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Gross-Pitaevskii hydrodynamics in Riemannian manifolds and application in Black Hole cosmology

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Introduction

This thesis focuses on the extension of the hydrodynamic formulation of the Gross-Pitaevskii equation to the case of a general Riemannian metric, its connection with privileged geometries and possible applications in black hole cosmology in the context of a Bose-Einstein condensate (BEC) at zero temperature. A BEC is an ultra cold and dilute gas of bosons where all particles are at the ground-state energy level. For this reason, in the mean field approximation, the whole system is completely determined by a single wave function Ψ, a tensor product of N identical wave functions, governed by a non-linear Schrödinger equation known as Gross-Pitaevskii equation (GPE) [Gr61, Pi61, Dal99]. The GPE arises from considering the ground-state of a second-quantized many-particle Hamiltonian for N interacting bosons, trapped by an external potential V, where only binary collisions are relevant and characterized by a single parameter called the scattering length a_s.

In 1927 Madelung introduced a transform which allows one to write the complex wave function Ψ in terms of a density ρ and a phase θ [Ma27]. This transform applied to the GPE provides a complete hydrodynamic interpretation, decomposing the GPE into a continuity equation for the density and an Euler type of equation for the velocity field, defined in terms of the gradient of the phase: $\boldsymbol{u} = \nabla\theta$ (see [BP16] for the standard hydrodynamic derivation). In this context, we define a topological vortex defect as the locus of points where the density ρ is zero, which coincides with the locus of points where the phase θ is ill-defined. These defects are just a few Ångströms wide (distance known as healing length) and they are long-lived, since their strength is fixed by the quantization of angular momentum in the absence of dissipative mechanisms. The GPE approach guarantees a natural dynamical setup for studying the interaction between vortices, and thus it is widely used in the context of superfluidity and quantum turbulence [BP16, PS16]. As we shall see, these vortices can experience forces as a result of geometric aspects of the ambient space, and this is one reason for considering the GPE on manifolds with generic metrics [RR21]. In particular, we shall show that the hydrodynamic formulation emerges analogously in any metric, and therefore it is in a one to one correspondence with the GPE itself; but we will also see the emergence of a new force depending essentially on two parameters: the geometry of the manifold and the first derivatives of the density profile. First, we notice that the new term becomes particularly relevant in the vicinity of the defects, since in the tubular region of width approximately one healing length surrounding the nodal line the derivatives of the density attain their maximum value. Secondly, we find that we can manipulate the geometry of the manifold in order to impose specific stationary conditions; in particular, we can attain a steady state condition when considering negative curvature surfaces [Ro21]. Notice that such surfaces have been previously related to BECs theory; indeed we will show how to get the sine-Gordon equation (sGE) from the GPE through a system of coupled BECs, and we will recall how the sine-Gordon equation was firstly introduced when studying Gauss-Codazzi equation for negative curvature surfaces [RR22].

In this work we will focus on the relationship naturally emerging between geometry and cosmology since 1915, when Einstein's theory of general relativity was published [Ei15]. Such a relationship became particularly relevant when Penrose introduced differential geometry and topological methods in the study of general relativity, to show that black holes, firstly theorized as highly symmetric solutions of Einstein's equation, could exist even without symmetry [Pe65]. His seminal paper made black holes become a real physical phenomenon leading to the first observational evidence by the Event Horizon Telescope [EHT19] (for this work Penrose was awarded the 2020 Nobel Prize in Physics). In this direction we show that, when imposing steady state conditions in the context of BECs on negative scalar curvature manifolds, we obtain a new type of Einstein field equations that governs the dynamics of the BEC [Ro21].

Surprisingly we find that there is a very strict relationship also between general relativity and the study of BECs. First of all because relativistic BECs on generic spacetime metrics are promising models for studying the behavior of the early universe and its expansion, as in the case of the two-phase model [FM06]–[FM09], where BECs are directly seen as dark energy and dark matter, and also in [Oal13], where they are numerically used for simulations. Indeed, in this context we make use of tools acquired for generic Riemannian (Lorentzian) manifolds, obtaining a new multi-dimensional form of an Einstein type equation. Moreover, BECs appear to be extremely promising analogue systems for probing kinematic aspects of general relativity, thus producing many experimental probes, such as (analogue) Hawking radiation from a black hole, as presented by Hawking in 1974

[Ha74]. The first observation of the possibility to simulate gravitational configurations in BECs was made by Unruh [Un81, Un95]. In particular, the analogue of a gravitational black hole in a dilute Bose-Einstein condensate has been investigated and experimentally proposed by Garay *et al.* [Gal00]–[Ga02], showing that in the hydrodynamic limit there exist configurations that exhibit behaviors analogous to that of a gravitational black hole (a sonic black hole is simulated by solving numerically the GPE for a condensate subject to some particular trapping potential). Important research has been done by Visser, Barceló and Liberati ([BLV00]–[BLV01], [VBL02a, VBL02b]), putting focus on finding an effective relativistic curved spacetime with metric entirely determined by the physical properties of the fluid (density and velocity field) that describes the evolution of phase perturbations by a d'Alembertian equation. They proved that BECs provide useful models of an approximate Lorentz invariance (in the low-momentum limit), and showed that in principle, the existence of an effective Lorentzian metric is what is needed to obtain simulations of the Hawking radiation, thus providing a crude estimate of the Hawking temperature, found to be of the order of 10^{-9} K, only three orders of magnitude less than the temperature of the BEC itself. This has been numerically proved by Carusotto and collaborators in 2008 [Cal08], and then physically detected in 2010 by Steinhauer and collaborators [Lal10, St16]. Following this line of research, we study the emergence of the effective Lorentzian metric in the specific case of a straight vortex defect placed along the $z-$axis, obtaining that, under particular approximations, such a metric can be explicitly determined [Ro22].

In summary, this work attempts to bridge physics and geometry, showing that many interesting ideas arise from the observation of analogies between different fields of study.

The structure of the work is as follows: in Chapter 1 we consider a generic manifold M with metric g and carry out the hydrodynamic derivation of the GPE, in order to focus on the purely geometrical aspects that arise from this generic setting. We focus on the forces arising in the vicinity of the vortex, deriving a new type of Einstein's field equations by imposing steady state conditions on the BEC. This will lead us to consider in Chapter 2 some specific manifolds with negative curvature, in particular the pseudosphere, and its relationship with the sine-Gordon equation, which is also related to the GPE through the study of coupled BECs. Then in Chapter 3 we focus on the relativistic version of the GPE in Klein-Gordon form and we determine the Einstein field equations in a generic spacetime. Finally in Chapter 4 we go through the linearization procedure introduced by Visser *et al.*

[BLV00], and we consider the specific case of a straight vortex placed along the $z-$axis, showing how to obtain the effective Lorentzian metric in this case. We conclude this work by presenting a discussion on possible future directions of research.

Chapter 1

Gross-Pitaevskii equation on a Riemannian manifold

In this chapter we start from the Gross-Pitaevskii equation (GPE) on flat $3-$dimensional space and then consider a curved space pointing out the emergence of some new terms.

1.1 Flat-space Gross-Pitaevskii equation

We start by considering the standard Gross-Pitaevskii equation [Gr61, Pi61]

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + \mathfrak{g}|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t)\,. \tag{1.1}$$

This is a non-linear Schrödinger equation governing the zero temperature Bose-Einstein condensate. Here m is the mass of the bosons, $\hbar$ is reduced Planck's constant, V is the external potential and $\mathfrak{g} = 4\pi\hbar^2 a_s/m$ is the coupling constant proportional to the scattering length a_s of two interacting bosons [BP16]. The wave function Ψ describing the whole BEC satisfies the normalization condition $N = \int|\Psi|^2\,\mathrm{d}^3\mathbf{r}$, and we can define the total mass of the condensate

$$M = mN = \int m|\Psi|^2\,\mathrm{d}^3\mathbf{r}\,, \tag{1.2}$$

the total energy

$$E = \int\left[\frac{\hbar^2}{2m}|\nabla\Psi|^2 + V|\Psi|^2 + \frac{\mathfrak{g}}{2}|\Psi|^4\right]\mathrm{d}^3\mathbf{r} = E_{kin} + E_{pot} + E_{int}\,, \tag{1.3}$$

and the normalized momentum density

$$P = \frac{i\hbar}{2}\frac{\Psi\nabla\Psi^* - \Psi^*\nabla\Psi}{|\Psi|^2}\,. \tag{1.4}$$

In order to derive the standard hydrodynamic form of the GPE (1.15) we apply the Madelung transformation [Ma27] and write the wave function Ψ in terms of density ρ and phase θ as

$$\Psi = \sqrt{\rho}\, e^{i\left(\frac{m}{\hbar}\theta\right)}\,, \tag{1.5}$$

where ρ satisfies $N = \int \rho\, \mathrm{d}^3\mathbf{r}$. We substitute Madelung's expression (1.5) into the GPE (1.1) and equate real and imaginary parts on both sides. Noticing that the velocity $\boldsymbol{u}$ can be written as the gradient of the phase $\boldsymbol{u} = \nabla\theta$ we obtain a continuity equation for the density ρ and an equation for the phase θ

$$\begin{cases} \partial_t\rho = -\mathrm{div}(\rho\boldsymbol{u})\,, \\ \partial_t\theta = \dfrac{\hbar^2}{2m^2\sqrt{\rho}}\nabla^2\sqrt{\rho} - \dfrac{1}{2}|\nabla\theta|^2 - \dfrac{\mathfrak{g}}{m}\rho - \dfrac{V}{m}\,. \end{cases} \tag{1.6}$$

Taking the gradient of the phase equation we obtain an Euler equation for the velocity

$$\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{\mathfrak{g}}{m}\nabla\rho + \nabla\left(\frac{\hbar^2}{2m^2\sqrt{\rho}}\nabla^2\sqrt{\rho}\right)\,, \tag{1.7}$$

where we set the external potential to zero $V = 0$. Multiplying the Euler equation times the density ρ we can recast (1.7) in the form of a Navier-Stokes equation

$$\rho(\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = -\nabla\mathrm{p} + \mathrm{div}\,\tau\,, \tag{1.8}$$

that in components reads:

$$\rho\left(\partial_t u_i + u^k\partial_k u_i\right) = -\partial_i\left(\frac{\mathfrak{g}\rho^2}{2m}\right) + \partial^k\left(\frac{\hbar^2}{4m^2}\rho\,\partial_k\partial_i(\ln\rho)\right)\,, \tag{1.9}$$

where $\mathrm{p} = \dfrac{\mathfrak{g}\rho^2}{2m}$ is pressure and $\tau_{ik} = \dfrac{\hbar^2}{4m^2}\rho\,\partial_k\partial_i(\ln\rho)$ is stress-tensor, and where we have a summation over repeated indices k. Finally, we can write (1.9) in the form of a conservation law for the momentum $\rho\boldsymbol{u}$

$$\partial_t(\rho\boldsymbol{u}) = -\mathrm{div}\,\Pi\,, \tag{1.10}$$

that in components reads:

$$\partial_t(\rho u_i) = -\partial^k(\rho u_k u_i + \delta_{ik}\mathrm{p} - \tau_{ik}) = -\partial^k\Pi_{ik}\,, \tag{1.11}$$

again with the summation on indices k.

We now consider the total energy of the condensate (1.3) and substitute Madelung's transform (1.5); we obtain

$$E = \int \left(\frac{\hbar^2}{2m} |\nabla \sqrt{\rho}|^2 + \frac{1}{2} m\rho |\nabla \theta|^2 + V\rho + \frac{1}{2} \mathfrak{g} \rho^2 \right) \mathrm{d}^3\mathbf{r} \,, \tag{1.12}$$

where we explicitly have quantum kinetic energy, conventional kinetic energy, potential energy and interaction energy, and where we can write $\int |\nabla\sqrt{\rho}|^2 \,\mathrm{d}^3\mathbf{r} = \frac{1}{4}\int |\nabla\rho|^2/\rho \,\mathrm{d}^3\mathbf{r}$. We now define the healing length ξ as the length scale where the quantum kinetic energy of the bosons equals the interaction energy, and we obtain

$$\frac{\hbar^2}{4m\xi^2}\rho = \frac{1}{2}\mathfrak{g}\rho^2 \quad \Longrightarrow \quad \xi = \frac{\hbar}{\sqrt{2\mathfrak{g}m\rho}} = \frac{1}{\sqrt{8\pi a_s \rho}} \,. \tag{1.13}$$

This ξ represents a fundamental length in BECs, giving the shortest distance over which the wave function can change; it is much smaller than any length scale in the solution of the single-particle wave function, and it also determines the size of the vortices that can form in a superfluid; indeed, in the presence of such defects, the healing length measures the distance over which the wave function grows from zero, at the center of the vortex, to its full value in the bulk of the superfluid.

1.2 Hydrodynamic formulation on Riemannian manifold

In order to derive the hydrodynamic formulation of the GPE on a Riemannian manifold, current consensus [BD82] is that one should replace the Cartesian Laplacian with the Laplace operator on the Riemannian manifold, plus a dimensionless curvature coupling term involving the Ricci scalar of the manifold. We end up with:

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)\Psi \longrightarrow \frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|}\, g^{ij} \partial_j \Psi \right) + \kappa R \Psi \,, \tag{1.14}$$

where the first term on the right-hand side is the Laplacian operator on the manifold, g^{ij} being the metric inverse matrix and $|g| = |\det(g_{ij})|$, and where R denotes Ricci scalar curvature (A.62). Hence, from now on we will consider the curved-space form of the GPE to be

$$i\hbar\partial_t \Psi = \left(-\frac{\hbar^2}{2m}(\nabla^2 + \kappa R) + V + \mathfrak{g}|\Psi|^2 \right) \Psi \,, \tag{1.15}$$

and we will go through the hydrodynamic formulation.

We use the Madelung transform (1.5), but now we have that the normalization condition is $N = \int \sqrt{|g|}\,\rho\,\mathrm{d}^3\mathbf{r}$ and the total energy is

$$E = \int \sqrt{|g|}\left[\frac{\hbar^2}{2m}|\nabla\Psi|^2 + V|\Psi|^2 + \frac{\mathfrak{g}}{2}|\Psi|^4\right]\mathrm{d}^3\mathbf{r}\,, \tag{1.16}$$

where $|\nabla\Psi|^2 = g^{ij}\partial_i\Psi\partial_j\Psi$, while the momentum density is still defined as in (1.4). By substituting Madelung's expression into the total energy we find the same result as in the flat space; hence, we can define the healing length as in (1.13).
We now substitute Madelung (1.5) into the curved-space GPE (1.15) and obtain that the left-hand side becomes

$$i\hbar\partial_t\Psi = \frac{i\hbar}{2\sqrt{\rho}}(\partial_t\rho)e^{i\left(\frac{m}{\hbar}\theta\right)} - m\sqrt{\rho}e^{i\left(\frac{m}{\hbar}\theta\right)}\partial_t\theta\,, \tag{1.17}$$

while the Laplacian term on the Riemannian manifold becomes

$$\begin{aligned}
\nabla^2\Psi &= \frac{1}{\sqrt{|g|}}\partial_i\left[\sqrt{|g|}g^{ij}\partial_j(\sqrt{\rho}e^{i\left(\frac{m}{\hbar}\theta\right)})\right]\\
&= \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\left(\partial_j\sqrt{\rho}e^{i\left(\frac{m}{\hbar}\theta\right)} + i\frac{m}{\hbar}\sqrt{\rho}\partial_j\theta e^{i\left(\frac{m}{\hbar}\theta\right)}\right)\right)\\
&= \frac{1}{\sqrt{|g|}}\partial_i\left((\sqrt{|g|}g^{ij}\partial_j\sqrt{\rho})e^{i\left(\frac{m}{\hbar}\theta\right)} + i\frac{m}{\hbar}\sqrt{\rho}(\sqrt{|g|}g^{ij}\partial_j\theta)e^{i\left(\frac{m}{\hbar}\theta\right)}\right)\\
&= \left(\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j\sqrt{\rho}) + i\frac{m}{2\hbar\sqrt{\rho}}g^{ij}\partial_j\rho\partial_i\theta + i\frac{m}{2\hbar\sqrt{\rho}}g^{ij}\partial_i\rho\partial_j\theta + \right.\\
&\quad \left. + i\frac{m}{\hbar}\sqrt{\rho}\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j\theta) - \frac{m^2}{\hbar^2}\sqrt{\rho}\,g^{ij}\partial_i\theta\partial_j\theta\right)e^{i\left(\frac{m}{\hbar}\theta\right)}\\
&= \left(\nabla^2\sqrt{\rho} + i\frac{m}{\hbar\sqrt{\rho}}\nabla\rho\cdot\nabla\theta + i\frac{m}{\hbar}\sqrt{\rho}\,\mathrm{div}(\nabla\theta) - \frac{m^2}{\hbar^2}\sqrt{\rho}|\nabla\theta|^2\right)e^{i\left(\frac{m}{\hbar}\theta\right)}\,,
\end{aligned} \tag{1.18}$$

where we have used the fact that the scalar product of two gradients can be written as $\nabla\rho\cdot\nabla\theta = g^{ij}\partial_i\rho\partial_j\theta$, and the divergence of a gradient is $\mathrm{div}(\nabla\theta) = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\theta\right)$.
The new curvature coupling term reads

$$\kappa R\,\Psi = \kappa R\sqrt{\rho}e^{i\left(\frac{m}{\hbar}\theta\right)}\,, \tag{1.19}$$

and the last two terms on the right-hand side of (1.15) become

$$(V + \mathfrak{g}\,|\Psi|^2)\Psi = (V + \mathfrak{g}\rho)\sqrt{\rho}e^{i\left(\frac{m}{\hbar}\theta\right)}\,. \tag{1.20}$$

By equating the imaginary parts we obtain an equation for the density

$$\frac{\hbar}{2\sqrt{\rho}}\partial_t\rho = -\frac{\hbar}{2\sqrt{\rho}}\nabla\rho\cdot\nabla\theta - \frac{\hbar}{2}\sqrt{\rho}\,\mathrm{div}(\nabla\theta)\,, \tag{1.21}$$

which can be recast in the form of a continuity equation similar to the one in the flat space

$$\partial_t\rho = -\nabla\rho\cdot\nabla\theta - \rho\,\mathrm{div}\nabla\theta = -\mathrm{div}\left(\rho\nabla\theta\right)\,. \tag{1.22}$$

By equating the real parts we obtain the equation for the phase θ

$$-m\sqrt{\rho}\partial_t\theta = -\frac{\hbar^2}{2m}\nabla^2\sqrt{\rho} + \frac{m}{2}\sqrt{\rho}|\nabla\theta|^2 - \frac{\hbar^2}{2m}\kappa R\sqrt{\rho} + \mathfrak{g}\rho\sqrt{\rho} + V\sqrt{\rho}\,, \tag{1.23}$$

which can be written as a Hamilton-Jacobi equation, obtaining

$$\partial_t\theta = \frac{\hbar^2}{2m^2\sqrt{\rho}}\nabla^2\sqrt{\rho} - \frac{1}{2}|\nabla\theta|^2 + \frac{\hbar^2\kappa}{2m^2}R - \frac{\mathfrak{g}}{m}\rho - \frac{V}{m}\,. \tag{1.24}$$

Here the last two terms represent the classical potential $U = \dfrac{V}{m} + \dfrac{\mathfrak{g}}{m}\rho$, the first term on the right-hand side $Q = \dfrac{\hbar^2}{2m^2\sqrt{\rho}}\nabla^2\sqrt{\rho}$ is known as quantum potential and it differs from the flat case only by replacing the flat-space Laplacian with the curved-space Laplace operator defined on the Riemannian manifold (A.31). The new term $C = \dfrac{\hbar^2\kappa}{2m^2}R$ can be defined as the curvature potential.

1.3 Euler form

We now define the velocity field as

$$\boldsymbol{u} = \frac{P}{m} = \frac{i\hbar}{2m}\frac{\Psi\nabla\Psi^* - \Psi^*\nabla\Psi}{|\Psi|^2}\,, \tag{1.25}$$

where P is the normalized momentum density (1.4). If we substitute Madelung's expression for Ψ we still obtain that the velocity field can be written as the gradient of the phase θ, indeed

$$\boldsymbol{u} = \frac{i\hbar}{2m\rho}\left(\frac{1}{2}\nabla\rho - i\frac{m}{\hbar}\rho\nabla\theta - \frac{1}{2}\nabla\rho - i\frac{m}{\hbar}\rho\nabla\theta\right) = \nabla\theta\,. \tag{1.26}$$

For the sake of clarity we write the velocity field in curved space as $u^i = g^{ij}\partial_j\theta$ and $u_i = \partial_i\theta$.

