Journal of Functional Analysis 288 (2025) 110817

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Quantitative spectral stability for the Neumann Laplacian in domains with small holes

Veronica Felli [a,*], Lorenzo Liverani [b], Roberto Ognibene [c]

[a] *Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano–Bicocca, Via Cozzi 55, 20125 Milano, Italy*
[b] *Department Mathematik, Chair of Dynamics, Control, Machine Learning and Numerics, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstraße 11, 91058 Erlangen, Germany*
[c] *Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo, 5, 56127 Pisa, Italy*

A R T I C L E I N F O

Article history:
Received 19 January 2024
Accepted 29 December 2024
Available online 6 January 2025
Communicated by J. Wei

MSC:
35J20
35B25
35P15

Keywords:
Neumann eigenvalues
Spectral stability
Singularly perturbed domains
Torsional rigidity

A B S T R A C T

The aim of the present paper is to investigate the behavior of the spectrum of the Neumann Laplacian in domains with little holes excised from the interior. More precisely, we consider the eigenvalues of the Laplacian with homogeneous Neumann boundary conditions on a bounded, Lipschitz domain. Then, we singularly perturb the domain by removing Lipschitz sets which are "small" in a suitable sense and satisfy a uniform extension property. In this context, we provide an asymptotic expansion for all the eigenvalues of the perturbed problem which are converging to simple eigenvalues of the limit one. The eigenvalue variation turns out to depend on a geometric quantity resembling the notion of (boundary) torsional rigidity: understanding this fact is one of the main contributions of the present paper. In the particular case of a hole shrinking to a point, through a fine blow-up analysis, we identify the exact vanishing order of such a quantity and we

* Corresponding author.
E-mail addresses: veronica.felli@unimib.it (V. Felli), lorenzo.liverani@fau.de (L. Liverani), roberto.ognibene@dm.unipi.it (R. Ognibene).

https://doi.org/10.1016/j.jfa.2024.110817

establish some connections between the location of the hole and the sign of the eigenvalue variation.

Contents

1. Introduction

In the present paper, we investigate the stability of the spectrum of the Neumann Laplacian under singular perturbations, consisting in the removal of small holes from a bounded domain.

Eigenvalues and eigenfunctions of differential operators are ubiquitous in the theory of partial differential equations. Understanding how these are sensitive to small perturbations, such as variations in the domain, is of interest in several fields of physical applications, e.g. quantum mechanics, material sciences, heat conduction, climate modeling and acoustics. See, in particular, [37] for perturbation theory in acoustics, [13, Chapter V] for eigenvalue problems in connection with vibrating systems and heat conduction and [17] (see also [42]) for links to climate analysis. We also quote [22] for a thorough survey on the dependence of eigenvalues and eigenfunctions on smooth and nonsmooth perturbations of the domain. Furthermore, a comprehensive understanding of the shape of eigenfunctions holds great significance in many numerical analysis problems. Nonetheless, the computational cost of determining eigenelements in domains with minute cavities is considerably high: indeed, to ensure precision in such cases, i.e. to discern even small variations, dense mesh structures are needed around these cavities. Consequently, theoretical approximation results in this specific context assume a pivotal role. We refer to [7] and references therein for a wide discussion of the topic. We also mention [32] for some recent applications to machine learning of spectral stability of the Neumann Laplacian under domain deformations. Finally, as pointed out in [24, Section 1.4], asymptotic expansions of eigenvalues in domains with small holes might find applications in shape optimization, e.g. in the proof of non-existence of minimizers.

The problem of spectral stability for differential operators in perforated domains has a long history, and presents intrinsically different features depending on which kind of boundary conditions are taken into account. Let us consider a bounded open set $\Omega \subseteq \mathbb{R}^N$, which we call *unperturbed domain*, and a compact subset $K \subseteq \Omega$, which we call *hole*; we refer to the set $\Omega \setminus K$ as the *perturbed domain*. In order for $\Omega \setminus K$ to be regarded as a perturbation of Ω, the hole K needs to be sufficiently small, in a suitable sense depending on the operator under investigation. In this regard, a key role is played by the conditions prescribed on the boundaries of both the unperturbed domain and the hole; among the most studied cases, we find homogeneous Dirichlet and Neumann boundary conditions, as well as Robin-type ones. Under each of the boundary conditions mentioned above and under suitable regularity assumptions on the sets, the eigenvalue problem for the Laplace operator on the perturbed domain $\Omega \setminus K$ typically admits a sequence of diverging eigenvalues

$$\Lambda_0(\Omega \setminus K) \le \Lambda_1(\Omega \setminus K) \le \Lambda_2(\Omega \setminus K) \le \cdots \le \Lambda_n(\Omega \setminus K) \le \cdots ,$$

by classical spectral theory. Analogously, the unpertubed problem (corresponding to the case $K = \emptyset$) typically admits a sequence of diverging eigenvalues

$$\Lambda_0(\Omega) \le \Lambda_1(\Omega) \le \Lambda_2(\Omega) \le \cdots \le \Lambda_n(\Omega) \le \cdots .$$

In this setting, the stability of the spectrum is a main object of investigation. More precisely, a major question is the following:

Question 1. Under which conditions on the hole K, are the perturbed eigenvalues $\Lambda_n(\Omega \setminus K)$ arbitrarily close to the corresponding unperturbed ones $\Lambda_n(\Omega)$?

Once conditions on K that ensure spectral stability are found, the further following question naturally arises:

Question 2. Is it possible to quantify the difference $\Lambda_n(\Omega \setminus K) - \Lambda_n(\Omega)$ in terms of some *measurement* of K?

In the case of homogeneous Neumann boundary conditions on both the external boundary $\partial\Omega$ and the hole's boundary ∂K, question 1 has been answered in [38]. In the present paper we focus on question 2.

We precede the presentation of our results with a brief overview of the literature dealing with the spectral stability for the Laplacian in perforated domains. This problem is widely investigated. In particular, the case of Dirichlet boundary conditions is one of the most studied and, being the literature on the topic so vast, we cite here just some of the most relevant papers. In the Dirichlet case, it is well known that a key quantity in the study of spectral stability is the capacity of the hole. Some first estimates of the variation of the Dirichlet eigenvalues in terms of the capacity of the removed set date back to [39].

The paper [38], published in 1975, still stands as a pivotal reference in this research field; it contains a more systematic study of spectral stability in domains with small holes, taking also into account more general boundary conditions. Subsequent studies are carried out in a series of papers by Ozawa in the 80s, deriving sharp asymptotic expansions of perturbed eigenvalues, especially in small dimensions, see e.g. [33]. Another relevant result is contained in [16] (recently revisited in [1]), which provides an asymptotic expansion for any perturbed (possibly multiple) eigenvalue; in particular, if $\Lambda_n(\Omega)$ is a simple Dirichlet eigenvalue and u_n is a corresponding L^2-normalized eigenfunction, then

$$\Lambda_n(\Omega \setminus K_\varepsilon) = \Lambda_n(\Omega) + \mathrm{cap}_\Omega(K_\varepsilon, u_n) + o(\mathrm{cap}_\Omega(K_\varepsilon, u_n)) \quad \text{as } \varepsilon \to 0, \tag{1.1}$$

where $\{K_\varepsilon\}_{\varepsilon>0}$ is a family of compact sets concentrating to a zero-capacity set as $\varepsilon \to 0$, and

$$\mathrm{cap}_\Omega(K_\varepsilon, u_n) := \inf \left\{ \int_\Omega |\nabla u|^2 \,\mathrm{d}x \colon u \in H^1_0(\Omega),\ u - u_n \in H^1_0(\Omega \setminus K_\varepsilon) \right\}.$$

We also cite [3], treating the case of multiple limit eigenvalues. For simple eigenvalues, an analogue of (1.1) is derived in [2] in a fractional setting and in [20] for polyharmonic operators. The results of [18] seem to suggest that only the boundary conditions prescribed on the hole essentially play a role in the asymptotics of eigenvalues when the hole disappears: indeed, in [18], in the case of Neumann conditions prescribed on the outer boundary and Dirichlet conditions on the hole, an asymptotic expansion similar to (1.1) is proved. Again a suitable notion of capacity of the hole comes into play.

As for Neumann boundary conditions prescribed on the hole, less is known, and a richer phenomenology can be observed. After the work of Rauch and Taylor [38], where sufficient conditions for stability of the Neumann spectrum are provided, several papers investigate the asymptotic behavior of perturbed eigenvalues. In dimension 2 and in the case of a disk-shaped hole, Ozawa [34] proves that, if $\Lambda_n(\Omega)$ is a simple eigenvalue of the Dirichlet Laplacian in Ω, and $\Lambda_{n,\varepsilon}$ is the n-th eigenvalue of

$$\begin{cases} -\Delta u = \Lambda u, & \text{in } \Omega_\varepsilon := \Omega \setminus \overline{\Sigma_\varepsilon}, \\ u = 0, & \text{on } \partial\Omega, \\ \partial_{\boldsymbol{\nu}} u = 0, & \text{on } \partial\Sigma_\varepsilon, \end{cases} \tag{1.2}$$

with $\Sigma_\varepsilon = \{x \in \mathbb{R}^2 : |x - x_0| < \varepsilon\}$ for some $x_0 \in \Omega$, then

$$\Lambda_{n,\varepsilon} = \Lambda_n(\Omega) - \pi\varepsilon^2 \left(2\,|\nabla u_n(x_0)|^2 - \Lambda_n(\Omega) u_n^2(x_0) \right) + O(\varepsilon^3 |\log \varepsilon|^2) \quad \text{as } \varepsilon \to 0,$$

where $u_n \in H^1_0(\Omega)$ is a L^2-normalized Dirichlet eigenfunction corresponding to $\Lambda_n(\Omega)$. See also [35] for asymptotic properties of eigenfunctions of (1.2) in dimension 2. For

$N=3$, asymptotic expansions for the perturbed Neumann eigenvalues $\mu_n(\Omega_\varepsilon)$ are obtained in [31] in the case of a hole shrinking to a point: more precisely, the expansion

$$\mu_n(\Omega_\varepsilon) = \mu_n(\Omega) + C_n\varepsilon^3 + o(\varepsilon^3) \quad \text{as } \varepsilon \to 0,$$

is proved, where $C_n \in \mathbb{R}$ is explicitly characterized in [31, (2.46)]. A more general framework is taken into account in [26]: here, the removed holes are tubular neighborhoods of d-dimensional manifolds $\mathcal{M} \subseteq \Omega$, i.e.

$$\Sigma_\varepsilon = \{x \in \Omega \colon \operatorname{dist}(x, \mathcal{M}) < \varepsilon\}, \tag{1.3}$$

and both Neumann and Robin conditions on $\partial\Sigma_\varepsilon$ are considered. Denoting $q := N-d \geq 2$, if $\Lambda_n(\Omega)$ is a simple eigenvalue of the Dirichlet Laplacian in Ω (with a L^2-normalized eigenfunction u_n) and $\Lambda_{n,\varepsilon}$ is the n-th eigenvalue of (1.2) with Σ_ε as in (1.3), then [26] proves the expansion

$$\Lambda_{n,\varepsilon} = \Lambda_n(\Omega) - \omega_q\,\varepsilon^q \int_{\mathcal{M}} \left[\frac{q}{q-1}\left|\nabla_\perp u_n\right|^2 + \left|\nabla_{\mathcal{M}} u_n\right|^2 - \Lambda_n(\Omega)u_n^2 + u_n H[u_n]\right] \mathrm{d}\mathcal{H}^d + o(\varepsilon^q)$$

as $\varepsilon \to 0$, where

$$H[u_n](x) := \lim_{t\to 0} \frac{u_n(x + tH(x)) - u_n(x)}{t}$$

and $H(x)$ denotes the mean curvature vector field on $\mathcal{M}$. For Neumann conditions prescribed on both $\partial\Omega$ and the hole's boundary $\partial\Sigma_\varepsilon$, full asymptotic expansions in terms of analytic functions are obtained in [29], in the case of a hole shrinking to a point. We also cite [27], where Neumann eigenvalues are studied for a zonal subdomain of the N-sphere, which converges to the whole sphere itself. Finally, it is worth mentioning [6,9,30] for other quantitative spectral stability results for the Neumann Laplacian and [5,10,11,14,23,28] for qualitative studies for more general types of holes.

As emerged from the previous discussion, the stability of Neumann eigenvalues in domains with small holes is not understood as well as in the Dirichlet case and presents different and peculiar features. For instance, stability of the spectrum of the Neumann Laplacian is not guaranteed under assumptions that would otherwise ensure stability in the Dirichlet case, see e.g. [13, p. 420] or [40]. In [38] (see also [5]) it is proved that, by removing sets $\overline{\Sigma_\varepsilon}$ whose measure tends to zero as $\varepsilon \to 0$, a sufficient condition for stability is a uniform extension property (in the Sobolev sense) inside the hole, which rules out too wild behaviors of the disappearing hole itself; see assumption **(H)** in section 2. The main novelty of the present paper lies in the identification of a geometric quantity, related to the shape of the hole $\overline{\Sigma_\varepsilon}$ and to the limit eigenfunction φ_n, that plays in the Neumann context the same role as the capacity does for the Dirichlet case, concerning quantitative spectral stability. This quantity, denoted as $\mathcal{T}_{\overline{\Omega}\setminus\Sigma_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n)$, is introduced in Definition 2.1 and resembles a notion of torsional rigidity.

Now, we provide a description of the most significant contents of the present paper, referring to section 2 for the rigorous statements. Our first main result Theorem 2.4 contains an asymptotic expansion for an eigenvalue $\lambda_n(\Omega\setminus\overline{\Sigma_\varepsilon})$ of the perturbed Neumann problem (2.3), in case it converges to a simple eigenvalue $\lambda_n(\Omega)$ of the unperturbed one (2.1). In the asymptotic expansion of the variation $\lambda_n(\Omega \setminus \overline{\Sigma_\varepsilon}) - \lambda_n(\Omega)$, the sum of two contributions appears: the geometric quantity

$$-\mathcal{T}_{\overline{\Omega}\setminus\Sigma_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) < 0 \tag{1.4}$$

which always has a negative sign, and the additional term

$$-\int_{\Sigma_\varepsilon} \left(|\nabla\varphi_n|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2\right) \mathrm{d}x, \tag{1.5}$$

whose sign depends on where the hole is located, with respect to the nodal, regular and singular sets of φ_n. The presence of this additional term causes the eigenvalue variation not to have a fixed sign, in stark contrast to what happens in the Dirichlet setting, where the monotonicity of the eigenvalues with respect to the inclusion of domains always results in positive differences $\Lambda_n(\Omega \setminus \overline{\Sigma_\varepsilon}) - \Lambda_n(\Omega)$.

Next, we focus on holes shrinking to a point by maintaining the same fixed shape, that is of the form

$$\Sigma_\varepsilon := x_0 + \varepsilon\Sigma = \{x_0 + \varepsilon x \colon x \in \Sigma\}, \tag{1.6}$$

for some $x_0 \in \Omega$ and $\Sigma \subseteq \mathbb{R}^N$. In this case, if $N \geq 3$, we succeed in performing a blow-up analysis, which provides the explicit rate of convergence of the quantity (1.4). Moreover, by analyzing the behavior of the limit eigenfunction φ_n near the point x_0, we determine the explicit rate of convergence of the additional term (1.5). Combining these two sharp expansions, we obtain our second and third main results, namely Theorem 2.8 and Theorem 2.9, which provide, for $N \geq 3$ and the hole being as in (1.6), a precise description of the asymptotic behavior of eigenvalues and eigenfunctions, respectively.

From Theorem 2.8 we can deduce some interesting information about the sign of the eigenvalue variation and the sharpness of the derived expansion. Notably, these aspects appear to depend on whether x_0 is or is not located on the singular set of the limit eigenfunction, and, in the latter case, on its specific positioning relative to the interface Γ introduced in (2.22). See Remark 2.10 for details.

Finally, in Theorems 2.11 and 2.12 we derive more explicit expansions in the case of spherical holes, in dimensions $N \geq 3$ and $N = 2$ respectively.

2. Statement of the main results

For any open, bounded, connected, Lipschitz set $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, we consider the following eigenvalue problem:

$$\begin{cases} -\Delta\varphi + \varphi = \lambda\varphi, & \text{in } \Omega, \\ \partial_{\boldsymbol{\nu}}\varphi = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\boldsymbol{\nu}$ denotes the outer unit normal vector to $\partial\Omega$. Problem (2.1) is meant in a weak sense; i.e., $\lambda \in \mathbb{R}$ is an *eigenvalue* if there exists $u \in H^1(\Omega) \setminus \{0\}$, called *eigenfunction*, such that

$$\int_\Omega (\nabla u \cdot \nabla\varphi + u\varphi)\,\mathrm{d}x = \lambda \int_\Omega u\varphi\,\mathrm{d}x, \quad \text{for all } \varphi \in H^1(\Omega). \tag{2.2}$$

In view of the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, classical spectral theory ensures the existence of a diverging sequence of eigenvalues

$$0 < 1 = \lambda_0(\Omega) < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \le \lambda_n(\Omega) \le \cdots .$$

It is evident that the eigenvalues $\{\mu_n(\Omega)\}$ of the standard Neumann Laplacian can be obtained from those of (2.1) with a translation, i.e.

$$\mu_n(\Omega) := \lambda_n(\Omega) - 1,$$

the eigenfunctions being exactly the same. Moreover, since $\lambda_0(\Omega)$ is equal to 1 for any choice of the set Ω, in the following we only consider eigenvalues with index 1 or higher.

Let us now perturb Ω, by removing a small hole from the interior. More precisely, we consider a family $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ of subsets of $\mathbb{R}^N$ satisfying the following assumption.

Assumption (H). We assume that, for some $\varepsilon_0 > 0$,

for every $\varepsilon \in (0,\varepsilon_0)$, Σ_ε is an open, Lipschitz set such that $\overline{\Sigma_\varepsilon} \subseteq \Omega$; (H1)

for every $\varepsilon \in (0,\varepsilon_0)$, there exists $\mathsf{E}_\varepsilon\colon H^1(\Omega \setminus \overline{\Sigma_\varepsilon}) \to H^1(\Omega)$ such that (H2)

$$(\mathsf{E}_\varepsilon u)|_{\Omega\setminus\overline{\Sigma_\varepsilon}} = u \quad \text{and} \quad \|\mathsf{E}_\varepsilon u\|_{H^1(\Omega)} \le \mathfrak{C}\,\|u\|_{H^1(\Omega\setminus\overline{\Sigma_\varepsilon})} \quad \text{for all } u \in H^1(\Omega \setminus \overline{\Sigma_\varepsilon}),$$

where $\mathfrak{C} > 0$ is a constant independent of $\varepsilon \in (0,\varepsilon_0)$;

$|\Sigma_\varepsilon| \to 0$ as $\varepsilon \to 0$ (where $|\cdot|$ denotes the N-dimensional Lebesgue measure). (H3)

For every $\varepsilon \in (0,\varepsilon_0)$, we denote the perturbed domain by

$$\Omega_\varepsilon := \Omega \setminus \overline{\Sigma_\varepsilon},$$

and consider the perturbed problem

$$\begin{cases} -\Delta\varphi + \varphi = \lambda\varphi, & \text{in } \Omega_\varepsilon, \\ \partial_{\boldsymbol{\nu}}\varphi = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.3}$$

meant in a weak sense as in (2.2). This produces the perturbed spectrum, which consists of an increasing diverging sequence $\{\lambda_j(\Omega_\varepsilon)\}_{j\in\mathbb{N}}$. In [38, Theorem 3.1] Rauch and Taylor prove that, under assumption **(H)**,

$$\lambda_j(\Omega_\varepsilon) \to \lambda_j(\Omega) \quad \text{as } \varepsilon \to 0, \quad \text{for all } j \in \mathbb{N}. \tag{2.4}$$

Moreover, by classical spectral theory, there exist

$$\{\varphi_j\}_{j\geq 0} \subseteq H^1(\Omega) \quad \text{and} \quad \{\varphi_j^\varepsilon\}_{j\geq 0} \subseteq H^1(\Omega_\varepsilon) \tag{2.5}$$

orthonormal bases of $L^2(\Omega)$, respectively $L^2(\Omega_\varepsilon)$, such that, for every j, φ_j and φ_j^ε are eigenfunctions associated to $\lambda_j(\Omega)$ and $\lambda_j(\Omega_\varepsilon)$, respectively.

Hereafter, we fix $n \in \mathbb{N} \setminus \{0\}$ such that

$$\lambda_n(\Omega) \text{ is simple.} \tag{2.6}$$

A key role in our asymptotic expansion is played by the geometric quantity defined below, which provides a measurement of the hole Σ_ε and resembles the notion of torsional rigidity of a set; see e.g. [25,36] for the classical notion of *torsional rigidity* and [8] for the *boundary torsional rigidity*.

Definition 2.1. Let $E \subseteq \mathbb{R}^N$ be an open Lipschitz set such that $\overline{E} \subset \Omega$ and $f \in L^2(\partial E)$. Let

$$J_{\Omega,E,f} : H^1(\Omega \setminus \overline{E}) \to \mathbb{R}, \quad J_{\Omega,E,f}(u) := \frac{1}{2}\int_{\Omega\setminus\overline{E}} (|\nabla u|^2 + u^2)\,\mathrm{d}x - \int_{\partial E} uf\,\mathrm{d}S.$$

We define the *Sobolev f-torsional rigidity of ∂E relative to $\overline{\Omega}\setminus E$* (briefly, the *$f$-torsional rigidity of ∂E*) as

$$\mathcal{T}_{\overline{\Omega}\setminus E}(\partial E, f) := -2\inf\left\{J_{\Omega,E,f}(u) \colon u \in H^1(\Omega\setminus\overline{E})\right\}.$$

By standard minimization arguments, there exists a unique $U_{\Omega,E,f} \in H^1(\Omega\setminus\overline{E})$ achieving the infimum defining $\mathcal{T}_{\overline{\Omega}\setminus E}(\partial E, f)$, i.e. such that

$$\mathcal{T}_{\overline{\Omega}\setminus E}(\partial E, f) = -2J_{\Omega,E,f}(U_{\Omega,E,f}), \tag{2.7}$$

see Proposition 3.4. We also recall the definition of Sobolev capacity of a set.

Definition 2.2. Let $K \subseteq \mathbb{R}^N$ be a compact set. The *Sobolev capacity* of K is defined as

$$\mathrm{Cap}\,(K) := \inf\Bigg\{ \int_{\mathbb{R}^N} \big(|\nabla u|^2 + u^2\big)\,\mathrm{d}x \colon u \in H^1(\mathbb{R}^N),\ u = 1 \text{ a.e. in an open neighborhood of } K \Bigg\}.$$

If the family $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ satisfies **(H)** and

$$\lim_{\varepsilon\to 0} \mathrm{Cap}\,(\overline{\Sigma_\varepsilon}) = 0, \tag{2.8}$$

under assumption (2.6) it is possible to uniquely choose the n-th eigenfunction of the orthonormal basis $\{\varphi_j^\varepsilon\}_{j\geq 0}$ in (2.5) in such a way that

$$\int_{\Omega_\varepsilon} \varphi_n^\varepsilon \varphi_n\,\mathrm{d}x \geq 0 \quad \text{for } \varepsilon \text{ sufficiently small.} \tag{2.9}$$

If φ_n^ε is chosen as above, one can prove that

$$\|\varphi_n^\varepsilon - \varphi_n\|_{H^1(\Omega_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0, \tag{2.10}$$

see Lemma 3.7. Furthermore, assumption **(H)** implies that

$$\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) \to 0 \quad \text{as } \varepsilon \to 0,$$

see Corollary 3.3.

