Multiple solutions for Schrödinger equations on Riemannian manifolds via ∇-theorems*

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Abstract

We consider a smooth, complete and non-compact Riemannian manifold $(\mathcal{M}, g)$ of dimension $d \geq 3$, and we look for solutions to the semilinear elliptic equation

$$-\Delta_g w + V(\sigma)w = \alpha(\sigma)f(w) + \lambda w \quad \text{in } \mathcal{M}.$$

The potential $V: \mathcal{M} \to \mathbb{R}$ is a continuous function which is coercive in a suitable sense, while the nonlinearity f has a subcritical growth in the sense of Sobolev embeddings. By means of ∇-Theorems introduced by Marino and Saccon, we prove that at least three non-trivial solutions exist as soon as the parameter λ is sufficiently close to an eigenvalue of the operator $-\Delta_g$.

1 Introduction

The study of solutions to semilinear partial differential equations of Schrödinger type is by far one of the richest field in Nonlinear Analysis, where Variational Methods and Critical Point Theory provide a powerful setting for existence results. The occurrence of more than one solution to such equations is guaranteed, at a basic level, by some symmetry condition together with the use of topological indices like the genus or the relative category. We refer to the classical monograph [**?**] for a survey.
Semilinear elliptic equations of Schrödinger type are typically set in the whole Euclidean space $\mathbb{R}^d$, $d \geq 3$, which has a rather poor geometric structure. Multiplicity results may then appear as a consequence of the presence of potential functions with suitable

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properties. The situation is much different if $\mathbb{R}^d$ is replaced by a more general Riemannian manifold $\mathcal{M}$, since the geometry of $\mathcal{M}$ may influence the existence of one or more solutions to the equation. Analysis on Manifolds and Geometric Analysis become the necessary language to work with these problems: we refer to [?, ?, ?, ?, ?] and the references therein for an introduction. For the sake of brevity, we will assume that the reader is familiar with the basic definitions of Riemannian Geometry.
We will consider d-dimensional smooth complete non-compact Riemannian manifold $(\mathcal{M}, g)$ with $d \geq 3$. The aim of this paper is to study the existence of solutions for problem

$$\begin{cases} -\Delta_g w + V(\sigma)w = \alpha(\sigma)f(w) + \lambda w & \text{in } \mathcal{M} \\ w(\sigma) \to 0 & \text{as } d_g(\sigma_0, \sigma) \to \infty, \end{cases}$$

where $\alpha \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M})$, $\alpha > 0$ a.e. in $\mathcal{M}$, $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda \in \mathbb{R}$ is a real parameter. We assume that $V \colon \mathcal{M} \to \mathbb{R}$ is a continuous function such that

(V_1) $\upsilon_0 := \inf_{\sigma \in \mathcal{M}} V(\sigma) > 0$;

(V_2) there exists $\sigma_0 \in \mathcal{M}$ such that

$$\lim_{d_g(\sigma_0,\sigma)\to\infty} V(\sigma) = +\infty,$$

where $d_g \colon \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ is the distance associated to the Riemannian metric g. Finally, Δ_g denotes the Laplace-Beltrami operator. This operator is defined in local coordinates by

$$\Delta_g h = \sum_{i,j} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial h}{\partial x^j} \right).$$

We point out that we have defined Δ_g with the "analyst's sign convention", so that $-\Delta_g$ coincides with $-\Delta$ in $\mathbb{R}^d$ with its flat metric.

The nonlinearity $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies

(f_1)

$$\lim_{t\to 0} \frac{f(t)}{|t|} = 0;$$

(f_2) there results

$$\lim_{t\to+\infty} \frac{f(t)}{|t|^{r-1}} < \infty$$

where $r \in \left(2, \dfrac{2d}{d-2}\right)$;

(f_3) $0 < rF(t) < f(t)t$ for all $t \in \mathbb{R} \setminus \{0\}$ where $F(t) := \int_0^t f(\tau)\, d\tau$.

To introduce the main assumption on the manifold $(\mathcal{M}, d)$, we suppose that there exists a function $H\colon\ [0,\infty) \to \mathbb{R}$ of class C^1 such that

$$\int_0^\infty tH(t)\,dt < \infty$$

and

(Ric) for some $\bar{\sigma}_0 \in \mathcal{M}$ there results

$$\mathrm{Ric}_{(\mathcal{M},g)}(\sigma) \geq (1-d)H(d_g(\bar{\sigma}_0,\sigma)).$$

Moreover, we will assume throughout the paper that

$$\inf_{\sigma\in\mathcal{M}} \mathrm{Vol}_g\left(B_\sigma(1)\right) > 0$$

where

$$B_\sigma(1) := \{\xi \in \mathcal{M} \mid d_g(\xi,\sigma) < 1\}$$

and

$$\mathrm{Vol}_g\left(B_\sigma(1)\right) := \int_{B_\sigma(1)} dv_g.$$

Since we want to prove a multiplicity result for (**??**), a natural approach could be based on Morse Theory, see [**?**, **?**]. Unfortunately, Morse Theory requires in general more regularity of the Euler functional associated to the variational problem, and this would require a more regular nonlinearity f in (**??**).
We propose here a different approach via ∇-Theorems, a family of variational tools which were introduced by Marino and Saccon in [**?**] to study the multiplicity of solutions of some asymptotically non-symmetric semilinear elliptic problems with jumping nonlinearities. More precisely, we will make use of the sphere-torus linking Theorem with mixed type assumptions (see [**?**, Theorem 2.10]). The main condition of this theorem can be roughly summarized in these terms: the Euler functional constrained on a closed subspace must not have critical values in a certain prescribed range with "some uniformity". A rigorous definition is as follows.

Definition 1. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{I}\colon \mathcal{H} \to \mathbb{R}$ a C^1 functional. Let also $\mathcal{X}$ be a closed subspace of $\mathcal{H}$, a, $b \in \mathbb{R}\cup\{-\infty,\infty\}$; we say that $\mathcal{I}$ satisfies the condition $(\nabla)\,(\mathcal{I},\mathcal{X},a,b)$ if there exists $\gamma > 0$ such that

$$\inf\left\{\|P_{\mathcal{X}}\nabla\mathcal{I}(w)\| \mid a \leq \mathcal{I}(w) \leq b,\ \mathrm{dist}(w,\mathcal{X}) \leq \gamma\right\} > 0$$

where $P_{\mathcal{X}}\colon \mathcal{H} \to \mathcal{X}$ denotes the standard orthogonal projection. In the following, we will refer to it as (∇)-condition for short.

In order to make the paper self-contained, we also write the statement of the ∇-theorem.

Theorem 1. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{X}_i$, $i = 1,2,3$ three subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ and $\dim \mathcal{X}_i < \infty$ for $i = 1,2$. Denote with $P_{\mathcal{X}_i} \colon \mathcal{H} \to \mathcal{X}_i$ the standard orthogonal projection. Let $\mathcal{I} \colon \mathcal{H} \to \mathbb{R}$ a C^1 functional. Let $\rho, \rho', \rho'', \rho_1$ be such that $\rho_1 > 0$, $0 \leq \rho' < \rho < \rho''$ and define*

$$\varDelta = \{ w \in \mathcal{X}_1 \oplus \mathcal{X}_2 \mid \rho' \leq \|P_{\mathcal{X}_2}\| \leq \rho'',\ \|P_{\mathcal{X}_2}\| \leq \rho_1 \} \quad \textit{and} \quad T = \partial_{\mathcal{X}_1 \oplus \mathcal{X}_2} \varDelta,$$

$$S_{23} = \{ w \in \mathcal{X}_2 \oplus \mathcal{X}_3 \mid \|w\| = \rho \} \quad \textit{and} \quad B_{23} = \{ w \in \mathcal{X}_2 \oplus \mathcal{X}_3 \mid \|w\| \leq \rho \}.$$

Assume that

$$a' = \sup \mathcal{I}(T) < \inf \mathcal{I}(S_{23}) = a''.$$

Let a and b be such that $a' < a < a''$ and $b > \sup \mathcal{I}(\varDelta)$. Assume $(\nabla)\,(\mathcal{I}, \mathcal{X}_1 \oplus \mathcal{X}_3, a, b)$ holds and that $(PS)_c$ is verified for all $c \in [a,b]$. Then $\mathcal{I}$ has at least two critical points in $\mathcal{I}^{-1}([a,b])$. Moreover, if $a_1 < \inf \mathcal{I}(B_{23}) > -\infty$ and $(PS)_c$ holds for all $c \in [a_1, b]$, then $\mathcal{I}$ has another critical level in $[a_1, a']$.

