Polarized orbifolds associated to quantized Hamiltonian torus actions

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Abstract

Suppose given an holomorphic and Hamiltonian action of a compact torus T on a polarized Hodge manifold M. Assume that the action lifts to a quantizing line bundle, so that there is an induced unitary representation of T on the associated Hardy space. If in addition the moment map is nowhere zero, for each weight $\boldsymbol{\nu}$ the $\boldsymbol{\nu}$-th isotypical component in the Hardy space of the polarization is finite-dimensional. Assuming that the moment map is transverse to the ray through $\boldsymbol{\nu}$, we give a gometric interpretation of the isotypical components associated to the weights $k\,\boldsymbol{\nu}$, $k \to +\infty$, in terms of certain polarized orbifolds associated to the Hamiltonian action and the weight. These orbifolds are generally not reductions of M in the usual sense, but arise rather as quotients of certain loci in the unit circle bundle of the polarization; this construction generalizes the one of weighted projective spaces as quotients of the unit sphere, viewed as the domain of the Hopf map.

1 Introduction

Let M be a d-dimensional connected complex projective manifold, with complex structure J. Let (A, h) be a positive holomorphic line bundle on (M, J); the curvature of the unique covariant derivative on A compatible with both the Hermitian metric h and the complex structures has the form $\Theta = -2\pi\,\imath\,\omega$, where ω is a Kähler form on (M, J). Let $\mathrm{d}V_M := \omega^{\wedge d}/d!$ be the associated volume form on M.

Let $A^\vee$ be the dual line bundle of A, endowed with the dual Hermitian metric $h^\vee$. As is well-known, positivity of (A, h) is equivalent to the unit

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disc bundle $D \subset A^\vee$ being a strictly pseudoconvex domain [Gr]. We shall denote by $X := \partial D \subset A^\vee$ the unit circle bundle of $h^\vee$, and by $\alpha \in \Omega^1(X)$ the (normalized) connection 1-form on X. Thus, X is a principal S^1-bundle on M, with the structure S^1-action $\rho^X : S^1 \times X \to X$ given by clockwise fiber rotation. If $\pi : X \to M$ is the bundle projection, and $-\partial_\theta \in \mathfrak{X}(X)$ is the generator of ρ^X, then

$$\mathrm{d}\alpha = 2\,\pi^*(\omega), \quad \alpha(\partial_\theta) = 1. \tag{1}$$

Let $\mathrm{d}V_X := (\alpha/2\pi) \wedge \pi^*(\mathrm{d}V_M)$ be the associated volume form on X

Then α is a contact form on X, and X is a CR manifold, with CR structure supported by the horizontal tangent bundle

$$Hor(X) := \ker(\alpha) \subset TX. \tag{2}$$

Let $H(X) \subseteq L^2(X)$ denote the Hardy space of X. Since ρ^X preserves α and the CR structure, it induces a unitary representation $\hat{\rho}^X$ of S^1 on $H(X)$, given by

$$\hat{\rho}^X_{e^{\imath\vartheta}}(s)(x) := s\left(\rho^X_{e^{-\imath\vartheta}}(x)\right) = s\left(e^{\imath\vartheta}\,x\right) \quad \left(x \in X,\, e^{\imath\vartheta} \in S^1,\, s \in H(X)\right).$$

The induced isotypical decomposition is the Hilbert space direct sum

$$H(X) = \bigoplus_{k=0}^{+\infty} H(X)_k, \tag{3}$$

where

$$H(X)_k := \left\{s \in H(X) \,:\, s\left(e^{\imath\theta}\,x\right) = e^{\imath\,k\,\theta}\,s(x) \quad \forall\, x \in X,\, e^{\imath\theta} \in S^1\right\}.$$

It is well-known that there are natural unitary isomorphisms $H(X)_k \cong H^0(M, A^{\otimes k})$, the latter being the space of global holomorphic sections of $A^{\otimes k}$.

Furthermore, let $T \cong (S^1)^r$ be an r-dimensional compact torus, with Lie algebra and coalgebra $\mathfrak{t}$ and $\mathfrak{t}^\vee$, respectively. We shall equivariantly identify $\mathfrak{t} \cong \mathfrak{t}^\vee \cong \imath\,\mathbb{R}^r$. Suppose given an Hamiltonian and holomorphic action $\mu^M : T \times M \to M$ of T on the Kähler manifold $(M, J, 2\,\omega)$. Let $\Phi : M \to \mathfrak{t}^\vee \cong \imath\,\mathbb{R}^r$ be the moment map.

It is standard that μ^M and Φ generate an infinitesimal contact and CR action of $\mathfrak{t}$ on X, so defined [Ko]. If $\xi \in \mathfrak{t}$, let $\xi_M \in \mathfrak{X}(M)$ be the Hamiltonian vector field induced by ξ on M, and define a vector field $\xi_X \in \mathfrak{X}(X)$ by setting

$$\xi_X := \xi_M^\sharp - \langle \Phi \circ \pi, \xi\rangle\,\partial_\theta \in \mathfrak{X}(X); \tag{4}$$

here $V^\sharp \in \mathfrak{X}(X)$ denotes the horizontal lift to X of a vector field $V \in \mathfrak{X}(M)$, with respect to α. The ξ_X's are commuting contact vector fields on X, whose flow preserves the CR structure, and the map $\xi \mapsto \xi_X$ is a morphism of Lie algebras $\mathfrak{t} \to \mathfrak{X}(X)$.

Let us make the stronger hypothesis that μ^M lifts to an actual contact and CR action of T on X, $\mu^X : T \times X \to X$, and that $\mathrm{d}\mu^X(\xi) = \xi_X$ for any $\xi \in \mathfrak{t}$. Then μ^X determines a unitary representation $\hat{\mu}^X$ of T on $H(X)$, given by

$$\hat{\mu}^X_{\mathbf{t}}(s)(x) := s\left(\mu^X_{\mathbf{t}^{-1}}(x)\right) \quad (x \in X,\, \mathbf{t} \in T,\, s \in H(X)). \tag{5}$$

By the Peter-Weyl Theorem [St], $\hat{\mu}^X$ induces a unitary and equivariant splitting of $H(X)$ into isotypical components.

Let us regard any $\boldsymbol{\nu} \in \mathbb{Z}^r$ as an integral weight on T, associated to the character

$$\chi_{\boldsymbol{\nu}}(\mathbf{t}) := \mathbf{t}^{\boldsymbol{\nu}},$$

where for $\mathbf{t} = (t_1, \ldots, t_r) \in T$ we set $\mathbf{t}^{\boldsymbol{\nu}} := \prod_{j=1}^r t_j^{\nu_j}$. For any $\boldsymbol{\nu} \in \mathbb{Z}^r$, let us consider the $\boldsymbol{\nu}$-th isotypical component

$$H(X)^{\hat{\mu}}_{\boldsymbol{\nu}} := \{s \in H(X) \,:\, \hat{\mu}_{\mathbf{t}}(s) = \chi_{\boldsymbol{\nu}}(\mathbf{t}) \cdot s \quad \forall\, \mathbf{t} \in T\}.$$

Then we have an equivariant Hilbert space direct sum

$$H(X) = \bigoplus_{\boldsymbol{\nu} \in \mathbb{Z}^r} H(X)^{\hat{\mu}}_{\boldsymbol{\nu}}. \tag{6}$$

In the special case where $T = S^1$, μ^M is trivial, and $\Phi = \imath$, $\imath \in \mathfrak{t}$ is mapped to $-\partial_\theta$, and so $\mu^X = \rho^X$; hence (6) reduces to (3), that is, $H(X)_k = H(X)^{\hat{\rho}}_k$ with k in place of $\boldsymbol{\nu}$.

In general, it may happen that $H(X)^{\hat{\mu}}_{\boldsymbol{\nu}} \cap H(X)_k \neq (0)$ for several k's, so that $H(X)^{\hat{\mu}}_{\boldsymbol{\nu}}$ does not correspond to a space of holomorphic sections of some power of A. Furthermore, $H(X)^{\hat{\mu}}_{\boldsymbol{\nu}}$ may be infinite-dimensional. The latter circumstance does not occur, however, if $\mathbf{0} \notin \Phi(M)$ (see §2 of [P1]). We shall make the following Basic Assumption (henceforth referred to as BA):

Basic Assumption 1.1. Φ and $\boldsymbol{\nu}$ satisfy the following properties:

1. $\boldsymbol{\nu} \neq \mathbf{0}$ is coprime, that is, l.c.d.$(\nu_1, \ldots, \nu_r) = 1$;
2. Φ is nowhere vanishing, that is, $\mathbf{0} \notin \Phi(M)$;
3. Φ is transverse to the ray $\mathbb{R}_+ \cdot \imath\boldsymbol{\nu}$, and $M_{\boldsymbol{\nu}} := \Phi^{-1}(\mathbb{R}_+ \cdot \imath\boldsymbol{\nu}) \neq \emptyset$.

If the previous properties are satisfied, then μ^X is generically locally free [P1]; perhaps after replacing T with its quotient by a finite subgroup, we may and will assume without loss of generality that μ^X is generically free.

Let us assume that BA holds. Then $H(X)^{\hat{\mu}}_{k\boldsymbol{\nu}} = (0)$ for all $k \leq 0$ (§2 of [P1]). We are interested in the sequence of spaces of finite-dimensional vector spaces $\big(H(X)^{\hat{\mu}}_{k\boldsymbol{\nu}}\big)_{k=1}^{+\infty}$ associated to the weights on the ray $\mathbb{R}_+ \cdot \boldsymbol{\nu}$. The corresponding 'equivariant Szegö projectors' $\Pi^{\hat{\mu}}_{k\boldsymbol{\nu}} : L^2(X) \to H(X)^{\hat{\mu}}_{k\boldsymbol{\nu}}$ are smoothing operators (that is, they have $\mathcal{C}^\infty$ integral kernels). Furthemore, $M_{\boldsymbol{\nu}} \subseteq M$ is a T-invariant coisotropic connected compact submanifold of real codimension $r-1$ [P1]. The local and global asymptotics for $k \to +\infty$ of the integral kernels $\Pi^{\hat{\mu}}_{k\boldsymbol{\nu}}$ and their concentration behaviour along $M_{\boldsymbol{\nu}}$ were studied in [P1], [P2], and related variants in the presense of additional symmetries where investigated in [Ca].

Our present aim is to clarify the geometric significance of the sequence $\big(H(X)^{\hat{\mu}}_{k\boldsymbol{\nu}}\big)_{k=1}^{+\infty}$, generalizing the interpretation of the sequence $\big(H(X)_k\big)$ in terms of the spaces $H^0(M, A^{\otimes k})$. We shall prove the following:

Theorem 1.1. *Assume BA holds. Then there exists a $(d+1-r)$-dimensional compact complex orbifold $N_{\boldsymbol{\nu}}$, and a positive holomorphic orbifold line bundle $B_{\boldsymbol{\nu}}$ on $N_{\boldsymbol{\nu}}$, naturally constructed from A, $\boldsymbol{\nu}$ and Φ, such that the following holds:*

1. *for $k \geq 1$, there is a natural injection $\delta_k : H(X)^{\hat{\mu}}_{k\boldsymbol{\nu}} \hookrightarrow H^0\big(N_{\boldsymbol{\nu}}, B_{\boldsymbol{\nu}}^{\otimes k}\big)$;*
2. *δ_k is an isomorphism if $k \gg 0$.*

Corollary 1.1. *If $k \gg 0$,*

$$\dim H(X)^{\hat{\mu}}_{k\boldsymbol{\nu}} = \chi\big(N_{\boldsymbol{\nu}}, B_{\boldsymbol{\nu}}^{\otimes k}\big).$$

Obviously with no pretense of exhaustiveness, discussions of orbifolds and orbifold line bundles (also known as V-manifolds and line V-bundles) can be found in [S1], [S2], [B], [Ka], [ALR], [BG]; specific treatments of Hamiltonian actions on symplectic orbifolds can be found in [LT] and [MS].

The geometric significance of the Theorem lies in the relation between the polarized orbifold $(N_{\boldsymbol{\nu}}, B_{\boldsymbol{\nu}})$ and the 'prequantum data' $(A, \Phi, \boldsymbol{\nu})$. It is therefore in order to outline how the former is constructed from the latter. The following statements will be clarified and proved in §2.

Let $\tilde{T} \cong (\mathbb{C}^*)^r$ be the complexification of T. Then μ^X extends to an holomorphic line bundle action $\tilde{\mu}^{A^\vee} : \tilde{T} \times A^\vee \to A^\vee$. Let $A^\vee_0$ be the complement of the zero section in $A^\vee$, and let $A^\vee_{\boldsymbol{\nu}} \subset A^\vee_0$ be the inverse image of $M_{\boldsymbol{\nu}}$. Let $\tilde{A}^\vee_{\boldsymbol{\nu}} := \tilde{T} \cdot A^\vee_{\boldsymbol{\nu}}$ be its saturation under $\tilde{\mu}^{A^\vee}$.

Then $\tilde{\mu}^{A^\vee}$ is proper and locally free on $\tilde{A}^\vee_{\boldsymbol{\nu}}$, and $N_{\boldsymbol{\nu}} = \tilde{A}^\vee_{\boldsymbol{\nu}} / \tilde{T}$. Thus the projection $p_{\boldsymbol{\nu}} : \tilde{A}^\vee_{\boldsymbol{\nu}} \to N_{\boldsymbol{\nu}}$ is a principal V-bundle with structure group $\tilde{T}$ over $N_{\boldsymbol{\nu}}$ [S2].

Furthermore, $\chi_{\boldsymbol{\nu}} : T \to S^1$ extends to a character $\tilde{\chi}_{\boldsymbol{\nu}} : \tilde{T} \to \mathbb{C}^*$; the datum of $p_{\boldsymbol{\nu}}$ and $\tilde{\chi}_{\boldsymbol{\nu}}$ determines the orbifold line bundle $B_{\boldsymbol{\nu}}$. Similarly, $B_{\boldsymbol{\nu}}^{\otimes k}$ (or $B_{k\boldsymbol{\nu}}$) denotes the orbifold line bundle associated to $p_{\boldsymbol{\nu}}$ and $\tilde{\chi}_{k\boldsymbol{\nu}} = \tilde{\chi}_{\boldsymbol{\nu}}^k$.

We can give the following alternative algebro-geometric characterization of $\tilde{A}_{\boldsymbol{\nu}}^{\vee}$. Let $\boldsymbol{\nu}^{\perp} \subset \mathbb{R}^r$ be the orthocomplement of $\boldsymbol{\nu}$ with respect to the standard scalar product, and consider the (Abelian) subalgebra $\imath\,\boldsymbol{\nu}^{\perp} \leqslant \mathfrak{t}$. Let $T_{\boldsymbol{\nu}^{\perp}}^{r-1} \leqslant T$ be the corresponding subtorus, $\tilde{T}_{\boldsymbol{\nu}^{\perp}}^{r-1} \leqslant \tilde{T}$ be its complexification. The restriction of $\tilde{\mu}^M$ to $\tilde{T}_{\boldsymbol{\nu}^{\perp}}^{r-1}$ is an holomorphic action $\tilde{\gamma}^M$ of $\tilde{T}_{\boldsymbol{\nu}^{\perp}}^{r-1}$ on $(M,\,J)$, with a built-in complex linearization $\tilde{\gamma}^{A^{\vee}} : \tilde{T}_{\boldsymbol{\nu}^{\perp}}^{r-1} \times A^{\vee} \to A^{\vee}$. Let $\tilde{M}_{\boldsymbol{\nu}} \subseteq M$ be the locus of (semi)stable points of $\tilde{\gamma}^M$; then $\tilde{A}_{\boldsymbol{\nu}}^{\vee}$ is the inverse image of $\tilde{M}_{\boldsymbol{\nu}}$ in $A_0^{\vee}$.

Up to a natural isomorphism, an alternative description of $N_{\boldsymbol{\nu}}$ is as follows. Let $X_{\boldsymbol{\nu}} := \pi^{-1}(M_{\boldsymbol{\nu}})$. Then T acts locally freely on $X_{\boldsymbol{\nu}}$, and $N_{\boldsymbol{\nu}} \cong X_{\boldsymbol{\nu}}/T$. This description is instrumental in describing the positivity of $B_{\boldsymbol{\nu}}$ and the Kähler structure of $N_{\boldsymbol{\nu}}$.

When $r = 1$, $M = \mathbb{P}^d$, and A is the hyperplane line bundle with the standard metric, we have $X_{\boldsymbol{\nu}} = X = S^{2d+1}$; thus the previous construction generalizes the one of weighted projective spaces (see also the discussions in in [P2] and [P3]).

2 Preliminaries

This section is devoted to a closer description of the geometric setting, and to the statement and proof of a series of geometric results that will combine into the proof of Theorem 1.1.