Remark 2.3. It could happen that (2.4) holds true while (2.8) fails (in contrast to what happens in the Dirichlet case). An example of this phenomenon can be found in [38, Section 4]. More precisely, for every $j \in \mathbb{N} \setminus \{0\}$, let E_j be the union of j disjoint open balls of radius $r_j > 0$, evenly spaced inside a bounded region $\mathcal{U} \subset \mathbb{R}^N$. Let us choose the radii r_j in such a way that $\lim_{j\to\infty} j r_j^N = 0$ (so that $|E_j| \to 0$) and

$$\begin{cases} \lim_{j\to\infty} j r_j^{N-2} = +\infty, & \text{if } N \geq 3,\\ \lim_{j\to\infty} \frac{j}{|\log r_j|} = +\infty, & \text{if } N = 2. \end{cases} \tag{2.11}$$

In [38, Section 4] it is observed that, under condition (2.11), $K_j = \overline{E_j}$ *becomes solid* in $\mathcal{U}$ as $j \to \infty$, i.e.

$$\alpha_j := \inf\left\{ \frac{\int_{\mathcal{U}\setminus\overline{E_j}} |\nabla v|^2\,\mathrm{d}x}{\int_{\mathcal{U}\setminus\overline{E_j}} v^2\,\mathrm{d}x} : v \in H^1(\mathcal{U} \setminus \overline{E_j}), v = 0 \text{ on } \partial E_j \right\} \underset{j\to\infty}{\longrightarrow} +\infty.$$

This implies that it cannot happen that $\lim_{j\to\infty} \mathrm{Cap}\,(\overline{E_j}) = 0$. Indeed, let us argue by contradiction and assume that $\lim_{j\to\infty} \mathrm{Cap}\,(\overline{E_j}) = 0$. Then, for every $j \in \mathbb{N} \setminus \{0\}$ there exists $u_j \in H^1(\mathbb{R}^N)$ such that $u_j = 1$ a.e. in an open neighborhood of $\overline{E_j}$ and $\lim_{j\to\infty} \|u_j\|_{H^1(\mathbb{R}^N)} = 0$. Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta \equiv 1$ in $\mathcal{U}$. Then

$$\frac{\int_{\mathcal{U}} |\nabla(u_j - \eta)|^2 \,\mathrm{d}x}{\int_{\mathcal{U}} (u_j - \eta)^2 \,\mathrm{d}x} = \frac{\int_{\mathcal{U}\setminus\overline{E_j}} |\nabla(u_j - \eta)|^2 \,\mathrm{d}x}{\int_{\mathcal{U}\setminus\overline{E_j}} (u_j - \eta)^2 \,\mathrm{d}x} \geq \alpha_j$$

and a contradiction arises letting $j \to \infty$, since the left hand side converges to $\frac{1}{|\mathcal{U}|}\int_{\mathcal{U}} |\nabla\eta|^2 \,\mathrm{d}x$.

On the other hand, the sequence of sets $\{E_j\}_j$ satisfies assumption **(H)**, so that [38, Theorem 3.1] ensures spectral stability as $j \to \infty$ for the Neumann problem under removal of the sets E_j.

Our first result provides an asymptotic expansion of a perturbed eigenvalue (and its corresponding eigenfunction) in the case it converges to a simple eigenvalue of the limit problem.

Theorem 2.4. *Let $n \geq 1$ be such that (2.6) is satisfied. Let $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ satisfy assumptions* **(H)** *and (2.8). Then*

$$\begin{aligned}\lambda_n(\Omega_\varepsilon) &= \lambda_n(\Omega) - \mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) - \int_{\Sigma_\varepsilon} \left(|\nabla\varphi_n|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2\right) \mathrm{d}x \\ &\quad + o\left(\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n)\right) + o\left(\int_{\Sigma_\varepsilon} \left(|\nabla\varphi_n|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2\right) \mathrm{d}x\right) \quad \text{as } \varepsilon \to 0. \qquad (2.12)\end{aligned}$$

In addition, if $U_\varepsilon := U_{\Omega,\Sigma_\varepsilon,\partial_{\boldsymbol{\nu}}\varphi_n} \in H^1(\Omega_\varepsilon)$ denotes the function achieving $\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n)$ as in (2.7) and φ_n^ε is chosen as in (2.9), then

$$\|\varphi_n^\varepsilon - (\varphi_n - U_\varepsilon)\|_{H^1(\Omega_\varepsilon)}^2 = o\left(\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n)\right) + O(\|\varphi_n\|_{L^2(\Sigma_\varepsilon)}^4) \quad \text{as } \varepsilon \to 0 \qquad (2.13)$$

and

$$\begin{aligned}\|\varphi_n^\varepsilon - \varphi_n\|_{H^1(\Omega_\varepsilon)}^2 &= \mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) + o\left(\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n)\right) \\ &\quad + O(\|\varphi_n\|_{L^2(\Sigma_\varepsilon)}^4) + O\left(\|\varphi_n\|_{L^2(\Sigma_\varepsilon)}^2 \sqrt{\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n)}\right) \quad \text{as } \varepsilon \to 0. \qquad (2.14)\end{aligned}$$

Let us briefly describe the basic idea behind the proof of Theorem 2.4. We consider the function

$$f_\varepsilon := \varphi_n - U_\varepsilon.$$

It turns out that f_ε is a good approximation of the perturbed eigenfunction φ_n^ε, while encoding information from given quantities (the unperturbed eigenfunction φ_n and the hole Σ_ε). Indeed, since by assumption

$$\|U_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} = \mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) \to 0 \quad \text{as } \varepsilon \to 0,$$

see Remark 3.5, then f_ε is close to φ_n for small ε. Moreover, it satisfies an equation rather similar to that of φ_n^ε, i.e.

$$\begin{cases} -\Delta f_\varepsilon + f_\varepsilon = \lambda_n(\Omega)\varphi_n, & \text{in } \Omega_\varepsilon, \\ \partial_{\boldsymbol{\nu}} f_\varepsilon = 0, & \text{on } \partial\Sigma_\varepsilon. \end{cases}$$

By estimating the difference $\varphi_n^\varepsilon - f_\varepsilon$, through an abstract result known in the literature as *Lemma on small eigenvalues* (originally proved in [15]), see Lemma A.1, we obtain the expansion of the eigenvalue variation stated in Theorem 2.4.

Theorem 2.4 applies to a fairly general framework and provides an expansion in terms of $\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n)$ and $\int_{\Sigma_\varepsilon}(|\nabla\varphi_n|^2 - (\lambda_n(\Omega)-1)\varphi_n^2)\,\mathrm{d}x$. We now direct our attention towards the asymptotic behavior of these quantities, with the aim of deriving an explicit expansion of the eigenvalue variation in some relevant examples: the case of a hole shrinking to an interior point in dimension $N \geq 3$ and the case of a disk-shaped hole in dimension $N = 2$.

Let us fix $x_0 \in \Omega$ and an open, bounded, Lipschitz set $\Sigma \subseteq \mathbb{R}^N$, and consider a hole $\Sigma_\varepsilon = x_0 + \varepsilon\Sigma$ as in (1.6). In this case, the family $\{\Sigma_\varepsilon\}$ is concentrating to the point x_0 by shrinking and maintaining the same shape Σ. Since Ω is open and Σ is bounded, there exist $r_0 > 0$ and $\varepsilon_0 > 0$ such that

$$x_0 + \overline{B_{r_0}} \subset \Omega \quad \text{and} \quad x_0 + \varepsilon\overline{\Sigma} \subset x_0 + B_{r_0} \quad \text{for all } \varepsilon \in (0, \varepsilon_0), \tag{2.15}$$

where $B_{r_0} := \{x \in \mathbb{R}^N : |x| < r_0\}$ is the ball in $\mathbb{R}^N$ with center at 0 and radius r_0. The family $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ turns out to satisfy assumptions **(H)** and (2.8), see Lemma A.2; therefore, Theorem 2.4 applies. Hence, if $\lambda_n(\Omega)$ is simple, the problem of finding explicit asymptotic expansions for the perturbed eigenvalue $\lambda_n(\Omega_\varepsilon)$ boils down to the analysis of the behavior of the quantities

$$\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) \quad \text{and} \quad \int_{\Sigma_\varepsilon} \left(|\nabla\varphi_n|^2 - (\lambda_n(\Omega)-1)\varphi_n^2\right) \mathrm{d}x$$

as $\varepsilon \to 0$. Similarly to what happens in other singularly perturbed spectral problems (see e.g. [1,2,4,18,19]), the local behavior of the normalized eigenfunction φ_n (which is unique, up to a sign) near the point x_0 plays a crucial role, as described below.

For every $u \in C^\infty(\Omega)$, $y \in \Omega$ and $i \in \mathbb{N}$, we consider the polynomial

$$P^u_{y,i}(x) := \sum_{\substack{\beta\in\mathbb{N}^N\\ |\beta|=i}} \frac{1}{\beta!} D^\beta u(y)\, x^\beta, \quad x \in \mathbb{R}^N, \tag{2.16}$$

where $|\beta| = \beta_1 + \ldots + \beta_N$ and $\beta! = \beta_1! \cdot \ldots \cdot \beta_N!$ for all $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{N}^N$, with the tacit convention that

$$P^u_{y,0}(x) := u(y) \quad \text{for all } x \in \mathbb{R}^N. \tag{2.17}$$

In the case $y = 0$, we drop the index and write

$$P^u_i := P^u_{0,i}. \tag{2.18}$$

Definition 2.5. Let $u \in C^\infty(\Omega)$. We say that u *vanishes of order* $k \in \mathbb{N}$ *at* y if

$$P^u_{y,k}(x) \not\equiv 0 \quad \text{and} \quad P^u_{y,i}(x) \equiv 0 \quad \text{for all } i < k.$$

We observe that every nontrivial solution to problem (2.1) is analytic in Ω, hence, at any point $y \in \Omega$, it vanishes with some finite order $k \in \mathbb{N}$ (which is 0 if $u(y) \neq 0$) in the sense of Definition 2.5. For every analytic function $u : \Omega \to \mathbb{R}$, the nodal set of u is defined as

$$\mathcal{Z}(u) := \bigcup_{k=1}^{\infty} \mathcal{Z}_k(u),$$

where, for every $k \in \mathbb{N} \setminus \{0\}$,

$$\mathcal{Z}_k(u) := \{x \in \Omega \colon u \text{ vanishes of order } k \text{ at } x\}.$$

We define the *regular part* of the nodal set as

$$\mathrm{Reg}\,(u) := \mathcal{Z}_1(u) = \{x \in \Omega \colon u(x) = 0 \text{ and } \nabla u(x) \neq 0\}$$

and the *singular part* as

$$\mathrm{Sing}\,(u) = \mathcal{Z}(u) \setminus \mathrm{Reg}\,(u).$$

Our second main result establishes that, in the case of a shrinking hole, the rate of convergence of the perturbed eigenvalue to the unperturbed one depends on whether the hole is made on the singular part or not. In order to state the result, we need a notion of limit boundary torsional rigidity, to introduce which we recall the definition of Beppo Levi spaces.

Definition 2.6. Let $N \geq 3$ and $E \subseteq \mathbb{R}^N$ be an open Lipschitz set. The space $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus E)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N \setminus E)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N\setminus E)} := \left(\int_{\mathbb{R}^N\setminus E} |\nabla u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

By classical Sobolev's inequality,

$$\mathcal{D}^{1,2}(\mathbb{R}^N \setminus E) = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N \setminus E) : \nabla u \in L^2(\mathbb{R}^N \setminus E) \right\}.$$

Definition 2.7. Let $N \geq 3$, $E \subseteq \mathbb{R}^N$ be a bounded open Lipschitz set, and $f \in L^2(\partial E)$. Let

$$\tilde{J}_{E,f} : \mathcal{D}^{1,2}(\mathbb{R}^N \setminus E) \to \mathbb{R}, \quad \tilde{J}_{E,f}(u) := \frac{1}{2} \int_{\mathbb{R}^N\setminus E} |\nabla u|^2 \, \mathrm{d}x - \int_{\partial E} uf \, \mathrm{d}S.$$

We define the *f-torsional rigidity of ∂E relative to* $\mathbb{R}^N \setminus E$ as

$$\tau_{\mathbb{R}^N\setminus E}(\partial E, f) := -2 \inf \left\{ \tilde{J}_{E,f}(u) \colon u \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus E) \right\}.$$

By standard minimization arguments, there exists a unique $\tilde{U}_{E,f} \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus E)$ achieving the infimum defining $\tau_{\mathbb{R}^N\setminus E}(\partial E, f)$, i.e.

$$\tau_{\mathbb{R}^N\setminus E}(\partial E, f) = -2\tilde{J}_{E,f}(\tilde{U}_{E,f}), \tag{2.19}$$

see Proposition 3.4.

We are now ready to state our second main result, which is based on a blow-up analysis for the quantities appearing in the asymptotic expansion in Theorem 2.4. This provides the explicit rate of convergence of the perturbed eigenvalues, in terms of the behavior of the limit eigenfunction near the point where the hole is excised.

Theorem 2.8. *Let $N \geq 3$, $x_0 \in \Omega$, $\Sigma \subseteq \mathbb{R}^N$ be an open, bounded, Lipschitz set, $\varepsilon_0 > 0$ be as in (2.15) and, for every $\varepsilon \in (0, \varepsilon_0)$, $\Sigma_\varepsilon := x_0 + \varepsilon\Sigma$. Let $n \geq 1$ be such that $\lambda_n(\Omega)$ is simple.*

(i) *If $x_0 \in \Omega \setminus \mathrm{Sing}\,(\varphi_n)$, then, as $\varepsilon \to 0$,*

$$\lambda_n(\Omega_\varepsilon) = \lambda_n(\Omega) - \varepsilon^N \left(\tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, \nabla\varphi_n(x_0) \cdot \boldsymbol{\nu}) + |\Sigma|(|\nabla\varphi_n(x_0)|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2(x_0)) \right) + o(\varepsilon^N).$$

(ii) *If* $x_0 \in \mathrm{Sing}\,(\varphi_n)$*, then, as* $\varepsilon \to 0$*,*

$$\lambda_n(\Omega_\varepsilon) = \lambda_n(\Omega) - \varepsilon^{N+2k-2}\left(\tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, \partial_{\boldsymbol{\nu}} P^{\varphi_n}_{x_0,k}) + \int_\Sigma |\nabla P^{\varphi_n}_{x_0,k}|^2\,\mathrm{d}x\right) + o(\varepsilon^{N+2k-2}),$$

where $k \geq 2$ *is the vanishing order of* φ_n *at* x_0 *and* $P^{\varphi_n}_{x_0,k}$ *is as in* (2.16)*.*

Thanks to the estimates for the norm convergence of perturbed eigenfunctions, see (2.13), we are able to obtain the explicit rate of convergence in the case of a shrinking hole.

Theorem 2.9. *Let* $N \geq 3$*,* $x_0 \in \Omega$*,* $\Sigma \subseteq \mathbb{R}^N$ *be an open, bounded, Lipschitz set,* $\varepsilon_0 > 0$ *be as in* (2.15)*, and, for every* $\varepsilon \in (0, \varepsilon_0)$*,* $\Sigma_\varepsilon := x_0 + \varepsilon\Sigma$*. Let* $n \geq 1$ *be such that* $\lambda_n(\Omega)$ *is simple and* $k \geq 1$ *be the vanishing order of* $\varphi_n - \varphi_n(x_0)$ *at* x_0*. Let*

$$\Phi_\varepsilon(x) := \frac{\varphi_n^\varepsilon(\varepsilon x + x_0) - \varphi_n(x_0)}{\varepsilon^k},$$

where φ_n^ε *is chosen as in* (2.9)*. Then, for all* $R > 0$ *such that* $\overline{\Sigma} \subseteq B_R$*,*

$$\Phi_\varepsilon \to P^{\varphi_n}_{x_0,k} - \tilde{U}_{\Sigma, \partial_{\boldsymbol{\nu}} P^{\varphi_n}_{x_0,k}} \quad \text{in } H^1(B_R \setminus \overline{\Sigma}) \text{ as } \varepsilon \to 0, \tag{2.20}$$

where $\tilde{U}_{\Sigma, \partial_{\boldsymbol{\nu}} P^{\varphi_n}_{x_0,k}} \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma)$ *is the function achieving* $\tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, \partial_{\boldsymbol{\nu}} P^{\varphi_n}_{x_0,k})$ *as in* (2.19)*. Moreover*

$$\lim_{\varepsilon\to 0} \varepsilon^{-(N+2k-2)} \|\varphi_n^\varepsilon - \varphi_n\|^2_{H^1(\Omega_\varepsilon)} = \tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, \partial_{\boldsymbol{\nu}} P^{\varphi_n}_{x_0,k}). \tag{2.21}$$

We observe that, in Theorem 2.8–(ii), k is actually equal to the vanishing order of $\varphi_n - \varphi_n(x_0)$, since $\varphi_n(x_0) = 0$ when $x_0 \in \mathrm{Sing}\,(\varphi_n)$, consistently with the notation used in Theorem 2.9. We refer to Remark 5.2 for further discussion on vanishing orders of eigenfunctions.

From Theorem 2.8, one can see that the sign of the leading term in the asymptotic expansion of $\lambda_n(\Omega_\varepsilon) - \lambda_n(\Omega)$ might change depending on the position of the hole. Indeed, the function $f \mapsto \tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, f)$ is continuous from $L^2(\partial\Sigma)$ into $\mathbb{R}$; hence $\tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, \nabla\varphi_n(x_0)\cdot\boldsymbol{\nu})$ is small if $|\nabla\varphi_n(x_0)|$ is small. It follows that, if x_0 is close to critical points of φ_n which are not zeroes, then the coefficient of the leading term in the expansion is strictly positive (since, for $n \geq 1$, we have $\lambda_n(\Omega) > 1$), while close to the nodal set $\mathcal{Z}(\varphi_n)$ the coefficient is negative. A more detailed discussion is contained in the following remark.

Remark 2.10. In the case of holes of type (1.6) shrinking to a point x_0, the vanishing order of $\lambda_n(\Omega_\varepsilon) - \lambda_n(\Omega)$ is strongly influenced by the position of the point $x_0 \in \Omega$. If x_0

lies on the singular part of the nodal set of φ_n, which is known to be at most $(N-2)$-dimensional (see [12]), the eigenvalue variation vanishes with the same order as ε^{N+2k-2}, being $k \geq 2$ the vanishing order of φ_n at x_0, and the coefficient of the term ε^{N+2k-2} in the expansion of $\lambda_n(\Omega_\varepsilon) - \lambda_n(\Omega)$ is strictly negative; this implies that the expansion is sharp and

$$\lambda_n(\Omega_\varepsilon) < \lambda_n(\Omega), \quad \text{for } \varepsilon \text{ sufficiently small.}$$

On the other hand, if x_0 is outside the singular set of φ_n and outside the set

$$\Gamma = \Gamma_{\Sigma,n} := \Big\{x \in \Omega\colon \tau_{\mathbb{R}^N \setminus \Sigma}(\partial\Sigma, \nabla\varphi_n(x) \cdot \boldsymbol{\nu}) + |\Sigma|(|\nabla\varphi_n(x)|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2(x)) = 0\Big\} \setminus \operatorname{Sing}(\varphi_n), \tag{2.22}$$

the rate of convergence is ε^N. If $x_0 \in \Gamma$, Theorem 2.8 just lets us know that

$$\lambda_n(\Omega_\varepsilon) - \lambda_n(\Omega) = o(\varepsilon^N) \quad \text{as } \varepsilon \to 0,$$

without further information about the next non-zero term in the expansion or about the sign. The complement of the set Γ in Ω is the disjoint union of the two regions

$$\Omega^+ := \big\{x \in \Omega\colon \tau_{\mathbb{R}^N \setminus \Sigma}(\partial\Sigma, \nabla\varphi_n(x) \cdot \boldsymbol{\nu}) + |\Sigma|(|\nabla\varphi_n(x)|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2(x)) < 0\big\}$$

and

$$\Omega^- := \{x \in \Omega\colon \tau_{\mathbb{R}^N \setminus \Sigma}(\partial\Sigma, \nabla\varphi_n(x) \cdot \boldsymbol{\nu}) + |\Sigma|(|\nabla\varphi_n(x)|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2(x)) > 0\} \cup \operatorname{Sing}(\varphi_n),$$

in each of which the mutual position of the perturbed eigenvalue and the limit one is different. Indeed, recalling that $\lambda_n(\Omega) > 1$, if $x_0 \in \Omega^+$, then

$$\lambda_n(\Omega_\varepsilon) > \lambda_n(\Omega), \quad \text{for } \varepsilon \text{ sufficiently small,}$$

while, if $x_0 \in \Omega^-$, then

$$\lambda_n(\Omega_\varepsilon) < \lambda_n(\Omega), \quad \text{for } \varepsilon \text{ sufficiently small.}$$

In particular, $\mathcal{Z}(\varphi_n) \subseteq \Omega^-$, while $\operatorname{Crit}(\varphi_n) \subseteq \Omega^+$, where

$$\operatorname{Crit}(\varphi_n) := \{x \in \Omega\colon \varphi_n(x) \neq 0 \text{ and } \nabla\varphi_n(x) = 0\}$$

denotes the set of critical points outside $\mathcal{Z}(\varphi_n)$.

The asymptotic expansion obtained in (2.12) can be made completely explicit in the case of a spherical hole. In dimension $N \geq 3$ this can be done by calculating the limit quantity $\tau_{\mathbb{R}^N \setminus \Sigma}(\partial\Sigma, \partial_{\boldsymbol{\nu}} P^{\varphi_n}_{x_0,k})$ that appears in Theorem 2.8.

Theorem 2.11. *Let $N \geq 3$, $x_0 \in \Omega$, and $\Sigma_\varepsilon := x_0 + \varepsilon B_1$. Let $n \geq 1$ be such that $\lambda_n(\Omega)$ is simple.*

(i) *If $x_0 \in \Omega \setminus \mathrm{Sing}\,(\varphi_n)$, then*

$$\lambda_n(\Omega_\varepsilon) = \lambda_n(\Omega) - \omega_N \varepsilon^N \left(\frac{N}{N-1} |\nabla \varphi_n(x_0)|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2(x_0) \right) + o(\varepsilon^N)$$

as $\varepsilon \to 0$, where $\omega_N = |B_1|$.

(ii) *If $x_0 \in \mathrm{Sing}\,(\varphi_n)$, then*

$$\lambda_n(\Omega_\varepsilon) = \lambda_n(\Omega) - \frac{k(N+2k-2)}{N+k-2} \varepsilon^{N+2k-2} \int_{\partial B_1} Y^2 \,\mathrm{d}S + o(\varepsilon^{N+2k-2}) \quad \text{as } \varepsilon \to 0,$$

where $k \geq 2$ is the vanishing order of φ_n at x_0 and Y is the spherical harmonic of degree k given by $Y = P^{\varphi_n}_{x_0,k}\big|_{\partial B_1}$, $P^{\varphi_n}_{x_0,k}$ being as in (2.16).

In the case $N = 2$, the blow-up argument is not helpful due to the unavailability of Hardy-type inequalities, which prevents us from identifying a concrete functional space to which the blow-up limits belong. In this case, direct computations, carried out by expanding the torsion function for the perturbed problem in Fourier series, allow us to prove the following result.

Theorem 2.12. *Let $N = 2$, $x_0 \in \Omega$, and $\Sigma_\varepsilon := x_0 + \varepsilon B_1$. Let $n \geq 1$ be such that $\lambda_n(\Omega)$ is simple.*

(i) *If $x_0 \in \Omega \setminus \mathrm{Sing}\,(\varphi_n)$, then*

$$\lambda_n(\Omega_\varepsilon) = \lambda_n(\Omega) - \pi\varepsilon^2 \left(2 |\nabla \varphi_n(x_0)|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2(x_0) \right) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

(ii) *If $x_0 \in \mathrm{Sing}\,(\varphi_n)$, then*

$$\lambda_n(\Omega_\varepsilon) = \lambda_n(\Omega) - 2k\pi\varepsilon^{2k} \left(\left| \frac{\partial^k \varphi_n}{\partial x_1^k}(x_0) \right|^2 + \frac{1}{k^2} \left| \frac{\partial^k \varphi_n}{\partial x_1^{k-1} \partial x_2}(x_0) \right|^2 \right) + o(\varepsilon^{2k})$$

as $\varepsilon \to 0$, where $k \geq 2$ is the vanishing order of $\varphi_n - \varphi_n(x_0)$ at x_0.

Theorem 2.11 and Theorem 2.12 provide a more explicit expression for the interface Γ introduced in Remark 2.10 and its 2-dimensional counterpart, in the case of a spherical hole: if $\Sigma = B_1$, we have

$$\Gamma = \left\{ x \in \Omega \colon \frac{N}{N-1} |\nabla \varphi_n(x)|^2 - (\lambda_n(\Omega) - 1)\varphi_n^2(x) = 0 \right\} \setminus \operatorname{Sing}(\varphi_n).$$

Some examples of interfaces Γ are described in Section 6, for Ω being a 3-dimensional box or a 2-dimensional disk.