With this expression for the velocity we can again write the continuity equation (1.22) in the usual form

$$\partial_t \rho = -\mathrm{div}(\rho \boldsymbol{u})\,. \tag{1.27}$$

Taking the gradient of the phase equation (1.24) we end up with the following result:

Theorem 1 (GPE in Euler form)**.** *In general Riemannian metric the Gross-Pitaevskii equation admits hydrodynamic description in the form of an Euler equation, given by*

$$\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{u}} \boldsymbol{u} = \nabla Q + \nabla C - \nabla U\,. \tag{1.28}$$

Here $\nabla_{\boldsymbol{u}} \boldsymbol{u}$ is the covariant derivative of the velocity field along itself (often called the advective derivative in a fluid dynamical setting), $Q = \dfrac{\hbar^2}{2m^2\sqrt{\rho}} \nabla^2 \sqrt{\rho}$ is the quantum potential, $C = \dfrac{\hbar^2 \kappa}{2m^2} R$ is the curvature potential and $U = \dfrac{V}{m} + \dfrac{\mathfrak{g}}{m}\rho$ is the classical potential.

Proof. In order to prove this fact, we take the gradient of the phase equation (1.24) and we prove that for $\boldsymbol{u} = \nabla\theta$, one has

$$\nabla \left(\frac{1}{2} |\boldsymbol{u}|^2 \right) = \nabla_{\boldsymbol{u}} \boldsymbol{u}\,, \tag{1.29}$$

indeed

$$\begin{aligned} \nabla^i \left(\frac{1}{2} |\boldsymbol{u}|^2 \right) &= \nabla^i \left(\frac{1}{2} g^{jk} \nabla_j \theta \nabla_k \theta \right) = g^{jk} \nabla_j \theta \nabla^i \nabla_k \theta \\ &= g^{jk} \nabla_j \theta \nabla_k \nabla^i \theta = u^k \nabla_k u^i = \left(\nabla_{\boldsymbol{u}} \boldsymbol{u} \right)^i\,, \end{aligned} \tag{1.30}$$

where we used the fact that second covariant derivatives of scalars commute. □

In the end, we find that the equation (1.28) is completely equivalent to the gradient of the phase equation (1.24), indeed we can take the inverse of the gradient operator finding the equation for the phase in the form of a Bernoulli type equation

$$\partial_t \theta + \frac{1}{2} |\nabla \theta|^2 + U - C - Q = \mathit{constant}\,. \tag{1.31}$$

Hence we have proved that there is a one-to-one correspondence between the curved-space GPE and curved-space zero-vorticity hydrodynamics (with a quantum potential) on any manifold M with generic metric g.

1.4 Navier-Stokes form

Without loss of generality we set the external potential to zero ($V = 0$). Indeed, in order to consider the free expansion of the gas from its original state, it is worth exploring the case of an unconstrained state of matter in the absence of a trapping potential. Under such condition we can prove the following result:

Theorem 2 (GPE in Navier-Stokes Form). *In general Riemannian metric the Gross-Pitaevskii equation admits, in absence of a trapping potential ($V = 0$), hydrodynamic description in the form of a Navier-Stokes equation, given by*

$$\rho\big(\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{u}} \boldsymbol{u}\big) = -\nabla \mathrm{p} + (\operatorname{div} \tau)^\sharp + \mathbf{f}\,. \tag{1.32}$$

On the right-hand side we have

$$\mathrm{p} = \frac{\mathfrak{g}\rho^2}{2m} \tag{1.33}$$

that is a pressure-like term and

$$\tau = \frac{\hbar^2}{4m^2}\rho\,\mathrm{Hess}(\ln \rho) \tag{1.34}$$

which is a stress tensor term with the Hessian defined by

$$[\mathrm{Hess}(\ln\rho)]_{ij} = \nabla_i \nabla_j (\ln \rho) = \Big[\partial_i \partial_j (\ln\rho) - \Gamma^k{}_{ij}\partial_k(\ln\rho)\Big]\,, \tag{1.35}$$

with its divergence being

$$\operatorname{div}\tau = g^{ik}\nabla_k \tau_{ij}\,\mathrm{d}x^j\,. \tag{1.36}$$

The new external force term $\mathbf{f}$ *is a density curvature vector given by*

$$\mathbf{f} = -\frac{\hbar^2}{4m^2}\boldsymbol{R}(\nabla\rho,\,\cdot\,) + \frac{\hbar^2\kappa}{2m^2}\rho\nabla R\,, \tag{1.37}$$

depending on the geometry of the manifold M through Ricci's tensor $\boldsymbol{R}$ and Ricci's scalar R, that can be written in components by

$$\mathrm{f}^i = -\frac{\hbar^2}{4m^2}R^{ij}\partial_j\rho + \frac{\hbar^2\kappa}{2m^2}\rho\, g^{ij}\partial_j R\,. \tag{1.38}$$

Proof. In order to prove this, we rewrite Euler equation (1.28) multiplying everything by the density ρ and setting, without loss of generality, $V = 0$; we have

$$\rho\,(\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{u}} \boldsymbol{u}) = -\frac{\mathfrak{g}}{m}\rho\,\nabla\rho + \frac{\hbar^2\kappa}{2m^2}\rho\nabla R + \frac{\hbar^2}{m^2}\rho\,\nabla\left(\frac{\nabla^2\sqrt{\rho}}{2\sqrt{\rho}}\right), \tag{1.39}$$

where we consider the Bose-Einstein coupling constant $\mathfrak{g}$ to be spatially constant.
For simplicity, we consider the correspondent equation for 1-forms instead of vectors, where the correspondence is given by the musical isomorphism (A.8). For the first term on the right-hand side, noticing that $\mathfrak{g}$ and m are constants, we evidently have

$$-\frac{\mathfrak{g}}{m}\rho\,\mathrm{d}\,(\rho) = -\,\mathrm{d}\left(\frac{\mathfrak{g}}{m}\frac{\rho^2}{2}\right) = -\,\mathrm{dp}\,, \tag{1.40}$$

obtaining exactly the same pressure term as in flat space. The second term is simply given by taking the gradient (or differential) of the curvature potential C multiplied by the density ρ; and for the last term we want to prove that

$$\frac{\hbar^2}{m^2}\rho\,\mathrm{d}\left(\frac{1}{2\sqrt{\rho}}\nabla^2\sqrt{\rho}\right) = \frac{\hbar^2}{m^2}\mathrm{div}\left(\frac{1}{4}\rho\,\mathrm{Hess}(\ln\rho)\right) + \frac{\hbar^2}{4m^2}\mathcal{E}^\flat\,. \tag{1.41}$$

In order to do that, first notice that we can write

$$\begin{aligned}
\frac{1}{2\sqrt{\rho}}\nabla^2\sqrt{\rho} &= \frac{1}{2\sqrt{\rho}}\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\sqrt{\rho}\right)\\
&= \frac{1}{4\rho}\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\rho\right) + \frac{1}{2\sqrt{\rho}}g^{ij}\left(-\frac{1}{4\rho\sqrt{\rho}}\partial_i\rho\,\partial_j\rho\right)\\
&= \frac{1}{4\rho}\nabla^2\rho - \frac{1}{8\rho^2}|\nabla\rho|^2\,,
\end{aligned} \tag{1.42}$$

hence we can multiply equation (1.41) times $4m^2/\hbar^2$ and get to demonstrate

$$\rho\,\mathrm{d}\left(\frac{1}{\rho}\nabla^2\rho - \frac{1}{2\rho^2}|\nabla\rho|^2\right) = \mathrm{div}\left(\rho\mathrm{Hess}(\ln\rho)\right) + \mathcal{E}^\flat\,. \tag{1.43}$$

We consider each term separately and write them in coordinates. For the left term we have

$$\begin{aligned}
\rho\,\mathrm{d}\mathcal{Q} &= \rho\,\mathrm{d}\left(\frac{1}{\rho}\nabla^2\rho - \frac{1}{2\rho^2}|\nabla\rho|^2\right)\\
&= \rho\,\partial_j\left(\frac{1}{\rho}g^{ik}\partial_i\partial_k\rho - \frac{1}{\rho}g^{ik}\Gamma^r{}_{ik}\partial_r\rho - \frac{1}{2\rho^2}g^{ik}\partial_i\rho\,\partial_k\rho\right)\,\mathrm{d}x^j
\end{aligned} \tag{1.44}$$

where $\Gamma^i{}_{jk}$ denotes Christoffel's symbol. Let us rewrite this term by noticing that the derivative of the metric can be expressed – using metric compatibility (A.35) – as

$$\partial_j g^{ik} = -g^{ir}(\partial_j g_{rs})g^{sk} = -g^{ir}\left(g_{as}\Gamma^a{}_{jr} + g_{rb}\Gamma^b{}_{js}\right)g^{sk} = -g^{ir}\Gamma^k{}_{jr} - g^{sk}\Gamma^i{}_{js}\,. \tag{1.45}$$

Because we are summing over a symmetric matrix $S_{(ik)}$ we can compactly write

$$(\partial_j g^{ik})S_{(ik)} = (-g^{ir}\Gamma^k{}_{jr} - g^{sk}\Gamma^i{}_{js})S_{(ik)} = -2g^{ir}\Gamma^k{}_{jr}S_{(ik)} = -2g^{is}\Gamma^k{}_{js}S_{(ik)}\,, \tag{1.46}$$

that for all practical purposes implies $(\partial_j g^{ik}) \to -2g^{is}\Gamma^k{}_{jr}$. We get

$$\begin{aligned}\rho\,\mathrm{d}\mathcal{Q} = \Bigg[g^{ik}\bigg(&-\frac{1}{\rho}(\partial_j\rho)\partial_i\partial_k\rho + \partial_j\partial_i\partial_k\rho + \frac{1}{\rho}\Gamma^r{}_{ik}(\partial_r\rho)\partial_j\rho - (\partial_j\Gamma^r{}_{ik})\partial_r\rho \\ &- \Gamma^r{}_{ik}\partial_r\partial_j\rho + \frac{1}{\rho^2}(\partial_j\rho)(\partial_i\rho)\partial_k\rho - \frac{1}{\rho}(\partial_i\rho)\partial_j\partial_k\rho\bigg) \\ &- 2g^{is}\Gamma^k{}_{js}\partial_i\partial_k\rho + 2g^{is}\Gamma^k{}_{js}\Gamma^r{}_{ik}\partial_r\rho + \frac{1}{\rho}g^{is}\Gamma^k{}_{js}(\partial_i\rho)\partial_k\rho\Bigg]\,\mathrm{d}x^j\,.\end{aligned} \tag{1.47}$$

For the right-hand side term of (1.43), using the definition of Hessian on a generic manifold (A.49), we can write

$$\begin{aligned}\rho\mathrm{Hess}(\ln\rho) &= \rho\left(\partial_i\partial_j\ln\rho - \Gamma^r{}_{ij}\partial_r\ln\rho\right)\,\mathrm{d}x^i\otimes\mathrm{d}x^j \\ &= \left(-\frac{1}{\rho}\partial_i\rho\,\partial_j\rho + \partial_i\partial_j\rho - \Gamma^r{}_{ij}\partial_r\rho\right)\,\mathrm{d}x^i\otimes\mathrm{d}x^j \\ &= H_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j = \boldsymbol{\mathcal{H}}\,.\end{aligned} \tag{1.48}$$

Remembering the definition of the divergence of a tensor (A.52) we end up with

$$\begin{aligned}\mathrm{div}\boldsymbol{\mathcal{H}} &= g^{ik}\left(\partial_k H_{ij} - H_{lj}\Gamma^l{}_{ki} - H_{il}\Gamma^l{}_{kj}\right)\,\mathrm{d}x^j \\ &= g^{ik}\Bigg[\frac{1}{\rho^2}(\partial_k\rho)(\partial_i\rho)\partial_j\rho - \frac{1}{\rho}(\partial_k\partial_i\rho)\partial_j\rho - \frac{1}{\rho}(\partial_i\rho)\partial_k\partial_j\rho + \partial_k\partial_i\partial_j\rho \\ &\quad - (\partial_k\Gamma^r{}_{ij})\partial_r\rho - \Gamma^r{}_{ij}\partial_k\partial_r\rho + \frac{1}{\rho}\Gamma^l{}_{ki}(\partial_l\rho)\partial_j\rho - \Gamma^l{}_{ki}\partial_l\partial_j\rho \\ &\quad + \Gamma^l{}_{ki}\Gamma^r{}_{lj}\,\partial_r\rho + \frac{1}{\rho}\Gamma^l{}_{kj}(\partial_l\rho)\partial_i\rho - \Gamma^l{}_{kj}\partial_l\partial_i\rho + \Gamma^l{}_{kj}\Gamma^r{}_{li}\,\partial_r\rho\Bigg]\,\mathrm{d}x^j\,.\end{aligned} \tag{1.49}$$

After substituting (1.47) and (1.49) into (1.43) we can check that

$$\begin{aligned}&-g^{ik}(\partial_j\Gamma^r{}_{ik})\partial_r\rho - 2g^{ik}\Gamma^l{}_{jk}\partial_i\partial_l\rho + 2g^{ik}\Gamma^l{}_{jk}\Gamma^r{}_{il}\partial_r\rho \\ &= -g^{ik}(\partial_k\Gamma^r{}_{ij})\partial_r\rho - g^{ik}\Gamma^l{}_{ij}\partial_k\partial_l\rho + g^{ik}\Gamma^l{}_{ki}\Gamma^r{}_{lj}\partial_r\rho - g^{ik}\Gamma^l{}_{kj}\partial_l\partial_i\rho + g^{ik}\Gamma^l{}_{kj}\Gamma^r{}_{li}\partial_r\rho + \mathcal{E}_j\,.\end{aligned} \tag{1.50}$$

By canceling the equal terms on both sides and relabeling some indices, what is left over is $\mathcal{E}^\flat = \mathcal{E}_j \, \mathrm{d}x^j$:

$$\mathcal{E}_j = g^{ik} \left(\partial_k \Gamma^r{}_{ij} - \partial_j \Gamma^r{}_{ik} + \Gamma^l{}_{jk} \Gamma^r{}_{il} - \Gamma^l{}_{ki} \Gamma^r{}_{lj} \right) \partial_r \rho = g^{ik} R^r{}_{ikj} \, \partial_r \rho \,, \tag{1.51}$$

where $R^r{}_{ikj}$ is the Riemann tensor (A.58) of our manifold M and describes its intrinsic geometry. This term can be written equivalently in terms of Ricci curvature tensor R_{ij} (A.60) as

$$\mathcal{E}_j = g^{ik} R^r{}_{ikj} \, \partial_r \rho = g^{ik} g^{lr} R_{likj} \, \partial_r \rho = -g^{lr} R_{lj} \, \partial_r \rho = -g^{ik} R_{ij} \, \partial_k \rho \,. \tag{1.52}$$

Using again the musical isomorphism, we obtain the corresponding vector

$$\mathcal{E} = \mathcal{E}^l \partial_l = -g^{lj} g^{ik} R_{ij} \partial_k \rho \, \partial_l = -R^{lk} \partial_k \rho \, \partial_l \,. \tag{1.53}$$

By simply letting

$$\tilde{\mathcal{E}} = \frac{\hbar^2}{4m^2} \mathcal{E} = -\frac{\hbar^2}{4m^2} \boldsymbol{R}(\nabla \rho, \, \cdot \,) \,, \tag{1.54}$$

we end up with

$$\rho(\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{u}} \boldsymbol{u}) = -\nabla \left(\frac{\mathfrak{g} \rho^2}{2m} \right) + \operatorname{div} \left(\frac{\hbar^2}{4m^2} \rho \operatorname{Hess}(\ln \rho) \right)^\sharp - \frac{\hbar^2}{4m^2} \boldsymbol{R}(\nabla \rho, \, \cdot \,) + \frac{\hbar^2 \kappa}{2m^2} \rho \nabla R \,, \tag{1.55}$$

where R is Ricci scalar and $\boldsymbol{R}$ is Ricci tensor, getting the Navier-Stokes equation (1.32). □

The first two terms on the right-hand side were already present in flat space, while the new density curvature vector $\mathbf{f}$ is present only in curved space. We now give a possible physical interpretation of $\mathbf{f}$ [RR21]. Notice that it enters in the Navier-Stokes equation as a sort of external force, depending both on the geometry of the initial configuration, through the Ricci curvature tensor and scalar, and on the gradient of the density profile, hence on the shape of the BEC wave function. This term becomes particularly relevant when we approach a defect region, because density must go rapidly to zero towards the singularity. It is well-known that in the tubular region around the nodal line the density profile gradient must go through an inflection point [CZ17], with outwardly pointing gradient everywhere positive. In this point, which is located at approximately a healing length from the line defect, the density first derivative takes on its maximum value, so that we can say that this term becomes significant at distances of the order of ξ from the vortex core. This observation is strictly related to the study of vortex defects as analogue black holes. Since the speed of sound plays an important role in the analogue cosmological models of black hole theory, it is useful to relate this quantity to the fundamental physical properties of the boson gas through the following

Corollary 1. *Having defined the pressure* p*, we can define the speed of sound using the standard definition, obtaining*

$$c^2 = \frac{\mathrm{dp}}{\mathrm{d}\rho} = \frac{\mathfrak{g}\rho}{m} \quad \Longrightarrow \quad c = \sqrt{\frac{\mathfrak{g}\rho}{m}}\,. \tag{1.56}$$

1.5 Euler equation for momentum

By combining the continuity equation (1.22) and the Navier-Stokes form of GPE (1.32) we also obtain an Euler equation for the momentum $\rho\boldsymbol{u}$.

