[(C^*, \alpha^*, p_1^*, ..., p_2^*)]$. For the special point we have $h^0(C^*, T_{d_2}^*) = 1$ since $d_1 > d_2$ by construction and $\text{gon}(C^*) > \min(d_1, 2d_2) > d_2$ by Proposition 4.3.6. Next we show that $h^0(C^*, T_{d_2}^* + \alpha^*) = 0$. Consider:

$$0 \to \mathcal{O}_X(\text{pr}_1^* p_{1,1} + \text{pr}_1^* \alpha'(-C^*)) \to \mathcal{O}_X(\text{pr}_1^* p_{1,1} + \text{pr}_1^* \alpha') \to \mathcal{O}_{C^*}(\text{pr}_1^* p_{1,1} + \text{pr}_1^* \alpha') \to 0. \quad (4.3.19)$$

By Künneth formula we have that $H^1(X, \mathcal{O}_X(\mathrm{pr}_1^* p_{1,1} + \mathrm{pr}_1^* \alpha'(-C^*)) \simeq$

$$H^0(C_1, \mathcal{O}_{C_1}(p_{1,1} + \alpha'(-D_1)) \otimes H^1(C_2, \mathcal{O}_{C_2}(-D_2))$$
$$\oplus$$
$$H^1(C_1, \mathcal{O}_{C_1}(p_{1,1} + \alpha'(-D_1)) \otimes H^0(C_2, \mathcal{O}_{C_2}(-D_2)).$$

and notice that h^0 terms are zero by the hypothesis on the degree of D_i. Now observe that choosing $p_{1,1} \in \mathrm{supp}(D_1)$ general in the construction, we can assume that $h^0(X, \mathcal{O}_X(\mathrm{pr}_1^* p_{1,1} + \mathrm{pr}_1^* \alpha')) = h^0(C_1, p_{1,1} + \alpha') = 0$. Therefore from 4.3.19 we get $h^0(C^*, \mathcal{O}_{C^*}(\mathrm{pr}_1^* p_{1,1} + \mathrm{pr}_1^* \alpha')) = 0$. By construction $h^0(C^*, \mathcal{O}_{C^*}(\mathrm{pr}_1^* p_{1,1} + \mathrm{pr}_1^* \alpha')) = h^0(C^*, T^*_{d_2} + \alpha)$ and so we conclude.

Now observe that since $g = (g_1-1)d_2+d_1(d_2-1)+1 \geq d_2$, if $[(C, p_1, ..., p_{d_2}, \alpha)]$ is a general point R_{g,d_2}, $h^0(C, T_{d_2}) = 1$. Analogously, since $g-1 \geq d_2$, $h^0(C, T_{d_2} + \alpha) = 0$. Hence we are done: the dimensions of the spaces of global sections of the line bundles we are considering are locally constant in a neighborhood of the special point. Moreover Φ^0 is surjective for the special point $[(C^*, \alpha^*, p_1^*, ..., p_{d_2}^*)]$ by Proposition 4.3.5. Then the kernel of the multiplication map on global sections $R(\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha)$ has constant dimension in a neighborhood of the special point. By Riemann-Roch, also $h^0(C, 3\omega_C - 2T_{d_2} + \alpha)$ is locally constant. Since, by Corollary 4.3.4 Φ is surjective for the special point, by semi-continuity it is surjective in a neighborhood. For the Gaussian map $\Phi_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha} : R(\omega_C, \omega_C - T_{d_2} + \alpha) \to H^0(C, 3\omega_C - 2T_{d_2} + \alpha)$ the proof is very similar. □

We observe that the previous result requires $d_2 \geq 4$ and hence we don't still have a surjectivity result for the general Prym curve with 2 or 3 marked points. We overcome this problem in the final result (Theorem 4.1.6). We recall it

Theorem. *Let (g_1, d_1, d_2) be in S (4.3.18), and $g = (g_1-1)d_2+d_1(d_2-1)+1$. Let d be an integer such that $2 \leq d \leq d_2$. If $[(C, \alpha, (p_1, ..., p_d))]$ is a general element in $R_{g,d}$, then the Gaussian maps*

$$\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$$

and

$$\Phi_{\omega_C, \omega_C - T_d + \alpha}$$

are surjective.

Proof. Let's deal with the first map. Let $(C, \alpha, p_1, ..., p_{d_2})$ be a d_2-pointed Prym curve such that the Gaussian map $\Phi_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha}$ is surjective, $\mathrm{gon}(C) =$

$[\frac{g+3}{2}]$, $h^0(C, T_{d_2}) = 1$ and $h^0(C, T_{d_2} + \alpha) = 0$. Observe that there exists such $(C, \alpha, p_1, ..., p_{d_2})$ by Proposition 4.3.16, since asking for $gon(C) = [\frac{g+3}{2}]$, $h^0(C, T_{d_2}) = 1$ and $h^0(C, T_{d_2} + \alpha) = 0$ are open conditions (the last two conditions because $g - 1 \geq d_2$).

Let $(C, \alpha, p_1, ..., p_d)$ be the $d-$pointed Prym curve obtained from $(C, \alpha, p_1, ..., p_{d_2})$ by considering the first d-points: i.e. $(p_1, ..., p_{d_2}) = (p_1, ..., p_d, ..., p_{d_2})$. An easy calculation shows that $[\frac{g+3}{2}] > 2(d_2+2)$, and so we have that $\omega_C - T_{d_2} + \alpha$ is very ample. In fact, if $\omega_C - T_{d_2} + \alpha$ is not very ample, by Lemma 4.3.9 there exists a $g^1_{2(d_2+2)}$. Observe that also $\omega_C - T_{d_2}$ and $\omega_C - T_d$ are very ample since otherwise the curve would admit a $g^1_{d_2+2}$ and a g^1_{d+2} respectively. Moreover, observe that $h^1(\omega_C - T_d) = h^1(\omega_C - T_{d_2})$ since $1 \leq h^0(T_d) \leq h^0(T_{d_2}) = 1$. Denote by T_n the divisor of $n-$(distinct) points $p_{d+1} + ... + p_{d_2}$, i.e. the one such that $T_{d_2} = T_d + T_n$. Then we can apply Proposition 4.2.2 with $L = \omega_C - T_d$, $n = d_2 - d$ (then $L - T_n = \omega_C - T_{d_2}$) and $M = \omega_C - T_{d_2} + \alpha$ and obtain a surjective map

$$\text{coker}(\Phi_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha}) \to \text{coker}(\Phi_{\omega_C - T_d, \omega_C - T_{d_2} + \alpha}).$$