Notation. In what follows, for any family $\{\Sigma_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ satisfying assumption **(H)**, we denote

$$\lambda_i := \lambda_i(\Omega) \quad \text{and} \quad \lambda_i^\varepsilon := \lambda_i(\Omega_\varepsilon)$$

for all $i \in \mathbb{N}$, where $\Omega_\varepsilon := \Omega \setminus \overline{\Sigma}_\varepsilon$. Moreover, we fix an index $n \in \mathbb{N}$, $n \geq 1$, such that (2.6) is satisfied; we recall that φ_n is a corresponding eigenfunction such that $\int_\Omega \varphi_n^2 \,\mathrm{d}x = 1$. We may also denote by

$$\mathcal{T}_\varepsilon := \mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial \Sigma_\varepsilon, \partial_{\boldsymbol{\nu}} \varphi_n) \tag{2.23}$$

the Sobolev $\partial_{\boldsymbol{\nu}}\varphi_n$-torsional rigidity of $\partial\Sigma_\varepsilon$ relative to $\overline{\Omega}_\varepsilon$, and by

$$U_\varepsilon := U_{\Omega,\Sigma_\varepsilon,\partial_{\boldsymbol{\nu}}\varphi_n} \in H^1(\Omega_\varepsilon) \tag{2.24}$$

the function achieving it, see (2.7) and Proposition 3.4.

3. Preliminaries

The first part of this section is devoted to some basic properties of the f-torsional rigidity of a set.

Lemma 3.1. *Let $E \subseteq \mathbb{R}^N$ be an open Lipschitz set such that $\overline{E} \subseteq \Omega$ and let $f \in L^2(\partial E)$. Then*

$$\mathcal{T}_{\overline{\Omega}\setminus E}(\partial E, f) = \sup \left\{ \frac{\left(\displaystyle\int_{\partial E} u f \,\mathrm{d}S \right)^2}{\displaystyle\int_{\Omega\setminus E} (|\nabla u|^2 + u^2)\,\mathrm{d}x} : u \in H^1(\Omega \setminus \overline{E}) \setminus \{0\} \right\}. \tag{3.1}$$

Proof. By the substitution $u \mapsto tu$, the characterization of $\mathcal{T}_{\overline{\Omega}\setminus E}(\partial E, f)$ as in Definition 2.1 is equivalent to

$$\begin{aligned}
&\mathcal{T}_{\overline{\Omega}\setminus E}(\partial E, f)\\
&\quad = -2\inf\left\{\frac{t^2}{2}\int_{\Omega\setminus E}(|\nabla u|^2+u^2)\,\mathrm{d}x - t\int_{\partial E} uf\,\mathrm{d}S\colon u\in H^1(\Omega\setminus\overline{E})\setminus\{0\},\, t\in\mathbb{R}\right\}\\
&\qquad = -2\inf_{u\in H^1(\Omega\setminus\overline{E})\setminus\{0\}}\inf\left\{\frac{t^2}{2}\int_{\Omega\setminus E}(|\nabla u|^2+u^2)\,\mathrm{d}x - t\int_{\partial E} uf\,\mathrm{d}S : t\in\mathbb{R}\right\}. \quad (3.2)
\end{aligned}$$

Minimizing in t for a fixed $u \not\equiv 0$, we find that

$$\inf_{t\in\mathbb{R}}\left\{\frac{t^2}{2}\int_{\Omega\setminus E}(|\nabla u|^2+u^2)\,\mathrm{d}x - t\int_{\partial E} uf\,\mathrm{d}S\right\}$$

is attained for

$$t = \frac{\int_{\partial E} uf\,\mathrm{d}S}{\int_{\Omega\setminus E}(|\nabla u|^2+u^2)\,\mathrm{d}x}.$$

Thus, substituting this into (3.2) we complete the proof. □

The characterization of $\mathcal{T}_{\overline{\Omega}\setminus E}(\partial E, f)$ given in (3.1) easily implies the following monotonicity property with respect to domain inclusion.

Corollary 3.2. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be two connected open bounded Lipschitz sets and $E \subset \mathbb{R}^N$ be an open Lipschitz set such that $\overline{E} \subset \Omega_1 \subset \Omega_2$. Then, for any $f \in L^2(\partial E)$,*

$$\mathcal{T}_{\overline{\Omega_2}\setminus E}(\partial E, f) \le \mathcal{T}_{\overline{\Omega_1}\setminus E}(\partial E, f).$$

Proof. If $u \in H^1(\Omega_2 \setminus \overline{E}) \setminus \{0\}$, then its restriction, still denoted as u, belongs to $H^1(\Omega_1 \setminus \overline{E})$. If $u \equiv 0$ in $\Omega_1 \setminus \overline{E}$ then u has null trace on ∂E so that

$$\frac{\left(\int_{\partial E} uf\,\mathrm{d}S\right)^2}{\int_{\Omega_2\setminus E}(|\nabla u|^2+u^2)\,\mathrm{d}x} = 0.$$

If $u \not\equiv 0$ in $\Omega_1 \setminus \overline{E}$, then $u \in H^1(\Omega_1 \setminus \overline{E}) \setminus \{0\}$ and, by (3.1),

$$\frac{\left(\int_{\partial E} uf\,\mathrm{d}S\right)^2}{\int_{\Omega_2\setminus E}(|\nabla u|^2+u^2)\,\mathrm{d}x} \le \frac{\left(\int_{\partial E} uf\,\mathrm{d}S\right)^2}{\int_{\Omega_1\setminus E}(|\nabla u|^2+u^2)\,\mathrm{d}x} \le \mathcal{T}_{\overline{\Omega_1}\setminus E}(\partial E, f).$$

In both cases, we have

$$\frac{\left(\int_{\partial E} uf\,\mathrm{d}S\right)^2}{\int_{\Omega_2\setminus E}(|\nabla u|^2+u^2)\,\mathrm{d}x} \le \mathcal{T}_{\overline{\Omega_1}\setminus E}(\partial E, f) \quad \text{for all } u \in H^1(\Omega_2 \setminus \overline{E}) \setminus \{0\},$$

which yields the conclusion by taking the supremum over $H^1(\Omega_2 \setminus \overline{E}) \setminus \{0\}$. □

Another relevant consequence of the characterization (3.1) is the vanishing of the $\partial_{\boldsymbol{\nu}}\varphi$-torsional rigidity of $\partial\Sigma_\varepsilon$ as $\varepsilon \to 0$, whenever the family $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ satisfies assumption **(H)** and φ is any eigenfunction of problem (2.1).

Corollary 3.3. *Let $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ be a family of sets satisfying assumptions* **(H)** *and φ be an eigenfunction of problem (2.1). Then*

$$\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi) \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.3}$$

Proof. For every $u \in H^1(\Omega_\varepsilon)$, the Divergence Theorem, Hölder's inequality, and assumption (H2) yield

$$\begin{aligned}\left|\int_{\partial\Sigma_\varepsilon} u\,\partial_{\boldsymbol{\nu}}\varphi\,\mathrm{d}S\right| &= \left|\int_{\Sigma_\varepsilon} \operatorname{div}((\mathsf{E}_\varepsilon u)\nabla\varphi)\,\mathrm{d}x\right| = \left|\int_{\Sigma_\varepsilon}\Big((\Delta\varphi)(\mathsf{E}_\varepsilon u) + \nabla(\mathsf{E}_\varepsilon u)\cdot\nabla\varphi\Big)\,\mathrm{d}x\right| \\ &\le \big(\|\Delta\varphi\|_{L^2(\Sigma_\varepsilon)} + \|\nabla\varphi\|_{L^2(\Sigma_\varepsilon;\mathbb{R}^N)}\big)\|\mathsf{E}_\varepsilon u\|_{H^1(\Omega)} \\ &\le \mathfrak{C}\|u\|_{H^1(\Omega_\varepsilon)}\big((|\lambda-1|\|\varphi\|_{L^2(\Sigma_\varepsilon)} + \|\nabla\varphi\|_{L^2(\Sigma_\varepsilon;\mathbb{R}^N)}\big),\end{aligned}$$

where λ is the eigenvalue corresponding to the eigenfunction φ.

The characterization of $\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi)$ given in (3.1) then implies

$$\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial\Sigma_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi) = \sup_{\substack{u\in H^1(\Omega_\varepsilon)\\ u\neq 0}} \frac{\Big(\int_{\partial\Sigma_\varepsilon} u\,\partial_{\boldsymbol{\nu}}\varphi\,\mathrm{d}S\Big)^2}{\|u\|^2_{H^1(\Omega_\varepsilon)}} \le \mathfrak{C}^2\big(|\lambda-1|\|\varphi\|_{L^2(\Sigma_\varepsilon)} + \|\nabla\varphi\|_{L^2(\Sigma_\varepsilon;\mathbb{R}^N)}\big)^2,$$

so that the conclusion follows from assumption (H3) and the absolute continuity of Lebesgue integral. □

The following proposition states that the infimum appearing in the definition of the torsional rigidity of a set is actually achieved.

Proposition 3.4.

(i) *Let $E \subseteq \mathbb{R}^N$ be an open Lipschitz set such that $\overline{E} \subseteq \Omega$ and let $f \in L^2(\partial E)$. Then, there exists a unique $U = U_{\Omega,E,f} \in H^1(\Omega \setminus \overline{E})$ such that*

$$\mathcal{T}_{\overline{\Omega}\setminus E}(\partial E, f) = -2J_{\Omega,E,f}(U),$$

with $J_{\Omega,E,f}$ *being as in Definition 2.1. In addition,* $U \in H^1(\Omega \setminus \overline{E})$ *is the unique function weakly satisfying*

$$\begin{cases} -\Delta U + U = 0, & \text{in } \Omega \setminus \overline{E}, \\ \partial_{\boldsymbol{\nu}} U = 0, & \text{on } \partial\Omega, \\ \partial_{\boldsymbol{\nu}} U = f, & \text{on } \partial E, \end{cases}$$

that is

$$\int_{\Omega\setminus E} (\nabla U \cdot \nabla v + Uv)\,\mathrm{d}x = \int_{\partial E} vf\,\mathrm{d}S, \quad \text{for all } v \in H^1(\Omega \setminus \overline{E}). \tag{3.4}$$

(ii) *Let* $N \geq 3$, $E \subseteq \mathbb{R}^N$ *be an open bounded Lipschitz set and* $f \in L^2(\partial E)$. *Then, there exists a unique* $\tilde{U} = \tilde{U}_{E,f} \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus E)$ *such that*

$$\tau_{\mathbb{R}^N\setminus E}(\partial E, f) = -2\tilde{J}_{E,f}(\tilde{U}),$$

where $\tilde{J}_{E,f}$ *is as in Definition 2.7. In addition,* $\tilde{U} \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus E)$ *is the unique function weakly satisfying*

$$\begin{cases} -\Delta \tilde{U} = 0, & \text{in } \mathbb{R}^N \setminus \overline{E}, \\ \partial_{\boldsymbol{\nu}} \tilde{U} = f, & \text{on } \partial E, \end{cases}$$

that is

$$\int_{\mathbb{R}^N\setminus E} \nabla \tilde{U} \cdot \nabla v\,\mathrm{d}x = \int_{\partial E} vf\,\mathrm{d}S \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus E).$$

Proof. The proof is a direct application of the Lax-Milgram lemma. In particular, concerning the proof of point (ii), we observe that the functional $v \mapsto \int_{\partial E} vf\,\mathrm{d}S$ is linear and continuous on $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus E)$. Indeed, since E is bounded, $\overline{E} \subset B$ for some ball B, hence the restriction map $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus E) \to H^1(B \setminus \overline{E})$ is continuous and there exists a continuous trace operator from $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus E)$ to $L^2(\partial E)$. $\square$

Remark 3.5. We observe that

$$\mathcal{T}_{\overline{\Omega}\setminus E}(\partial E, f) = \int_{\Omega\setminus E} (|\nabla U_{\Omega,E,f}|^2 + U_{\Omega,E,f}^2)\,\mathrm{d}x = \int_{\partial E} fU_{\Omega,E,f}\,\mathrm{d}S,$$

as one easily obtains by choosing $v = U_{\Omega,E,f}$ in (3.4). Similarly,

$$\tau_{\mathbb{R}^N\setminus E}(\partial E, f) = \int\limits_{\mathbb{R}^N\setminus E} |\nabla \tilde{U}_{E,f}|^2 \,\mathrm{d}x = \int\limits_{\partial E} f\tilde{U}_{E,f}\,\mathrm{d}S.$$

The following lemma provides a comparison between the L^2-norm of the torsion function and the torsional rigidity as $\varepsilon \to 0$.

Lemma 3.6. *Let $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ satisfy assumptions* **(H)** *and* (2.8). *If $\mathcal{T}_\varepsilon \to 0$ as $\varepsilon \to 0$, then*

$$\int\limits_{\Omega_\varepsilon} U_\varepsilon^2 \,\mathrm{d}x = o(\mathcal{T}_\varepsilon), \quad \text{as } \varepsilon \to 0,$$

with $\mathcal{T}_\varepsilon$ and U_ε being as in (2.23) *and* (2.24) *respectively.*

Proof. Let us assume by contradiction that there exist a constant $C > 0$ and a sequence $\{\varepsilon_j\}_{j\geq 1}$ such that $\lim_{j\to\infty} \varepsilon_j = 0$, $U_{\varepsilon_j} \not\equiv 0$ and

$$\frac{\int\limits_{\Omega_{\varepsilon_j}} U_{\varepsilon_j}^2 \,\mathrm{d}x}{\mathcal{T}_{\varepsilon_j}} = \frac{\int\limits_{\Omega_{\varepsilon_j}} U_{\varepsilon_j}^2 \,\mathrm{d}x}{\left\|U_{\varepsilon_j}\right\|_{H^1(\Omega_{\varepsilon_j})}^2} \geq C \quad \text{for all } j \geq 1,$$

see Remark 3.5. For any ε, let us consider the extension to the whole Ω of U_ε, i.e.

$$\tilde{U}_\varepsilon := \mathsf{E}_\varepsilon U_\varepsilon \in H^1(\Omega),$$

being E_ε as in (H2). Letting $W_j := \tilde{U}_{\varepsilon_j}/\|\tilde{U}_{\varepsilon_j}\|_{L^2(\Omega)}$, we have $\|W_j\|_{L^2(\Omega)} = 1$ and

$$\|W_j\|_{H^1(\Omega)} = \frac{\|\tilde{U}_{\varepsilon_j}\|_{H^1(\Omega)}}{\|\tilde{U}_{\varepsilon_j}\|_{L^2(\Omega)}} \leq \frac{\mathfrak{C}\left\|U_{\varepsilon_j}\right\|_{H^1(\Omega_{\varepsilon_j})}}{\left\|U_{\varepsilon_j}\right\|_{L^2(\Omega_{\varepsilon_j})}} \leq \frac{\mathfrak{C}}{\sqrt{C}}.$$

Therefore, there exists $W \in H^1(\Omega)$ such that, along a subsequence (still denoted by $\{W_j\}$),

$$W_j \rightharpoonup W \quad \text{weakly in } H^1(\Omega) \quad \text{and} \quad W_j \to W \quad \text{strongly in } L^2(\Omega)$$

as $j \to \infty$. From the strong $L^2(\Omega)$-convergence we immediately infer that $\|W\|_{L^2(\Omega)} = 1$, which in turn tells us that $W \not\equiv 0$.

Let $v \in C^\infty(\overline{\Omega})$. By assumption (2.8), there exists $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset C_c^\infty(\mathbb{R}^N)$ such that $u_\varepsilon = 1$ in a neighborhood of $\overline{\Sigma}_\varepsilon$ and $\|u_\varepsilon\|_{H^1(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$. Letting $v_j = v(1-u_{\varepsilon_j}\big|_\Omega)$, we observe that $v_j \in C^\infty(\overline{\Omega})$, $v_j \equiv 0$ in a neighborhood of $\overline{\Sigma}_{\varepsilon_j}$, and $v_j \to v$ strongly in $H^1(\Omega)$, as $j \to \infty$. Then, from the weak convergence $W_j \rightharpoonup W$ in $H^1(\Omega)$ it follows that

$$\int_\Omega (\nabla W_j \cdot \nabla v_j + W_j v_j)\,\mathrm{d}x \to \int_\Omega (\nabla W \cdot \nabla v + W v)\,\mathrm{d}x, \quad \text{as } j \to \infty.$$

On the other hand, equation (3.4) and the fact that $v_j \equiv 0$ in a neighborhood of $\overline{\Sigma}_{\varepsilon_j}$ imply that

$$\begin{aligned}\int_\Omega (\nabla W_j \cdot \nabla v_j + W_j v_j)\,\mathrm{d}x &= \frac{1}{\|\tilde{U}_{\varepsilon_j}\|_{L^2(\Omega)}} \int_{\Omega_{\varepsilon_j}} (\nabla U_{\varepsilon_j} \cdot \nabla v_j + U_{\varepsilon_j} v_j)\,\mathrm{d}x \\ &= \frac{1}{\|\tilde{U}_{\varepsilon_j}\|_{L^2(\Omega)}} \int_{\partial\Sigma_{\varepsilon_j}} v_j \partial_{\boldsymbol{\nu}} \varphi_n \,\mathrm{d}S = 0,\end{aligned}$$

for all $j \in \mathbb{N}$. Therefore, we conclude that

$$\int_\Omega (\nabla W \cdot \nabla v + W v)\,\mathrm{d}x = 0$$

for every $v \in C^\infty(\overline{\Omega})$, and, by density, for every $v \in H^1(\Omega)$. This implies that $W = 0$, thus giving rise to a contradiction. □

We conclude this section by proving (2.10).

Lemma 3.7. *Let $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ satisfy* **(H)** *and (2.8) and $n \geq 1$ be such that (2.6) holds. If, for every $\varepsilon \in (0, \varepsilon_0)$, φ_n^ε is an eigenfunction of (2.3) associated to the eigenvalue λ_n^ε and chosen in such a way that $\int_{\Omega_\varepsilon} |\varphi_n^\varepsilon|^2\,\mathrm{d}x = 1$ and (2.9) is satisfied, then $\lim_{\varepsilon\to 0} \|\varphi_n^\varepsilon - \varphi_n\|_{H^1(\Omega_\varepsilon)} = 0$.*

Proof. Since φ_n^ε solve (2.3) with $\lambda = \lambda_n^\varepsilon$, from (2.4) and (H2) it follows that (possibly choosing ε_0 smaller) $\{\mathsf{E}_\varepsilon \varphi_n^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ is bounded in $H^1(\Omega)$. Therefore, for every sequence $\varepsilon_j \to 0^+$, there exist a subsequence (still denoted as ε_j) and $\tilde{\varphi} \in H^1(\Omega)$ such that $\mathsf{E}_{\varepsilon_j} \varphi_n^{\varepsilon_j} \rightharpoonup \tilde{\varphi}$ weakly in $H^1(\Omega)$ as $j \to \infty$.

Let $v \in C^\infty(\overline{\Omega})$. Arguing as in the proof of Lemma 3.6, thanks to assumption (2.8) we can find a sequence $\{v_j\}$ such that $v_j \in C^\infty(\overline{\Omega})$, $v_j \equiv 0$ in a neighborhood of $\overline{\Sigma}_{\varepsilon_j}$, and $v_j \to v$ strongly in $H^1(\Omega)$, as $j \to \infty$. From the equation satisfied by φ_n^ε we have

$$\int_\Omega \left(\nabla(\mathsf{E}_{\varepsilon_j}\varphi_n^{\varepsilon_j}) \cdot \nabla v_j + (\mathsf{E}_{\varepsilon_j}\varphi_n^{\varepsilon_j}) v_j\right)\,\mathrm{d}x = \lambda_n^{\varepsilon_j} \int_\Omega (\mathsf{E}_{\varepsilon_j}\varphi_n^{\varepsilon_j}) v_j \,\mathrm{d}x,$$

passing to the limit in which we obtain, taking into account (2.4),

$$\int_\Omega (\nabla\tilde{\varphi} \cdot \nabla v + \tilde{\varphi} v)\,\mathrm{d}x = \lambda_n \int_\Omega \tilde{\varphi} v \,\mathrm{d}x, \tag{3.5}$$

for every $v \in C^\infty(\overline{\Omega})$ and hence, by density, for every $v \in H^1(\Omega)$.

Since, for any $p > 2$,

$$\int_{\Sigma_\varepsilon} |\mathsf{E}_\varepsilon \varphi_n^\varepsilon|^2 \,\mathrm{d}x \le \left(\int_{\Omega} |\mathsf{E}_\varepsilon \varphi_n^\varepsilon|^p \,\mathrm{d}x \right)^{2/p} |\Sigma_\varepsilon|^{(p-2)/p},$$

by assumption (H3), Sobolev embeddings and boundedness of $\{\mathsf{E}_\varepsilon \varphi_n^\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ in $H^1(\Omega)$ we deduce that

$$\lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} |\mathsf{E}_\varepsilon \varphi_n^\varepsilon|^2 \,\mathrm{d}x = 0. \tag{3.6}$$

Hence

$$\begin{aligned}\int_{\Omega} |\tilde{\varphi}|^2 \,\mathrm{d}x &= \lim_{j \to \infty} \int_{\Omega} |\mathsf{E}_{\varepsilon_j} \varphi_n^{\varepsilon_j}|^2 \,\mathrm{d}x \\ &= \lim_{j \to \infty} \left(\int_{\Omega_{\varepsilon_j}} |\varphi_n^{\varepsilon_j}|^2 \,\mathrm{d}x + \int_{\Sigma_{\varepsilon_j}} |\mathsf{E}_{\varepsilon_j} \varphi_n^{\varepsilon_j}|^2 \,\mathrm{d}x \right) = \lim_{j \to \infty} (1 + o(1)) = 1 \end{aligned} \tag{3.7}$$

and, in view of (2.9),

$$\int_{\Omega} \tilde{\varphi} \varphi_n \,\mathrm{d}x = \lim_{j \to \infty} \int_{\Omega} (\mathsf{E}_{\varepsilon_j} \varphi_n^{\varepsilon_j}) \varphi_n \,\mathrm{d}x = \lim_{j \to \infty} \left(\int_{\Omega_{\varepsilon_j}} \varphi_n^{\varepsilon_j} \varphi_n \,\mathrm{d}x + o(1) \right) \ge 0. \tag{3.8}$$

In view of assumption (2.6), (3.5), (3.7), and (3.8) imply that $\tilde{\varphi} = \varphi_n$. In view of Urysohn's subsequence principle, we conclude that

$$\mathsf{E}_\varepsilon \varphi_n^\varepsilon \rightharpoonup \varphi_n \quad \text{as } \varepsilon \to 0 \quad \text{weakly in } H^1(\Omega). \tag{3.9}$$

By (3.9) and compactness of the embedding $H^1(\Omega) \subset L^2(\Omega)$ we have $\lim_{\varepsilon \to 0} \|\mathsf{E}_\varepsilon \varphi_n^\varepsilon - \varphi_n\|_{L^2(\Omega)} = 0$, hence

$$\|\varphi_n^\varepsilon - \varphi_n\|_{L^2(\Omega_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.10}$$

Testing the equation satisfied by φ_n^ε with $\varphi_n^\varepsilon - \varphi_n$ and taking into account (3.6) and (3.9) we obtain

$$\int_{\Omega_\varepsilon} \nabla \varphi_n^\varepsilon \cdot \nabla (\varphi_n^\varepsilon - \varphi_n) \,\mathrm{d}x = (\lambda_n^\varepsilon - 1) \int_{\Omega_\varepsilon} \varphi_n^\varepsilon (\varphi_n^\varepsilon - \varphi_n) \,\mathrm{d}x$$

$$= (\lambda_n^\varepsilon - 1)\left(1 - \int_{\Omega_\varepsilon} \varphi_n^\varepsilon \varphi_n \, dx\right)$$

$$= (\lambda_n^\varepsilon - 1)\left(1 - \int_{\Omega} (\mathsf{E}_\varepsilon \varphi_n^\varepsilon)\varphi_n \, dx + \int_{\Sigma_\varepsilon} (\mathsf{E}_\varepsilon \varphi_n^\varepsilon)\varphi_n \, dx\right) = o(1) \tag{3.11}$$

as $\varepsilon \to 0$. Furthermore,

$$\left| \int_{\Sigma_\varepsilon} \nabla\varphi_n \cdot \nabla(\mathsf{E}_\varepsilon \varphi_n^\varepsilon - \varphi_n) \, dx \right| \le \|\mathsf{E}_\varepsilon \varphi_n^\varepsilon - \varphi_n\|_{H^1(\Omega)} \|\nabla\varphi_n\|_{L^2(\Sigma_\varepsilon)} = o(1) \quad \text{as } \varepsilon \to 0,$$

so that, in view of (3.9),

$$\int_{\Omega_\varepsilon} \nabla\varphi_n \cdot \nabla(\varphi_n^\varepsilon - \varphi_n) \, dx$$

$$= \int_{\Omega} \nabla\varphi_n \cdot \nabla(\mathsf{E}_\varepsilon \varphi_n^\varepsilon - \varphi_n) \, dx - \int_{\Sigma_\varepsilon} \nabla\varphi_n \cdot \nabla(\mathsf{E}_\varepsilon \varphi_n^\varepsilon - \varphi_n) \, dx = o(1) \tag{3.12}$$

as $\varepsilon \to 0$. Combining (3.11) and (3.12) we obtain

$$\int_{\Omega_\varepsilon} |\nabla(\varphi_n^\varepsilon - \varphi_n)|^2 \, dx \to 0 \quad \text{as } \varepsilon \to 0 \tag{3.13}$$

The conclusion follows from (3.10) and (3.13). □

4. Asymptotics of simple eigenvalues

The aim of this section is to prove Theorem 2.4. To this end, we apply the "Lemma on small eigenvalues" due to Colin de Verdiére [15], which is stated in the Appendix, see Lemma A.1. The underlying idea is that good approximations of perturbed eigenfunctions induce good approximations of perturbed eigenvalues.