Theorem 3 (Momentum Conservation Law)**.** *The momentum* $\rho\boldsymbol{u}$ *associated with the hydrodynamic form of GPE satisfies the following conservation law*

$$\partial_t(\rho\boldsymbol{u}^\flat) = -\mathrm{div}\,\boldsymbol{\Pi} + \chi \tag{1.57}$$

where χ *is a geometrical correction term related to the geometry of the manifold* M *through the Ricci scalar curvature, given by*

$$\chi = -\frac{\hbar^2}{8m^2} R\,\mathrm{d}\rho + \frac{\hbar^2\kappa}{2m^2}\rho\,\mathrm{d}R\,, \tag{1.58}$$

while $\boldsymbol{\Pi} = \Pi_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ *is the momentum flux tensor, and can be decomposed into*

$$\Pi_{ij} = \mathcal{D}_{ij} + \mathrm{p}g_{ij} - \tau_{ij} - \mathcal{G}_{ij}\,, \tag{1.59}$$

with $\mathcal{D}_{ij} = \rho u_i u_j$*,* $\boldsymbol{u}^\flat = u_j\,\mathrm{d}x^j$*,* τ_{ij} *being the stress tensor and* $\mathcal{G}_{ij}$ *being an element of tensor*

$$\boldsymbol{\mathcal{G}} = \frac{\hbar^2\rho}{4m^2}\boldsymbol{G}\,, \tag{1.60}$$

proportional to Einstein's tensor $\boldsymbol{G}$*.*

Proof. Starting from the left-hand side and using the continuity equation (1.22) and Euler equation (1.28) we get

$$\partial_t(\rho \boldsymbol{u}) = \partial_t \rho\, \boldsymbol{u} + \rho\, \partial_t \boldsymbol{u} = -\mathrm{div}(\rho \boldsymbol{u})\, \boldsymbol{u} - \rho\, \nabla_{\boldsymbol{u}} \boldsymbol{u} - \nabla \mathrm{p} + (\mathrm{div}\, \tau)^\sharp + \mathbf{f}\,. \tag{1.61}$$

If we consider the corresponding equation for the 1-forms in coordinates, we get

$$\partial_t(\rho u_j)\, \mathrm{d}x^j = \mathcal{A} - \partial_j \mathrm{p}\, \mathrm{d}x^j + g^{ik} \nabla_k \tau_{ij}\, \mathrm{d}x^j - \frac{\hbar^2}{4m^2} g^{ik} R_{ij} \partial_k \rho\, \mathrm{d}x^j - \frac{\kappa}{m} \rho\, \partial_j R\, \mathrm{d}x^j\,, \tag{1.62}$$

where

$$\mathcal{A} = -\rho\, (\partial_k u^k + u^k \Gamma^l{}_{lk}) u_j\, \mathrm{d}x^j - (\partial_k \rho) u^k u_j\, \mathrm{d}x^j - \rho (u^k \partial_k u_j - u^k u_l \Gamma^l{}_{kj})\, \mathrm{d}x^j\,. \tag{1.63}$$

By writing $u^k = g^{ik} u_i$ we can express $\mathcal{A}$ in the form

$$\begin{aligned} \mathcal{A} &= -\left[\rho \partial_k (u^k) u_j + \rho u^k (\partial_k u_j) + (\partial_k \rho) u^k u_j + \rho u^k \Gamma^l{}_{lk} u_j - \rho u^k u_l \Gamma^l{}_{kj}\right] \mathrm{d}x^j \\ &= -\left[\rho (\partial_k g^{ik}) u_i u_j + g^{ik} \left(\partial_k (\rho u_i u_j) + \rho u_i u_j \Gamma^l{}_{lk} - \rho u_i u_l \Gamma^l{}_{kj}\right)\right] \mathrm{d}x^j\,. \end{aligned} \tag{1.64}$$

Let us rewrite the derivative of the metric as

$$\partial_k g^{ik} = -g^{ir} \Gamma^k{}_{kr} - g^{kr} \Gamma^i{}_{rk} = -g^{ik} \Gamma^l{}_{lk} - g^{kr} \Gamma^i{}_{rk}\,, \tag{1.65}$$

hence, by substituting the latter into (1.64) and relabeling some repeated indices, we get

$$\mathcal{A} = -g^{ik} \left[\partial_k (\rho u_i u_j) - \rho u_i u_l \Gamma^l{}_{kj} - \rho u_l u_j \Gamma^l{}_{ik}\right] \mathrm{d}x^j\,. \tag{1.66}$$

Defining the $(0,2)-$tensor $\boldsymbol{\mathcal{D}} = \rho \boldsymbol{u}^\flat \otimes \boldsymbol{u}^\flat$ (namely $\boldsymbol{\mathcal{D}} = \mathcal{D}_{ij}\ \mathrm{d}x^i \otimes \mathrm{d}x^j$ with $\mathcal{D}_{ij} = \rho u_i u_j$), which we see as the standard stress tensor for dust of density ρ and velocity $\boldsymbol{u}$, we find that

$$\mathcal{A} = -g^{ik} \left(\partial_k \mathcal{D}_{ij} - \mathcal{D}_{il} \Gamma^l{}_{kj} - \mathcal{D}_{lj} \Gamma^l{}_{ki}\right) \mathrm{d}x^j = -g^{ik} \nabla_k \mathcal{D}_{ij}\, \mathrm{d}x^j = -\mathrm{div} \boldsymbol{\mathcal{D}}\,. \tag{1.67}$$

As regards the second term on the right-hand side of (1.62), we define the $(0,2)-$tensor $\mathrm{p} g = \mathrm{p} g_{ij}\ \mathrm{d}x^i \otimes \mathrm{d}x^j$ so to have

$$\mathrm{div}(\mathrm{p} g) = g^{ik} \nabla_k (\mathrm{p} g_{ij})\, \mathrm{d}x^j = g^{ik} (\partial_k \mathrm{p}) g_{ij}\, \mathrm{d}x^j = \partial_j \mathrm{p}\, \mathrm{d}x^j\,, \tag{1.68}$$

where we used the metric compatibility property $\nabla_k g_{ij} = 0$.

Finally, as regards the last term on the right-hand side of (1.62), we want to express the Ricci tensor part of the density curvature vector term $\mathcal{E}_j = -g^{ik}R_{ij}\partial_k\rho$ as divergence of a $(0,2)-$tensor; hence, let us consider $\boldsymbol{G}\rho = G_{ij}\rho\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ where $\boldsymbol{G}$ denotes Einstein's tensor given by

$$G_{ij} = R_{ij} - \frac{1}{2}R\,g_{ij}\,, \tag{1.69}$$

where R_{ij} is Ricci's tensor and R is Ricci scalar curvature; then the divergence is given by

$$\begin{aligned} \operatorname{div}(\boldsymbol{G}\rho) &= g^{ik}\nabla_k(G_{ij}\rho)\,\mathrm{d}x^j = \left(g^{ik}(\partial_k\rho)G_{ij} + \rho g^{ik}\nabla_k G_{ij}\right)\mathrm{d}x^j \\ &= g^{ik}(\partial_k\rho)R_{ij}\,\mathrm{d}x^j - \frac{1}{2}R(\partial_j\rho)\,\mathrm{d}x^j\,, \end{aligned} \tag{1.70}$$

where we have used the fact that Einstein's tensor satisfies $\nabla_k G^k_j = 0$. Hence we end up with

$$\mathcal{E}_j\,\mathrm{d}x^j = -\operatorname{div}(G\rho) - \frac{1}{2}R(\partial_j\rho)\,\mathrm{d}x^j\,. \tag{1.71}$$

Thus, we can finally rewrite (1.62) as

$$\begin{aligned} \partial_t(\rho\boldsymbol{u}^\flat) &= -g^{ik}\nabla_k\!\left(\rho u_i u_j + \mathrm{p}g_{ij} - \tau_{ij} + \frac{\hbar^2}{4m^2}G_{ij}\rho\right)\mathrm{d}x^j - \frac{\hbar^2}{8m^2}R(\partial_j\rho)\,\mathrm{d}x^j + \frac{\hbar^2\kappa}{2m^2}\rho\,(\partial_j R)\,\mathrm{d}x^j \\ &= -\operatorname{div}\left(\boldsymbol{\mathcal{D}} + \mathrm{p}g - \tau + \frac{\hbar^2}{4m^2}\boldsymbol{G}\rho\right) - \frac{\hbar^2}{8m^2}R\,\mathrm{d}\rho + \frac{\hbar^2\kappa}{2m^2}\rho\,\mathrm{d}R \\ &= -\operatorname{div}\boldsymbol{\Pi} + \chi\,, \end{aligned} \tag{1.72}$$

proving the theorem. □

Notice that new term χ breaks the standard form of a conservation law. As a point of digression it is perhaps worth noticing that there is a formal analogy with the right-hand side term that appears in the energy–momentum conservation law in general relativity ([Mo55], see section 126, eq.143). As pointed out by Penrose [Pe82], though, since gravitational energy contributes non-locally to the total energy of the system and does not play an integral part in the energy and momentum conservation law associated with the Einstein field equation, a strict parallel with a standard conservation law does not hold in general relativity. In any case, the possibility of identifying a more stringent similarity between χ and the term mentioned above by writing χ in terms of divergence of a pseudo-tensor is certainly intriguing. This situation can arise in the case of constant Ricci curvature, where χ can be absorbed into a divergence operator, as we will show in the next subsection.

1.5.1 Constant curvature case

Notice that, if the scalar curvature is constant on the manifold, the correction term χ loses the curvature coupling part and can be written as the divergence of a tensor, analogously to the pressure term, as

$$\chi = -\frac{\hbar^2}{8m^2} R\, \mathrm{d}\rho = -\mathrm{div}\boldsymbol{\mathcal{X}}\,, \tag{1.73}$$

where $\boldsymbol{\mathcal{X}} = \mathcal{X}_{ij}\ \mathrm{d}x^i \otimes \mathrm{d}x^j$ with

$$\mathcal{X}_{ij} = \frac{\hbar^2}{8m^2} R\rho\, g_{ij}\,. \tag{1.74}$$

Hence we can write equation (1.57) in the form of a standard conservation law for the momentum, as in flat space

$$\begin{aligned}\partial_t(\rho\boldsymbol{u}) = -\mathrm{div}\,(\boldsymbol{\Pi} + \boldsymbol{\mathcal{X}}) &= -\mathrm{div}\left(\boldsymbol{\mathcal{D}} + \mathrm{p}g - \tau + \frac{\hbar^2}{4m^2}\boldsymbol{G}\rho + \frac{\hbar^2}{8m^2}R\rho\, g\right)\\ &= -\mathrm{div}\left(\boldsymbol{\mathcal{D}} + \mathrm{p}g - \tau + \frac{\hbar^2}{4m^2}\boldsymbol{R}\rho\right)\,,\end{aligned} \tag{1.75}$$

where we used the fact that the Ricci tensor can be written as $\boldsymbol{R} = \boldsymbol{G} + \frac{1}{2}Rg$. In this particular case we can impose a steady state by asking $\boldsymbol{\Pi} + \boldsymbol{\mathcal{X}} = 0$, which is a sufficient but not necessary condition, and we obtain a constraint on the Ricci curvature tensor

$$R_{ij} = -\frac{4m^2}{\hbar^2}u_i u_j - \frac{2m\mathfrak{g}\rho}{\hbar^2}g_{ij} + \mathrm{Hess}_{ij}(\ln\rho)\,. \tag{1.76}$$

1.5.2 Special case: $\kappa = -1/4$

Using the fact that

$$\mathrm{d}(R\rho) = \rho\,\mathrm{d}R + R\,\mathrm{d}\rho\,, \tag{1.77}$$

we can rewrite

$$\chi = -\frac{\hbar^2}{8m^2}\,\mathrm{d}(R\rho) + \frac{\hbar^2}{8m^2}\rho\,\mathrm{d}R + \frac{\hbar^2\kappa}{2m^2}\rho\,\mathrm{d}R\,, \tag{1.78}$$

hence, when $\kappa = -1/4$, the χ term can be written as the divergence of a tensor, obtaining again the continuity equation. We can write

$$\chi = -\mathrm{div}\boldsymbol{\mathcal{X}}\,, \tag{1.79}$$

with same conclusions as in (1.76).

1.5.3 Stationary case: Einstein field equations – negative curvature

Going back to the general case, re-writing expression (1.72) using (1.77) and taking the pressure term out of the divergence, we obtain

$$\begin{aligned}\partial_t(\rho \boldsymbol{u}^\flat) = &- g^{ik}\nabla_k \left(\rho u_i u_j - \frac{\hbar^2}{4m^2}\rho\,(\mathrm{Hess}_{ij}(\ln\rho) - G_{ij}) - \frac{\hbar^2\kappa}{2m^2}R\rho\, g_{ij}\right)\mathrm{d}x^j \\ &- \left(\frac{\mathfrak{g}}{m}\rho + \frac{\hbar^2}{8m^2}R + \frac{\hbar^2\kappa}{2m^2}R\right)\partial_j\rho\,\mathrm{d}x^j\,.\end{aligned} \tag{1.80}$$

We see that a stationary state can be achieved in the very special situation in which the two terms in brackets are simultaneously zero (sufficient but again not necessary condition). By imposing the first bracket to be zero we derive a form of Einstein's field equation [Ro21], given by

$$G_{ij} = -\frac{4m^2}{\hbar^2}u_i u_j + \mathrm{Hess}_{ij}(\ln\rho) + 2\kappa R\, g_{ij}\,, \tag{1.81}$$

and, by imposing the second bracket to be zero, we determine the Ricci scalar curvature

$$R = -\frac{8m\mathfrak{g}}{\hbar^2(1+4\kappa)}\rho = -\frac{32\pi a_s\rho}{(1+4\kappa)}\,. \tag{1.82}$$

Under such conditions we obtain a negative Gaussian curvature, given by

$$K = \frac{R}{2} = -\frac{4m\mathfrak{g}}{\hbar^2(1+4\kappa)}\rho = -\frac{16\pi a_s\rho}{(1+4\kappa)}\,. \tag{1.83}$$

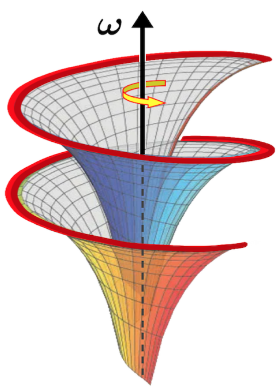

Figure 1.1: An example of possible geometry given by a twisted elliptic pseudosphere (Dini's surface) associated with the presence of a central twisted vortex defect $\boldsymbol{\omega}$.

Using the definition of healing length (1.13), we can write Ricci scalar curvature (1.82) and

Gaussian curvature (1.83) as

$$R = -\frac{4}{\xi^2} \quad \Longrightarrow \quad K = -\frac{2}{\xi^2}\,, \tag{1.84}$$

and hence we can rewrite equation (1.81) as

$$G_{ij} = -\frac{4m^2}{\hbar^2} u_i u_j + \mathrm{Hess}_{ij}(\ln\rho) - \frac{8\kappa}{\xi^2}\, g_{ij}\,. \tag{1.85}$$

We notice that condition (1.84) is automatically satisfied when considering negative curvature manifolds. An example of surface satisfying such possibility is represented by the pseudosphere and the twisted pseudosphere, also known as Dini surface (see the example in Figure 1.1).

Chapter 2

Negative curvature surfaces and sine-Gordon

Suppose $\sigma(w,\theta)$ is a generic surface defined by the complete revolution of a parametric curve $\gamma(w) = (f(w), 0, g(w))$ around the $z-$axis. Its Gaussian curvature is given by

$$K = \frac{g'\left(f'g'' - f''g'\right)}{f\left(f'^{\,2} + g'^{\,2}\right)}\,, \tag{2.1}$$

hence, if the curve is parameterized by arc length ($f'^{\,2} + g'^{\,2} = 1$), then we have $K = -\dfrac{f''}{f}$. Let us consider a generic constant negative Gaussian curvature surface of revolution

$$K = -\frac{1}{a^2} \iff f'' = \frac{f}{a^2}\,, \tag{2.2}$$

with a solution given by $f(w) = ae^{-w/a}$. Hence, by solving the differential equation coming from the unitary speed condition $g'(w) = \pm\sqrt{1 - e^{-2w/a}}$, we find

$$g(w) = \pm a\left(\cosh^{-1}(e^{w/a}) - \sqrt{1 - e^{-2w/a}}\right)\,. \tag{2.3}$$

According to this we obtain the unit speed tractrix curve given by

$$\gamma(w) = \left(ae^{-w/a}, 0, \pm a\left(\cosh^{-1}(e^{w/a}) - \sqrt{1 - e^{-2w/a}}\right)\right)\,. \tag{2.4}$$

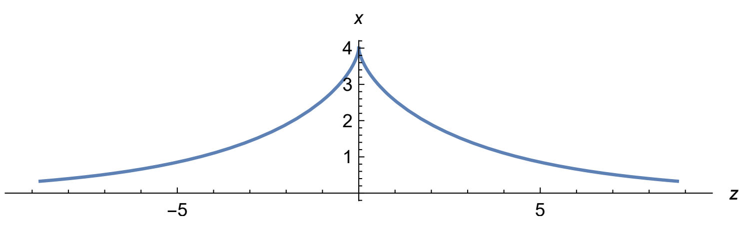

Figure 2.1: Tractrix defined by (2.4) with $a = 4$.

2.1 Pseudosphere

Taking a full rotation of the tractrix about its asymptote we obtain a pseudosphere. This surface was first proposed by E. Beltrami [Be68] in 1868 as a model to develop hyperbolic geometry. Writing the surface generated by the complete rotation of curve (2.4) about the $z-$axis, we obtain

$$\sigma_1(w,\theta) = \begin{pmatrix} ae^{-w/a}\cos\theta \\ ae^{-w/a}\sin\theta \\ \pm\left(\cosh^{-1}(e^{w/a}) - \sqrt{1-e^{-2w/a}}\right) \end{pmatrix}, \tag{2.5}$$

with induced 2-metric

$$\mathrm{d}s^2 = \mathrm{d}w^2 + f(w)^2\,\mathrm{d}\theta^2 = \mathrm{d}w^2 + a^2 e^{-2w/a}\,\mathrm{d}\theta^2\,. \tag{2.6}$$

We can compute the mean and Gaussian curvatures by applying the formulas

$$H = \frac{1}{2}\left(\frac{g'}{f} - \frac{f''}{g'}\right) = \frac{1}{2a}\left(\sqrt{e^{2w/a}-1} - \frac{1}{\sqrt{e^{2w/a}-1}}\right), \tag{2.7}$$

$$K = -\frac{f''}{f} = \frac{-\frac{1}{a}e^{-w/a}}{ae^{-w/a}} = -\frac{1}{a^2}\,. \tag{2.8}$$

The Gaussian curvature is given by the product of the principal curvatures; in particular, we see in Figure 2.2 that $a^2 = r_1 r_2$, where r_1 is the radius of the exterior tangent circle and r_2 is the radius of the interior tangent circle.

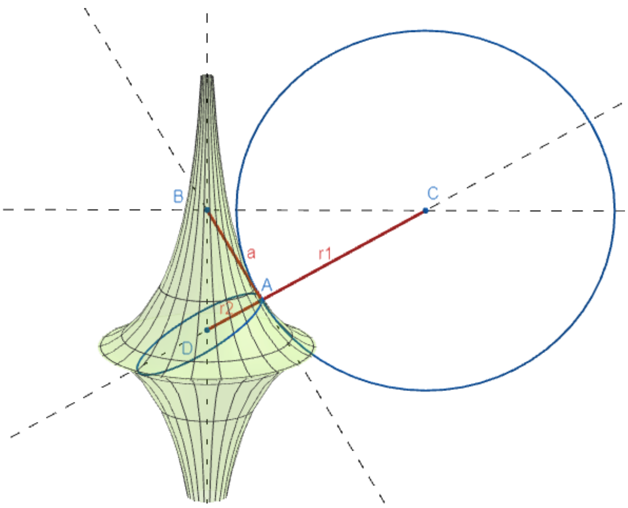

Figure 2.2: Principal curvatures of the pseudosphere for the calculation of the Gaussian curvature.

If now we set

$$\cosh t = e^{w/a} \implies w = a\ln(\cosh t)\,, \quad t = \cosh^{-1}\left(e^{w/a}\right)\,, \quad \sinh t = \sqrt{e^{2w/a}-1}\,, \tag{2.9}$$

we can rewrite (2.4) as

$$\gamma(w(t)) = (a\,\mathrm{sech}\,t, 0, a(t-\tanh t)) \tag{2.10}$$

and we have that the pseudosphere can be also written in parametric form as

$$\sigma_2(t,\theta) = \begin{pmatrix} a\,\mathrm{sech}\,t\cos\theta \\ a\,\mathrm{sech}\,t\sin\theta \\ a(t-\tanh t) \end{pmatrix}\,, \quad t\in\mathbb{R},\ \theta\in[0,2\pi] \tag{2.11}$$

with induced 2-metric

$$\mathbf{g} = \mathrm{d}s^2 = E\,\mathrm{d}t^2 + 2F\,\mathrm{d}t\,\mathrm{d}\theta + G\,\mathrm{d}\theta^2 = a^2\left(\tanh^2 t\,\mathrm{d}t^2 + \frac{1}{\cosh^2 t}\,\mathrm{d}\theta^2\right)\,, \tag{2.12}$$

and second fundamental form

$$\mathbf{h} = e\,\mathrm{d}t^2 + 2f\,\mathrm{d}t\,\mathrm{d}\theta + g\,\mathrm{d}\theta^2 = a\,\mathrm{sech}\,t\tanh t\,(\,\mathrm{d}t^2 - \,\mathrm{d}\theta^2)\,. \tag{2.13}$$

Using the first and second fundamental forms we can again calculate the mean curvature

$$H = \frac{eG-2fF+gE}{2(EG-F^2)} = \frac{1}{2}\left(\frac{e}{E}+\frac{g}{G}\right) = \frac{1}{2a}\left(\sinh t - \mathrm{csch}\,t\right)\,, \tag{2.14}$$

and the Gaussian curvature

$$K = \frac{\det\mathbf{h}}{\det\mathbf{g}} = \frac{eg-f^2}{EG-F^2} = -\frac{1}{a^2}\,. \tag{2.15}$$

Christoffel symbols

Let us compute the Christoffel symbols for the pseudosphere σ_2 (2.11) using the definition

$$\Gamma^m{}_{jk} = \frac{1}{2}g^{im}\left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}\right)\,.$$

Because of (2.12) we can write the metric matrix

$$g_{ij} = a^2 \begin{bmatrix} \tanh^2 t & 0 \\ 0 & \dfrac{1}{\cosh^2 t} \end{bmatrix} , \tag{2.16}$$

and its inverse

$$g^{ij} = \frac{1}{a^2} \begin{bmatrix} \coth^2 t & 0 \\ 0 & \cosh^2 t \end{bmatrix} . \tag{2.17}$$

Computing the Christoffel symbols we obtain

$$\Gamma^t = \operatorname{csch} t \operatorname{sech} t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \qquad \Gamma^\theta = -\tanh t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{2.18}$$

Ricci tensor, Ricci scalar and Einstein tensor

Now we calculate the Ricci tensor components using the definition $R_{kj} = R^i{}_{kij}$ where

$$R^i{}_{kij} = \partial_i \Gamma^i{}_{jk} - \partial_j \Gamma^i{}_{ik} + \Gamma^a{}_{jk} \Gamma^i{}_{ia} - \Gamma^b{}_{ik} \Gamma^i{}_{jb} ,$$

the Ricci scalar $R = g^{ij} R_{ij}$ and the Einstein tensor $G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$.