Sincer $\text{coker}((\Phi_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha}) = 0$, we conclude that $\text{coker}(\Phi_{\omega_C - T_d, \omega_C - T_{d_2} + \alpha})$ is zero. Now we use again Proposition 4.2.2 with $L = \omega_C - T_d + \alpha$, $n = d_2 - d$ and $M = \omega_C - T_d$ (in particular $L - T_n = \omega_C - T_{d_2} + \alpha$). Notice that $h^1(L) = h^1(L - T_n)$ since $0 \leq h^0(T_d + \alpha) \leq h^0(T_{d_2} + \alpha) = 0$. We then obtain a surjective map:

$$\text{coker}(\Phi_{\omega_C - T_{d_2} + \alpha, \omega_C - T_d}) \to \text{coker}(\Phi_{\omega_C - T_d + \alpha, \omega_C - T_d}),$$

and so we conclude that $\text{coker}(\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}) = 0$. So we have shown that starting from a general d_2-pointed Prym curve in R_{g,d_2} with surjective Gaussian map $\Phi_{\omega_C - T_{d_2}, \omega_C - T_{d_2} + \alpha}$, its image in $R_{g,d}$ (obtained by forgetting the last $n-$points) is such that the Gaussian map $\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$ is surjective. To conclude that this surjectivity holds for a general element in $R_{g,d}$ one can observe that the morphism $R_{g,d_2} \to R_{g,d}$ is open. The proof for

$$\Phi_{\omega_C, \omega_C - T_d + \alpha}$$

is analogous. □

Example 4.3.1. Observe that choosing $d_2 = 4$, $d_1 = g_1 + l + 5$ with $g_1 \geq 3$ and $0 \leq l + 14 \leq 3g_1$, all the conditions are satisfied and in this case $g = 7g_1 + 3l + 12$. Choosing $(g_1, l) \in \{(7+k, 5), (8+k, 3), (9+k, 1), (7+k, 6), (8+k, 4), (9+k, 2), (7+k, 7), (8+k, 5), (9+k, 3), (10+k, 1), (8+k, 6), (9+k, 4),\ k \geq 0\}$, we get all the genera greater or equal than 76. Then, by Theorem 4.1.6, for all $g \geq 76$ the Gaussian maps with 2,3 or 4$-$marked points are surjective.

Remark 4.3.17. We expect our results regarding the surjectivity of the marked Gaussian-Prym maps to be not sharp. In this remark, we compute the expected numerical range of degrees d and genus g such that one can expect the surjectivity of the Gaussian maps for the general element $[(C, \alpha, (p_1, ..., p_d))]$. Denote by Φ^0, Φ ($\Phi^{0'}$, Φ') respectively $\Phi^0_{\omega_C, \omega_C - T_d + \alpha}$ and $\Phi_{\omega_C, \omega_C - T_d + \alpha}$ ($\Phi^0_{\omega_C - T_d, \omega_C - T_d + \alpha}$ and $\Phi_{\omega_C - T_d, \omega_C - T_d + \alpha}$), and denote by $R(g, d)$ ($R'(g, d)$) the kernel of Φ^0 ($\Phi^{0'}$).

We first observe that a necessary condition for surjectivity is $d \geq g - 3$. Indeed, let $[(C, \alpha, (p_1, ..., p_d))]$ be a general element in $R_{g,d}$. Observe that $h^0(C, \omega_C + \alpha - T_d) = \max\{g - 1 - d, 0\}$. Then, if $d \geq g - 1$, $R(g, d) = 0$. If $d = g - 2$, $h^0(C, \omega_C + \alpha - T_d) = 1$ and Φ^0 ($\Phi^{0'}$) is injective in both cases. Then suppose $d \leq g - 3$. An easy calculation shows that to have the surjectivity of Φ, we need $d \leq g - 7 - \frac{6}{g-2} + \frac{\mathrm{cork}(\Phi^0)}{g-2}$. In particular, one can expect to have surjectivity of Φ for every $g \geq 9$ and $d \leq g - 8$.

Analogously, an easy calculation shows that to have the surjectivity of Φ', we need $d \leq g - 3$, if $g = 4$ or $g = 5$, and $d \leq g - \frac{5}{2} - \sqrt{8g - 7} + \mathrm{cork}(\Phi^{0'})$ if $g \geq 6$.

Chapter 5

Higher Wahl maps on hyperelliptic curves

Historically, the motivation to study Gaussian maps on special classes of curves was to understand the relation between the stratification of the moduli space of curves M_g given by the Wahl map and some special loci in M_g. More recently, this problem appears to be even more interesting either in connection with the study of the local geometry of M_g in A_g at a point $x = [J(C)]$ (see subsection 1.3.4) or in connection with higher dimensional extendability (Theorem 1.3.14). Here we briefly present the results giving the (co)rank of Wahl maps on curves with fixed gonality.

Recall that if $d \geq 1$ is an integer, a curve has gonality d if d is the minimum among the integers such that there exists a g^1_d on C. We start by recalling the following result of Wahl([68], Theorem 4.3) and Ciliberto and Miranda (Proposition 1 [18]).

Theorem 5.0.1. *If $g \geq 4$, then*

$$\mathrm{corank}(\Phi_{\omega_C}) \leq 3g - 2$$

with equality if and only if C is hyperelliptic.

For $g = 3$ the corank is 7 for all curves (see [68]). For trigonal curves, it has been proven by Brawner the following ([11], Theorem 3.2)

Theorem 5.0.2. *If C is a smooth trigonal curve then the corank of Φ_{ω_C} is $g + 5$.*

In [11], Theorem 6.8, the author also computes the corank of the Wahl map for a general tetragonal ($d = 4$) curve.

Theorem 5.0.3. *If C is a general tetragonal curve of genus $g = 15$ or $g \geq 17$ then the corank of Φ_{ω_C} is 9.*

Finally, the answer for the general $d-$gonal curve for $d \geq 5$ is given by Ciliberto and Lopez ([17], Theorem 1.2).

Theorem 5.0.4. *Let C be a general $d-$gonal curve of genus $g \geq 12$. Then the Wahl map Φ_{ω_C} is surjective as soon as*

(i) $d = 5$ *and* $g \geq 15$*;*

(ii) $d = 6$ *and* $g \geq 13$*;*

(iii) $d \geq 7$.