Proof of Theorem 2.4. We first observe that, in view of (3.3) and Remark 3.5,

$$\lim_{\varepsilon\to 0} \|\varphi_n - U_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = \|\varphi_n\|^2_{L^2(\Omega)} = 1; \tag{4.1}$$

hence, possibly choosing ε_0 smaller from the beginning, $\varphi_n - U_\varepsilon \not\equiv 0$ in Ω_ε and

$$2 \ge \|\varphi_n - U_\varepsilon\|_{L^2(\Omega_\varepsilon)} \ge \frac{1}{2} \tag{4.2}$$

for all $\varepsilon \in (0, \varepsilon_0)$. In order to apply Lemma A.1 in our setting, we fix $\varepsilon \in (0, \varepsilon_0)$ and define:

$$\mathcal{H} := L^2(\Omega_\varepsilon), \text{ with } (\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega_\varepsilon)} \text{ and } \|\cdot\| := \|\cdot\|_{L^2(\Omega_\varepsilon)};$$

$$\mathcal{D} := H^1(\Omega_\varepsilon);$$

$$q(u,v) := \int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v + uv)\, \mathrm{d}x - \lambda_n \int_{\Omega_\varepsilon} uv\, \mathrm{d}x, \quad \text{for every } u, v \in \mathcal{D};$$

$$f := \frac{\varphi_n - U_\varepsilon}{\|\varphi_n - U_\varepsilon\|}.$$

We observe that $\lambda_n^\varepsilon - \lambda_n$ is an eigenvalue of q and an associated normalized eigenfunction is given by φ_n^ε; hence assumption (i) in Lemma A.1 is satisfied with

$$\lambda := \lambda_n^\varepsilon - \lambda_n, \quad \phi := \varphi_n^\varepsilon.$$

Letting $H_1 = \mathrm{span}\{\varphi_k^\varepsilon : 0 \le k < n\}$ and $H_2 = \overline{\mathrm{span}\{\varphi_k^\varepsilon : k > n\}}$, we observe that H_1, H_2 are mutually orthogonal in $L^2(\Omega_\varepsilon)$, $\{\phi\}^\perp = H_1 \oplus H_2$, and condition (A.1) is satisfied.

We are going to estimate the corresponding values δ, γ_1, and γ_2 defined in (A.4), (A.2), and (A.3), respectively. For what concerns the former, for any $v \in \mathcal{D} \setminus \{0\}$ we have

$$\begin{aligned}
q(f,v) &= \frac{1}{\|\varphi_n - U_\varepsilon\|}\, q(\varphi_n - U_\varepsilon, v) \\
&= \frac{1}{\|\varphi_n - U_\varepsilon\|} \int_{\Omega_\varepsilon} \left(\nabla(\varphi_n - U_\varepsilon) \cdot \nabla v + (1-\lambda_n)(\varphi_n - U_\varepsilon) v\right)\, \mathrm{d}x \\
&= \frac{1}{\|\varphi_n - U_\varepsilon\|} \Bigg(\int_{\Omega_\varepsilon} (\nabla \varphi_n \cdot \nabla v + \varphi_n v)\, \mathrm{d}x - \int_{\Omega_\varepsilon} \big(\nabla U_\varepsilon \cdot \nabla v + U_\varepsilon v\big)\, \mathrm{d}x \\
&\qquad - \lambda_n \int_{\Omega_\varepsilon} (\varphi_n - U_\varepsilon) v\, \mathrm{d}x \Bigg) \\
&= \frac{\lambda_n}{\|\varphi_n - U_\varepsilon\|} \int_{\Omega_\varepsilon} U_\varepsilon v\, \mathrm{d}x,
\end{aligned}$$

where the last equality follows from the equations satisfied by φ_n and U_ε respectively, see (2.2) and (3.4). Combining this with (4.2) and the Cauchy-Schwarz inequality, we obtain that

$$\delta \le 2\lambda_n \|U_\varepsilon\| \quad \text{for every } \varepsilon \in (0, \varepsilon_0).$$

Since λ_n is simple and $\lim_{\varepsilon \to 0} \lambda_i^\varepsilon = \lambda_i$ for all $i \in \mathbb{N}$, if ε is sufficiently small we have

$$\gamma_1 = \inf\left\{\frac{|q(v,v)|}{\|v\|^2} : v \in H_1 \setminus \{0\}\right\} = \lambda_n - \lambda_{n-1}^\varepsilon > 0,$$

$$\gamma_2 = \inf\left\{\frac{|q(v,v)|}{\|v\|^2} : v \in (H_2 \cap \mathcal{D}) \setminus \{0\}\right\} = \lambda_{n+1}^\varepsilon - \lambda_n > 0,$$

so that, if ε is sufficiently small,

$$\gamma = \min\{\gamma_1, \gamma_2\} \geq \gamma_0,$$

where

$$\gamma_0 = \frac{1}{2}\min\{\lambda_{n+1} - \lambda_n, \lambda_n - \lambda_{n-1}\}$$

is a positive number independent of ε. Hence, with these estimates for δ and γ and denoting as Π_ε the orthogonal projection onto $\operatorname{span}\{\varphi_n^\varepsilon\}$, i.e.

$$\Pi_\varepsilon \colon L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon), \quad \Pi_\varepsilon(v) = (\varphi_n^\varepsilon, v)_{L^2(\Omega_\varepsilon)}\, \varphi_n^\varepsilon, \tag{4.3}$$

from Lemma A.1 and (4.2) we obtain

$$\|\varphi_n - U_\varepsilon - \Pi_\varepsilon(\varphi_n - U_\varepsilon)\| = \|f - \Pi_\varepsilon f\|\, \|\varphi_n - U_\varepsilon\| \leq \frac{4\sqrt{2}\lambda_n}{\gamma_0}\, \|U_\varepsilon\| \tag{4.4}$$

and

$$|\lambda_n^\varepsilon - \lambda_n - \xi_\varepsilon| \leq \frac{8\lambda_n^2 \|U_\varepsilon\|^2}{\gamma_0}\left(\frac{|\lambda_n^\varepsilon - \lambda_n|}{\gamma_0} + 1\right), \tag{4.5}$$

for ε sufficiently small, where

$$\xi_\varepsilon := q(f,f) = \frac{q(\varphi_n - U_\varepsilon, \varphi_n - U_\varepsilon)}{\|\varphi_n - U_\varepsilon\|^2}.$$

At this point, we analyze what happens asymptotically as $\varepsilon \to 0$. Bearing in mind that Lemma 3.6 ensures that $\|U_\varepsilon\|^2 = \|U_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = o(\mathcal{T}_\varepsilon)$ as $\varepsilon \to 0$, estimates (4.4) and (4.5) yield

$$\|\varphi_n - U_\varepsilon - \Pi_\varepsilon(\varphi_n - U_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)} = o(\mathcal{T}_\varepsilon) \tag{4.6}$$

and

$$\lambda_n^\varepsilon = \lambda_n + \xi_\varepsilon + o(\mathcal{T}_\varepsilon), \tag{4.7}$$

as $\varepsilon \to 0$. We are now ready to establish expansions (2.12) and (2.13)–(2.14).

Proof of (2.12). We begin by expanding ξ_ε as $\varepsilon \to 0$. By (4.1) we have

$$\xi_\varepsilon = q(\varphi_n - U_\varepsilon, \varphi_n - U_\varepsilon)(1 + o(1)) \quad \text{as } \varepsilon \to 0. \tag{4.8}$$

Furthermore

$$\begin{aligned} q(\varphi_n - U_\varepsilon, \varphi_n - U_\varepsilon) &= \int_{\Omega_\varepsilon} (|\nabla \varphi_n|^2 + \varphi_n^2)\, dx - \lambda_n \int_{\Omega_\varepsilon} \varphi_n^2\, dx \\ &\quad + \int_{\Omega_\varepsilon} (|\nabla U_\varepsilon|^2 + U_\varepsilon^2)\, dx - \lambda_n \int_{\Omega_\varepsilon} U_\varepsilon^2\, dx \\ &\quad - 2\left(\int_{\Omega_\varepsilon} (\nabla U_\varepsilon \nabla \varphi_n + U_\varepsilon \varphi_n)\, dx - \lambda_n \int_{\Omega_\varepsilon} U_\varepsilon \varphi_n\, dx \right). \end{aligned} \tag{4.9}$$

Since φ_n is an eigenfunction associated to λ_n we have

$$\int_{\Omega_\varepsilon} (|\nabla \varphi_n|^2 + \varphi_n^2)\, dx - \lambda_n \int_{\Omega_\varepsilon} \varphi_n^2\, dx = -\int_{\Sigma_\varepsilon} \left(|\nabla \varphi_n|^2 - (\lambda_n - 1)\varphi_n^2 \right) dx.$$

In view of Remark 3.5 and Lemma 3.6, the term on the second line of (4.9) satisfies

$$\int_{\Omega_\varepsilon} (|\nabla U_\varepsilon|^2 + U_\varepsilon^2)\, dx - \lambda_n \int_{\Omega_\varepsilon} U_\varepsilon^2\, dx = \mathcal{T}_\varepsilon + o(\mathcal{T}_\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Finally, an integration by parts and Remark 3.5 allow us to rewrite the term on the last line of (4.9) as

$$\int_{\Omega_\varepsilon} (\nabla U_\varepsilon \cdot \nabla \varphi_n + U_\varepsilon \varphi_n)\, dx - \lambda_n \int_{\Omega_\varepsilon} U_\varepsilon \varphi_n\, dx = \int_{\partial \Sigma_\varepsilon} U_\varepsilon \partial_{\boldsymbol{\nu}} \varphi_n\, dS = \mathcal{T}_\varepsilon.$$

Plugging these identities into (4.8) and (4.7), we conclude the proof of (2.12).

Proof of (2.13). Let Π_ε be as in (4.3). We claim that

$$\|h_\varepsilon - \Pi_\varepsilon h_\varepsilon\|_{H^1(\Omega_\varepsilon)}^2 = o(\mathcal{T}_\varepsilon) \quad \text{as } \varepsilon \to 0, \tag{4.10}$$

where $h_\varepsilon = \varphi_n - U_\varepsilon$. We observe that

$$\begin{cases} -\Delta(h_\varepsilon - \Pi_\varepsilon h_\varepsilon) + (h_\varepsilon - \Pi_\varepsilon h_\varepsilon) = \lambda_n^\varepsilon (h_\varepsilon - \Pi_\varepsilon h_\varepsilon) + \lambda_n U_\varepsilon + (\lambda_n - \lambda_n^\varepsilon) h_\varepsilon, & \text{in } \Omega_\varepsilon, \\ \partial_{\boldsymbol{\nu}}(h_\varepsilon - \Pi_\varepsilon h_\varepsilon) = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

in a weak sense. By testing the above equation with $h_\varepsilon - \Pi_\varepsilon h_\varepsilon$ itself, we obtain

$$\|h_\varepsilon - \Pi_\varepsilon h_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} = \lambda_n^\varepsilon \|h_\varepsilon - \Pi_\varepsilon h_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \lambda_n (U_\varepsilon, h_\varepsilon - \Pi_\varepsilon h_\varepsilon)_{L^2(\Omega_\varepsilon)} + (\lambda_n - \lambda_n^\varepsilon)(h_\varepsilon, h_\varepsilon - \Pi_\varepsilon h_\varepsilon)_{L^2(\Omega_\varepsilon)}. \tag{4.11}$$

We are going to estimate each of the three terms on the right-hand side. Concerning the first one, thanks to (4.6) and (2.4), we have

$$\lambda_n^\varepsilon \|h_\varepsilon - \Pi_\varepsilon h_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = o(\mathcal{T}_\varepsilon) \quad \text{as } \varepsilon \to 0.$$

To estimate the second term on the right hand side of (4.11), we use the Cauchy-Schwarz inequality, Lemma 3.6 and (4.6). This leads to

$$\lambda_n (U_\varepsilon, h_\varepsilon - \Pi_\varepsilon h_\varepsilon)_{L^2(\Omega_\varepsilon)} = o(\mathcal{T}_\varepsilon) \quad \text{as } \varepsilon \to 0.$$

As far as the third term is concerned, we preliminarily observe that, by Lemma 3.1,

$$\mathcal{T}_\varepsilon = \sup_{u \in H^1(\Omega_\varepsilon)\setminus\{0\}} \frac{\left(\int_{\partial \Sigma_\varepsilon} u \partial_{\boldsymbol{\nu}} \varphi_n \, \mathrm{d}S \right)^2}{\int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2) \, \mathrm{d}x} \geq \frac{\left(\int_{\partial \Sigma_\varepsilon} \varphi_n \partial_{\boldsymbol{\nu}} \varphi_n \, \mathrm{d}S \right)^2}{\lambda_n},$$

which, by an integration by parts, implies that

$$\left| \int_{\Sigma_\varepsilon} \left(|\nabla \varphi_n|^2 - (\lambda_n - 1)\varphi_n^2 \right) \mathrm{d}x \right| = \left| \int_{\partial \Sigma_\varepsilon} \varphi_n \partial_{\boldsymbol{\nu}} \varphi_n \, \mathrm{d}S \right| = O(\sqrt{\mathcal{T}_\varepsilon}) \quad \text{as } \varepsilon \to 0. \tag{4.12}$$

Combining (2.12) and (4.12) we obtain the rough estimate

$$\lambda_n^\varepsilon - \lambda_n = O(\sqrt{\mathcal{T}_\varepsilon}) \quad \text{as } \varepsilon \to 0. \tag{4.13}$$

The last term in (4.11) can be estimated using (4.13), the Cauchy-Schwarz inequality, (4.1) and (4.6), thus obtaining

$$(\lambda_n - \lambda_n^\varepsilon)(h_\varepsilon, h_\varepsilon - \Pi_\varepsilon h_\varepsilon)_{L^2(\Omega_\varepsilon)} = o(\mathcal{T}_\varepsilon) \quad \text{as } \varepsilon \to 0.$$

This concludes the proof of (4.10).

By the triangle inequality, Lemma 3.6, and (4.10), we have

$$\|\Pi_\varepsilon h_\varepsilon - \varphi_n\|_{L^2(\Omega_\varepsilon)} \leq \|h_\varepsilon - \varphi_n\|_{L^2(\Omega_\varepsilon)} + \|\Pi_\varepsilon h_\varepsilon - h_\varepsilon\|_{L^2(\Omega_\varepsilon)} = o(\sqrt{\mathcal{T}_\varepsilon}) \tag{4.14}$$

as $\varepsilon \to 0$, which implies that

$$\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)} = \Big(1 - \|\varphi_n\|^2_{L^2(\Sigma_\varepsilon)} + o(\sqrt{\mathcal{T}_\varepsilon})\Big)^{1/2}$$
$$= 1 - \frac{1}{2}\|\varphi_n\|^2_{L^2(\Sigma_\varepsilon)} + o(\sqrt{\mathcal{T}_\varepsilon}) + o(\|\varphi_n\|^2_{L^2(\Sigma_\varepsilon)}) \quad \text{as } \varepsilon \to 0. \tag{4.15}$$

From (4.14) and the fact that $\lim_{\varepsilon\to 0} \|\varphi_n\|_{H^1(\Sigma_\varepsilon)} = 0$ we also deduce that

$$\int_{\Omega_\varepsilon} \varphi_n \, \Pi_\varepsilon h_\varepsilon \, \mathrm{d}x = 1 + o(1) \quad \text{as } \varepsilon \to 0,$$

which, combined with (4.15), implies that

$$\int_{\Omega_\varepsilon} \varphi_n \frac{\Pi_\varepsilon h_\varepsilon}{\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)}} \, \mathrm{d}x > 0$$

for ε sufficiently small. Hence, since $\varphi_n^\varepsilon \in H^1(\Omega_\varepsilon)$ is uniquely determined by the condition above, see (2.9), then necessarily

$$\varphi_n^\varepsilon = \frac{\Pi_\varepsilon h_\varepsilon}{\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)}},$$

for ε sufficiently small. We finally observe that

$$\begin{aligned}
&\|\varphi_n^\varepsilon - \varphi_n + U_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \\
&= \frac{1}{\|\Pi_\varepsilon h_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}} \Big\|\Pi_\varepsilon h_\varepsilon - \|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)} \varphi_n + \|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)} U_\varepsilon\Big\|^2_{H^1(\Omega_\varepsilon)} \\
&= \frac{1}{\|\Pi_\varepsilon h_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}} \Big\|\Pi_\varepsilon h_\varepsilon - h_\varepsilon + \Big(1 - \|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)}\Big) \varphi_n \\
&\quad + \Big(\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)} - 1\Big) U_\varepsilon\Big\|^2_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

By the previous identity, (4.10) and (4.15) we obtain (2.13). To prove (2.14) we observe that, since $\|U_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} = \mathcal{T}_\varepsilon$, see Remark 3.5,

$$\begin{aligned}
&(U_\varepsilon, \varphi_n^\varepsilon - \varphi_n + U_\varepsilon)_{H^1(\Omega_\varepsilon)} \\
&= \frac{1}{\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)}} \Big(U_\varepsilon, \Pi_\varepsilon h_\varepsilon - h_\varepsilon + \Big(1 - \|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)}\Big) \varphi_n \\
&\quad + \Big(\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)} - 1\Big) U_\varepsilon\Big)_{H^1(\Omega_\varepsilon)} \\
&= \frac{\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)} - 1}{\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)}} \mathcal{T}_\varepsilon + \frac{(U_\varepsilon, \Pi_\varepsilon h_\varepsilon - h_\varepsilon)_{H^1(\Omega_\varepsilon)}}{\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)}} + \frac{1 - \|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)}}{\|\Pi_\varepsilon h_\varepsilon\|_{L^2(\Omega_\varepsilon)}} (U_\varepsilon, \varphi_n)_{H^1(\Omega_\varepsilon)},
\end{aligned}$$

hence, in view of (4.15) and (4.10),

$$(U_\varepsilon, \varphi_n^\varepsilon - \varphi_n + U_\varepsilon)_{H^1(\Omega_\varepsilon)} = o(\mathcal{T}_\varepsilon) + O\left(\|\varphi_n\|^2_{L^2(\Sigma_\varepsilon)}\sqrt{\mathcal{T}_\varepsilon}\right) \tag{4.16}$$

as $\varepsilon \to 0$. Writing $\|\varphi_n^\varepsilon - \varphi_n\|^2_{H^1(\Omega_\varepsilon)}$ as

$$\|\varphi_n^\varepsilon - \varphi_n\|^2_{H^1(\Omega_\varepsilon)} = \|U_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} + \|\varphi_n^\varepsilon - \varphi_n + U_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} - 2(U_\varepsilon, \varphi_n^\varepsilon - \varphi_n + U_\varepsilon)_{H^1(\Omega_\varepsilon)},$$

estimate (2.14) follows from (2.13), (4.16), and the fact that $\|U_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} = \mathcal{T}_\varepsilon$. □

5. Blow-up analysis

In the present section, we focus on a particular choice of holes Σ_ε. More precisely, let

$$N \geq 3, \quad x_0 \in \Omega, \quad \text{and} \quad \Sigma \text{ be a bounded open Lipschitz set,} \tag{5.1}$$

so that (2.15) is satisfied for some $\varepsilon_0, r_0 > 0$. Then, for every $\varepsilon \in (0, \varepsilon_0)$, we consider the hole $\Sigma_\varepsilon := x_0 + \varepsilon\Sigma$ as in (1.6) and the corresponding perforated domain

$$\Omega_\varepsilon = \Omega \setminus \overline{\Sigma_\varepsilon} = \Omega \setminus (x_0 + \varepsilon\overline{\Sigma}). \tag{5.2}$$

Without loss of generality, we can assume that $x_0 = 0$. We observe that the family $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ defined as above satisfies assumption **(H)**. Indeed, (H1) and (H3) follow directly from the definition of Σ_ε and (2.15). Condition (H2) is, instead, a consequence of Lemma A.2 in the Appendix.

The local behavior of the eigenfunction φ_n near 0 is described in the following proposition.

Proposition 5.1. *If φ_n vanishes of order $k \geq 0$ at 0, then, for every $R > 0$,*

$$\frac{\varphi_n(rx)}{r^k} \to P_k^{\varphi_n}(x) \text{ uniformly in } \overline{B_R} \text{ and in } H^1(B_R)$$

as $r \to 0$; furthermore, $P_k^{\varphi_n}$ is a harmonic polynomial, homogeneous of degree k.

If $\varphi_n - \varphi_n(0)$ vanishes of order $k \geq 1$ at 0, then, for every $R > 0$,

$$\frac{\varphi_n(rx) - \varphi_n(0)}{r^k} \to P_k^{\varphi_n}(x) \quad \text{uniformly in } \overline{B_R} \text{ and in } H^1(B_R)$$

and

$$\frac{\nabla\varphi_n(rx)}{r^{k-1}} \to \nabla P_k^{\varphi_n}(x) \quad \text{uniformly in } \overline{B_R} \text{ and in } H^1(B_R;\mathbb{R}^N)$$

as $r \to 0$.

Proof. The proof of the convergences follows from the analyticity of φ_n. Moreover, the fact that $P_k^{\varphi_n}$ is harmonic follows from standard scaling arguments, together with the fact that φ_n is an eigenfunction. □

Remark 5.2. It is obvious that φ_n vanishes at 0 of order $k \geq 1$ if and only if $\varphi_n(0) = 0$; in such a case, $\varphi_n - \varphi_n(0)$ vanishes of the same order k. On the other hand, if $\varphi_n(0) \neq 0$, the vanishing order of $\varphi_n - \varphi_n(0)$ is necessarily equal to either $k = 1$ or $k = 2$; this can be easily verified by taking into account that φ_n is a solution to (2.1) (which is not constant since $n \geq 1$) and comparing the Taylor expansions of $-\Delta\varphi_n + \varphi_n$ and $\lambda_n\varphi_n$. In the case $\varphi_n(0) \neq 0$ and $k = 2$, 0 is a critical point for the function φ_n, whereas, if $\varphi_n(0) \neq 0$ and $k = 1$, 0 is a regular point outside the nodal set.

The following Hardy-type inequality on perforated balls will be crucial to identify the limit blow-up profiles.

Lemma 5.3 *(Hardy-type inequality). Let $N \geq 3$ and $\Sigma \subseteq \mathbb{R}^N$ be an open, Lipschitz set such that $\overline{\Sigma} \subset B_{R_0}$ for some $R_0 > 0$. There exists $C_{\mathrm{H}} > 0$, depending only on N and Σ, such that*

$$\int_{B_R\setminus\Sigma} \frac{u^2}{|x|^2}\,\mathrm{d}x \leq C_{\mathrm{H}} \left[\int_{B_R\setminus\Sigma} |\nabla u|^2\,\mathrm{d}x + \frac{1}{R^2}\int_{B_R\setminus\Sigma} u^2\,\mathrm{d}x \right] \tag{5.3}$$

for all $u \in H^1(B_R \setminus \overline{\Sigma})$ and $R > 2R_0$. Moreover,

$$\int_{\mathbb{R}^N\setminus\Sigma} \frac{u^2}{|x|^2}\,\mathrm{d}x \leq C_{\mathrm{H}} \int_{\mathbb{R}^N\setminus\Sigma} |\nabla u|^2\,\mathrm{d}x \tag{5.4}$$

for all $u \in C_c^\infty(\mathbb{R}^N \setminus \Sigma)$.

Inequality (5.4) allows us to characterize the space $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma)$ introduced in Definition 2.6 as

$$\mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma) = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \Sigma) \colon \int_{\mathbb{R}^N\setminus\Sigma} \left(|\nabla u|^2 + \frac{u^2}{|x|^2} \right) \mathrm{d}x < \infty \right\}. \tag{5.5}$$

Furthermore,

$$u \mapsto \left(\int_{\mathbb{R}^N\setminus\Sigma} \left(|\nabla u|^2 + \frac{u^2}{|x|^2} \right) \mathrm{d}x \right)^{\frac{1}{2}}$$

is an equivalent norm on $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma)$.