For the pseudosphere the only non-vanishing terms of the Riemann tensor are

$$R^\theta{}_{t\theta t} = -\tanh^2 t \quad \text{and} \quad R^t{}_{\theta t \theta} = -\operatorname{sech}^2 t , \tag{2.19}$$

from which we deduce

$$R_{tt} = R^\theta{}_{t\theta t} = -\tanh^2 t \quad \text{and} \quad R_{\theta\theta} = R^t{}_{\theta t\theta} = -\operatorname{sech}^2 t \tag{2.20}$$

hence we write the Ricci scalar as

$$R = g^{tt} R_{tt} + g^{\theta\theta} R_{\theta\theta} = -\frac{1}{a^2} - \frac{1}{a^2} = -\frac{2}{a^2} , \tag{2.21}$$

obtaining twice the Gaussian curvature. We finally observe that all components of the Einstein tensor are zero, as expected for $2-$dimensional manifolds.

2.1.1 Gauss-Weingarten equations

Let the pseudosphere be described in parametric form by $\mathbf{r}(t,\theta)$ as in (2.11), and let $\mathbf{n} = \dfrac{\mathbf{r}_t \times \mathbf{r}_\theta}{||\mathbf{r}_t \times \mathbf{r}_\theta||}$ be the unit normal. The Gauss-Weingarten equations can be written as

$$\begin{pmatrix} \mathbf{r}_t \\ \mathbf{r}_\theta \\ \mathbf{n} \end{pmatrix}_k = \begin{bmatrix} \Gamma^t{}_{kt} & \Gamma^\theta{}_{kt} & h_{kt} \\ \Gamma^t{}_{k\theta} & \Gamma^\theta{}_{k\theta} & h_{k\theta} \\ -s^t{}_k & -s^\theta{}_k & 0 \end{bmatrix} \begin{pmatrix} \mathbf{r}_t \\ \mathbf{r}_\theta \\ \mathbf{n} \end{pmatrix} = \Omega_k \begin{pmatrix} \mathbf{r}_t \\ \mathbf{r}_\theta \\ \mathbf{n} \end{pmatrix} \tag{2.22}$$

where h_{ij} are the components of the second fundamental form (2.13)

$$h_{ij} = a \operatorname{sech} t \tanh t \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.23}$$

$\Gamma^m{}_{ij}$ are the Christoffel symbols and $s^i{}_k = g^{ip} h_{pk}$, hence we obtain

$$-s^i{}_k = \frac{1}{a} \begin{bmatrix} \operatorname{csch} t & 0 \\ 0 & -\sinh t \end{bmatrix}. \tag{2.24}$$

With this in mind the Gauss-Weingarten equations can be written through the matrices

$$\Omega_t = \begin{bmatrix} \operatorname{csch} t \operatorname{sech} t & 0 & -a \operatorname{sech} t \tanh t \\ 0 & -\tanh t & 0 \\ \dfrac{\operatorname{csch} t}{a} & 0 & 0 \end{bmatrix}, \quad \Omega_\theta = \begin{bmatrix} 0 & -\tanh t & 0 \\ \operatorname{csch} t \operatorname{sech} t & 0 & a \operatorname{sech} t \tanh t \\ 0 & -\dfrac{\sinh t}{a} & 0 \end{bmatrix}, \tag{2.25}$$

and the Mainardi-Codazzi consistency condition

$$\begin{pmatrix} \mathbf{r}_t \\ \mathbf{r}_\theta \\ \mathbf{n} \end{pmatrix}_{t\theta} = \begin{pmatrix} \mathbf{r}_t \\ \mathbf{r}_\theta \\ \mathbf{n} \end{pmatrix}_{\theta t}, \tag{2.26}$$

can be written in the form

$$W = \partial_\theta \Omega_t - \partial_t \Omega_\theta + \Omega_t \Omega_\theta - \Omega_\theta \Omega_t = \mathbb{0}\,. \tag{2.27}$$

In the next section we shall see that the above equation is strictly related to the sine-Gordon equation.

2.2 From pseudosphere to sine-Gordon equation

The study of geometric properties of surfaces of constant negative Gaussian curvature led to the discovery (1862) of a type of non-linear hyperbolic, partial differential equation in $(1+1)-$dimension known as the sine-Gordon equation (sGE). This equation results from the Codazzi–Mainardi compatibility condition between the first and the second fundamental form, and it involves the d'Alembert operator and the sine of the unknown function. It was re-discovered much later in many other fields, for example in the study of crystal dislocations [FK39], and as non-linear integrable field theory, due to the presence of soliton solutions [LR76, NSW92]. In recent years it was proved that the sine-Gordon model on an analogue curved spacetime arises as the effective quantum field theory for phase fluctuations of a weakly imperfect Bose gas on an incompressible background superfluid flow [VFU16]. The equation can be written in two equivalent forms. In physics we usually find it in spacetime coordinates, denoted by (x,t), in the form

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0\,, \tag{2.28}$$

which can be easily reconducted to the well-known Klein–Gordon equation in the low-amplitude limit ($\sin\varphi \approx \varphi$). This expression for the sGE can be derived as the Euler-Lagrange equation with Lagrangian density given by

$$\mathcal{L}_{\text{sG}}(\varphi) = \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + \cos\varphi\,. \tag{2.29}$$

The second form of the sGE is obtained by passing to light-cone (asymptotic) coordinates (u,v): setting advanced and retarded time

$$u = \frac{x+t}{2}, \quad v = \frac{x-t}{2} \quad \Longrightarrow \quad x = u+v, \quad t = u-v\,. \tag{2.30}$$

The equation takes the form

$$\varphi_{uv} = \sin\varphi\,, \tag{2.31}$$

which was indeed the original form of the sGE, obtained while investigating pseudospherical surfaces of constant negative Gaussian curvature $K=-1$.

2.2.1 Derivation of the sine-Gordon equation

Let us consider the asymptotically parametrized pseudosphere with $a = 1$, namely let $t = u + v$ and $\theta = u - v$. Substituting into (2.11) we obtain

$$\mathbf{r}(u,v) = \begin{pmatrix} \operatorname{sech}(u+v)\cos(u-v) \\ \operatorname{sech}(u+v)\sin(u-v) \\ (u+v) - \tanh(u+v) \end{pmatrix} . \tag{2.32}$$

The metric matrix can be written as

$$g_{ij}(u,v) = \begin{bmatrix} 1 & 1 - 2\operatorname{sech}^2(u+v) \\ 1 - 2\operatorname{sech}^2(u+v) & 1 \end{bmatrix} , \tag{2.33}$$

from which we see that the asymptotic tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are normalized and the off-diagonal term gives the cosine of the angle between them

$$\omega(u,v) = \arccos(1 - 2\operatorname{sech}^2(u+v)) . \tag{2.34}$$

Hence, we can compactly write

$$g_{ij}(u,v) = \begin{bmatrix} 1 & \cos\omega(u,v) \\ \cos\omega(u,v) & 1 \end{bmatrix} , \tag{2.35}$$

from which we see that $\det g = \sin^2\omega(u,v)$. We calculate the inverse metric

$$g^{ij} = \frac{1}{4}\coth^2(u+v) \begin{bmatrix} \cosh^2(u+v) & 2 - \cosh^2(u+v) \\ 2 - \cosh^2(u+v) & \cosh^2(u+v) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sin^2\omega} & -\frac{\cos\omega}{\sin^2\omega} \\ -\frac{\cos\omega}{\sin^2\omega} & \frac{1}{\sin^2\omega} \end{bmatrix} , \tag{2.36}$$

and the second fundamental form

$$h_{ij}(u,v) = \begin{bmatrix} 0 & \sin\omega(u,v) \\ \sin\omega(u,v) & 0 \end{bmatrix} , \tag{2.37}$$

where we used the fact that $\sin\omega(u,v) = 2\operatorname{sech}(u+v)\tanh(u+v)$.

Now, in order to derive the sine-Gordon equation, we want to write the Gauss-Weingarten equations for the pseudosphere, hence we need the Christoffel symbols

$$\Gamma^u = \begin{bmatrix} \dfrac{2-\cosh^2(u+v)}{\sinh(u+v)\cosh(u+v)} & 0 \\ 0 & \coth(u+v) \end{bmatrix} = \begin{bmatrix} \cot\omega\cdot\omega_u & 0 \\ 0 & -\csc\omega\cdot\omega_v \end{bmatrix},$$

$$\Gamma^v = \begin{bmatrix} \coth(u+v) & 0 \\ 0 & \dfrac{2-\cosh^2(u+v)}{\sinh(u+v)\cosh(u+v)} \end{bmatrix} = \begin{bmatrix} -\csc\omega\cdot\omega_u & 0 \\ 0 & \cot\omega\cdot\omega_v \end{bmatrix}, \tag{2.38}$$

and the matrix

$$s^i{}_k = g^{ip}h_{pk} = \begin{bmatrix} -\cot\omega & \csc\omega \\ \csc\omega & -\cot\omega \end{bmatrix}. \tag{2.39}$$

We find that

$$\Omega_u = \begin{bmatrix} \omega_u\cot\omega & -\omega_u\csc\omega & 0 \\ 0 & 0 & \sin\omega \\ \cot\omega & -\csc\omega & 0 \end{bmatrix}, \quad \Omega_v = \begin{bmatrix} 0 & 0 & \sin\omega \\ -\omega_v\csc\omega & \omega_v\cot\omega & 0 \\ -\csc\omega & \cot\omega & 0 \end{bmatrix}. \tag{2.40}$$

Imposing the consistency condition

$$W = \partial_v\Omega_u - \partial_u\Omega_v + \Omega_u\Omega_v - \Omega_v\Omega_u = \mathbb{0}\,, \tag{2.41}$$

we obtain

$$W = \begin{bmatrix} \cos\omega\left(\dfrac{\omega_{uv}}{\sin\omega}-1\right) & \left(1-\dfrac{\omega_{uv}}{\sin\omega}\right) & 0 \\ \left(\dfrac{\omega_{uv}}{\sin\omega}-1\right) & \cos\omega\left(1-\dfrac{\omega_{uv}}{\sin\omega}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.42}$$

from which we get that the angle ω satisfies the sine-Gordon equation

$$\omega_{uv} = \sin\omega\,. \tag{2.43}$$

In particular, we can write it explicitly as

$$\omega_{uv} = 2|\operatorname{sech}(u+v)\tanh(u+v)|\,. \tag{2.44}$$

Hence we have proved that by choosing a coordinate system for which the asymptotic lines are parameterized with respect to the arc length, the first and second fundamental forms of the surface take the specific form

$$\mathbf{g} = \mathrm{d}s^2 = \mathrm{d}u^2 + 2\cos\omega \;\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2 \qquad \text{and} \qquad \mathbf{h} = 2\sin\omega \;\mathrm{d}u\,\mathrm{d}v\,, \tag{2.45}$$

where ω expresses the angle between the asymptotic lines, and the Mainardi-Codazzi equation results in the sine-Gordon equation. The study of this equation and of the associated transformations of pseudospherical surfaces in the 19th century by Bianchi and Bäcklund led to the discovery of Bäcklund transformations, that we will briefly recall in what follows.

2.2.2 The solution through Bäcklund transform

Bäcklund transform is an important tool relating partial differential equations to their solutions. The procedure is well known [Ba80]: (i) find two solutions for the same PDE $P(f) = 0$ and $P(g) = 0$; (ii) find two relations on the solutions

$$\begin{aligned} R_1(f, g, f_x, g_x, f_y, g_y, \ldots; x, y) = 0\,, \\ R_2(f, g, f_x, g_x, f_y, g_y, \ldots; x, y) = 0\,, \end{aligned} \tag{2.46}$$

such that they are integrable for g when $P(f) = 0$; the resulting g is a solution for $P(g) = 0$ and vice versa. This technique is widely used to find new solutions starting from trivial ones; in particular we will show that pseudospherical surfaces can be described as solutions of the sine-Gordon equation.

Let us start with the sGE as a PDE, taking solutions f and g

$$f_{xy} = \sin f \quad \text{and} \quad g_{xy} = \sin g\,. \tag{2.47}$$

Two appropriate relations for the sGE are

$$R_x: \;(f+g)_x = 2a\sin\left(\frac{f-g}{2}\right) \quad \text{and} \quad R_y: \;(f-g)_y = \frac{2}{a}\sin\left(\frac{f+g}{2}\right)\,. \tag{2.48}$$

This gives a valid Bäcklund transform, since

$$\partial_y R_x = (f+g)_{xy} = 2\cos\left(\frac{f-g}{2}\right)\sin\left(\frac{f+g}{2}\right)\,, \tag{2.49}$$

and

$$\partial_x R_y = (f-g)_{yx} = 2\sin\left(\frac{f-g}{2}\right)\cos\left(\frac{f+g}{2}\right)\,, \tag{2.50}$$

and by adding and subtracting the two conditions, we get that f and g satisfy the sine-Gordon equation.

Now we set $g = 0$ as a trivial solution, hence the Bäcklund transform takes the form

$$f_x = 2a\sin\left(\frac{f}{2}\right) \quad \text{and} \quad f_y = \frac{2}{a}\sin\left(\frac{f}{2}\right)\,. \tag{2.51}$$

Integrating the two equations we find

$$\tan\left(\frac{f}{4}\right) = C\,e^{(ax+\frac{y}{a})}\,, \tag{2.52}$$

which leads to the $a-$parameterized class of solutions

$$f(x,y;a) = 4\arctan\left(Ce^{(ax+\frac{y}{a})}\right)\,. \tag{2.53}$$

2.2.3 Going back to the pseudosphere

Let us take the simplest solutions of the sGE in asymptotic coordinates $\omega_{uv} = \sin\omega$, namely consider the solution (2.53) with $C = a = 1$

$$\omega(u,v) = 4\arctan\left(e^{u+v}\right)\,; \tag{2.54}$$

then $\omega(u,v) \in (0,\pi)$ if (u,v) are in the simply connected domain $D = \{(u,v)\,|\,u+v<0\}$.
Now going back to coordinates $t = u+v$ and $\theta = u-v$, we have

$$\cos\omega = \cos\left(4\arctan e^t\right) = 2\tanh^2 t - 1\,, \tag{2.55}$$

where we used the fact that, if $\tan x = e^t$, then $\cos^2 x = (e^{2t}+1)^{-1}$, and the fact that

$$\tanh^2 t = \left(\frac{2}{e^{2t}+1} - 1\right)^2 .$$

With this in mind we can now rewrite the first and second fundamental forms (2.45) as

$$\mathbf{g} = \mathrm{d}s^2 = \tanh^2 t\,\mathrm{d}t^2 + \mathrm{sech}^2 t\,\mathrm{d}\theta^2 \qquad \text{and} \qquad \mathbf{h} = \tanh t\,\mathrm{sech}\,t\left(\mathrm{d}t^2 - \mathrm{d}\theta^2\right)\,, \tag{2.56}$$

where we used

$$\operatorname{sech}^2 t = \left(\frac{4}{e^{2t}+1} - \frac{4}{(e^{2t}+1)^2}\right) \quad \text{and} \quad \sin\omega = 2\tanh t \operatorname{sech} t\,,$$

obtaining precisely the fundamental forms of the pseudosphere as written in (2.12) and (2.13), proving that

Theorem 4. *For a function $\omega = \omega(u,v)$ defined on a simply connected region D on $\mathbb{R}^2$ satisfying*

$$\omega_{uv} = \sin\omega \tag{2.57}$$

and

$$\omega(u,v) \in (0,\pi) \quad \forall\,(u,v) \in D\,, \tag{2.58}$$

there exists a unique immersion $f : D \to \mathbb{R}^3$ (up to congruence of $\mathbb{R}^3$) with first and second fundamental forms as

$$\boldsymbol{g} = \mathrm{d}s^2 = \mathrm{d}u^2 + 2\cos\omega\; du\,\mathrm{d}v + \mathrm{d}v^2 \qquad \textit{and} \qquad \boldsymbol{h} = 2\sin\omega\; \mathrm{d}u\,\mathrm{d}v\,. \tag{2.59}$$

Conversely, any surfaces in $\mathbb{R}^3$ with constant curvature -1 can be obtained in this way. Such $(u,v)-$coordinates are known as an asymptotic Chebyshev net [Ch78].

2.3 From Gross-Pitaevskii equation to sine-Gordon equation

In this section I want to highlight the relationship between sine-Gordon equation and Gross-Pitaevskii equation, showing how the first one can emerge when we consider two coupled non-relativistic BECs [Oal13, Sal15] governed by coupled GPE equations that can be written, using the non-dimensional form of the GPE (B.6), as

$$\begin{cases} i\partial_t\psi_1 = \left(-\frac{1}{2}\nabla^2 + \frac{1}{2}(|\psi_1|^2 - 1)\right)\psi_1 - \nu\psi_2 \\ i\partial_t\psi_2 = \left(-\frac{1}{2}\nabla^2 + \frac{1}{2}(|\psi_2|^2 - 1)\right)\psi_2 - \nu\psi_1 \end{cases} \tag{2.60}$$

We can recover these equations from the stationary action principle $\delta S = 0$, with

$$S = \int L\,\mathrm{d}t\,\mathrm{d}^3\mathbf{r} = \int \Big(\mathcal{L} - w\Big)\,\mathrm{d}t\,\mathrm{d}^3\mathbf{r}\,, \tag{2.61}$$

where

$$\mathcal{L} = \Re(\psi_1 i\partial_t \psi_1^* + \psi_2 i\partial_t \psi_2^*)\,, \tag{2.62}$$

and w is the energy density

$$w = \sum_{j=1,2} \left(\frac{1}{2}|\nabla\psi_j|^2 + \frac{1}{4}|\psi_j|^4 - \frac{1}{2}|\psi_j|^2\right) - \nu\,(\psi_1^*\psi_2 + \psi_2^*\psi_1) = \mathcal{K} + \mathcal{I}\,, \tag{2.63}$$

with $\mathcal{K} = \frac{1}{2}(|\nabla\psi_1|^2 + |\nabla\psi_2|^2)$ being the kinetic energy density and $\mathcal{I}$ the remaining interaction energy density. We now follow the technique proposed in [Oal13] and write

$$\begin{cases} \psi_1 = \sqrt{\rho}e^{i(\phi_s+\phi_a)/2}\cos\theta \\ \psi_2 = \sqrt{\rho}e^{i(\phi_s-\phi_a)/2}\sin\theta \end{cases} \tag{2.64}$$

where ρ is the total density, ϕ_s is the total phase, ϕ_a is the relative phase and θ is the density mixing angle. When substituting into (2.62) and (2.63), we obtain

$$\begin{aligned} \mathcal{L} &= \frac{\rho}{2}\,\partial_t\phi_s + \frac{\rho}{2}\,\cos(2\theta)\,\partial_t\phi_a\,, \\ \mathcal{K} &= \frac{1}{2}\,|\nabla\sqrt{\rho}|^2 + \frac{1}{2}\,\rho\,|\nabla\theta|^2 + \frac{\rho}{8}\left(|\nabla\phi_s|^2 + |\nabla\phi_a|^2\right) + \frac{\rho}{4}\,\nabla\phi_s\cdot\nabla\phi_a\,\cos(2\theta)\,, \\ \mathcal{I} &= -\nu\rho\cos(\phi_a)\sin(2\theta) - \frac{\rho}{2} + \frac{\rho^2}{8}(\cos(4\theta)+3)\,, \end{aligned} \tag{2.65}$$

having used the fact that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos(4\theta)+3)$.

