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A Nečas-Lions inequality with symmetric gradients on star-shaped domains based on a first order Babuška-Aziz inequality

Michele Botti [a], Lorenzo Mascotto [b,c,d,*]

[a] MOX, Department of Mathematics, Politecnico di Milano, 20133 Milan, Italy
[b] Department of Mathematics and Applications, University of Milano-Bicocca, 20125 Milan, Italy
[c] Faculty of Mathematics, University of Vienna, 1090 Vienna, Austria
[d] IMATI-CNR, 27100 Pavia, Italy

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ABSTRACT

We prove a Nečas-Lions inequality with symmetric gradients on two and three dimensional domains of diameter R that are star-shaped with respect to a ball of radius ρ; we exhibit a bound for the constant appearing in that inequality, which is explicit with respect to R and ρ. Crucial tools in the derivation of such a bound are a first order Babuška-Aziz inequality based on Bogovskiĭ's construction of a right-inverse of the divergence and Fourier transform techniques proposed by Durán. As a byproduct, we derive arbitrary order estimates in arbitrary dimension for Bogovskiĭ's operator, with upper bounds on the corresponding constants that are explicit with respect to R and ρ.

1. Introduction

We derive a Nečas-Lions inequality with symmetric gradients on star-shaped domains in two and three dimensions; first order Babuška-Aziz and Nečas-Lions inequalities are crucial tools to show an upper bound on the constant appearing in that inequality, which is explicit with respect to certain geometric quantities of the domain. As a byproduct, we also derive arbitrary order Babuška-Aziz inequalities in arbitrary dimension with explicit bounds on the corresponding constant.

Outline of the introduction After introducing the functional setting and the domains of interest, we review the literature and the main concepts related to the lowest order Babuška-Aziz and Nečas-Lions inequalities in Sections 1.1 and 1.2. In Sections 1.3 and 1.4, we discuss the generalisation of these two results to the

* Corresponding author.
E-mail addresses: michele.botti@polimi.it (M. Botti), lorenzo.mascotto@unimib.it (L. Mascotto).

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first order case. The Nečas-Lions inequality with symmetric gradients, based on all the foregoing results, is shown in Section 1.5. Finally, we describe the outline of the remainder of the paper.

Functional spaces and notation In what follows, ∇, $\nabla\times$, and $\nabla\cdot$ denote the gradient, curl, and divergence operators. The operators ∇_S and ∇_{SS} are the symmetric and skew-symmetric parts of ∇:

$$\nabla = \nabla_S + \nabla_{SS}. \tag{1}$$

We use standard notation [14] for Sobolev spaces on Lipschitz domains Ω with boundary $\partial\Omega$. The outward unit normal vector to $\partial\Omega$ is $\mathbf{n}_\Omega$. $H^s(\Omega)$ denotes the Sobolev space of order $s \geq 0$, which we equip with inner product $(\cdot,\cdot)_s$, seminorm $|\cdot|_{s,\Omega}$, and norm $\|\cdot\|_{s,\Omega}$. The case $s = 0$ corresponds to $H^0(\Omega) = L^2(\Omega)$. The space of functions in $L^2(\Omega)$ with zero average over Ω is denoted by $L^2_0(\Omega)$.

For s positive, we define $H^s_0(\Omega)$ as the closure of $\mathcal{C}^\infty_0(\Omega)$ with respect to the $H^s(\Omega)$ norm. In what follows, we shall particularly use the spaces $H^1_0(\Omega)$ and $H^2_0(\Omega)$, which coincide [22] with the spaces of functions with zero trace, and functions with zero trace and whose gradients have zero trace over $\partial\Omega$, respectively.

Negative order Sobolev spaces are defined by duality. We introduce the spaces $H^{-1}(\Omega) := [H^1_0(\Omega)]^*$ and $H^{-2}(\Omega) := [H^2_0(\Omega)]^*$ equipped with the norms

$$\|u\|_{-1,\Omega} := \sup_{v\in H^1_0(\Omega)} \frac{{}_{-1}\langle u,v\rangle_1}{|v|_{1,\Omega}}, \qquad \|u\|_{-2,\Omega} := \sup_{v\in H^2_0(\Omega)} \frac{{}_{-2}\langle u,v\rangle_2}{|v|_{2,\Omega}}, \tag{2}$$

where ${}_{-\ell}\langle\cdot,\cdot\rangle_\ell$ is the duality pairing between $H^{-\ell}(\Omega)$ and $H^\ell_0(\Omega)$.

The definitions above extend to the case of vector fields and tensors. With an abuse of notation, the norms on scalar, vector fields, and tensors are denoted with the same symbols.

For positive a and b, by $a \lesssim b$, we shall occasionally mean that there exists a positive constant c independent of relevant geometric parameters such that $a \leq c\, b$. An extra subscript makes it explicit a hidden dependence on a parameter of interest.

Domains of interest Henceforth, Ω in $\mathbb{R}^n$ is a

$$\text{domain of diameter } R \text{ that is star-shaped with respect to a ball } B_\rho \text{ of radius } \rho. \tag{3}$$

1.1. The lowest order Babuška-Aziz inequality

The standard, lowest order version of the Babuška-Aziz inequality was proven as early as 1961 by Cattabriga [10]. However, the name is associated to the authors of the later work [5] and was assigned by Horgan and Payne [19]; see also [11].

The inequality reads as follows: there exists a positive constant $C_{\mathcal{BA},0}$ such that for any f in $L^2_0(\Omega)$ one can construct $\mathbf{u}$ in $[H^1_0(\Omega)]^n$ satisfying

$$\nabla\cdot \mathbf{u} = f, \qquad |\mathbf{u}|_{1,\Omega} \leq C_{\mathcal{BA},0}\|f\|_{0,\Omega}. \tag{4}$$

The subscript in $C_{\mathcal{BA},0}$ relates to Babuška-Aziz. Explicit constructions of a vector field $\mathbf{u}$ as in (4) may be performed in different ways; here, we shall follow the approach by Bogovskiĭ in [6,7], where he showed a particular construction of the right inverse of the divergence based on integral kernels. Alternative avenues, which give less information on the constant $C_{\mathcal{BA},0}$ and construct less smooth right-inverses of the divergence, are based on solving curl-div, diffusion, or Stokes problems; see, e.g., [4] and [9, Lemma 11.2.3]; as such, higher order estimates based on this approach require extra regularity assumptions on the boundary of the domain, which are instead not needed following Bogovskiĭ's approach.

Minimal literature on the lowest order Babuška-Aziz inequality The literature associated with inequality (4) is widespread. We refer to [15, pp. 227-228], [18], and [11] for a thorough historical review.

Here, we only mention that the divergence problem in (4) was raised as early as in 1961 by Cattabriga [10]; see also the later works [20,23]. Bogovskiĭ introduced an explicit representation for $\mathbf{v}$ solving (4) in [6, 7]. Several references discuss the validity of similar estimates; for instance, implicit constants for general Lipschitz domains are available in [8, Theorems 2.4 and 2.9]. The case of negative Sobolev norms is described in [16].

The explicit dependence of the constant $C_{\mathcal{BA},0}$ on geometric parameters of Ω is studied in fewer references; see [1] for a list. For R and ρ as in (3), Galdi [15] gives estimates of the form

$$C_{\mathcal{BA},0} \lesssim_n \left(\frac{R}{\rho}\right)^{n+1}. \tag{5}$$

The main tool in the analysis is the one discussed originally by Bogovskiĭ [6,7], i.e., the Calderón-Zygmund singular integral operator theory.

Improved estimates of the form

$$C_{\mathcal{BA},0} \lesssim_n \frac{R}{\rho}\left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n-2}{2(n-1)}} \left(\log\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n}{2(n-1)}} \tag{6}$$

were proven by Durán [13] based on the properties of the Fourier transform. In the same reference, it is shown that the estimates are optimal up to the logarithmic factor for $n = 2$. More precisely, a 2D counterexample is exhibited showing that the following holds true:

$$C_{\mathcal{BA},0} \gtrsim \frac{R}{\rho}.$$

For the two dimensional case, Costabel and Dauge [11, Theorem 2.3] proved that the logarithmic factor in Durán's estimates can be removed again for $n = 2$.

Minimal literature on higher order Babuška-Aziz inequalities Higher order Babuška-Aziz inequalities are far less investigated. They are stated in the original paper by Bogovskiĭ [6] without mention on the behaviour of the constants. Galdi [15, Remark III.3.2] claims that similar bounds to the lowest order case can be derived; however, no explicit constants are given in that case as well; the analysis hinges upon the Calderón-Zygmund theory. Costabel and McIntosh prove arbitrary order estimates [12] without explicit dependence on the geometry on the domain. Guzmán and Salgado [18] prove an explicit first order generalised Poincaré inequality, which is related to Bogovskiĭ's operator without imposition of boundary conditions, and give a road map on how to prove higher order explicit estimates; tools as those in [13] are employed; no estimates are given for Bogovskiĭ's operator.

1.2. The lowest order Nečas-Lions inequality

The standard, lowest order Nečas-Lions inequality is a very well known result in the theory of Sobolev spaces. It is proven in the book by Nečas [23, Lemma 3.7.1]; the connection to the name of Lions is less clear, and is probably due to [21, Note 27, page 320], where the result is mentioned as a private communication by Lions himself to Magenes and Stampacchia, yet without an explicit proof.

Given $\Pi^0 : L^1(\Omega) \to \mathbb{R}$ the average operator over Ω, the inequality reads as follows: there exists a positive constant $C_{\mathcal{NL},0}$ such that

$$\left\|f - \Pi^0 f\right\|_{0,\Omega} \leq C_{\mathcal{NL},0}\|\nabla f\|_{-1,\Omega} \qquad \forall f \in L^2(\Omega). \tag{7}$$

The subscript in $C_{\mathcal{NL},0}$ relates to Nečas-Lions. An equivalent statement for (7) is that the following constants are bounded from above and below, respectively:

$$C_{\mathcal{NL},0} := \sup_{f\in L^2(\Omega)} \frac{\left\|f-\Pi^0 f\right\|_{0,\Omega}}{\|\nabla f\|_{-1,\Omega}}, \qquad C_{\mathcal{NL},0}^{-1} := \inf_{f\in L^2(\Omega)} \frac{\|\nabla f\|_{-1,\Omega}}{\left\|f-\Pi^0 f\right\|_{0,\Omega}}. \tag{8}$$

The constants $C_{\mathcal{BA},0}$ and $C_{\mathcal{NL},0}$ in (4) and (7) are related to constants in other relevant inequalities in Sobolev spaces as well, including the standard inf-sup constant β_0 defined as

$$\inf_{f\in L^2_0(\Omega)} \sup_{\mathbf{u}\in[H^1_0(\Omega)]^n} \frac{(\nabla\cdot\mathbf{u}, f)_{0,\Omega}}{\|\mathbf{u}\|_{1,\Omega}\|f\|_{0,\Omega}} =: \beta_0. \tag{9}$$

Proposition 1.1. *Let $C_{\mathcal{BA},0}$, $C_{\mathcal{NL},0}$, and β_0 be given in (4), (7), and (9). Then, the following holds true:*

$$C_{\mathcal{BA},0} \geq C_{\mathcal{NL},0} = \beta_0^{-1}.$$

Proof. For all f in $L^2_0(\Omega)$, the definition of negative norms in (2) implies

$$\sup_{\mathbf{u}\in[H^1_0(\Omega)]^n} \frac{(\nabla\cdot\mathbf{u}, f)_{0,\Omega}}{\|\mathbf{u}\|_{1,\Omega}\|f\|_{0,\Omega}} = \sup_{\mathbf{u}\in[H^1_0(\Omega)]^n} \frac{{}_{-1}\langle\nabla f,\mathbf{u}\rangle_1}{\|\mathbf{u}\|_{1,\Omega}\|f\|_{0,\Omega}} = \frac{\|\nabla f\|_{-1,\Omega}}{\|f\|_{0,\Omega}}.$$

We take the inf over all possible f in $L^2_0(\Omega)$, use (8), and deduce that $\beta_0 = C_{\mathcal{NL},0}^{-1}$.