Proof of Lemma 5.3. Let $R > 2R_0$ and $u \in H^1(B_R \setminus \overline{\Sigma})$. We define the scaled function

$$u_R(x) := u(Rx) \in H^1\big(B_1 \setminus \tfrac{1}{R}\overline{\Sigma}\big),$$

as well as its extension to the whole B_1

$$v_R := \mathsf{E}_{\frac{1}{R}} u_R \in H^1(B_1).$$

Lemma A.2 ensures that the norm of the extension operator $\mathsf{E}_{\frac{1}{R}}$ does not depend on R. Moreover

$$\int_{B_1} \frac{v_R^2}{|x|^2}\,\mathrm{d}x \le C_N \left(\int_{B_1} |\nabla v_R|^2\,\mathrm{d}x + \int_{B_1} v_R^2\,\mathrm{d}x \right), \tag{5.6}$$

for some constant $C_N > 0$ depending only on N. The above Hardy-type inequality is classical, see, for instance, [18, Lemma 6.7] for a proof in half-balls. In view of (5.6), we have

$$\int_{B_1\setminus\frac{1}{R}\Sigma} \frac{v_R^2}{|x|^2}\,\mathrm{d}x \le C_N \mathfrak{C}^2 \left(\int_{B_1\setminus\frac{1}{R}\Sigma} |\nabla v_R|^2\,\mathrm{d}x + \int_{B_1\setminus\frac{1}{R}\Sigma} v_R^2\,\mathrm{d}x \right),$$

with $\mathfrak{C}$ being as in Lemma A.2 with $\Omega = B_1$, $\varepsilon_0 = \frac{1}{2R_0}$, and $r_0 = \frac{1}{2}$. Being v_R the extension of u_R, the above inequality holds for u_R as well. Scaling back the inequality to B_R yields

$$R^{2-N} \int_{B_R\setminus\Sigma} \frac{u^2(x)}{|x|^2}\,\mathrm{d}x \le C_N \mathfrak{C}^2 R^{-N} \left(\int_{B_R\setminus\Sigma} R^2 |\nabla u(x)|^2\,\mathrm{d}x + \int_{B_R\setminus\Sigma} u^2(x)\,\mathrm{d}x \right),$$

which, after a straightforward simplification, is precisely (5.3). Inequality (5.4) follows from (5.3) by letting $R \to \infty$. □

The following result provides a first rough estimate of $\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma), \partial_{\boldsymbol{\nu}}\varphi_n)$.

Lemma 5.4. *Under assumptions* (5.1)–(5.2) *with* $x_0 = 0$*, let* $k \ge 1$ *be the vanishing order of* $\varphi_n - \varphi_n(0)$ *at* 0*. Then*

$$\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma), \partial_{\boldsymbol{\nu}}\varphi_n) = O(\varepsilon^{N+2k-2}) \quad \text{as } \varepsilon \to 0.$$

Proof. For every $u \in H^1(\Omega_\varepsilon)$, by the Divergence Theorem, Hölder's inequality, and Lemma A.2 we have

$$\begin{aligned}\left|\int_{\partial(\varepsilon\Sigma)} u\,\partial_{\boldsymbol{\nu}}\varphi_n\,\mathrm{d}S\right| &= \left|\int_{\varepsilon\Sigma} \operatorname{div}((\mathsf{E}_\varepsilon u)\nabla\varphi_n)\,\mathrm{d}x\right| = \left|\int_{\varepsilon\Sigma}\Big((\Delta\varphi_n)(\mathsf{E}_\varepsilon u)+\nabla(\mathsf{E}_\varepsilon u)\cdot\nabla\varphi_n\Big)\,\mathrm{d}x\right| \\ &\le \|\Delta\varphi_n\|_{L^2(\varepsilon\Sigma)}\|\mathsf{E}_\varepsilon u\|_{L^{2^*}(\varepsilon\Sigma)}|\varepsilon\Sigma|^{1/N} \\ &\quad + \|\nabla\varphi_n\|_{L^2(\varepsilon\Sigma;\mathbb{R}^N)}\|\nabla(\mathsf{E}_\varepsilon u)\|_{L^2(\varepsilon\Sigma;\mathbb{R}^N)} \\ &\le \|\mathsf{E}_\varepsilon u\|_{H^1(\Omega)}\Big(S_{N,\Omega}\|\Delta\varphi_n\|_{L^2(\varepsilon\Sigma)}\varepsilon|\Sigma|^{1/N}+\|\nabla\varphi_n\|_{L^2(\varepsilon\Sigma;\mathbb{R}^N)}\Big) \\ &\le \mathfrak{C}\|u\|_{H^1(\Omega_\varepsilon)}\Big(S_{N,\Omega}\|\Delta\varphi_n\|_{L^2(\varepsilon\Sigma)}\varepsilon|\Sigma|^{1/N}+\|\nabla\varphi_n\|_{L^2(\varepsilon\Sigma;\mathbb{R}^N)}\Big)\end{aligned}$$

where $2^*=\frac{2N}{N-2}$ is the critical Sobolev exponent (remember that in the present section we are assuming $N\ge 3$) and $S_{N,\Omega}$ is the operator norm of the embedding $H^1(\Omega)\hookrightarrow L^{2^*}(\Omega)$. In view of the characterization of $\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma),\partial_{\boldsymbol{\nu}}\varphi_n)$ given in (3.1), the above estimate yields

$$\begin{aligned}\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma),\partial_{\boldsymbol{\nu}}\varphi_n) &= \sup_{u\in H^1(\Omega_\varepsilon)\setminus\{0\}}\frac{\Big(\int_{\partial(\varepsilon\Sigma)}u\,\partial_{\boldsymbol{\nu}}\varphi_n\,\mathrm{d}S\Big)^2}{\|u\|^2_{H^1(\Omega_\varepsilon)}} \\ &\le \mathfrak{C}^2\Big(S_{N,\Omega}\|\Delta\varphi_n\|_{L^2(\varepsilon\Sigma)}\varepsilon|\Sigma|^{1/N}+\|\nabla\varphi_n\|_{L^2(\varepsilon\Sigma;\mathbb{R}^N)}\Big)^2. \qquad (5.7)\end{aligned}$$

Since $\varphi_n-\varphi_n(0)$ vanishes at 0 with order $k\ge 1$, we have

$$\Delta\varphi_n(x)=O(|x|^{k-2}) \quad \text{and} \quad |\nabla\varphi_n(x)|=O(|x|^{k-1}) \quad \text{as } x\to 0,$$

which implies that

$$\|\Delta\varphi_n\|_{L^2(\varepsilon\Sigma)}=O\left(\varepsilon^{k-2+\frac{N}{2}}\right) \quad \text{and} \quad \|\nabla\varphi_n\|_{L^2(\varepsilon\Sigma;\mathbb{R}^N)}=O\left(\varepsilon^{k-1+\frac{N}{2}}\right) \qquad (5.8)$$

as $\varepsilon\to 0$. The conclusion follows by combining (5.7) and (5.8). □

Remark 5.5. Arguing as in the proof of Lemma 5.4, we can prove that, if $N=2$,

$$\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma),\partial_{\boldsymbol{\nu}}\varphi_n)=O(\varepsilon^{2(k-\delta)}) \quad \text{as } \varepsilon\to 0,$$

for every $\delta\in(0,1)$. To prove this, it is sufficient to retrace the steps of the previous proof, using the Sobolev embedding $H^1(\Omega)\hookrightarrow L^p(\Omega)$ with $p=2/\delta$. In particular, we have, even in dimension $N=2$, $\lim_{\varepsilon\to 0}\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma),\partial_{\boldsymbol{\nu}}\varphi_n)=0$.

We are now in position to state and prove the main result of this section.

Theorem 5.6 *(Blow-up). Under assumptions* (5.1)–(5.2) *with $x_0=0$, let $k\ge 1$ be the vanishing order of $\varphi_n-\varphi_n(0)$ at 0 and $P_k^{\varphi_n}$ be as in* (2.16)–(2.18)*. Then*

$$\lim_{\varepsilon\to 0}\varepsilon^{-N-2k+2}\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma),\partial_{\boldsymbol{\nu}}\varphi_n)=\tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}).$$

Furthermore, if $U_\varepsilon := U_{\Omega,\varepsilon\Sigma,\partial_{\boldsymbol{\nu}}\varphi_n}\in H^1(\Omega_\varepsilon)$ *is the function achieving* $\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma),\partial_{\boldsymbol{\nu}}\varphi_n)$, *see* (2.7), *and*

$$\tilde{U}_\varepsilon(x):=\varepsilon^{-k}U_\varepsilon(\varepsilon x),\quad x\in\left(\tfrac{1}{\varepsilon}\Omega\right)\setminus\Sigma,\tag{5.9}$$

then

$$\tilde{U}_\varepsilon\to\tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}}\qquad\text{in } H^1(B_R\setminus\overline{\Sigma}),\ \text{as } \varepsilon\to 0,$$

for all $R>0$ *such that* $\overline{\Sigma}\subseteq B_R$, *where* $\tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}}\in\mathcal{D}^{1,2}(\mathbb{R}^N\setminus\Sigma)$ *is the function achieving* $\tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n})$ *as in* (2.19).

Proof. Let $r_0,R_0>0$ be such that $\overline{B_{r_0}}\subset\Omega$ and $\overline{\Sigma}\subset B_{R_0}$, so that (2.15) is satisfied with $x_0=0$ and $\varepsilon_0=r_0/R_0$. Let $R>R_0$ and $\varepsilon<\frac{r_0}{2R}$. Since $R<\frac{r_0}{\varepsilon}$ and $\frac{r_0}{\varepsilon}>2R_0$, by Lemma 5.3 and a change of variable, we have

$$\begin{aligned}\int_{B_R\setminus\Sigma}\left(|\nabla\tilde{U}_\varepsilon|^2+\frac{\tilde{U}_\varepsilon^2}{|x|^2}\right)\mathrm{d}x&\le\int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma}\left(|\nabla\tilde{U}_\varepsilon|^2+\frac{\tilde{U}_\varepsilon^2}{|x|^2}\right)\mathrm{d}x\\&\le\int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma}|\nabla\tilde{U}_\varepsilon|^2\,\mathrm{d}x+C_{\mathrm{H}}\int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma}\left(|\nabla\tilde{U}_\varepsilon|^2+\frac{\varepsilon^2}{r_0^2}\tilde{U}_\varepsilon^2\right)\mathrm{d}x\\&=\varepsilon^{-N-2k+2}\left(\int_{B_{r_0}\setminus\varepsilon\Sigma}|\nabla U_\varepsilon|^2\,\mathrm{d}x+C_{\mathrm{H}}\int_{B_{r_0}\setminus\varepsilon\Sigma}\left(|\nabla U_\varepsilon|^2+\frac{1}{r_0^2}U_\varepsilon^2\right)\mathrm{d}x\right).\end{aligned}$$

Hence, by Lemma 5.4 we have

$$\int_{B_R\setminus\Sigma}\left(|\nabla\tilde{U}_\varepsilon|^2+\frac{\tilde{U}_\varepsilon^2}{|x|^2}\right)\mathrm{d}x\le C_1\varepsilon^{-N-2k+2}\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma),\partial_{\boldsymbol{\nu}}\varphi_n)\le C_2,\tag{5.10}$$

where $C_1,C_2>0$ are constants independent of R and ε. Therefore, by a diagonal argument, for every sequence $\varepsilon_j\to 0^+$ there exists a subsequence (still denoted as $\{\varepsilon_j\}$) and a limit profile $\tilde{U}\in L^1_{\mathrm{loc}}(\mathbb{R}^N\setminus\Sigma)$ such that, for all $R>R_0$, $\tilde{U}\in H^1(B_R\setminus\overline{\Sigma})$ and

$$\tilde{U}_{\varepsilon_j}\rightharpoonup\tilde{U}\quad\text{as } j\to\infty \text{ weakly in } H^1(B_R\setminus\overline{\Sigma}).\tag{5.11}$$

Furthermore, by compactness of the embedding $H^1(B_R\setminus\overline{\Sigma})\hookrightarrow L^2(B_R\setminus\overline{\Sigma})$ and of the trace map from $H^1(B_R\setminus\overline{\Sigma})$ into $L^2(\partial\Sigma)$, we also have, as $j\to\infty$,

$$\tilde{U}_{\varepsilon_j} \to \tilde{U} \quad \text{strongly in } L^2(B_R \setminus \overline{\Sigma}) \text{ for all } R > R_0, \tag{5.12}$$

$$\tilde{U}_{\varepsilon_j} \to \tilde{U} \quad \text{strongly in } L^2(\partial\Sigma). \tag{5.13}$$

From (5.10) and the weak lower semicontinuity of the norm we deduce that

$$\int_{B_R\setminus\Sigma} \left(|\nabla \tilde{U}|^2 + \frac{\tilde{U}^2}{|x|^2}\right) \mathrm{d}x \le C_2 \quad \text{for all } R > R_0,$$

which implies that

$$\int_{\mathbb{R}^N\setminus\Sigma} \left(|\nabla \tilde{U}|^2 + \frac{\tilde{U}^2}{|x|^2}\right) \mathrm{d}x < +\infty$$

and, consequently, that $\tilde{U} \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma)$, see (5.5).

For any $v \in C_c^\infty(\mathbb{R}^N \setminus \Sigma)$ fixed, let j be sufficiently large in order to ensure that

$$\operatorname{supp} v \subseteq B_{R_v} \setminus \Sigma \subseteq \tfrac{1}{\varepsilon_j}\Omega \setminus \Sigma,$$

for some $R_v > R_0$. From the equation satisfied by U_ε, see (3.4), and a change of variable it follows that

$$\int_{B_{R_v}\setminus\Sigma} (\nabla \tilde{U}_{\varepsilon_j} \cdot \nabla v + \varepsilon_j^2 \tilde{U}_{\varepsilon_j} v)\,\mathrm{d}x - \int_{\partial\Sigma} v \frac{\partial_{\boldsymbol{\nu}} \varphi_n(\varepsilon_j x)}{\varepsilon_j^{k-1}}\,\mathrm{d}S = 0. \tag{5.14}$$

In view of (5.11), (5.12), and Proposition 5.1, we can pass to the limit as $j \to \infty$ in (5.14). Hence, by density, we obtain

$$\int_{\mathbb{R}^N\setminus\Sigma} \nabla \tilde{U} \cdot \nabla v\,\mathrm{d}x - \int_{\partial\Sigma} v\,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}\,\mathrm{d}S = 0 \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma),$$

which, together with Proposition 3.4, implies that $\tilde{U} = \tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}}$. On the other hand, by (5.13) and Proposition 5.1 we have

$$\begin{aligned} \varepsilon_j^{-N-2k+2} \mathcal{T}_{\overline{\Omega}_{\varepsilon_j}}(\partial(\varepsilon_j\Sigma), \partial_{\boldsymbol{\nu}}\varphi_n) &= \int_{\partial\Sigma} \tilde{U}_{\varepsilon_j} \frac{\partial_{\boldsymbol{\nu}} \varphi_n(\varepsilon_j x)}{\varepsilon_j^{k-1}}\,\mathrm{d}S \\ &\to \int_{\partial\Sigma} \tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}}\,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n} = \tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, \partial_{\boldsymbol{\nu}} P_k^{\varphi_n}) \end{aligned}$$

as $j \to \infty$. Being the limit profile uniquely determined, by Urysohn's subsequence principle we conclude that the convergence statements above hold as $\varepsilon \to 0$, independently of the sequence $\{\varepsilon_j\}$ and of the subsequence.

In order to prove the strong H^1-convergence, we observe that, in view of the equations satisfied by $\tilde{U}_\varepsilon$ and $\tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}}$, for $R > R_0$ we have

$$\begin{aligned}\int_{B_R\setminus\Sigma} \left|\nabla(\tilde{U}_\varepsilon - \tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}})\right|^2 \mathrm{d}x &= \int_{\partial\Sigma} (\tilde{U}_\varepsilon - \tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}})(\partial_{\boldsymbol{\nu}}\tilde{\varphi}_\varepsilon - \partial_{\boldsymbol{\nu}} P_k^{\varphi_n})\,\mathrm{d}S \\ &+ \int_{\partial B_R} (\tilde{U}_\varepsilon - \tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}})(\partial_{\boldsymbol{\nu}}\tilde{U}_\varepsilon - \partial_{\boldsymbol{\nu}}\tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}})\,\mathrm{d}S \\ &- \varepsilon^2 \int_{B_R\setminus\Sigma} (\tilde{U}_\varepsilon^2 - \tilde{U}_\varepsilon\tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}})\,\mathrm{d}x, \end{aligned} \tag{5.15}$$

where

$$\tilde{\varphi}_\varepsilon(x) := \frac{\varphi_n(\varepsilon x) - \varphi_n(0)}{\varepsilon^k}. \tag{5.16}$$

Since $\tilde{U}_\varepsilon$ weakly solves the equation $-\Delta\tilde{U}_\varepsilon = -\varepsilon^2\tilde{U}_\varepsilon$ in $B_{2R} \setminus B_{R_0}$ and the family $\{\varepsilon^2\tilde{U}_\varepsilon\}_{0<\varepsilon<\frac{r_0}{2R}}$ is bounded in $L^2(B_{2R} \setminus B_{R_0})$, by classical elliptic regularity theory $\{\tilde{U}_\varepsilon\}_{0<\varepsilon<\frac{r_0}{2R}}$ is bounded in $H^2(B_{\frac{3}{2}R}\setminus B_{(R_0+R)/2})$, so that, by continuity of the trace operator, $\{\partial_{\boldsymbol{\nu}}\tilde{U}_\varepsilon\}_{0<\varepsilon<\frac{r_0}{2R}}$ is bounded in $L^2(\partial B_R)$. This, combined with Proposition 5.1, convergences (5.12)–(5.13), and the compactness of the embedding $H^1(B_R \setminus \overline{\Sigma}) \hookrightarrow L^2(B_R \setminus \overline{\Sigma})$, allows us to pass to the limit in (5.15), proving that $\nabla\tilde{U}_\varepsilon \to \nabla\tilde{U}_{\Sigma,\partial_{\boldsymbol{\nu}} P_k^{\varphi_n}}$ strongly in $L^2(B_R \setminus \overline{\Sigma})$ and completing the proof in view of (5.12). □

We finally have all the necessary ingredients for the proofs of Theorem 2.8 and Theorem 2.9.

Proof of Theorem 2.8. By translation, it is not restrictive to assume $x_0 = 0$. We first observe that the family $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)} = \{\varepsilon\Sigma\}_{\varepsilon\in(0,\varepsilon_0)}$ satisfies the assumptions of Theorem 2.4. Indeed, by scaling arguments, one can easily verify that $|\varepsilon\Sigma| \to 0$ and $\mathrm{Cap}\,(\varepsilon\overline{\Sigma}) \to 0$ as $\varepsilon \to 0$. Moreover, (H2) follows from Lemma A.2. In view of Theorem 2.4 and Theorem 5.6, to obtain an explicit expansion for the perturbed eigenvalue we only have to analyze the asymptotic behavior, as $\varepsilon \to 0$, of the term

$$\int_{\Sigma_\varepsilon} \left(|\nabla\varphi_n|^2 - (\lambda_n - 1)\varphi_n^2\right) \mathrm{d}x.$$

To start, let us consider the case $0 \in \Omega \setminus \mathrm{Sing}\,(\varphi_n)$. Since φ_n is smooth, we have

$$\varphi_n(x) = \varphi_n(0) + O(|x|) \quad \text{and} \quad \nabla\varphi_n(x) = \nabla\varphi_n(0) + O(|x|) \quad \text{as } |x| \to 0,$$

which directly yields

$$\int_{\varepsilon\Sigma} \left(|\nabla\varphi_n|^2 - (\lambda_n - 1)\varphi_n^2\right) \mathrm{d}x = \varepsilon^N \; |\Sigma| \left(|\nabla\varphi_n(0)|^2 - (\lambda_n - 1)\varphi_n^2(0)\right) + o(\varepsilon^N) \quad (5.17)$$

as $\varepsilon \to 0$. On the other hand, to identify the order of the term $\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma), \partial_{\boldsymbol{\nu}}\varphi_n)$ appearing in the expansion (2.12), we distinguish two cases: $\nabla\varphi_n(0) \neq 0$ and $\nabla\varphi_n(0) = 0$. If $\nabla\varphi_n(0) \neq 0$, we can apply Theorem 5.6 with $k = 1$ and, since $P_1^{\varphi_n}(x) = \nabla\varphi_n(0) \cdot x$, we obtain

$$\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma), \partial_{\boldsymbol{\nu}}\varphi_n) = \varepsilon^N \tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, \nabla\varphi_n(0)\cdot\boldsymbol{\nu}) + o(\varepsilon^N) \quad \text{as } \varepsilon \to 0. \quad (5.18)$$

If, instead, $\nabla\varphi_n(0) = 0$, then Theorem 5.6 applies with some $k \geq 2$, thus implying that

$$\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma), \partial_{\boldsymbol{\nu}}\varphi_n) = o(\varepsilon^N) \quad \text{as } \varepsilon \to 0. \quad (5.19)$$

Moreover, trivially,

$$\tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, \nabla\varphi_n(0)\cdot\boldsymbol{\nu}) = \tau_{\mathbb{R}^N\setminus\Sigma}(\partial\Sigma, 0) = 0. \quad (5.20)$$

Combining (5.18), (5.19), and (5.20) with (5.17) we obtain (i).

If $0 \in \mathrm{Sing}\,(\varphi_n)$ and $k \geq 2$ is the vanishing order of φ_n at 0, then

$$\varphi_n(x) = O(|x|^k) \quad \text{and} \quad \nabla\varphi_n(x) = \nabla P_k^{\varphi_n}(x) + O(|x|^k) \quad \text{as } |x| \to 0,$$

thus implying that, as $\varepsilon \to 0$,

$$\int_{\varepsilon\Sigma} |\nabla\varphi_n(x)|^2 \,\mathrm{d}x = \int_{\varepsilon\Sigma} |\nabla P_k^{\varphi_n}|^2 \,\mathrm{d}x + O(\varepsilon^{2k-1+N}) = \varepsilon^{N+2k-2} \left(\int_{\Sigma} |\nabla P_k^{\varphi_n}|^2 \,\mathrm{d}x + o(1) \right),$$
$$\int_{\varepsilon\Sigma} |\varphi_n(x)|^2 \,\mathrm{d}x = O(\varepsilon^{2k+N}) = o(\varepsilon^{N+2k-2}),$$

and hence

$$\int_{\varepsilon\Sigma} \left(|\nabla\varphi_n|^2 - (\lambda_n - 1)\varphi_n^2\right) \mathrm{d}x = \varepsilon^{N+2k-2} \left(\int_{\Sigma} |\nabla P_k^{\varphi_n}|^2 \,\mathrm{d}x + o(1) \right) \quad \text{as } \varepsilon \to 0.$$

Combining this and Theorem 5.6 with Theorem 2.4 we obtain (ii). □

Proof of Theorem 2.9. By translation, it is not restrictive to assume $x_0 = 0$.