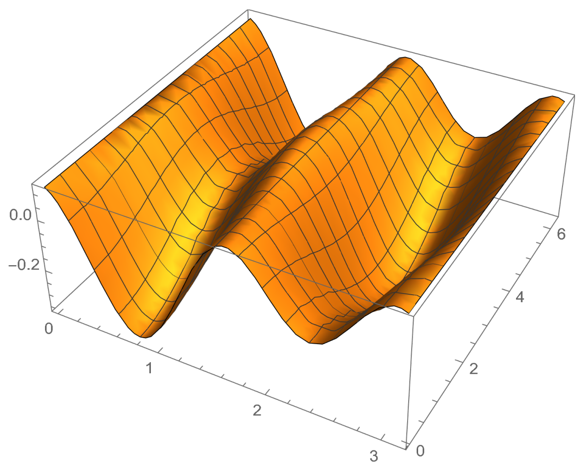

Figure 2.3: Plot of $\mathcal{I}$ for $\nu = 0.1$ and $\rho = 1.2$, being the value of ρ that minimizes $\mathcal{I}$. Notice that $\mathcal{I}$ presents a valley for $\theta = \frac{\pi}{4}$.

We see that $\mathcal{I}$ attains its minimum at $\theta = \frac{\pi}{4}$ (see Figure 2.3); hence we can consider

$\theta = \dfrac{\pi}{4} + y$ and expand to leading order in y, noticing that:

$$\cos 2\theta = -2y + O(y^3)\,, \quad \sin 2\theta = 1 - 2y^2 + O(y^4)\,, \quad \cos 4\theta = 8y^2 - 1 + O(y^4)\,. \tag{2.66}$$

We obtain

$$\begin{aligned}
\tilde{\mathcal{L}} &= \frac{1}{2}\,\rho\,\partial_t\phi_s - \rho\,\partial_t\phi_a\,y\,,\\
\tilde{\mathcal{K}} &= \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}\,\rho\,|\nabla y|^2 + \frac{\rho}{8}\left(|\nabla\phi_s|^2 + |\nabla\phi_a|^2\right) - \frac{\rho}{2}\,\nabla\phi_s\cdot\nabla\phi_a\,y\,,\\
\tilde{\mathcal{I}} &= -\nu\rho\cos\phi_a(1-2y^2) - \frac{\rho}{2} + \rho^2\,y^2 + \frac{\rho^2}{4}\,.
\end{aligned} \tag{2.67}$$

Taking only relevant terms in the action we get

$$\tilde{L} = \tilde{\mathcal{L}} - \tilde{\mathcal{K}} - \tilde{\mathcal{I}} \approx -\rho\,\partial_t\phi_a\,y - \frac{\rho}{8}|\nabla\phi_a|^2 + \nu\rho\cos\phi_a - \rho^2\,y^2\,. \tag{2.68}$$

Taking the Euler-Lagrange equations we find

$$\partial_t\phi_a = -2\rho\,y \quad \Longrightarrow \quad y = -\frac{\partial_t\phi_a}{2\rho}\,, \tag{2.69}$$

hence (2.68) can be written as

$$\tilde{L} = \frac{1}{4}\left((\partial_t\phi_a)^2 - \frac{\rho}{2}|\nabla\phi_a|^2 + 4\nu\rho\cos\phi_a\right)\;; \tag{2.70}$$

and noticing that $c^2 = \dfrac{\rho}{2}$ (see Corollary 1), we can write it in the form of the sine-Gordon Lagrangian

$$\tilde{L} = \frac{c^2}{4}\left(\frac{1}{c^2}(\partial_t\phi_a)^2 - |\nabla\phi_a|^2 + 8\nu\cos\phi_a\right)\,. \tag{2.71}$$

This chapter has highlighted the relation between the geometry of negative curvature surfaces and the sine-Gordon equation (sGE), and in its final part the connection between the GPE and the sGE. Since in the low-amplitude limit sGE reduces to the Klein-Gordon equation (KG), evidently we have also a direct connection between GPE and KG. In the next chapter we will explore another strict relationship between the GPE and the KG that emerges when considering the relativistic case.

Chapter 3

Relativistic Bose-Einstein condensates

Our study of the GPE on generic Riemannian manifolds becomes useful when we move to the relativistic case, where the main interest lies in the geometry of the considered spacetime. In order to extend the GPE to relativistic BECs, it is sufficient to think of the Schrödinger equation as a non-relativistic approximation of the Klein-Gordon equation. We consider the relativistic GPE

$$\Box\phi = \frac{m^2c^2}{\hbar^2}\phi - \kappa R\phi + \frac{2\mathfrak{g}m}{\hbar^2}|\phi|^2\phi + \frac{2m}{\hbar^2}V\phi\,, \tag{3.1}$$

where m is the mass of the bosons, $\hbar$ the reduced Planck's constant, V the external potential, $\mathfrak{g} = 4\pi\hbar^2 a_s/m$ the coupling constant proportional to the scattering length a_s of two interacting boson, R the Ricci scalar and κ the coupling constant between the scalar field and the gravitational field [BD82]. Here the D'Alembertian is defined by

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right)\,, \tag{3.2}$$

with $g^{\mu\nu}$ being the inverse spacetime metric.

Lemma 1. *The equation governing a relativistic BEC* (3.1) *can be derived from the Lagrangian*

$$L = \frac{1}{2}\sqrt{|g|}\left(-g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - \frac{m^2c^2}{\hbar^2}|\phi|^2 + \kappa R|\phi|^2 - \frac{\mathfrak{g}m}{\hbar^2}|\phi|^4 - \frac{2m}{\hbar^2}V|\phi|^2\right)\,. \tag{3.3}$$

Proof. We apply the Euler-Lagrange equation

$$\frac{\partial L}{\partial\phi^*} = \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi^*)}\,; \tag{3.4}$$

in particular, for the left-hand side we have

$$\frac{\partial L}{\partial \phi^*} = \frac{1}{2}\sqrt{|g|}\left(-\frac{m^2c^2}{\hbar^2}\phi + \kappa R\phi - \frac{2\mathfrak{g}m}{\hbar^2}|\phi|^2\phi - \frac{2m}{\hbar^2}V\phi\right), \tag{3.5}$$

and for the right-hand side

$$\frac{\partial L}{\partial(\partial_\mu \phi^*)} = -\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\,, \tag{3.6}$$

so that, equating and dividing both sides by $\sqrt{|g|}$, we obtain the result. $\square$

3.1 Semi-relativistic Gross-Pitaevskii equation

The equation (3.1) is the correct relativistic version of the GPE since we can extract the standard GPE (1.15) by a simple procedure. Consider a generic ultra-relativistic spacetime where the metric can be written in block-diagonal form

$$g_{\mu\nu} = \left[\begin{array}{c|c} -c^2 & 0 \\ \hline 0 & h_{ij} \end{array}\right], \tag{3.7}$$

and the D'Alembertian can be decomposed into $\square = -\frac{1}{c^2}\partial_t^2 + \nabla^2$, where ∇^2 is the spatial Laplace operator. In order to derive the standard GPE we now substitute $\phi = \psi e^{-imc^2t/\hbar}$ into the relativistic GPE (3.1) and multiply all terms times $\hbar^2/2m$. We obtain an equation that we may refer to as the semi-relativistic GPE

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}(\square + \kappa R)\psi + \mathfrak{g}|\psi|^2\psi + V\psi\,, \tag{3.8}$$

which can be considered as a 4−dimensional standard GPE, since the D'Alembertian is nothing but a 4−dimensional Laplace operator. From (3.8) we can recover the 3−dimensional GPE (1.15) by the standard procedure of considering the formal limit $c \to \infty$. We will show that all the results obtained in Chapter 1 are still valid and can be used under these conditions.

3.1.1 Minkowski spacetime

As an example of block-diagonal metric, we consider the Minkowski spacetime metric with signature $(-,+,+,+)$: $\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. We see that (3.1) takes the form of a standard Klein-Gordon equation

$$-\frac{1}{c^2}\partial_t^2\phi + \nabla^2\phi = \frac{m^2c^2}{\hbar^2}\phi + \frac{2\mathfrak{g}m}{\hbar^2}|\phi|^2\phi + \frac{2m}{\hbar^2}V\phi\,, \tag{3.9}$$

where we suppressed the Ricci term since Minkowski spacetime is flat, namely has zero Riemann tensor and, as a consequence, zero Ricci scalar. Substituting $\phi = \psi e^{-imc^2t/\hbar}$ into the Minkowski KG equation (3.9) we get the Minkowski semi-relativistic GPE

$$i\hbar\partial_t\psi = +\frac{\hbar^2}{2mc^2}\partial_t^2\psi - \frac{\hbar^2}{2m}\nabla^2\psi + \mathfrak{g}|\psi|^2\psi + V\psi\,. \tag{3.10}$$

From this we can easily recover exactly the standard non-relativistic GPE (1.15) by simply taking the limit $c \to \infty$; to be precise, we assume that the wave function is slowly evolving, namely

$$\frac{\hbar}{mc^2}\partial_t^2\psi \ll \partial_t\psi\,. \tag{3.11}$$

3.1.2 Madelung representation of the semi-relativistic GPE

We now substitute Madelung's transform $\psi = \sqrt{\rho}e^{i\left(\frac{m}{\hbar}\theta\right)}$ into the semi-relativistic GPE (3.8). By equating the imaginary parts we find a form of continuity equation

$$\partial_t\rho = -\mathbf{div}(\rho\boldsymbol{\nabla}\theta)\,, \tag{3.12}$$

where we are using the bold symbols to indicate the 4−dimensional operators.
In the specific example of Minkowski spacetime, such a continuity equation can be written as

$$\partial_t\rho = \frac{1}{c^2}\partial_t(\rho\partial_t\theta) - \mathrm{div}(\rho\nabla\theta)\,. \tag{3.13}$$

Evidently, the usual continuity equation (1.22) is recovered from (3.13) by taking the formal limit $c \to \infty$.
After some algebraic manipulation, by equating the real parts we have the phase equation

$$\partial_t\theta = \frac{\hbar^2}{2m^2}\frac{\Box\sqrt{\rho}}{\sqrt{\rho}} - \frac{1}{2}|\boldsymbol{\nabla}\theta|^2 + \frac{\hbar^2\kappa}{2m^2}R - \frac{\mathfrak{g}}{m}\rho - \frac{V}{m}\,, \tag{3.14}$$

which assumes the following form in Minkowski spacetime

$$\partial_t\theta = \frac{\hbar^2}{2m^2}\frac{-\frac{1}{c^2}\partial_t^2\sqrt{\rho} + \nabla^2\sqrt{\rho}}{\sqrt{\rho}} + \frac{1}{2}\left(\frac{1}{c^2}(\partial_t\theta)^2 - |\nabla\theta|^2\right) - \frac{\mathfrak{g}}{m}\rho - \frac{V}{m}\,. \tag{3.15}$$

Again the standard phase equation (1.24) is obtained from (3.15) in the formal limit $c \to \infty$. Under our conditions we can write the 4−dimensional gradient as $\boldsymbol{\nabla}\theta = \left(-\frac{1}{c}\partial_t\theta, \nabla\theta\right)$. Taking the spatial gradient of (3.15) and defining $\frac{1}{c}\partial_t\theta = v$ and $\nabla\theta = \boldsymbol{u}$ we obtain the Euler equation in the form presented in [FMT08]:

$$\left(1 - \frac{v}{c}\right)\partial_t\boldsymbol{u} + \nabla_{\boldsymbol{u}}\boldsymbol{u} = \nabla\left(\frac{\hbar^2}{2m^2}\frac{\Box\sqrt{\rho}}{\sqrt{\rho}}\right) - \nabla\left(\frac{\mathfrak{g}}{m}\rho\right) - \nabla\left(\frac{V}{m}\right). \tag{3.16}$$

Note that in the formal limit $c \to \infty$ we have $\frac{v}{c} \to 0$ and $\Box \to \nabla^2$, where ∇^2 is the curved space Laplacian, so that (3.16) reduces to the Euler equation (1.28).

3.2 Hydrodynamic formulation of the relativistic GPE

We now substitute Madelung's transform $\phi = \sqrt{\rho}e^{i\left(\frac{m}{\hbar}\theta\right)}$ into the relativistic GPE in the KG formulation with generic spacetime metric (3.1), obtaining the Madelung representation of the relativistic GPE. We can see equation (3.1) as a stationary GPE (1.15) with 4−dimensional generic metric $g_{\mu\nu}$, hence we can make use of the formalism introduced for generic Riemannian manifolds in Chapter 1.

By equating the imaginary parts we see that the continuity equation takes the 4−dimensional stationary form

$$\mathbf{div}(\rho\boldsymbol{\nabla}\theta) = 0\,, \quad \text{namely} \quad \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\rho g^{\mu\nu}\partial_\nu\theta) = 0\,, \tag{3.17}$$

where again we are using the bold symbol to indicate 4−dimensional operators.

By equating the real parts we obtain the equation for the phase in the stationary form

$$\frac{\hbar^2}{2m^2\sqrt{\rho}}\Box\sqrt{\rho} - \frac{1}{2}|\boldsymbol{\nabla}\theta|^2 - \frac{c^2}{2} + \frac{\hbar^2\kappa}{2m^2}R - \frac{\mathfrak{g}}{m}\rho - \frac{V}{m} = 0\,. \tag{3.18}$$

3.2.1 4−gradient pseudo-velocity hydrodynamics

Using what we have from Chapter 1, we can consider the generic metric g_{ij} to be the 4−dimensional spacetime metric $g_{\mu\nu}$ and we can define the 4−gradient pseudo-velocity $\boldsymbol{V} = \boldsymbol{\nabla}\theta$, namely $V^\mu = g^{\mu\nu}\partial_\nu\theta$, to obtain the hydrodynamic formulation. Notice that $\boldsymbol{V}$ is not the standard relativistic 4−velocity because it is not a unit vector.

Euler equation

Taking the 4−gradient $\boldsymbol{\nabla}$ of the relativistic phase equation (3.18) we find the Euler equation for the 4−gradient pseudo-velocity $\boldsymbol{V}$

$$\boldsymbol{\nabla}_{\boldsymbol{V}}\boldsymbol{V} = \boldsymbol{\nabla}\left(\frac{\hbar^2}{2m^2}\frac{\Box\sqrt{\rho}}{\sqrt{\rho}} + \frac{\hbar^2\kappa}{2m^2}R - \frac{\mathfrak{g}}{m}\rho - \frac{V}{m}\right), \tag{3.19}$$

where again we used the fact that $\boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\nabla}\theta|^2\right) = \boldsymbol{\nabla}_{\boldsymbol{V}}\boldsymbol{V}$.

Navier-Stokes equation

Multiplying (3.19) times the density ρ and considering the absence of external potential ($V=0$), we get the Navier-Stokes formulation

$$\rho\,\boldsymbol{\nabla}_{\boldsymbol{V}}\boldsymbol{V} = -\boldsymbol{\nabla}\mathfrak{p} + (\textbf{div}\,\boldsymbol{\tau})^\sharp + \mathbf{f}\,. \tag{3.20}$$

Pressure is $\mathfrak{p} = \frac{\mathfrak{g}\rho^2}{2m}$, stress-tensor is $\boldsymbol{\tau}_{\mu\nu} = \frac{\hbar^2}{4m^2}\rho\,\mathrm{Hess}_{\mu\nu}(\ln\rho)$, with $\mathrm{Hess}_{\mu\nu}(\ln\rho) = \nabla_\mu\nabla_\nu(\ln\rho)$. We denote by **div** the 4−divergence, and the density curvature vector is given by

$$\mathbf{f} = -\frac{\hbar^2}{4m^2}\boldsymbol{R}(\boldsymbol{\nabla}\rho,\,\cdot\,) + \frac{\hbar^2\kappa}{2m^2}\rho\boldsymbol{\nabla}R\,, \tag{3.21}$$

where $\boldsymbol{R}$ denotes the Ricci tensor and R the Ricci scalar.

Momentum equation

Finally we can get the momentum equation in the form

$$\partial_t(\rho\,\boldsymbol{V}^\flat) = -\textbf{div}\,\boldsymbol{\Pi} + \boldsymbol{\chi}\,, \tag{3.22}$$

where $\boldsymbol{\Pi}_{\mu\nu} = \boldsymbol{\mathcal{D}}_{\mu\nu} + \mathfrak{p}g_{\mu\nu} - \boldsymbol{\tau}_{\mu\nu} - \boldsymbol{\mathcal{G}}_{\mu\nu}$, with

$$\boldsymbol{\mathcal{D}}_{\mu\nu} = \rho\,\boldsymbol{V}_\mu\,\boldsymbol{V}_\nu\,, \quad \text{and} \quad \boldsymbol{\mathcal{G}}_{\mu\nu} = \frac{\hbar^2\rho}{4m^2}\boldsymbol{G}_{\mu\nu}\,, \tag{3.23}$$

where $\boldsymbol{G}_{\mu\nu}$ denotes Einstein tensor; while the correction term is

$$\boldsymbol{\chi}_\mu = -\frac{\hbar^2}{8m^2}R\,\mathrm{d}\rho + \frac{\hbar^2\kappa}{2m^2}\rho\,\mathrm{d}R\,. \tag{3.24}$$

Also in the relativistic case we can impose the sufficient condition to have steady state, thus obtaining the Einstein-type field equations:

$$\begin{cases} \boldsymbol{G}_{\mu\nu} = -\dfrac{4m^2}{\hbar^2}\boldsymbol{V}_\mu \boldsymbol{V}_\nu + \mathbf{Hess}_{\mu\nu}(\ln\rho) + 2\kappa R\, g_{\mu\nu} \\ \\ R = -\dfrac{8m\mathfrak{g}}{\hbar^2(1+4\kappa)}\rho \end{cases} \tag{3.25}$$

3.3 Weak gravitational field

In order to study much more in detail the bond between GPE and cosmology we want to consider another example of relativistic GPE. We take into account a weak gravitational field, characterized by a metric being the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ plus a small perturbation term $h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. Under such condition we can simplify the expressions for the Ricci tensor and the Ricci scalar, obtaining

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\mu h^\sigma{}_\nu + \partial_\sigma\partial_\nu h^\sigma{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \quad \text{and} \quad R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h\,, \tag{3.26}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation and $\Box = -\partial_t^2 + \nabla^2$ is the Minkowski d'Alembert operator. The process of decomposing the spacetime metric is not unique, hence we have to choose a specific gauge.

3.3.1 Harmonic gauge

The gauge in which linearized gravity is mostly simplified is known as harmonic (or Lorenz) gauge, and is obtained by imposing the condition $\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h$. In such a situation the linearized Einstein equation can be solved exactly using wave solutions that define gravitational radiation.

An example of linearized gravity in the harmonic gauge is the Newtonian limit, where fluxes T_{0i} are negligible and stresses T_{ij} are isotropic. Such a situation is a perfectly good approximation for non-rotating cosmology and can be expressed writing the perturbations on the metric in terms of two scalar potentials Ψ and Φ which appear in the line element as in [MB95]:

$$\mathrm{d}s^2 \approx -(1+2\Psi)c^2\,\mathrm{d}t \otimes \mathrm{d}t + (1-2\Phi)\delta_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j\,, \tag{3.27}$$

where we are neglecting higher order terms.

We then have

$$g_{\mu\nu} \approx \left[\begin{array}{c|c} -c^2(1+2\Psi) & 0 \\ \hline 0 & (1-2\Phi)\delta_{ij} \end{array}\right], \quad g^{\mu\nu} \approx \left[\begin{array}{c|c} -\frac{1}{c^2}(1-2\Psi) & 0 \\ \hline 0 & (1+2\Phi)\delta^{ij} \end{array}\right], \tag{3.28}$$

where we used the fact that the field is weak to invert the metric. In particular, calculating the determinant $|g| \approx -c^2(1+2\Psi)(1-2\Phi)^3 \approx -c^2(1+2\Psi-6\Phi)$, we find that the D'Alembertian takes the new (approximated) form

$$\begin{aligned} \Box_{new}\phi &\approx \frac{1}{c^2}(3\dot{\Phi}+\dot{\Psi})\partial_t\phi - \frac{1}{c^2}(1-2\Psi)\partial_t^2\phi + (1+2\Phi)\nabla^2\phi + (\nabla\Psi - \nabla\Phi)\cdot\nabla\phi \\ &\approx \Box_{Minkowski} + \frac{1}{c^2}(3\dot{\Phi}+\dot{\Psi})\partial_t\phi + \frac{2\Psi}{c^2}\partial_t^2\phi + 2\Phi\nabla^2\phi + (\nabla\Psi - \nabla\Phi)\cdot\nabla\phi\,, \end{aligned} \tag{3.29}$$

where the dot represents time derivative.