Notation 2.1. We shall adopt the following notation and conventions.

1. If a Lie group G with Lie algebra $\mathfrak{g}$ acts smoothly on a manifold R, for any $\xi \in \mathfrak{g}$ we shall denote by $\xi_R \in \mathfrak{X}(R)$ the vector field on R generated by ξ.

2. If $r \in R$ and $\mathfrak{l} \subseteq \mathfrak{g}$ is a vector subspace, we shall set
$$\mathfrak{l}_R(r) := \big\{\xi_R(r) \,:\, \xi \in \mathfrak{l}\big\} \subseteq T_rR.$$

3. Given an isomorphism $T \cong (S^1)^r$, we have $\mathfrak{t} \cong \imath\,\mathbb{R}^r$. If we identify the Lie algebra $\tilde{\mathfrak{t}}$ of $\tilde{T} \cong \mathbb{R}_+^r \times T$ with $\mathbb{C}^r \cong \mathbb{R}^r \oplus \imath\,\mathbb{R}^r$, $\mathfrak{t}$ corresponds to the imaginary summand $\imath\,\mathbb{R}^r$. For $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_r \end{pmatrix} \in \mathbb{R}^r$, we have $e^{\mathbf{x}} := \begin{pmatrix} e^{x_1} & \cdots & e^{x_r} \end{pmatrix} \in \mathbb{R}_+^r \leqslant \tilde{T}^r$, while $e^{\imath\,\mathbf{x}} := \begin{pmatrix} e^{\imath\,x_1} & \cdots & e^{\imath\,x_r} \end{pmatrix} \in T^r$.

4. We shall equivariantly identify $\mathfrak{t} \cong \mathfrak{t}^\vee$, and view Φ as $\mathfrak{t}$-valued.

5. If V is any Euclidean vector space and $\epsilon > 0$, $V(\epsilon) \subset V$ will denote the open ball in V centered at the origin and of radius ϵ.

6. $g(\cdot, \cdot) := \omega\big(\cdot, J(\cdot)\big)$ is the Riemannian metric associated to ω.

7. J' is the complex structure of $A^\vee$.

8. The superscript $\sharp$ will denote horizontal lifts from M to either X or $A^\vee$, according to the context, and will be applied to both tangent vectors and vector subspaces of tangent spaces.

9. $\pi : X \to M$ and $\pi' : A_0^\vee \to M$ are the projections.

10. If $\beta^Z : G \times Z \to Z$ is an action of the group G on the set Z, and if $S \subseteq Z$ is G-invariant, we shall often denote by $\beta^S : G \times S \to S$ the restricted action. Thus, for example, $\tilde{T}$ acts on $A^\vee$ by $\tilde{\mu}^{A^\vee}$, on $A_0^\vee \subset A^\vee$ by $\tilde{\mu}^{A_0^\vee}$, on $\tilde{A}_{\boldsymbol{\nu}}^\vee \subseteq A_0^\vee$ by $\tilde{\mu}^{\tilde{A}_{\boldsymbol{\nu}}^\vee}$.

2.1 The locus $M_{\boldsymbol{\nu}} \subseteq M$

Let $\gamma^M : T_{\boldsymbol{\nu}^\perp}^{r-1} \times M \to M$ be the action induced by restriction of μ^M. Then γ^M is Hamiltonian with respect to $2\,\omega$, and its moment map $\Phi_{\boldsymbol{\nu}^\perp} : M \to \imath\,\boldsymbol{\nu}^\perp$ is the composition of Φ with the orthogonal projection $\mathfrak{t} \to \imath\,\boldsymbol{\nu}^\perp$. Assuming BA, we can draw the following conclusions:

1. $\mathbf{0} \in \imath\,\boldsymbol{\nu}^\perp$ is a regular value of $\Phi_{\boldsymbol{\nu}^\perp}$;

2. $M_{\boldsymbol{\nu}} = \Phi_{\boldsymbol{\nu}^\perp}^{-1}(\mathbf{0})$ is a compact and connected coisotropic submanifold of M, of (real) codimension $r-1$;

3. γ^M is locally free along $M_{\boldsymbol{\nu}^\perp}$, that is,
$$\dim\big(\imath\,\boldsymbol{\nu}^\perp\big)_M(m) = r-1 \quad \forall\, m \in M_{\boldsymbol{\nu}^\perp};$$

4. for every $m \in M_{\boldsymbol{\nu}}$, we have
$$T_m M_{\boldsymbol{\nu}} = \big(\imath\,\boldsymbol{\nu}^\perp\big)_M(m)^{\perp_{\omega_m}} = J_m\left((\imath\,\boldsymbol{\nu}^\perp)_M(m)\right)^{\perp_{g_m}}.$$

This implies the following statement. Let us define

$$\Psi : (\mathbf{x}, m) \in (\imath\,\boldsymbol{\nu}^\perp) \times M_{\boldsymbol{\nu}} \mapsto \tilde{\mu}_{e^{\mathbf{x}}}(m) \in M. \tag{7}$$

Lemma 2.1. *Given Basic Assumption 1.1, the following holds:*

1. $\tilde{\gamma}^M$ *is locally free along* $M_{\boldsymbol{\nu}}$*;*

2. *for any sufficiently small* $\epsilon > 0$, Ψ *in (7) restricts to a diffeomorphism between* $(\imath\,\boldsymbol{\nu}^\perp)(\epsilon) \times M_{\boldsymbol{\nu}}$ *and an open tubular neighborhood* U_ϵ *of* $M_{\boldsymbol{\nu}}$ *in* M.

2.2 The locus $X_{\boldsymbol{\nu}} \subseteq X$ and its saturation $\tilde{X}_{\boldsymbol{\nu}}$ in $A^\vee$

Let us set:

$$X_{\boldsymbol{\nu}} := \pi^{-1}(M_{\boldsymbol{\nu}}) \subseteq X. \tag{8}$$

If $x \in X_{\boldsymbol{\nu}}$, then in view of (4)

$$\big(\imath\,\boldsymbol{\nu}^\perp\big)_X(x) = \big(\imath\,\boldsymbol{\nu}^\perp\big)_M(m)^\sharp. \tag{9}$$

We have the following analogue of Lemma 2.1.

Lemma 2.2. *Given BA 1.1, the following holds:*

1. μ^X *is locally free along* $X_{\boldsymbol{\nu}}$*;*

2. *for any* $x \in X_{\boldsymbol{\nu}}$,
$$T_x X_{\boldsymbol{\nu}} \cap J'_x\big(\mathfrak{t}_{A^\vee}(x)\big) = (0);$$

3. *for all suitably small* $\epsilon > 0$, *the map*
$$\Psi' : (\mathbf{x}, x) \in (\imath\,\mathfrak{t}) \times X_{\boldsymbol{\nu}} \mapsto \tilde{\mu}^{A_0^\vee}_{e^{\mathbf{x}}}(x) \in A_0^\vee$$
determines a diffemorphism from $(\imath\,\mathfrak{t})(\epsilon) \times X_{\boldsymbol{\nu}}$ *to a tubular neighborhood of* $X_{\boldsymbol{\nu}}$ *in* $A^\vee$*;*

4. $\tilde{\mu}^{A^\vee}$ *is locally free along* $X_{\boldsymbol{\nu}}$.

Proof of Lemma 2.2. That μ^X is locally free on $X_{\boldsymbol{\nu}}$ under the transversality assumption in BA is proved in §2 of [P1].

By the discussion in §2.1, if $x \in X_{\boldsymbol{\nu}}$ and $m = \pi(x)$ then

$$J_m\big(\imath\,\boldsymbol{\nu}^\perp\big)_M(m)^\sharp \subseteq T_x X$$

is the normal space of $X_{\boldsymbol{\nu}}$ in X at x; hence given (9) we have

$$T_x X_{\boldsymbol{\nu}} \cap J'_x\big(\imath\,\boldsymbol{\nu}^\perp\big)_X(x) = T_x X_{\boldsymbol{\nu}} \cap J_m\big(\imath\,\boldsymbol{\nu}^\perp\big)_M(m)^\sharp = (0). \tag{10}$$

Furthermore, by definition of $X_{\boldsymbol{\nu}}$ there exists a smooth function $\lambda_{\boldsymbol{\nu}} : M_{\boldsymbol{\nu}} \to \mathbb{R}_+$ such that

$$(\imath\,\boldsymbol{\nu})_X(x) = (\imath\,\boldsymbol{\nu})_M(m)^\sharp - \lambda_{\boldsymbol{\nu}}(m)\,\|\boldsymbol{\nu}\|^2\ \partial_\theta|_x \not\in Hor(X)_x, \quad \forall\, x \in X_{\boldsymbol{\nu}}. \quad (11)$$

If r denotes the radial coordinate along the fibers of $A^\vee$, this implies

$$J'_x\big((\imath\,\boldsymbol{\nu})_X(x)\big) = J_m\big((\imath\,\boldsymbol{\nu})_M(m)\big)^\sharp + \lambda_{\boldsymbol{\nu}}(m)\,\|\boldsymbol{\nu}\|^2\ \partial_r|_x \in T_xA^\vee \setminus T_xX. \quad (12)$$

The second statement follows from (10), (11) and (12).

The third statement is an immediate consequence of the second.

Since $X_{\boldsymbol{\nu}}$ is a T-invariant submanifold of $A^\vee$, $\left(\imath\,\boldsymbol{\nu}^\perp\right)_X(x) \subseteq T_xX_{\boldsymbol{\nu}}$ for any $x \in X_{\boldsymbol{\nu}}$. Hence if $x \in X_{\boldsymbol{\nu}}$ and $m = \pi(x)$ then by (10)

$$\left(\imath\,\boldsymbol{\nu}^\perp\right)_X(x) \cap J'_x\left(\imath\,\boldsymbol{\nu}^\perp\right)_X(x) = \left(\imath\,\boldsymbol{\nu}^\perp\right)_M(m)^\sharp \cap J_m\left(\imath\,\boldsymbol{\nu}^\perp\right)_M(m)^\sharp = (0). \quad (13)$$

Together with the first statement, this implies $\dim_{\mathbb{C}}\left(\tilde{\mathfrak{t}}_{A^\vee}(x)\right) = r$ for any $x \in X_{\boldsymbol{\nu}}$.

□

Let us consider the saturation

$$\tilde{X}_{\boldsymbol{\nu}} := \tilde{T} \cdot X_{\boldsymbol{\nu}} \subseteq A^\vee. \quad (14)$$

Corollary 2.1. *$\tilde{X}_{\boldsymbol{\nu}}$ is open in $A^\vee$.*

Corollary 2.2. *If Basic Assumption 1.1 holds, then μ^X is generically free on $X_{\boldsymbol{\nu}}$.*

Proof of Corollary 2.2. If the general $x \in X_{\boldsymbol{\nu}}$ had non-trivial stabilizer in T, the same would hold of the general $\ell \in \tilde{X}_{\boldsymbol{\nu}}$; since the latter is open in $A_0^\vee$, this contradicts the assumption that $\mu^{A^\vee}$ is generically free. □

Corollary 2.3. *If BA 1.1 holds, then $N'_{\boldsymbol{\nu}} := X_{\boldsymbol{\nu}}/T$ is a compact orbifold of real dimension $2\,(d+1-r)$, and the projection*

$$p'_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \to N'_{\boldsymbol{\nu}} \quad (15)$$

is a principal V-bundle with structure group T.

Remark 2.1. Associated to $p'_{\boldsymbol{\nu}}$ and the character $\chi_{\boldsymbol{\nu}}$ there is a orbifold complex line bundle $B'_{\boldsymbol{\nu}}$ on $N'_{\boldsymbol{\nu}}$.

2.3 The Kähler structure of $A_0^\vee$

Let $\varrho : A_0^\vee \to \mathbb{R}$ denote the square norm function in the Hermitian metric h, and set

$$\tilde{\omega} := 2\,\mathrm{d}\left(\Im\left(\partial\varrho^{1/2}\right)\right) = 2\,\imath\,\partial\overline{\partial}\left(\varrho^{1/2}\right). \tag{16}$$

If $\pi' : A_0^\vee \to M$ is the projection, then

$$\tilde{\omega} = 2\,\varrho^{\,1/2}\,\pi'^*(\omega) + \frac{\imath}{2\,\varrho^{3/2}}\,\partial\varrho \wedge \overline{\partial}\varrho. \tag{17}$$

The contact action $\mu^X : T \times X \to X$ extends to an holomorphic unitary action $\mu^{A^\vee} : T \times A_0^\vee \to A_0^\vee$.

Proposition 2.1. *$\tilde{\omega}$ is a $\mu^{A^\vee}$-invariant exact Kähler form on $A_0^\vee$.*

Proof. Since $\mu^{A^\vee}$ preserves both ϱ and the complex structure, by its definition $\tilde{\omega}$ is a $\mu^{A^\vee}$-invariant closed $(1,1)$-form. Thus we need only prove that $\tilde{\omega}$ is non-degenerate.

The unique connection compatible with both h and the holomorphic structure determines an invariant decomposition

$$TA_0^\vee = Hor(A_0^\vee) \oplus Ver(A_0^\vee), \tag{18}$$

where

$$Hor(A_0^\vee) := \ker(\partial\varrho), \quad Ver(A_0^\vee) := \ker(\mathrm{d}\pi') \subset TA^\vee \tag{19}$$

denote the horizontal and vertical tangent bundles. Then $Hor(A_0^\vee)$ and $Ver(A_0^\vee)$ are complex vector subbundles of $TA_0^\vee$, and by (17) they are orthogonal for $\tilde{\omega}$. Furthermore, the first summand on the right hand side of (17) is symplectic on $Hor(A_0^\vee)$ and vanishes on $Ver(A_0^\vee)$, and conversely for the second summand. Hence $\tilde{\omega}$ is non-degenerate.

□

Corollary 2.4. *$\mu^{A^\vee}$ is Hamiltonian on $(A_0^\vee, \tilde{\omega})$, with moment map*

$$\tilde{\Phi} := \varrho^{1/2} \cdot \Phi \circ \pi' : A_0^\vee \to \mathfrak{t}, \tag{20}$$

where $\pi' : A_0^\vee \to M$ is the projection.

Proof of Corollary 2.4. Given an exact symplectic manifold (R, η) with $\eta = -\mathrm{d}\lambda$, and a smooth Lie group action $\varsigma : G \times N \to N$ preserving λ, it is well-known that ς is Hamiltonian, with moment map $\Upsilon : R \to \mathfrak{g}^\vee$ determined by the relation

$$\upsilon^\xi := \langle \Upsilon, \xi \rangle = \iota(\xi_R)\,\lambda \in \mathcal{C}^\infty(M).$$

In our setting, $R = A_0^\vee$, $\varsigma = \mu^{A^\vee}$, $\eta = \tilde{\omega}$ and, in view of (16),

$$\lambda = -2\left(\Im\left(\partial\varrho^{1/2}\right)\right) = \imath\left(\partial\varrho^{1/2} - \overline{\partial}\varrho^{1/2}\right);$$

furthermore, for any $\xi \in \mathfrak{t}$ we have

$$\xi_{A^\vee} = \xi_M^\sharp - \langle \Phi \circ \pi', \xi\rangle\, \partial_\vartheta.$$

Since $\xi_M^\sharp$ is a section of $Hor(A_0^\vee)$, it follows from (19) that $\iota(\xi_M^\sharp)\,\partial\varrho = 0$. Furthermore, one can verify that $\imath(\partial_\theta)\,\partial\rho = \imath\,\rho$. Putting this together, we conclude that $\mu^{A^\vee}$ is Hamiltonian, and furthermore the component $\tilde{\phi}^\xi = \langle\tilde{\Phi}, \xi\rangle$ of the moment map is

$$\begin{aligned} \tilde{\phi}^\xi &= \imath \cdot \varrho^{-1/2}\,\iota\left(\xi_M^\sharp - (\varphi^\xi \circ \pi')\,\partial_\theta\right)\left(\partial\varrho - \overline{\partial}\varrho\right) \\ &= (\varphi^\xi \circ \pi')\,\varrho^{1/2}. \end{aligned}$$

□

Let $\{\cdot,\cdot\}_{A_0^\vee}$ denote by Poisson brackets on $(A_0^\vee, \tilde{\omega})$. Since $\mu^{A^\vee}$ is unitary, in view of (20) we conclude the the following.