Let $u_\varepsilon := \Phi_\varepsilon - \tilde{\varphi}_\varepsilon + \tilde{U}_\varepsilon$, where $\tilde{\varphi}_\varepsilon$ and $\tilde{U}_\varepsilon$ are defined in (5.16) and (5.9), respectively. From (2.13) and the change of variable $x \mapsto \varepsilon x$ it follows that, as $\varepsilon \to 0$,

$$\int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} |\nabla u_\varepsilon|^2\, dx + \varepsilon^2 \int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} u_\varepsilon^2\, dx = o\big(\varepsilon^{-N-2k+2}\mathcal{T}_{\overline{\Omega}_\varepsilon}(\partial(\varepsilon\Sigma), \partial_{\boldsymbol{\nu}}\varphi_n)\big) + O\big(\varepsilon^{-N-2k+2}\|\varphi_n\|^4_{L^2(\varepsilon\Sigma)}\big).$$

If $\varphi_n(0) = 0$, $\varphi_n(x) = O(|x|^k)$ as $|x| \to 0$, hence $\|\varphi_n\|^4_{L^2(\varepsilon\Sigma)} = O(\varepsilon^{2N+4k}) = o(\varepsilon^{N+2k-2})$ as $\varepsilon \to 0$. If $\varphi_n(0) \neq 0$, either $k = 1$ or $k = 2$ by Remark 5.2, so that

$$\|\varphi_n\|^4_{L^2(\varepsilon\Sigma)} = O(\varepsilon^{2N}) = o(\varepsilon^{N+2k-2})$$

as $\varepsilon \to 0$. In both cases we have

$$\|\varphi_n\|^4_{L^2(\varepsilon\Sigma)} = o(\varepsilon^{N+2k-2}) \quad \text{as } \varepsilon \to 0. \tag{5.21}$$

From this and Theorem 5.6 we deduce that

$$\int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} |\nabla u_\varepsilon|^2\, dx + \varepsilon^2 \int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} u_\varepsilon^2\, dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.22}$$

Let $r_0, R_0 > 0$ be such that $\overline{B_{r_0}} \subset \Omega$ and $\overline{\Sigma} \subset B_{R_0}$; let $R > R_0$ and $\varepsilon < \frac{r_0}{2R}$. By Lemma 5.3 we have

$$\begin{aligned}
\int_{B_R\setminus\Sigma} \left(|\nabla u_\varepsilon|^2 + \frac{u_\varepsilon^2}{|x|^2}\right) dx &\leq \int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma} \left(|\nabla u_\varepsilon|^2 + \frac{u_\varepsilon^2}{|x|^2}\right) dx \\
&\leq \int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma} |\nabla u_\varepsilon|^2\, dx + C_{\mathrm{H}} \int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma} \left(|\nabla u_\varepsilon|^2 + \frac{\varepsilon^2}{r_0^2} u_\varepsilon^2\right) dx \\
&\leq (C_{\mathrm{H}} + 1)\left(\int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} |\nabla u_\varepsilon|^2\, dx + \frac{\varepsilon^2}{r_0^2}\int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} u_\varepsilon^2\, dx\right).
\end{aligned}$$

From this estimate and (5.22) we deduce that

$$\int_{B_R\setminus\Sigma} \left(|\nabla u_\varepsilon|^2 + \frac{u_\varepsilon^2}{|x|^2}\right) dx \to 0 \quad \text{as } \varepsilon \to 0$$

for any $R > 0$ such that $\overline{\Sigma} \subseteq B_R$, which implies that $u_\varepsilon \to 0$ strongly in $H^1(B_R \setminus \overline{\Sigma})$ as $\varepsilon \to 0$. Combining this with Proposition 5.1 and Theorem 5.6 we obtain (2.20).

Finally, (2.21) follows from (2.14), Theorem 5.6, and (5.21). □

6. The case of a spherical hole

In this section, we focus on spherical holes, deriving in this specific situation more explicit expressions for the coefficients of the asymptotic expansions obtained above. We distinguish between the cases $N \geq 3$ and $N = 2$.

6.1. The case $N \geq 3$

As proved in Theorem 2.8, different behaviors occur depending on the vanishing order or φ_n at x_0. The most interesting and diverse phenomena are observed when $x_0 \in \Omega \setminus \mathrm{Sing}(\varphi_n)$, as in this situation the sign of the leading term in the asymptotic expansion is not always the same regardless of where the domain is perforated. In view of Theorem 2.8-(i), the interface Γ defined in (2.22) divides the points of Ω where a hole produces a positive sign of the eigenvalue variation $\lambda_n^\varepsilon - \lambda_n$ from those where there would be a negative sign, see Remark 2.10. Here we focus on the specific case

$$\Sigma = B_1,$$

providing the proof of Theorem 2.11, to which we precede the following preliminary lemma.

Lemma 6.1. *If $N \geq 3$ and $P : \mathbb{R}^N \to \mathbb{R}$ is a harmonic polynomial homogeneous of degree $k \in \mathbb{N} \setminus \{0\}$, then*

$$\tau_{\mathbb{R}^N \setminus B_1}(\partial B_1, \partial_{\boldsymbol{\nu}} P) = \frac{k^2}{N + k - 2} \int_{\partial B_1} Y^2 \, \mathrm{d}S,$$

where Y is the spherical harmonic of degree k given by $Y = P\big|_{\partial B_1}$.

Proof. To determine the torsion function $U := \tilde{U}_{B_1, \partial_{\boldsymbol{\nu}} P}$, we recall that $U \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus B_1)$ is the unique weak solution to

$$\begin{cases} -\Delta U = 0, & \text{in } \mathbb{R}^N \setminus \overline{B_1}, \\ \partial_{\boldsymbol{\nu}} U = \partial_{\boldsymbol{\nu}} P, & \text{on } \partial B_1. \end{cases} \tag{6.1}$$

We work in spherical coordinates $(r, \boldsymbol{\theta})$ and look for solutions to (6.1) of the form

$$U(r, \boldsymbol{\theta}) = u(r) Y(\boldsymbol{\theta}),$$

where $Y = P\big|_{\partial B_1}$. We observe that, since P is harmonic and k-homogeneous, Y is a spherical harmonic of degree k and solves

$$-\Delta_{\partial B_1} Y = k(N + k - 2) Y, \quad \text{on } \partial B_1,$$

where $\Delta_{\partial B_1}$ is the Laplace-Beltrami operator. Then, we can rewrite (6.1) as

$$\begin{cases} u''(r) + \dfrac{N-1}{r}u'(r) - \dfrac{k(N+k-2)}{r^2}u(r) = 0, & \text{in } (1,+\infty), \\ u'(1) = k. \end{cases} \tag{6.2}$$

The solutions to the equation in the first line of (6.2) are of the form

$$u(r) = c_1 r^k + c_2 r^{-(N+k-2)}$$

for some c_1, c_2. The fact that $U(r,\boldsymbol{\theta}) = u(r)Y(\boldsymbol{\theta}) \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus B_1)$ implies that necessarily $c_1 = 0$, whereas the condition $u'(1) = k$ yields $c_2 = -\frac{k}{N+k-2}$. Hence, by uniqueness of the torsion function,

$$U(r,\boldsymbol{\theta}) = -\frac{k}{N+k-2} r^{-(N+k-2)} Y(\boldsymbol{\theta}).$$

We conclude that

$$\tau_{\mathbb{R}^N \setminus B_1}(\partial B_1, \partial_{\boldsymbol{\nu}} P) = \int_{\partial B_1} U(\partial_{\boldsymbol{\nu}} P)\,\mathrm{d}S = \frac{k^2}{N+k-2}\int_{\partial B_1} Y^2(\boldsymbol{\theta})\,\mathrm{d}S,$$

thus completing the proof. □

Proof of Theorem 2.11. We observe that $P(x) = \nabla\varphi_n(x_0)\cdot x$ is a harmonic polynomial of degree $k = 1$. Then Lemma 6.1 applies and yields

$$\tau_{\mathbb{R}^N \setminus B_1}(\partial B_1, \nabla\varphi_n(x_0)\cdot\boldsymbol{\nu}) = \frac{1}{N-1}\int_{\partial B_1} |\nabla\varphi_n(x_0)\cdot\boldsymbol{\theta}|^2\,\mathrm{d}S.$$

Exploiting the symmetry of the domain of integration, a simple computation yields

$$\tau_{\mathbb{R}^N \setminus B_1}(\partial B_1, \nabla\varphi_n(x_0)\cdot\boldsymbol{\nu}) = \frac{\mathcal{H}^{N-1}(\partial B_1)}{N(N-1)}|\nabla\varphi_n(x_0)|^2 = \frac{\omega_N}{N-1}|\nabla\varphi_n(x_0)|^2,$$

where $\omega_N := |B_1|$ denotes the N-dimensional measure of B_1. Substituting the above expression for $\tau_{\mathbb{R}^N \setminus B_1}(\partial B_1, \nabla\varphi_n(x_0)\cdot\boldsymbol{\nu})$ in the expansion of Theorem 2.8-(i), we obtain (i).

If $x_0 \in \mathrm{Sing}(\varphi_n)$, φ_n vanishes at x_0 with order $k \geq 2$. Then, as observed in Proposition 5.1, $P^{\varphi_n}_{x_0,k}$ is a harmonic polynomial homogeneous of degree k. From Lemma 6.1 it follows that

$$\tau_{\mathbb{R}^N \setminus B_1}(\partial B_1, \partial_{\boldsymbol{\nu}} P^{\varphi_n}_{x_0,k}) = \frac{k^2}{N+k-2}\int_{\partial B_1} Y^2\,\mathrm{d}S \tag{6.3}$$

where $Y = P^{\varphi_n}_{x_0,k}\big|_{\partial B_1}$ is a spherical harmonic of degree k. Furthermore, by the fact that $\Delta P^{\varphi_n}_{x_0,k} = 0$ and the Divergence Theorem, we have

$$\int_{B_1} |\nabla P^{\varphi_n}_{x_0,k}(x)|^2 \, \mathrm{d}x = \int_{B_1} \mathrm{div}(P^{\varphi_n}_{x_0,k} \nabla P^{\varphi_n}_{x_0,k}) \, \mathrm{d}x$$
$$= \int_{\partial B_1} P^{\varphi_n}_{x_0,k} \nabla P^{\varphi_n}_{x_0,k} \cdot \boldsymbol{\theta} \, \mathrm{d}S = k \int_{\partial B_1} Y^2 \, \mathrm{d}S. \tag{6.4}$$

Substituting (6.3)–(6.4) in the expansion of Theorem 2.8-(ii), we obtain (ii). □

Thanks to Lemma 6.1 and Theorem 2.11, the interface Γ defined in (2.22) can be described quite explicitly in the case of spherical holes. More precisely, if $\Sigma = B_1$ we have

$$\Gamma = \{x \in \Omega \setminus \mathrm{Sing}\,(\varphi_n) \colon h(x) = 0\},$$

where

$$h(x) := \frac{N}{N-1} |\nabla \varphi_n(x)|^2 - (\lambda_n - 1)\varphi_n^2(x).$$

We present below the example of spherical holes excised from 3-dimensional boxes.

Example 6.2. Let us consider the 3-dimensional open box

$$\Omega = (0,1) \times (0, \sqrt[4]{2}) \times (0, \sqrt[4]{3}).$$

It is a well-known fact (see e.g. [21]) that the eigenvalues of problem (2.1) on Ω are simple and of the form

$$\lambda_{n_1,n_2,n_3} = \pi^2 n_1^2 + \frac{\pi^2 n_2^2}{\sqrt{2}} + \frac{\pi^2 n_3^2}{\sqrt{3}} + 1, \quad n_1, n_2, n_2 \in \mathbb{N},$$

and the associated eigenfunctions are, up to a normalization constant,

$$\varphi_{n_1,n_2,n_3}(x,y,z) = \cos(\pi n_1 x) \cos\left(\frac{\pi n_2}{\sqrt[4]{2}} y\right) \cos\left(\frac{\pi n_3}{\sqrt[4]{3}} z\right).$$

Then the interface Γ associated to φ_{n_1,n_2,n_3} is characterized by the equation

$$n_1^2 \tan^2(\pi n_1 x) + \frac{n_2^2}{\sqrt{2}} \tan^2\left(\frac{\pi n_2 y}{\sqrt[4]{2}}\right) + \frac{n_3^2}{\sqrt{3}} \tan^2\left(\frac{\pi n_3 z}{\sqrt[4]{3}}\right) - \frac{2}{3}\left(n_1^2 + \frac{n_2^2}{\sqrt{2}} + \frac{n_3^2}{\sqrt{3}}\right) = 0.$$

Let us consider two specific cases. The first situation of interest is the one corresponding to the smallest nontrivial eigenvalue, namely,

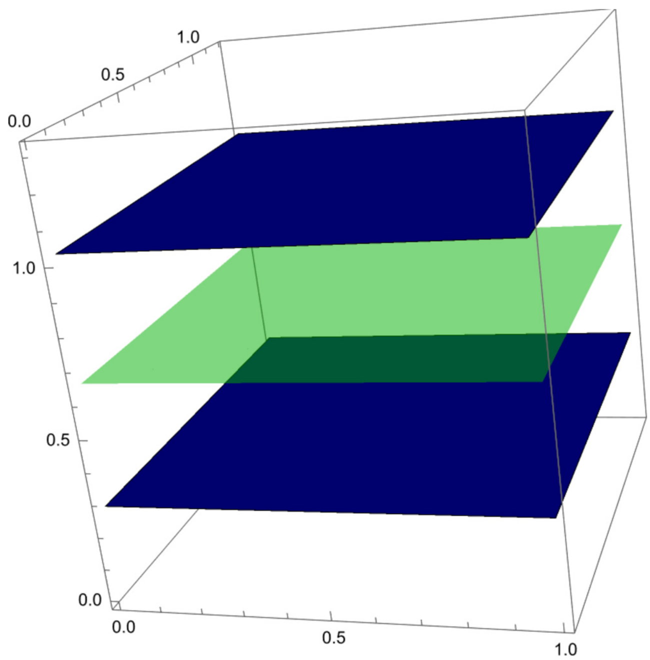

Fig. 1. The case $\lambda_{0,0,1}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\lambda_{0,0,1} = \frac{\pi^2}{\sqrt{3}} + 1.$$

Here, Γ turns out to be the union of two planes

$$\Gamma = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ z = \frac{\sqrt[4]{3}}{\pi} \arctan \sqrt{\frac{2}{3}} \right\}$$
$$\cup \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ z = \frac{\sqrt[4]{3}}{\pi} \left(\pi - \arctan \sqrt{\frac{2}{3}} \right) \right\}.$$

In Fig. 1 we can see the plot of Γ (in blue), along with the nodal set of the eigenfunction $\varphi_{0,0,1}$ (in green). By our analysis, if the hole is punctured between the green and a blue plane, then $\lambda_n^\varepsilon < \lambda_n$.

Finally, we describe Γ for $\lambda_n = \lambda_{1,1,1}$. In this case the situation is more complex, but the general picture does not change. With the help of Mathematica™, we can plot the set $\Gamma = \{h = 0\}$, along with the nodal set of $\varphi_{1,1,1}$ (once again in blue and green respectively). The resulting image is presented in Fig. 2.

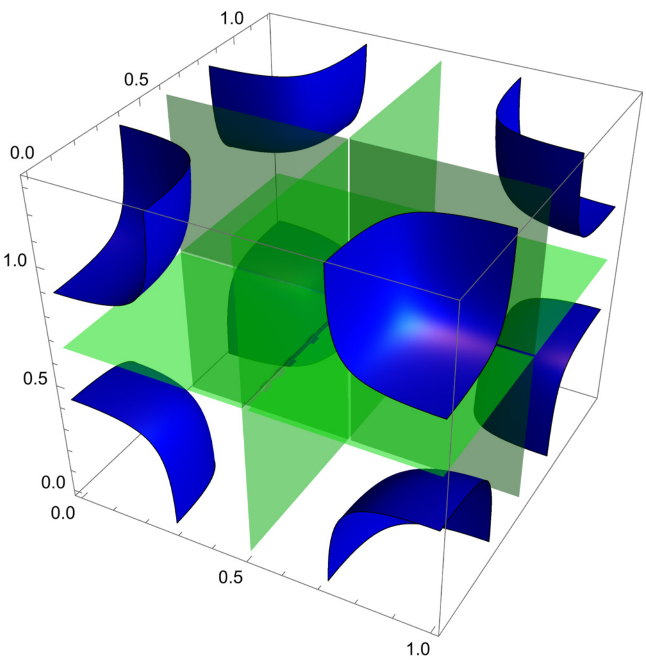

Fig. 2. The case $\lambda_{1,1,1}$.

6.2. *The case $N = 2$*

In this subsection we consider the case $N = 2$ and Σ_ε being of the form (1.6) with $\Sigma = B_1$ and $x_0 = 0$, i.e. $\Sigma_\varepsilon = B_\varepsilon$, proving the following asymptotic expansion for $\mathcal{T}_{\overline{\Omega}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n)$.

Proposition 6.3. *If $N = 2$ and the vanishing order of $\varphi_n - \varphi_n(0)$ at 0 is $k \geq 1$, then*

(i) *if $\varphi_n(0) \neq 0$ and 0 is a critical point of φ_n (hence, necessarily, $k = 2$), then*

$$\mathcal{T}_{\overline{\Omega}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) = \frac{\pi}{2}(\lambda_n - 1)^2(\varphi_n(0))^2\varepsilon^4|\log\varepsilon| + o(\varepsilon^4|\log\varepsilon|) \quad \text{as } \varepsilon \to 0;$$

(ii) *if either $\varphi_n(0) = 0$ or $\nabla\varphi_n(0) \neq (0,0)$, then*

$$\mathcal{T}_{\overline{\Omega}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) = \pi k \left(\left| \frac{\partial^k \varphi_n}{\partial x_1^k}(0) \right|^2 + \frac{1}{k^2} \left| \frac{\partial^k \varphi_n}{\partial x_1^{k-1}\partial x_2}(0) \right|^2 \right) \varepsilon^{2k} + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \to 0.$$

Let $k \geq 1$ be the vanishing order of $\varphi_n - \varphi_n(0)$ at 0. We observe that the polynomial $P_k^{\varphi_n}$ is harmonic in $\mathbb{R}^2$ in all cases except when $k = 2$ and $\varphi_n(0) \neq 0$. More precisely, recalling that in all critical points outside the nodal set $\varphi_n - \varphi_n(0)$ necessarily vanishes of order 2, we have

$$\Delta P_k^{\varphi_n} = \begin{cases} 0, & \text{if either } \varphi_n(0) = 0 \text{ or } \nabla\varphi_n(0) \neq (0,0), \\ (1-\lambda_n)\varphi_n(0), & \text{if } \varphi_n(0) \neq 0 \text{ and } 0 \text{ is a critical point of } \varphi_n. \end{cases} \tag{6.5}$$

By (2.16)-(2.17) we have, for all $j \geq 1$,

$$P_j^{\varphi_n}(r\cos t, r\sin t) = r^j f_j(t),$$

where

$$f_j(t) = \sum_{i=0}^{j} \frac{\partial^j \varphi_n}{\partial x_1^i \partial x_2^{j-i}}(0)(\cos t)^i (\sin t)^{j-i}.$$

Let us consider the Fourier coefficients of f_j:

$$a_i^j = \frac{1}{\pi}\int_0^{2\pi} f_j(t)\cos(it)\,\mathrm{d}x, \quad i \geq 0, \tag{6.6}$$

$$b_i^j = \frac{1}{\pi}\int_0^{2\pi} f_j(t)\sin(it)\,\mathrm{d}x, \quad i \geq 1. \tag{6.7}$$

We observe that

$$a_i^j = b_i^j = 0 \quad \text{if } i > j, \tag{6.8}$$

and, by the Divergence Theorem,

$$a_0^j = \frac{1}{\pi}\int_0^{2\pi} f_j(t)\,\mathrm{d}x = \frac{1}{\pi j}\int_{\partial B_1} \nabla P_j^{\varphi_n} \cdot \frac{x}{|x|}\,\mathrm{d}S = \frac{1}{\pi j}\int_{B_1} \Delta P_j^{\varphi_n}\,\mathrm{d}S.$$

Remark 6.4. In particular, for $j = k$ we have

$$a_0^k = \begin{cases} 0, & \text{if either } \varphi_n(0) = 0 \text{ or } \nabla\varphi_n(0) \neq (0,0), \\ \dfrac{1-\lambda_n}{k}\varphi_n(0), & \text{if } \varphi_n(0) \neq 0 \text{ and } 0 \text{ is a critical point of } \varphi_n. \end{cases} \tag{6.9}$$

Furthermore, if $j = k$ and if either $\varphi_n(0) = 0$ or $\nabla\varphi_n(0) \neq (0,0)$, then, by (6.5), $P_k^{\varphi_n}$ is harmonic and, consequently, there exist $c_1, c_2 \in \mathbb{R}$ such that $(c_1, c_2) \neq (0,0)$ and

$$P_k^{\varphi_n}(r\cos t, r\sin t) = r^k\big(c_1\cos(kt) + c_2\sin(kt)\big), \quad r \geq 0, \ t \in [0, 2\pi].$$

Since

$$P_k^{\varphi_n}(r\cos t, r\sin t) = r^k \sum_{i=0}^{k} \frac{\partial^k \varphi_n}{\partial x_1^i \partial x_2^{k-i}}(0)(\cos t)^i (\sin t)^{k-i},$$

direct computations yield

$$c_1 = \frac{\partial^k \varphi_n}{\partial x_1^k}(0) \quad \text{and} \quad c_2 = \frac{1}{k}\frac{\partial^k \varphi_n}{\partial x_1^{k-1}\partial x_2}(0). \tag{6.10}$$

Therefore, if either $\varphi_n(0) = 0$ or $\nabla\varphi_n(0) \neq (0,0)$, for $i \geq 1$ we have

$$a_i^k = \begin{cases} 0, & \text{if } i \neq k, \\ \frac{\partial^k \varphi_n}{\partial x_1^k}(0), & \text{if } i = k, \end{cases} \qquad b_i^k = \begin{cases} 0, & \text{if } i \neq k, \\ \frac{1}{k}\frac{\partial^k \varphi_n}{\partial x_1^{k-1}\partial x_2}(0), & \text{if } i = k. \end{cases} \tag{6.11}$$

For every $j \geq 1$, $R > 0$, and $\varepsilon \in (0, R)$, we define

$$T_{\varepsilon,R}^j = -2\inf\left\{ \frac{1}{2}\int_{B_R\setminus B_\varepsilon} |\nabla u|^2 \,\mathrm{d}x - \int_{\partial B_\varepsilon} (\partial_{\boldsymbol{\nu}} P_j^{\varphi_n})\, u \,\mathrm{d}S \colon u \in H^1(B_R \setminus \overline{B_\varepsilon}), \int_{B_R\setminus B_\varepsilon} u \,\mathrm{d}x = 0 \right\}.$$

The above infimum is achieved by a unique function $W_{\varepsilon,R,j} \in H^1(B_R \setminus \overline{B_\varepsilon})$ satisfying

$$\int_{B_R\setminus B_\varepsilon} W_{\varepsilon,R,j} \,\mathrm{d}x = 0,$$

and

$$\begin{aligned} \int_{B_R\setminus B_\varepsilon} \nabla W_{\varepsilon,R,j} \cdot \nabla v \,\mathrm{d}x &= \int_{\partial B_\varepsilon} \partial_{\boldsymbol{\nu}} P_j^{\varphi_n} \left(v - \frac{1}{|B_R \setminus B_\varepsilon|} \int_{B_R\setminus B_\varepsilon} v \,\mathrm{d}x \right) \mathrm{d}S \\ &= \int_{\partial B_\varepsilon} (\partial_{\boldsymbol{\nu}} P_j^{\varphi_n})\, v \,\mathrm{d}S + \frac{j a_0^j \varepsilon^j}{R^2 - \varepsilon^2} \int_{B_R\setminus B_\varepsilon} v \,\mathrm{d}x \end{aligned} \tag{6.12}$$

for every $v \in H^1(B_R \setminus \overline{B_\varepsilon})$, i.e. $W_{\varepsilon,R,j}$ is the unique zero-average weak solution to

$$\begin{cases} -\Delta W_{\varepsilon,R,j} = \frac{j a_0^j \varepsilon^j}{R^2 - \varepsilon^2}, & \text{in } B_R \setminus B_\varepsilon, \\ \partial_{\boldsymbol{\nu}} W_{\varepsilon,R,j} = 0, & \text{on } \partial B_R, \\ \partial_{\boldsymbol{\nu}} W_{\varepsilon,R,j} = \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}, & \text{on } \partial B_\varepsilon. \end{cases} \tag{6.13}$$

Lemma 6.5. *For every $j \geq 1$ and $R > 0$*

$$T^j_{\varepsilon,R} = \begin{cases} \frac{1}{2}\pi j^2 (a^j_0)^2 \varepsilon^{2j} |\log \varepsilon| + o(\varepsilon^{2j} |\log \varepsilon|), & \text{if } a^j_0 \neq 0, \\ \pi j^2 \left(\sum_{i=1}^{j} \frac{(a^j_i)^2 + (b^j_i)^2}{i} \right) \varepsilon^{2j} + o(\varepsilon^{2j}), & \text{if } a^j_0 = 0, \end{cases} \tag{6.14}$$

as $\varepsilon \to 0$, with a^j_i, b^j_i being as in (6.6)–(6.7). Moreover,

$$\int_{B_R \setminus B_\varepsilon} W^2_{\varepsilon,R,j} \, \mathrm{d}x = o(T^j_{\varepsilon,R}) \quad \text{as } \varepsilon \to 0. \tag{6.15}$$

Proof. For $j \geq 1$ and $R > 0$ fixed, let us expand $W_{\varepsilon,R,j}$ in Fourier series:

$$W_{\varepsilon,R,j}(r\cos t, r\sin t) = \frac{\varphi_{0,\varepsilon}(r)}{2} + \sum_{i=1}^{\infty} \Big(\varphi_{i,\varepsilon}(r)\cos(it) + \psi_{i,\varepsilon}(r)\sin(it) \Big),$$

where

$$\varphi_{i,\varepsilon}(r) = \frac{1}{\pi} \int_0^{2\pi} W_{\varepsilon,R,j}(r\cos t, r\sin t)\cos(it)\, \mathrm{d}x, \quad i \geq 0,$$

$$\psi_{i,\varepsilon}(r) = \frac{1}{\pi} \int_0^{2\pi} W_{\varepsilon,R,j}(r\cos t, r\sin t)\sin(it)\, \mathrm{d}x, \quad i \geq 1.$$