Therefore, we need a suitable Hilbert space in which (**??**) can be associated to the critical points of a C^1 functional $\mathcal{I}$. In order to respond to this need, we introduce here the variational framework in which problem (**??**) is set.
The Sobolev space $H^1_g(\mathcal{M})$ is obtained as the closure of $C^\infty_0(\mathcal{M})$ with respect to the norm

$$\|w\|_g := \left(\int_{\mathcal{M}} |\nabla_g w(\sigma)|^2 \, dv_g + \int_{\mathcal{M}} |w(\sigma)|^2 \, dv_g \right)^{\frac{1}{2}},$$

where $C^\infty_0(\mathcal{M})$ denotes the space of smooth compactly supported function in $\mathcal{M}$. We further set

$$H^1_V(\mathcal{M}) := \{ w \in H^1_g(\mathcal{M}) \mid \|w\|^2 < \infty \}$$

where

$$\|w\| := \left(\int_{\mathcal{M}} |\nabla_g w(\sigma)|^2 \, dv_g + \int_{\mathcal{M}} V(\sigma) \, |w(\sigma)|^2 \, dv_g \right)^{1/2}$$

is the norm induced by the scalar product

$$\langle w_1, w_2 \rangle := \int_{\mathcal{M}} \langle \nabla_g w_1(\sigma), \nabla_g w_2(\sigma) \rangle_g \, dv_g + \int_{\mathcal{M}} V(\sigma) w_1(\sigma) w_2(\sigma) \, dv_g.$$

We recall that under the assumptions we made on the potential and the manifold, the embedding $H^1_V(\mathcal{M}) \hookrightarrow L^q(\mathcal{M})$ is continuous for any $q \in [2, 2^*]$. Furthermore, as a result of the Hypothesis V_1 and V_2 we also have the following Lemma, whose proof can be found in [**?**, Lemma 2.1].

Lemma 1. *Let $\mathcal{M}$ be a complete, non-compact d-dimensional Riemannian manifold satisfying the curvature condition* (Ric) *and* $\inf_{\sigma \in \mathcal{M}} \operatorname{Vol}_g (B_\sigma(1)) > 0$. *If V satisfies V_1 and V_2 the embedding $H^1_V(\mathcal{M}) \hookrightarrow L^q(\mathcal{M})$ is compact for all $q \in [2, 2^*)$.*

∇-Theorems turned out to be a powerful tool when one is interested in studying the multiplicity of solutions for nonlinear equations. In particular, in [?] Pistoia proved the existence of four solutions for a superlinear elliptic problem on a bounded domain of $\mathbb{R}^d$. At a later time, in the same spirit of the paper of Pistoia, Mugnai proved in [?] the existence of three solutions for a superlinear boundary problem with a more general nonlinearity. ∇-Theorems are useful also when one deal with problems with higher order operators as showed in [?] by Micheletti, Pistoia and Saccon. It is also worth mentioning [?] where Molica Bisci, Mugnai and Servadei showed the existence of three solutions for an equation driven by the fractional Laplacian on a bounded domain of $\mathbb{R}^d$ with Dirichlet condition and a general nonlinearity. When one draws his attentions to problems settled in unbounded domains, the situation is completely different. Indeed, in order to apply the sphere-torus linking Theorem it is necessary to split the space on which is defined the functional in three linear subspaces, two of them finite dimensional, while the third infinite dimensional. When Ω is a bounded domain of $\mathbb{R}^d$ it is well known that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. As a consequence of that, the resolvent of the Schrödinger operator or the Laplacian is compact and with standard arguments it is possible to prove that the spectrum of these operators is discrete and that the eigenfunctions are dense in the space under considerations. So, a common approach to select the three subspaces is to consider the whole space as a direct sum of eigenspaces. Unfortunately, this strategy fails in the case of unbounded domains, since the spectrum of the Schrödinger operator or the Laplacian is not even discrete in general. A contribution in this direction was given by Tehrani in [?] where the existence of two solution for the Nonlinear Schrödinger equation in $\mathbb{R}^d$. Following the characterization of the essential spectrum of a Schrödinger operator present in [?], they are able to decompose the space and apply the theorem. The drawback of their approach is that they don't give sufficient condition on the potential to ensure the existence of eigenvalues subsequent to the first one. A recent result was also obtained by Mugnai in [?] proving the existence of at least two solutions for an equation in which the nonlinearity is allowed to have an exponential growth in $\mathbb{R}^2$. In the present paper, we want to extend the results quoted previously in two directions. The first one is to give sufficient condition that will enable us to completely characterize the spectrum of the operator taken into account. Secondly, the problem we want to investigate is settled in a non-compact Riemannian manifold and, as far as we know, results as the one we are going to prove are not present in literature. One of the first contribute for the Nonlinear Schrödinger equation on Riemannian manifolds was given in [?], where Faraci and Farkas established a necessary and sufficient condition for the existence of non-trivial solutions with hypothesis on the manifold equal to the ones we will assume. More recently, Molica Bisci and Secchi in [?] showed the existence of at least two solutions for (**??**) requiring λ large enough under similar assumptions on f. We also quote [?] where the same authors of this paper proved the existence of infinitely many solutions for a Schödinger equation set on a Cartan-Hadamard manifold.

The main result of the paper is a multiplicity result for problem (**??**) whenever λ is sufficiently close to an eigenvalue of $-\Delta_g$.

Theorem 2. *Assume $f\colon \mathbb{R} \to \mathbb{R}$ and $V\colon \mathcal{M} \to \mathbb{R}$ are continuous functions that verify respectively (f_1) – (f_3) and (V_1) – (V_2). For every eigenvalue λ_k of $-\Delta_g$, there exists $\mu > 0$ such that if $\lambda_k - \mu < \lambda < \lambda_k$, then problem (**??**) admits at least three non-trivial weak solutions w_1, w_2 and w_3. Furthermore, these solutions belong to $L^\infty(\mathcal{M})$ and for each $i \in \{1, 2, 3\}$ there results*

$$\lim_{d_g(\sigma,\sigma_0)\to+\infty} w_i(\sigma) = 0.$$

The proof of the previous Theorem is based on a precise description of the spectral properties of the operator $-\Delta_g + V$ which governs (P_λ). In Section **??** we list in detail these properties, since they seem to be new in the setting of a non-compact manifold $\mathcal{M}$.

Remark 1. The boundedness of our solutions and their decay at infinity (**??**) follow from [**?**, Theorem 3.1]. This remark applies to the eigenfunctions considered in Theorem **??** as well.

To the best of our knowledge, our results are new even in the Euclidean case $\mathcal{M} = \mathbb{R}^d$, $d \geq 3$. In this case, our assumptions on V can be relaxed, and we can rely on some conditions introduced in [**?**] which ensure both the discreteness of the spectrum of the operator $-\Delta + V$ and the necessary compact embedding of the Sobolev space $H^1_V(\mathbb{R}^d)$. In our setting, the compactness of the embedding of $H^1_V(\mathcal{M})$ into $L^p(\mathcal{M})$ for all $p \in [2, 2^*)$ follows from [**?**, Lemma 2.1]. As a concrete example, we propose the following result.

Theorem 3. *Assume that $V\colon \mathbb{R}^d \to \mathbb{R}$ is a function in $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ which verifies $V(x) \geq V_0 > 0$ for almost every $x \in \mathbb{R}^d$ and*

$$\lim_{|x|\to+\infty} \int_{B_1(x)} \frac{dy}{V(y)} = 0.$$

*Then the same conclusions as in Theorem **??** hold for*

$$\begin{cases} -\Delta w + V(x)w = \left(1 + |x|^d\right)^{-2} |w|^{r-2}w + \lambda w & \text{in } \mathbb{R}^d \\ w(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $r \in \left(2, \frac{2d}{d-2}\right)$.

2 A setting for (P_λ)

Let us consider

$$\begin{cases} -\Delta_g w + V(\sigma)w = \alpha(\sigma)f(w) + \lambda w & \text{in } \mathcal{M} \\ w(\sigma) \to 0 & \text{as } d_g(\sigma_0, \sigma) \to \infty, \end{cases}$$

where $\alpha \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M}) \setminus \{0\}$ is a non-negative function and f satisfies assumptions (f_1) – (f_3).