On the other hand, for all f in $L^2_0(\Omega)$, we can consider a specific $\mathbf{u}$ satisfying (4), which gives

$$\sup_{\mathbf{u}\in[H^1_0(\Omega)]^n} \frac{(\nabla\cdot\mathbf{u}, f)_{0,\Omega}}{\|\mathbf{u}\|_{1,\Omega}\|f\|_{0,\Omega}} \geq \frac{\|f\|^2_{0,\Omega}}{\|\mathbf{u}\|_{1,\Omega}\|f\|_{0,\Omega}} \geq C_{\mathcal{BA},0}^{-1}.$$

Taking the inf over all possible f in $L^2_0(\Omega)$ and recalling the standard inf-sup condition (9) give $\beta_0 \geq C_{\mathcal{BA},0}^{-1}$. The assertion follows. □

Since an upper bound on $C_{\mathcal{BA},0}$ is available from Lemma (4), which is explicit in terms of n, R, and ρ as in (3), then Proposition 1.1 implies an upper bound for $C_{\mathcal{NL},0}$ and a lower bound for β_0 with the same explicit dependence. The relation with constants appearing in other inequalities is discussed, amongst others, in [2,11,19].

1.3. Main result 1: a first order Babuška-Aziz inequality

An important tool in the proof of Theorem 1.4 below is the proof of a Babuška-Aziz inequality, based on first order estimates for Bogovskiĭ's construction of the right-inverse of the divergence. More precisely, there exist positive constants $C_{\mathcal{BA},1}^A$ and $C_{\mathcal{BA},1}^B$ such that for all f in $H^1_0(\Omega)\cap L^2_0(\Omega)$, one can construct $\mathbf{u}$ in $[H^2_0(\Omega)]^n$ satisfying

$$\nabla\cdot\mathbf{u} = f, \qquad |\mathbf{u}|_{2,\Omega} \leq C_{\mathcal{BA},1}^A\|f\|_{0,\Omega} + C_{\mathcal{BA},1}^B|f|_{1,\Omega}. \tag{10}$$

We state the result here and postpone its proof to Section 2 below.

Theorem 1.2 *(A first order Babuška-Aziz inequality). Let $\mathbf{u}$ and f be as in (10), and Ω, B_ρ, R, and ρ be as in (3). Then, inequality (10) holds true with*

$$C_{\mathcal{BA},1}^A \lesssim \frac{R}{\rho^2}\left[1+\left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n-2}{2(n-1)}}\left(\log\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n}{2(n-1)}}\right], \qquad C_{\mathcal{BA},1}^B \lesssim \frac{R}{\rho}. \tag{11}$$

We provide the reader with some comments on the optimality of the constant $C^A_{\mathcal{BA},1}$ in (11) in Section 2.5 below.

1.4. Main result 2: a first order Nečas-Lions inequality

Introduce the space $H^{-1}(\Omega)/\mathbb{R}$, which is the space $H^{-1}(\Omega)$ equipped with the norm

$$\|f\|_{H^{-1}(\Omega)/\mathbb{R}} := \inf_{c\in\mathbb{R}} \|f-c\|_{-1,\Omega}.$$

Recall the negative norms in (2). We discuss a first order Nečas-Lions inequality (for vectors): there exists a positive constant $C_{\mathcal{NL},1}$ such that

$$\|f\|_{H^{-1}(\Omega)/\mathbb{R}} \le C_{\mathcal{NL},1}\|\nabla f\|_{-2,\Omega} \qquad \forall f\in H^{-1}(\Omega)/\mathbb{R}. \tag{12}$$

An equivalent statement for (12) is that the following constants are bounded from above and below, respectively:

$$C_{\mathcal{NL},1} = \sup_{f\in H^{-1}(\Omega)/\mathbb{R}} \frac{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}}{\|\nabla f\|_{-2,\Omega}}, \qquad C^{-1}_{\mathcal{NL},1} = \inf_{f\in H^{-1}(\Omega)/\mathbb{R}} \frac{\|\nabla f\|_{-2,\Omega}}{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}}. \tag{13}$$

The constants $C^A_{\mathcal{BA},1}$ and $C^B_{\mathcal{BA},1}$, and $C_{\mathcal{NL},1}$ in (10) and (11), and (12) are related to constants in other relevant inequalities in Sobolev spaces as well, including the first order inf-sup constant β_1 defined as

$$\inf_{f\in H^{-1}(\Omega)/\mathbb{R}} \sup_{\mathbf{u}\in[H^2_0(\Omega)]^n} \frac{{}_{-1}\langle f, \boldsymbol{\nabla}\cdot \mathbf{u}\rangle_1}{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}|\mathbf{u}|_{2,\Omega}} =: \beta_1 \tag{14}$$

and the positive constant C_P appearing in the Poincaré inequality

$$\|f\|_{0,\Omega} \le C_P R|f|_{1,\Omega} \qquad \forall f\in H^1_0(\Omega). \tag{15}$$

The constant C_P is independent of R and ρ in (3); see, e.g., [14, Section 3.3].

The following result is the first order version of Proposition 1.1.

Proposition 1.3. *Let $C^A_{\mathcal{BA},1}$ and $C^B_{\mathcal{BA},1}$, $C_{\mathcal{NL},1}$, β_1, and C_P be given in (10) and (11), (12), (14), and (15). Then, the following holds true:*

$$C^A_{\mathcal{BA},1} C_P R + C^B_{\mathcal{BA},1} \ge C_{\mathcal{NL},1} = \beta_1^{-1}. \tag{16}$$

Proof. For all f in $H^{-1}(\Omega)/\mathbb{R}$, an integration by parts and the definition of negative Sobolev norms in (2) imply

$$\sup_{\mathbf{u}\in[H^2_0(\Omega)]^n} \frac{{}_{-1}\langle f, \nabla\cdot \mathbf{u}\rangle_1}{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}|\mathbf{u}|_{2,\Omega}} = \sup_{\mathbf{u}\in[H^2_0(\Omega)]^n} \frac{{}_{-2}\langle \nabla f, \mathbf{u}\rangle_2}{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}|\mathbf{u}|_{2,\Omega}} =: \frac{\|\nabla f\|_{-2,\Omega}}{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}}.$$

We take the infimum over all such possible f and exploit the identities

$$\beta_1 \overset{(14)}{=} \inf_{f\in H^{-1}(\Omega)/\mathbb{R}} \frac{\|\nabla f\|_{-2,\Omega}}{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}} = \left(\sup_{f\in H^{-1}(\Omega)/\mathbb{R}} \frac{\|\nabla f\|_{-2,\Omega}}{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}}\right)^{-1} \overset{(13)}{=} C^{-1}_{\mathcal{NL},1},$$

which implies $\beta_1 = C^{-1}_{\mathcal{NL},1}$.

On the other hand, (10) guarantees for all $\widetilde{f}$ in $H^1_0(\Omega)$ the existence of $\mathbf{u}$ in $[H^2_0(\Omega)]^n$ such that

$$\nabla\cdot \mathbf{u} = \widetilde{f}, \qquad |\mathbf{u}|_{2,\Omega} \le C^A_{\mathcal{BA},1}\left\|\widetilde{f}\right\|_{0,\Omega} + C^B_{\mathcal{BA},1}\left|\widetilde{f}\right|_{1,\Omega}.$$

This and the Poincaré inequality (15) give

$$\sup_{\mathbf{u}\in[H^2_0(\Omega)]^n} \frac{{}_{-1}\langle f, \nabla\cdot \mathbf{u}\rangle_1}{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}|\mathbf{u}|_{2,\Omega}} \ge \sup_{\widetilde{f}\in H^1_0(\Omega)} \frac{{}_{-1}\langle f, \widetilde{f}\rangle_1}{\|f\|_{H^{-1}(\Omega)/\mathbb{R}}[C^A_{\mathcal{BA},1}\left\|\widetilde{f}\right\|_{0,\Omega} + C^B_{\mathcal{BA},1}\left|\widetilde{f}\right|_{1,\Omega}]}$$

$$\ge \sup_{\widetilde{f}\in H^1_0(\Omega)} \frac{{}_{-1}\langle f, \widetilde{f}\rangle_1}{(C^A_{\mathcal{BA},1}C_P R + C^B_{\mathcal{BA},1})\|f\|_{H^{-1}(\Omega)/\mathbb{R}}\left|\widetilde{f}\right|_{1,\Omega}} \overset{(2)}{=} (C^A_{\mathcal{BA},1}C_P R + C^B_{\mathcal{BA},1})^{-1}.$$

We take the infimum over all f in $H^{-1}(\Omega)/\mathbb{R}$, recall the first order inf-sup condition (14), and deduce $\beta_1 \ge (C^A_{\mathcal{BA},1}C_P R + C^B_{\mathcal{BA},1})^{-1}$. The assertion follows. □

Since an upper bound on $C^A_{\mathcal{BA},1}$ and $C^B_{\mathcal{BA},1}$ is available from Theorem 1.2, which is explicit in terms of n, R, and ρ as in (3), then Proposition 1.3 implies an upper bound for $C_{\mathcal{NL},1}$ and a lower bound for β_1 with the same explicit dependence. For more general Nečas-Lions inequalities, yet with unknown constants, see [3] and the references therein.

1.5. Main result 3: a Nečas-Lions inequality with symmetric gradients

The spaces $\mathbb{RM}(\Omega)$ of rigid body motions in two and three dimensions have cardinality 3 and 6, and are given by

$$\mathbb{RM}(\Omega) := \begin{cases} \left\{\mathbf{r}(x) = \boldsymbol{\alpha} + b(x_2, -x_1)^{\mathrm{T}} \text{ for any } \boldsymbol{\alpha}\in\mathbb{R}^2,\ b\in\mathbb{R}\right\} & \text{in 2D} \\ \left\{\mathbf{r}(x) = \boldsymbol{\alpha} + \boldsymbol{\omega}\times(x_1,x_2,x_3)^{\mathrm{T}} \text{ for any } \boldsymbol{\alpha},\boldsymbol{\omega}\in\mathbb{R}^3\right\} & \text{in 3D.} \end{cases}$$

Let $\mathbf{\Pi}_{\mathbb{RM}}$ denote the $L^2(\Omega)$ projection onto $\mathbb{RM}(\Omega)$. We further introduce the space of symmetric tensors

$$\mathbf{\Sigma} := \{\boldsymbol{\tau}\in H(\nabla\cdot,\Omega) \mid \boldsymbol{\tau} \text{ is symmetric}\},$$

which we endow with the norm

$$\|\boldsymbol{\tau}\|^2_{\mathbf{\Sigma}} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + R^2\|\nabla\cdot\boldsymbol{\tau}\|^2_{0,\Omega}. \tag{17}$$

Note that

$${}_{-1}\langle\nabla\mathbf{v},\boldsymbol{\tau}\rangle_1 = {}_{-1}\langle\nabla_S\mathbf{v},\boldsymbol{\tau}\rangle_1 \qquad \forall\mathbf{v}\in[L^2(\Omega)]^2,\ \boldsymbol{\tau}\in\mathbf{\Sigma}. \tag{18}$$

We state a Nečas-Lions inequality with symmetric gradients on two and three dimensional domains, which is explicit in terms of R, ρ, and n as in (3).

We state the inequality here and postpone its proof to Section 3 below.