Corollary 2.5. *$\{\tilde{\phi}^\xi, \tilde{\phi}^\eta\}$ vanishes, $\forall\,\xi,\eta \in \mathfrak{t}$. In particular, the orbits of $\mu^{A^\vee}$ in $A_0^\vee$ are isotropic for $\tilde{\omega}$.*

Therefore:

Corollary 2.6. *For every $\ell \in A_0^\vee$, $\mathfrak{t}_{A^\vee}(\ell) \subseteq T_\ell A^\vee$ is totally real, that is,*

$$\mathfrak{t}_{A^\vee}(\ell) \cap J'_\ell\big(\mathfrak{t}_{A^\vee}(\ell)\big) = (0).$$

By Proposition 1.6 and Theorem 1.12 in [Sj], Corollary 2.4 has the following consequences.

Corollary 2.7. *For every $\ell \in A_0^\vee$, the following holds:*

1. *the stabilizer $\tilde{T}_\ell \leqslant \tilde{T}$ of ℓ for $\tilde{\mu}^{A^\vee}$ is the complexification of of the stabilizer $T_\ell \leqslant T$ of ℓ for $\mu^{A^\vee}$;*
2. *there exists an holomorphic slice for $\tilde{\mu}^{A_0^\vee}$ at ℓ.*

Let $A_{lf}^\vee \subseteq A_0^\vee$ be the open subset where $\tilde{\mu}^{A_0^\vee}$ is locally free. It follows from Proposition 1.6 of [Sj] and Corollary 2.5 above that the stabilizer in $\tilde{T}$ of any $\ell \in A_{lf}^\vee$ is finite and contained in T.

Definition 2.1. Following [Sj], we shall call $\tilde{\mu}^{A_0^\vee}$ *proper at* $\ell \in A_0^\vee$ if for all sequences $(\ell_j) \subset A_0^\vee$ and $(\tilde{t}_j) \subset \tilde{T}$ such that $\ell_j \to \ell$ and $\tilde{\mu}_{\tilde{t}_j}^{A_0^\vee}(\ell_j)$ converges to some point in $A_0^\vee$, $(\tilde{t}_j)$ is convergent in $\tilde{T}$.

Remark 2.2. Let $U \subseteq A_0^\vee$ be a $\tilde{T}$-invariant open set. Then:

1. if $\tilde{\mu}^{A_0^\vee}$ is proper at every $\ell \in U$, then so is *a fortiori* $\tilde{\mu}^U$;
2. if $\tilde{\mu}^U$ is proper at every $\ell \in U$, $\tilde{\mu}^U$ is (globally) proper.

Corollary 2.8. *Given BA, the following holds:*

1. $\tilde{\mu}^{A_0^\vee}$ *is proper on* $A_{lf}^\vee$ *(that is,* $\tilde{\mu}^{A_{lf}^\vee}$ *is proper);*
2. $\tilde{X}_{\boldsymbol{\nu}} \subseteq A_{lf}^\vee$;
3. $\tilde{\mu}^{A_0^\vee}$ *is proper on* $\tilde{X}_{\boldsymbol{\nu}}$ *(that is,* $\tilde{\mu}^{\tilde{X}_{\boldsymbol{\nu}}}$ *is proper).*

Proof of Corollary 2.8. We have remarked that the stabilizer in $\tilde{T}$ of any $\ell \in A_{lf}^\vee$ coincides with the stabilizer of ℓ in T, and therefore it is finite. In view of Theorem 1.22 of [Sj], $\tilde{\mu}^{A_0^\vee}$ is proper at any $\ell \in A_{lf}^\vee$, and therefore it is proper on $A_{lf}^\vee$. This proves the first statement.

We know that $\mu^{X_{\boldsymbol{\nu}}}$ is locally free; in other words, $\mu^{A_0^\vee}$ is locally free along $X_{\boldsymbol{\nu}}$. Therefore, $\mu^{A_0^\vee}$ is locally free on $\tilde{X}_{\boldsymbol{\nu}} = \tilde{T} \cdot X_{\boldsymbol{\nu}}$, because $\tilde{T}$ is Abelian. Therefore, by Corollary 2.7, the stabilizer of any $\ell \in \tilde{X}_{\boldsymbol{\nu}}$ in $\tilde{T}$ for $\tilde{\mu}^{A_0^\vee}$ is finite, since it coincides with the stabilizermof ℓ in T for $\mu^{A_0^\vee}$. This proves the second statement.

The third statement is a straighforward consequence of the first two. ☐

The structure S^1-action ρ^X extends to the holomorphic action

$$\tilde{\rho}^{A^\vee} : (z,\, \ell) \in \mathbb{C}^* \times A_0^\vee \mapsto z^{-1}\, \ell \in A_0^\vee,$$

whose orbits are the fibers of $A_0^\vee$ over M.

Lemma 2.3. $\tilde{X}_{\boldsymbol{\nu}}$ *is* $\tilde{\rho}^{A^\vee}$*-invariant.*

Proof of Lemma 2.3. Since $\tilde{\mu}^{A^\vee}$ and $\tilde{\rho}^{A^\vee}$ commute, it suffices to show that for any $x \in X_{\boldsymbol{\nu}}$ and $z \in \mathbb{C}^*$ we have $z\, x \in \tilde{X}_{\boldsymbol{\nu}}$.

Let us set

$$C_x := \left\{ z \in \mathbb{C}^* \,:\, z\, x \in \tilde{X}_{\boldsymbol{\nu}} \right\}.$$

Then $1 \in C_x$ and C_x is open in $\mathbb{C}^*$ because scalar multiplication is continuous and $\tilde{X}_{\boldsymbol{\nu}}$ is open in $A^\vee$ (Corollary 2.1).

Suppose $z_\infty \in \mathbb{C}^*$ is a limit point of C_x. Then there exist $z_1, z_2, \ldots \in C_x$ such that $z_i \to z_\infty$. By definition of C_x, for any $i = 1, 2, \ldots$ we have $z_i\, x \in \tilde{X}_{\boldsymbol{\nu}}$ for any $i = 1, 2, \ldots$. By definition of $\tilde{X}_{\boldsymbol{\nu}}$, therefore, there exist $\tilde{t}_i \in \tilde{T}$ and $x_i \in X_{\boldsymbol{\nu}}$ such that $z_i\, x = \tilde{\mu}^{A^\vee}_{\tilde{t}_i}(x_i)$. Since $A^\vee_{lf}$ in Corollary 2.8 is clearly $\mathbb{C}^*$-invariant, we have $z_\infty\, x \in A^\vee_{lf}$. Thus

$$\tilde{\mu}^{A^\vee}_{\tilde{t}_i}(x_i) = z_i\, x \to z_\infty\, x \in A^\vee_{lf}.$$

By Corollary 2.8 and the compactness of $X_{\boldsymbol{\nu}}$, perhaps replacing $(\tilde{t}_i)$ and (x_i) by subsequences, we may assume that $\tilde{t}_i \to \tilde{t}_\infty \in \tilde{T}$ and $x_i \to x_\infty \in X_{\boldsymbol{\nu}}$. Hence

$$z_\infty\, x = \tilde{\mu}^{A^\vee}_{\tilde{t}_\infty}(x_\infty) \in \tilde{X}_{\boldsymbol{\nu}} \;\Rightarrow\; z_\infty \in C_x.$$

We conclude that $C_x = \mathbb{C}^*$ for any $x \in X_{\boldsymbol{\nu}}$. □

Let us set, as in the Introduction,

$$A^\vee_{\boldsymbol{\nu}} := (\pi')^{-1}(M_{\boldsymbol{\nu}}), \quad \tilde{A}^\vee_{\boldsymbol{\nu}} := \tilde{T} \cdot A^\vee_{\boldsymbol{\nu}}.$$

Corollary 2.9. $\tilde{X}_{\boldsymbol{\nu}} = \tilde{A}^\vee_{\boldsymbol{\nu}}$.

Proof of Corollary 2.9. Since $X_{\boldsymbol{\nu}} \subset A^\vee_{\boldsymbol{\nu}}$, clearly $\tilde{X}_{\boldsymbol{\nu}} \subseteq \tilde{A}^\vee_{\boldsymbol{\nu}}$. On the other hand, $\tilde{X}_{\boldsymbol{\nu}}$ is $\mathbb{C}^*$-invariant by Lemma 2.3 and contains $X_{\boldsymbol{\nu}}$, hence $\tilde{X}_{\boldsymbol{\nu}} \supseteq A_{\boldsymbol{\nu}}$. Since $\tilde{X}_{\boldsymbol{\nu}}$ is $\tilde{T}$-invariant, we also have $\tilde{X}_{\boldsymbol{\nu}} \supseteq \tilde{A}^\vee_{\boldsymbol{\nu}}$. □

It follows from Lemma 2.3 that $\tilde{X}_{\boldsymbol{\nu}}$ is the inverse image of a $\tilde{T}$-invariant open set of M. More precisely, let

$$M'_{\boldsymbol{\nu}} := \tilde{T} \cdot M_{\boldsymbol{\nu}} \subseteq M.$$

Since π' is a submersion and intertwines $\tilde{\mu}^M$ and $\tilde{\mu}^{A^\vee}$, we conclude the following:

Corollary 2.10. *$M'_{\boldsymbol{\nu}}$ is open in M and $\tilde{X}_{\boldsymbol{\nu}} = (\pi')^{-1}(M'_{\boldsymbol{\nu}})$.*

As in the Introduction, let $\tilde{M}_{\boldsymbol{\nu}} \subseteq M$ be the dense open subset of stable points for γ^M. Obviously $\tilde{M}_{\boldsymbol{\nu}}$ is $\tilde{\gamma}^M$-invariant (notation is as in the Introduction and §2.1).

Lemma 2.4. $\tilde{M}_{\boldsymbol{\nu}} = M'_{\boldsymbol{\nu}}$.

Proof of Lemma 2.4. Since $\mathbf{0} \in \imath\, \boldsymbol{\nu}^\perp$ is a regular value of $\Phi_{\boldsymbol{\nu}^\perp}$, $\tilde{M}_{\boldsymbol{\nu}} = \tilde{T}^{r-1}_{\boldsymbol{\nu}^\perp} \cdot M_{\boldsymbol{\nu}}$. Hence trivially $\tilde{M}_{\boldsymbol{\nu}} = \tilde{T}^{r-1}_{\boldsymbol{\nu}^\perp} \cdot M_{\boldsymbol{\nu}} \subseteq \tilde{T} \cdot M_{\boldsymbol{\nu}} = M'_{\boldsymbol{\nu}}$.

To prove the converse inclusion it suffices to check that $\tilde{M}_{\boldsymbol{\nu}}$ is $\tilde{T}$-invariant. For $k = 1, 2, \ldots$, let $\tilde{\tilde{\mu}}^{(k)}$ be the representation of $\tilde{T}$ on $H^0(M, A^{\otimes k})$ induced by $\tilde{\mu}^{A^\vee}$, and let $H^0(M, A^{\otimes k})^{T_{\nu^\perp}^{r-1}} \subseteq H^0(M, A^{\otimes k})$ be the subspace of those sections that are invariant under $T_{\nu^\perp}^{r-1}$ (equivalently, $\tilde{T}_{\nu^\perp}^{r-1}$). Then $m \in \tilde{M}_\nu$ if and only if for some $k = 1, 2, \ldots$ there exists $\sigma \in H^0(M, A^{\otimes k})^{T_{\nu^\perp}^{r-1}}$ such that $\sigma(m) \neq 0$. Since $\tilde{T}$ is Abelian, $\tilde{\mu}_{\tilde{t}}^{(k)}(\sigma) \in H^0(M, A^{\otimes k})^{T_{\nu^\perp}^{r-1}}$ for any $\tilde{t} \in \tilde{T}$; therefore, if $m \in M$ is stable for γ^M, then so is $\tilde{\mu}_{\tilde{t}}^M(m)$, for any $\tilde{t} \in \tilde{T}$.

□

In the following, we shall write $\tilde{A}_\nu^\vee$ for $\tilde{X}_\nu$. Since $\tilde{\mu}^{A^\vee}$ is holomorphic, proper, effective and locally free on $\tilde{A}_\nu^\vee$, we reach the following conclusion.

Corollary 2.11. *If BA 1.1 holds, then $N_\nu := \tilde{A}_\nu^\vee/\tilde{T}$ is a compact and connected orbifold of complex dimension $d + 1 - r$, and the projection*

$$p_\nu : \tilde{A}_\nu^\vee \to N_\nu \tag{21}$$

is a principal V-bundle with structure group $\tilde{T}$.

Proof. Since $\tilde{T}$ acts properly, holomorphically and locally freely on $\tilde{X}_\nu$, N_ν is a connected complex orbifold of dimension $d + 1 - r$. Furthermore, by definition of $\tilde{X}_\nu$, $p_\nu(X_\nu) = N_\nu$. Hence N_ν is compact.

□

Remark 2.3. The holomorphic slices in Corollary 2.7 provide local uniformizing charts for N_ν. Associated to p_ν and the character $\tilde{\chi}_\nu$ there is an holomorphic orbifold line bundle B_ν on N_ν.

2.4 The isomorphism between N'_ν and N_ν

We shall see that N'_ν has a natural complex structure, and that the pairs (N'_ν, B'_ν) and (N_ν, B_ν) in Corollaries 2.3 and 2.11 are naturally isomorphic as complex orbifolds and orbifold line bundles.

If $F \subseteq A_0^\vee$ is an holomorphic slice for $\tilde{\mu}^{A^\vee}$ as in Corollary 2.7, let J^F be its complex structure. Then (F, J^F) is a complex submanifold of $(A_0^\vee, J')$, and provides a local uniformizing chart for the complex orbifold N_ν.

On the other hand, given $x \in X_\nu$ let $F \subseteq X_\nu$ be a slice at x for the action $\mu^{X_\nu} : T \times X_\nu \to X_\nu$ induced by μ^X. The stabilizer $T_x \leqslant T$ of x in T is a finite subgroup of T, and by Corollary 2.7 $T_x = \tilde{T}_x$ (the stabilizer in $\tilde{T}$).

If $\epsilon > 0$, let $F_\epsilon \subseteq F$ be the intersection of F with an open ball centered at x and radius ϵ, in the Kähler metric on $A_0^\vee$ associated to $\tilde{\omega}$ in (17).

The proof of the following will be omitted.

Proposition 2.2. *If $\epsilon > 0$ is suitably small, F_ϵ is a slice for of $\tilde{\mu}^{A_0^\vee}$.*

Certainly F is not a complex submanifold of $A_0^\vee$, and in fact it does not contain any complex submanifold of positive dimension. Nonetheless, there is a natural complex structure J^F on it, that may be described as follows.

If $\ell \in \tilde{A}_{\boldsymbol{\nu}}^\vee$, the tangent space to the $\tilde{T}$-orbit of ℓ, $\tilde{\mathfrak{t}}_{A_0^\vee}(\ell) \subseteq T_\ell A_0^\vee$, is an r-dimensional complex subspace; let $S_\ell \subset T_\ell A_0^\vee$ be the orthocomplement of $\tilde{\mathfrak{t}}_{A_0^\vee}(\ell)$ for the Riemannian metric associated to (17). Thus S_ℓ is a complex subspace of $T_\ell A_0^\vee$, of dimension $d+1-r$, and we have a smoothly varying direct sum decomposition $T_\ell A^\vee = \tilde{t}_{A^\vee}(\ell) \oplus S_\ell$. Globally on $\tilde{A}_{\boldsymbol{\nu}}^\vee$, this yields a vector bundle decomposition $TA^\vee = \tilde{\mathfrak{t}}_{A^\vee} \oplus S$. Projecting along $\tilde{\mathfrak{t}}_{A^\vee}$, we obtain a morphism of vector bundles $\Pi : TA^\vee \to S$ (on $A_{\boldsymbol{\nu}}^\vee$).

Let F by any slice for $\tilde{\mu}^{A_0^\vee}$ in $\tilde{A}_{\boldsymbol{\nu}}^\vee$; in particular, by Proposition 2.2, F might be a slice for $\mu^{X_{\boldsymbol{\nu}}}$. At any $\ell \in F$, the restriction of Π_ℓ is an isomorphism of real vector spaces $\Pi_\ell^F : T_\ell F \to S_\ell$. We may define an almost complex structure J^F on F by declaring Π_ℓ^F to be an isomorphism of complex vector spaces for each $\ell \in F$. If F is an holomorphic slice, J^F clearly coincides with the complex structure of F as a submaifold of $A_0^\vee$.

It is clear that the same J^F would be defined, if instead of S one had chosen another complementary complex subundle S' to $\tilde{\mathfrak{t}}_{A_0^\vee}$. The following characterization does not involve the choice of a specific sub-bundle.

Lemma 2.5. *If $\ell \in F$ and $v \in T_\ell F$, then $J_\ell^F(v)$ is uniquely determined by the conditions:*

- $J_\ell^F(v) \in T_\ell L$;
- $J_\ell^F(v) - J'_\ell(v) \in \tilde{\mathfrak{t}}_{A^\vee}(\ell)$.