In order to find solutions for problem (**??**) we introduce the energy functional associated to the problem. Namely, let $J_\lambda \colon H^1_V(\mathcal{M}) \to \mathbb{R}$ be such that

$$J_\lambda(w) = \frac{1}{2}\|w\|^2 - \frac{\lambda}{2}\|w\|^2_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma)F(w(\sigma))\, dv_g.$$

By virtue of the embedding results presented in the previous sections, this functional is well-defined, and it is standard to prove that it is of class C^1. Moreover, as is well known, critical points of J_λ correspond to weak solutions of problem (**??**), i.e.

$$\langle w, \varphi\rangle = \lambda\langle w, \varphi\rangle_{L^2(\mathcal{M})} + \int_{\mathcal{M}} \alpha(\sigma)f(w(\sigma))\varphi(\sigma)\, dv_g$$

for any $\varphi \in H^1_V(\mathcal{M})$. More in general, one can show that the derivative of the functional J_λ along a function $v \in H^1_V(\mathcal{M})$ is

$$J'_\lambda(w)\,[w] = \langle w, v\rangle - \lambda\langle w, v\rangle_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma)f(w(\sigma))v(\sigma)\, dv_g.$$

Now, take $s \in [2, 2^*)$ and consider its conjugate exponent s' such that $1/s + 1/s' = 1$. We select a function $h \in L^{s'}(\mathcal{M})$ and we focus on the equation

$$-\Delta_g w = h(\sigma), \quad \sigma \in \mathcal{M}.$$

By applying the classical Riesz or Lax-Milgram Theorem, one can easily show that the problem above has a unique weak solution. By virtue of that, we are able to define

$$\begin{aligned} \Delta_g^{-1} \colon L^{s'}(\mathcal{M}) &\to H^1_V(\mathcal{M}) \\ h &\mapsto w = \Delta_g^{-1}h \end{aligned}$$

where $\Delta_g^{-1}h$ is the only weak solution of (**??**), which means

$$\langle \Delta_g^{-1}h, \varphi\rangle = \langle h, \varphi\rangle_{L^2(\mathcal{M})}.$$

Remark 2. We emphasize that the operator Δ_g^{-1} is compact. Indeed, it is possible to write it by the composition of two maps

$$L^{s'}(\mathcal{M}) \xrightarrow{\ j\ } (H^1_V(\mathcal{M}))^* \xrightarrow{\ \Delta_g^{-1}\ } H^1_V(\mathcal{M})$$

where the first is compact, recalling that $H^1_V(\mathcal{M}) \hookrightarrow L^s(\mathcal{M})$ is compact and applying [**?**, Theorem 6.4]. Since $H^1_V(\mathcal{M})$ is a Hilbert space, there is a unique element called the gradient of J_λ and denoted ∇J_λ such that

$$\langle \nabla J_\lambda(w), v\rangle = J'_\lambda(u)\,[v]\,.$$

It is also possible to verify that the gradient of J_λ can be written as

$$\nabla J_\lambda(w) = w - \Delta_g^{-1}\left(\lambda w + \alpha f(w)\right).$$

We begin our analysis by proving a technical lemma that will provide some useful estimates we will use throughout the paper.

Lemma 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies (f_1) – (f_3), then we have the following estimates:*

(i) for any $\varepsilon > 0$ there exists a constant $A_1^\varepsilon > 0$ such that

$$|f(t)| \leq 2\varepsilon\,|t| + rA_1^\varepsilon\,|t|^{r-1}$$

and

$$F(t) \leq \varepsilon t^2 + A_1^\varepsilon\,|t|^r$$

for every $t \in \mathbb{R}$;

(ii) for any $\varepsilon > 0$ there exist $A_2, A_2^{\tilde{\varepsilon}} > 0$ such that

$$|f(t)| \leq A_2 + A_2^\varepsilon\,|t|^{r-1}$$

for every $t \in \mathbb{R}$;

(iii) there exists $A_3, A_4 > 0$ such that

$$F(t) \geq A_3\,|t|^r - A_4$$

for every $t \in \mathbb{R}$.

Proof. The verification of the three inequalities is standard, and we omit the details. □

We end this section by proving that the functional J_λ satisfies a good compactness condition in Critical Point Theory.

Definition 2. We say that a sequence $(w_j)_j \subset H_V^1(\mathcal{M})$ is a Palais-Smale sequence at level $c \in \mathbb{R}$, $(PS)_c$ sequence for short, if $J_\lambda(w_j) \to c$ in $\mathbb{R}$ and $J'_\lambda(w_j) \to 0$ in $\left(H_V^1(\mathcal{M})\right)^*$ as $j \to \infty$. Furthermore, the functional J_λ is said to satisfy the $(PS)_c$ condition if every $(PS)_c$ sequence for J_λ admits a strongly convergent subsequence in $H_V^1(\mathcal{M})$.

Proposition 1. *Let f be a map that satisfies (f_1)–(f_3) and $\lambda > 0$ a real parameter. Then, $(PS)_c$ condition holds for every $c \in \mathbb{R}$ for functional J_λ.*

Proof. Let $(w_j)_j \subset H_V^1(\mathcal{M})$ a $(PS)_c$ sequence for functional J_λ, i.e.

$$J_\lambda(w_j) \to c \quad \text{in } \mathbb{R}$$

and

$$J'_\lambda(w_j) \to 0 \quad \text{in } H_V^1(\mathcal{M})$$

as $j \to \infty$. We first prove that $(w_j)_j$ is bounded in $H^1_V(\mathcal{M})$, adapting the ideas of [**?**, Proof of Theorem 6.1]. We proceed by contradiction, assuming without loss of generality, that $\rho_j = \|w_j\| \to +\infty$ as $j \to +\infty$. Let us set $v_j = w_j/\rho_j$, so that we may assume that $v_j \rightharpoonup v$ in $H^1_V(\mathcal{M})$ and $v_j \to v$ strongly in $L^2(\mathcal{M})$.
Now,

$$c + o(1) = J_\lambda(w_j) = \frac{1}{2}\|w_j\|^2 - \frac{\lambda}{2}\|w_j\|_2^2 - \int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\, dv_g,$$

hence

$$o(1) = \frac{1}{2} - \frac{\lambda}{2}\|v_j\|_2^2 - \int_{\mathcal{M}} \alpha(\sigma)\frac{F(w_j(\sigma))}{\rho_j^2}\, dv_g,$$

and

$$\lim_{j\to+\infty} \int_{\mathcal{M}} \alpha(\sigma)\frac{F(w_j(\sigma))}{\rho_j^2}\, dv_g = \frac{1}{2} - \frac{\lambda}{2}\|v\|_2^2.$$

We consider

$$\mathcal{M}_0 = \{\sigma \in \mathcal{M} \mid v(\sigma) \neq 0\},$$

and we notice that $w_j(\sigma) \to +\infty$ when $\sigma \in \mathcal{M}_0$. From **??** (iii) it is straightforward to verify

$$\lim_{t\to\infty} \frac{F(t)}{t^2} = \infty$$

thus, applying the Fatou's Lemma, we get

$$\lim_{j\to\infty} \int_{\mathcal{M}_0} \alpha(\sigma)\frac{F(w_j(\sigma))}{\|w_j\|^2}\, dv_g = \infty.$$

This obviously implies that

$$\lim_{j\to+\infty} \int_{\mathcal{M}} \alpha(\sigma)\frac{F(w_j(\sigma))}{\rho_j^2}\, dv_g = +\infty.$$

Comparing (**??**) and (**??**) we must conclude that $\mathrm{Vol}_g(\mathcal{M}_0) = 0$, which means that $v = 0$ a.e. on $\mathcal{M}$ and in particular $v_j \to 0$ strongly in $L^2(\mathcal{M})$. From

$$C\|w_j\| \geq \langle \nabla J_\lambda(w_j), w_j\rangle = \|w_j\|^2 - \lambda\|w_j\|_2^2 - \int_{\mathcal{M}} \alpha(\sigma)f(w_j(\sigma))w_j(\sigma)\, dv_g$$

we see that

$$\lim_{j\to+\infty} \int_{\mathcal{M}} \alpha(\sigma)\frac{f(w_j(\sigma))w_j(\sigma)}{\rho_j^2}\, dv_g = 1 - \lambda\|v\|_2^2 = 1.$$

Therefore

$$\lim_{j\to+\infty} \int_{\mathcal{M}} \alpha(\sigma)\frac{rF(w_j(\sigma)) - f(w_j(\sigma))w_j(\sigma)}{\rho_j^2} dv_g = \frac{r}{2} - \frac{\lambda r}{2}\|v\|_2^2 - 1 + \lambda\|v\|_2^2 = \frac{r}{2} - 1.$$

Coupling this with assumption (f_3), we conclude that $\frac{r}{2} \leq 1$, against the assumption that $r > 2$. This contradiction implies that $(w_j)_j$ is a bounded sequence in $H^1_V(\mathcal{M})$.
We can now use (**??**) and Remark **??** (see also [**?**, Proposition 2.2] for a general approach) to conclude the proof. □

3 Geometry of the ∇-Theorem

As mentioned at the beginning of the paper, our aim is to prove an existence result through the so-called ∇-Theorem. In order to apply this tool, it is necessary to split the space in three closed subspaces, two of finite dimension and one of infinite dimension. Furthermore, the functional is required to have a precise geometrical structure. A standard decomposition of $H_V^1(\mathcal{M})$ into three subspaces can be made through an adequate selection of some eigenspaces associated to the operator Δ_g. The following theorem characterizes completely the spectrum of the resolvent of the Laplace-Beltrami operator under the assumptions that guarantees the compact embedding $H_V^1(\mathcal{M}) \hookrightarrow L^s(\mathcal{M})$ for $s \in (2, 2^*)$.