Theorem 1.4 *(A Nečas-Lions inequality with symmetric gradients). There exists a positive constant $C^*_{\mathcal{NL},0}$ depending only on n, R, and ρ as in (3) through $C_{\mathcal{NL},0}$ in (7), $C^A_{\mathcal{BA},1}$ and $C^B_{\mathcal{BA},1}$ in (10), and C_P in (15), such that*

$$
\begin{aligned}
\|\mathbf{v}-\mathbf{\Pi}_{\mathbb{RM}}\mathbf{v}\|_{0,\Omega} \le & C^*_{\mathcal{NL},0}\|\nabla_S\mathbf{v}\|_{-1,\Omega} \\
\le & C_{\mathcal{NL},0}\left[1+\sqrt{2}\left(C^A_{\mathcal{BA},1}C_P R + C^B_{\mathcal{BA},1}\right)\right]\|\nabla_S\mathbf{v}\|_{-1,\Omega} \qquad \forall \mathbf{v}\in[L^2(\Omega)]^n.
\end{aligned} \tag{19}
$$

Since upper bounds on $C^A_{\mathcal{BA},1}$ and $C^B_{\mathcal{BA},1}$, and $C_{\mathcal{NL},0}$ are available from Lemma (10), and Proposition 1.1 and display (6), which are explicit in terms of n, R, and ρ as in (3), (bounds on C_P are standard) then Theorem 1.4 implies an upper bound for $C^*_{\mathcal{NL},0}$ and a lower bound for β_1 with the same explicit dependence. Roughly speaking, Theorem 1.4 is a Korn-type version of the standard lowest order Nečas-Lions inequality.

Introduce the spaces

$$
\widetilde{\mathbf{\Sigma}} := \{\boldsymbol{\tau}\in\mathbf{\Sigma} \mid \langle\boldsymbol{\tau}\,\mathbf{n},\boldsymbol{w}\rangle_{\partial\Omega}=0 \quad \forall \boldsymbol{w}\in[H^1(\Omega)]^n\}; \qquad \widetilde{\mathbf{V}} := \{\mathbf{v}\in[L^2(\Omega)]^n \mid \mathbf{\Pi}_{\mathbb{RM}}\mathbf{v}=0\}.
$$

A consequence of Theorem 1.4 is an inf-sup condition, which is of great importance in the analysis of the mixed (Hellinger-Reissner) formulation of linear elasticity problems: there exists a positive constant β_0^* such that

$$
\inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\sup_{\boldsymbol{\tau}\in\widetilde{\mathbf{\Sigma}}}\frac{(\boldsymbol{\nabla}\cdot\boldsymbol{\tau},\mathbf{v})_{0,\Omega}}{\|\boldsymbol{\tau}\|_{\mathbf{\Sigma}}\|\mathbf{v}\|_{0,\Omega}} =: \beta_0^* R^{-1}. \tag{20}
$$

Proposition 1.5. *Let $C^*_{\mathcal{NL},0}$ and β_0^* be given in (19) and (20). Then, the inf-sup condition (20) holds true with*

$$
\beta_0^* \ge (C^*_{\mathcal{NL},0})^{-1}(1+C_P^2)^{-\frac{1}{2}}.
$$

Proof. For all tensors $\boldsymbol{\tau}$, let $\boldsymbol{\tau}_S$ denote its symmetric part. We have

$$
\begin{aligned}
\beta_0^* &= \inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\sup_{\boldsymbol{\tau}\in\widetilde{\mathbf{\Sigma}}}\frac{(\boldsymbol{\nabla}\cdot\boldsymbol{\tau},\mathbf{v})_{0,\Omega}}{\|\boldsymbol{\tau}\|_{\mathbf{\Sigma}}\|\mathbf{v}\|_{0,\Omega}} \ge \inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\sup_{\boldsymbol{\tau}\in\widetilde{\mathbf{\Sigma}}\cap[H^1_0(\Omega)]^{n\times n}}\frac{(\boldsymbol{\nabla}\cdot\boldsymbol{\tau},\mathbf{v})_{0,\Omega}}{\|\boldsymbol{\tau}\|_{\mathbf{\Sigma}}\|\mathbf{v}\|_{0,\Omega}} \\
&\overset{\text{IBP}}{=} \inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\sup_{\boldsymbol{\tau}\in\widetilde{\mathbf{\Sigma}}\cap[H^1_0(\Omega)]^{n\times n}}\frac{{}_{-1}\langle\nabla\mathbf{v},\boldsymbol{\tau}\rangle_1}{\|\boldsymbol{\tau}\|_{\mathbf{\Sigma}}\|\mathbf{v}\|_{0,\Omega}} \overset{(18)}{=} \inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\sup_{\boldsymbol{\tau}\in\widetilde{\mathbf{\Sigma}}\cap[H^1_0(\Omega)]^{n\times n}}\frac{{}_{-1}\langle\nabla_S\mathbf{v},\boldsymbol{\tau}\rangle_1}{\|\boldsymbol{\tau}\|_{\mathbf{\Sigma}}\|\mathbf{v}\|_{0,\Omega}} \\
&\overset{(17)}{\ge} \inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\sup_{\boldsymbol{\tau}\in\widetilde{\mathbf{\Sigma}}\cap[H^1_0(\Omega)]^{n\times n}}\frac{{}_{-1}\langle\nabla_S\mathbf{v},\boldsymbol{\tau}\rangle_1}{(\|\boldsymbol{\tau}\|^2_{0,\Omega}+R^2|\boldsymbol{\tau}|^2_{1,\Omega})^{\frac{1}{2}}\|\mathbf{v}\|_{0,\Omega}} \\
&\overset{(15)}{\ge} \inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\sup_{\boldsymbol{\tau}\in\widetilde{\mathbf{\Sigma}}\cap[H^1_0(\Omega)]^{n\times n}}\frac{{}_{-1}\langle\nabla_S\mathbf{v},\boldsymbol{\tau}\rangle_1}{(1+C_P^2)^{\frac{1}{2}}R|\boldsymbol{\tau}|_{1,\Omega}\|\mathbf{v}\|_{0,\Omega}} = \inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\sup_{\boldsymbol{\tau}\in[H^1_0(\Omega)]^{n\times n}}\frac{{}_{-1}\langle\nabla_S\mathbf{v},\boldsymbol{\tau}_S\rangle_1}{(1+C_P^2)^{\frac{1}{2}}R|\boldsymbol{\tau}_S|_{1,\Omega}\|\mathbf{v}\|_{0,\Omega}}.
\end{aligned}
$$

Since $|\boldsymbol{\tau}_S|_{1,\Omega}\le|\boldsymbol{\tau}|_{1,\Omega}$ for all tensors $\boldsymbol{\tau}$ and the numerator involves $\nabla_S\mathbf{v}$, we deduce the assertion:

$$
\begin{aligned}
\beta_0^* &\ge \inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\sup_{\boldsymbol{\tau}\in[H^1_0(\Omega)]^{n\times n}}\frac{{}_{-1}\langle\nabla_S\mathbf{v},\boldsymbol{\tau}\rangle_1}{(1+C_P^2)^{\frac{1}{2}}R|\boldsymbol{\tau}|_{1,\Omega}\|\mathbf{v}\|_{0,\Omega}} \\
&\overset{(2)}{=} (1+C_P^2)^{-\frac{1}{2}}R^{-1}\inf_{\mathbf{v}\in\widetilde{\mathbf{V}}}\frac{\|\nabla_S\mathbf{v}\|_{-1,\Omega}}{\|\mathbf{v}\|_{0,\Omega}} \overset{(19)}{\ge} (1+C_P^2)^{-\frac{1}{2}}R^{-1}(C^*_{\mathcal{NL},0})^{-1}. \quad \square
\end{aligned}
$$

Outline of the remainder of the paper In Section 2, we prove Theorem 1.2, whereas in Section 3, we prove Theorem 1.4. We also prove an arbitrary order version of Theorem 1.2 in Appendix A.

2. Proof of a first order Babuška-Aziz inequality

In this section, we prove Theorem 1.2 in several steps and further discuss the optimality of the bounds on the constants therein. To this aim, we follow Bogovskiĭ's construction [6] of a right-inverse of the divergence and generalise Durán's analysis [13] to the first order case.

Explicit construction of Bogovskiĭ's right-inverse of the divergence Consider ω in $\mathcal{C}_0^\infty(\Omega)$ with

$$\int_\Omega \omega(x)\,\mathrm{dx} = 1, \qquad \operatorname{supp}(\omega) \subset B_\rho.$$

Given

$$\mathbf{G} : \Omega \times \Omega \to \mathbb{R}^n, \qquad \mathbf{G}(x,y) := \int_0^1 \frac{x-y}{t}\,\omega\left(y + \frac{x-y}{t}\right)\frac{\mathrm{dt}}{t^n}, \tag{21a}$$

we define

$$\mathbf{u}(x) := \int_\Omega \mathbf{G}(x,y) f(y)\,\mathrm{dy}\,. \tag{21b}$$

2.1. Preliminary results

We recall basic properties of the Fourier transform. Given f in $L^1(\mathbb{R}^n)$, we define its Fourier transform as

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi \mathrm{i} x\cdot\xi} f(x)\,\mathrm{dx}\,. \tag{22}$$

If f is in $L^2(\Omega)$, we have the isometry

$$\|f\|_{0,\mathbb{R}^n} = \left\|\widehat{f}\right\|_{0,\mathbb{R}^n} \tag{23}$$

and the following property on the derivatives of the Fourier transform:

$$\widehat{\partial_{x_j} f}(\xi) = 2\pi \mathrm{i}\xi_j \widehat{f}(\xi) \qquad \forall j = 1,\ldots,n. \tag{24}$$

We consider the following splitting of each component k, $k = 1,\ldots,n$, of $\mathbf{u}$:

$$\begin{aligned}\mathbf{u}_k &:= \mathbf{u}_{k,1} - \mathbf{u}_{k,2}\\ &:= \int_0^1\int_{\mathbb{R}^n}\left(y_k + \frac{x_k-y_k}{t}\right)\omega\left(y+\frac{x-y}{t}\right) f(y)\,\mathrm{dy}\,\frac{\mathrm{dt}}{t^n} - \int_0^1\int_{\mathbb{R}^n} y_k\,\omega\left(y+\frac{x-y}{t}\right) f(y)\,\mathrm{dy}\,\frac{\mathrm{dt}}{t^n}.\end{aligned}$$

In order to take derivatives of $\mathbf{u}_k$, it is convenient to take a limit in the sense of distributions [13, Section 2]:

$$\mathbf{u}_{k,1} = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} \int_{\mathbb{R}^n} \left(y_k + \frac{x_k - y_k}{t}\right) \omega\left(y + \frac{x-y}{t}\right) f(y)\,\mathrm{dy}\,\frac{\mathrm{dt}}{t^n},$$

$$\mathbf{u}_{k,2} = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} \int_{\mathbb{R}^n} y_k\ \omega\left(y + \frac{x-y}{t}\right) f(y)\,\mathrm{dy}\,\frac{\mathrm{dt}}{t^n}.$$

By doing that, we can interchange the derivative with the limit $\varepsilon \to 0$ and then pass with the derivative under the integral symbol; in fact, the functions under the integral are in L^1 and admit L^1 derivatives.

We take the second derivatives of $\mathbf{u}_{k,1}$ and $\mathbf{u}_{k,2}$ with respect to the j-th and ℓ-th directions (without loss of generality we assume j different from ℓ), and get

$$(\partial^2_{x_j,x_\ell} \mathbf{u}_k)(x) = [\widetilde{T}_{k,j\ell,1}(f(y)) - \widetilde{T}_{k,j\ell,2}(y_k f(y))](x). \tag{25}$$

In (25), given $\widetilde{T}$ any of the two operators $\widetilde{T}_{k,j\ell,1}$ and $\widetilde{T}_{k,j\ell,2}$, we let

$$\widetilde{T}(g)(x) := \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} \int_{\mathbb{R}^n} \partial^2_{x_j,x_\ell} \left[\varphi\left(y + \frac{x-y}{t}\right)\right] g(y)\,\mathrm{dy}\,\frac{\mathrm{dt}}{t^n},$$

where, for all $j, \ell = 1, \dots, n$, we have set

$$g(y) := \begin{cases} f(y) & \text{if } \widetilde{T} = \widetilde{T}_{k,j\ell,1}, \\ y_k f(y) & \text{if } \widetilde{T} = \widetilde{T}_{k,j\ell,2}, \end{cases} \qquad \varphi(x) := \begin{cases} x_k \omega(x) & \text{if } \widetilde{T} = \widetilde{T}_{k,j\ell,1}, \\ \omega(x) & \text{if } \widetilde{T} = \widetilde{T}_{k,j\ell,2}. \end{cases} \tag{26}$$

In the forthcoming sections, we shall prove the continuity of the operators in (25). To this aim, we henceforth fix j and ℓ, and consider the decomposition

$$\widetilde{T}g := \widetilde{T}_\alpha g + \widetilde{T}_\beta g, \tag{27}$$

where

$$\widetilde{T}_\alpha g(x) := \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\frac{1}{2}} \int_{\mathbb{R}^n} \partial^2_{x_j,x_\ell} \left[\varphi\left(y + \frac{x-y}{t}\right)\right] g(y)\,\mathrm{dy}\,\frac{\mathrm{dt}}{t^n} \tag{28}$$

and

$$\widetilde{T}_\beta g(x) := \int_{\frac{1}{2}}^{1} \int_{\mathbb{R}^n} \partial^2_{x_j,x_\ell} \left[\varphi\left(y + \frac{x-y}{t}\right)\right] g(y)\,\mathrm{dy}\,\frac{\mathrm{dt}}{t^n}. \tag{29}$$

The continuity estimates will follow summing over all j and ℓ.