Proof of Lemma 2.5. For $v \in T_\ell A^\vee$, let $v_t \in \tilde{\mathfrak{t}}_{A^\vee}(\ell)$ and $v_s \in S_\ell$ be its components. As both $\tilde{\mathfrak{t}}_{A^\vee}(\ell)$ and S_ℓ are complex subspaces for J'_ℓ,

$$J'_\ell(v_s) = J'_\ell(v)_s, \quad J'_\ell(v_t) = J'_\ell(v)_t.$$

By definition of J^F if $v \in T_\ell A^\vee$ then

$$J_\ell^F(v)_s = J'_\ell(v_s) = J'_\ell(v)_s.$$

Hence,

$$\big(J_\ell^F(v) - J'_\ell(v)\big)_s = J'_\ell(v)_s - J'_\ell(v)_s = 0 \quad \Rightarrow \quad J_\ell^F(v) - J'_\ell(v) \in \tilde{\mathfrak{t}}_{A^\vee}(\ell).$$

Suppose that $I_\ell^F : T_\ell F \to T_\ell F$ is another operator such that $I_\ell^F(v) - J'_\ell(v) \in \tilde{\mathfrak{t}}_{A^\vee}(\ell)$ for every $v \in T_\ell F$. Then (by definition of slice) $\forall\, v \in T_\ell F$ we have

$$I_\ell^F(v) - J_\ell^F(v) = \big(I_\ell^F(v) - J'_\ell(v)\big) - \big(J_\ell^F(v) - J'_\ell(v)\big) \in T_\ell F \cap \tilde{\mathfrak{t}}_{A^\vee}(\ell) = (0).$$

□

Consider two slices F_1, $F_2 \subset \tilde{A}^\vee_{\boldsymbol{\nu}}$ for $\tilde{\mu}^{A_0^\vee}$ such that $p_{\boldsymbol{\nu}}(F_1) \subseteq p_{\boldsymbol{\nu}}(F_2)$. Let $\ell_j \in F_j$ be such that $p_{\boldsymbol{\nu}}(\ell_1) = p_{\boldsymbol{\nu}}(\ell_2)$. Hence there exists $\tilde{t} \in \tilde{T}$ such that $\ell_2 = \tilde{\mu}^{A_0^\vee}_{\tilde{t}}(\ell_1)$. Perhaps after restricting F_1, we may find a unique $\mathcal{C}^\infty$ function $f : F_1 \to \tilde{\mathfrak{t}}$, such that $f(\ell_1) = \mathbf{0}$ and $\jmath(\ell) := \tilde{\mu}^{A_0^\vee}_{\tilde{t}e^{f(\ell)}}(\ell) \in F_2$, for all $\ell \in F_1$. Thus $\jmath : F_1 \to F_2$ is an injection in the sense of Satake ([S1], [S2]).

Lemma 2.6. *$\jmath : F_1 \to F_2$ is $\left(J^{F_1}, J^{F_2}\right)$-holomorphic.*

Proof of Lemma 2.6. By local uniqueness, it suffices to prove that

$$\mathrm{d}_{\ell_1}\jmath : (T_{\ell_1}F_1, J^{F_1}_{\ell_1}) \to (T_{\ell_2}F_2, J^{F_2}_{\ell_2})$$

is $\mathbb{C}$-linear. If $v \in T_{\ell_1}F$, we have

$$\mathrm{d}_{\ell_1}f(v) \in \mathfrak{t}, \quad \mathrm{d}_{\ell_1}f(v)_{A^\vee} \in \mathfrak{X}(A^\vee), \quad \mathrm{d}_{\ell_1}f(v)_{A^\vee}(\ell_2) \in \tilde{\mathfrak{t}}_{A^\vee}(\ell_2) \subseteq T_{\ell_2}A^\vee,$$

and

$$\mathrm{d}_{\ell_1}\jmath(v) = \mathrm{d}_{\ell_1}f(v)_{A^\vee}(\ell_2) + \mathrm{d}_{\ell_1}\tilde{\mu}^{A_0^\vee}_{\tilde{t}}(v). \tag{22}$$

If $w, w' \in T_{\ell_2}A^\vee$, we shall write $w \equiv w'$ to mean that $w - w' \in \tilde{\mathfrak{t}}_{A^\vee}(\ell_2)$. By (22), we have $\mathrm{d}_{\ell_1}\jmath(v) \equiv \mathrm{d}_{\ell_1}\tilde{\mu}^{A_0^\vee}_{\tilde{t}}(v)$ for every $v \in T_{\ell_1}F_1$. Replacing v with $J^{F_1}_{\ell_1}(v)$, in view of Lemma 2.5 we obtain

$$\begin{aligned} \mathrm{d}_{\ell_1}\jmath\left(J^{F_1}_{\ell_1}(v)\right) &\equiv \mathrm{d}_{\ell_1}\tilde{\mu}^{A_0^\vee}_{\tilde{t}}\left(J^{F_1}_{\ell_1}(v)\right) \equiv \mathrm{d}_{\ell_1}\tilde{\mu}^{A_0^\vee}_{\tilde{t}}\left(J'_{\ell_1}(v)\right) \\ &= J'_{\ell_2}\left(\mathrm{d}_{\ell_1}\tilde{\mu}^{A_0^\vee}_{\tilde{t}}(v)\right) \equiv J'_{\ell_2}\left(\mathrm{d}_{\ell_1}\jmath(v)\right) \equiv J^{F_2}_{\ell_2}\left(\mathrm{d}_{\ell_1}\jmath(v)\right). \end{aligned} \tag{23}$$

The first and the last vector in (23) belong to $T_{\ell_2}F_2$; hence by Lemma 2.5 $\mathrm{d}_{\ell_1}\jmath\left(J^{F_1}_{\ell_1}(v)\right) = J^{F_2}_{\ell_2}\left(\mathrm{d}_{\ell_1}\jmath(v)\right)$, for all $v \in T_{\ell_1}F_1$.

□

In Lemma 2.6, we may assume by Corollary 2.7 that F_2, say, is holomorphic; hence Lemma 2.6 implies the following.

Corollary 2.12. *For any slice $F \subset \tilde{A}^\vee_{\boldsymbol{\nu}}$ for $\tilde{\mu}^{A_0^\vee}$, J^F is integrable.*

We may also take $F = F_1 = F_2$ be a slice at $\ell \in A_0^\vee$, and consider the self-injections of F induced by the stabilizer $T_\ell \leqslant T$ of ℓ.

Corollary 2.13. *If $\ell \in A^\vee_{\boldsymbol{\nu}}$ and $F \subset A^\vee_{\boldsymbol{\nu}}$ is a slice for $\tilde{\mu}^{A_0^\vee}$ at ℓ, then $\tilde{T}_\ell$ acts holomorphically on (F, J^F).*

If we apply these considerations to the slices $F \subseteq X_{\boldsymbol{\nu}}$ for $\mu^{X_{\boldsymbol{\nu}}}$, we conclude the following.

Corollary 2.14. *The V-manifold $N'_{\boldsymbol{\nu}}$ in Corollary 2.3 is complex.*

Since every T-orbit in $X_{\boldsymbol{\nu}}$ is obviously contained in a unique $\tilde{T}$-orbit in $\tilde{A}_{\boldsymbol{\nu}}$, there is a well-defined map

$$\psi : T \cdot x \in N'_{\boldsymbol{\nu}} \mapsto \tilde{T} \cdot x \in N_{\boldsymbol{\nu}}.$$

Let $J^{N'_{\boldsymbol{\nu}}}$ and $J^{N_{\boldsymbol{\nu}}}$ be the orbifold complex structures of $N'_{\boldsymbol{\nu}}$ and $N_{\boldsymbol{\nu}}$, respectively.

Proposition 2.3. *ψ is an isomorphism of complex orbifolds $\left(N'_{\boldsymbol{\nu}},\, J^{N'_{\boldsymbol{\nu}}}\right) \rightarrow \left(N_{\boldsymbol{\nu}},\, J^{N_{\boldsymbol{\nu}}}\right)$.*

Proof of Proposition 2.3. By Corollary 2.9, any $\tilde{T}$-orbit in $\tilde{A}^{\vee}_{\boldsymbol{\nu}}$ intersects $X_{\boldsymbol{\nu}}$; thus ψ is surjective.

To prove that ψ is injective, suppose by contradiction that there exist $x_1, x_2 \in X_{\boldsymbol{\nu}}$ such that $x_2 \in \tilde{T} \cdot x_1$ (i.e., $\psi(T \cdot x_1) = \psi(T \cdot x_2)$), but $x_2 \notin T \cdot x_1$ (i.e., $T \cdot x_1 \neq T \cdot x_2$). Perhaps after replacing x_2 with another point in $T \cdot x_2$, we may assume that $x_2 = \tilde{\mu}^{A^{\vee}_0}_{e^{-\boldsymbol{\xi}}}(x_1)$ for some $\boldsymbol{\xi} \in \mathbb{R}^r \setminus \{\mathbf{0}\}$. We may write uniquely $\boldsymbol{\xi} = \boldsymbol{\xi}' + a\,\boldsymbol{\nu}$, where $\boldsymbol{\xi}' \in \boldsymbol{\nu}^{\perp}$ and $a \in \mathbb{R}$. Perhaps interchanging x_1 and x_2, we may assume without loss that $a \geq 0$.

Let us set $\boldsymbol{\eta} := \imath\,\boldsymbol{\xi} \in \mathfrak{t}$. Considering the associated vector fields $\boldsymbol{\xi}_{A^{\vee}}$, $\boldsymbol{\eta}_{A^{\vee}} \in \mathfrak{X}(A^{\vee})$ we have $-\boldsymbol{\xi}_{A^{\vee}} = J'(\boldsymbol{\eta}_{A^{\vee}})$; hence $-\boldsymbol{\xi}_{A^{\vee}}$ is the gradient vector field of the Hamiltonian function $\tilde{\Phi}^{\boldsymbol{\eta}} = \langle \tilde{\Phi}, \boldsymbol{\eta} \rangle$, where $\tilde{\Phi}$ is as in (20).

Since $x_1 \in X_{\boldsymbol{\nu}}$, we have $\tilde{\Phi}(x_1) = \imath\,\lambda\,\boldsymbol{\nu}$ for some $\lambda > 0$, hence $\tilde{\Phi}^{\boldsymbol{\eta}}(x_1) = \lambda\, a\, \|\boldsymbol{\nu}\|^2 \geq 0$. Since $\tilde{\Phi}^{\boldsymbol{\eta}}$ is strictly increasing along its gradient flow where the gradient is non-vanishing,

$$\tilde{\Phi}^{\boldsymbol{\eta}}\left(\tilde{\mu}^{A^{\vee}_0}_{e^{-t\,\boldsymbol{\xi}}}(x_1)\right) > \tilde{\Phi}^{\boldsymbol{\eta}}(x_1) \geq 0 \quad \forall\, t > 0. \tag{24}$$

On the other hand, we have

$$\boldsymbol{\eta}_{A^{\vee}_0} = \boldsymbol{\eta}^{\sharp}_M - \tilde{\Phi}^{\boldsymbol{\eta}}\,\partial_\theta \quad \Rightarrow \quad -\boldsymbol{\xi}_{A^{\vee}} = \left(J\boldsymbol{\eta}_M\right)^{\sharp} + \tilde{\Phi}^{\boldsymbol{\eta}}\, r\,\partial_r.$$

Here $r\,\partial_r$ is the generator of the 1-parameter group of diffeomorphisms $\ell \mapsto e^t\,\ell$. With ϱ as in (16), for every $t > 0$ we have

$$-\boldsymbol{\xi}_{A^{\vee}}(\varrho)\left(\tilde{\mu}^{A^{\vee}_0}_{e^{-t\,\boldsymbol{\xi}}}(x_1)\right) = \tilde{\Phi}^{\boldsymbol{\eta}}\left(\tilde{\mu}^{A^{\vee}_0}_{e^{-t\,\boldsymbol{\xi}}}(x_1)\right)\, r\,\partial_r \varrho\left(\tilde{\mu}^{A^{\vee}_0}_{e^{-t\,\boldsymbol{\xi}}}(x_1)\right) > 0.$$

It follows that $\varrho\left(\tilde{\mu}^{A^{\vee}_0}_{e^{-t\,\boldsymbol{\xi}}}(x_1)\right) > \varrho(x_1) = 1$ for $t > 0$; taking $t = 1$, we conclude that $x_2 \notin X$, a contradiction. Hence ψ is a bijection.

Let us verify that ψ is a homeomorphism. The open sets of $N'_{\boldsymbol{\nu}}$ have the form U/T, where $U \subseteq X_{\boldsymbol{\nu}}$ is open and T-invariant, and the open sets of $N_{\boldsymbol{\nu}}$ have the form $\tilde{U}/\tilde{T}$, where $\tilde{U} \subseteq A^{\vee}_{\boldsymbol{\nu}}$ is open and $\tilde{T}$-invariant. The previous argument shows that each $\tilde{T}$-orbit in $\tilde{A}^{\vee}_{\boldsymbol{\nu}}$ intersects $X_{\boldsymbol{\nu}}$ in a single T-orbit. One can see from this (and the definition of $\tilde{A}^{\vee}_{\boldsymbol{\nu}}$) that there is a bijection between the family of $\tilde{T}$-invariant open sets $\tilde{U}$ in $\tilde{A}^{\vee}_{\boldsymbol{\nu}}$ and the family of T-invariant open sets U in $X_{\boldsymbol{\nu}}$ given by $\tilde{U} \mapsto U := \tilde{U} \cap X_{\boldsymbol{\nu}}$, with inverse $U \mapsto \tilde{U} := \tilde{T} \cdot U$.

Given any such $\tilde{U}$, we have $\psi^{-1}(\tilde{U}/\tilde{T}) = U/T \subseteq N'_{\boldsymbol{\nu}}$, implying that ψ is continuous. Similarly, given any such U we have $\psi(U/T) = \tilde{U}/\tilde{T}$, implying that ψ is open. Hence ψ is a homeomorphism.

To conclude that ψ is an isomorphism of complex orbifolds, it suffices to verify that its local expressions in uniformizing charts are biholomorphisms; actually, it suffices to do so for corresponding defining families in the sense of [S1] and [S2] that cover $N'_{\boldsymbol{\nu}}$ and $N_{\boldsymbol{\nu}}$. Let F be a slice at x for $\mu^{X_{\boldsymbol{\nu}}}$ at some $x \in X_{\boldsymbol{\nu}}$; by Proposition 2.2, perhaps after shrinking F if necessary, we may assume that F is also a slice at x for $\tilde{\mu}^{\tilde{A}^{\vee}_{\boldsymbol{\nu}}}$. Hence (F, J^F) is a uniformizing chart of both $N'_{\boldsymbol{\nu}}$ and $N_{\boldsymbol{\nu}}$. By definition of ψ and the previous considerations, the identity $\mathrm{id}_F : F \to F$ is a local representative map of ψ, and it is obviously biholomorphic $(F, J^F) \to (F, J^F)$. □

The sheaf of holomorphic functions on $N_{\boldsymbol{\nu}}$ is defined equivalently by the $\tilde{T}$-invariant holomorphic functions on $\tilde{A}^{\vee}_{\boldsymbol{\nu}}$ or the T_ℓ-invariant holomorphic functions on the slices (F, J^F). Let us briefly clarify this point.

Since (F, J^F) is generally not a complex submanifold of $(A^{\vee}, J')$, arbitrary holomorphic functions on the saturation $\tilde{T} \cdot F$ needn't restrict to holomorphic functions on (F, J^F). However, this does happens if we restrict to invariant holomorphic functions.

Definition 2.2. Suppose that $\ell \in A^{\vee}_{\boldsymbol{\nu}}$ and that $F \subseteq A^{\vee}_{\boldsymbol{\nu}}$ is a slice at ℓ for $\tilde{\mu}^{\tilde{A}^{\vee}_{\boldsymbol{\nu}}}$. Let us adopt the following notation.

1. $\mathcal{O}(F)$ is the ring of J^F-holomorphic functions on F;
2. $\mathcal{O}(F)^{T_\ell} \subseteq \mathcal{O}(F)$ is the subring of T_ℓ-invariant functions in $\mathcal{O}(F)$;
3. $\mathcal{O}(\tilde{T} \cdot F)$ is the ring of J'-holomorphic functions on the saturation of F under $\tilde{\mu}^{A^{\vee}}$;
4. $\mathcal{O}(\tilde{T} \cdot F)^{\tilde{T}} \subseteq \mathcal{O}(\tilde{T} \cdot F)$ is the subring of $\tilde{\mu}^{\tilde{A}^{\vee}_{\boldsymbol{\nu}}}$-invariant functions.