Theorem 4. *The following statements hold true:*

(*a*) *the smallest eigenvalue of problem* (**??**) *is positive, and it can be characterized as*

$$\lambda_1 := \min_{\substack{w \in H_V^1(\mathcal{M}) \\ \|w\|_{L^2(\mathcal{M})}=1}} \|w\|^2$$

or analogously

$$\lambda_1 := \min_{w \in H_V^1(\mathcal{M}) \setminus \{0\}} \frac{\|w\|^2}{\|w\|_{L^2(\mathcal{M})}^2};$$

(*b*) *there is a non-negative eigenfunction $e_1 \in H_V^1(\mathcal{M})$ that is an associated eigenfunction to λ_1 where the minimum in* (**??**) *is attained. Moreover, $\|e_1\|_{L^2(\mathcal{M})} = 1$ and $\lambda_1 = \|e_1\|^2$;*

(*c*) *the eigenvalue λ_1 is simple, i.e. if $w \in H_V^1(\mathcal{M})$ is such that*

$$\int_{\mathcal{M}} \langle \nabla_g w(\sigma), \nabla_g \varphi(\sigma) \rangle_g \, dv_g + \int_{\mathcal{M}} V(\sigma) w(\sigma) \varphi(\sigma) \, dv_g = \lambda_1 \int_{\mathcal{M}} w(\sigma) \varphi(\sigma) \, dv_g$$

for any $\varphi \in H_V^1(\mathcal{M})$ then there exists $\xi \in \mathbb{R}$ such that $w = \xi e_1$;

(*d*) *the set of eigenvalues of problem* (**??**) *can be arranged into a sequence $(\lambda_k)_k$ such that*

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \dots \le \lambda_k \le \lambda_{k+1} \le \dots$$

where $\lim_{k \to \infty} \lambda_k = +\infty$. Moreover, every eigenvalue can be characterized as

$$\lambda_{k+1} := \min_{\substack{w \in E_k^\perp \\ \|w\|_{L^2(\mathcal{M})}=1}} \|w\|$$

or equivalently

$$\lambda_{k+1} := \min_{w \in E_k^\perp} \frac{\|w\|}{\|w\|_{L^2(\mathcal{M})}^2}$$

where

$$E_k := \operatorname{span}\{e_1, \dots, e_k\};$$

(e) *for any $k \in \mathbb{N}$ there is an eigenfunction $e_k \in E_{k-1}^{\perp}$ associated to the eigenvalue λ_k such that the minimum in (??) is attained, i.e.* $\|e_k\|_{L^2(\mathcal{M})} = 1$ *and*

$$\lambda_k = \|e_k\|^2;$$

(f) *the eigenfunctions $(e_k)_k$ are an orthonormal basis for $L^2(\mathcal{M})$ and an orthogonal basis for $H^1_V(\mathcal{M})$;*

(g) *each eigenvalue has finite multiplicity. Namely, if λ_k is such that*

$$\lambda_{k-1} < \lambda_k = \ldots = \lambda_{k+h} < \lambda_{k+h+1}$$

for some $h \in \mathbb{N}_0$, then $\operatorname{span}\{e_k, \ldots, e_{k+h}\}$ *is the eigenspace associated to λ_k.*

Proof. All these results are a by-product of the classical theorems of functional analysis on the basic properties of compact self-adjoint operators defined on Hilbert spaces. As a consequence of that, we will omit the proof, and we remind the interested reader to [?] where an elementary proof is presented that can be easily adapted to our new setting. □

We point out that the previous Theorem completely describes the set of solutions of the eigenvalues problem

$$\begin{cases} -\Delta_g w + V(\sigma) w = \lambda w & \text{in } \mathcal{M} \\ w(\sigma) \to 0 & \text{as } d_g(\sigma_0, \sigma) \to \infty. \end{cases}$$

The condition $w(\sigma) \to 0$ as $d_g(\sigma, \sigma_0) \to +\infty$ follows from Remark **??**.
In this section, we are going to show that the functional J_λ associated to problem (??) possesses the geometrical structure required by the (∇)-Theorem under the assumption we made on the nonlinearity f and the potential V. Before doing that, for the sake of simplicity, we fix some notation. Henceforth, k and h will be positive integers such that

$$\lambda_{k-1} < \lambda_k = \ldots = \lambda_{k+h} < \lambda_{k+h+1}.$$

We define

$$X_1 := E_{k-1}, \quad X_2 := \operatorname{span}\{e_k, \ldots e_{k+h}\}, \quad X_3 := E_{k+h}^{\perp}.$$

We point out that the existence of such integers h and k is guaranteed by Theorem **??**.
Next Lemma generalize the Poincaré inequality to the case in which the functions belong to eigenspaces or its orthogonal.

Lemma 3. *Let $k \in \mathbb{N}$. The following inequalities hold:*

(a) *if $w \in E_k^{\perp}$ then*

$$\|w\|^2 \geq \lambda_{k+1} \|w\|^2_{L^2(\mathcal{M})};$$

(b) *if $w \in E_k$ then*

$$\|w\|^2 \leq \lambda_k \|w\|^2_{L^2(\mathcal{M})}.$$

Proof. We start with the case (a). Since $w \in E_k^{\perp}$ we can write

$$w = \sum_{j=k+1}^{\infty} \alpha_j e_j$$

for some coefficients $\alpha_j \in \mathbb{R}$. Thus, we compute

$$\|w\|^2 = \langle w, w \rangle = \sum_{j=k+1}^{\infty} \alpha_j^2 \lambda_j \geq \lambda_{k+1} \|w\|^2_{L^2(\mathcal{M})}$$

where we used Theorem **??** (f), (**??**) and the Bessel-Parseval's identity (see for instance [**?**, Theorem 5.9]). On the other hand, when $w \in E_k$ we have

$$w = \sum_{j=1}^{k} \alpha_j e_j.$$

As a consequence, similarly as we did above we get

$$\|w\|^2 = \sum_{j=1}^{k} \alpha_j^2 \lambda_j \leq \lambda_k \|w\|^2_{L^2(\mathcal{M})}.$$

□

Next proposition will show the functional J_λ verifies the desired geometrical property we need to apply the ∇-Theorem.

Proposition 2. *If assumptions (f_1) – (f_3) hold and $\lambda \in (\lambda_{k-1}, \lambda_k)$ then there are $\rho, R \in \mathbb{R}$, with $R > \rho > 0$ such that*

$$\sup_{\{w \in X_1 \ | \ \|w\| \leq R\} \cup \{w \in X_1 \oplus X_2 \ | \ \|w\| = R\}} J_\lambda < \inf_{\{w \in X_2 \oplus X_3 \ | \ \|w\| = \rho\}} J_\lambda$$

Proof. We start showing

$$\inf_{\{w \in X_2 \oplus X_3 \ | \ \|w\| = \rho\}} J_\lambda > 0$$

choosing ρ adequately and observing that $X_2 \oplus X_3 = E_{k-1}^{\perp}$. Applying twice the Hölder inequality, we get

$$\int_{\mathcal{M}} \alpha(\sigma) |w(\sigma)|^2 \, dv_g \leq \|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})} \|w\|^2_{L^{2^*}(\mathcal{M})}$$

and

$$\int_{\mathcal{M}} \alpha(\sigma) |w(\sigma)|^r \, dv_g \leq \|\alpha\|_{L^{\frac{2^*}{2^*-r}}(\mathcal{M})} \|w\|^r_{L^{2^*}(\mathcal{M})}.$$