We also mention that in [14] it is computed the corank of $\Phi_{\omega_C,L}$, for a line bundle L, in some interesting cases. In particular, it is proven that if C is a hyperelliptic curve of genus $g \geq 2$, and L is a line bundle on C of degree $d \geq 2g+3$, or a general line bundle of degree $\geq g+4$, then $\text{cork}(\Phi_{\omega_C,L}) = 2g+2$ (see [14], Theorem 1.3). See also Remark 1.3.15.

In the case of higher Gaussian maps, Colombo and Frediani have determined the rank of the second Gaussian map on hyperelliptic curves [26], Proposition 4.2).

Theorem 5.0.5. *Let C be a hyperelliptic curve of genus $g \geq 3$. Then the rank of $\Phi^2_{\omega_C}$ is $2g - 5$.*

In this chapter we are going to compute the rank of higher Gaussian-Wahl maps $\Phi^{2k}_{\omega_C}$ for any $k \geq 2$ on any hyperelliptic curve C, extending the previous known results for Φ_{ω_C} and $k = 1$ (Theorem 5.0.1 and Proposition 5.0.5). The motivation for studying this problem is twofold: on one hand, it is nice to understand the behavior of (higher) Gaussian maps on some special classes of curves, on the other hand, this result will be hopefully applied to the study of the local geometry of $j(M_g)$ in A_g, where j is the Torelli map. We refer to subsection 1.3.4 for a brief exposition of the connection between the local geometry of $j(M_g)$ in A_g and the second Gaussian map Φ^2. Moreover, we refer to [37] for an interesting example of how higher Gaussian-Wahl maps play a role in the study of special subvarieties of A_g generically contained in $j(M_g)$.

The results contained in this chapter are part of an ongoing project with Paola Frediani and Antonio Lacopo.

5.1 Local description of higher Gaussian maps

Let $k \geq 1$ be an integer. In this section we recall the local description of the kth Gaussian maps. The case $k = 1$ has already been presented in subsection 1.3.1. The main reference for this section is Frediani [37].

Let L be a line bundle on a curve C and consider the multiplication map:

$$\Phi_L^0 : H^0(L) \otimes H^0(L) \to H^0(L^{\otimes 2}). \tag{5.1.1}$$

Recall that $H^0(L)^{\otimes 2} = S^2H^0(L) \oplus \Lambda^2 H^0(L)$ and denote by μ_L^0 the restriction of Φ_L^0 to the symmetric tensors:

$$\mu_L^0 := \Phi^0_{L|_{S^2H^0(L)}} : S^2H^0(L) \to H^0(L^{\otimes 2}). \tag{5.1.2}$$

Moreover denote denote by $I_2(L)$ the kernel of μ_L^0.

Recall from Section 1.3 that the domain of the kth Gaussian map, $k \geq 1$ is the kernel of the previous one and in particular

$$\Phi_L^1 : R(L, L) \to H^0(\omega_C \otimes L^{\otimes 2}). \tag{5.1.3}$$

where $R(L, L) = I_2(L) \oplus \Lambda^2 H^0(L)$ is the kernel of Φ_L^0. Also recall that if $\alpha = \sum l_i \otimes m_i \in R(L, L)$, $l_i = f_i l$, $m_i = g_i l$, - where l is a local generator of L and f_i, g_i are local holomorphic functions - Φ_L^1 is locally given by

$$\Phi_L^1(\alpha) = \sum (f_i dg_i - g_i df_i) \otimes l \otimes l \tag{5.1.4}$$

$$= \sum (f_i g_i' - g_i f_i') dz \otimes l^{\otimes 2}. \tag{5.1.5}$$

where we can take z to be a local coordinate on C. Moreover recall that since Φ^1 identically vanishes on symmetric tensors, then we can equivalently study μ_L^1, i.e. the restriction of Φ_L^1 to $\Lambda^2 H^0(L)$, which is locally given by

$$\mu_L^1(\sum l_i \wedge m_i) = \sum (f_i g_i' - g_i f_i') dz \otimes l^{\otimes 2}. \tag{5.1.6}$$

where f_i' and g_i' are derivatives. Now consider Φ_L^2. It is defined on $\ker(\Phi_L^1)$ which decomposes, because of what we have just said, as $I_2(L) \oplus \ker(\mu_L^1)$. The local expression is given by the equation

$$\Phi_L^2(\alpha) = \sum f_i' g_i' \, dz^{\otimes 2} \otimes l^{\otimes 2}. \tag{5.1.7}$$

From the local description, it immediately follows that Φ_L^2 vanishes on anti-symmetric tensors. Then we can equivalently study its restriction to $I_2(L)$, which it is usually denoted by μ_L^2:

$$\mu_L^2 : I_2(L) \to H^0(C, \omega_C^{\otimes 2} \otimes L^{\otimes 2}). \tag{5.1.8}$$

If $\alpha = \sum l_i \otimes m_i \in I_2(L)$ with $l_i = f_i l$ and $m_i = g_i l$, using that $\mu_L^0(\alpha) = 0$ and $\Phi_L^1(\alpha) = 0$, one finds the identities

$$\sum f_i g_i \equiv 0 \quad \sum f_i' g_i - f_i g_i' \equiv 0. \tag{5.1.9}$$

Taking derivatives of the first equation in 5.1.9 and using the second one we obtain the identities:

$$\sum f_i' g_i \equiv \sum f_i g_i' \equiv 0, \tag{5.1.10}$$

Deriving the equations in 5.1.10 we have:

$$\sum f_i'' g_i = -\sum f_i' g_i' = \sum f_i g_i''. \tag{5.1.11}$$

Hence μ_L^2 can be equivalently locally expressed as

$$\mu_L^2(\alpha) = -\sum f_i'' g_i \; dz^{\otimes 2} \otimes l^{\otimes 2} = -\sum f_i g_i'' \; dz^{\otimes 2} \otimes l^{\otimes 2}. \tag{5.1.12}$$