2.2. Continuity of $\widetilde{T}_\alpha$

We discuss the continuity of the operator in (28). We proceed in several steps. First, we prove some properties of the Fourier transform of $\widetilde{T}_\alpha(g)$.

Lemma 2.1. *If g in (26) belongs to $\mathcal{C}_0^\infty(\mathbb{R}^n)$, then the following identity is valid:*

$$\widehat{\widetilde{T}_\alpha g}(\xi) = (2\pi \mathrm{i}\xi_j) \lim_{\varepsilon\to 0} \left[\int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\ \xi)\, \widehat{\partial_{x_\ell} g}(\xi)\, \mathrm{d}t + \int_\varepsilon^{\frac12} \widehat{\partial_{x_\ell}\varphi}(t\xi)\widehat{g}((1-t)\xi)\, \mathrm{d}t \right] =: \widehat{\widetilde{T}_{\alpha,1} g}(\xi) + \widehat{\widetilde{T}_{\alpha,2} g}(\xi). \tag{30}$$

Proof. By definition, we write

$$\widetilde{T}_\alpha g(x) = \lim_{\varepsilon\to 0} \widetilde{T}_{\alpha,\varepsilon} g(x).$$

For all positive ε, we write

$$\widehat{\widetilde{T}_{\alpha,\varepsilon} g}(\xi) = \int_{\mathbb{R}^n} \int_\varepsilon^{\frac12} \int_{\mathbb{R}^n} \partial^2_{x_j,x_\ell} \left[\varphi\left(y + \frac{x-y}{t} \right) \right] e^{-2\pi \mathrm{i} x\cdot\xi} g(y)\, \mathrm{d}y\, \frac{\mathrm{d}t}{t^n}\, \mathrm{d}x\,.$$

Due to the regularity of g and φ, the integral exists. Therefore, we can change the order of the integrals, integrate by parts twice, use the change of variable $z = y+(x-y)/t$, the definition of the Fourier transform (22) twice, and (24), recall that the support of φ is compact, and arrive at

$$\begin{aligned}
\widehat{\widetilde{T}_{\alpha,\varepsilon} g}(\xi) &= (2\pi \mathrm{i}\xi_j)(2\pi \mathrm{i}\xi_\ell) \int_\varepsilon^{\frac12} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi\left(y + \frac{x-y}{t} \right) g(y) e^{-2\pi \mathrm{i} x\cdot\xi}\, \mathrm{d}x\, \mathrm{d}y\, \frac{\mathrm{d}t}{t^n} \\
&= (2\pi \mathrm{i}\xi_j)(2\pi \mathrm{i}\xi_\ell) \int_\varepsilon^{\frac12} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi\,(z)\, e^{-2\pi \mathrm{i}(t\ z+(1-t)y)\cdot\xi} g(y)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}t \\
&= (2\pi \mathrm{i}\xi_j)(2\pi \mathrm{i}\xi_\ell) \int_\varepsilon^{\frac12} \int_{\mathbb{R}^n} \widehat{\varphi}\,(t\ \xi)\, e^{-2\pi \mathrm{i}(1-t)y\cdot\xi} g(y)\, \mathrm{d}y\, \mathrm{d}t = (2\pi \mathrm{i}\xi_j)(2\pi \mathrm{i}\xi_\ell) \int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\ \xi)\, \widehat{g}((1-t)\xi)\, \mathrm{d}t \\
&= (2\pi \mathrm{i}\xi_j) \int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\ \xi)\, [2\pi \mathrm{i}(1-t)\xi_\ell] \widehat{g}((1-t)\xi)\, \mathrm{d}t + (2\pi \mathrm{i}\xi_j) \int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\ \xi)\, [2\pi \mathrm{i} t\xi_\ell] \widehat{g}((1-t)\xi)\, \mathrm{d}t \\
&= (2\pi \mathrm{i}\xi_j) \int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\ \xi)\, \widehat{\partial_{x_\ell} g}((1-t)\xi)\, \mathrm{d}t + (2\pi \mathrm{i}\xi_j) \int_\varepsilon^{\frac12} \widehat{\partial_{x_\ell}\varphi}(t\xi) \widehat{g}((1-t)\xi)\, \mathrm{d}t\,.
\end{aligned}$$

The assertion follows taking the limit $\varepsilon \to 0$. □

Next, we prove a technical result.

Lemma 2.2. *Let φ be any of the two options in (26). Then, the two following inequalities hold true:*

$$2\pi|\xi_j| \int_0^\infty |\widehat{\varphi}(t\xi)|\, \mathrm{d}t \le C_{\varphi,\rho,0} := \rho^{-1}\|\varphi\|_{L^1(\mathbb{R}^n)} + \rho \left\| \partial^2_{x_j^2}\varphi \right\|_{L^1(\mathbb{R}^n)} \qquad \forall j = 1,\ldots,n, \tag{31a}$$

$$2\pi|\xi_j| \int_0^\infty \left| \widehat{\partial_{x_\ell}\varphi}(t\xi) \right| \mathrm{d}t \le C_{\varphi,\rho,1} := \rho^{-1}\|\partial_{x_\ell}\varphi\|_{L^1(\mathbb{R}^n)} + \rho \left\| \partial^3_{x_j^2 x_\ell}\varphi \right\|_{L^1(\mathbb{R}^n)} \qquad \forall j = 1,\ldots,n. \tag{31b}$$

Proof. The proof of (31a) is given in [13, Lemma 2.3] and is therefore omitted here. On the other hand, inequality (31b) may be shown as an application of (31a) to $\partial_{\xi_\ell}\varphi$. □

We are now in a position to prove the continuity of the operator $\widetilde{T}_\alpha$.

Proposition 2.3. *Let φ be any of the two options in (26). Then, for all g in $H^1(\mathbb{R}^n)$, the operator $\widetilde{T}_\alpha$ defined in (28) satisfies the following continuity property:*

$$\left\|\widetilde{T}_\alpha g\right\|_{0,\mathbb{R}^n} \le 2^{\frac{n-1}{2}}\left[C_{\varphi,\rho,0}\|\partial_{x_\ell} g\|_{0,\mathbb{R}^n} + C_{\varphi,\rho,1}\|g\|_{0,\mathbb{R}^n}\right].$$

If g vanishes outside Ω, we also have

$$\left\|\widetilde{T}_\alpha g\right\|_{0,\Omega} \le 2^{\frac{n-1}{2}}\left[C_{\varphi,\rho,0}\|\partial_{x_\ell} g\|_{0,\Omega} + C_{\varphi,\rho,1}\|g\|_{0,\Omega}\right].$$

Proof. We only prove the second assertion and focus on functions g in $\mathcal{C}_0^\infty(\Omega)$; the general statement follows then from a density argument.

We consider splitting (30) and show separate bounds for the two terms on the right-hand side. The first one can be handled as in [13, Lemma 2.4] and its proof is therefore omitted:

$$\left\|\widehat{\widetilde{T}_{\alpha,1} g}\right\|_{0,\mathbb{R}^n} \le 2^{\frac{n-1}{2}} C_{\varphi,\rho,0}\left\|\widehat{\partial_{x_\ell} g}\right\|_{0,\mathbb{R}^n}. \tag{32}$$

Thus, we focus on the second term. By the definition of $\widehat{\widetilde{T}_{\alpha,2} g}$, the Cauchy-Schwarz inequality implies

$$\left|\widehat{\widetilde{T}_{\alpha,2} g}(\xi)\right|^2 \le \left(\int_0^{\frac12} 2\pi|\xi_j|\left|\widehat{\partial_{x_\ell}\varphi}(t\xi)\right|\mathrm{d}t\right)\left(\int_0^{\frac12} 2\pi|\xi_j|\left|\widehat{\partial_{x_\ell}\varphi}(t\xi)\right||\widehat{g}((1-t)\xi)|^2\,\mathrm{d}t\right).$$

Using (31b), we deduce

$$\left|\widehat{\widetilde{T}_{\alpha,2} g}(\xi)\right|^2 \le C_{\varphi,\rho,1}\int_0^{\frac12} 2\pi|\xi_j|\left|\widehat{\partial_{x_\ell}\varphi}(t\xi)\right||\widehat{g}((1-t)\xi)|^2\,\mathrm{d}t\,.$$

Integrating over ξ and employing the change of variable $\eta = (1-t)\xi$ give

$$\int_{\mathbb{R}^n}\left|\widehat{\widetilde{T}_{\alpha,2} g}(\xi)\right|^2\mathrm{d}\xi \le C_{\varphi,\rho,1}\int_0^{\frac12}\int_{\mathbb{R}^n}\frac{2\pi}{(1-t)^{n+1}}|\eta_j|\left|\widehat{\partial_{x_\ell}\varphi}\left(\frac{t\eta}{1-t}\right)\right||\widehat{g}(\eta)|^2\,\mathrm{d}\eta\,\mathrm{d}t$$

If we consider the change of variable $s = t/(1-t)$, which entails

$$\mathrm{d}t = (1+s)^{-2}\,\mathrm{d}s, \qquad\qquad \frac{1}{(1-t)^{n+1}} = \left(\frac{s}{t}\right)^{n+1} = (1+s)^{n+1},$$

then we arrive at

$$\int_{\mathbb{R}^n}\left|\widehat{\widetilde{T}_{\alpha,2} g}(\xi)\right|^2\mathrm{d}\xi \le 2^{n-1}C_{\varphi,\rho,1}\int_{\mathbb{R}^n}\left(\int_0^1 2\pi|\eta_j|\left|\widehat{\partial_{x_\ell}\varphi}(s\eta)\right|\mathrm{d}s\right)|\widehat{g}(\eta)|^2\,\mathrm{d}\eta\,.$$

We apply again (31b):

$$\left\|\widehat{\widetilde{T}_{\alpha,2}g}\right\|^2_{0,\mathbb{R}^n} = \int_{\mathbb{R}^n} \left|\widehat{\widetilde{T}_{\alpha,2}g}(\xi)\right|^2 \mathrm{d}\xi \leq 2^{n-1} C^2_{\varphi,\rho,1} \int_{\mathbb{R}^n} |\widehat{g}(\eta)|^2 \,\mathrm{d}\eta = 2^{n-1} C^2_{\varphi,\rho,1} \|\widehat{g}\|^2_{0,\mathbb{R}^n}. \tag{33}$$

The assertion follows using the Fourier isometry (23) in (32), identity (33), and the properties of ω detailed in Section 2.1. □

2.3. Continuity of $\widetilde{T}_\beta$

We discuss the continuity of the operator in (29). As discussed in [13], a direct application of the Hölder and Cauchy-Schwarz inequalities would end up with suboptimal estimates as those in [15]. Therefore, finer estimates are in order. To this aim, we extend [13, Section 3] to the first order case.