From Lemma **??** (i), (**??**) and (**??**) we obtain

$$\begin{aligned}J_\lambda(w) &\geq \frac{1}{2}\|w\|^2 - \frac{\lambda}{2}\|w\|^2_{L^2(\mathcal{M})} - \varepsilon\int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^2\,dv_g - A_1^\varepsilon\int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^r\,dv_g \\ &\geq \frac{1}{2}\|w\|^2 - \frac{\lambda}{2}\|w\|^2_{L^2(\mathcal{M})} - \varepsilon\|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})}\|w\|^2_{L^{2^*}(\mathcal{M})} - A_1^\varepsilon\|\alpha\|_{L^{\frac{2^*}{2^*-r}}(\mathcal{M})}\|w\|^r_{L^{2^*}(\mathcal{M})}.\end{aligned}$$

Now, recalling $H^1_V(\mathcal{M}) \hookrightarrow L^s(\mathcal{M})$ for every $s \in [2, 2^*]$ continuously, it is possible to find $C > 0$ such that

$$J_\lambda(w) \geq \frac{1}{2}\|w\|^2 - \frac{\lambda}{2}\|w\|^2_{L^2(\mathcal{M})} - \varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})}\|w\|^2 - A_1^\varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-r}}(\mathcal{M})}\|w\|^r.$$

Finally, Lemma **??** yields

$$J_\lambda(w) \geq \left[\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right) - \varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})}\right]\|w\|^2 - A_1^\varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-r}}(\mathcal{M})}\|w\|^r.$$

At this point, choosing $\varepsilon > 0$ such that

$$\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right) - \varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})} > 0$$

and ρ sufficiently small, the desired assertion is proved. On the other hand, it is possible to prove

$$\sup_{\{w\in X_1\ |\ \|w\|\leq R\}\cup\{w\in X_1\oplus X_2\ |\ \|w\|=R\}} J_\lambda \leq 0.$$

Indeed, in the case $w \in X_1$, from Lemma **??** and (f_3), recalling $\alpha > 0$ for a.e. $\sigma \in \mathcal{M}$, it follows that

$$J_\lambda(w) \leq \frac{\lambda_{k-1} - \lambda}{2}\|w\|^2_{L^2(\mathcal{M})} \leq 0.$$

Instead, when $w \in X_1 \oplus X_2$ it suffices to use Lemma **??** (iii) to obtain

$$J_\lambda(w) \leq \frac{1}{2}\|w\|^2 - A_3\int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^r\,dv_g + A_4\|\alpha\|_{L^1(\mathcal{M})}.$$

Since $X_1 \oplus X_2$ has finite dimension all norms are equivalent, then choosing $R > 0$ big enough it is straightforward to see that $r > 2$ implies $J_\lambda(w) \leq 0$. □

4 Validity of the (∇)-condition

This section is devoted to show the validity of the (∇)-condition introduced in Definition **??**. Before proving the main result of this section, we need two preliminary lemmas.

Proposition 3. *Assume Hypotheses (f_1) – (f_3) hold. Then for every $\varrho > 0$ there exists $\delta_\varrho > 0$ such that for each $\lambda \in [\lambda_{k-1} + \varrho, \lambda_{k+h+1} - \varrho]$ the only critical point of J_λ constrained on $X_1 \oplus X_3$ with $J_\lambda \in [-\delta_\varrho, \delta_\varrho]$ is the trivial one.*

Proof. By contradiction, we suppose the statement false. So, we assume the existence of $\tilde{\varrho} > 0$, two sequences $\mu_j \subset [\lambda_{k-1} + \tilde{\varrho}, \lambda_{k+h+1} - \tilde{\varrho}]$ and $(w_j)_j \subset X_1 \oplus X_3$ of critical points, i.e.

$$\langle \nabla J_{\mu_j}(w_j), \varphi \rangle = 0 \quad \text{for any } \varphi \in X_1 \oplus X_3$$

such that

$$\lim_{j \to +\infty} J_{\mu_j}(w_j) = 0.$$

Since $(w_j)_j \subset X_1 \oplus X_3$, we can choose $\varphi = w_j$ in (**??**). As a consequence we have

$$0 = \|w_j\|^2 - \mu_j \|w_j\|^2_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g.$$

Then, we notice that (**??**) can be rewritten as

$$0 = 2J_{\mu_j}(w_j) + 2\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g.$$

Exploiting (f_3) in (**??**) we obtain

$$0 \leq 2J_{\mu_j}(w_j) + (2 - r)\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g.$$

Reordering the terms in (**??**) we get

$$0 \leq (r - 2)\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g \leq 2J_{\mu_j}(w_j).$$

Putting together (**??**) and (**??**) we obtain

$$\lim_{j \to \infty} \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g = 0.$$

Now, recalling $w_j \in X_1 \oplus X_3$ for all $j \in \mathbb{N}$, we are able to find $w_{1,j} \in X_1$ and $w_{3,j} \in X_3$ such that $w_j = w_{1,j} + w_{3,j}$. At this point, on the one hand, we test (**??**) with $\varphi = w_{1,j} - w_{3,j}$ and exploiting the properties of orthogonality of $w_{1,j}$ and $w_{3,j}$ we have

$$\begin{aligned} 0 &= \langle \nabla J_{\mu_j}(w_j), w_{1,j} - w_{3,j} \rangle \\ &= \|w_{1,j}\|^2 - \|w_{3,j}\|^2 - \mu_j \|w_{1,j}\|^2_{L^2(\mathcal{M})} + \mu_j \|w_{3,j}\|^2_{L^2(\mathcal{M})} \\ &\quad - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma))(w_{1,j}(\sigma) - w_{3,j}(\sigma))\, dv_g. \end{aligned}$$

Rearranging (**??**) and applying Lemma **??** we get

$$\begin{aligned}
\int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma))(w_{1,j}(\sigma) - w_{3,j}(\sigma))\, dv_g &= \|w_{1,j}\|^2 - \|w_{3,j}\|^2 - \mu_j \|w_{1,j}\|^2_{L^2(\mathcal{M})} \\
&\quad + \mu_j \|w_{3,j}\|^2_{L^2(\mathcal{M})} \\
&\leq \|w_{1,j}\|^2 - \|w_{3,j}\|^2 - \frac{\mu_j}{\lambda_{k-1}} \|w_{1,j}\|^2 \\
&\quad + \frac{\mu_j}{\lambda_{k+h+1}} \|w_{3,j}\|^2 \\
&= \frac{\lambda_{k-1} - \mu_j}{\lambda_{k-1}} \|w_{1,j}\|^2 + \frac{\mu_j - \lambda_{k+h+1}}{\lambda_{k+h+1}} \|w_{3,j}\|^3 \\
&< -\frac{\tilde{\varrho}}{\lambda_{k-1}} \|w_{1,j}\|^2 - \frac{\tilde{\varrho}}{\lambda_{k+h+1}} \|w_{3,j}\|^2 \\
&< -\frac{2\tilde{\varrho}}{\lambda_{k+h+1}} \|w_j\|^2.
\end{aligned}$$

On the other hand, thanks to Hölder and the continuous embedding $H^1_V(\mathcal{M}) \hookrightarrow L^r(\mathcal{M})$, we have

$$\begin{aligned}
\left| \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma))(w_{1,j}(\sigma) - w_{3,j}(\sigma))\, dv_g \right| &\leq \|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})} \|w_{1,j} - w_{3,j}\|_{L^r(\mathcal{M})} \\
&\leq C \|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})} \|w_j\|
\end{aligned}$$

for some $C > 0$, where we used

$$\langle w_{1,j} - w_{3,j}, w_{1,j} - w_{3,j} \rangle = \|w_{1,j}\|^2 - \|w_{3,j}\|^2 = \|w_j\|^2.$$

Coupling (**??**) and (**??**), we have

$$-C \|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})} \|w_j\| \leq -\frac{2\tilde{\varrho}}{\lambda_{k+h+1}} \|w_j\|^2$$

from which it follows that

$$\frac{2\tilde{\varrho}}{\lambda_{k+h+1}} \|w_j\| \leq C \|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})}.$$

Then, we use Lemma **??** *(ii)* and we obtain

$$\int_{\mathcal{M}} |\alpha(\sigma) f(w_j(\sigma))|^{r'}\, dv_g \leq \int_{\mathcal{M}} \left[\alpha(\sigma) \left(A_2 + A_2^{\varepsilon} |w_j|^{r-1} \right) \right]^{\frac{r}{r-1}}.$$