From (6.13) and the fact that $\int_{B_R \setminus B_\varepsilon} W_{\varepsilon,R,j} \, \mathrm{d}x = 0$ it follows that the function $\varphi_{0,\varepsilon}$ solves the problem

$$\begin{cases} -\varphi''_{0,\varepsilon}(r) - \frac{1}{r}\varphi'_{0,\varepsilon}(r) = \frac{2ja^j_0\varepsilon^j}{R^2 - \varepsilon^2}, & \text{in } (\varepsilon, R), \\ \varphi'_{0,\varepsilon}(\varepsilon) = j\varepsilon^{j-1}a^j_0, \\ \varphi'_{0,\varepsilon}(R) = 0, \\ \int_\varepsilon^R r\varphi_{0,\varepsilon}(r)\, \mathrm{d}r = 0, \end{cases} \tag{6.16}$$

while the functions $\varphi_{i,\varepsilon}$ and $\psi_{i,\varepsilon}$ with $i \geq 1$ solve

$$\begin{cases} -\varphi''_{i,\varepsilon}(r) - \frac{1}{r}\varphi'_{i,\varepsilon}(r) + \frac{i^2}{r^2}\varphi_{i,\varepsilon}(r) = 0, & \text{in } (\varepsilon, R), \\ \varphi'_{i,\varepsilon}(\varepsilon) = j\varepsilon^{j-1}a^j_i, \\ \varphi'_{i,\varepsilon}(R) = 0, \end{cases}$$

and

$$\begin{cases} -\psi_{i,\varepsilon}''(r) - \frac{1}{r}\psi_{i,\varepsilon}'(r) + \frac{i^2}{r^2}\psi_{i,\varepsilon}(r) = 0, & \text{in } (\varepsilon, R), \\ \psi_{i,\varepsilon}'(\varepsilon) = j\varepsilon^{j-1}b_i^j, \\ \psi_{i,\varepsilon}'(R) = 0, \end{cases}$$

respectively. For $i \geq 1$, direct computations yield

$$\varphi_{i,\varepsilon}(r) = -\frac{ja_i^j\varepsilon^{i+j}}{i(R^{2i} - \varepsilon^{2i})}(r^i + R^{2i}r^{-i}), \quad \psi_{i,\varepsilon}(r) = -\frac{jb_i^j\varepsilon^{i+j}}{i(R^{2i} - \varepsilon^{2i})}(r^i + R^{2i}r^{-i}). \tag{6.17}$$

In particular, by (6.8) we have $\varphi_{i,\varepsilon} \equiv \psi_{i,\varepsilon} \equiv 0$ if $i > j$. Moreover, the unique solution to (6.16) is

$$\varphi_{0,\varepsilon}(r) = \frac{ja_0^j\varepsilon^j}{1 - \left(\frac{\varepsilon}{R}\right)^2}\left(\log r - \frac{r^2}{2R^2} + \frac{1}{2} + \frac{\varepsilon^2\log\varepsilon - R^2\log R}{R^2 - \varepsilon^2} + \frac{1}{4R^2}(R^2 + \varepsilon^2)\right). \tag{6.18}$$

We conclude that the unique zero-average weak solution to (6.13) is given by

$$W_{\varepsilon,R,j}(r\cos t, r\sin t) = \frac{\varphi_{0,\varepsilon}(r)}{2} + \sum_{i=1}^{j}\Big(\varphi_{i,\varepsilon}(r)\cos(it) + \psi_{i,\varepsilon}(r)\sin(it)\Big),$$

with $\varphi_{0,\varepsilon}$ as in (6.18) and $\varphi_{i,\varepsilon}, \psi_{i,\varepsilon}$ as in (6.17). Furthermore, by Parseval's Theorem,

$$\begin{aligned} T_{\varepsilon,R}^j &= \int_{\partial B_\varepsilon} (\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}) W_{\varepsilon,R,j}\, \mathrm{d}S \\ &= \varepsilon\int_0^{2\pi} \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}(\varepsilon\cos t, \varepsilon\sin t) W_{\varepsilon,R,j}(\varepsilon\cos t, \varepsilon\sin t)\, dt \\ &= \varepsilon(-j\varepsilon^{j-1})\pi\left(\frac{a_0^j\varphi_{0,\varepsilon}(\varepsilon)}{2} + \sum_{i=1}^{j}(a_i^j\varphi_{i,\varepsilon}(\varepsilon) + b_i^j\psi_{i,\varepsilon}(\varepsilon))\right). \end{aligned}$$

We observe that, by (6.18),

$$\frac{a_0^j\varphi_{0,\varepsilon}(\varepsilon)}{2} \sim \frac{1}{2}j(a_0^j)^2\varepsilon^j\log\varepsilon \quad \text{as } \varepsilon \to 0,$$

while (6.17) implies

$$\sum_{i=1}^{j}(a_i^j\varphi_{i,\varepsilon}(\varepsilon) + b_i^j\psi_{i,\varepsilon}(\varepsilon)) \sim -j\varepsilon^j\sum_{i=1}^{j}\frac{(a_i^j)^2 + (b_i^j)^2}{i} \quad \text{as } \varepsilon \to 0,$$

thus proving (6.14).

By Parseval's Theorem we have

$$\int_{B_R\setminus B_\varepsilon} W^2_{\varepsilon,R,j}\,\mathrm{d}x = \pi\int_\varepsilon^R r\left(\frac{\varphi^2_{0,\varepsilon}(r)}{2} + \sum_{i=1}^j \left(\varphi^2_{i,\varepsilon}(r)+\psi^2_{i,\varepsilon}(r)\right)\mathrm{d}r\right). \tag{6.19}$$

By (6.14) and (6.18)

$$\int_\varepsilon^R r\varphi^2_{0,\varepsilon}(r)\,\mathrm{d}r = \begin{cases} 0, & \text{if } a_0^j = 0,\\ O(\varepsilon^{2j}) = o(T^j_{\varepsilon,R}), & \text{if } a_0^j \neq 0,\end{cases}$$
$$= o(T^j_{\varepsilon,R}) \quad \text{as } \varepsilon\to 0,$$

and, by (6.14) and (6.17),

$$\int_\varepsilon^R r(\varphi^2_{i,\varepsilon}(r)+\psi^2_{i,\varepsilon}(r))\,\mathrm{d}r = \frac{j^2\varepsilon^{2i+2j}((a_i^j)^2+(b_i^j)^2)}{i^2(R^{2i}-\varepsilon^{2i})^2}\int_\varepsilon^R r(r^{2i}+R^{4i}r^{-2i}+2R^{2i})\,\mathrm{d}r$$
$$= \frac{j^2\varepsilon^{2i+2j}((a_i^j)^2+(b_i^j)^2)}{i^2(R^{2i}-\varepsilon^{2i})^2}\left(\frac{R^{2i+2}-\varepsilon^{2i+2}}{2i+2} + R^{4i}\frac{R^{2-2i}-\varepsilon^{2-2i}}{2-2i} + R^{2i}(R^2-\varepsilon^2)\right)$$
$$= O(\varepsilon^{2j+2}) = o(T^j_{\varepsilon,R}) \quad \text{as } \varepsilon\to 0,$$

if $i\geq 2$, while, for $i=1$,

$$\int_\varepsilon^R r(\varphi^2_{1,\varepsilon}(r)+\psi^2_{1,\varepsilon}(r))\,\mathrm{d}r = \frac{j^2\varepsilon^{2+2j}((a_1^j)^2+(b_1^j)^2)}{(R^2-\varepsilon^2)^2}\int_\varepsilon^R r(r^2+R^4r^{-2}+2R^2)\,\mathrm{d}r$$
$$= \frac{j^2\varepsilon^{2+2j}((a_1^j)^2+(b_1^j)^2)}{(R^2-\varepsilon^2)^2}\left(\frac{R^4-\varepsilon^4}{4} + R^4(\log R-\log\varepsilon) + R^2(R^2-\varepsilon^2)\right)$$
$$= O(\varepsilon^{2j+2}|\log\varepsilon|) = o(T^j_{\varepsilon,R}) \quad \text{as } \varepsilon\to 0.$$

Therefore (6.15) follows from (6.19). □

Remark 6.6. In view of (6.9) and (6.11), in the case $j=k$ Lemma 6.5 provides the following information:

(i) if $\varphi_n(0)\neq 0$ and 0 is a critical point of φ_n (hence, necessarily, $k=2$), then

$$T^k_{\varepsilon,R} = T^2_{\varepsilon,R} = \frac{\pi}{2}(\lambda_n-1)^2(\varphi_n(0))^2\varepsilon^4|\log\varepsilon| + o(\varepsilon^4|\log\varepsilon|) \quad \text{as } \varepsilon\to 0;$$

(ii) if either $\varphi_n(0) = 0$ or $\nabla\varphi_n(0) \neq (0,0)$, then

$$T^k_{\varepsilon,R} = \pi k \left(\left| \frac{\partial^k \varphi_n}{\partial x_1^k}(0) \right|^2 + \frac{1}{k^2} \left| \frac{\partial^k \varphi_n}{\partial x_1^{k-1}\partial x_2}(0) \right|^2 \right) \varepsilon^{2k} + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \to 0.$$

Lemma 6.7. *For every $j \geq 1$ and $R > 0$,*

$$\mathcal{T}_{\overline{B_R}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}) = T^j_{\varepsilon,R} + o(T^j_{\varepsilon,R}) \quad \textit{as } \varepsilon \to 0.$$

Proof. By (3.4) we have

$$\begin{aligned}
&\int_{B_R\setminus B_\varepsilon} (\nabla U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} \cdot \nabla W_{\varepsilon,R,j} + U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} W_{\varepsilon,R,j}) \,\mathrm{d}x \\
&\quad = \int_{\partial B_\varepsilon} (\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}) W_{\varepsilon,R,j} \,\mathrm{d}S = T^j_{\varepsilon,R},
\end{aligned}$$

while (6.12) and Remark 3.5 yield

$$\begin{aligned}
&\int_{B_R\setminus B_\varepsilon} \nabla W_{\varepsilon,R,j} \cdot \nabla U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} \,\mathrm{d}x \\
&\quad = \int_{\partial B_\varepsilon} (\partial_{\boldsymbol{\nu}} P_j^{\varphi_n})\, U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} \,\mathrm{d}S + \frac{j a_0^j \varepsilon^j}{R^2-\varepsilon^2} \int_{B_R\setminus B_\varepsilon} U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} \,\mathrm{d}x \\
&\quad = \mathcal{T}_{\overline{B_R}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}) + \frac{j a_0^j \varepsilon^j}{R^2-\varepsilon^2} \int_{B_R\setminus B_\varepsilon} U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} \,\mathrm{d}x.
\end{aligned}$$

From the above identities we deduce that

$$\begin{aligned}
&\mathcal{T}_{\overline{B_R}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}) - T^j_{\varepsilon,R} \\
&\qquad = -\int_{B_R\setminus B_\varepsilon} U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} W_{\varepsilon,R,j} \,\mathrm{d}x - \frac{j a_0^j \varepsilon^j}{R^2-\varepsilon^2} \int_{B_R\setminus B_\varepsilon} U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} \,\mathrm{d}x.
\end{aligned}$$

From Cauchy-Schwarz's inequality, (6.15), and Lemma 3.6 (Remark 5.5 guaranteeing the validity of assumption $\lim_{\varepsilon\to 0} \mathcal{T}_{\overline{B_R}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}) = 0$) it follows that

$$\int_{B_R\setminus B_\varepsilon} U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} W_{\varepsilon,R,j} \,\mathrm{d}x = o\left(\sqrt{T^j_{\varepsilon,R}\, \mathcal{T}_{\overline{B_R}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n})} \right)$$

as $\varepsilon \to 0$. Moreover, since $\varepsilon^{2j} = O(T^j_{\varepsilon,R})$ as $\varepsilon \to 0$ in view of (6.14), from Cauchy-Schwarz's inequality and Lemma 3.6 we deduce that

$$\varepsilon^j \left| \int_{B_R \setminus B_\varepsilon} U_{B_R, B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} \, dx \right| \leq \varepsilon^j \sqrt{\pi(R^2 - \varepsilon^2)} \sqrt{\int_{B_R \setminus B_\varepsilon} U^2_{B_R, B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}} \, dx}$$
$$= O\left(\sqrt{T^j_{\varepsilon,R}}\right) o\left(\sqrt{\mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n})}\right)$$
$$= o\left(\sqrt{T^j_{\varepsilon,R} \mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n})}\right)$$

as $\varepsilon \to 0$. Hence $\mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}) = T^j_{\varepsilon,R} + o(T^j_{\varepsilon,R}) + o(\mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}))$, i.e.

$$(1 + o(1))\mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}) = (1 + o(1))T^j_{\varepsilon,R} \quad \text{as } \varepsilon \to 0.$$

The lemma is thereby proved. $\square$

Combining Lemmas 6.7 and (6.14) we derive the following asymptotic expansion as $\varepsilon \to 0$

$$\mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_j^{\varphi_n}) = \begin{cases} \frac{1}{2}\pi j^2 (a_0^j)^2 \varepsilon^{2j} |\log \varepsilon| + o(\varepsilon^{2j} |\log \varepsilon|), & \text{if } a_0^j \neq 0, \\ \pi j^2 \left(\sum_{i=1}^{j} \frac{(a_i^j)^2 + (b_i^j)^2}{i} \right) \varepsilon^{2j} + o(\varepsilon^{2j}), & \text{if } a_0^j = 0, \end{cases} \tag{6.20}$$

for all $j \geq 1$. If $j = k$, in view of Remark 6.6, we have, more precisely,

(i) if $\varphi_n(0) \neq 0$ and $\nabla \varphi_n(0) = (0, 0)$ (hence, necessarily, $k = 2$), then, as $\varepsilon \to 0$,

$$\mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_k^{\varphi_n}) = \frac{\pi}{2}(\lambda_n - 1)^2 (\varphi_n(0))^2 \varepsilon^4 |\log \varepsilon| + o(\varepsilon^4 |\log \varepsilon|); \tag{6.21}$$

(ii) if either $\varphi_n(0) = 0$ or $\nabla \varphi_n(0) \neq (0, 0)$, then, as $\varepsilon \to 0$,

$$\mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_k^{\varphi_n}) = \pi k \left(\left| \frac{\partial^k \varphi_n}{\partial x_1^k}(0) \right|^2 + \frac{1}{k^2} \left| \frac{\partial^k \varphi_n}{\partial x_1^{k-1} \partial x_2}(0) \right|^2 \right) \varepsilon^{2k} + o(\varepsilon^{2k}). \tag{6.22}$$

Lemma 6.8. *For every $R > 0$*

$$\mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} \varphi_n)$$
$$= \mathcal{T}_{\overline{B_R} \setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}} P_k^{\varphi_n}) + \begin{cases} O(\varepsilon^{9/2} |\log \varepsilon|^{3/4}), & \text{if } \varphi_n(0) \neq 0 \text{ and } \nabla \varphi_n(0) = (0, 0), \\ O(\varepsilon^{2k + \frac{1}{2}}), & \text{if either } \varphi_n(0) = 0 \text{ or } \nabla \varphi_n(0) \neq (0, 0), \end{cases}$$

$$= \begin{cases} \frac{\pi}{2}(\lambda_n - 1)^2(\varphi_n(0))^2\varepsilon^4|\log\varepsilon| + o(\varepsilon^4|\log\varepsilon|), & \textit{if } \varphi_n(0) \neq 0 \textit{ and} \\ & \nabla\varphi_n(0) = (0,0), \\ \pi k\Big(\big|\frac{\partial^k\varphi_n}{\partial x_1^k}(0)\big|^2 + \frac{1}{k^2}\big|\frac{\partial^k\varphi_n}{\partial x_1^{k-1}\partial x_2}(0)\big|^2\Big)\varepsilon^{2k} + o(\varepsilon^{2k}), & \textit{if either } \varphi_n(0) = 0 \textit{ or} \\ & \nabla\varphi_n(0) \neq (0,0), \end{cases}$$

as $\varepsilon \to 0$.

Proof. We first observe that, if $N = 2$, by Lemma A.2 and Sobolev trace theorems, there exists $C_R > 0$ (depending on R but independent of ε) such that

$$\int_{\partial B_\varepsilon} u^2 \, dS \leq \frac{C_R}{\varepsilon}\|u\|^2_{H^1(B_R\setminus\overline{B_\varepsilon})} \quad \text{for all } u \in H^1(B_R \setminus \overline{B_\varepsilon}). \tag{6.23}$$

Let $\Psi_\varepsilon = U_{B_R,B_\varepsilon,\partial_{\boldsymbol\nu}\varphi_n} - U_{B_R,B_\varepsilon,\partial_{\boldsymbol\nu}P_k^{\varphi_n}} - U_{B_R,B_\varepsilon,\partial_{\boldsymbol\nu}P_{k+1}^{\varphi_n}}$. From (3.4) it follows that

$$\int_{B_R\setminus B_\varepsilon} (\nabla\Psi_\varepsilon\cdot\nabla v + \Psi_\varepsilon v)\, dx = \int_{\partial B_\varepsilon} v\,\partial_{\boldsymbol\nu}(\varphi_n - P_k^{\varphi_n} - P_{k+1}^{\varphi_n})\, dS$$

for every $v \in H^1(B_R \setminus \overline{B_\varepsilon})$, so that (6.23) yields

$$\begin{aligned} \int_{B_R\setminus B_\varepsilon} (|\nabla\Psi_\varepsilon|^2 + \Psi_\varepsilon^2)\, dx &= \int_{\partial B_\varepsilon} \Psi_\varepsilon\,\partial_{\boldsymbol\nu}(\varphi_n - P_k^{\varphi_n} - P_{k+1}^{\varphi_n})\, dS \leq \text{const}\,\varepsilon^{k+\frac{3}{2}}\sqrt{\int_{\partial B_\varepsilon}\Psi_\varepsilon^2\, dS} \\ &\leq \text{const}\,\varepsilon^{k+1}\|\Psi_\varepsilon\|_{H^1(B_R\setminus\overline{B_\varepsilon})} \end{aligned}$$

for some const > 0 independent of ε which varies from line to line. Hence

$$\|\Psi_\varepsilon\|_{H^1(B_R\setminus\overline{B_\varepsilon})} = O(\varepsilon^{k+1}) \quad \text{as } \varepsilon \to 0. \tag{6.24}$$

From (6.24), Remark 3.5, (6.20), (6.21), and (6.22) it follows that

$$\begin{aligned} &\|U_{B_R,B_\varepsilon,\partial_{\boldsymbol\nu}\varphi_n}\|_{H^1(B_R\setminus\overline{B_\varepsilon})} \\ &\quad\leq \|\Psi_\varepsilon\|_{H^1(B_R\setminus\overline{B_\varepsilon})} + \|U_{B_R,B_\varepsilon,\partial_{\boldsymbol\nu}P_k^{\varphi_n}}\|_{H^1(B_R\setminus\overline{B_\varepsilon})} + \|U_{B_R,B_\varepsilon,\partial_{\boldsymbol\nu}P_{k+1}^{\varphi_n}}\|_{H^1(B_R\setminus\overline{B_\varepsilon})} \\ &\quad= \begin{cases} O(\varepsilon^2|\log\varepsilon|^{1/2}), & \text{if } \varphi_n(0) \neq 0 \text{ and } \nabla\varphi_n(0) = (0,0), \\ O(\varepsilon^k), & \text{if either } \varphi_n(0) = 0 \text{ or } \nabla\varphi_n(0) \neq (0,0), \end{cases} \end{aligned} \tag{6.25}$$

as $\varepsilon \to 0$. Cauchy-Schwarz's inequality and estimates (6.24)–(6.25), (6.20), (6.21), and (6.22) imply

$$\begin{aligned}
&\|U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}\varphi_n} - U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}}\|^2_{H^1(B_R\setminus\overline{B_\varepsilon})} - \|U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_{k+1}^{\varphi_n}}\|^2_{H^1(B_R\setminus\overline{B_\varepsilon})} \\
&\quad = (\Psi_\varepsilon, U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}\varphi_n} - U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}} + U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_{k+1}^{\varphi_n}})_{H^1(B_R\setminus\overline{B_\varepsilon})} \\
&\quad \le \|\Psi_\varepsilon\|_{H^1(B_R\setminus\overline{B_\varepsilon})}\Big(\|U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}\varphi_n}\|_{H^1(B_R\setminus\overline{B_\varepsilon})} + \|U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}}\|_{H^1(B_R\setminus\overline{B_\varepsilon})} \\
&\qquad\qquad + \|U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_{k+1}^{\varphi_n}}\|_{H^1(B_R\setminus\overline{B_\varepsilon})}\Big) \\
&\quad = \begin{cases} O(\varepsilon^5|\log\varepsilon|^{1/2}), & \text{if } \varphi_n(0)\neq 0 \text{ and } \nabla\varphi_n(0)=(0,0), \\ O(\varepsilon^{2k+1}), & \text{if either } \varphi_n(0)=0 \text{ or } \nabla\varphi_n(0)\neq(0,0), \end{cases}
\end{aligned}$$

as $\varepsilon \to 0$. Hence, in view of (6.20),

$$\begin{aligned}
&\|U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}\varphi_n} - U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}}\|_{H^1(B_R\setminus\overline{B_\varepsilon})} \\
&\qquad = \begin{cases} O(\varepsilon^{5/2}|\log\varepsilon|^{1/4}), & \text{if } \varphi_n(0)\neq 0 \text{ and } \nabla\varphi_n(0)=(0,0), \\ O(\varepsilon^{k+\frac12}), & \text{if either } \varphi_n(0)=0 \text{ or } \nabla\varphi_n(0)\neq(0,0), \end{cases}
\end{aligned} \tag{6.26}$$

as $\varepsilon \to 0$. From Remark 3.5, Cauchy-Schwarz's inequality, (6.26), (6.25), (6.21), and (6.22) it follows that

$$\begin{aligned}
&\mathcal{T}_{\overline{B_R}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) - \mathcal{T}_{\overline{B_R}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}}P_k^{\varphi_n}) \\
&\quad = \|U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}\varphi_n}\|^2_{H^1(B_R\setminus\overline{B_\varepsilon})} - \|U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}}\|^2_{H^1(B_R\setminus\overline{B_\varepsilon})} \\
&\quad = (U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}\varphi_n} - U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}}, U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}\varphi_n} + U_{B_R,B_\varepsilon,\partial_{\boldsymbol{\nu}}P_k^{\varphi_n}})_{H^1(B_R\setminus\overline{B_\varepsilon})} \\
&\quad = \begin{cases} O(\varepsilon^{9/2}|\log\varepsilon|^{3/4}), & \text{if } \varphi_n(0)\neq 0 \text{ and } \nabla\varphi_n(0)=(0,0), \\ O(\varepsilon^{2k+\frac12}), & \text{if either } \varphi_n(0)=0 \text{ or } \nabla\varphi_n(0)\neq(0,0), \end{cases}
\end{aligned}$$

as $\varepsilon \to 0$, thus completing the proof in view of (6.21) and (6.22). □

Proof of Proposition 6.3. Since $0 \in \Omega$, there exist $R_1, R_2 > 0$ such that $B_{R_1} \subset \Omega \subset B_{R_2}$. From Corollary 3.2 it follows that

$$\mathcal{T}_{\overline{B_{R_2}}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) \le \mathcal{T}_{\overline{\Omega}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n) \le \mathcal{T}_{\overline{B_{R_1}}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n),$$

so that the conclusion follows from Lemma 6.8. □

Proposition 6.9. *Let $N = 2$.*

(i) *If $0 \in \Omega \setminus \operatorname{Sing}(\varphi_n)$, then*

$$\int_{B_\varepsilon} \Big(|\nabla\varphi_n|^2 - (\lambda_n - 1)\varphi_n^2\Big)\, dx = \pi\varepsilon^2\Big(|\nabla\varphi_n(0)|^2 - (\lambda_n - 1)|\varphi_n(0)|^2\Big) + o(\varepsilon^2)$$

as $\varepsilon \to 0$.

(ii) *If* $0 \in \operatorname{Sing}(\varphi_n)$, *then*

$$\int_{B_\varepsilon} \left(|\nabla \varphi_n|^2 - (\lambda_n - 1)\varphi_n^2\right) \mathrm{d}x = k\pi\varepsilon^{2k}\left(\left|\frac{\partial^k \varphi_n}{\partial x_1^k}(0)\right|^2 + \frac{1}{k^2}\left|\frac{\partial^k \varphi_n}{\partial x_1^{k-1}\partial x_2}(0)\right|^2\right) + o(\varepsilon^{2k})$$

as $\varepsilon \to 0$, *where* $k \geq 2$ *is the vanishing order at* 0 *of* $\varphi_n - \varphi_n(0)$.