More generally the local expression for Φ_L^{2k}, defined up to a sign, when $k \geq 1$ is given by

$$\Phi_L^{2k}(\alpha) = \sum f_i^{(k)} g_i^{(k)} \; dz^{\otimes 2k} \otimes l^{\otimes 2}, \tag{5.1.13}$$

and since it vanishes on antisymmetric tensors one defines

$$\mu_L^{2k} := \Phi^{2k}_{L_{|_{\ker(\mu_L^{2k-2})}}}. \tag{5.1.14}$$

Analogously the local expression for Φ_L^{2k+1} when $k \geq 0$ is given by

$$\Phi_L^{2k+1}(\alpha) = \sum (f_i^{(k+1)} g_i^{(k)} - f_i^{(k)} g_i^{(k+1)}) \; dz^{\otimes 2k+1} \otimes l^{\otimes 2}, \tag{5.1.15}$$

and since it vanishes on symmetric tensors one defines for every $k \geq 2$

$$\mu_L^{2k+1} := \Phi^{2k}_{|_{\ker(\mu_L^{2k-1})}}. \tag{5.1.16}$$

Moreover since $\alpha = \sum l_i \otimes m_i \in \ker(\mu_L^{2k-2})$ if and only if

$$\mu_L^0(\alpha) = \Phi_L^1(\alpha) = ..., = \mu_L^{2k-2}(\alpha) = \Phi_L^{2k-1}(\alpha) = 0, \tag{5.1.17}$$

and these conditions give identities

$$\sum f_i^{(h)} g_i^{(r)} \equiv 0, \forall h, r \text{ such that } h + r \leq 2k - 1, \tag{5.1.18}$$

the local expression for μ_L^{2k} is equivalent to

$$\mu_L^{2k}(\alpha) = (-1)^{m+1} \sum f_i^{(2k-m)} g_i^{(m)} dz^{\otimes 2k} \otimes l^{\otimes 2}, \tag{5.1.19}$$

for every $m = 0, ..., 2k$. Analogously $\alpha = \sum l_i \otimes m_i \in \ker(\mu_L^{2k-1})$ if and only if

$$\Phi_L^0(\alpha) = \mu_L^1(\alpha) = ... = \mu_L^{2k-1}(\alpha) = \Phi_L^{2k}(\alpha) = 0, \tag{5.1.20}$$

and these conditions give the identities

$$\sum f_i^{(h)} g_i^{(r)} \equiv 0, \forall h, r \text{ such that } h + r \leq 2k, \tag{5.1.21}$$

and from this, it follows that we can equivalently express 5.1.15 as

$$\mu_L^{2k+1}(\alpha) = (-1)^{m+1} \sum (f_i^{(2k+1-m)} g_i^{(m)} - f_i^{(m)} g_i^{(2k+1-m)}) dz^{\otimes 2k+1} \otimes l^{\otimes 2} \tag{5.1.22}$$

for every $m = 0, ..., k$.

Remark 5.1.1. Observe that for every $k \geq 1$

$$\ker(\Phi_L^{2k}) = \ker(\mu_L^{2k}) \oplus \ker(\mu_L^{2k-1}), \tag{5.1.23}$$

and from the definition of μ_L^k we have the inclusion:

$$... \subset \ker(\mu_L^{2k}) \subset \ker(\mu_L^{2(k-1)}) \subset ... \subset \ker(\mu_L^2) \subset I_2 = \ker(\mu_L^0); \tag{5.1.24}$$

$$... \subset \ker(\mu_L^{2k+1}) \subset \ker(\mu_L^{2k-1}) \subset ... \subset \ker(\mu_L^1). \tag{5.1.25}$$

5.2 Rank of higher order Wahl maps on hyperelliptic curves

In this section we will denote the Gaussian-Wahl maps $\Phi_{\omega_C}^k$ and $\mu_{\omega_C}^k$ by Φ^k and μ^k respectively. Moreover, when $k = 1$ we will write Φ and μ. The main result of this chapter is the computation of the rank of higher Wahl maps for any hyperelliptic curve of any genus. More precisely we are going to prove the following:

Theorem 5.2.1. *Let C be a hyperelliptic curve of genus $g \geq 3$. Then for every $2 \leq k \leq \frac{g-1}{2}$*

$$\operatorname{rank}(\mu^{2k}) = 2g - (4k + 1), \tag{5.2.1}$$

and is zero for every $k > \frac{g-1}{2}$,

As explained in the introduction of chapter, the result was already known for μ and μ^2 by works of Wahl and Frediani-Colombo respectively. The strategy of the proof is based on the fact that for a hyperelliptic curve we have an explicit description of a basis of $H^0(\omega_C)$, and the strategy used by Colombo and Frediani in [26] connecting the μ^2 to the μ_L where $L \simeq \omega_C(-F)$ and F is the g^1_2 on C.

The starting point is the following lemma (see for example the discussion just above Lemma 4.1, [26]).

Lemma 5.2.2. *Let C be a hyperelliptic curve of genus $g \geq 3$. Let $|F|$ be the g^1_2. Set $L = \omega_C(-F)$ and let $\omega_1, ..., \omega_{g-1}$ be a basis for $H^0(L)$ and let $\langle s, t\rangle$ be a basis for $H^0(F)$. Then the map defined by*

$$\begin{aligned} \Lambda^2 H^0(L) &\xrightarrow{\psi} I_2 \\ \omega_i \wedge \omega_j &\to Q_{ij} := t\omega_i \odot s\omega_j - t\omega_j \odot s\omega_i, \end{aligned}$$

is an isomorphism. In particular, observe that Q_{ij} gives a basis for I_2.

This isomorphism is fundamental in [26] to show the following (see [26], Lemma 4.1).

Lemma 5.2.3. *For any Q_{ij} as in the previous lemma:*

$$\mu^2(Q_{ij}) = \mu^1_F(s \wedge t)\mu^1_L(\psi^{-1}(Q_{ij})),$$

where with the expression

$$\mu^1_F(s \wedge t)\mu^1_L(\psi^{-1}(Q_{ij})) \tag{5.2.2}$$

we mean the image of $\mu^1_F(s\wedge t)\otimes\mu^1_L(\psi^{-1}(Q_{ij}))$ under the multiplication map

$$H^0(\omega_C \otimes F^{\otimes 2}) \otimes H^0(\omega_C \otimes L^{\otimes 2}) \to H^0(\omega_C^{\otimes 4}). \tag{5.2.3}$$

Observe that from Lemma 5.2.3 it follows that $Rk(\mu^2) = Rk(\mu^1_L)$ since $\mu^1_F(s \wedge t)$ is not the zero section in $H^0(\omega_C \otimes F^{\otimes 2})$. Indeed the zero locus of $\mu^1_F(s \wedge t)$ is given by the base locus of $|F|$ togeter with the ramification divisor of the induced morphism (see [26], section 2).