Then we have the following, whose prooof will be omitted (see the argument for Proposition 2.4).

Lemma 2.7. *In the situation of Definition 2.2, restriction yields an isomorphism* $\mathcal{O}(\tilde{T} \cdot F)^{\tilde{T}} \to \mathcal{O}(F)^{T_\ell}$.

2.5 Holomorphic and CR functions on $\tilde{A}^\vee_{\boldsymbol{\nu}}$ and $X_{\boldsymbol{\nu}}$

$M_{\boldsymbol{\nu}}$ is a CR submanifold of M, and the maximal complex sub-bundle $\mathcal{H}(M_{\boldsymbol{\nu}}) \subseteq TM_{\boldsymbol{\nu}}$ has complex dimension $d+1-r$, and is as follows. If $m \in M_{\boldsymbol{\nu}}$, $(\tilde{\mathfrak{t}}^{r-1}_{\boldsymbol{\nu}^\perp})_M(m) \subseteq T_m M_{\boldsymbol{\nu}}$ is a complex subspace of dimension $r-1$, since $\tilde{\gamma}^M$ is locally free at m. Then

$$\mathcal{H}(M_{\boldsymbol{\nu}})_m = (\tilde{\mathfrak{t}}^{r-1}_{\boldsymbol{\nu}^\perp})_M(m)^{\perp_{h_m}},$$

where $h_m = g_m - \imath\,\omega_m$ is the Hermitian product on $T_m M$ associated to the Kähler metric.

Similarly, $X_{\boldsymbol{\nu}}$ is a CR submanifold of $A^\vee$. The maximal complex subbundle $\mathcal{H}(X_{\boldsymbol{\nu}}) \subset TX_{\boldsymbol{\nu}}$ is as follows. If $x \in X_{\boldsymbol{\nu}}$ and $m = \pi(x)$, then

$$\mathcal{H}(X_{\boldsymbol{\nu}})_x = \mathcal{H}(M_{\boldsymbol{\nu}})^\sharp_m. \tag{25}$$

Definition 2.3. Let be given $\boldsymbol{\lambda} \in \mathbb{Z}^r$.

For any $\tilde{T}$-invariant open subset $\tilde{U} \subseteq \tilde{A}^\vee_{\boldsymbol{\nu}}$, let $\mathcal{O}(\tilde{U})_{\boldsymbol{\lambda}}$ be the ring of holomorphic functions $\tilde{S} : \tilde{U} \to \mathbb{C}$ such that

$$\tilde{S}\left(\tilde{\mu}^{A^\vee}_{\tilde{\mathbf{t}}^{-1}}(\ell)\right) = \tilde{\chi}_{\boldsymbol{\lambda}}(\tilde{\mathbf{t}})\,\tilde{S}(\ell) \qquad (\tilde{\mathbf{t}} \in \tilde{T},\ \ell \in \tilde{U}). \tag{26}$$

For any T-invariant open subset $U \subseteq X_{\boldsymbol{\nu}}$, let let $\mathcal{CR}(U)_{\boldsymbol{\lambda}}$ be the ring of CR functions on U satisfying

$$S\left(\mu^c_{\mathbf{t}^{-1}}(x)\right) = \chi_{\boldsymbol{\lambda}}(\mathbf{t})\,S(x) \qquad (\mathbf{t} \in T,\ x \in X_{\boldsymbol{\nu}}). \tag{27}$$

Proposition 2.4. *With notation in Definition 2.3, suppose that* $U = \tilde{U} \cap X_{\boldsymbol{\nu}}$. *Then restriction yields a ring isomorphism* $\mathcal{O}(\tilde{U})_{\boldsymbol{\lambda}} \to \mathcal{CR}(U)_{\boldsymbol{\lambda}}$.

Corollary 2.15. *Restriction yields an isomorphism* $\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{\boldsymbol{\lambda}} \to \mathcal{CR}(X_{\boldsymbol{\nu}})_{\boldsymbol{\lambda}}$.

Proof of Proposition 2.4. Clearly if $\tilde{S} \in \mathcal{O}(\tilde{U})_{\boldsymbol{\lambda}}$ then $S := \tilde{S}\Big|_U \in \mathcal{CR}(U)_{\boldsymbol{\lambda}}$. Thus the ring homomorphim in the statement is well-defined and obviously injective.

To prove surjectivity, suppose conversely that $S \in \mathcal{CR}(U)_{\boldsymbol{\lambda}}$. Let us define $\tilde{S} : \tilde{U} = \tilde{T} \cdot U \to \mathbb{C}$ by setting

$$\tilde{S}\left(\tilde{\mu}^{A_0^\vee}_{\tilde{\mathbf{t}}}(x)\right) = \tilde{\chi}_{-\boldsymbol{\lambda}}(\tilde{\mathbf{t}})\,S(x) = \tilde{\chi}_{\boldsymbol{\lambda}}(\tilde{\mathbf{t}})^{-1}\,S(x) \qquad (\tilde{\mathbf{t}} \in \tilde{T}, \quad x \in X_{\boldsymbol{\nu}}). \tag{28}$$

To verify that (28) is well-defined, suppose that $\tilde{\mu}^{A_0^\vee}_{\tilde{\mathbf{t}}_1}(x_1) = \tilde{\mu}^{A_0^\vee}_{\tilde{\mathbf{t}}_2}(x_2)$ with $\tilde{\mathbf{t}}_j \in \tilde{T}$ and $x_j \in U$. By the argument in the proof of Proposition 2.3, $\tilde{\mathbf{t}}_2^{-1}\,\tilde{\mathbf{t}}_1 \in T$. Therefore

$$S(x_2) = \chi_{-\boldsymbol{\lambda}}\left(\tilde{\mathbf{t}}_2^{-1}\,\tilde{\mathbf{t}}_1\right)\, S(x_1) = \tilde{\chi}_{-\boldsymbol{\lambda}}\left(\tilde{\mathbf{t}}_2\right)^{-1}\,\tilde{\chi}_{-\boldsymbol{\lambda}}\left(\tilde{\mathbf{t}}_1\right)\, S(x_1).$$

By construction, $\tilde{S}$ satisfies (26) and restricts to S on U. To prove that $S \mapsto \tilde{S}$ inverts restriction it remains to verify that $\tilde{S}$ is holomorphic, i.e. that $\mathrm{d}_\ell \tilde{S}$ is $\mathbb{C}$-linear for any $\ell \in \tilde{U}$.

By Corollary 2.9, the map

$$\mathcal{F} : (\tilde{\mathbf{t}}, x) \in \tilde{T} \times X_{\boldsymbol{\nu}} \mapsto \tilde{\mu}^{A_0^\vee}_{\tilde{\mathbf{t}}}(x) \in \tilde{A}^\vee_{\boldsymbol{\nu}}$$

is surjective. In fact, $\mathcal{F}$ exhibits $\tilde{A}^\vee_{\boldsymbol{\nu}}$ as the quotient of $\tilde{T} \times X_{\boldsymbol{\nu}}$ by the free action of T given by

$$\mathbf{t} \cdot \left(\tilde{\mathbf{t}},\, x\right) := \left(\tilde{\mathbf{t}}\,\mathbf{t}^{-1},\, \mu^X_{\mathbf{t}}(x)\right). \tag{29}$$

Furthermore, $\mathcal{F}(\tilde{T} \times U) = \tilde{U}$.

For every $\tilde{\mathbf{t}} \in \tilde{T}$ let us set

$$X^{\tilde{\mathbf{t}}}_{\boldsymbol{\nu}} := \mathcal{F}\left(\left\{\tilde{\mathbf{t}}\right\} \times X_{\boldsymbol{\nu}}\right) = \tilde{\mu}^{A^\vee}_{\tilde{\mathbf{t}}}\left(X_{\boldsymbol{\nu}}\right).$$

Again by the proof of Proposition 2.3 we have $X^{\tilde{\mathbf{t}}_1}_{\boldsymbol{\nu}} = X^{\tilde{\mathbf{t}}_2}_{\boldsymbol{\nu}}$ if $\tilde{\mathbf{t}}_1^{-1}\,\tilde{\mathbf{t}}_2 \in T$, and $X^{\tilde{\mathbf{t}}_1}_{\boldsymbol{\nu}} \cap X^{\tilde{\mathbf{t}}_2}_{\boldsymbol{\nu}} = \emptyset$ otherwise. Clearly $X^{\tilde{\mathbf{t}}}_{\boldsymbol{\nu}}$ is a CR submanifold of $A^\vee$, and its CR bundle $\mathcal{H}(X^{\tilde{\mathbf{t}}}_{\boldsymbol{\nu}})$ is as follows. If $\ell = \mathcal{F}(\tilde{\mathbf{t}}, x) \in X^{\tilde{\mathbf{t}}}_{\boldsymbol{\nu}}$, then

$$\mathcal{H}(X^{\tilde{\mathbf{t}}}_{\boldsymbol{\nu}})_\ell = \mathrm{d}_x \tilde{\mu}^{A^\vee}_{\tilde{\mathbf{t}}}\big(\mathcal{H}(X_{\boldsymbol{\nu}})_x\big).$$

If $\tilde{\mathbf{t}} \in \tilde{T}$, let us identify $T_{\tilde{\mathbf{t}}}\tilde{T} \cong \tilde{\mathfrak{t}}$ in the standard manner. For $(\tilde{\mathbf{t}},\, x) \in \tilde{T} \times X_{\boldsymbol{\nu}}$, let us consider the vector subspace

$$\mathcal{K}(\tilde{\mathbf{t}}, x) := \tilde{\mathfrak{t}} \times \mathcal{H}(X_{\boldsymbol{\nu}})_x \subseteq T_{\tilde{\mathbf{t}}}\tilde{T} \times T_x X_{\boldsymbol{\nu}} \cong T_{(\tilde{\mathbf{t}},x)}(\tilde{T} \times X_{\boldsymbol{\nu}}).$$

The distribution $\mathcal{K} \subseteq T(\tilde{T} \times X_{\boldsymbol{\nu}})$ is invariant under (29) and is naturally a complex vector bundle; furthermore, $\mathrm{d}\mathcal{F}$ yields an isomorphism of complex vector bundles $\mathcal{K} \to \mathcal{F}^*(T\tilde{A}^\vee_{\boldsymbol{\nu}})$. More explicitly, if $\ell = \mathcal{F}(\tilde{\mathbf{t}}, x)$ then

$$\mathrm{d}_{(\tilde{\mathbf{t}},x)}\mathcal{F}\big|_{\mathcal{K}(\tilde{t},x)} : \tilde{\mathfrak{t}} \times \mathcal{H}(X_{\boldsymbol{\nu}})_x \to \tilde{\mathfrak{t}}_{A^\vee}(\ell) \oplus \mathcal{H}(X^{\tilde{\mathbf{t}}}_{\boldsymbol{\nu}})_\ell = T_\ell A^\vee \tag{30}$$

is an isomorphism of complex vector spaces, respecting the direct sum decompositions on both sides.

Given $S \in \mathcal{CR}(U)_{\boldsymbol{\lambda}}$, let us consider the complex function $\hat{S}$ on $\tilde{T} \times X_{\boldsymbol{\nu}}$ given by

$$\hat{S}\left(\tilde{\mathbf{t}},\, x\right) := \tilde{\chi}_{\boldsymbol{\nu}}(\tilde{\mathbf{t}})^{-1}\, S(x). \tag{31}$$

Then $\hat{S} = \tilde{S} \circ \mathcal{F}$.

Let us assume that $\ell = \mathcal{F}(\tilde{\mathbf{t}}, x)$. We have

$$\mathrm{d}_{(\tilde{\mathbf{t}},x)}\hat{S}\Big|_{\mathcal{H}(\tilde{\mathbf{t}},x)} = \mathrm{d}_\ell \tilde{S} \circ \mathrm{d}_{(\tilde{\mathbf{t}},x)}\mathcal{F}\big|_{\mathcal{K}(\tilde{\mathbf{t}},x)} : \mathcal{K}(\tilde{\mathbf{t}}, x) \to \mathbb{C}.$$

Hence to prove that $\mathrm{d}_\ell \tilde{S} : T_\ell A^\vee_{\boldsymbol{\nu}} \to \mathbb{C}$ is $\mathbb{C}$-linear it suffices to show that $\mathrm{d}_{(\tilde{\mathbf{t}},x)}\hat{S}$ is $\mathbb{C}$-linear on $\mathcal{K}_{(\tilde{\mathbf{t}},x)} = \tilde{\mathfrak{t}} \times \mathcal{H}(X_{\boldsymbol{\nu}})_x$; to do so, in turn it is sufficient to verify $\mathbb{C}$-linearity on each summand $\tilde{\mathfrak{t}}$ and $\mathcal{H}(X_{\boldsymbol{\nu}})_x$ separately. This follows from (31), since $\tilde{\chi}_{\boldsymbol{\nu}}$ is holomorphic (implying $\mathbb{C}$-linearity on the first summand) and S is CR (implying $\mathbb{C}$-linearity on the second summand).

□

2.6 The orbifold line bundles $B_{\boldsymbol{\nu}}$ and $B'_{\boldsymbol{\nu}}$

We have seen that the restrictions of μ^X to $X_{\boldsymbol{\nu}}$ and of $\tilde{\mu}^{A^\vee}$ to $\tilde{A}^\vee_{\boldsymbol{\nu}}$ are locally free, effective and proper actions of T and $\tilde{T}$, respectively, and that the corresponding quotients $N'_{\boldsymbol{\nu}} := X_{\boldsymbol{\nu}}/T$ and $N_{\boldsymbol{\nu}} := \tilde{A}^\vee_{\boldsymbol{\nu}}/\tilde{T}$ are naturally isomorphic complex orbifolds. Furthermore, the projections $p'_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \to N'_{\boldsymbol{\nu}}$ and $p_{\boldsymbol{\nu}} : \tilde{A}^{\boldsymbol{\nu}}_{\boldsymbol{\nu}} \to N_{\boldsymbol{\nu}}$ are principal V-bundles with structure group T and $\tilde{T}$, respectively.

Associated to the characters $\chi_{\boldsymbol{\nu}} : T \to S^1$ and $\tilde{\chi}_{\boldsymbol{\nu}} : \tilde{T} \to \mathbb{C}^*$, we have 1-dimensional representations of T and $\tilde{T}$, respectively; we shall denote either one by $\mathbb{C}_{\boldsymbol{\nu}}$. The product actions $\mu^{X\times\mathbb{C}_{\boldsymbol{\nu}}}$ and $\tilde{\mu}^{A^\vee\times\mathbb{C}_{\boldsymbol{\nu}}}$ are therefore also locally free, effective and proper on $X_{\boldsymbol{\nu}} \times \mathbb{C}_{\boldsymbol{\nu}}$ and $\tilde{A}^\vee_{\boldsymbol{\nu}} \times \mathbb{C}_{\boldsymbol{\nu}}$ respectively. Hence the quotients $B'_{\boldsymbol{\nu}} := X_{\boldsymbol{\nu}} \times_T \mathbb{C}_{\boldsymbol{\nu}}$ and $B_{\boldsymbol{\nu}} := \tilde{A}^\vee_{\boldsymbol{\nu}} \times_{\tilde{T}} \mathbb{C}_{\boldsymbol{\nu}}$ are orbifold line bundles on $N'_{\boldsymbol{\nu}}$ and $N_{\boldsymbol{\nu}}$. Let us denote by

$$P'_{\boldsymbol{\nu}} : B'_{\boldsymbol{\nu}} \to N'_{\boldsymbol{\nu}} \quad \text{and} \quad P_{\boldsymbol{\nu}} : B_{\boldsymbol{\nu}} \to N_{\boldsymbol{\nu}} \tag{32}$$

the respective projections.

Lemma 2.8. *Suppose $x \in X_{\boldsymbol{\nu}}$ and let $F \subseteq X_{\boldsymbol{\nu}}$ be a slice for the restriction of μ^X to $X_{\boldsymbol{\nu}}$. Then $F \times \mathbb{C}_{\boldsymbol{\nu}}$ is a slice at $(x, 0)$ for the restriction of $\mu^{X\times\mathbb{C}_{\boldsymbol{\nu}}}$ to $X_{\boldsymbol{\nu}} \times \mathbb{C}_{\boldsymbol{\nu}}$. The collection of all these slices yields a defining family for $B'_{\boldsymbol{\nu}}$.*

Similarly, suppose $\ell \in \tilde{A}^\vee_{\boldsymbol{\nu}}$ and let $F \subseteq \tilde{A}^\vee_{\boldsymbol{\nu}}$ be a slice at ℓ for the restriction of $\tilde{\mu}^{A^\vee}$ to $\tilde{A}^\vee_{\boldsymbol{\nu}}$. Then $F \times \mathbb{C}_{\boldsymbol{\nu}}$ is a slice at $(\ell, 0)$ for the restriction of $\tilde{\mu}^{A^\vee\times\mathbb{C}_{\boldsymbol{\nu}}}$ to $\tilde{A}^\vee_{\boldsymbol{\nu}} \times \mathbb{C}_{\boldsymbol{\nu}}$. The collection of all these slices yields a defining family for $B_{\boldsymbol{\nu}}$.