Hence, from (3.1) we obtain the gravitational relativistic GPE in Klein-Gordon form [MG15]

$$\Box_{new}\phi = \frac{m^2c^2}{\hbar^2}\phi - \kappa R\phi + \frac{2\mathfrak{g}m}{\hbar^2}|\phi|^2\phi + \frac{2m}{\hbar^2}V\phi\,. \tag{3.30}$$

In order to calculate the Ricci scalar we use coordinates (ct, x, y, z), so that the spacetime metric is written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu} = -2\,\mathrm{diag}(\Psi, \Phi, \Phi, \Phi)$. Hence we have the trace $h = \eta^{\mu\nu}h_{\mu\nu} = 2\Psi - 6\Phi$ and $h^{\mu\nu} = \eta^{\mu\rho}h_{\rho\sigma}\eta^{\sigma\nu} = h_{\mu\nu}$, from which we get

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h = -6\,\partial_t^2\Phi + 2\,\nabla^2(2\Phi - \Psi)\,. \tag{3.31}$$

Gravitational semi-relativistic GPE

If we substitute again $\phi = \psi e^{-imc^2t/\hbar}$ into the gravitational relativistic GPE (3.30), we find the semi-relativistic GPE in a gravitational field

$$i(1-2\Psi)\hbar\partial_t\psi - i\hbar\frac{3\dot{\Phi}+\dot{\Psi}}{2}\psi = -\frac{\hbar^2}{2m}\left(\Box_{new} + \kappa R\right)\psi + mc^2\Psi\psi + \mathfrak{g}|\psi|^2\psi + V\psi\,, \tag{3.32}$$

which again allows us to recover the classical GPE (1.15) by considering no gravitational field, namely setting $\Phi = \Psi = 0$, and taking the formal limit $c \to \infty$.

Now equation (3.32) can be further simplified if we consider the case of a single gravitational potential $\Psi = \Phi$ (which is a perfectly good approximation for all solar system physics), and a static gravitational field ($\dot{\Phi} = 0$).

In this case (3.32) takes the form

$$i(1-2\Phi)\hbar\partial_t\psi = \frac{\hbar^2}{2mc^2}(1-2\Phi)\partial_t^2\psi - \frac{\hbar^2}{2m}(1+2\Phi)\nabla^2\psi - \frac{\hbar^2\kappa}{2m}R\psi + mc^2\Phi\psi + \mathfrak{g}|\psi|^2\psi + V\psi\,. \tag{3.33}$$

We now substitute Madelung's expression $\psi = \sqrt{\rho}e^{i\frac{m}{\hbar}\theta}$ into the static gravitational semi-relativistic GPE (3.33). Equating the imaginary parts we find a 4−dimensional continuity equation

$$(1-2\Phi)\partial_t\rho = -\mathbf{div}(\rho\boldsymbol{\nabla}\theta) = -\frac{1}{\sqrt{|g|}}\partial_\mu(\rho\sqrt{|g|}g^{\mu\nu}\partial_\nu\theta)\,, \tag{3.34}$$

that we can write in terms of $v = \frac{1}{c}\partial_t\theta$ and $\boldsymbol{u} = \nabla\theta$ as

$$(1-2\Phi)\partial_t\rho = -(1+2\Phi)\mathrm{div}(\rho\boldsymbol{u}) + \frac{(1-2\Phi)}{c}\partial_t\left(\rho v\right)\,. \tag{3.35}$$

Equating instead the real parts, we get the equation for the phase θ

$$(1-2\Phi)\partial_t\theta = \frac{\hbar^2}{2m^2}\frac{\bar{\Box}\sqrt{\rho}}{\sqrt{\rho}} + \frac{(1-2\Phi)}{c^2}\frac{(\partial_t\theta)^2}{2} - (1+2\Phi)\frac{|\nabla\theta|^2}{2} - c^2\Phi + \frac{\hbar^2\kappa}{2m}R - \frac{\mathfrak{g}}{m}\rho - \frac{V}{m}\,, \tag{3.36}$$

where

$$\bar{\Box} = -\frac{1}{c^2}(1-2\Phi)\partial_t^2 + (1+2\Phi)\nabla^2\,.$$

Dividing both sides by c^2, we have

$$\frac{(1+2\Phi)}{2}\frac{|\boldsymbol{u}|^2}{c^2} + (1-2\Phi)\frac{v}{c} - \frac{(1-2\Phi)}{2}\frac{v^2}{c^2} = \frac{\hbar^2}{2m^2c^2}\frac{\bar{\Box}\sqrt{\rho}}{\sqrt{\rho}} - \Phi + \frac{\hbar^2\kappa}{2mc^2}R - \frac{\mathfrak{g}}{mc^2}\rho - \frac{V}{mc^2}\,. \tag{3.37}$$

By taking the spatial gradient of (3.36), we obtain the gravitational semi-relativistic version of Euler equation

$$\begin{aligned}(1-2\Phi)\left(1-\frac{v}{c}\right)\partial_t\boldsymbol{u} + (1+2\Phi)\nabla_{\boldsymbol{u}}\boldsymbol{u} =& \nabla\left(\frac{\hbar^2}{2m^2}\frac{\bar{\Box}\sqrt{\rho}}{\sqrt{\rho}} + \frac{\hbar^2\kappa}{2m}R - \frac{\mathfrak{g}}{m}\rho - \frac{V}{m}\right)\\ &- c^2\nabla\Phi\left(\frac{v^2}{c^2} + \frac{|\mathbf{u}|^2}{c^2} - 2\frac{v}{c} + 1\right)\,.\end{aligned} \tag{3.38}$$

Finally, multiplying (3.38) times density ρ, we get the static gravitational semi-relativistic Navier-Stokes equation

$$\begin{aligned}\rho\left[(1-2\Phi)\left(1-\frac{v}{c}\right)\partial_t\boldsymbol{u} + (1+2\Phi)\nabla_{\boldsymbol{u}}\boldsymbol{u}\right] =& \rho\nabla\left(\frac{\hbar^2}{2m^2}\frac{\bar{\Box}\sqrt{\rho}}{\sqrt{\rho}} + \frac{\hbar^2\kappa}{2m}R - \frac{\mathfrak{g}}{m}\rho - \frac{V}{m}\right)\\ &- \rho c^2\nabla\Phi\left(\frac{v^2}{c^2} + \frac{|\mathbf{u}|^2}{c^2} - 2\frac{v}{c} + 1\right)\,.\end{aligned} \tag{3.39}$$

4−gradient pseudo-velocity hydrodynamics

We now consider the gravitational relativistic GPE (3.30) under the hypothesis of a static single gravitational potential ($\Psi = \Phi$ and $\dot{\Phi} = 0$)

$$-\frac{1}{c^2}(1-2\Phi)\partial_t^2\phi + (1+2\Phi)\nabla^2\phi = \frac{m^2c^2}{\hbar^2}\phi - \kappa R\phi + \frac{2\mathfrak{g}m}{\hbar^2}|\phi|^2\phi + \frac{2m}{\hbar^2}V\phi\,, \tag{3.40}$$

and substitute Madelung's transform $\phi = \sqrt{\rho}e^{i\left(\frac{m}{\hbar}\theta\right)}$. Using the results in Chapter 1, from the imaginary parts we get $\mathbf{div}(\rho\boldsymbol{\nabla}\theta) = 0$, and from the real parts we obtain

$$\frac{\hbar^2}{2m^2\sqrt{\rho}}\bar{\Box}\sqrt{\rho} - \frac{1}{2}|\boldsymbol{\nabla}\theta|^2 = \frac{c^2}{2} - \frac{\hbar^2\kappa}{2m^2}R + \frac{\mathfrak{g}}{m}\rho + \frac{V}{m}\,, \tag{3.41}$$

where again the d'Alembertian takes the simplified form $\bar{\Box} = -\frac{1}{c^2}(1-2\Phi)\partial_t^2 + (1+2\Phi)\nabla^2$ and the Ricci scalar is $R = 2\nabla^2\Phi$.

We take the 4−gradient ($\boldsymbol{\nabla}$) of (3.41) and we get an equation for the 4−gradient pseudo-velocity $\boldsymbol{V} = \boldsymbol{\nabla}\theta$

$$\boldsymbol{\nabla}_{\boldsymbol{V}}\boldsymbol{V} = -\boldsymbol{\nabla}\mathfrak{p} + (\mathbf{div}\,\boldsymbol{\tau})^\sharp + \mathbf{f} - \boldsymbol{\nabla}\left(\frac{V}{m}\right)\,, \tag{3.42}$$

where we have the standard definitions for

$$\begin{aligned}
\mathfrak{p} &= \frac{\mathfrak{g}\rho^2}{2m}\,,\\
\boldsymbol{\tau}_{\mu\nu} &= \frac{\hbar^2}{4m^2}\rho\mathrm{Hess}_{\mu\nu}(\ln\rho)\,,\\
\mathbf{f}^\mu &= -\frac{\hbar^2}{4m^2}R^{\mu\nu}\partial_\nu\rho + \frac{\hbar^2\kappa}{2m^2}\rho g^{\mu\nu}\partial_\nu R\,.
\end{aligned} \tag{3.43}$$

As shown by the discussion above this new formulation of the GPE on generic Riemannian manifolds can be very useful; even more so if we think that BECs are one of the most important toy models for creating analogue black holes in cosmology, allowing the study of the Hawking radiation, of the black holes' horizons, and of the gravity as a large-scale phenomenon. In the next chapter we shall examine in more detail features of the analogue models used in black hole theory.

Chapter 4

Linearization and effective metric

In this chapter we follow the line of the work done by Barceló, Liberati and Visser in [BLV00] in order to underline the relation between general relativity and condensed matter systems. We first analyze some known cases and then focus of a new situation in presence of a straight vortex on the $z-$axis [Ro22].

Let us start from the non-dimensional GPE (B.6) and its hydrodynamic formulation in terms of continuity equation (B.9) for the non-dimensional density ρ (with $\lim_{r\to\infty} \rho = 1$), and phase equation (B.10) for the non-dimensional phase θ. The speed of sound has been defined as $c = \sqrt{\rho/2}$, and this definition becomes relevant in analogue models, since it leads to the formal definition of many analogue surfaces that play a central role in cosmology. In particular, Visser in [Vi98] defined the ergo-sphere as the surface where the magnitude of the flow velocity equals exactly the speed of sound: $c^2 - v^2 = 0$; in other terms, it is the boundary of the ergo-region, where $|v| > c$, namely the flow is supersonic. We can also define a trapped surface as a surface where the fluid velocity is everywhere inward pointing and the normal component is everywhere grater than the local speed of sound. Finally, we can define the event horizon as the boundary of the region from which null geodesics, namely phonons, cannot escape; in particular, it is a null surface and its generators are null geodesics. It is a fact that for stationary geometries and highly symmetric fluid flows the ergo-sphere may coincide with the trapped surface and even with the event horizon, namely if we are 'inside' the horizon ($r < r_H$), we have a supersonic flow and acoustic perturbations remain trapped inside this apparent surface.

Another important definition in analogue models is surface gravity. It was first proposed by Unruh in 1981 [Un81] and then slightly modified by Visser in 1998 [Vi98], considering a position-dependent speed of sound and obtaining the definition

$$g_H = \frac{1}{2} \frac{\partial (c^2 - v^2)}{\partial n}\Big|_H , \tag{4.1}$$

where n is the normal to the surface. This makes us obtain a formula for the Hawking radiation emitted by the acoustic event horizon, as discussed by Unruh, in the exact same form that we find in cosmology:

$$k_B T_H = \frac{\hbar g_H}{2\pi c_H} , \tag{4.2}$$

which gives us a flavor of the importance of analogue models, and in particular of BEC models, in the experimental detection of this phenomenon that, seen in its cosmological context, we wouldn't have the technology to prove.

Most of the techniques and considerations in this field are summed up in [BLV11], which gives a useful synthesis of the state of art in this field of research up to 2010. In particular, in this chapter I will make use of the linearization process as presented in [BLV00], where the authors put much emphasis on the generality of the procedure and where the quasi-linear, low-momentum and eikonal approximations are widely analyzed.

4.1 Linearization process

Now we perform a linearization by considering the density and the phase as the sum of a background quantity (ρ_0 and θ_0) and a fluctuation (ρ_1 and θ_1) controlled by a small parameter ϵ, specifically we write

$$\rho = \rho_0 + \epsilon \rho_1 \quad \text{and} \quad \theta = \theta_0 + \epsilon \theta_1 . \tag{4.3}$$

Our aim is to obtain a D'Alembertian equation for the phase perturbation θ_1 in the form

$$\nabla^2 \theta_1 = 0 , \quad \text{namely} \quad \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \theta \right) = 0 , \tag{4.4}$$

from which we can extract $g = g_{\mu\nu}(t, \mathbf{x})$, being the effective Lorentzian acoustic metric. In particular, this equation describes a massless minimally coupled quantum scalar field over a curved background and, as we will see, this equation can emerge only if some invertibility conditions are fulfilled. The aim of this chapter is to show some cases in which these conditions are indeed satisfied.

By substituting (4.3) into the non-dimensional continuity equation (B.9) we find:

$$\begin{aligned}\partial_t\rho_0+\epsilon\partial_t\rho_1&=-\mathrm{div}((\rho_0+\epsilon\rho_1)(\nabla\theta_0+\epsilon\nabla\theta_1))\\&=-\mathrm{div}(\rho_0\nabla\theta_0+\epsilon(\rho_1\nabla\theta_0+\rho_0\nabla\theta_1))+O(\epsilon^2)\,,\end{aligned}\tag{4.5}$$

from which we get

$$\begin{cases}\partial_t\rho_0=-\mathrm{div}(\rho_0\nabla\theta_0)\\ \partial_t\rho_1=-\mathrm{div}(\rho_1\nabla\theta_0+\rho_0\nabla\theta_1)\end{cases}\tag{4.6}$$

where the first equation is nothing but the continuity equation for the background quantities, while the second one is what comes from the ϵ terms.

Now we substitute (4.3) into the non-dimensional equation for the phase (B.10), getting

$$\begin{aligned}\partial_t\theta_0+\epsilon\partial_t\theta_1&=\frac{1}{2}(1-\rho_0)-\epsilon\frac{\rho_1}{2}-\frac{1}{2}|\nabla\theta_0+\epsilon\nabla\theta_1|^2+\underbrace{\frac{1}{2\sqrt{\rho_0+\epsilon\rho_1}}\nabla^2\sqrt{\rho_0+\epsilon\rho_1}}\\&=\frac{1}{2}(1-\rho_0)-\epsilon\frac{\rho_1}{2}-\frac{1}{2}|\nabla\theta_0|^2-\epsilon\nabla\theta_0\cdot\nabla\theta_1+\qquad\tilde{Q}\qquad+O(\epsilon^2)\,.\end{aligned}\tag{4.7}$$

Let us focus on the quantum potential term $\tilde{Q}$:

$$\begin{aligned}\tilde{Q}&=\frac{1}{2}\left(\frac{1}{\sqrt{\rho_0}}-\epsilon\frac{\rho_1}{2\sqrt{\rho_0^3}}\right)\nabla^2\left(\sqrt{\rho_0}+\epsilon\frac{\rho_1}{2\sqrt{\rho_0}}\right)\\&=\frac{1}{2\sqrt{\rho_0}}\nabla^2\sqrt{\rho_0}-\epsilon\left(\frac{\rho_1}{4\sqrt{\rho_0^3}}\nabla^2\sqrt{\rho_0}-\frac{1}{4\sqrt{\rho_0}}\nabla^2\left(\frac{\rho_1}{\sqrt{\rho_0}}\right)\right)+O(\epsilon^2)\,,\end{aligned}\tag{4.8}$$

hence, we obtain the two equations

$$\begin{cases}\partial_t\theta_0=\frac{1}{2}(1-\rho_0)-\frac{1}{2}|\nabla\theta_0|^2+\frac{1}{2\sqrt{\rho_0}}\nabla^2\sqrt{\rho_0}\\ \partial_t\theta_1=-\frac{\rho_1}{2}-\nabla\theta_0\cdot\nabla\theta_1-\frac{\rho_1}{4\sqrt{\rho_0^3}}\nabla^2\sqrt{\rho_0}+\frac{1}{4\sqrt{\rho_0}}\nabla^2\left(\frac{\rho_1}{\sqrt{\rho_0}}\right)\end{cases}\tag{4.9}$$

where the first one is the Bernoulli type equation for the background field, while the second one is the equation obtained from the first order in ϵ.

In order to find a Lorentzian effective metric we need to isolate ρ_1 from the second equation in (4.9). One way of doing that, at least formally, is by defining the inverse of an operator

$$\rho_1=-\left(\frac{1}{2}+\frac{1}{4\sqrt{\rho_0^3}}\nabla^2\sqrt{\rho_0}-\frac{1}{4\sqrt{\rho_0}}\nabla^2\left(\frac{\cdot}{\sqrt{\rho_0}}\right)\right)^{-1}(\partial_t\theta_1+\nabla\theta_0\cdot\nabla\theta_1)\,,\tag{4.10}$$

so that, substituting this expression into the second equation in (4.6), we get

$$\begin{aligned} 0 = &- \partial_t \left[\left(\frac{1}{2} + \frac{1}{4\sqrt{\rho_0^3}} \nabla^2 \sqrt{\rho_0} - \frac{1}{4\sqrt{\rho_0}} \nabla^2 \left(\frac{\cdot}{\sqrt{\rho_0}} \right) \right)^{-1} (\partial_t \theta_1 + \nabla\theta_0 \cdot \nabla\theta_1) \right] \\ &+ \operatorname{div} \left(\rho_0 \nabla\theta_1 - \nabla\theta_0 \left[\left(\frac{1}{2} + \frac{1}{4\sqrt{\rho_0^3}} \nabla^2 \sqrt{\rho_0} - \frac{1}{4\sqrt{\rho_0}} \nabla^2 \left(\frac{\cdot}{\sqrt{\rho_0}} \right) \right)^{-1} (\partial_t \theta_1 + \nabla\theta_0 \cdot \nabla\theta_1) \right] \right), \end{aligned} \tag{4.11}$$

which we can regard as a wave-like equation for the propagation of the phase perturbation θ_1, that can be written in the form

$$\partial_\mu (f^{\mu\nu} \partial_\nu \theta_1) = 0 \,, \tag{4.12}$$

by simply setting

$$\begin{aligned} f^{00} &= - \left(\frac{1}{2} + \frac{1}{4\sqrt{\rho_0^3}} \nabla^2 \sqrt{\rho_0} - \frac{1}{4\sqrt{\rho_0}} \nabla^2 \left(\frac{\cdot}{\sqrt{\rho_0}} \right) \right)^{-1} , \\ f^{0j} &= - \left(\frac{1}{2} + \frac{1}{4\sqrt{\rho_0^3}} \nabla^2 \sqrt{\rho_0} - \frac{1}{4\sqrt{\rho_0}} \nabla^2 \left(\frac{\cdot}{\sqrt{\rho_0}} \right) \right)^{-1} \nabla_j \theta_0 \,, \\ f^{i0} &= -\nabla_i \theta_0 \left(\frac{1}{2} + \frac{1}{4\sqrt{\rho_0^3}} \nabla^2 \sqrt{\rho_0} - \frac{1}{4\sqrt{\rho_0}} \nabla^2 \left(\frac{\cdot}{\sqrt{\rho_0}} \right) \right)^{-1} , \\ f^{ij} &= \rho_0 - \nabla_i \theta_0 \left(\frac{1}{2} + \frac{1}{4\sqrt{\rho_0^3}} \nabla^2 \sqrt{\rho_0} - \frac{1}{4\sqrt{\rho_0}} \nabla^2 \left(\frac{\cdot}{\sqrt{\rho_0}} \right) \right)^{-1} \nabla_j \theta_0 \,, \end{aligned} \tag{4.13}$$

from which we can finally extract the effective Lorentzian metric $g_{\mu\nu}$. The emergence of such a metric is proved to be a general consequence of any non-linear Schrödinger equation or, even more in general, it is possible to write such an equation for every system described by a scalar field φ whose dynamics is governed by a Lagrangian which depends only on the field and its first derivatives (see [BLV01] for a precise derivation). However, it is important to remark that $f^{\mu\nu}$ is a matrix of differential operators and no metric can arise if invertibility conditions on the operator that defines it are not fulfilled. To avoid this problem we can perform some approximations [BLV00].