Proposition 2.4. *Let g and $\widetilde{T}_\beta$ be as in (26) and (29). Assume that g belongs to $L^2(\mathbb{R}^n)$ and has support contained in Ω. Given $1 \leq p < n/(n-1)$ and p' the conjugate index of p, the following inequality holds true:*

$$\left\|\widetilde{T}_\beta g\right\|_{0,\Omega} \leq \frac{2^{\frac{n}{2}}}{(1-\frac{n}{p'})^{\frac{p}{2}}} |\Omega|^{1-\frac{p}{2}} \left\|\partial^2_{x_j x_\ell}\varphi\right\|^{\frac{p}{2}}_{L^1(\Omega)} \left\|\partial^2_{x_j x_\ell}\varphi\right\|^{1-\frac{p}{2}}_{L^\infty(\Omega)} \|g\|_{0,\Omega}.$$

Proof. The proof follows along the same lines of [13, Lemma 3.2], the only difference being the number of derivatives of φ. □

2.4. Continuity of $\widetilde{T}$

We prove the continuity of the operator in (27).

Theorem 2.5. *Let g and $\widetilde{T}$ be as in (26) and (27). Assume that g belongs to $H^1_0(\Omega)$. Given $1 \leq p < n/(n-1)$ and p' the conjugate index of p, the following inequality holds true:*

$$\begin{aligned}\left\|\widetilde{T} g\right\|_{0,\Omega} &\leq 2^{\frac{n-1}{2}} (\rho^{-1}\|\varphi\|_{L^1(\Omega)} + \rho\left\|\partial^2_{x_j^2}\varphi\right\|_{L^1(\Omega)}) \|\partial_{x_\ell} g\|_{0,\Omega} \\ &+ 2^{\frac{n-1}{2}} (\rho^{-1}\|\partial_{x_\ell}\varphi\|_{L^1(\Omega)} + \rho\left\|\partial^3_{x_j^2 x_\ell}\varphi\right\|_{L^1(\Omega)}) \|g\|_{0,\Omega} \\ &+ \frac{2^{\frac{n}{2}}}{(1-\frac{n}{p'})^{\frac{p}{2}}} |\Omega|^{1-\frac{p}{2}} \left\|\partial^2_{x_j x_\ell}\varphi\right\|^{\frac{p}{2}}_{L^1(\Omega)} \left\|\partial^2_{x_j x_\ell}\varphi\right\|^{1-\frac{p}{2}}_{L^\infty(\Omega)} \|g\|_{0,\Omega}.\end{aligned} \tag{34}$$

Proof. The assertion follows combining Lemmas 2.3 and 2.4, and the explicit representation of the constants $C_{\varphi,\rho,0}$ and $C_{\varphi,\rho,1}$ in (31). □

Next, we derive explicit constants with respect to R and ρ for inequality (34), i.e., we are in a position for proving one of the main results of the manuscript.

Proof of Theorem 1.2. For all $j,\ \ell = 1, \ldots, n$, we have to bound the two terms on the right-hand side of splitting (25). We only prove bounds for $\widetilde{T}_{k,j\ell,2}(y_k f(y))$, as the bounds for $\widetilde{T}_{k,j\ell,1}(f(y))$ are analogous. For the term $\widetilde{T}_{k,j\ell,1}(y_k f(y))$, we have that $\varphi(x)$ in (26) is given by $\omega(x)$, which is supported, with integral 1, in the ball B_ρ of radius ρ. Without loss of generality, the ball can be centred at the origin.

We can write

$$\varphi(x) = \rho^{-n}\psi(\rho^{-1}x),$$

where ψ has the same smoothness as φ, is supported in the unitary ball $B_1(0)$ with integral 1, and is fixed once and for all.

The following properties of φ and its derivatives are valid:

$$\begin{aligned}
& & \|\varphi\|_{L^1(\mathbb{R}^n)} \approx 1, & \qquad \|\varphi\|_{L^\infty(\mathbb{R}^n)} \approx \rho^{-n};\\
\partial_{x_j}\varphi(x) &= \rho^{-n-1}\partial_{x_j}\varphi(\rho^{-1}x), & \left\|\partial_{x_j}\varphi\right\|_{L^1(\mathbb{R}^n)} \approx \rho^{-1}, & \qquad \left\|\partial_{x_j}\varphi\right\|_{L^\infty(\mathbb{R}^n)} \approx \rho^{-n-1};\\
\partial^2_{x_jx_\ell}\varphi(x) &= \rho^{-n-2}\partial^2_{x_jx_\ell}\varphi(\rho^{-1}x), & \left\|\partial^2_{x_jx_\ell}\varphi\right\|_{L^1(\mathbb{R}^n)} \approx \rho^{-2}, & \qquad \left\|\partial^2_{x_jx_\ell}\varphi\right\|_{L^\infty(\mathbb{R}^n)} \approx \rho^{-n-2};\\
\partial^3_{x_j^2x_\ell}\varphi(x) &= \rho^{-n-3}\partial^3_{x_j^2x_\ell}\varphi(\rho^{-1}x), & \left\|\partial^3_{x_j^2x_\ell}\varphi\right\|_{L^1(\mathbb{R}^n)} \approx \rho^{-3}, & \qquad \left\|\partial^3_{x_j^2x_\ell}\varphi\right\|_{L^\infty(\mathbb{R}^n)} \approx \rho^{-n-3}.
\end{aligned}$$

The definition of $\widetilde{T}_{k,j\ell,2}(y_kf)$, the chain rule, the inequality $|y_k| \le R$, and the fact that $\rho^{-1} \le R\rho^{-2}$ imply

$$\begin{aligned}
|\mathbf{u}|_{2,\Omega} &\lesssim_n R\left[\rho^{-1} + \rho\,\rho^{-2}\right]|f|_{1,\Omega} + \left[\rho^{-1}+\rho\rho^{-2}\right]\|f\|_{0,\Omega}\\
&\qquad + R[\rho^{-1}\rho^{-1}+\rho\rho^{-3}]\|f\|_{0,\Omega} + R\frac{2^{\frac{n}{2}}}{\left(1-\frac{n}{p'}\right)^{\frac{p}{2}}}|\Omega|^{1-\frac{p}{2}}(\rho^{-2})^{\frac{p}{2}}(\rho^{-n-2})^{1-\frac{p}{2}}\|f\|_{0,\Omega}\\
&\lesssim R\left[\rho^{-1}|f|_{1,\Omega} + \rho^{-2}\|f\|_{0,\Omega} + \frac{2^{\frac{n}{2}}}{\left(1-\frac{n}{p'}\right)^{\frac{p}{2}}}|\Omega|^{1-\frac{p}{2}}\rho^{-2-n(1-\frac{p}{2})}\|f\|_{0,\Omega}\right]\\
&\lesssim_n \frac{R}{\rho}|f|_{1,\Omega} + \frac{R}{\rho^2}\|f\|_{0,\Omega}\left[1+\left(1-\frac{n}{p'}\right)^{-\frac{p}{2}}|\Omega|^{1-\frac{p}{2}}\rho^{-n(1-\frac{p}{2})}\right].
\end{aligned} \tag{35}$$

We focus on the last coefficient on the right-hand side. Using that

$$|B_\rho|^{1-\frac{p}{2}}\rho^{-n(1-\frac{p}{2})} \approx 1,$$

we write

$$\begin{aligned}
\left(1-\frac{n}{p'}\right)^{-\frac{p}{2}}|\Omega|^{1-\frac{p}{2}}\rho^{-n(1-\frac{p}{2})} &= \left(1-\frac{n}{p'}\right)^{-\frac{p}{2}}\left(\frac{|\Omega|}{|B_\rho|}\right)^{1-\frac{p}{2}}|B_\rho|^{1-\frac{p}{2}}\rho^{-n(1-\frac{p}{2})}\\
&\approx \left(1-\frac{n}{p'}\right)^{-\frac{p}{2}}\left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n-2}{2(n-1)}}\left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{1}{2}\left(\frac{n}{n-1}-p\right)}.
\end{aligned} \tag{36}$$

For $|\Omega|/|B_\rho|$ sufficiently large (the ball B_ρ is anyhow fixed once and for all in the reference framework), we choose p such that

$$\frac{1}{2}\left(\frac{n}{n-1}-p\right) = \frac{1}{\log\left(\frac{|\Omega|}{|B_\rho|}\right)}.$$

Equivalently, we pick p such that

$$\log\left(\frac{|\Omega|}{|B_\rho|}\right) = \frac{1}{\frac{1}{2}\left(\frac{n}{n-1}-p\right)} \quad\Longrightarrow\quad \frac{|\Omega|}{|B_\rho|} = e^{\frac{1}{\frac{1}{2}\left(\frac{n}{n-1}-p\right)}} \quad\Longrightarrow\quad \left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{1}{2}\left(\frac{n}{n-1}-p\right)} = e^1.$$

We plug the last identity in (36) and deduce

$$\left(1-\frac{n}{p'}\right)^{-\frac{p}{2}} |\Omega|^{1-\frac{p}{2}} \rho^{-n(1-\frac{p}{2})} \approx \left(1-\frac{n}{p'}\right)^{-\frac{p}{2}} \left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n-2}{2(n-1)}} e^1.$$

Going back to (35), we write

$$|\mathbf{u}|_{2,\Omega} \lesssim_n \frac{R}{\rho}|f|_{1,\Omega} + \frac{R}{\rho^2}\|f\|_{0,\Omega} \left[1 + \left(1-\frac{n}{p'}\right)^{-\frac{p}{2}} \left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n-2}{2(n-1)}}\right].$$

We further note that

$$1-\frac{n}{p'} = \left(\frac{n-1}{p}\right)\left(\frac{n}{n-1}-p\right) = \frac{2(n-1)}{p\log\left(\frac{|\Omega|}{|B_\rho|}\right)}.$$

We combine the two above displays:

$$|\mathbf{u}|_{2,\Omega} \lesssim_n \frac{R}{\rho}|f|_{1,\Omega} + \frac{R}{\rho^2}\|f\|_{0,\Omega} \left[1 + \left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n-2}{2(n-1)}} \frac{p^{\frac{p}{2}}}{2(n-1)^{\frac{p}{2}}} \log\left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{p}{2}}\right].$$

Using $p < n/(n-1)$, the following quantity is uniformly bounded in n and thus in p:

$$\frac{p^{\frac{p}{2}}}{2(n-1)^{\frac{p}{2}}}.$$

Moreover, we know that $p/2 < n/(2(n-1))$. We deduce that

$$|\mathbf{u}|_{2,\Omega} \lesssim_n \frac{R}{\rho}|f|_{1,\Omega} + \frac{R}{\rho^2}\|f\|_{0,\Omega} \left[1 + \left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n-2}{2(n-1)}} \log\left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n}{2(n-1)}}\right],$$

which is the assertion. □

Compared to the lowest order estimate (6), we have an extra term involving the gradient of f and an extra ρ^{-1} scaling factor for the term involving f.

Remark 1. The issue on whether estimates as in Theorem 1.2 can be extended to union of star-shaped domains was addressed in [18]. Their proof relies on partition of unity techniques; this entails that estimates have constants that are not fully explicit with respect to the shape of the domain [17]. A simpler open problem is whether one may be able to prove arbitrary order Babuška-Aziz inequalities with explicit constants on the union of simpler star-shaped domains, e.g., on simplicial patches.

Remark 2. We have that $\mathbf{u}$ in (21) satisfies the boundary conditions, i.e., that $\mathbf{u}$ belongs to $[H_0^2(\Omega)]^n$; this is shown for instance in [15, Lemma III.3.1] for smooth functions; the corresponding result for Sobolev spaces is achieved via density arguments. As it is of independent interest, we discuss an alternative proof of this fact in Appendix B, which also holds in the non-Hilbertian setting.