Proof of Lemma 2.8. Let us consider the former statement, the proof of the latter being similar.

Since $F \times \mathbb{C}_{\boldsymbol{\nu}}$ is transverse to the T-orbits in $X_{\boldsymbol{\nu}} \times \mathbb{C}_{\boldsymbol{\nu}}$, the map $h : T \times (F \times \mathbb{C}_{\boldsymbol{\nu}}) \to X_{\boldsymbol{\nu}} \times \mathbb{C}_{\boldsymbol{\nu}}$ induced by the diagonal action is a local diffeomorphism onto the open saturation $T \cdot (F \times \mathbb{C}_{\boldsymbol{\nu}}) \subseteq X_{\boldsymbol{\nu}} \times \mathbb{C}$.

Clearly we have the equality of stabilizers $T_{(x,0)} = T_x$. Furthermore, suppose $(y, w) \in F \times \mathbb{C}_{\boldsymbol{\nu}}$, $\mathbf{t} \in T$. Then

$$\mu_{\mathbf{t}}^{X \times \mathbb{C}_{\boldsymbol{\nu}}}(y, w) = \big(\mu_{\mathbf{t}}^{X}(y), \chi_{\boldsymbol{\nu}}(\mathbf{t})\, w\big) \in F \times \mathbb{C}_{\boldsymbol{\nu}}$$

if and only if $\mu_{\mathbf{t}}^{X}(y) \in F$, that is, if and only if $\mathbf{t} \in T_x$. Hence h descends to a diffeomorphism

$$\overline{h} : \big(T \times (F \times \mathbb{C}_{\boldsymbol{\nu}})\big)/T_x \to T \cdot (F \times \mathbb{C}_{\boldsymbol{\nu}}),$$

where T_x acts antidiagonally on $T \times (F \times \mathbb{C}_{\boldsymbol{\nu}})$.

□

Corollary 2.16. *$F \times \mathbb{C}_{\boldsymbol{\nu}}$ with the diagonal action of T_x uniformizes the open set $(F \times \mathbb{C}_{\boldsymbol{\nu}})/T_x \subseteq B_{\boldsymbol{\nu}'}$. The collection of all these uniformizing charts is a defnining family for the orbifold line bundle $B'_{\boldsymbol{\nu}}$. A similar statement holds fo $B_{\boldsymbol{\nu}}$.*

Given the complex structure J^F on each F (Lemma 2.5), we obtain a product complex structure on $F \times \mathbb{C}_{\boldsymbol{\nu}}$. Hence both $B'_{\boldsymbol{\nu}}$ and $B_{\boldsymbol{\nu}}$ are complex orbifolds of complex dimension $d + 2 - r$.

Since any T-orbit in $X_{\boldsymbol{\nu}} \times \mathbb{C}$ is contained in a unique $\tilde{T}$-orbit in $\tilde{A}^{\vee}_{\boldsymbol{\nu}} \times \mathbb{C}$, there is a natural continuous map $\tilde{\psi} : B'_{\boldsymbol{\nu}} \to B_{\boldsymbol{\nu}}$. The proof of Proposition 2.3 can be adapted to yield the following:

Proposition 2.5. *$\tilde{\psi}$ is an isomorphism of complex orbifolds, and $\psi \circ P'_{\boldsymbol{\nu}} = P_{\boldsymbol{\nu}} \circ \tilde{\psi}$.*

2.7 The orbifold circle bundle $Y_{\boldsymbol{\nu}}$

We need an alternative description of $B'_{\boldsymbol{\nu}}$. Consider the intermediate quotient

$$Y_{\boldsymbol{\nu}} := X_{\boldsymbol{\nu}}/T^{r-1}_{\boldsymbol{\nu}^{\perp}};$$

then $Y_{\boldsymbol{\nu}}$ is compact orbifold, of (real) dimension $2\,(d + 1 - r) + 1$, and the integrable and invariant CR structure on $X_{\boldsymbol{\nu}}$ descends to an integrable CR structure on $Y_{\boldsymbol{\nu}}$. We shall denote by $\mathcal{H}(Y_{\boldsymbol{\nu}})$ the CR bundle of $Y_{\boldsymbol{\nu}}$.

Let $T^1_{\boldsymbol{\nu}} \leqslant T$ be the connected compact subgroup of T associated to the Lie subalgebra $\mathrm{span}(\imath\,\boldsymbol{\nu}) \subseteq \mathfrak{t}$. Given that $\boldsymbol{\nu}$ is coprime, we have a Lie group isomorphism

$$\kappa_{\boldsymbol{\nu}} : e^{\imath\,\vartheta} \in S^1 \mapsto e^{\imath\,\vartheta\,\boldsymbol{\nu}} := \left(e^{\imath\,\vartheta\,\nu_1}, \ldots, e^{\imath\,\vartheta\,\nu_r}\right) \in T^1_{\boldsymbol{\nu}}. \tag{33}$$

Let us set $\overline{T}^1_{\boldsymbol{\nu}} := T/T^{r-1}_{\boldsymbol{\nu}^\perp} \cong T^1_{\boldsymbol{\nu}}/(T^1_{\boldsymbol{\nu}} \cap T^{r-1}_{\boldsymbol{\nu}^\perp})$.

Suppose $x \in X_{\boldsymbol{\nu}}$, and let $F \subseteq X_{\boldsymbol{\nu}}$ be a slice at x for the restriction of γ^X to $X_{\boldsymbol{\nu}}$. We can view $T \times F$ as a uniformizing chart for the smooth orbifold $X_{\boldsymbol{\nu}}$, with uniformized open set $T \cdot F = (T \times F)/T_x$. Then $\overline{T}^1_{\boldsymbol{\nu}} \times F$ is a uniformizing chart for $Y_{\boldsymbol{\nu}}$, covering the open set $(T \cdot F)/T^{r-1}_{\boldsymbol{\nu}^\perp}$.

Explicitly, T_x act effectively on $\overline{T}^1_{\boldsymbol{\nu}} \times F$ by

$$t_0 \cdot \left(\overline{t}, f\right) := \left(\overline{t}\,\overline{t}_0^{-1}, \mu^X_{t_0}(f)\right),$$

where for any $t \in T$ we have set $\overline{t} = t\,T^{r-1}_{\boldsymbol{\nu}^\perp} \in \overline{T}^1_{\boldsymbol{\nu}}$. Then the map

$$\gamma : (\overline{t}, f) \in \overline{T}^1_{\boldsymbol{\nu}} \times F \mapsto T^{r-1}_{\boldsymbol{\nu}^\perp} \cdot \mu^X_t(f) \in (T \cdot F)/T^{r-1}_{\boldsymbol{\nu}^\perp} \subseteq Y_{\boldsymbol{\nu}}$$

induces a homeomorphism $(T \cdot F)/T^{r-1}_{\boldsymbol{\nu}^\perp} = (\overline{T}^1 \times F)/T_\ell$. Letting F vary, we obtain a defining family for $Y_{\boldsymbol{\nu}}$.

Furthermore, $\overline{T}^1_{\boldsymbol{\nu}}$ acts effectively on $Y_{\boldsymbol{\nu}}$, and $N'_{\boldsymbol{\nu}} = Y_{\boldsymbol{\nu}}/\overline{T}^1_{\boldsymbol{\nu}}$; let $\sigma_{\boldsymbol{\nu}} : Y_{\boldsymbol{\nu}} \to N'_{\boldsymbol{\nu}}$ be the projection. For each slice $F \subseteq X_{\boldsymbol{\nu}}$, as above, the local representation of $\sigma_{\boldsymbol{\nu}}$ is the projection $\overline{T}^1_{\boldsymbol{\nu}} \times F \to F$. Thus $Y_{\boldsymbol{\nu}}$ is a principal V-bundle over $N'_{\boldsymbol{\nu}}$, with structure group $\overline{T}^1_{\boldsymbol{\nu}}$.

Being trivial on $T^{r-1}_{\boldsymbol{\nu}^\perp}$, $\chi_{\boldsymbol{\nu}}$ descends to a character $\chi'_{\boldsymbol{\nu}} : \overline{T}^1_{\boldsymbol{\nu}} \to S^1$.

Lemma 2.9. *Given that $\boldsymbol{\nu}$ is coprime, $\chi'_{\boldsymbol{\nu}}$ is a Lie group isomorphism.*

Proof of Lemma 2.9. Since $\overline{T}^1_{\boldsymbol{\nu}} \cong T^1_{\boldsymbol{\nu}}/\left(T^1_{\boldsymbol{\nu}} \cap T^{r-1}_{\boldsymbol{\nu}^\perp}\right)$, the statement is equivalent to the equality

$$\ker(\chi_{\boldsymbol{\nu}}|_{T^1_{\boldsymbol{\nu}}}) = T^1_{\boldsymbol{\nu}} \cap T^{r-1}_{\boldsymbol{\nu}^\perp}; \tag{34}$$

since clearly $T^{r-1}_{\boldsymbol{\nu}^\perp} \subseteq \ker(\chi_{\boldsymbol{\nu}})$, we need only prove that $\ker(\chi_{\boldsymbol{\nu}}|_{T^1_{\boldsymbol{\nu}}}) \subseteq T^1_{\boldsymbol{\nu}} \cap T^{r-1}_{\boldsymbol{\nu}^\perp}$.

Since $\boldsymbol{\nu}$ is coprime, there exists $\mathbf{k} = \begin{pmatrix} k_1 & \cdots & k_r \end{pmatrix} \in \mathbb{Z}^r$ such that $\langle \boldsymbol{\nu}, \mathbf{k} \rangle = \sum_{j=1}^r k_j\,\nu_j = 1$.

Let $\kappa_{\boldsymbol{\nu}}$ be as in (33). Then

$$\chi_{\boldsymbol{\nu}} \circ \kappa_{\boldsymbol{\nu}}\left(e^{\imath\,\vartheta}\right) = \chi_{\boldsymbol{\nu}}\left(e^{\imath\,\vartheta\,\boldsymbol{\nu}}\right) = e^{\imath\,\vartheta\,\|\boldsymbol{\nu}\|^2} \quad \left(e^{\imath\,\vartheta} \in S^1\right). \tag{35}$$

Hence if $e^{\imath\,\vartheta\,\boldsymbol{\nu}} \in \ker(\chi_{\boldsymbol{\nu}})$, then we may assume $\vartheta = \vartheta_j := 2\,\pi\,j/\|\boldsymbol{\nu}\|^2$ for some $j = 0, \ldots, \|\boldsymbol{\nu}\|^2 - 1$. We have

$$\langle \vartheta_j\,\boldsymbol{\nu}, \boldsymbol{\nu}\rangle = \frac{2\,\pi\,j}{\|\boldsymbol{\nu}\|^2}\,\langle\,\boldsymbol{\nu}, \boldsymbol{\nu}\rangle = 2\,\pi\,j = 2\,\pi\,j\,\langle \mathbf{k}, \boldsymbol{\nu}\rangle,$$

so that $\vartheta_j\,\boldsymbol{\nu} - 2\,\pi\,j\,\mathbf{k} \in \boldsymbol{\nu}^\perp$. Thus

$$e^{\imath\,\vartheta_j\,\boldsymbol{\nu}} = e^{\imath\,[\vartheta_j\,\boldsymbol{\nu} - 2\,\pi\,j\,\mathbf{k}]} \in T^1_{\boldsymbol{\nu}} \cap T^{r-1}_{\boldsymbol{\nu}^\perp}.$$

□

Since $k_{\boldsymbol{\nu}}$ is an isomorphism, (35) implies the following.

Corollary 2.17. *Assuming that $\boldsymbol{\nu}$ is coprime,*

$$\left|T^{r-1}_{\boldsymbol{\nu}^\perp} \cap T^1_{\boldsymbol{\nu}}\right| = \|\boldsymbol{\nu}\|^2.$$

Given the isomorphism $\chi'_{\boldsymbol{\nu}} : \overline{T}^1_{\boldsymbol{\nu}} \cong S^1$, we shall view $Y_{\boldsymbol{\nu}}$ as a principal V-bundle over $N'_{\boldsymbol{\nu}}$ with structure group S^1. Let us denote by

$$\sigma^{Y_{\boldsymbol{\nu}}} : S^1 \times Y_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}} \tag{36}$$

the corresponding action.

Let

$$Q_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}} \tag{37}$$

be the projection. Then U is a T-invariant open subset of $X_{\boldsymbol{\nu}}$ if and only if its image $Q_{\boldsymbol{\nu}}(U)$ is a $\overline{T}^1_{\boldsymbol{\nu}} \cong S^1$-invariant open subset of $Y_{\boldsymbol{\nu}}$. It follows (recall the proof of Proposition 2.3) that there is a bijective correspondence between $\tilde{T}$-invariant open subsets $\tilde{U} \subseteq \tilde{A}^\vee_{\boldsymbol{\nu}}$, T-invariant open subsets $U \subseteq X_{\boldsymbol{\nu}}$, S^1-invariant open subsets $\overline{U} \subseteq Y_{\boldsymbol{\nu}}$, given by $U; = \tilde{U} \cap X_{\boldsymbol{\nu}}$, $\overline{U} := Q_{\boldsymbol{\nu}}(U)$.

The character $\chi'_{k\,\boldsymbol{\nu}} = (\chi'_{\boldsymbol{\nu}})^k : \overline{T}^1_{\boldsymbol{\nu}} \to S^1$ corresponds to the endomorphism $\chi_k : g \in S^1 \mapsto g^k \in S^1$. Let us denote by $\mathcal{CR}(Y_{\boldsymbol{\nu}})$ the collection of all CR functions on $Y_{\boldsymbol{\nu}}$, and for any $k \in \mathbb{Z}$ let us set

$$\mathcal{CR}(Y_{\boldsymbol{\nu}})_k = \left\{f \in \mathcal{CR}(Y_{\boldsymbol{\nu}})\ :\ f \circ \sigma^{Y_{\boldsymbol{\nu}}}_{e^{-\imath\,\theta}} = e^{\imath\,k\,\theta}\,f, \quad \forall\, e^{\imath\,\theta} \in S^1\right\}. \tag{38}$$

Using that the CR structure of $Y_{\boldsymbol{\nu}}$ is obtained by descending the invariant CR structure of $X_{\boldsymbol{\nu}}$, we can complement Proposition 2.4 and Corollary 2.15 by the following isomorphisms induced by pull-back:

$$\mathcal{O}\big(\tilde{U}\big)_{k\,\boldsymbol{\nu}} \cong \mathcal{CR}(U)_{k\,\boldsymbol{\nu}} \cong \mathcal{CR}\left(\overline{U}\right)_k. \tag{39}$$

Letting $H^0(N_{\boldsymbol{\nu}}, B_{k\,\boldsymbol{\nu}})$ denote the space of holomorphic sections of the orbifold line bundle $B_{k\,\boldsymbol{\nu}}$, we conclude that

$$H^0(N_{\boldsymbol{\nu}}, B_{k\,\boldsymbol{\nu}}) \cong \mathcal{O}\big(\tilde{A}^\vee_{\boldsymbol{\nu}}\big)_{k\,\boldsymbol{\nu}} \cong \mathcal{CR}(X_{\boldsymbol{\nu}})_{k\,\boldsymbol{\nu}} \cong \mathcal{CR}\left(Y_{\boldsymbol{\nu}}\right)_k. \tag{40}$$

2.8 The induced Kähler structure of $N_{\boldsymbol{\nu}}$

We shall see that $P_{\boldsymbol{\nu}} : B_{\boldsymbol{\nu}} \to N_{\boldsymbol{\nu}}$ is a positive holomorphic V-line bundle. In view of Propositions 2.3 and 2.5 we may equivalently consider $P'_{\boldsymbol{\nu}} : B'_{\boldsymbol{\nu}} \to N'_{\boldsymbol{\nu}}$. With α as in (1), let $\alpha^{X_{\boldsymbol{\nu}}} := \jmath_{\boldsymbol{\nu}}^*(\alpha)$, where

$$\jmath_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \hookrightarrow X \tag{41}$$

is the inclusion. Then $\alpha^{X_{\boldsymbol{\nu}}}$ is T-invariant, and by definition of $X_{\boldsymbol{\nu}}$ for any $\boldsymbol{\xi} \in \boldsymbol{\nu}^\perp$ we have

$$\iota\big((\imath\,\boldsymbol{\xi})_{X_{\boldsymbol{\nu}}}\big)\,\alpha^{X_{\boldsymbol{\nu}}} = \jmath_{\boldsymbol{\nu}}^*\Big(\iota\big((\imath\,\boldsymbol{\xi})_X\big)\,\alpha\Big) = -\,\langle \Phi, \imath\,\boldsymbol{\xi}\rangle \circ \jmath_{\boldsymbol{\nu}} = 0.$$

Hence $\alpha^{X_{\boldsymbol{\nu}}}$ is the pull-back of an orbifold 1-form $\alpha^{Y_{\boldsymbol{\nu}}}$ on $Y_{\boldsymbol{\nu}}$. Similarly, being T-invariant, Φ descends to a smooth function $\overline{\Phi} : Y_{\boldsymbol{\nu}} \to \mathfrak{t}^\vee$; hence $\Phi^{\boldsymbol{\nu}} = \langle \Phi, \imath\,\boldsymbol{\nu}\rangle$ descends to a smooth function $\overline{\Phi}^{\boldsymbol{\nu}} : Y_{\boldsymbol{\nu}} \to \mathbb{R}$.