Recalling that for any $a, b \geq 0$ we have

$$(a+b)^r \leq 2^r (a^r + b^r),$$

it follows from (**??**) that

$$\int_{\mathcal{M}} |\alpha(\sigma) f(w_j(\sigma))|^{r'}\, dv_g \leq \left(2A_2 \|\alpha\|_{L^{r'}(\mathcal{M})} \right)^{r'} + (2A_2^{\varepsilon})^{r'} \int_{\mathcal{M}} (\alpha(\sigma))^{r'} |w_j|^r\, dv_g.$$

Finally, we exploit Lemma **??** in (**??**) and we obtain

$$\int_{\mathcal{M}} |\alpha(\sigma) f(w_j(\sigma))|^{r'}\, dv_g \le \left(2A_2 \|\alpha\|_{L^{r'}(\mathcal{M})}\right)^{r'} + \frac{A_4}{A_3} (2A_2^{\varepsilon})^{r'} \|\alpha\|_{L^{r'}(\mathcal{M})}^{r'} + (2A_2^{\varepsilon})^{r'} \frac{A_4}{A_3} \|\alpha\|_{L^{\infty}(\mathcal{M})}^{r'-1} \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g.$$

From (**??**), (**??**) and (**??**), we can deduce that $(w_j)_j$ is bounded in $H_V^1(\mathcal{M})$. Hence, up to a subsequence

$$w_j \rightharpoonup w_\infty \quad \text{in } H_V^1(\mathcal{M}).$$

Furthermore, recalling that $H_V^1(\mathcal{M}) \hookrightarrow L^r(\mathcal{M})$ is compact, we have

$$\begin{aligned} &w_j \to w_\infty \quad \text{in } L^r(\mathcal{M}), \\ &w_j(\sigma) \to w_\infty(\sigma) \quad \text{for a.e. } \sigma \in \mathcal{M} \end{aligned}$$

as $j \to \infty$. Now, from (**??**), Lemma **??** (i) and the Minkowski inequality it follows

$$\begin{aligned} 0 < \frac{2\tilde{\varrho}}{C\lambda_{k+h+1}} &\le \frac{\|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})}}{\|w_j\|} \\ &\le \frac{\left(\int_{\mathcal{M}} \left[\alpha(\sigma)\left(2\varepsilon|w_j| + rA_1^{\varepsilon}|w_j|^{r-1}\right)\right]^{\frac{r}{r-1}} dv_g\right)^{\frac{r-1}{r}}}{\|w_j\|} \\ &\le \frac{4\varepsilon\left(\int_{\mathcal{M}} \alpha(\sigma)^{\frac{r}{r-1}} |w_j|^{\frac{r}{r-1}}\, dv_g\right)^{\frac{r-1}{r}} + 2rA_1^{\varepsilon}\left(\int_{\mathcal{M}} \alpha(\sigma)^{\frac{r}{r-1}} |w_j|^{r}\, dv_g\right)^{\frac{r-1}{r}}}{\|w_j\|}. \end{aligned}$$

Recalling that the embedding $H_V^1(\mathcal{M}) \hookrightarrow L^s(\mathcal{M})$ is continuous for every $s \in [2, 2^*]$ we deduce from (**??**) that

$$0 < \frac{2\tilde{\varrho}}{C\lambda_{k+h+1}} \le \tilde{C}\left(2\varepsilon + rA_1^{\varepsilon}\|w_j\|^{r-2}\right)$$

for some optimal $\tilde{C} > 0$. With similar estimates, it is straightforward to check that

$$|\alpha(\sigma) f(w_j(\sigma))|^{\frac{r}{r-1}} \le C_1^{\varepsilon} |\alpha(\sigma)|^{\frac{r}{r-1}} + C_2^{\varepsilon} |w_j(\sigma)|^r$$

and

$$|\alpha(\sigma) F(w_j(\sigma))| \le C_3^{\varepsilon} |w_j(\sigma)|^2 + C_4^{\varepsilon} |w_j(\sigma)|^r$$

choosing adequately $C_1^{\varepsilon}, C_2^{\varepsilon}, C_3^{\varepsilon}, C_4^{\varepsilon} > 0$. Hence, the general Lebesgue dominated convergence Theorem [**?**, Section 4.4, Theorem 19] implies

$$\lim_{j\to\infty} \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g = \int_{\mathcal{M}} \alpha(\sigma) F(w_\infty(\sigma))\, dv_g$$

and

$$\lim_{j\to\infty} \int_{\mathcal{M}} |\alpha(\sigma) f(w_j(\sigma))|^{\frac{r}{r-1}}\, dv_g = \int_{\mathcal{M}} |\alpha(\sigma) f(w_\infty(\sigma))|^{\frac{r}{r-1}}\, dv_g.$$

Coupling (**??**) and (**??**), keeping into account (f_3), we see that $w_\infty = 0$ is the only admissible case. At this point, only two possible scenarios are possible. The first one is that $w_j \to 0$ in $H^1_V(\mathcal{M})$, but if that were true, letting $j \to \infty$, then we would have

$$0 < \frac{2\tilde{\varrho}}{C\lambda_{k+h+1}} \leq 2\varepsilon\tilde{C}$$

which is impossible since $\varepsilon > 0$ is arbitrary. The second one is that there exist $\eta > 0$ such that $\|w_j\| \geq \eta$ for each $j \in \mathbb{N}$. In this case, firstly we notice that from $w_\infty = 0$ and $f(0) = 0$ it follows

$$\lim_{j\to\infty} \int_{\mathcal{M}} |\alpha(\sigma) f(w_j(\sigma))|^{\frac{r}{r-1}}\, dv_g = 0.$$

Then, thanks to (**??**), (**??**) becomes

$$0 < \frac{2\tilde{\varrho}\eta}{\lambda_{k+h+1}} \leq 0,$$

which is clearly a contradiction. □

In the sequel, given a closed subspace Y of $H^1_V(\mathcal{M})$ we will denote with $P_Y \colon H^1_V(\mathcal{M}) \to Y$ the usual orthogonal projection.

Proposition 4. *Suppose f satisfies (f_1) – (f_3), $\lambda \in \mathbb{R}$ and let $(w_j)_j \subset H^1_V(\mathcal{M})$ be a sequence such that*

$$\left(J_\lambda(w_j)\right)_j \quad \textit{is bounded}$$

$$P_{X_2} w_j \to 0 \quad \textit{in } H^1_V(\mathcal{M})$$

$$P_{X_1 \oplus X_3} \nabla J_\lambda(w_j) \to 0 \quad \textit{in } H^1_V(\mathcal{M}).$$

Then $(w_j)_j$ is bounded in $H^1_V(\mathcal{M})$.

Proof. We argue by contradiction, and we suppose that

$$\|w_j\| \to \infty$$

as $j \to \infty$. Normalizing, we assume up to a subsequence

$$\frac{w_j}{\|w_j\|} \rightharpoonup w_\infty \quad \text{in } H^1_V(\mathcal{M})$$

and

$$\frac{w_j}{\|w_j\|} \to w_\infty \quad \text{in } L^s(\mathcal{M})$$

as $j \to \infty$ for all $s \in [2, 2^*)$.
Clearly, we can write

$$w_j = P_{X_2} w_j + P_{X_1 \oplus X_3} w_j$$

with $P_{X_2} w_j \to 0$. Recalling (**??**), (**??**) and (**??**) we have

$$\begin{aligned}
\langle P_{X_1 \oplus X_3} \nabla J_\lambda(w_j), w_j \rangle &= \langle \nabla J_\lambda(w_j), w_j \rangle - \langle P_{X_2} \nabla J_\lambda(w_j), w_j \rangle \\
&= \|w_j\|^2 - \lambda \|w_j\|^2_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g \\
&\quad - \langle P_{X_2} \Big(w_j - \Delta_g^{-1} \left(\lambda w_j + \alpha f(w_j) \right) \Big), w_j \rangle
\end{aligned}$$

By orthogonality, we get

$$\langle P_{X_2} w, v \rangle = \langle P_{X_2} w, P_{X_1 \oplus X_3} v + P_{X_2} v \rangle = \langle P_{X_2} w, P_{X_2} v \rangle$$

and

$$\langle w, P_{X_2} v \rangle = \langle P_{X_1 \oplus X_3} w + P_{X_2} w, P_{X_2} v \rangle = \langle P_{X_2} w, P_{X_2} v \rangle$$

for every $w, v \in H^1_V(\mathcal{M})$, which means that P_{X_2} is a symmetric operator. In virtue of that, we have

$$\begin{aligned}
\langle P_{X_2} \Big(w_j - \Delta_g^{-1} \left(\lambda w_j + \alpha f(w_j) \right) \Big), w_j \rangle &= \|P_{X_2} w_j\|^2 - \lambda \langle \Delta_g^{-1} w_j, P_{X_2} w_j \rangle \\
&\quad - \langle \Delta_g^{-1} \left(\alpha f(w_j) \right), P_{X_2} w_j \rangle.
\end{aligned}$$