We start generalizing Lemma 5.2.3 to any higher-order Wahl maps. We use the same notations as in Lemma 5.2.2

Lemma 5.2.4. *Let $k \geq 0$ be an integer and let*

$$Q = \sum_{1 \leq i < j \leq g-1} a_{ij} Q_{ij} \in \ker(\mu^{2k}).$$

Then

(i) for any $k \geq 1$

$$\sum_{1\leq i<j\leq g-1} a_{ij}(\omega_i \wedge \omega_j) \in \ker(\mu_L^{2k-1}),$$

(ii) for any $k \geq 0$

$$\mu^{2k+2}(Q) = (k+1)\mu_F^1(s\wedge t)\mu_L^{2k+1}(\sum_{1\leq i<j\leq g-1} a_{ij}(\omega_i \wedge \omega_j)),$$

where with this last expression we mean the image of

$$(k+1)\mu_F^1(s\wedge t) \otimes \mu_L^{2k+1}(\sum_{1\leq i<j\leq g-1} a_{ij}(\omega_i \wedge \omega_j))$$

under the multiplication map

$$H^0(\omega_C \otimes F^{\otimes 2}) \otimes H^0(\omega_C^{2k+1} \otimes L^{\otimes 2}) \to H^0(\omega_C^{\otimes(2k+4)}).$$

Proof. Let us proceed by induction. When $k = 0$ (ii) is Lemma 5.2.3, and when $k = 1$, (i) is an immediate consequence of the hypothesis $\mu^2(Q) = 0$ together with Lemma 5.2.3.

Now take $n \geq 2$ and suppose that (ii) holds for every $0 \leq k < n-1$ and (i) holds for every $1 \leq k < n$. We are going to prove that (ii) holds for $k = n-1$, which automatically implies that (i) holds for $k = n$. Set $k = n-1$ and suppose that

$$Q = \sum_{1\leq 1<j\leq g-1} a_{ij}Q_{ij} \in \ker(\mu^{2k}).$$

On some open sets write

$$\omega_i = f_i dz',\ t = fT \text{ and } s = gT,$$

where f_i is a holomorphic function, dz' is a local generator on $L = \omega_C(-F)$ and T is a local generator on F. From the definition of Q_{ij} in Lemma 5.2.2, it follows that we can write locally Q as

$$\sum_{1\leq 1<j\leq g-1} a_{ij}Q_{ij} = \sum a_{ij}(ff_i dz \odot gf_j dz - ff_j dz, \odot gf_i dz), \tag{5.2.4}$$

where we can take z to be a local coordinate on C. Then, by definition

$$\mu^{2k+2}(\sum_{1\leq 1<j\leq g-1} a_{ij}Q_{ij})$$
$$= \sum_{1\leq 1<j\leq g-1} a_{ij}((ff_i)^{(k+1)}(gf_j)^{(k+1)} - (ff_j)^{(k+1)}(gf_i)^{k+1})dz^{\otimes(2k+4)}.$$

Now we proceed with some algebraic manipulations. The latter expression can be written as

$$= \left[\sum_{1\leq i<j\leq g-1} a_{ij} \sum_{h=0}^{k+1} \binom{k+1}{h} f^{(k+1-h)} f_i^{(h)} \sum_{l=0}^{k+1} \binom{k+1}{l} g^{(k+1-l)} f_j^{(l)}\right] dz^{\otimes(2k+4)} \tag{5.2.5}$$

$$- \left[\sum_{1\leq i<j\leq g-1} a_{ij} \sum_{r=0}^{k+1} \binom{k+1}{r} f^{(k+1-r)} f_j^{(r)} \sum_{e=0}^{k+1} \binom{k+1}{e} g^{(k+1-e)} f_i^{(e)}\right] dz^{\otimes(2k+4)}$$

$$= \left[\sum_{1\leq i<j\leq l-1} a_{ij} \sum_{h,l=0}^{k+1} \binom{k+1}{h}\binom{k+1}{l} f^{(k+1-h)} g^{(k+1-l)} f_i^{(h)} f_j^{(l)}\right] dz^{\otimes(2k+4)} \tag{5.2.6}$$

$$- \left[\sum_{1\leq i<j\leq l-1} a_{ij} \sum_{r,e=0}^{k+1} \binom{k+1}{r}\binom{k+1}{e} f^{(k+1-r)} g^{(k+1-e)} f_j^{(r)} f_i^{(e)}\right] dz^{\otimes(2k+4)} \tag{5.2.7}$$

$$= \left[\sum_{1\leq i<j\leq l-1} a_{ij} \sum_{h,l=0}^{k+1} \binom{k+1}{h}\binom{k+1}{l} f^{(k+1-h)} g^{(k+1-l)} (f_i^{(h)} f_j^{(l)} - f_j^{(h)} f_i^{(l)})\right] dz^{\otimes(2k+4)} \tag{5.2.8}$$

$$= \left[\sum_{h,l=0}^{k+1} \binom{k+1}{h}\binom{k+1}{l} f^{(k+1-h)} g^{(k+1-l)} \sum_{1\leq i<j\leq l-1} a_{ij} (f_i^{(h)} f_j^{(l)} - f_j^{(h)} f_i^{(l)})\right] dz^{\otimes(2k+4)} \tag{5.2.9}$$

By hypothesis $Q \in \ker(\mu^{2k})$ and hence from the inductive hypothesis

$$\mu_L^{2k-1}(\sum_{1\leq i<j\leq g-1} a_{ij}(\omega_i \wedge \omega_j)) = 0.$$

Then using 5.1.21 it follows that for every $h + l \leq 2k$

$$\sum_{1\leq i<j\leq g-1} a_{ij}(f_i^{(h)} f_j^{(l)} - f_j^{(h)} f_i^{(l)}) \equiv 0.$$

Then (5.2.9) becomes

$$\left[\binom{k+1}{k} f g^{(1)} \Big(\sum_{1\le i<j\le l-1} a_{ij} (f_i^{(k+1)} f_j^{(k)} - f_j^{(k+1)} f_i^{(k)} \Big)\right] dz^{\otimes(2k+4)} +$$
$$\left[\binom{k+1}{k} f^{(1)} g \Big(\sum_{1\le i<j\le l-1} a_{ij} (f_i^{(k)} f_j^{(k+1)} - f_j^{(k)} f_i^{(k+1)} \Big)\right] dz^{\otimes(2k+4)}.$$