Clearly,

$$\iota\big((\imath\,\boldsymbol{\nu})_{Y_{\boldsymbol{\nu}}}\big)\,\alpha^{Y_{\boldsymbol{\nu}}} = -\overline{\Phi}^{\boldsymbol{\nu}}. \tag{42}$$

Let us define

$$\beta_{\boldsymbol{\nu}} := \frac{\|\boldsymbol{\nu}\|^2}{\overline{\Phi}^{\boldsymbol{\nu}}}\,\alpha^{Y_{\boldsymbol{\nu}}},$$

and let $-\delta^{Y_{\boldsymbol{\nu}}} \in \mathfrak{X}(Y_{\boldsymbol{\nu}})$ be the infintesimal generator of $\sigma^{Y_{\boldsymbol{\nu}}}$ in (36). Thus by Corollary 2.17

$$-\,\delta^{Y_{\boldsymbol{\nu}}} = \frac{1}{\|\boldsymbol{\nu}\|^2}\,(\imath\,\boldsymbol{\nu})_{Y_{\boldsymbol{\nu}}}. \tag{43}$$

Given (43) and (42), we conclude the following.

Corollary 2.18. *$\beta_{\boldsymbol{\nu}}$ is $\sigma^{Y_{\boldsymbol{\nu}}}$-invariant, and $\beta_{\boldsymbol{\nu}}(\delta^{Y_{\boldsymbol{\nu}}}) = 1$.*

Hence $\beta_{\boldsymbol{\nu}}$ is a connection 1-form for the principal V-bundle $P'_{\boldsymbol{\nu}} : B'_{\boldsymbol{\nu}} \to N'_{\boldsymbol{\nu}}$. Explicitly,

$$\mathrm{d}\beta_{\boldsymbol{\nu}} = \|\boldsymbol{\nu}\|^2\,\left[\frac{1}{\overline{\Phi}^{\boldsymbol{\nu}}}\,\mathrm{d}\alpha^{Y_{\boldsymbol{\nu}}} - \frac{1}{(\overline{\Phi}^{\boldsymbol{\nu}})^2}\,\mathrm{d}\overline{\Phi}^{\boldsymbol{\nu}} \wedge \alpha^{Y_{\boldsymbol{\nu}}}\right], \tag{44}$$

and one can also verify that $\iota(\delta^{Y_{\boldsymbol{\nu}}})\,\mathrm{d}\beta_{\boldsymbol{\nu}} = 0$ by direct inspection using (43) and (44). Furthermore, the kernel of $\beta_{\boldsymbol{\nu}}$ is the CR bundle of $Y_{\boldsymbol{\nu}}$:

$$\ker(\beta_{\boldsymbol{\nu}}) = \ker(\alpha^{Y_{\boldsymbol{\nu}}}) = \mathcal{H}(Y_{\boldsymbol{\nu}}).$$

Hence we reach the following conclusion. Let $\pi_{\boldsymbol{\nu}} : Y_{\boldsymbol{\nu}} \to N'_{\boldsymbol{\nu}}$ be the projection.

Lemma 2.10. *There exists a $(1,1)$-form $\eta'_{\boldsymbol{\nu}}$ on $N'_{\boldsymbol{\nu}}$ such that $\mathrm{d}(\beta_{\boldsymbol{\nu}}) = 2\,\pi_{\boldsymbol{\nu}}^*(\eta'_{\boldsymbol{\nu}})$.*

We shall denote by $\eta_{\boldsymbol{\nu}}$ the corresponding form on $N_{\boldsymbol{\nu}}$. With the notation of Proposition 2.3, we have the following.

Proposition 2.6. *$(N'_{\boldsymbol{\nu}}, J^{N'_{\boldsymbol{\nu}}}, \eta'_{\boldsymbol{\nu}})$ and $(N_{\boldsymbol{\nu}}, J^{N_{\boldsymbol{\nu}}}, \eta_{\boldsymbol{\nu}})$ are isomorphic Kähler orbifolds. In particular, $(Y_{\boldsymbol{\nu}},\, \beta_{\boldsymbol{\nu}})$ is a contact orbifold.*

Proof of Proposition 2.6. It suffices to prove that $(N'_{\boldsymbol{\nu}}, J^{N'_{\boldsymbol{\nu}}}, \eta'_{\boldsymbol{\nu}})$ is a Kähler orbifold, since the other statements follow directly.

The uniformized tangent space of $Y_{\boldsymbol{\nu}}$ splits as the direct sum $V(Y_{\boldsymbol{\nu}}) \oplus H(Y_{\boldsymbol{\nu}})$, where $V(Y_{\boldsymbol{\nu}})$ is the tangent space to the orbits of $\sigma^{Y_{\boldsymbol{\nu}}}$. To check that $\eta_{\boldsymbol{\nu}}$ is Kähler, it suffices therefore to verify that the restriction of $\mathrm{d}\beta_{\boldsymbol{\nu}}$ to $H(Y_{\boldsymbol{\nu}})$ is compatible with the complex structure. In view of (44) and T-invariance, we need only check that the form

$$Q_{\boldsymbol{\nu}}^*(\mathrm{d}\beta_{\boldsymbol{\nu}}) = \|\boldsymbol{\nu}\|^2\, \jmath_{\boldsymbol{\nu}}^* \left(\frac{1}{\Phi^{\boldsymbol{\nu}}}\, \mathrm{d}\alpha - \frac{1}{(\Phi^{\boldsymbol{\nu}})^2}\, \mathrm{d}\Phi^{\boldsymbol{\nu}} \wedge \alpha \right), \tag{45}$$

where $Q_{\boldsymbol{\nu}}$ is as in (37) and $\jmath_{\boldsymbol{\nu}}$ as in (41), is compatible with the complex structure of the CR bundle $\mathcal{H}(X_{\boldsymbol{\nu}})$.

Suppose $x \in X_{\boldsymbol{\nu}}$ and let $m := \pi(x) \in M_{\boldsymbol{\nu}}$. The general vector in $\mathcal{H}(X_{\boldsymbol{\nu}})_x$ has the form $v^\sharp$ for some $v \in \mathcal{H}(M_{\boldsymbol{\nu}})_m$ (see (25)), and then $J'_x(v^\sharp) = J_m(v)^\sharp$. By (45), for any $v, w \in \mathcal{H}(M_{\boldsymbol{\nu}})_m$

$$Q_{\boldsymbol{\nu}}^*(\mathrm{d}\beta_{\boldsymbol{\nu}})_x(v^\sharp, w^\sharp) \;=\; \frac{\|\boldsymbol{\nu}\|^2}{\Phi^{\boldsymbol{\nu}}(m)}\, \mathrm{d}_x\alpha(v^\sharp, w^\sharp) = \frac{\|\boldsymbol{\nu}\|^2}{\Phi^{\boldsymbol{\nu}}(m)}\, 2\,\omega_m(v, w). \tag{46}$$

The statement follows, since $\Phi^{\boldsymbol{\nu}}(m) > 0$ by definition of $M_{\boldsymbol{\nu}}$, $\mathcal{H}_m(M_{\boldsymbol{\nu}}) \subseteq T_mM$ is a complex subspace, and ω is Kähler.

□

Corollary 2.19. *$(N_{\boldsymbol{\nu}}, B_{\boldsymbol{\nu}})$ is polarized Kähler orbifold.*

Here notation is as in Chaper 4 of [BG]. By the Kodaira-Baily Vanishing Theorem ([B], [BG]), we obtain the following conclusion.

Corollary 2.20. *$H^i(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}}) = 0$, $\forall\, i > 0$, $k \gg 0$.*

2.9 An Hamiltonian circle action on $N_{\boldsymbol{\nu}}$

The action $\rho^X : S^1 \times X \to X$ with infinitesimal generator $-\partial_\theta$ in (1) is the contact lift of the trivial circle action on M corresponding to the moment map $\Phi = 1$ (recall (4)). We shall see that ρ^X determines a contact action on $(Y_{\boldsymbol{\nu}},\, \beta_{\boldsymbol{\nu}})$ and an holomorphic Hamiltonian action on $(N'_{\boldsymbol{\nu}}, J^{N'_{\boldsymbol{\nu}}}, 2\,\eta'_{\boldsymbol{\nu}})$, such that the former is the contact lift of the latter by (the orbifold version of) the procedure in (4), when we regard $Y_{\boldsymbol{\nu}}$ as an orbifold circle bundle on $X_{\boldsymbol{\nu}}$.

Clearly, ρ^X commutes with μ^X. Hence ρ^X leaves $X_{\boldsymbol{\nu}}$ invariant and determines a restricted action $\rho^{X_{\boldsymbol{\nu}}} : S^1 \times X_{\boldsymbol{\nu}} \to X_{\boldsymbol{\nu}}$. For the same reason $\rho^{X_{\boldsymbol{\nu}}}$ passes to the quotients $Y_{\boldsymbol{\nu}}$ and $N_{\boldsymbol{\nu}}$. In other words, $\rho^{X_{\boldsymbol{\nu}}}$ descends to actions $\rho^{Y_{\boldsymbol{\nu}}} : S^1 \times Y_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}}$ and $\rho^{N'_{\boldsymbol{\nu}}} : S^1 \times N'_{\boldsymbol{\nu}} \to N'_{\boldsymbol{\nu}}$, so that the projections $Q_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}}$ and $\pi_{\boldsymbol{\nu}} : Y_{\boldsymbol{\nu}} \to N'_{\boldsymbol{\nu}}$ are equivariant.

In particular, if $-\partial_\theta^{X_{\boldsymbol{\nu}}}$ is the restriction of $-\partial_\theta$ to $X_{\boldsymbol{\nu}}$, $-\partial_\theta^{Y_{\boldsymbol{\nu}}}$ is the infinitesimal generator of $\rho^{Y_{\boldsymbol{\nu}}}$, and $-\partial_\theta^{N_{\boldsymbol{\nu}}}$ is the infinitesimal generator of $\rho^{N_{\boldsymbol{\nu}}}$, then $\partial_\theta^{X_{\boldsymbol{\nu}}}$ and $\partial_\theta^{Y_{\boldsymbol{\nu}}}$ are $Q_{\boldsymbol{\nu}}$-related, and similarly $\partial_\theta^{Y_{\boldsymbol{\nu}}}$ and $\partial_\theta^{N_{\boldsymbol{\nu}}}$ are $\pi_{\boldsymbol{\nu}}$-related.

Lemma 2.11. *$\rho^{N'_{\boldsymbol{\nu}}}$ is Hamiltonian on $(N'_{\boldsymbol{\nu}}, 2\,\eta'_{\boldsymbol{\nu}})$, with moment map $\|\boldsymbol{\nu}\|^2/\overline{\Phi}^{\boldsymbol{\nu}} + c$, for any $c \in \mathbb{R}$.*

Proof of Lemma 2.11. By T-invariance of all terms involved, and the previous remark about the correlations of the generating vector fields, we need only prove that

$$-\iota\left(\partial_\theta^{X_{\boldsymbol{\nu}}}\right)\, Q_{\boldsymbol{\nu}}^*(\mathrm{d}\beta_{\boldsymbol{\nu}}) = \mathrm{d}\left(\|\boldsymbol{\nu}\|^2/\Phi^{\boldsymbol{\nu}} \circ \jmath_{\boldsymbol{\nu}}\right),$$

where $\jmath_{\boldsymbol{\nu}}$ is as in (41) and $Q_{\boldsymbol{\nu}}^*(\mathrm{d}\beta_{\boldsymbol{\nu}})$ as in (45). We have on a neighborhood of $X_{\boldsymbol{\nu}}$:

$$-\iota\left(\partial_\theta\right)\, \|\boldsymbol{\nu}\|^2 \left(\frac{1}{\Phi^{\boldsymbol{\nu}}}\,\mathrm{d}\alpha - \frac{1}{(\Phi^{\boldsymbol{\nu}})^2}\,\mathrm{d}\Phi^{\boldsymbol{\nu}} \wedge \alpha\right) = -\frac{\|\boldsymbol{\nu}\|^2}{(\Phi^{\boldsymbol{\nu}})^2}\,\mathrm{d}\Phi^{\boldsymbol{\nu}} = \mathrm{d}\left(\frac{\|\boldsymbol{\nu}\|^2}{\Phi^{\boldsymbol{\nu}}}\right),$$

establishing the claim. □

Thus $-\partial_\theta^{N'_{\boldsymbol{\nu}}}$ is a Hamiltonian vector field on $(N'_{\boldsymbol{\nu}},\, 2\,\eta'_{\boldsymbol{\nu}})$, and every choice of $c \in \mathbb{R}$ in Lemma 2.11 determines a contact lift $-\widetilde{\partial_\theta^{N'_{\boldsymbol{\nu}}}}$ (implicitly depending on c) to $(Y_{\boldsymbol{\nu}},\, \beta_{\boldsymbol{\nu}})$ of $-\partial_\theta^{N'_{\boldsymbol{\nu}}}$, as in (4).

Here $Y_{\boldsymbol{\nu}}$ plays the role of X, $\beta_{\boldsymbol{\nu}}$ the role of α, and $-\partial_\theta^{N'_{\boldsymbol{\nu}}}$ the one of ξ_M. The role of $-\partial_\theta$ (the infinitesimal generator of ρ^X) is played by $-\delta^{Y_{\boldsymbol{\nu}}}$ (the infinitesimal generator of $\sigma^{Y_{\boldsymbol{\nu}}}$).

We need to determine the 'correct choice' of c that determines $\rho^{Y_{\boldsymbol{\nu}}}$ as the contact lift of $\rho^{N'_{\boldsymbol{\nu}}}$.

Lemma 2.12. *We have $\widetilde{\partial_\theta^{N'_{\boldsymbol{\nu}}}} = \partial_\theta^{Y_{\boldsymbol{\nu}}}$ if and only if $c = 0$.*

Given an (orbifold) vector field V on $N_{\boldsymbol{\nu}}$, we shall denote by $V^\natural \in \mathfrak{X}(Y_{\boldsymbol{\nu}})$ its horizontal lift to $Y_{\boldsymbol{\nu}}$ with respect to $\beta_{\boldsymbol{\nu}}$.

Proof of Lemma 2.12. On $X_{\boldsymbol{\nu}}$ we have by (4)

$$\frac{1}{\|\boldsymbol{\nu}\|^2}\,\boldsymbol{\nu}_{X_\nu} = \frac{1}{\|\boldsymbol{\nu}\|^2}\,\boldsymbol{\nu}^{\sharp}_{M_\nu} - \frac{\Phi^{\boldsymbol{\nu}}}{\|\boldsymbol{\nu}\|^2}\,\partial_\theta^{X_\nu}. \tag{47}$$

Here $\boldsymbol{\nu}_{M_\nu}$ is the restriction of $\boldsymbol{\nu}_M$ to $M_{\boldsymbol{\nu}}$ (a vector field on $M_{\boldsymbol{\nu}}$), and $\boldsymbol{\nu}^{\sharp}_{M_\nu}$ is its horizontal lift to $X_{\boldsymbol{\nu}}$.

Given that ρ^X and μ^X commute, $[\boldsymbol{\nu}_X, \partial_\theta] = 0$ on X; this implies $[\boldsymbol{\nu}^{\sharp}_M, \boldsymbol{\nu}_X] = [\boldsymbol{\nu}^{\sharp}_M, \partial_\theta] = 0$. Furthermore, one has $[\boldsymbol{\nu}_X, \boldsymbol{\gamma}_X] = 0$ for every $\boldsymbol{\gamma} \in \mathfrak{t}$, and this implies also $[\boldsymbol{\nu}^{\sharp}_M, \boldsymbol{\gamma}_X] = 0$.