Recalling (**??**) we get

$$\begin{aligned}
&\lambda \langle P_{X_2} w_j, \Delta_g^{-1} w_j \rangle + \langle P_{X_2} w_j, \Delta_g^{-1} \left(\alpha f(w_j) \right) \rangle \\
&\qquad = \lambda \|P_{X_2} w_j\|^2_{L^2(\mathcal{M})} + \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j(\sigma)\, dv_g
\end{aligned}$$

Inserting (**??**) and (**??**) in (**??**) we obtain

$$\begin{aligned}
\langle P_{X_1 \oplus X_3} \nabla J_\lambda(w_j), w_j \rangle &= 2 J_\lambda(w_j) + 2 \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g \\
&\quad - \|P_{X_2} w_j\|^2 + \lambda \|P_{X_2} w_j\|^2_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g \\
&\quad + \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j(\sigma)\, dv_g.
\end{aligned}$$

Reordering the terms in (**??**) and using (**??**), (**??**), (**??**) and (**??**) we get

$$\begin{aligned}
&\frac{1}{\|w_j\|^r} \Big(2 \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g \\
&\qquad\qquad + \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j\, dv_g \Big) \to 0
\end{aligned}$$

as $j \to \infty$.
Claim: $w_\infty = 0$
We first need to show

$$\frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j \, dv_g}{\|w_j\|^r} \to 0$$

as $j \to \infty$. As a first step, observe that all eigenfunctions are bounded by [**?**, Theorem 3.1]. Moreover, having X_2 finite dimension, all norms are equivalent. Therefore, from (**??**) it follows that

$$\|P_{X_2} w_j\|_{L^\infty(\mathcal{M})} \to 0$$

as $j \to \infty$. Then, from Lemma **??** (i)

$$\left| \frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j(\sigma) \, dv_g}{\|w_j\|^r} \right| \\ \leq \frac{2\varepsilon \displaystyle\int_{\mathcal{M}} \alpha(\sigma) w_j(\sigma) \, dv_g + r A_1^\varepsilon \|P_{X_2} w_j\|_{L^\infty(\mathcal{M})} \int_{\mathcal{M}} \alpha(\sigma) |w_j(\sigma)|^{r-1} \, dv_g}{\|w_j\|^r}.$$

Applying the Hölder inequality twice and recalling $H^1_V(\mathcal{M}) \hookrightarrow L^2(\mathcal{M})$ it follows

$$\left| \frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j(\sigma) \, dv_g}{\|w_j\|^r} \right| \\ \leq \frac{2\varepsilon C \|\alpha\|_{L^2(\mathcal{M})}}{\|w_j\|^{r-2}} + \frac{r A_1^\varepsilon \|P_{X_2} w_j\|_{L^\infty(\mathcal{M})} \|\alpha\|^r_{L^r(\mathcal{M})} \left\| \frac{w_j}{\|w_j\|} \right\|^{r-1}_{L^r(\mathcal{M})}}{\|w_j\|}$$

for some $C > 0$. Now, the validity of (**??**) follows from the boundedness of the sequence $w_j/\|w_j\|$ in $L^r(\mathcal{M})$. In virtue of (**??**), combining (**??**) with (f_3), we obtain

$$o(1) = \frac{2 \displaystyle\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma)) \, dv_g - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma) \, dv_g}{\|w_j\|^r} \\ \leq \frac{(2 - r) \displaystyle\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma)) \, dv_g}{\|w_j\|^r} \leq 0$$

from which we deduce

$$\lim_{j \to \infty} \frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma)) \, dv_g}{\|w_j\|^r} = 0.$$

At this point, Lemma **??** (iii) implies that

$$\frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma)|w_j|^r\, dv_g}{\|w_j\|^r} \le \frac{A_4\|\alpha\|_{L^1(\mathcal{M})}}{A_3\|w_j\|^r} + \frac{1}{A_3\|w_j\|^r}\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\, dv_g.$$

Combining this with (**??**) we get that $\alpha(\sigma)|w_j(\sigma)|^r \to 0$ a.e. in $\mathcal{M}$ as $j \to \infty$, but then the claim follows because of the positivity a.e of α. Now, we observe that

$$0 \leftarrow \frac{J_\lambda(w_j)}{\|w_j\|^2} = \frac{1}{2} - \frac{\lambda}{2}\left\|\frac{w_j}{\|w_j\|}\right\|^2_{L^2(\mathcal{M})} - \frac{1}{\|w_j\|^2}\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\, dv_g.$$

Recalling $w_j/\|w_j\| \to 0$ in $L^2(\mathcal{M})$ we obtain

$$\frac{1}{\|w_j\|^2}\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\, dv_g \to \frac{1}{2}$$

as $j \to \infty$. Furthermore, from Lemma **??** (iii) it follows

$$\frac{1}{\|w_j\|^2}\int_{\mathcal{M}} \alpha(\sigma)|w_j(\sigma)|^r\, dv_g \le \frac{A_4\|\alpha\|_{L^1(\mathcal{M})}}{A_3\|w_j\|^2} + \frac{1}{A_3\|w_j\|^2}\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\, dv_g.$$

Because of (**??**), the second member of (**??**) is bounded and so there exist a $\tilde{C} > 0$ such that

$$\int_{\mathcal{M}} \alpha(\sigma)|w_j(\sigma)|^r\, dv_g \le \tilde{C}\|w_j\|^2.$$

At this point, applying Lemma **??** (ii), the Hölder inequality and (**??**), we notice

$$\begin{aligned}
&\frac{\displaystyle\int_{\mathcal{M}} |\alpha(\sigma)f(w_j(\sigma))P_{X_2}w_j(\sigma)|\, dv_g}{\|w_j\|^2} \\
&\le \frac{\|P_{X_2}w_j\|_{L^\infty(\mathcal{M})}}{\|w_j\|^2}\left(A_2\|\alpha\|_{L^1(\mathcal{M})} + A_2^\varepsilon\int_{\mathcal{M}} |\alpha(\sigma)|^{\frac{1}{r}}|\alpha(\sigma)|^{\frac{r-1}{r}}|w_j(\sigma)|^{r-1}\right) \\
&\le \|P_{X_2}w_j\|_{L^\infty}\left[\frac{A_2\|\alpha\|_{L^1(\mathcal{M})}}{\|w_j\|^2} + \frac{A_2^\varepsilon\|\alpha\|^{\frac{1}{r}}_{L^1(\mathcal{M})}}{\|w_j\|^{\frac{2}{r}}}\left(\frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma)|w_j(\sigma)|^r\, dv_g}{\|w_j\|^2}\right)^{\frac{r-1}{r}}\right] \\
&\le \|P_{X_2}w_j\|_{L^\infty}\left[\frac{A_2\|\alpha\|_{L^1(\mathcal{M})}}{\|w_j\|^2} + \frac{A_2^\varepsilon\tilde{C}^{1-\frac{1}{r}}\|\alpha\|^{\frac{1}{r}}_{L^1(\mathcal{M})}}{\|w_j\|^{\frac{2}{r}}}\right],
\end{aligned}$$

which implies

$$\lim_{j\to\infty}\frac{\displaystyle\int_{\mathcal{M}} |\alpha(\sigma)f(w_j(\sigma))P_{X_2}w_j(\sigma)|\, dv_g}{\|w_j\|^2} = 0.$$

Dividing (**??**) by $\|w_j\|^2$ and using (**??**), (**??**), (**??**) and (**??**) we get

$$\frac{1}{\|w_j\|^2}\left(\int_{\mathcal{M}}\alpha(\sigma)F(w_j(\sigma))\,dv_g-\int_{\mathcal{M}}\alpha(\sigma)f(w_j(\sigma))w_j(\sigma)\,dv_g\right)\to 0$$

as $j\to\infty$. To conclude the proof, we argue as did in (**??**) to obtain

$$\lim_{j\to\infty}\frac{1}{2}\int_{\mathcal{M}}\alpha(\sigma)F(w_j(\sigma))\,dv_g=0.$$