This is just

$$\left[\binom{k+1}{k} (f g^{(1)} - f^{(1)} g) \Big(\sum_{1\le i<j\le l-1} a_{ij} (f_i^{(k+1)} f_j^{(k)} - f_j^{(k+1)} f_i^{(k)} \Big)\right] dz^{\otimes(2k+4)},$$

which is equal to

$$(k+1)\mu_F^1(t\wedge s)\mu_L^{2k+1}\Big(\sum_{1\le i<j\le l-1} a_{ij}(\omega_i\wedge\omega_j)\Big),$$

and so (ii) holds. □

In the following lemma, we are going to describe the equations of the loci

$$... \subset \ker(\mu^{2k}) \subset \ker(\mu^{2(k-1)}) \subset ... \subset \ker(\mu^2) \subset I_2 = \ker(\mu_L^0).$$

The equation of $\ker(\mu^2)$ in I_2 have already been described in the proof of Proposition 4.2, [26]. Indeed we have the following

Lemma 5.2.5. *Let*

$$Q = \sum_{1\le i<j\le g-1} a_{ij} Q_{ij} \in I_2,$$

Q_{ij} *as in 5.2.2. Then* $Q \in \ker(\mu^2)$ *if and only if for all* $3 \le l \le 2g-3$

$$\sum_{\substack{1\le i<j \\ i+j=l}} a_{ij}(j-i) = 0. \tag{5.2.10}$$

Now we generalize the approach in [26]. The strategy is to use Lemma 5.2.4 together the well-known description of a basis of the $H^0(C,\omega_C)$ for a hyperelliptic curve C. More precisely since we want to describe when a section $s \in \mu^{2k}(\ker(\mu^{2(k-1)})$ is zero, we can reason locally and suppose that on some open set with coordinate x a basis for $H^0(C,\omega_C)$ is given by

$$\Big(\frac{dx}{y}, x\frac{dx}{y}, ..., x^{g-1}\frac{dx}{y}\Big).$$

Moreover we can suppose that the morphism to $\mathbb{P}^1$ given by $|F|$(the g^1_2) is given (on this open set) by $(x,y) \mapsto x$. Hence a local description of a basis for $H^0(C,L)$ is given by

$$(x\frac{dx}{y}, ..., x^{g-1}\frac{dx}{y}). \tag{5.2.11}$$

Lemma 5.2.6. *Let*

$$Q = \sum_{1\leq i<j\leq g-1} a_{ij}Q_{ij} \in I_2,$$

Q_{ij} as in 5.2.2 (recall that they are a basis of I_2), and let $k \geq 2$. For any $k \geq 2$, $Q \in \ker(\mu^{2k})$,if and only if

- *$\forall\ 3 \leq l \leq 2g-3$,*

$$\sum_{\substack{1\leq i<j\\ i+j=l}} a_{ij}(j-i) = 0,$$

which is the condition of belonging in $\ker(\mu^2)$ *(5.2.10);*

- *$\forall\ 2 \leq m \leq k$, $\forall\ 2m-1 \leq l \leq 2g-3$,*

$$\sum_{\substack{1\leq i<j\\ i+j=l\\ i\geq m-1}} a_{ij}(j-i)ij(i-1)(j-1)...(i-(m-2))(j-(m-2)) = 0$$

Proof. As we have said we will reason locally and assume that a basis of $H^0(C,\omega_C)$ is given by 5.2 and a basis for $H^0(C,L)$ is given by 5.2.11. We will proceed by induction on k. The thesis holds for $k=1$ by Lemma 5.2.5. Now we suppose that the thesis holds for $k-1$, $k\geq 2$ and take $Q \in \ker(\mu^{2k-2})$. Since Q is also in the previous kernels (recall the inclusions in 5.1.19 proving the lemma is then equivalent to prove that $Q \in \ker(\mu^{2k})$ if and only if

$$\sum_{\substack{1\leq i<j\\ i+j=l\\ i\geq k-1}} a_{ij}(j-i)ij(i-1)(j-1)...(i-(k-2))(j-(k-2)) = 0. \tag{5.2.12}$$

for all $2k-1 \leq l \leq 2g-3$. Using lemma 5.2.4 we have that $\mu^{2k}(Q) = 0$ if and only if $\mu_L^{2k-1}(\sum_{1\leq i<j\leq g-1} a_{ij}(\omega_i \wedge \omega_j)) = 0$, that is iff

$$\left[\sum_{1\leq i<j\leq g-1} a_{ij}((\frac{x^i}{y})^{(k-1)}(\frac{x^j}{y})^{(k)} - (\frac{x^i}{y})^{(k)}(\frac{x^j}{y})^{(k-1)})\right] dx^{\otimes(2k-1)} \otimes dz'^{\otimes 2} = 0, \tag{5.2.13}$$

where as usual dz' is a local generator for L. Of course, this is equivalent to show that

$$\sum_{1\leq i<j\leq g-1} a_{ij}((x^i)^{(k-1)}(x^j)^{(k)} - (x^i)^{(k)}(x^j)^{(k-1)}) \equiv 0. \qquad (5.2.14)$$

Observe that 5.2.14 is equal to

$$\sum_{\substack{1\leq i<j\leq g-1\\ i\geq k-1}} a_{ij} i...(i-(k-2))x^{(i-(k-1))} j...(j-(k-1))x^{j-k} \qquad (5.2.15)$$

$$- \sum_{\substack{1\leq i<j\leq g-1\\ i\geq k}} a_{ij} i...(i-(k-1))x^{(i-k)} j...(j-(k-2))x^{j-(k-1)} \qquad (5.2.16)$$

$$= \sum_{\substack{1\leq i<j\leq g-1\\ i\geq k-1}} x^{(i+j-(2k-1))} a_{ij} i...(i-(k-2)) j...(j-(k-1)) \qquad (5.2.17)$$

$$- \sum_{\substack{1\leq i<j\leq g-1\\ i\geq k}} x^{(i+j-(2k-1))} a_{ij} i...(i-(k-1)) j...(j-(k-2)) \qquad (5.2.18)$$

$$= \sum_{\substack{1\leq i<j\leq g-1\\ i\geq k}} x^{(i+j-(2k-1))} a_{ij} i...(i-(k-2)) j...(j-(k-2))(j-(k-1)-(i-(k-1)) \qquad (5.2.19)$$

$$+ \sum_{\substack{1\leq i<j\leq g-1\\ i=k-1}} x^{(i+j-(2k-1))} a_{ij} i...(i-(k-2)) j...(j-(k-1)) \qquad (5.2.20)$$

$$= \sum_{\substack{1\leq i<j\leq g-1\\ i\geq k}} x^{(i+j-(2k-1))} a_{ij} i...(i-(k-2)) j...(j-(k-2))(j-i) \qquad (5.2.21)$$

$$+ \sum_{\substack{1\leq i<j\leq g-1\\ i=k-1}} x^{(i+j-(2k-1))} a_{ij} i...(i-(k-2)) j...(j-(k-1)) \qquad (5.2.22)$$

$$= \sum_{\substack{1\leq i<j\leq g-1\\ i\geq k-1}} x^{(i+j-(2k-1))} a_{ij} i...(i-(k-2)) j...(j-(k-2))(j-i). \qquad (5.2.23)$$

So we have shown that for $Q \in \ker \mu^{2k-2}$, $\mu^{2k}(Q) = 0$ if and only if for every $2k-1 \leq l \leq 2g-3$

$$= \sum_{\substack{1\leq i<j\leq g-1\\ i+j=l\\ i\geq k-1}} a_{ij}(j-i) i...(i-(k-2)) j...(j-(k-2)) = 0.$$

and so we have the thesis.