Being horizontal and $T^{r-1}_{\boldsymbol{\nu}^\perp}$-invariant, $\boldsymbol{\nu}^{\sharp}_{M_\nu}/\Phi^{\boldsymbol{\nu}}$ is $\pi_{\boldsymbol{\nu}}$-related to a horizontal vector field on $Y_{\boldsymbol{\nu}}$; the latter is $\sigma^{Y^{\boldsymbol{\nu}}}$-invariant by the above, and therefore it is the horizontal lift $-\upsilon^{\natural}$ to $Y_{\boldsymbol{\nu}}$ of a vector field $-\upsilon$ on $N_{\boldsymbol{\nu}}$.

Multiplying both sides of (47) by $\|\boldsymbol{\nu}\|^2/\Phi^{\boldsymbol{\nu}}$ and then pushing down to $Y_{\boldsymbol{\nu}}$ we obtain

$$\upsilon^{\natural} - \frac{\|\boldsymbol{\nu}\|^2}{\overline{\Phi^{\boldsymbol{\nu}}}}\,\delta^{Y_\nu} = -\frac{1}{\overline{\Phi^{\boldsymbol{\nu}}}}\,\boldsymbol{\nu}^{\sharp}_{M_\nu} - \frac{\|\boldsymbol{\nu}\|^2}{\overline{\Phi^{\boldsymbol{\nu}}}}\,\delta^{Y_\nu} = -\partial_\theta^{Y_\nu}, \tag{48}$$

and pushing down to $N_{\boldsymbol{\nu}}$ this yields

$$\upsilon = -\partial_\theta^{N_\nu}. \tag{49}$$

In view of Lemma 2.11, (49) implies that υ is the Hamiltonian vector field on $(N'_{\boldsymbol{\nu}}, 2\,\eta'_{\boldsymbol{\nu}})$ of $\|\boldsymbol{\nu}\|^2/\overline{\Phi^{\boldsymbol{\nu}}} + c$; then (48) implies that $-\partial_\theta^{Y_\nu}$ is its contact lift corresponding to $c = 0$. It is clear that any other choice of c yields a different lift.

□

In the following, we shall identify the pairs $(N'_{\boldsymbol{\nu}}, B'_{k\,\boldsymbol{\nu}}) \cong (N_{\boldsymbol{\nu}}, B_{k\,\boldsymbol{\nu}})$.

2.10 The Fourier decomposition of $\mathcal{O}(\tilde{A}^{\vee}_{\boldsymbol{\nu}})_{k\,\boldsymbol{\nu}}$

Consider the holomorphic action

$$\rho^{A_0^\vee} : (e^{\imath\,\theta},\, \ell) \in S^1 \times A_0^\vee \to e^{-\imath\,\theta}\,\ell \in A_0^\vee.$$

Thus $\rho^{A_0^\vee}$ extends ρ^X. Similarly, let $\mu^{A_0^\vee} : T \times A_0^\vee \to A_0^\vee$ be the holomorphic action extending μ^X. Clearly, $\rho^{A_0^\vee}$ and $\mu^{A_0^\vee}$ commute.

The dense open subset $\tilde{A}^{\vee}_{\boldsymbol{\nu}} \subseteq A_0^\vee$ is invariant under both $\rho^{A_0^\vee}$ and $\mu^{A_0^\vee}$, which therefore restrict to commuting holomorphic actions $\rho^{\tilde{A}^\vee_\nu}$ and $\mu^{\tilde{A}^\vee_\nu}$ on $\tilde{A}^{\vee}_{\boldsymbol{\nu}}$.

Therefore, $\rho^{\tilde{A}^\vee_{\boldsymbol{\nu}}}$ and $\mu^{\tilde{A}^\vee_{\boldsymbol{\nu}}}$ determine commuting representations $\hat{\rho}^{\tilde{A}^\vee_{\boldsymbol{\nu}}}$ of S^1 and $\hat{\mu}^{\tilde{A}^\vee_{\boldsymbol{\nu}}}$ of T on the space $\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})$ of holomorphic functions on $\tilde{A}^\vee_{\boldsymbol{\nu}}$, given by

$$\hat{\rho}^{\tilde{A}^\vee_{\boldsymbol{\nu}}}_{e^{\imath\theta}}(s) := s \circ \rho^{\tilde{A}^\vee_{\boldsymbol{\nu}}}_{e^{-\imath\theta}}, \quad \hat{\mu}^{\tilde{A}^\vee_{\boldsymbol{\nu}}}_{\mathbf{t}}(s) := s \circ \mu^{\tilde{A}^\vee_{\boldsymbol{\nu}}}_{\mathbf{t}^{-1}} \qquad \big(s \in \mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}}),\, e^{\imath\theta} \in S^1,\, \mathbf{t} \in T.\big).$$

For every $l \in \mathbb{Z}$ and $\boldsymbol{\lambda} \in \mathbb{Z}^r$, let $\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_l$ and $\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{\boldsymbol{\lambda}}$ be the l-th and $\boldsymbol{\lambda}$-th isotypical components of $\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})$, respectively, for $\hat{\rho}^{\tilde{A}^\vee_{\boldsymbol{\nu}}}$ and $\hat{\mu}^{\tilde{A}^\vee_{\boldsymbol{\nu}}}$, respectively. Hence $\hat{\rho}^{\tilde{A}^\vee_{\boldsymbol{\nu}}}$ restricts to a subrepresentation on $\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{\boldsymbol{\lambda}}$. In particular, for every $k = 1, 2, \dots$ the vector space $\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{k\boldsymbol{\nu}}$ is finite dimensional by (40), and we have an $S^1 \times T$-equivariant decomposition

$$\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{k\boldsymbol{\nu}} = \bigoplus_{l \in \mathbb{Z}} \mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{k\boldsymbol{\nu},l}, \tag{50}$$

where $\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{k\boldsymbol{\nu},l} = \mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{k\boldsymbol{\nu}} \cap \mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_l$. Since the isomorphisms in (40) are by construction S^1-equivariant, (50) may be interpreted in terms of $H^0(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}})$:

$$H^0(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}}) = \bigoplus_{l \in \mathbb{Z}} H^0(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}})_l. \tag{51}$$

Lemma 2.13. *If $k \gg 0$, $H^0(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}})_l = 0$ for all $l \leq 0$.*

Proof of Lemma 2.13. In the terminology of [MS], the datum of the Hamiltonian action $\rho^{N_{\boldsymbol{\nu}}}$, with moment map $\|\boldsymbol{\nu}\|^2/\overline{\Phi^{\boldsymbol{\nu}}}$, makes $B_{k\boldsymbol{\nu}}$ into a prequantum S^1-equivariant orbibundle, hence into a moment line bundle. By Corollary 2.11 of [MS], and given that $\|\boldsymbol{\nu}\|^2/\overline{\Phi^{\boldsymbol{\nu}}} > 0$, we conclude that the Fourier decomposition of $\mathrm{RR}(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}})$ (viewed as a virtual character of S^1) has the form

$$\mathrm{RR}(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}}) = \sum_{l>0} \mathrm{RR}(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}})_l \cdot \chi_l, \tag{52}$$

where where $\chi_l(e^{\imath\theta}) = e^{\imath l \theta}$. In view of Corollary 2.20 this means that, as a representation of S^1,

$$H^0(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}}) = \bigoplus_{l>0} H^0(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}})_l \qquad \forall\, k \gg 0. \tag{53}$$

□

By the S^1-equivariance in (40), we can now sharpen (50) as follows.

Corollary 2.21. *If $k \gg 0$, then*

$$\mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{k\boldsymbol{\nu}} = \bigoplus_{l>0} \mathcal{O}(\tilde{A}^\vee_{\boldsymbol{\nu}})_{k\boldsymbol{\nu},l}. \tag{54}$$

3 Proof of Theorem 1.1

We can now give the proof of Theorem 1.1. First, however, let us consider the following statement.

Lemma 3.1. *For every $\boldsymbol{\lambda} \in \mathbb{Z}$, restriction yields an isomorphism $\mathcal{O}(A_0^\vee)_{\boldsymbol{\lambda}} \cong H(X)^{\hat{\mu}}_{\boldsymbol{\lambda}}$.*

Proof of Lemma 3.1. Clearly, restriction yields a morphism $\zeta_{\boldsymbol{\lambda}} : \mathcal{O}(A_0^\vee)_{\boldsymbol{\lambda}} \to H(X)^{\hat{\mu}}_{\boldsymbol{\lambda}}$. If $f \in \mathcal{O}(A_0^\vee)$ is non-zero, then the locus where its differential vanishes has real codimension ≥ 2; if it vanishes on X, therefore, $f = 0$. Hence $\zeta_{\boldsymbol{\lambda}}$ is injective.

Since by assumption $\mathbf{0} \notin \Phi(M)$, we have $\dim H(X)^{\hat{\mu}}_{\boldsymbol{\lambda}} < +\infty$ for every $\boldsymbol{\lambda}$. Hence we have a finite direct sum

$$H(X)^{\hat{\mu}}_{\boldsymbol{\lambda}} = \bigoplus_{l=a(\boldsymbol{\lambda})}^{b(\boldsymbol{\lambda})} H(X)^{\hat{\mu}}_{\boldsymbol{\lambda},l},$$

where

$$0 \leq a(\boldsymbol{\lambda}) \leq b(\boldsymbol{\lambda}) < +\infty, \qquad H(X)^{\hat{\mu}}_{\boldsymbol{\lambda},l} := H(X)^{\hat{\mu}}_{\boldsymbol{\lambda}} \cap H(X)_l.$$

Hence, to verify that $\zeta_{\boldsymbol{\lambda}}$ is surjective, it suffices to show that any $s \in H(X)^{\hat{\mu}}_{\boldsymbol{\lambda},l}$ is the restriction of some $\tilde{s} \in \mathcal{O}(A_0^\vee)_{\boldsymbol{\lambda}}$. Any $s \in H(X)^{\hat{\mu}}_l$ is the restriction of an holomorphic homogeneous function of degree l, $\tilde{s} \in \mathcal{O}(A_0^\vee)_l$. Since $\rho^{A_0^\vee}$ and $\gamma^{A_0^\vee}$ commute, one sees that $\tilde{s}$ is in the $\boldsymbol{\lambda}$-th isotype for T, and therefore for $\tilde{T}$ as well. Hence $\zeta_{\boldsymbol{\lambda}}$ is surjective.

□

Proof of Theorem 1.1. By Lemma 3.1, for every $k = 1, 2, \ldots$ we have a natural equivariant injective linear map

$$F_{k\boldsymbol{\nu}} := res_{k\boldsymbol{\nu}} \circ \zeta_{k\boldsymbol{\nu}}^{-1} : H(X)^{\hat{\mu}}_{k\boldsymbol{\nu}} \to \mathcal{O}(\tilde{A}_{\boldsymbol{\nu}}^\vee)_{k\boldsymbol{\nu}} \cong H^0(N_{\boldsymbol{\nu}}, B_{k\boldsymbol{\nu}}), \tag{55}$$

where $res_{k\boldsymbol{\nu}} : \mathcal{O}(A_0^\vee)_{k\boldsymbol{\nu}} \to \mathcal{O}(\tilde{A}_{\boldsymbol{\nu}}^\vee)_{k\boldsymbol{\nu}}$ denotes restriction, and is obviously injective since $\tilde{A}_{\boldsymbol{\nu}}^\vee$ is open and dense in $A_0^\vee$; this proves the first statement of Theorem 1.1.

To prove the second statement, it suffices to verify that $res_{k\boldsymbol{\nu}}$ is surjective for $k \gg 0$. We have for some $c(k, \boldsymbol{\nu})$, $d(k, \boldsymbol{\nu}) \in \mathbb{Z}$ with $c(k, \boldsymbol{\nu}) \leq d(k, \boldsymbol{\nu})$:

$$\mathcal{O}(\tilde{A}_{\boldsymbol{\nu}}^\vee)_{k\boldsymbol{\nu}} = \bigoplus_{l=c(k,\boldsymbol{\nu})}^{d(k,\boldsymbol{\nu})} \mathcal{O}(\tilde{A}_{\boldsymbol{\nu}}^\vee)_{k\boldsymbol{\nu},l};$$

hence

$$res_{k\boldsymbol{\nu}} = \bigoplus_{l=c(k,\boldsymbol{\nu})}^{d(k,\boldsymbol{\nu})} res_{k\boldsymbol{\nu},l},$$

where

$$res_{k\boldsymbol{\nu},l} : \mathcal{O}(A_0^\vee)_{k\boldsymbol{\nu},l} \to \mathcal{O}(\tilde{A}_{\boldsymbol{\nu}}^\vee)_{k\boldsymbol{\nu},l}$$

and we need to check that $res_{k\boldsymbol{\nu},l}$ is surjective for every $l = c(k,\boldsymbol{\nu}), \ldots, d(k,\boldsymbol{\nu})$ and $k \gg 0$.

By Corollary 2.21, we may assume that $c(k,\boldsymbol{\nu}) > 0$. Furthermore, by Lemma 2.4 $\tilde{A}_{\boldsymbol{\nu}}^\vee = (\pi')^{-1}(\tilde{M}_{\boldsymbol{\nu}})$ and therefore $res_{k\boldsymbol{\nu},l}$ may canonically reinterpreted in terms of the restriction of holomorphic sections:

$$\widetilde{res}_{k\boldsymbol{\nu},l} : H^0\left(M, A^{\otimes l}\right)_{k\boldsymbol{\nu}} \to H^0\big(\tilde{M}_{\boldsymbol{\nu}}, A^{\otimes l}\big)_{k\boldsymbol{\nu}}. \tag{56}$$

Hence we are reduced to proving that $\widetilde{res}_{k\boldsymbol{\nu},l}$ in (56) is surjective for all $l > 0$.

Suppose $s \in H^0\left(M, A^{\otimes l}\right)$. Then $s \in H^0\left(M, A^{\otimes l}\right)_{k\boldsymbol{\nu}}$ if and only if the following two conditions hold:

1. s is γ^X-invariant, i.e., $s \in H^0(M, A^{\otimes l})^{T^{r-1}_{\boldsymbol{\nu}^\perp}}$;
2. for any $e^{\imath\vartheta} \in S^1$,
$$\hat{\mu}_{e^{\imath\vartheta\boldsymbol{\nu}}}(s) = e^{\imath k\|\boldsymbol{\nu}\|^2\vartheta}\, s.$$

In other words, we can identify $H^0\left(M, A^{\otimes l}\right)_{k\boldsymbol{\nu}}$ with the $k\|\boldsymbol{\nu}\|^2$-isotypical component for the representation of $T^1_{\boldsymbol{\nu}} \cong S^1$ on $H^0(M, A^{\otimes l})^{T^{r-1}_{\boldsymbol{\nu}^\perp}}$. The same considerations apply to $H^0\big(\tilde{M}_{\boldsymbol{\nu}}, A^{\otimes l}\big)_{k\boldsymbol{\nu}}$. We shall express this by writing

$$H^0(M, A^{\otimes l})_{k\boldsymbol{\nu}} = H^0(M, A^{\otimes l})^{T^{r-1}_{\boldsymbol{\nu}^\perp}}_{k\|\boldsymbol{\nu}\|^2}, \quad H^0(\tilde{M}_{\boldsymbol{\nu}}, A^{\otimes l})_{k\boldsymbol{\nu}} = H^0(\tilde{M}_{\boldsymbol{\nu}}, A^{\otimes l})^{T^{r-1}_{\boldsymbol{\nu}^\perp}}_{k\|\boldsymbol{\nu}\|^2}.$$

It is well-known that the restriction map

$$f_{l,\boldsymbol{\nu}} : H^0\left(M, A^{\otimes l}\right)^{T^{r-1}_{\boldsymbol{\nu}^\perp}} \to H^0\big(\tilde{M}_{\boldsymbol{\nu}}, A^{\otimes l}\big)^{T^{r-1}_{\boldsymbol{\nu}^\perp}} \tag{57}$$

is an isomorphism (§5 of [GS], Theorem 2,18 of [Sj]), and it is clearly $T^1_{\boldsymbol{\nu}}$-equivariant. The claim follows, since $\widetilde{res}_{k\boldsymbol{\nu},l}$ is the restriction of $f_{l,\boldsymbol{\nu}}$ to the $k\|\boldsymbol{\nu}\|^2$-isotypical component, hence by equivariance it induces an isomorphism $H^0(M, A^{\otimes l})^{T^{r-1}_{\boldsymbol{\nu}^\perp}}_{k\|\boldsymbol{\nu}\|^2} \cong H^0(\tilde{M}_{\boldsymbol{\nu}}, A^{\otimes l})^{T^{r-1}_{\boldsymbol{\nu}^\perp}}_{k\|\boldsymbol{\nu}\|^2}$. □

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