2.5. On the optimality of the estimates in Theorem 1.2 in 2D

We comment on the optimality of the estimates in Theorem 1.2 for planar domains based on a counterexample in [13, Section 3]. Introduce the domain

$$\Omega_{a,\varepsilon} := (-a,a)\times(-\varepsilon,\varepsilon)$$

and the function

$$f(x_1, x_2) = x_1.$$

Let $\mathbf{u}$ be the solution to the divergence problem (10). We have

$$\|x_1\|^2_{0,\Omega_{a,\varepsilon}} = \int\limits_{\Omega_{a,\varepsilon}} x_1 \nabla\cdot \mathbf{u} = -\int\limits_{\Omega_{a,\varepsilon}} \mathbf{u}_1 = -\frac{1}{2}\int\limits_{\Omega_{a,\varepsilon}} x_2^2 \partial^2_{x_2^2} \mathbf{u}_1 \leq \frac{1}{2}\|x_2^2\|_{0,\Omega_{a,\varepsilon}} \left\|\partial^2_{x_2^2}\mathbf{u}_1\right\|_{0,\Omega_{a,\varepsilon}}.$$

We use estimates as in (10) for the last term on the right-hand side: there exist positive constants C_1 and C_2 depending on $R = 2a$ and $\rho = \varepsilon$ such that

$$\|x_1\|^2_{0,\Omega_{a,\varepsilon}} \leq \frac{1}{2}\|x_2^2\|_{0,\Omega_{a,\varepsilon}} \left[C^A_{\mathcal{BA},1}\|x_1\|_{0,\Omega_{a,\varepsilon}} + C^B_{\mathcal{BA},1}\|1\|_{0,\Omega_{a,\varepsilon}}\right].$$

We have

$$\|x_1\|^2_{0,\Omega_{a,\varepsilon}} = \frac{4}{3}a^3\varepsilon, \qquad \|x_2^2\|^2_{0,\Omega_{a,\varepsilon}} = \frac{4}{5}a\varepsilon^5, \qquad \|1\|^2_{0,\Omega_{a,\varepsilon}} = 4a\varepsilon.$$

Combining the two displays, we get

$$a^3\varepsilon \lesssim C^A_{\mathcal{BA},1}a^2\varepsilon^3 + C^B_{\mathcal{BA},1}a\varepsilon^3,$$

whence

$$1 \lesssim C^A_{\mathcal{BA},1}a^{-1}\varepsilon^2 + C^B_{\mathcal{BA},1}a^{-2}\varepsilon^2 \approx C^A_{\mathcal{BA},1}\frac{\rho^2}{R} + C^B_{\mathcal{BA},1}\frac{\rho^2}{R^2}.$$

This inequality implies that at least one of the following must hold true:

$$C^A_{\mathcal{BA},1} \gtrsim \frac{R}{\rho^2}, \qquad C^B_{\mathcal{BA},1} \gtrsim \frac{R^2}{\rho^2}.$$

Using (11), we deduce

$$\frac{R^2}{\rho^2} \lesssim C^B_{\mathcal{BA},1} \lesssim \frac{R}{\rho},$$

which cannot be valid in general with hidden constants independent of R and ρ as in (3). This entails that

$$C^A_{\mathcal{BA},1} \gtrsim \frac{R}{\rho^2},$$

i.e., the first bound in (11) is optimal up to a logarithmic factor.

We are not able to infer a clear statement on the optimality of $C^B_{\mathcal{BA},1}$ from the bound above. A heuristic argument based on scaling techniques, suggests however that bound (11) on $C^B_{\mathcal{BA},1}$ should be also optimal.

3. Proof of a Nečas-Lions inequality with symmetric gradients

We prove Theorem 1.4. To this aim, we first show an auxiliary result.

Lemma 3.1. *Let* $\mathbf{u}$ *be a sufficiently smooth vector field. Then, the following identities hold true:*

$$[\nabla(\nabla\times\mathbf{u})]^T = \nabla\times(\nabla\mathbf{u}+(\nabla\mathbf{u})^T) = 2\nabla\times(\nabla_S\mathbf{u}), \tag{37}$$

where, for a given tensor $\mathbf{A}$, $\nabla\times\mathbf{A}$ *denotes the matrix obtained by applying the curl operator row-wise to* $\mathbf{A}$.

Proof. Direct computations give

$$[\nabla(\nabla\times\mathbf{u})]^T = \begin{bmatrix} \partial_{x_1}(\partial_{x_2}\mathbf{u}_3-\partial_{x_3}\mathbf{u}_2) & \partial_{x_1}(\partial_{x_3}\mathbf{u}_1-\partial_{x_1}\mathbf{u}_3) & \partial_{x_1}(\partial_{x_1}\mathbf{u}_2-\partial_{x_2}\mathbf{u}_1) \\ \partial_{x_2}(\partial_{x_2}\mathbf{u}_3-\partial_{x_3}\mathbf{u}_2) & \partial_{x_2}(\partial_{x_3}\mathbf{u}_1-\partial_{x_1}\mathbf{u}_3) & \partial_{x_2}(\partial_{x_1}\mathbf{u}_2-\partial_{x_2}\mathbf{u}_1) \\ \partial_{x_3}(\partial_{x_2}\mathbf{u}_3-\partial_{x_3}\mathbf{u}_2) & \partial_{x_3}(\partial_{x_3}\mathbf{u}_1-\partial_{x_1}\mathbf{u}_3) & \partial_{x_3}(\partial_{x_1}\mathbf{u}_2-\partial_{x_2}\mathbf{u}_1) \end{bmatrix}.$$

Similarly, we may show that the above tensor coincides with $\nabla\times(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)$, which gives the first identity in (37); the second identity in (37) is a consequence of the definition of the symmetric gradient. □

We are now in a position to prove Theorem 1.4. For any generic $\mathbf{q}^{\mathrm{RM}}$ in the space of rigid body motions $\mathbb{RM}(\Omega)\cap[L^2_0(\Omega)]^n$ as discussed in Section 1.5, we have

$$\|\mathbf{v}-\boldsymbol{\Pi}_{\mathbb{RM}}\mathbf{v}\|_{0,\Omega} \le \left\|\mathbf{v}-\mathbf{q}^{\mathrm{RM}}-\boldsymbol{\Pi}^0(\mathbf{v}-\mathbf{q}^{\mathrm{RM}})\right\|_{0,\Omega} \qquad \forall\mathbf{v}\in[L^2(\Omega)]^n.$$

An immediate consequence of the standard, vector version, lowest order Nečas-Lions inequality (7) is that

$$\|\mathbf{v}-\boldsymbol{\Pi}_{\mathbb{RM}}\mathbf{v}\|_{0,\Omega} \le C_{\mathcal{NL},0}\left\|\nabla(\mathbf{v}-\mathbf{q}^{\mathrm{RM}})\right\|_{-1,\Omega} \qquad \forall\mathbf{q}^{\mathrm{RM}}\in\mathbb{RM}(\Omega)\cap[L^2_0(\Omega)]^n. \tag{38}$$

We are left to prove the existence of a positive constant C with explicit dependence on R and ρ as in (3), such that, for a specific choice of $\mathbf{q}^{\mathrm{RM}}$, the following inequality is valid:

$$\left\|\nabla(\mathbf{v}-\mathbf{q}^{\mathrm{RM}})\right\|_{-1,\Omega} \le C\|\nabla_S\mathbf{v}\|_{-1}.$$

Using splitting (1) of the gradient into symmetric and skew-symmetric parts, and the triangle inequality entails

$$\left\|\nabla(\mathbf{v}-\mathbf{q}^{\mathrm{RM}})\right\|_{-1,\Omega} \le \left\|\nabla_S(\mathbf{v}-\mathbf{q}^{\mathrm{RM}})\right\|_{-1,\Omega} + \left\|\nabla_{SS}(\mathbf{v}-\mathbf{q}^{\mathrm{RM}})\right\|_{-1,\Omega}. \tag{39}$$

Since $\mathbf{q}^{\mathrm{RM}}$ is a rigid body motion, $\nabla_S\mathbf{q}^{\mathrm{RM}}$ is the zero tensor in $\mathbb{R}^{n\times n}$: the first term on the right-hand side is equal to $\|\nabla_S\mathbf{v}\|_{-1,\Omega}$. As for the second term on the right-hand side, we define $A^{n\times n}$ as the space of $(n\times n)$ skew-symmetric matrices, $n=2,3$. We take $\mathbf{q}^{\mathrm{RM}}$ such that

$$\left\|\nabla_{SS}(\mathbf{v}-\mathbf{q}^{\mathrm{RM}})\right\|_{-1,\Omega} := \inf_{\widetilde{\mathbf{q}^{\mathrm{RM}}}\in\mathbb{RM}(\Omega)\cap[L^2_0(\Omega)]^{n\times n}} \left\|\nabla_{SS}(\mathbf{v}-\widetilde{\mathbf{q}^{\mathrm{RM}}})\right\|_{-1,\Omega} = \inf_{\mathbf{c}\in A^{n\times n}}\|\nabla_{SS}\mathbf{v}-\mathbf{c}\|_{-1,\Omega}.$$

With this at hand, elementary computations give

$$\left\|\nabla_{SS}(\mathbf{v}-\mathbf{q}^{\mathrm{RM}})\right\|_{-1,\Omega} = \inf_{\mathbf{c}\in A^{n\times n}}\|\nabla_{SS}\mathbf{v}-\mathbf{c}\|_{-1,\Omega} \le \frac{1}{\sqrt{2}}\inf_{\mathbf{c}\in\mathbb{R}^{2n-3}}\|\nabla\times\mathbf{v}-\mathbf{c}\|_{-1,\Omega}. \tag{40}$$

Note that the last two norms involve tensors and vectors, respectively.

An immediate consequence of the scalar and vector versions of (12), (15), and (16) is that

$$\inf_{\mathbf{c}\in\mathbb{R}^{2n-3}}\|\nabla\times\mathbf{v}-\mathbf{c}\|_{-1,\Omega} \le C_{\mathcal{NL},1}\|\nabla(\nabla\times\mathbf{v})\|_{-2,\Omega} \le (C^A_{\mathcal{BA},1}C_PR+C^B_{\mathcal{BA},1})\|\nabla(\nabla\times\mathbf{v})\|_{-2,\Omega}.$$

Lemma 3.1 reveals that

$$\begin{aligned}\inf_{\mathbf{c}\in\mathbb{R}^{2n-3}} \|\nabla\times\mathbf{v}-\mathbf{c}\|_{-1,\Omega} &\le 2(C^A_{\mathcal{BA},1}C_P R + C^B_{\mathcal{BA},1})\|\boldsymbol{\nabla}\times\nabla_S\mathbf{v}\|_{-2,\Omega}\\ &\le 2(C^A_{\mathcal{BA},1}C_P R + C^B_{\mathcal{BA},1})\|\nabla_S\mathbf{v}\|_{-1,\Omega}.\end{aligned} \tag{41}$$

Combining (38), (39), (40), and (41) yields the assertion.

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Appendix A. Arbitrary order Babuška-Aziz inequalities on star-shaped domains with explicit constants

In this appendix, we provide a road-map for proving arbitrary order Babuška-Aziz inequalities with upper bounds on the corresponding constants, which are explicit with respect to R and ρ in (3), based on a generalisation of the analysis in Section 2. In particular, we prove the following generalisation of Theorem 1.2.