Proof. If $0 \notin \operatorname{Sing}(\varphi_n)$, we can argue as in (5.17) to deduce (i).

Let $0 \in \operatorname{Sing}(\varphi_n)$. In this case $P_k^{\varphi_n}(r\cos t, r\sin t) = r^k\big(c_1\cos(kt) + c_2\sin(kt)\big)$ with c_1, c_2 as in (6.10), see Remark 6.4. Then

$$\begin{aligned}\int_{B_\varepsilon} \left(|\nabla \varphi_n|^2 - (\lambda_n - 1)\varphi_n^2\right) \mathrm{d}x &= -\int_{\partial B_\varepsilon} \varphi_n \partial_{\boldsymbol{\nu}} \varphi_n \, \mathrm{d}S = -\int_{\partial B_\varepsilon} P_k^{\varphi_n} \partial_{\boldsymbol{\nu}} P_k^{\varphi_n} \, \mathrm{d}S + o(\varepsilon^{2k}) \\ &= k\varepsilon^{2k}\int_0^{2\pi} (c_1\cos(kt) + c_2\sin(kt))^2 \, \mathrm{d}t + o(\varepsilon^{2k}) = k\pi\varepsilon^{2k}(c_1^2 + c_2^2) + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \to 0,\end{aligned}$$

thus proving (ii). □

We are now in position to prove Theorem 2.12.

Proof of Theorem 2.12. By translation, it is not restrictive to assume $x_0 = 0$. The conclusion follows from Theorem 2.4, expanding the torsional rigidity $\mathcal{T}_{\overline{\Omega}\setminus B_\varepsilon}(\partial B_\varepsilon, \partial_{\boldsymbol{\nu}}\varphi_n)$ as in Proposition 6.3 and $\int_{B_\varepsilon}\left(|\nabla \varphi_n|^2 - (\lambda_n - 1)\varphi_n^2\right) \mathrm{d}x$ as in Proposition 6.9. □

Example 6.10. We conclude this section with an example, in which the hole is excised from a disk. To this end, let us take $\Omega = B_2 \subset \mathbb{R}^2$. It is well known (see, e.g., [21]) that the eigenvalues of the unperturbed Neumann problem (2.1) are

$$\lambda_{nk} = \frac{\alpha_{nk}^2}{4} + 1,$$

α_{nk} being the positive roots, enumerated by k, of $J_n'(z)$, where $J_n(z)$ is the Bessel function of the first kind of order n. These eigenvalues are all simple for $n = 0$. In this case, the eigenfunctions read

$$\varphi_k(r, \theta) = J_0\left(\alpha_{0k}\frac{r}{2}\right).$$

Therefore, the 2-dimensional analogue of the interface Γ introduced in Remark 2.10 is characterized by the equation

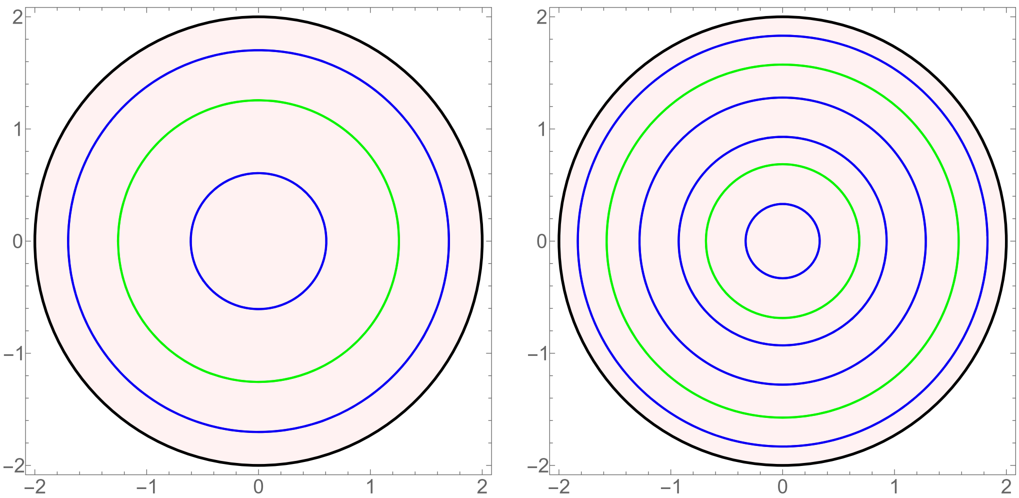

Fig. 3. Interface Γ and nodal lines of the eigenfunction for the cases α_{01} (left) and α_{02} (right).

$$2J_1^2\left(\alpha_{0k}\frac{r}{2}\right) - J_0^2\left(\alpha_{0k}\frac{r}{2}\right) = 0.$$

Relying again on the computational software Mathematica™ we can plot the interface (in blue), along with the nodal lines of φ_k (in green), for the cases

$$\alpha_{01} \approx 3.831, \qquad \alpha_{02} \approx 7.016.$$

The results can be seen in Fig. 3.

Acknowledgments

R. Ognibene is supported by the project ERC VAREG - *Variational approach to the regularity of the free boundaries* (grant agreement No. 853404) and by the INdAM-GNAMPA project 2023 CUP E53C22001930001. V. Felli is supported by the PRIN project no. 20227HX33Z - *Pattern formation in nonlinear phenomena*. Part of this work was developed while V. Felli and R. Ognibene were in residence at Institut Mittag-Leffler in Djursholm, Stockholm (Sweden) during the semester *Geometric Aspects of Nonlinear Partial Differential Equations* in 2022, supported by the Swedish Research Council under grant no. 2016-06596. L. Liverani is supported by the Alexander von Humboldt fellowship for Postdoctoral Researchers.

Appendix A

We recall here a known result about approximation of small eigenvalues of linear operators. This lemma, originally proved by Y. Colin de Verdiére in [15] and then revisited in [16] and [3], also applies to multiple eigenvalues. We present here a simplified version applicable to the case of simple eigenvalues and provide a short proof for the readers' convenience.

Lemma A.1 *(Lemma on small eigenvalues). Let $(\mathcal{H}, (\cdot,\cdot))$ be a real Hilbert space, $\mathcal{D} \subseteq \mathcal{H}$ a subspace, and $q\colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ a bilinear symmetric form. Let*

(i) *$\lambda \in \mathbb{R}$ and $\phi \in \mathcal{D}$ be such that*

$$\|\phi\| = 1 \quad \textit{and} \quad q(\phi, v) = \lambda(\phi, v) \quad \textit{for all } v \in \mathcal{D},$$

where $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ denotes the norm associated to the scalar product;

(ii) *$f \in \mathcal{D}$ be such that $\|f\| = 1$.*

Let us assume that $\{\phi\}^\perp = H_1 \oplus H_2$ for some subspaces H_1, H_2 mutually orthogonal such that $H_1 \subset \mathcal{D}$,

$$q(v_1, v_2) = 0 \quad \textit{for all } v_1 \in H_1 \textit{ and } v_2 \in H_2 \cap \mathcal{D}, \tag{A.1}$$

$$\gamma_1 := \inf\left\{ \frac{|q(v,v)|}{\|v\|^2} \colon v \in H_1 \setminus \{0\} \right\} > 0, \tag{A.2}$$

$$\gamma_2 := \inf\left\{ \frac{|q(v,v)|}{\|v\|^2} \colon v \in (H_2 \cap \mathcal{D}) \setminus \{0\} \right\} > 0, \tag{A.3}$$

and

$$\delta := \sup\left\{ \frac{|q(f,v)|}{\|v\|} \colon v \in \mathcal{D} \setminus \{0\} \right\} < +\infty. \tag{A.4}$$

Then

$$\|f - \Pi f\| \le \frac{\sqrt{2}\,\delta}{\gamma}, \tag{A.5}$$

where $\gamma := \min\{\gamma_1, \gamma_2\}$ and Π denotes the orthogonal projection onto span$\{\phi\}$, *i.e.*

$$\begin{aligned} \Pi\colon \mathcal{H} &\to \operatorname{span}\{\phi\} \\ v &\mapsto (\phi, v)\,\phi. \end{aligned}$$

Finally, if $\xi := q(f,f)$, then

$$|\lambda - \xi| \le 2|\lambda|\,\frac{\delta^2}{\gamma^2} + 2\,\frac{\delta^2}{\gamma}. \tag{A.6}$$

Proof. Let us denote

$$\mathsf{N}f := f - \Pi f.$$

We observe that $\mathsf{N}f$ is orthogonal to ϕ, i.e.

$$(\phi, \mathsf{N}f) = 0,$$

and, letting

$$\mathsf{N}_1 : H \to H_1 \quad \text{and} \quad \mathsf{N}_2 : H \to H_2$$

be the orthogonal projections on H_1 and H_2, respectively, we have

$$\mathsf{N}f = \mathsf{N}_1 f + \mathsf{N}_2 f. \tag{A.7}$$

Moreover

$$(\phi, \mathsf{N}_1 f) = (\phi, \mathsf{N}_2 f) = 0. \tag{A.8}$$

Since $H_1 \subset \mathcal{D}$ by assumption, we have $\mathsf{N}_1 f \in \mathcal{D}$; moreover $\mathsf{N}f \in \mathcal{D}$, hence $\mathsf{N}_2 f \in \mathcal{D} \cap H_2$ by (A.7). Therefore, taking into account (A.8),

$$q(\phi, \mathsf{N}_1 f) = \lambda(\phi, \mathsf{N}_1 f) = 0, \quad q(\phi, \mathsf{N}_2 f) = \lambda(\phi, \mathsf{N}_2 f) = 0,$$

so that

$$q(\Pi f, \mathsf{N}_1 f) = q(\Pi f, \mathsf{N}_2 f) = 0. \tag{A.9}$$

From (A.1), (A.7), and (A.9) it follows that

$$\begin{aligned} q(\mathsf{N}_1 f, \mathsf{N}_1 f) &= q(f - \Pi f - \mathsf{N}_2 f, \mathsf{N}_1 f) = q(f, \mathsf{N}_1 f) - q(\Pi f, \mathsf{N}_1 f) - q(\mathsf{N}_2 f, \mathsf{N}_1 f) \\ &= q(f, \mathsf{N}_1 f) \\ q(\mathsf{N}_2 f, \mathsf{N}_2 f) &= q(f - \Pi f - \mathsf{N}_1 f, \mathsf{N}_2 f) = q(f, \mathsf{N}_2 f) - q(\Pi f, \mathsf{N}_2 f) - q(\mathsf{N}_1 f, \mathsf{N}_2 f) \\ &= q(f, \mathsf{N}_2 f). \end{aligned}$$

Therefore, from the definition of δ, γ_1, and γ_2 we obtain

$$|q(\mathsf{N}_1 f, \mathsf{N}_1 f)| = |q(f, \mathsf{N}_1 f)| \le \delta \, \|\mathsf{N}_1 f\| \le \delta \sqrt{\frac{|q(\mathsf{N}_1 f, \mathsf{N}_1 f)|}{\gamma_1}},$$

$$|q(\mathsf{N}_2 f, \mathsf{N}_2 f)| = |q(f, \mathsf{N}_2 f)| \le \delta \, \|\mathsf{N}_2 f\| \le \delta \sqrt{\frac{|q(\mathsf{N}_2 f, \mathsf{N}_2 f)|}{\gamma_2}},$$

which yields

$$|q(\mathsf{N}_1 f, \mathsf{N}_1 f)| \le \frac{\delta^2}{\gamma_1} \quad |q(\mathsf{N}_1 f, \mathsf{N}_2 f)| \le \frac{\delta^2}{\gamma_2}. \tag{A.10}$$

Combining (A.10) with the definition of $\gamma_1, \gamma_2, \gamma$, we obtain the estimates

$$\|\mathsf{N}_1 f\|^2 \le \frac{|q(\mathsf{N}_1 f, \mathsf{N}_1 f)|}{\gamma_1} \le \frac{\delta^2}{\gamma_1^2} \le \frac{\delta^2}{\gamma^2}, \quad \|\mathsf{N}_2 f\|^2 \le \frac{|q(\mathsf{N}_2 f, \mathsf{N}_2 f)|}{\gamma_2} \le \frac{\delta^2}{\gamma_2^2} \le \frac{\delta^2}{\gamma^2}. \tag{A.11}$$

From (A.7), the orthogonality of H_1 and H_2 and (A.11), we deduce that

$$\|\mathsf{N} f\|^2 = \|\mathsf{N}_1 f\|^2 + \|\mathsf{N}_2 f\|^2 \le \frac{2\delta^2}{\gamma^2},$$

thus proving (A.5).

Now, the proof of (A.6) follows from direct estimates, making use of (A.5) and (A.10). More precisely, if $\Pi f \neq 0$, we first write $|\lambda - \xi|$ as

$$|\lambda - \xi| = \left| \frac{q(\Pi f, \Pi f)}{\|\Pi f\|^2} - \frac{q(f,f)}{\|f\|^2} \right| = \left| \frac{q(\Pi f, \Pi f)}{\|\Pi f\|^2} - \frac{q(\mathsf{N} f + \Pi f, \mathsf{N} f + \Pi f)}{\|\mathsf{N} f + \Pi f\|^2} \right|.$$

Then, by this, the orthogonality condition (A.9), assumption (A.1), and the fact that $\|f\| = 1$, we obtain

$$|\lambda - \xi| = \left| \lambda \|\mathsf{N} f\|^2 - q(\mathsf{N}_1 f, \mathsf{N}_1 f) - q(\mathsf{N}_2 f, \mathsf{N}_2 f) \right|. \tag{A.12}$$

On the other hand, (A.12) is trivially satisfied if $\Pi f = 0$, since, in this case, $f = \mathsf{N} f$. Combining (A.12) with the triangle inequality, (A.5) and (A.10), we obtain (A.6). □

The following lemma provides an uniform extension property in domains with small holes of the form (1.6), see [41] for the proof.

Lemma A.2 *(Extension operators). For $N \ge 2$, let $\Omega \subset \mathbb{R}^N$ and $\Sigma \subset \mathbb{R}^N$ be bounded, open Lipschitz sets. Let $\varepsilon_0 > 0$ and $r_0 > 0$ be such that (2.15) is satisfied for some $x_0 \in \Omega$. For every $\varepsilon \in (0, \varepsilon_0)$, let $\Sigma_\varepsilon := x_0 + \varepsilon\Sigma$ and $\Omega_\varepsilon = \Omega \setminus \overline{\Sigma_\varepsilon}$. Then, for every $\varepsilon \in (0, \varepsilon_0)$, there exists an (inner) extension operator*

$$\mathsf{E}_\varepsilon \colon H^1(\Omega_\varepsilon) \to H^1(\Omega)$$

such that, for all $u \in H^1(\Omega_\varepsilon)$,

$$(\mathsf{E}_\varepsilon u)|_{\Omega_\varepsilon} = u$$

and

$$\|\mathsf{E}_\varepsilon u\|_{H^1(\Omega)} \leq \mathfrak{C}\, \|u\|_{H^1(\Omega_\varepsilon)}\,,$$

for some constant $\mathfrak{C} > 0$ *independent of* $\varepsilon \in (0, \varepsilon_0)$.

Data availability

No data was used for the research described in the article.

References

[1] L. Abatangelo, V. Felli, L. Hillairet, C. Léna, Spectral stability under removal of small capacity sets and applications to Aharonov-Bohm operators, J. Spectr. Theory 9 (2) (2019) 379–427, https://doi.org/10.4171/JST/251.

[2] L. Abatangelo, V. Felli, B. Noris, On simple eigenvalues of the fractional Laplacian under removal of small fractional capacity sets, Commun. Contemp. Math. 22 (8) (2020) 32, https://doi.org/10.1142/S0219199719500718.

[3] L. Abatangelo, C. Léna, P. Musolino, Ramification of multiple eigenvalues for the Dirichlet-Laplacian in perforated domains, J. Funct. Anal. 283 (12) (2022) 50, https://doi.org/10.1016/j.jfa.2022.109718.

[4] L. Abatangelo, R. Ognibene, Sharp behavior of Dirichlet-Laplacian eigenvalues for a class of singularly perturbed problems, SIAM J. Math. Anal. 56 (1) (2024) 474–500, https://doi.org/10.1137/23M1564444.

[5] C. Anné, O. Post, Wildly perturbed manifolds: norm resolvent and spectral convergence, J. Spectr. Theory 11 (1) (2021) 229–279, https://doi.org/10.4171/JST/340.

[6] V.I. Burenkov, E.B. Davies, Spectral stability of the Neumann Laplacian, J. Differ. Equ. 186 (2) (2002) 485–508, https://doi.org/10.1016/S0022-0396(02)00033-5.

[7] D. Boffi, F. Gardini, L. Gastaldi, Virtual element approximation of eigenvalue problems, in: The Virtual Element Method and Its Applications, in: SEMA SIMAI Springer Ser., vol. 31, Springer, Cham, 2022, pp. 275–320.

[8] L. Brasco, M. González, M. Ispizua, A Steklov version of the torsional rigidity, Commun. Contemp. Math. 26 (7) (2024) 2350037, https://doi.org/10.1142/S0219199723500372.

[9] V.I. Burenkov, V. Gol'dshtein, A. Ukhlov, Conformal spectral stability estimates for the Neumann Laplacian, Math. Nachr. 289 (17-18) (2016) 2133–2146, https://doi.org/10.1002/mana.201500439.

[10] D. Barseghyan, B. Schneider, S. Bernstein, Magnetic Neumann Laplacian on a domain with a hole, Rep. Math. Phys. 92 (3) (2023) 259–278, https://doi.org/10.1016/S0034-4877(23)00079-4.

[11] D. Barseghyan, B. Schneider, L.H. Hai, Neumann Laplacian in a perturbed domain, Mediterr. J. Math. 19 (3) (2022) 17, https://doi.org/10.1007/s00009-022-02046-x.

[12] L.A. Caffarelli, A. Friedman, Partial regularity of the zero-set of solutions of linear and superlinear elliptic equations, J. Differ. Equ. 60 (3) (1985) 420–433, https://doi.org/10.1016/0022-0396(85)90133-0.

[13] R. Courant, D. Hilbert, Methods of Mathematical Physics, vol. I, Interscience Publishers, Inc., New York, 1953, pp. xv+561.

[14] G. Cardone, A. Khrabustovskyi, Spectrum of the Laplacian on a domain perturbed by small resonators, SIAM J. Math. Anal. 55 (4) (2023) 3677–3712, https://doi.org/10.1137/22M148207X.

[15] Y. Colin de Verdière, Sur la multiplicité de la première valeur propre non nulle du laplacien, Comment. Math. Helv. 61 (2) (1986) 254–270, https://doi.org/10.1007/BF02621914.

[16] G. Courtois, Spectrum of manifolds with holes, J. Funct. Anal. 134 (1) (1995) 194–221, https://doi.org/10.1006/jfan.1995.1142.

[17] T. DelSole, M.K. Tippett, Laplacian eigenfunctions for climate analysis, J. Climate 28 (18) (2015) 7420–7436, https://doi.org/10.1175/JCLI-D-15-0049.1.

[18] V. Felli, B. Noris, R. Ognibene, Eigenvalues of the Laplacian with moving mixed boundary conditions: the case of disappearing Dirichlet region, Calc. Var. Partial Differ. Equ. 60 (1) (2021) 12, https://doi.org/10.1007/s00526-020-01878-3.

[19] V. Felli, B. Noris, R. Ognibene, Eigenvalues of the Laplacian with moving mixed boundary conditions: the case of disappearing Neumann region, J. Differ. Equ. 320 (2022) 247–315, https://doi.org/10.1016/j.jde.2022.02.052.

[20] V. Felli, G. Romani, Perturbed eigenvalues of polyharmonic operators in domains with small holes, Calc. Var. Partial Differ. Equ. 62 (4) (2023) 128, https://doi.org/10.1007/s00526-023-02467-w.

[21] D.S. Grebenkov, B.-T. Nguyen, Geometrical structure of Laplacian eigenfunctions, SIAM Rev. 55 (4) (2013) 601–667, https://doi.org/10.1137/120880173.

[22] J.K. Hale, Eigenvalues and perturbed domains, in: Ten Mathematical Essays on Approximation in Analysis and Topology, Elsevier B.V., Amsterdam, 2005, pp. 95–123.

[23] R. Hempel, On the lowest eigenvalue of the Laplacian with Neumann boundary condition at a small obstacle, J. Comput. Appl. Math. 194 (1) (2006) 54–74, https://doi.org/10.1016/j.cam.2005.06.014.

[24] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006, pp. x+202.

[25] A. Henrot, M. Pierre, Shape variation and optimization, in: European Mathematical Society (EMS), Zürich, in: EMS Tracts in Mathematics, vol. 28, 2018, pp. xi+365.

[26] S. Jimbo, Eigenvalues of the Laplacian in a domain with a thin tubular hole, J. Elliptic Parabolic Equ. 1 (1) (2015) 137–174, https://doi.org/10.1007/BF03377373.

[27] Y. Kabeya, Eigenvalues of the Laplace-Beltrami operator under the homogeneous Neumann condition on a large zonal domain in the unit sphere, Discrete Contin. Dyn. Syst. 40 (6) (2020) 3529–3559, https://doi.org/10.3934/dcds.2020040.

[28] V. Kozlov, J. Thim, Hadamard type asymptotics for eigenvalues of the Neumann problem for elliptic operators, J. Spectr. Theory 6 (1) (2016) 99–135, https://doi.org/10.4171/JST/120.

[29] M. Lanza de Cristoforis, Simple Neumann eigenvalues for the Laplace operator in a domain with a small hole, Rev. Mat. Complut. 25 (2) (2012) 369–412, https://doi.org/10.1007/s13163-011-0081-8.

[30] A. Lemenant, E. Milakis, L.V. Spinolo, Spectral stability estimates for the Dirichlet and Neumann Laplacian in rough domains, J. Funct. Anal. 264 (9) (2013) 2097–2135, https://doi.org/10.1016/j.jfa.2013.02.006.

[31] A. Laurain, S. Nazarov, J. Sokolowski, Singular perturbations of curved boundaries in three dimensions. The spectrum of the Neumann Laplacian, Z. Anal. Anwend. 30 (2) (2011) 145–180, https://doi.org/10.4171/ZAA/1429.

[32] N.F. Marshall, The stability of the first Neumann Laplacian eigenfunction under domain deformations and applications, Appl. Comput. Harmon. Anal. 49 (3) (2020) 916–934, https://doi.org/10.1016/j.acha.2019.05.001.

[33] S. Ozawa, Singular variation of domains and eigenvalues of the Laplacian, Duke Math. J. 48 (4) (1981) 767–778, https://doi.org/10.1215/S0012-7094-81-04842-0.

[34] S. Ozawa, Spectra of domains with small spherical Neumann boundary, J. Fac. Sci., Univ. Tokyo, Sect. 1A 30 (1983) 259–277.

[35] S. Ozawa, Asymptotic property of an eigenfunction of the Laplacian under singular variation of domains - the Neumann condition, Osaka J. Math. 22 (1985) 639–655.

[36] G. Pólya, G. Szegö, Isoperimetric Inequalities in Mathematical Physics, Ann. Math. Stud., vol. 27, Princeton University Press, Princeton, NJ, 1951.

[37] J.W.S. Rayleigh Baron, The Theory of Sound, Dover Publications, New York, 1945, Vol. I, xlii+480 pp., vol. II, xii+504.

[38] J. Rauch, M. Taylor, Potential and scattering theory on wildly perturbed domains, J. Funct. Anal. 18 (1975) 27–59, https://doi.org/10.1016/0022-1236(75)90028-2.

[39] A. Samarskii, On the influence of constraints on the characteristic frequencies of closed volumes, Dokl. Akad. Nauk SSSR 63 (6) (1948) 631–634.

[40] B. Schweizer, The low-frequency spectrum of small Helmholtz resonators, Proc. R. Soc. A, Math. Phys. Eng. Sci. 471 (2174) (2015) 18, https://doi.org/10.1098/rspa.2014.0339.

[41] S.A. Sauter, R. Warnke, Extension operators and approximation on domains containing small geometric details, East-West J. Numer. Math. 7 (1) (1999) 61–77.

[42] Q. Yang, C.-Y. Lee, M.K. Tippett, D.R. Chavas, T.R. Knutson, Machine learning–based hurricane wind reconstruction, Weather Forecast. 37 (4) (2022) 477–493, https://doi.org/10.1175/WAF-D-21-0077.1.