Clearly (**??**) and (**??**) are not compatible. $\square$

Proposition 5. *Assume that f satisfies (f_1) – (f_3). For any $\varrho>0$ there exists $\eta_\varrho>0$ such that for any $\eta',\eta''\in(0,\eta_\varrho)$, with $\eta'<\eta''$ we have that $\nabla(J_\lambda,X_1\oplus X_3,\eta',\eta'')$ is verified for all $\lambda\in(\lambda_{k-1}+\varrho,\lambda_{k+h+1}-\varrho)$.*

Proof. By contradiction, we suppose that there is $\tilde{\varrho}>0$ such that for any $\eta_{\tilde{\varrho}}>0$ we can find $\tilde{\lambda}\in[\lambda_{k-1}+\tilde{\varrho},\lambda_{k+h+1}-\tilde{\varrho})$ and $\eta'<\eta''$ such that

$$(\nabla)\,(J_\lambda,X_1\oplus X_3,\eta',\eta'')$$

does not hold. If so, it is possible to find a sequence $(w_j)_j\subset H^1_V(\mathcal{M})$ such that

$$J_{\tilde{\lambda}}(w_j)\in[\eta',\eta'']$$

$$\operatorname{dist}(w_j,X_1\oplus X_3)\to 0\quad\text{as } j\to\infty$$

$$P_{X_1\oplus X_3}\nabla J_{\tilde{\lambda}}(w_j)\to 0\quad\text{as } j\to\infty.$$

Because of that, Proposition **??** can be applied, thus $(w_j)_j$ is bounded in $H^1_V(\mathcal{M})$. Hence, up to a subsequence,

$$w_j\rightharpoonup w_\infty\quad\text{in } H^1_V(\mathcal{M})$$

$$w_j\to w_\infty\quad\text{in } L^s(\mathcal{M})\quad\text{for all } s\in[2,2^*)$$

$$w_j(\sigma)\to w_\infty(\sigma)\quad\text{a.e in } \mathcal{M}$$

as $j\to\infty$. Now, arguing as we did to obtain (**??**), we can find $\tilde{A}_1^\varepsilon,\tilde{A}_2^\varepsilon>0$ such that

$$\int_{\mathcal{M}}|\alpha(\sigma)f(w_j(\sigma))|^{\frac{r}{r-1}}\,dv_g\le\tilde{A}_1^\varepsilon+\tilde{A}_2^\varepsilon\int_{\mathcal{M}}|w_j(\sigma)|^r\,dv_g.$$

Since $w_j\to w_\infty$ in $L^r(\mathcal{M})$ there is $\tilde{C}>0$ such that

$$\int_{\mathcal{M}}|\alpha(\sigma)f(w_j(\sigma))|^{\frac{r}{r-1}}\,dv_g\le\tilde{C}.$$

Then, recalling that Δ_g^{-1} is a compact operator,

$$P_{X_1 \oplus X_3} \Delta_g^{-1} \left(\tilde{\lambda} w_j + \alpha f(w_j) \right) \to P_{X_1 \oplus X_3} \Delta_g^{-1} \left(\tilde{\lambda} w_\infty + \alpha f(w_\infty) \right).$$

Recalling (**??**), we have

$$P_{X_1 \oplus X_3} \nabla J_\lambda(w_j) = w_j - P_{X_2} w_j - P_{X_1 \oplus X_3} \Delta_g^{-1} \left(\tilde{\lambda} w_j + \alpha f(w_j) \right).$$

Since that, (**??**), (**??**) and (**??**) we deduce

$$w_j \to P_{X_1 \oplus X_3} \Delta_g^{-1} \left(\tilde{\lambda} w_\infty + \alpha f(w_\infty) \right)$$

in $H^1_V(\mathcal{M})$ as $j \to \infty$. Now, on the one hand from (**??**) and (**??**) it follows

$$\langle \nabla J_{\tilde{\lambda}}(w_j), \varphi \rangle = \langle w_j, \varphi \rangle - \tilde{\lambda} \langle w_j, \varphi \rangle_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) \varphi(\sigma)\, dv_g \to 0$$

for any $\varphi \in X_1 \oplus X_3$ as $j \to \infty$. □

On the other hand, from (**??**) and (**??**) we also have

$$\langle \nabla J_{\tilde{\lambda}}(w_j), \varphi \rangle \to \langle w_\infty, \varphi \rangle - \tilde{\lambda} \langle w_\infty, \varphi \rangle_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) \varphi(\sigma)\, dv_g$$

for any $\varphi \in X_1 \oplus X_3$. Coupling (**??**) and (**??**) we get that w_∞ is a critical point for $J_{\tilde{\lambda}}$ constrained on $X_1 \oplus X_3$. Then, we can apply Proposition **??** to obtain $w_\infty = 0$. But, since $J_{\tilde{\lambda}}(w_j) \geq \eta'$, $w_j \to w_\infty$ in $H^1_V(\mathcal{M})$, exploiting the continuity of $J_{\tilde{\lambda}}$ we obtain $J_{\tilde{\lambda}}(w_\infty) > 0$. This is a contradiction, as $J_{\tilde{\lambda}}(0) = 0$.

5 Proof of Theorem ??

We begin with a technical result.

Lemma 4. *If f verifies (f_1)–(f_3) then*

$$\lim_{\lambda \to \lambda_k} \sup_{w \in E_{k+h}} J_\lambda(w) = 0$$

Proof. We start noticing that from Lemma **??** (iii) it follows

$$\lim_{\xi \to \pm\infty} J_\lambda(\xi w) = -\infty$$

for all $w \in E_{k+h}$, thus

$$\sup_{w \in E_{k+h}} J_\lambda(w) \quad \text{is achieved.}$$

Now, by contradiction we suppose there is a sequence $\tau_j \to \lambda_k$ as $j \to \infty$ and a sequence $(w_j)_j \subset E_{k+h}$ such that

$$J_{\tau_j}(w_j) = \sup_{w \in E_{k+h}} J_\lambda(w) > \gamma$$

for some $\gamma > 0$. We split the proof analysing separately the case $(w_j)_j$ bounded and unbounded. In the first one, since the weak and the strong topology coincide, we can suppose $w_j \to w_\infty$ in E_{k+h}. In order to reach a contradiction, keeping into account (**??**) and letting $j \to \infty$, it suffices to apply Lemma **??** to obtain

$$\gamma \leq J_{\lambda_k}(w_\infty) = (\lambda_{k+h} - \lambda_k) - \int_{\mathcal{M}} \alpha(\sigma) F(w_\infty(\sigma))\, dv_g \leq 0.$$

Instead, if $(w_j)_j$ is unbounded, we can assume $\|w_j\| \to \infty$ as $j \to \infty$. From Lemma **??** (iii) it follows

$$0 < \gamma \leq J_{\tau_j}(w_j) \leq \frac{1}{2}\|w_j\|^2 - \frac{\tau_j}{2}\|w_j\|^2_{L^2(\mathcal{M})} - A_3 |w_j\|^r_{L^r(\mathcal{M})} + A_4\|\alpha\|_{L^1(\mathcal{M})}.$$

Exploiting again the fact that on the finite-dimensional subspace E_{h+k} all norms are equivalent, the right hand side of the above inequality goes to $-\infty$ concluding the proof. □

Proof of Theorem **??**. We want to apply [**?**, Theorem 2.10]. We start choosing $\varrho > 0$. In correspondence of that, thanks to Proposition **??** there are $\eta_\varrho, \eta', \eta'' > 0$, with $\eta' < \eta'' < \eta_\varrho$ such that $\nabla\left(J_\lambda, X_1 \oplus X_3, \eta', \eta''\right)$ is verified for all $\lambda \in (\lambda_{k-1} + \varrho, \lambda_{k+h+1} - \varrho)$. Exploiting Lemma **??** we also have the existence of $\overline{\varrho} > 0$, with $\overline{\varrho} \leq \varrho$ such that

$$\sup_{w \in E_{k+h}} J_\lambda(w) \leq \eta'$$

for $\lambda \in (\lambda_k - \overline{\varrho}, \lambda_k)$. At this point, recalling Propositions **??** and **??**, all hypothesis of Theorem 2.10 in [**?**] are satisfied, and we have the existence of two non-trivial critical points w_1 and w_2 such that

$$J_\lambda(w_i) \in [\eta', \eta''] \quad (i = 1, 2).$$

The third critical point w_3 is a consequence of the classical Linking Theorem. Furthermore, from Lemma **??**, choosing λ sufficiently close to λ_k, we can see that

$$J_\lambda(w_i) < \sup_{w \in E_{k+h}} J_\lambda(w) \leq J_\lambda(w_3), \quad (i = 1, 2)$$

proving that w_1, w_2, w_3 are distinct. □

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