Theorem A.1. *Let d in $\mathbb{N}$ be larger than or equal to 1, and R and ρ be as in (3). Assume that f is in $H^{d-1}_0(\Omega)\cap L^2_0(\Omega)$. Then, there exists $\mathbf{u}$ in $[H^d_0(\Omega)]^n$ such that $\nabla\cdot\mathbf{u}=f$ and*

$$|\mathbf{u}|_{d,\Omega} \lesssim_n \sum_{\ell=1}^{d-1}\frac{R}{\rho^{d-\ell}}|f|_{\ell,\Omega} + \frac{R}{\rho^d}\left[1+\left(\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n-2}{2(n-1)}}\left(\log\frac{|\Omega|}{|B_\rho|}\right)^{\frac{n}{2(n-1)}}\right]\|f\|_{0,\Omega}. \tag{42}$$

We give some details on how to prove (42). Essentially, we have to prove the continuity of the counterpart of the operator in (27) involving all the derivatives of order d. More precisely, for each multi-index $\mathbf{j}=(j_1,\dots,j_n)$ in $\{1,\dots,d\}^n$, $|\mathbf{j}|=d$, we study the continuity of the operator

$$\widetilde{T}g := \widetilde{T}_\alpha g + \widetilde{T}_\beta g, \tag{43}$$

where

$$\widetilde{T}_\alpha g(x) := \lim_{\varepsilon\to 0}\int_\varepsilon^{\frac12}\int_{\mathbb{R}^n}\partial^d_{x_1^{j_1},\dots,x_n^{j_n}}\left[\varphi\left(y+\frac{x-y}{t}\right)\right]g(y)\,\mathrm{d}y\,\frac{\mathrm{d}t}{t^n} \tag{44}$$

and

$$\widetilde{T}_\beta f(x) := \int_{\frac12}^{1}\int_{\mathbb{R}^n}\partial^d_{x_1^{j_1},\dots,x_n^{j_n}}\left[\varphi\left(y+\frac{x-y}{t}\right)\right]g(y)\,\mathrm{d}y\,\frac{\mathrm{d}t}{t^n}. \tag{45}$$

Above, g and φ are precisely as in (26). The general assertion then follows summing over all possible multi-indices $\mathbf{j}$. In the remainder of the appendix,

$$\mathbf{j} \text{ is fixed}, \qquad j_1 \neq 0. \tag{46}$$

A.1. Continuity of $\widetilde{T}_\alpha$

We discuss here the continuity of the operator in (44). We have the two following technical results.

Lemma A.2. *Let* $\mathbf{j}$ *be as in* (46)*. If* g *in* (26) *belongs to* $\mathcal{C}_0^\infty(\mathbb{R}^n)$*, then the following identity is valid:*

$$\widehat{\widetilde{T}_\alpha g}(\xi) = (2\pi \mathrm{i}\xi_1) \lim_{\varepsilon\to 0} \left[\sum_{\mathbf{k},\ell\in\{1,\dots,d\}^n, \mathbf{k}+\ell=(j_1-1,j_2,\dots,j_d)} \int_\varepsilon^{\frac12} \widehat{\partial^{|\mathbf{k}|}_{x_1^{k_1},\dots,x_n^{k_n}}\varphi}(t\xi) \widehat{\partial^{|\ell|}_{x_1^{\ell_1},\dots,x_n^{\ell_n}} g}(\xi)\,\mathrm{d}t \right].$$

Proof. The proof is a modification of that of Lemma 2.1. Since this result contains most of the differences compared to the first order case, we prove the assertion for the second order case, i.e., we assume that $\mathbf{j} = (j_1, j_2, j_3)$ with $j_1 + j_2 + j_3 = 3$. To further simplify the proof, we discuss the case $j_1 = j_2 = j_3 = 1$. The general assertion is proven analogously.

In analogy to the proof of Lemma 2.1, we write

$$\begin{aligned}
\widehat{\widetilde{T}_{\alpha,\varepsilon} g}(\xi) &= (2\pi\mathrm{i}\xi_1)(2\pi\mathrm{i}\xi_2)(2\pi\mathrm{i}\xi_3) \int_\varepsilon^{\frac12}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi\left(y + \frac{x-y}{t}\right) g(y) e^{-2\pi\mathrm{i}x\cdot\xi}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}t}{t^n} \\
&= (2\pi\mathrm{i}\xi_1)\Bigg[\int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\,\xi)\,[2\pi\mathrm{i}(1-t)\xi_2][2\pi\mathrm{i}(1-t)\xi_3]\widehat{g}((1-t)\xi)\,\mathrm{d}t + \int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\,\xi)\,[2\pi\mathrm{i}t\xi_2][2\pi\mathrm{i}(1-t)\xi_3]\widehat{g}((1-t)\xi)\,\mathrm{d}t \\
&\qquad + \int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\,\xi)\,[2\pi\mathrm{i}(1-t)\xi_2][2\pi\mathrm{i}t\xi_3]\widehat{g}((1-t)\xi)\,\mathrm{d}t + \int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\,\xi)\,[2\pi\mathrm{i}t\xi_2][2\pi\mathrm{i}t\xi_3]\widehat{g}((1-t)\xi)\,\mathrm{d}t\Bigg] \\
&= (2\pi\mathrm{i}\xi_1)\Bigg[\int_\varepsilon^{\frac12} \widehat{\varphi}\,(t\,\xi)\,\widehat{\partial^2_{x_2x_3} g}((1-t)\xi)\,\mathrm{d}t + \int_\varepsilon^{\frac12} \widehat{\partial_{x_2}\varphi}(t\xi)\widehat{\partial_{x_3} g}((1-t)\xi)\,\mathrm{d}t \\
&\qquad + \int_\varepsilon^{\frac12} \widehat{\partial_{x_3}\varphi}\,(t\,\xi)\,\widehat{\partial_{x_2} g}((1-t)\xi)\,\mathrm{d}t + \int_\varepsilon^{\frac12} \widehat{\partial^2_{x_2x_3}\varphi}(t\xi)\widehat{g}((1-t)\xi)\,\mathrm{d}t\Bigg].
\end{aligned}$$

This yields the assertion for the case $d = 3$, and $j_1 = j_2 = j_3 = 1$. □

From Lemma A.2, it is apparent that we have to bound the norm of several derivatives of φ, which is what we accomplish in the next result.

Lemma A.3. *Let* φ *and* $\mathbf{j}$ *be as in* (26) *and* (46)*. Then, the following inequalities hold true: for all multi-indices* $\mathbf{k}$ *in* $\{1,\dots,d\}^n$ *such that* $|\mathbf{k}| \le |\mathbf{j}| = d$*,*

$$\begin{aligned}
2\pi|\xi_1| \int_0^\infty \left| \widehat{\partial^{|\mathbf{k}|}_{x_1^{k_1-1},\dots,x_n^{k_n}}\varphi}(t\xi)\right| \mathrm{d}t &\le C_{\varphi,\rho,\mathbf{k}} \\
&:= \rho^{-1}\left\|\partial^{|\mathbf{k}|}_{x_1^{k_1-1},x_2^{k_2},\dots,x_n^{k_n}}\varphi\right\|_{L^1(\mathbb{R}^n)} + \rho\left\|\partial^{|\mathbf{k}|+2}_{x_1^{k_1+1},x_2^{k_2},\dots,x_n^{k_n}}\varphi\right\|_{L^1(\mathbb{R}^n)}.
\end{aligned} \tag{47}$$

Proof. The proof is a modification of that of Lemma 2.2. □

The two above technical lemmas give the following result.

Proposition A.4. *Let φ be any of the two options in (26). Then, for all g in $H^d(\mathbb{R}^n)$, the operator $\widetilde{T}_\alpha$ defined in (44) satisfies the following continuity property:*

$$\left\|\widetilde{T}_\alpha g\right\|_{0,\mathbb{R}^n} \leq 2^{\frac{n-1}{2}} \sum_{\mathbf{k},\boldsymbol{\ell}\in\{1,\dots,d\}^n,\mathbf{k}+\boldsymbol{\ell}=(j_1-1,\dots,j_d)} \left[C_{\varphi,\rho,\mathbf{k}} \left\|\partial^{|\boldsymbol{\ell}|}_{x_1^{\ell_1},\dots,x_n^{\ell_n}} g\right\|_{0,\mathbb{R}^n}\right],$$

where φ is any of the two options in (26).

Proof. The proof is a modification of that of Proposition 2.3, and further combines Lemmas A.2 and A.3. □

A.2. Continuity of $\widetilde{T}_\beta$

We discuss here the continuity of the operator in (45). We have the following result.

Proposition A.5. *Let g, $\widetilde{T}_\beta$, and $\mathbf{j}$ be as in (26), (45), and (46). Assume that g belongs to $L^2(\mathbb{R}^n)$ and has support contained in Ω. Given $1 \leq p < n/(n-1)$ and p' the conjugate index of p, the following inequality holds true:*

$$\left\|\widetilde{T}_\beta g\right\|_{0,\Omega} \leq \frac{2^{\frac{n}{2}}}{(1-\frac{n}{p'})^{\frac{p}{2}}} |\Omega|^{1-\frac{p}{2}} \left\|\partial^d_{x_1^{j_1}\dots x_n^{j_n}}\varphi\right\|^{\frac{p}{2}}_{L^1(\Omega)} \left\|\partial^d_{x_1^{j_1}\dots x_n^{j_n}}\varphi\right\|^{1-\frac{p}{2}}_{L^\infty(\Omega)} \|g\|_{0,\Omega}.$$

Proof. The proof is a modification of that of Proposition 2.4. □

A.3. Continuity of $\widetilde{T}$

We discuss here the continuity of the operator in (43). We have the following result.

Theorem A.6. *Let g, $\widetilde{T}$, and $\mathbf{j}$ be as in (26), (43), and (46). Assume that g belongs to $H_0^{d-1}(\Omega)$. Given $1 \leq p < n/(n-1)$ and p' the conjugate index of p, the following inequality holds true:*

$$\begin{aligned}
&\left\|\widetilde{T} g\right\|_{0,\Omega}\\
&\leq 2^{\frac{n-1}{2}} \sum_{\mathbf{k},\boldsymbol{\ell}\in\{1,\dots,d\}^n,\mathbf{k}+\boldsymbol{\ell}=(j_1-1,\dots,j_d)} \left[\left(\rho^{-1}\left\|\partial^{|\mathbf{k}|}_{x_1^{k_1-1},x_2^{k_2}\dots,x_n^{k_n}}\varphi\right\|_{L^1(\mathbb{R}^n)} + \rho\left\|\partial^{|\mathbf{k}|+2}_{x_1^{k_1+1},x_2^{k_2},\dots,x_n^{k_n}}\varphi\right\|_{L^1(\mathbb{R}^n)}\right)\left\|\partial^{|\boldsymbol{\ell}|}_{x_1^{\ell_1},\dots,x_n^{\ell_n}} g\right\|_{0,\mathbb{R}^n}\right]\\
&\quad + \frac{2^{\frac{n}{2}}}{(1-\frac{n}{p'})^{\frac{p}{2}}} |\Omega|^{1-\frac{p}{2}} \left\|\partial^d_{x_1^{j_1},\dots,x_n^{j_n}}\varphi\right\|^{\frac{p}{2}}_{L^1(\Omega)} \left\|\partial^d_{x_1^{j_1},\dots,x_n^{j_n}}\varphi\right\|^{1-\frac{p}{2}}_{L^\infty(\Omega)} \|g\|_{0,\Omega}.
\end{aligned}$$

Proof. The assertion follows combining Lemmas A.4 and A.5, and the explicit representation of the constants $C_{\varphi,\rho,\mathbf{k}}$ in (47). □

Theorem A.1 follows using Theorem A.6, the chain rule, and proceeding as in the proof of Theorem 1.2.

An arbitrary order Nečas-Lions, generalising the first order version in (12), may be shown based on Theorem A.1 following the proof of Proposition 1.3; for d in $\mathbb{N}$, it reads

$$\|f\|_{H^{-(d-1)}(\Omega)/\mathbb{P}_{d-1}(\Omega)} \leq C_{\mathcal{NL},d}\|\nabla f\|_{-d,\Omega} \qquad \forall f \in H^{-(d-1)}(\Omega)/\mathbb{P}_{d-1}(\Omega),$$

where $H^{-(d-1)}(\Omega)/\mathbb{P}_{d-1}(\Omega)$ is the space $H^{-(d-1)}(\Omega)$ equipped with the norm

$$\|f\|_{H^{-(d-1)}(\Omega)/\mathbb{P}_{d-1}(\Omega)} := \inf_{q_{d-1}\in\mathbb{P}_{d-1}(\Omega)} \|f - q_{d-1}\|_{-(d-1),\Omega}.$$

The constant $C_{\mathcal{NL},d}$ depends on the Babuška-Aziz constants of all orders up to d and the Poincaré constant through [25].