Semi-classical approximation

In the semi-classical approximation we just neglect all the terms coming from the quantum potential, since we let $\hbar \to 0$. We then have that (4.11) takes the simpler form

$$-\partial_t \Big(2(\partial_t \theta_1 + \nabla\theta_0 \cdot \nabla\theta_1) \Big) + \operatorname{div} \Big(\rho_0 \nabla\theta_1 - \nabla\theta_0 \left(2(\partial_t \theta_1 + \nabla\theta_0 \cdot \nabla\theta_1) \right) \Big) = 0 \,, \tag{4.14}$$

hence, in this case we have

$$\begin{aligned} f^{00} &= -2\,, \\ f^{0j} &= -2\nabla_j\theta_0\,, \\ f^{i0} &= -2\nabla_i\theta_0\,, \\ f^{ij} &= \rho_0 - 2\nabla_i\theta_0\nabla_j\theta_0\,. \end{aligned} \tag{4.15}$$

Now remember that $c^2 = \dfrac{\rho_0}{2}$ and $u_0^i = \nabla_i\theta_0$, so that the matrix becomes

$$f^{\mu\nu} = 2\left[\begin{array}{c|c} -1 & -u_0^j \\ \hline -u_0^i & c^2 \quad - \quad u_0^i u_0^j \end{array}\right]\,. \tag{4.16}$$

Finally, let $g_{\mu\nu}$ be a metric such that $\sqrt{|g|}g^{\mu\nu} = f^{\mu\nu}$, in order to make θ_1 satisfy the desired equation $\nabla^2\theta_1 = 0$. Since in $(3+1)-$dimension we have $|\det f^{\mu\nu}| = |\det g^{\mu\nu}| = |g| = 16c^6$, we obtain

$$g^{\mu\nu} = \frac{1}{4c^3} f^{\mu\nu} = \frac{1}{2c^3}\left[\begin{array}{c|c} -1 & -u_0^j \\ \hline -u_0^i & c^2 \quad - \quad u_0^i u_0^j \end{array}\right] = \frac{1}{c\rho_0}\left[\begin{array}{c|c} -1 & -u_0^j \\ \hline -u_0^i & c^2 \quad - \quad u_0^i u_0^j \end{array}\right]\,, \tag{4.17}$$

which is the effective Lorentzian metric seen by the perturbations of the phase θ of the condensate wave function, also known as condensate metric.
Inverting $g^{\mu\nu}$ we get the acoustic metric

$$g_{\mu\nu} = \frac{\rho_0}{c}\left[\begin{array}{c|c} -(c^2 \quad - \quad u_0^2) & -u_0^j \\ \hline -u_0^i & \mathbb{1} \end{array}\right]\,, \tag{4.18}$$

hence the acoustic interval is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\rho_0}{c}\left[-c^2\,\mathrm{d}t^2 + (\,\mathrm{d}x^i - u_0^i\,\mathrm{d}t)(\,\mathrm{d}x^j - u_0^j\,\mathrm{d}t)\right]\,, \tag{4.19}$$

and its null curves are of the form

$$\delta_{ij}\left(\frac{\mathrm{d}x^i}{\mathrm{d}t} - u_0^i\right)\left(\frac{\mathrm{d}x^j}{\mathrm{d}t} - u_0^j\right) = c^2\,. \tag{4.20}$$

Regions where the speed of the condensate u_0 exceeds the speed of sound c very closely resemble the kinematic properties of black hole physics.

Low-momentum approximation

Now we restore the quantum potential and go to the low-momentum approximation, namely we suppose that the gradients of the background quantities are much bigger than the gradients of the fluctuations (since these last terms contain both a factor of $\hbar$ and a factor of the linearization parameter ϵ). The second equation in (4.9) becomes

$$\begin{aligned} \partial_t \theta_1 &= -\frac{\rho_1}{2} - \nabla\theta_0 \cdot \nabla\theta_1 - \frac{\rho_1}{4\sqrt{\rho_0^3}} \nabla^2 \sqrt{\rho_0} + \frac{\rho_1}{4\sqrt{\rho_0}} \nabla^2 \left(\frac{1}{\sqrt{\rho_0}} \right) \\ &= -\nabla\theta_0 \cdot \nabla\theta_1 - \rho_1 \left(\frac{1}{2} + \frac{1}{4\rho_0^2} \nabla^2(\rho_0) - \frac{|\nabla\rho_0|^2}{4\rho_0^3} \right), \end{aligned} \tag{4.21}$$

from which we can easily isolate ρ_1

$$\rho_1 = -\left(\frac{1}{2} + \frac{1}{4\rho_0^2} \nabla^2(\rho_0) - \frac{|\nabla\rho_0|^2}{4\rho_0^3} \right)^{-1} (\partial_t\theta_1 + \nabla\theta_0 \cdot \nabla\theta_1), \tag{4.22}$$

hence now the matrix $f^{\mu\nu}$ has the form

$$\begin{aligned} f^{00} &= -\left(\frac{1}{2} + \frac{1}{4\rho_0^2} \nabla^2(\rho_0) - \frac{|\nabla\rho_0|^2}{4\rho_0^3} \right)^{-1}, \\ f^{0j} &= -\left(\frac{1}{2} + \frac{1}{4\rho_0^2} \nabla^2(\rho_0) - \frac{|\nabla\rho_0|^2}{4\rho_0^3} \right)^{-1} \nabla_j\theta_0, \\ f^{i0} &= -\nabla_i\theta_0 \left(\frac{1}{2} + \frac{1}{4\rho_0^2} \nabla^2(\rho_0) - \frac{|\nabla\rho_0|^2}{4\rho_0^3} \right)^{-1}, \\ f^{ij} &= \rho_0 - \nabla_i\theta_0 \left(\frac{1}{2} + \frac{1}{4\rho_0^2} \nabla^2(\rho_0) - \frac{|\nabla\rho_0|^2}{4\rho_0^3} \right)^{-1} \nabla_j\theta_0, \end{aligned} \tag{4.23}$$

and if we now set the speed of sound to be

$$c^2 = \rho_0 \left(\frac{1}{2} + \frac{1}{4\rho_0^2} \nabla^2(\rho_0) - \frac{|\nabla\rho_0|^2}{4\rho_0^3} \right) = c^2_{class} \left(1 + \frac{1}{2\rho_0^2} \nabla^2(\rho_0) - \frac{|\nabla\rho_0|^2}{2\rho_0^3} \right), \tag{4.24}$$

we can still recover the same conclusions as in the semi-classical approximation.

Eikonal approximation

The eikonal approximation is a high-momentum approximation where both θ_1 and ρ_1 are written in terms of a slowly varying complex amplitude and a rapidly varying phase, equal for both quantities:

$$\begin{aligned} \theta_1 &= \Re(\mathcal{A}_\theta e^{-i\varphi}); \\ \rho_1 &= \Re(\mathcal{A}_\rho e^{-i\varphi}). \end{aligned} \tag{4.25}$$

Neglecting gradients of the amplitude and of the background fields, small with respect to gradients of φ, and denoting the derivatives of this phase with

$$\omega = \frac{\partial \varphi}{\partial t} \qquad \text{and} \qquad \boldsymbol{k} = \nabla \varphi \,, \tag{4.26}$$

then again we obtain a function matrix $f^{\mu\nu}$

$$\begin{aligned} f^{00} &= -\left(\frac{1}{2} + \frac{k^2}{4\rho_0}\right)^{-1} , \\ f^{0j} &= -\left(\frac{1}{2} + \frac{k^2}{4\rho_0}\right)^{-1} \nabla^j \theta_0 \,, \\ f^{i0} &= -\nabla^i \theta_0 \left(\frac{1}{2} + \frac{k^2}{4\rho_0}\right)^{-1} , \\ f^{ij} &= \rho_0 \delta^{ij} - \nabla^i \theta_0 \left(\frac{1}{2} + \frac{k^2}{4\rho_0}\right)^{-1} \nabla^j \theta_0 \,, \end{aligned} \tag{4.27}$$

and the wave equation $\partial_\mu (f^{\mu\nu} \partial_\nu \theta_1) = 0$ becomes $f^{\mu\nu} \partial_\mu \partial_\nu \theta_1 = 0$, that simply takes the form of a non-linear dispersion relation

$$f^{00} \omega^2 - +(f^{0i} + f^{i0}) \omega k_i + f^{ij} k_i k_j = 0 \,, \tag{4.28}$$

from which we obtain

$$(\omega - u_0^i k_i)^2 = \frac{\rho_0 k^2}{2} + \frac{k^4}{4} \,, \tag{4.29}$$

hence

$$\omega = u_0^i k_i \pm \sqrt{c^2 k^2 + \left(\frac{k^2}{2}\right)^2} \,. \tag{4.30}$$

Notice that this dispersion relation is in agreement with the Bogoliubov dispersion relation for collective excitations of a homogeneous Bose gas as $T \to 0$ [Bo47], and notice that it admits two different regimes. For low-momenta k we obtain the standard phonon dispersion relation $\omega \approx ck$; while in the high-momentum limit we obtain $\omega \approx k^2/2$, telling us that the quasi-particle energy tends to the kinetic energy of an individual gas particle.

4.2 Straight vortex case

Here I prove that under the assumption of a background flow characterized by the presence of a straight vortex defect along the $z-$axis the operator appearing in (4.10) is indeed invertible, and thus it is possible to extract the effective Lorentzian metric seen by the velocity potential phononic perturbations [Ro22].

4.2.1 Straight vortex solution with Padé approximation

We now focus on the straight vortex steady state and fix the vortex line to lay on the $z-$axis. We know that the azimuthal velocity is of the form

$$u_\theta = \frac{\Gamma}{2\pi r}\,, \tag{4.31}$$

where Γ is the quantized circulation of the vortex. The quantization condition gives $\Gamma = 2\pi n$, where in principle n can be any integer. However, physical observation showed that vortex lines with $n > 1$ are unstable and decay into n vortices with $\Gamma = 2\pi$. For this reason we set $\Gamma = 2\pi$ and we get

$$\boldsymbol{u} = \nabla\theta = \frac{1}{r}\,\hat{\theta}\,, \tag{4.32}$$

so that the velocity vector field potential, namely Madelung's phase, will be just the azimuthal angle θ. This shows that the velocity field is divergenceless ($\nabla\cdot\boldsymbol{u} = 0$) so that, under steady conditions, the continuity equation ensures that $\nabla\theta\cdot\nabla\rho = 0$, which is satisfied if $\nabla\rho \perp \nabla\theta$, and this condition, together with symmetry considerations, ensures that $\rho = \rho(r)$, and let us conclude, using (B.10), that the density ρ will satisfy the equation

$$\frac{\nabla^2\sqrt{\rho}}{2\sqrt{\rho}} - \frac{1}{2}|\nabla\theta|^2 - \frac{\rho}{2} + \frac{1}{2} = 0 \quad\Longrightarrow\quad \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = \frac{1}{r^2} + \rho - 1\,. \tag{4.33}$$

Writing the Laplacian in cylindrical coordinates we easily get that, in order to find the density, we need to solve a second order differential equation:

$$\rho'' + \frac{\rho'}{r} - \frac{(\rho')^2}{2\rho} + 2\rho\left(1 - \rho - \frac{1}{r^2}\right) = 0\,, \tag{4.34}$$

with boundary conditions $\rho(0) = 0$ and $\lim_{r\to\infty}\rho = 1$.

A standard numerical technique to solve this equation is to use a Padé approximation [Be04]:

$$\rho(r) \approx \frac{\sum_{j=0}^{p} a_j r^{2j}}{1 + \sum_{k=1}^{q} b_k r^{2k}}\,. \tag{4.35}$$

In 2017 Caliari and Zuccher [CZ17] showed that the best fit in solving our differential equation is attained when $q = p = 4$. For this reason we will consider as our starting point the density profile ρ given by their particular solution (see Figure 4.1).

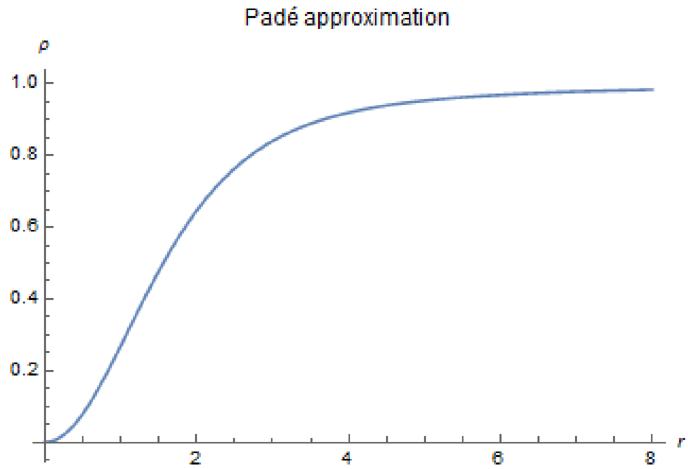

Figure 4.1: Density profile for the steady straight vortex along the $z-$axis given by the Padé approximation with $q = p = 4$. Notice that it presents an inflection point at $r = 1.08605$, comparable with the healing length.

4.2.2 Analysis of the operator

We consider the operator that arises in the equation of the phase fluctuation and that we need to invert in order to isolate ρ_1 as in (4.10) and obtain the metric (4.13):

$$\mathcal{D}\rho_1 = -\left(\frac{1}{2} + \frac{1}{4\rho_0\sqrt{\rho_0}}\nabla^2\sqrt{\rho_0} - \frac{1}{4\sqrt{\rho_0}}\nabla^2\left(\frac{\cdot}{\sqrt{\rho_0}}\right)\right)\rho_1 \,. \tag{4.36}$$

From now on we will consider a background density ρ_0 given by the steady straight vortex solution in Figure 4.1, and we will rewrite the operator in a clearer way.

$\mathcal{D}$ acting on ρ_1

If we want to rewrite the Laplacian term in the form of an operator acting only on ρ_1 then we need to use the fact that

$$\begin{aligned}\nabla^2\left(\frac{\rho_1}{\sqrt{\rho_0}}\right) &= \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\left(\frac{\rho_1}{\sqrt{\rho_0}}\right)\right) \\ &= \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\frac{\partial_j\rho_1\cdot\sqrt{\rho_0}-\partial_j\sqrt{\rho_0}\cdot\rho_1}{\rho_0}\right) \\ &= \frac{\nabla^2\rho_1}{\sqrt{\rho_0}} - \frac{1}{\rho_0\sqrt{\rho_0}}\nabla\rho_0\cdot\nabla\rho_1 - \frac{\nabla^2\sqrt{\rho_0}}{\rho_0}\rho_1 + \frac{|\nabla\rho_0|^2}{2\rho_0^2\sqrt{\rho_0}}\rho_1 \,.\end{aligned} \tag{4.37}$$

Hence, we can define a new operator $\tilde{\mathcal{D}}$ of the form

$$\tilde{\mathcal{D}}(\rho_1) = -\left(\frac{1}{2} + \frac{1}{2\rho_0\sqrt{\rho_0}}\nabla^2\sqrt{\rho_0} - \frac{1}{8\rho_0^3}|\nabla\rho_0|^2\right)\rho_1 + \frac{1}{4\rho_0}\nabla^2\rho_1 - \frac{1}{2\rho_0^2}\nabla\rho_0\cdot\nabla\rho_1 \,, \tag{4.38}$$

acting explicitly on ρ_1. However, the operator written in this form is not easy to interpret or to invert, hence we try to find a better expression for the same operator in order to get a simpler invertibility condition.

$\mathcal{D}$ acting on $\rho_1/\sqrt{\rho_0}$

We now rewrite the operator $\mathcal{D}$ as a new operator $\bar{\mathcal{D}}$ directly acting on $\rho_1/\sqrt{\rho_0}$, so that we have

$$\bar{\mathcal{D}}\left(\frac{\rho_1}{\sqrt{\rho_0}}\right) = -\left(\frac{\sqrt{\rho_0}}{2} + \frac{\nabla^2\sqrt{\rho_0}}{4\rho_0} - \frac{1}{4\sqrt{\rho_0}}\nabla^2\right)\left(\frac{\rho_1}{\sqrt{\rho_0}}\right) . \tag{4.39}$$

We can extract the term $\dfrac{1}{4\sqrt{\rho_0}}$ and we find

$$\bar{\mathcal{D}}\left(\frac{\rho_1}{\sqrt{\rho_0}}\right) = -\frac{1}{4\sqrt{\rho_0}}\left(\frac{\nabla^2\sqrt{\rho_0}}{\sqrt{\rho_0}} + 2\rho_0 - \nabla^2\right)\left(\frac{\rho_1}{\sqrt{\rho_0}}\right) , \tag{4.40}$$

which we can clearly see that is given in the form $-k(\mathcal{V} - \nabla^2)$ where $\mathcal{V}$ is a multiplicative term and ∇^2 is the Laplacian. Moreover, if we consider ρ_0 to be the vortex line solution of the GPE, then we can use the condition (4.33) and we can rewrite the operator as

$$\bar{\mathcal{D}}\left(\frac{\rho_1}{\sqrt{\rho_0}}\right) = -\frac{1}{4\sqrt{\rho_0}}\left[\left(\frac{1}{r^2} - 1 + 3\rho_0\right) - \nabla^2\right]\left(\frac{\rho_1}{\sqrt{\rho_0}}\right) , \tag{4.41}$$

thus obtaining the simplest form

$$\bar{\mathcal{D}} = -k(\mathcal{V} - \nabla^2) \quad \text{with} \quad k = \frac{1}{4\sqrt{\rho_0}} \quad \text{and} \quad \mathcal{V} = \left(\frac{1}{r^2} - 1 + 3\rho_0\right) . \tag{4.42}$$

4.2.3 Invertibility of the operator

After rearranging the operator in a more suitable way, we prove invertibility. To this end we need to prove first the following

Theorem 5. *Given a differential operator in the form $H = \mathcal{V} - \nabla^2$, if the multiplicative potential $\mathcal{V} = \mathcal{V}(r)$ is strictly positive and $\lim_{r\to\infty} \mathcal{V}(r) = \mathcal{V}^* > 0$, then the operator is injective and thus invertible.*

Proof. If the operator $H = \mathcal{V} - \nabla^2$ has no eigenvalues then it is injective since 0 is not an eigenvalue. Now, without loss of generality we can consider that H has eigenvalues; its lowest eigenvalue E_0 can be considered the inf of the following functional ([LL15], Ch. 11):

$$\mathcal{E}(\psi) = \int |\nabla\psi|^2 \, \mathrm{d}x + \int \mathcal{V}(x)|\psi(x)|^2 \, \mathrm{d}x = \langle \psi, -\nabla^2\psi \rangle + \langle \psi, \mathcal{V}\psi \rangle \,, \tag{4.43}$$

on the set $\{\psi \in L^2(D) : \int |\psi(x)|^2 \, \mathrm{d}x = 1\}$. Since $-\nabla^2$ is positive definite, if $\mathcal{V}$ is bounded from below by a positive constant c then we have

$$\mathcal{E}(\psi) \geq \langle \psi, \mathcal{V}\psi \rangle \geq c\langle \psi, \psi \rangle = c \,. \tag{4.44}$$

Taking the inf of both sides we obtain $E_0 \geq c$, so that 0 can't be an eigenvalue of H, hence the operator is injective and invertible. □

As a result, let us substitute the Padé approximation for ρ_0 given in Figure 4.1 into $\mathcal{V}$; the function we obtain is presented in the plot of Figure 4.2. $\mathcal{V}$ is evidently strictly positive with minimum given by 0.760347 at $r = 1.15929$, and limit $\mathcal{V}^* = 2$. Hence we have that the operator $\bar{\mathcal{D}}$ is indeed invertible.