□

Now we come to the main proposition of the section, from which it will follow Theorem 5.2.1.

Proposition 5.2.7. *Let C be a hyperelliptic curve of genus g. Let $2 \leq k \leq \frac{g-1}{2}$. Then* $\dim(\ker(\mu^{2k}) = \dim\ker(\mu^{2(k-1)}) - (2g-(4k+1)))$. *More precisely, the equations*

$$\sum_{\substack{1\leq i<j\leq g-1 \\ i+j=l \\ i\geq k-1}} a_{ij}(j-i)ij(i-1)(j-1)()(i-(k-2))(j-(k-2)) = 0, \quad 2k-1 \leq l \leq 2g-3,$$

impose $2g-(4k+1)$ *linearly independent conditions. The linearly independent equations are given by*

$$\sum_{\substack{1\leq i<j\leq g-1 \\ i+j=l \\ i\geq k-1}} a_{ij}(j-i)ij(i-1)(j-1)...(i-(k-2))(j-(k-2)) = 0, \quad 2k+1 \leq l \leq 2g-(2k+1).$$

Proof. We will start by introducing some notations. For every $l = 3, ..., 2g-3$, for every i, j such that $1 \leq i < j \leq g-1$, let $(w_l)^2$ the vector whose coordinate are

$$(w_l)^2_{ij} = \begin{cases} (j-i) & \text{if } i+j=l \\ 0 & \text{if } i+j \neq l \end{cases}$$

i.e. the coordinates vector of the lth equation of the set of equations which describe $\ker(\mu^2)$ in I_2, ordered by increasing values of i (see Lemma 5.2.6). For every $2 \leq r \leq k$ and $l = 2r-1, ..., 2g-3$, let $w_l^{(2r)}$ be the vector whose coordinate are

$$(w_l)^{(2r)}_{ij} = \begin{cases} (j-i)ij(i-1)(j-1)...(i-(r-2))(j-(r-2)) & \text{if } i+j=l, i, j \geq r-1 \\ 0 & \text{if } i+j \neq l, \text{ or } 1 \leq i \leq r-2. \end{cases} \tag{5.2.24}$$

i.e. the coordinates vector of the lth equation of the set of equations which describe $\ker(\mu^{2r})$ inside $(\ker(\mu^{2(r-1)}))$, ordered by increasing values of i.

For any $3 \leq l \leq 2g-3$ set

$$n_l = \#\{(i,j) : 1 \leq i < j \leq g-1, \ i+j=l\}. \tag{5.2.25}$$

and let $B'_{k,l}$ be the $k \times n_l$ matrix whose rth row, $1 \leq r \leq k$, are the n_l coordinates of $w_l^{(2r)}$ corresponding to the indices (i,j) such that $i+j=l$.

Observe that $B'_{k,l}$ is just the matrix with rows $w_l^{(2)}, ..., w_l^{(2k)}$ where we have removed the entries corresponding to $i + j \neq l$. Observe that these entries are all 0 for all the $w_l^{(r)}$ vectors.

Define $c_{k,l} = min\{n_l, k\}$ and let $B_{k,l}$ be the minor of order $c_{k,l}$ given by the first $c_{k,l}$ rows and $c_{k,l}$ columns. We are going to prove that it is not zero for all $3 \leq l \leq 2g - 3$ by induction on k.

Consider first the case $k = 2$. Observe that if $l = 3, 4, 2g - 4, 2g - 3$, then $n_l = 1$. In this case $B_{2,l} = j - i > 0$ where (i, j) is the only pair such that $i + j = l$. If $5 \leq l \leq 2g - 5$ then $n_l \geq 2$ and $c_{2,l} = min(2, n_l) = 2$ and we have

$$B_{2,l} = \begin{pmatrix} j - i & j - 1 - 2 \\ ij(j - i) & (j - i - 2)(i + 1)(j - 1). \end{pmatrix}$$

where i is the minimum such that $i + j = l$. Observe that

$$det(B_{2,l}) = (j - 1)(j - 1 - 2)det\left(\begin{pmatrix} 1 & 1 \\ ij & (i + 1)(j - 1). \end{pmatrix}\right)$$

Set

$$A_{2,l} := \left(\begin{pmatrix} 1 & 1 \\ ij & (i + 1)(j - 1). \end{pmatrix}\right).$$

Subtracting the first column of $A_{2,l}$ from the second we obtain the matrix

$$\begin{pmatrix} 1 & 0 \\ ij & j - i - 1. \end{pmatrix}.$$

whose determinant is $j - i - 1 > 0$. Now assume that for every $2 \leq k_0 \leq k$ and for every $3 \leq l \leq 2g - 3$, $B_{k,l}$ is not zero. We want to prove that the same holds for $k + 1$. If $n_l \leq k$, then $n_l < k + 1$, $c_{k+1,l} = min\{n_l, k\} = n_l$ and $B_{k+1,l} = B_{k,l}$. In this case, we have nothing to prove. Assume then $n_l > k$. Then $c_{k+1,l} = min\{n_l, k + 1\} = k + 1$. Then $B_{k+1,l}$ is the following $k + 1 \times k + 1$ matrix

$$\begin{pmatrix} j - i & j - i - 2 & \dots & j - i - 2k \\ (j - i)ji & (j - 2 - i)(i + 1)(j - 1) & \dots & (j - i - 2k)(j - k)(i + k) \\ \vdots & \vdots & \vdots & \vdots \\ (j - i)ji...(j - (k + 1 - 2))(i - (k + 1 - 2)) & & \dots & (j - i - 2k)(j - k)(i + k)...(j - 2k + 1)(i + 1) \end{pmatrix}$$

where i is the minimum such that $i + j = l$ and where we mean that every entries in the r row, $1 \leq r \leq k + 1$ (corresponding to an index (i, j)) is zero whenever $i \leq r - 1$ (recall the definition of $w_l^{(2r)}$, 5.2.24)