Appendix B. An alternative proof of the zero boundary conditions

We present here an alternative proof of the properties discussed in Remark 2, as it is based on some technical results that are stated in the literature, the proof of which we were not able to find. Such results may be useful in the derivation of explicit estimates for the right-inverse of the divergence in $W^{k,p}$ spaces; see Remark 3 below for additional comments on this point.

We begin by providing the reader with a detailed proof of an alternative expression for the first derivatives of $\mathbf{u}$, which has been stated in [15, Remark III.3.2], and then proceed along the same lines as in [1, Chapter 2].

We start by showing a preliminary result, which requires the definition of an operator $\widetilde{\mathbf{G}}_j : \Omega\times\Omega \to \mathbb{R}$ given by

$$\widetilde{\mathbf{G}}_j(x,y) := \int_0^1 \frac{x-y}{t}\, \partial_{x_j}\omega\left(y + \frac{x-y}{t}\right)\frac{\mathrm{d}t}{t^n} \qquad \forall j = 1,\dots,n. \tag{48}$$

Lemma B.1. *Let* $\mathbf{G}$ *and* $\widetilde{\mathbf{G}}_j$, $j = 1,\dots,n$, *be defined as in* (21a) *and* (48). *Then, for all positive* ε, *the following identity holds true:*

$$\partial_{x_j}\mathbf{G}(x,y) = -\partial_{y_j}\mathbf{G}(x,y) + \widetilde{\mathbf{G}}_j(x,y) \qquad \forall |x-y| > \varepsilon. \tag{49}$$

Proof. We fix two indices $j,k = 1,\dots,n$ and show the assertion on the k-th components of $\mathbf{G}$ and $\widetilde{\mathbf{G}}$. The fact that ω belongs to $\mathcal{C}_0^\infty(B_\rho)$ and direct computations reveal

$$\partial_{x_j}\mathbf{G}_k(x,y) = \int_0^1 \left[\delta_{kj}\ \omega\left(y+\frac{x-y}{t}\right) + \frac{(x-y)_k}{t}\ \partial_{x_j}\omega\left(y+\frac{x-y}{t}\right)\right]\frac{\mathrm{d}t}{t^{n+1}}$$

and

$$\begin{aligned}\partial_{y_j}\mathbf{G}_k(x,y) &= \int_0^1 \left[\frac{-\delta_{kj}}{t}\ \omega\left(y+\frac{x-y}{t}\right) + \frac{(x-y)_k}{t}\ \partial_{x_j}\omega\left(y+\frac{x-y}{t}\right)\left(1-\frac{1}{t}\right)\right]\frac{\mathrm{d}t}{t^n}\\ &= -\int_0^1 \left[\delta_{kj}\ \omega\left(y+\frac{x-y}{t}\right) + \frac{(x-y)_k}{t}\ \partial_{x_j}\omega\left(y+\frac{x-y}{t}\right)\right]\frac{\mathrm{d}t}{t^{n+1}} + (\widetilde{\mathbf{G}}_j)_k(x,y).\end{aligned}$$

A combination of the two previous displays gives the assertion. □

Next, we prove an identity involving the first derivatives of $\mathbf{u}$.

Proposition B.2 *(Galdi's formula). Let* $\mathbf{G}$ *and* $\widetilde{\mathbf{G}}_j$, $j = 1,\dots,n$, *be defined as in* (21a) *and* (48). *Given* f *in* $H_0^1(\Omega)$, *consider* $\mathbf{u}$ *as in* (21). *Then, the following identity holds true:*

$$\partial_{x_j}\mathbf{u}(x) = \int_\Omega \mathbf{G}(x,y)\ \partial_{y_j} f(y)\ \mathrm{d}y + \int_\Omega \widetilde{\mathbf{G}}_j(x,y)\ f(y)\ \mathrm{d}y \qquad \forall j = 1,\dots,n. \tag{50}$$

Proof. Without loss of generality, we assume that f belongs to $\mathcal{C}_0^\infty(\Omega)$; the general assertion follows from a density argument. Moreover, we prove the assertion on the k-th components of $\mathbf{u}$, $\mathbf{G}$, and $\widetilde{\mathbf{G}}$.

For any ϕ in $\mathcal{C}_0^\infty(\Omega)$ and $j,k = 1,\dots,n$, proceeding as in [1, Lemma 2.3], we have

$$\begin{aligned}
-\int_\Omega \mathbf{u}_k(x)\ \partial_{x_j}\phi(x)\ \mathrm{d}x &= -\int_\Omega\int_\Omega \mathbf{G}_k(x,y) f(y)\ \partial_{x_j}\phi(x)\ \mathrm{d}x\ \mathrm{d}y \\
&= -\int_\Omega f(y)\ \lim_{\varepsilon\to 0}\left(\int_{|y-x|>\varepsilon} \mathbf{G}_k(x,y)\ \partial_{x_j}\phi(x)\ \mathrm{d}x\right)\mathrm{d}y \\
&= \int_\Omega f(y)\ \lim_{\varepsilon\to 0}\left(\int_{|y-x|>\varepsilon} \partial_{x_j}\mathbf{G}_k(x,y)\ \phi(x)\ \mathrm{d}x - \int_{|y-\xi|=\varepsilon} \mathbf{G}_k(\xi,y)\ \phi(\xi)\frac{y_j-\xi_j}{|y_j-\xi_j|}\mathrm{d}\xi\right)\mathrm{d}y.
\end{aligned}$$

Using (49) and switching the order of integration lead us to

$$\begin{aligned}
-\int_\Omega \mathbf{u}_k(x)\ \partial_{x_j}\phi(x)\ \mathrm{d}x &= -\lim_{\varepsilon\to 0}\int_\Omega \phi(x)\left(\int_{|y-x|>\varepsilon} \partial_{y_j}\mathbf{G}_k(x,y)\ f(y)\ \mathrm{d}y\right)\mathrm{d}x \\
&\quad + \int_\Omega\int_\Omega (\widetilde{\mathbf{G}}_j)_k(x,y)\ f(y)\ \phi(x)\ \mathrm{d}x\ \mathrm{d}y - \lim_{\varepsilon\to 0}\int_\Omega\int_{|y-\xi|=\varepsilon} \mathbf{G}_k(\xi,y)\ \phi(\xi)\ f(y)\frac{y_j-\xi_j}{|y_j-\xi_j|}\mathrm{d}\xi\ \mathrm{d}y \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

Integrating by parts with respect to the y variable and recalling that f belongs to $\mathcal{C}_0^\infty(\Omega)$, we obtain

$$\begin{aligned}
\mathcal{I}_1 &= \lim_{\varepsilon\to 0}\int_\Omega \phi(x)\left(\int_{|x-y|>\varepsilon} \mathbf{G}_k(x,y)\ \partial_{y_j} f(y)\ \mathrm{d}y - \int_{|x-\zeta|=\varepsilon} \mathbf{G}_k(x,\zeta)\ f(\zeta)\frac{x_j-\zeta_j}{|x_j-\zeta_j|}\mathrm{d}\zeta\right)\mathrm{d}x \\
&= \int_\Omega\int_\Omega \mathbf{G}_k(x,y)\ \partial_{y_j} f(y)\ \phi(x)\ \mathrm{d}y\ \mathrm{d}x + \lim_{\varepsilon\to 0}\int_\Omega\int_{|\zeta-x|=\varepsilon} \mathbf{G}_k(x,\zeta)\ f(\zeta)\ \phi(x)\frac{\zeta_j-x_j}{|\zeta_j-x_j|}\mathrm{d}x\ \mathrm{d}\zeta \\
&= \int_\Omega\int_\Omega \mathbf{G}_k(x,y)\ \partial_{y_j} f(y)\ \phi(x)\ \mathrm{d}y\ \mathrm{d}x - \mathcal{I}_3.
\end{aligned}$$

Combining the two above displays, we infer

$$-\int_\Omega \mathbf{u}_k(x)\ \partial_{x_j}\phi(x)\ \mathrm{d}x = \int_\Omega\left(\int_\Omega \mathbf{G}_k(x,y)\ \partial_{y_j} f(y)\ \mathrm{d}y + \int_\Omega (\widetilde{\mathbf{G}}_j)_k(x,y)\ f(y)\ \mathrm{d}y\right)\phi(x)\ \mathrm{d}x,$$

which gives (50) for any f in $\mathcal{C}_0^\infty(\Omega)$. □

We are in a position to prove that $\mathbf{u}$ satisfies homogeneous boundary conditions.

Corollary B.3. *Given f in $H_0^1(\Omega)\cap L_0^2(\Omega)$, consider $\mathbf{u}$ as in (21). Then, $\mathbf{u}$ belongs to $[H_0^2(\Omega)]^n$.*

Proof. Step 1: a decomposition of $\nabla \mathbf{u}$. Proposition B.2 allows us to express the first derivatives of $\mathbf{u}$ as a sum of two contributions. The first correspond to Bogovskiĭ's constructions applied to the first derivatives of f; the second are equivalent to (21b) with ω in (21a) replaced by its first derivatives, which still belongs to $\mathcal{C}_0^\infty(\Omega)$.

In other words, we write $\nabla \mathbf{u} = \boldsymbol{\tau} + \boldsymbol{\eta}$, where the j-th columns, $j = 1, \dots, n$, of $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ are given by

$$\boldsymbol{\tau}_j(x) = \int_\Omega \mathbf{G}(x,y)\ \partial_{y_j} f(y)\ \mathrm{d}y; \qquad \boldsymbol{\eta}_j(x) = \int_\Omega \widetilde{\mathbf{G}}_j(x,y)\ f(y)\ \mathrm{d}y.$$

Step 2: treating $\boldsymbol{\tau}$. Let $j = 1, \dots, n$ be fixed. Consider a sequence g_m in $L^\infty(\Omega)$ such that $g_m \to \partial_{y_j} f$ in $L^2(\Omega)$ as $m \to \infty$. We consider a sequence of tensors $\boldsymbol{\tau}_m$ whose j-th columns are given by

$$(\boldsymbol{\tau}_m)_j(x) = \int_\Omega \mathbf{G}(x,y)\ g_m(y)\ \mathrm{d}y.$$

Applying [1, Proposition 2.1], it follows that $(\boldsymbol{\tau}_m)_j$ is continuous and vanishes on $\partial\Omega$; Aa a result of [24], we obtain that $(\boldsymbol{\tau}_m)_j$ belongs to $[W_0^{1,\infty}(\Omega)]^n$. Using [18, Corollary 19, part (iii)] [1] applied to $\partial_{y_j} f - g_m \in L^2(\Omega)$, we get $(\boldsymbol{\tau}_m)_j \to \boldsymbol{\tau}_j$ in $[H^1(\Omega)]^n$. Since $(\boldsymbol{\tau}_m)_j$ is in $[W_0^{1,\infty}(\Omega)]^n$ for all m, we deduce that $\boldsymbol{\tau}_j$ belongs to $[H_0^1(\Omega)]^n$.

Step 3: treating $\boldsymbol{\eta}$ and conclusion. The proof that $\boldsymbol{\eta}_j$ belongs to $[H_0^1(\Omega)]^n$ essentially boils down to the proof that $\mathbf{u}$ in (4) vanishes on the boundary of Ω; the only difference resides in the presence of $\widetilde{\mathbf{G}}$ in lieu of $\mathbf{G}$, which solely impacts the constant in (4). We conclude that $\nabla \mathbf{u}$ belongs to $[H_0^1(\Omega)]^{n\times n}$. In other words, $\mathbf{u}$ belongs to $[H_0^2(\Omega)]^n$, since (4) already gives that $\mathbf{u}$ is in $[H_0^1(\Omega)]^n$. □

Remark 3. Identity (50) may allow us to prove a first order Babuška-Aziz inequality also in a non-Hilbertian setting, namely we may substitute H^k-type spaces by $W^{k,p}$-type spaces, $p \neq 2$. However, this would come at the price of suboptimal estimates as those in (5). The reason for this is the use of the Calderón-Zygmund theory instead of the Fourier transform approach by Durán while handling the term $\widetilde{T}_\alpha$ in Section 2.2.

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