The determinant of the above matrix is equal to the product of $(j - i)...(j - i - 2k)$ by the determinant of the following matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ ji & (i+1)(j-1) & \ldots & (j-k)(i+k) \\ \vdots & \vdots & \vdots & \vdots \\ (j-i)ji\ldots(j-(k+1-2))(i-(k+1-2)) & & \ldots & (j-k)(i+k)\ldots(j-2k+1)(i+1) \end{pmatrix}$$

Now consider the matrix above and subtract each column to the previous one. Then one gets a matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ ji & (j-1)-i & \ldots & (j-1)-i-2(k-1) \\ \vdots & \vdots & \vdots & \vdots \\ (j-i)ji\ldots(j-(k+1-2))(i-(k+1-2)) & & \ldots & (j-1-i-2(k-1))(j-1-(k-1))(i+k-1)\ldots(j-2(k-1)+1)(i+1) \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} (j-1)-i & \ldots & (j-1)-i-2(k-1) \\ \vdots & \vdots & \vdots \\ (j-1-i)(j-1)i\ldots((j-1-(k-2)))(i-((k-2))) & \ldots & (j-1-i-2(k-1))(j-1-(k-1))(i+k-1)\ldots(j-2(k-1)+1)(i+1) \end{pmatrix}.$$

is $B_{k,l-1}$ (where as we have already said, we mean that every entry corresponding to a index (i,j) in the r row, $1 \leq r \leq k$ is zero whenever $i \leq r-1$. The matrix has not zero determinant by the inductive hypothesis. Hence until this point we have shown, by induction, that for any $3 \leq l \leq 2g-3$ and for any $k \geq 2$, $(w_l)^2$,...,$(w_l)^{2k}$ impose exactly $c_{k,l}$ linearly independent conditions.

Now let s_k be the number of $\{l\}$ such that $c_{k,l} > c_{k-1,l}$ (this equivalent to say that lth equation of $\ker(\mu^{2k})$ is independent from the lth equation of $\ker(\mu^2), ..., \ker(\mu^{2(k-1)})$). This is what we want to find since then we will have

$$\dim(\ker(\mu^{2k}) = \dim\ker(\mu^{2(k-1)}) - s_k. \tag{5.2.26}$$

Observe that we are also using that if $l \neq l'$, the equations associated with $(w_l)^{(2r)}$ and $(w_{l'})^{(2r')}$ involve different variables.

Now notice that $c_{k,l} > c_{k-1,l}$, if and only if $k-1 < n_l \iff k \leq n_l$. This happens if and only if $2k+1 \leq l \leq 2g-(2k+1)$. We explain it.

Indeed if $2k+1 \leq l \leq 2g-(2k+1)$, we have at least k coordinates (i,j) such that $i+j=l$. Indeed write l as $l = 2k+1+m$, $0 \leq m \leq 2g-2(2k+1)$ (where we are using that $2g-2(2k+1) \geq 0$, which we have by the hypothesis $2 \leq k \leq \frac{g-1}{2}$ on the statement of the Proposition).

If l is odd (equivalently m is even), these are given by

$$(i,j) = (1+\frac{m}{2}, 2k+\frac{m}{2}), ..., (k+\frac{m}{2}, k+1+\frac{m}{2})$$

These are easily seen to be admissible indices (i,j), that is they satisfy $1 \leq i < j$, $i+j = l$ and $j \leq g-1$. If l is even (equivalently m is odd),these are given by

$$(i,j) = (1+[\frac{m}{2}], 2k+[\frac{m}{2}])], ..., (k+[\frac{m}{2}], k+2+[\frac{m}{2}])$$

On the other hand, if $l \leq 2k$ or $l \geq 2g-2k$, it is easy to see that we have strictly less than k coordinates such that $i+j = l$ (using that (i,j) have to satisfy also $1 \leq i < j \leq g-1$). We then conclude that $s_k = 2g-(2k+1)-2k = 2g-(4k+1)$ and hence $\dim(\ker(\mu^{2k}) = \dim\ker(\mu^{2(k-1)}) - s_k = \dim\ker(\mu^{2(k-1)}) - (2g-(4k+1))$. □

As an immediate corollary, we get the Theorem 5.2.1.

Proof of Theorem 5.2.1. Since the domain of $\mu^{(2k)}$ is $\ker(\mu^{(2(k-1)})$, from Proposition 5.2.7 we immediately get that if $2 \leq k \leq \frac{g-1}{2}$, the rank of $\mu^{(2k)}$ is $2g-(4k+1)$. Now we show that if $k > \frac{g-1}{2}$ then $\mu^{(2k)}$ is zero.

First suppose that g is odd. By the usual inclusions (Remark 5.1.1) the thesis is equivalent to show that $\mu^{(2k)}$ is zero for $k = \frac{g-1}{2}+1$. Since the domain of $\mu^{(g+1)}$ is $\ker(\mu^{(g-1)})$, it is sufficient to show that the latter is zero. By proposition 5.2.7 we have that the rank of $\mu^{(2r)}$, $r = 1,..,\frac{g-1}{2}$ is $2g-(4r+1)$. Hence

$$\dim(\ker(\mu^{(g-1)})) = \dim I_2 - \sum_{i=1}^{\frac{g-1}{2}}(2g-(4i+1)) \tag{5.2.27}$$

$$= \frac{g(g+1)}{2} - (2g-1) - \sum_{i=1}^{\frac{g-1}{2}}(2g-(4i+1)), \tag{5.2.28}$$

which is easy to see that is zero.

Analogously, if g is even, the thesis is equivalent to show that $\mu^{(2k)}$ is zero for $k = \frac{g-2}{2}+1 = \frac{g}{2}$. Since the domain of $\mu^{(2k)}(=\mu^{(g)})$ is $\ker(\mu^{(g-2)})$, it is sufficient to show that the latter is zero. For this we have

$$\dim(\ker(\mu^{(g-2)})) = \dim I_2 - \sum_{i=1}^{\frac{g-2}{2}}(2g - (4i+1)) \tag{5.2.29}$$

$$= \frac{g(g+1)}{2} - (2g-1) - \sum_{i=1}^{\frac{g-2}{2}}(2g - (4i+1)), \tag{5.2.30}$$

which again is immediate to see that is zero.

□

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