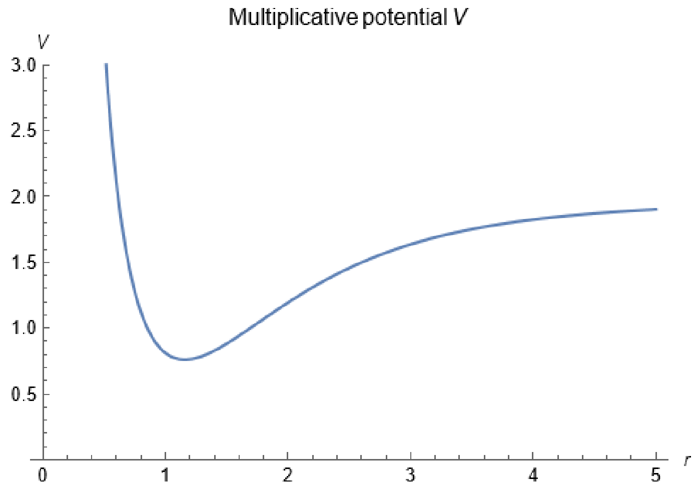

Figure 4.2: $\mathcal{V} = \left(1/r^2 - 1 + 3\rho_0\right)$ plotted against r; ρ_0 is given by the Padé approximation. The minimum 0.760347 is at $r = 1.15929$ and $\lim_{r\to\infty} \mathcal{V}(r) = 2$.

We conclude that, in the case of a steady straight vortex defect, we can actually find ρ_1 in terms of the phase fluctuation, that is

$$\rho_1 = \sqrt{\rho_0}(\mathcal{V} - \nabla^2)^{-1} \left[-4\sqrt{\rho_0}(\partial_t\theta_1 + \nabla\theta_0 \cdot \nabla\theta_1)\right] \,, \tag{4.45}$$

hence find a D'Alembertian equation for the phase perturbation θ_1 that yields the emergence of an effective acoustic Lorentzian metric.

4.2.4 Future developments

In order to explicitly invert the operator $H = \mathcal{V} - \nabla^2$ we can consider some approximations. In the vicinity of the vortex line we can take $\rho_0 \to 0$ as $r \to 0$ obtaining $H = (1/r^2 - \nabla^2)$. We can then assume that density perturbations depend only on r, that is $\rho_1 = \rho_1(r)$; hence, the Laplacian reduces to its $r-$dependent part, giving

$$H = \left(\frac{1}{r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\mathrm{d}^2}{\mathrm{d}r^2} \right) . \tag{4.46}$$

The operator (4.46) is a Bessel operator of dimension 2, whose inverse is given by the Green function obtained through the Bessel functions of first kind $J_1(x)$ and second kind $Y_1(x)$. This work represents a starting point in the direction of further analysis on linearization, which seems to be a very fruitful field of research in the context of analogue black holes.

Chapter 5

Concluding remarks

Here I want to recall our new results and outline some directions for future research. We started from deriving the Gross-Pitaevskii equation and its hydrodynamic description in a generic Riemannian manifold, obtaining that the geometry of the manifold enters the dynamics through an external force related to the first derivatives of the density profile and to the geometry of the manifold through its Ricci tensor and scalar. Obviously, no such term is present on a flat space; but it becomes particularly relevant on curved space when we approach a vortex defect at a distance of the healing length. Here the density profile has an inflection point and a maximum for the gradient. Moreover, in the case of negative scalar curvature, the density curvature vector becomes increasingly negative in the outer part of the healing region, thus contributing to the natural trapping of acoustic waves. Since acoustic waves in condensates are analogues to light waves in black hole theory, this mechanism seems to enforce the analogy between defect properties in condensates and black hole dynamics in cosmology.
Another relation with cosmology is given by the momentum equation, showing that it is no longer conserved in a general situation, because of the presence of a correction term related to the geometry of the manifold through the Ricci scalar and to the first derivatives of the density profile. If this correction term can be written as the divergence of a tensor or pseudo-tensor, as in the case of a constant Ricci scalar, then we recover the standard conservation law (the same situation can be found in [Pe82]). Finally, a relation with general relativity is obtained by imposing a steady state condition, which leads to a new type of Einstein field equations.
Studying these equations we realize that they are strictly related to negative scalar curvature surfaces; but this is not the only connection between BECs governed by the GPE and

negative constant curvature surfaces. These surfaces are in one to one correspondence with soliton solutions of the sine-Gordon equation, and the sine-Gordon model is widely used in the study of BECs; indeed, we obtain the sine-Gordon equation as an approximation of the GPE for a system of two coupled BECs.
Again studying the connections between BECs and cosmology we realize that many cosmological models make use of relativistic BECs; hence we go through the relativistic derivation of the GPE, using the tools obtained for generic Riemannian manifolds, extracting the generic Klein-Gordon equation and the corresponding semi-relativistic GPE, studying the hydrodynamic formulation both in the Minkowski spacetime and in presence of a weak gravitational field.
Because of the emergence of strong analogies between vortices in BECs and black hole dynamics, we move to the study of analogue models. After having analyzed the linearization procedure, necessary to extract the effective Lorentzian acoustic metric needed to have Hawking radiation and black hole formation, we decide to focus on the interesting case of a straight vortex placed along the $z-$axis. We study the invertibility conditions necessary for the Lorentzian metric and some possible approximations in order to extract a proper solution.
We conclude this chapter by outlining some interesting aspects that we intend to develop in the near future.

5.1 Einstein field equations on a pseudosphere

If we consider to be on a Riemannian pseudospherical manifold, then the Einstein field equations (1.82) and (1.81) become

$$\begin{cases} R = -\dfrac{2}{\xi^2} \\ \\ u_i u_j = \dfrac{\hbar^2}{4m^2} \mathrm{Hess}_{ij}(\ln \rho) \end{cases} \tag{5.1}$$

and we can easily see that the first equation is automatically satisfied, while the second simply gives a relationship between the density ρ and the velocity field. In the near future we intend to numerically prove that this request is satisfied by letting the condensate evolve under the GPE with some specific density curvature vector $\mathbf{f}$.

5.2 Relativistic Gross-Pitaevskii equation and sine-Gordon

If we consider the relativistic equation for the phase (3.14) and refer to an isophase surface, then the equation, written in terms of $A = \sqrt{\rho}$, becomes

$$\frac{1}{c^2}\partial_t^2 A - \nabla^2 A = -\frac{2\mathfrak{g}m}{\hbar^2}A^3 - \frac{2Vm}{\hbar^2}A + \kappa R A\,, \tag{5.2}$$

from which we can see the right-hand side as an approximation of a $-\sin A$. In particular if we impose the Einstein equation for the Ricci scalar (3.25) we can write

$$\begin{aligned}\frac{1}{c^2}\partial_t^2 A - \nabla^2 A &= -\frac{2Vm}{\hbar^2}A - \frac{2\mathfrak{g}m}{\hbar^2}\left(1 + \frac{4\kappa}{1+4\kappa}\right)A^3 \\ &= -\frac{2Vm}{\hbar^2}\left(A + \frac{\mathfrak{g}}{V}\frac{1+8\kappa}{1+4\kappa}A^3\right)\,.\end{aligned} \tag{5.3}$$

By setting

$$\frac{\mathfrak{g}}{V}\left(\frac{1+8\kappa}{1+4\kappa}\right) = -\frac{1}{6}\,, \tag{5.4}$$

we can compactly write

$$\frac{1}{c^2}\partial_t^2 A - \nabla^2 A \approx -\frac{2Vm}{\hbar^2}\sin A\,, \tag{5.5}$$

which is again a sine-Gordon equation. The approximation is valid as long as A is small, hence it is valid inside the vortex tube placed around defects where all the isophase surface are hinged.

5.3 Isophase surfaces and Klein-Gordon equation

Some observations can be done on the geometry of isophase surfaces generated by a vortex defect. If we consider the time-independent GPE in absence of trapping potential ($V = 0$) and in flat space ($R = 0$), we find that (1.15) becomes

$$\frac{\hbar^2}{2m}\nabla^2\Psi = \mathfrak{g}|\Psi|^2\Psi\,, \tag{5.6}$$

which can be written as

$$\nabla^2\Psi = \frac{2m\mathfrak{g}}{\hbar^2}|\Psi|^2\Psi = \frac{1}{\xi^2}\Psi\,, \tag{5.7}$$

obtaining a homogeneous screened Poisson equation.

If we instead suppose to be on a pseudospherical manifold with negative Gaussian curvature $K = -1/\xi^2$ we can write (1.15) as

$$\nabla^2\Psi = -K(1-2\kappa)\Psi \,. \tag{5.8}$$

Substituting the Madelung transform $\Psi = \sqrt{\rho}e^{i\theta}$ into (5.7) we obtain the hydrodynamic equations

$$\begin{cases} \text{div}(\rho\nabla\theta) = 0 \\ \dfrac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} - |\nabla\theta|^2 = \dfrac{1}{\xi^2}(1-2\kappa) \end{cases} \tag{5.9}$$

If we assume to be far from the defect we can take ρ to be constant ($\rho \to 1$), obtaining

$$\begin{cases} \nabla^2\theta = 0 \\ |\nabla\theta|^2 = -\dfrac{1}{\xi^2}(1-2\kappa) \end{cases} \tag{5.10}$$

This point of view allows us to study properties of isophase surfaces far from the defect. Our future aim is to study the motion of such surfaces by specifying tangential and normal velocity according to physics, thus obtaining time evolution equations for the metric $g_{\mu\nu}$, the second fundamental form $h_{\mu\nu}$, and Gaussian curvature K, in the way presented by Nakayama and Wadati [NW93] (extending the model proposed by Lund and Regge for the motion of curves [LR76], and then studied by Nakayama, Segur, Wadati [NSW92]). In particular, a possible direction for our research could be the study of the evolution of the minimal area surface hinged on the defect; indeed, it has been numerically observed that the area of such a surface decreases when evolving under GPE. This fact, if confirmed, would provide a condition for a privileged direction of evolution.

5.4 Superimposed twist and the modified GPE

Finally, we want to move further in the study of defects by adding a twist to the vortex, both locally and globally, as it was presented in [FR20]. In particular, adding a twisted phase θ_{tw} to the solution ψ_0 of the non-dimensional GPE (B.6), we can define

$$\psi_{tw} = \psi_0 \, e^{i\theta_{tw}(\mathbf{r},t)} \,, \tag{5.11}$$

which satisfies a modified GPE (given in non-dimensional form)

$$
\begin{aligned}
i\partial_t \psi_{tw} &= (i\partial_t \psi_0) e^{i\theta_{tw}} + i\psi_0 \partial_t (e^{i\theta_{tw}}) \\
&= -\frac{1}{2}(\nabla^2 \psi_0) e^{i\theta_{tw}} - \frac{1}{2}(1 - |\psi_{tw}|^2)\psi_{tw} - (\partial_t \theta_{tw})\psi_{tw} \\
&= -\frac{1}{2}(\nabla - i\nabla\theta_{tw})^2 \psi_{tw} - \frac{1}{2}(1 - |\psi_{tw}|^2)\psi_{tw} - (\partial_t \theta_{tw})\psi_{tw} \, ,
\end{aligned}
\tag{5.12}
$$

where we used the fact that

$$
\begin{aligned}
(\nabla - i\nabla\theta_{tw})^2 \psi_{tw} &= (\nabla - i\nabla\theta_{tw}) \left[\nabla\psi_0 \, e^{i\theta_{tw}} + i\nabla\theta_{tw}\psi_0 e^{i\theta_{tw}} - i\nabla\theta_{tw}\psi_0 e^{i\theta_{tw}} \right] \\
&= \nabla^2 \psi_0 \, e^{i\theta_{tw}} + i\nabla\psi_0 \cdot \nabla\theta_{tw} e^{i\theta_{tw}} - i\nabla\theta_{tw} \cdot \nabla\psi_0 e^{i\theta_{tw}} \\
&= (\nabla^2 \psi_0) \, e^{i\theta_{tw}} \, .
\end{aligned}
\tag{5.13}
$$

We can easily obtain the hydrodynamic formulation by noticing that it is still valid for $\psi_0 = \sqrt{\rho} e^{i(\theta - \theta_{tw})}$, hence we obtain

$$
\begin{cases}
\partial_t \rho = -\mathrm{div}(\rho \nabla(\theta - \theta_{tw})) \\
\partial_t (\theta - \theta_{tw}) + \frac{1}{2}|\nabla(\theta - \theta_{tw})|^2 = \frac{\nabla^2 \rho}{2\sqrt{\rho}} + \frac{1}{2}(1 - \rho)
\end{cases}
\tag{5.14}
$$

while the energy is given by the expression

$$
E_{tw} = \int \left(-\partial_t \theta_{tw} |\psi_{tw}|^2 + \frac{1}{2}|(\nabla - i\nabla\theta_{tw})\psi_{tw}|^2 - \frac{1}{2}|\psi_{tw}|^2 + \frac{1}{4}|\psi_{tw}|^4 \right) \mathrm{d}^3\mathbf{r} \, , \tag{5.15}
$$

that, through Madelung's transform $\psi_{tw} = \sqrt{\rho} e^{i\theta}$, takes the form

$$
E_{tw} = \int \left((-\partial_t \theta_{tw})\rho + \frac{1}{8\rho}|\nabla\rho|^2 + \frac{1}{2}\rho|\nabla\theta|^2 + \frac{1}{2}\rho|\nabla\theta_{tw}|^2 - \rho\nabla\theta \cdot \nabla\theta_{tw} - \frac{1}{2}\rho + \frac{1}{4}\rho^4 \right) \mathrm{d}^3\mathbf{r} \, . \tag{5.16}
$$

A possible future direction of research is the study of the relationship between superimposed twist and twisted negative curvature surfaces, also known as Dini surfaces as briefly presented in Appendix (C).

Appendix A

Topics in Riemannian geometry

To fix notation let M be a Riemannian manifold with metric g and let $\{x^i\}$ be a system of local coordinates on it. For each $p \in M$ we have a basis for the tangent space T_pM that we denote with $\{\partial_i\}$ and we have a basis for the cotangent space T_p^*M (namely for the 1-forms) that we denote with $\{\,\mathrm{d}x^i\,\}$, defined by: $\mathrm{d}x^i(\boldsymbol{v}) = v^i$, where v^i is the coefficient of ∂_i in the decomposition in $\boldsymbol{v}$ on that basis. Hence, for each vector $\boldsymbol{v}$ we can write

$$\boldsymbol{v} = v^i\,\partial_i = \mathrm{d}x^i(\boldsymbol{v})\,\partial_i\,. \tag{A.1}$$

In particular the metric g can be represented in these coordinates as a matrix whose entries are scalar products of the elements of the basis:

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\,, \tag{A.2}$$

and, since it is a $(0,2)-$tensor, it can be written as

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j\,. \tag{A.3}$$

Moreover we denote with g^{ij} the inverse matrix and with $|g|$ the modulus of its determinant. Hence we have $g^{ij} = g(\,\mathrm{d}x^i,\,\mathrm{d}x^j)$ and obviously $g^{im}g_{mj} = \delta^i{}_j\,$.
By taking the derivative on both sides of the last expression we get

$$(\partial_k\,g^{im})g_{mj} + g^{im}(\partial_k\,g_{mj}) = 0 \quad\Longrightarrow\quad \partial_k\,g^{ij} = -g^{im}\,(\partial_k\,g_{mn})\,g^{nj}\,. \tag{A.4}$$

Musical isomorphism

It is well known that there is a classical isomorphism between the tangent bundle TM and the cotangent bundle T^*M that takes vector fields into 1-forms by using the metric g. Specifically, given a vector field $\boldsymbol{v}$ we define the correspondent 1-form α as

$$\alpha = g(\boldsymbol{v}, \cdot) \quad \Longrightarrow \quad \alpha(\boldsymbol{u}) = g(\boldsymbol{v}, \boldsymbol{u})\,. \tag{A.5}$$

The function that maps a vector to its correspondent 1-form is the *flat* map

$$\flat : TM \to T^*M\,, \tag{A.6}$$

while its inverse is the *sharp* map

$$\sharp : T^*M \to TM\,. \tag{A.7}$$

For each vector field $\boldsymbol{v} = v^i \partial_i$ we define $\flat$ as

$$v^\flat = g_{ij} v^i \, \mathrm{d}x^j\,, \tag{A.8}$$

and for each 1-form $\alpha = \alpha_i \, \mathrm{d}x^i$ we define $\sharp$ as its inverse

$$\alpha^\sharp = g^{ij} \alpha_i \, \partial_j\,, \tag{A.9}$$

hence

$$v_j = g_{ij} v^i \qquad \text{and} \qquad \alpha^j = g^{ij} \alpha_i\,. \tag{A.10}$$

This procedure is also known as "lowering" or "raising" an index.
We can extend this procedure to all tensors, in fact the musical isomorphism allows one to take any covariant index into a contravariant one and vice versa by simply applying the metric tensor, in particular we lower an index by applying the metric and raise an index by applying the inverse of the metric.
Moreover, by using this isomorphism, we can define the trace of a $(0,2)-$tensor $T = T_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j$ as the trace of the corresponding matrix obtained by raising the second index

$$\mathrm{tr}(T) = \mathrm{tr}(g^{jk} T_{ij} \, \mathrm{d}x^i \otimes \partial_k) = g^{ji} T_{ij}\,. \tag{A.11}$$

Gradient of a function

Now let $f : M \to \mathbb{R}$ be a function. We first define its differential as a 1-form given in coordinates by

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i\,, \tag{A.12}$$

hence we define the gradient of f as the correspondent vector field (dual to the 1-form $\mathrm{d}f$) obtained by applying the musical isomorphism

$$\mathrm{d}f(\boldsymbol{v}) = g(\nabla f, \boldsymbol{v})\,. \tag{A.13}$$

In local coordinates it can be written as

$$\frac{\partial f}{\partial x^j}\,\mathrm{d}x^j(\boldsymbol{v}) = g_{ij}\,\mathrm{d}x^i(\nabla f)\,\mathrm{d}x^j(\boldsymbol{v})\,, \tag{A.14}$$

from which we find that

$$\frac{\partial f}{\partial x^j} = g_{ij}\,\mathrm{d}x^i(\nabla f) \qquad \Longrightarrow \qquad \mathrm{d}x^i(\nabla f) = g^{ij}\,\frac{\partial f}{\partial x^j}\,. \tag{A.15}$$

Hence we have that the gradient of a function in coordinates can be written as

$$\nabla f = g^{ij}\partial_j f\;\partial_i\,, \tag{A.16}$$

which can be compactly written as

$$\nabla f = (\,\mathrm{d}f)^\sharp\,. \tag{A.17}$$

Using this description we can now calculate the scalar product of the gradients of two given functions f and h in coordinates and we get

$$g(\nabla f, \nabla h) = \,\mathrm{d}f(\nabla h) = \partial_i f\,\mathrm{d}x^i\left(g^{ij}\partial_j h\;\partial_i\right) = \partial_i f\,g^{ij}\,\partial_j h\,. \tag{A.18}$$

In particular, for the square modulus of a gradient we have

$$|\nabla f|^2 = g^{ij}\,\partial_i f\,\partial_j f\,. \tag{A.19}$$

Lie-derivative

Now that we know how to take the gradient of functions, we learn how to derive vector fields on manifolds. Firstly we give the definition of a *Lie derivative*:

Definition 1. *A Lie derivative is an operator that takes in input two vector fields (*$\boldsymbol{v}$ *and* $\boldsymbol{u}$*) and gives as an output another vector field defined by the commutator of the two fields:*

$$\mathcal{L}_{\boldsymbol{v}}\boldsymbol{u} = [\boldsymbol{v}, \boldsymbol{u}] = \boldsymbol{v}(\boldsymbol{u}) - \boldsymbol{u}(\boldsymbol{v})\,. \tag{A.20}$$

Writing our vector fields in coordinates and making the Lie derivative act on a function f we get

$$\begin{aligned}\mathcal{L}_{(v^i\partial_i)}(u^j\partial_j)(f) &= v^i\partial_i(u^j\partial_j)f - u^j\partial_j(v^i\partial_i)f \\ &= v^i\frac{\partial u^j}{\partial x^i}\frac{\partial f}{\partial x^j} + v^i u^j\frac{\partial^2 f}{\partial x^i x^j} - u^j\frac{\partial v^i}{\partial x^j}\frac{\partial f}{\partial x^i} - u^j v^i\frac{\partial^2 f}{\partial x^j x^i} \\ &= \left[\left(v^j\frac{\partial u^i}{\partial x^j} - u^j\frac{\partial v^i}{\partial x^j}\right)\partial_i\right] f\,,\end{aligned} \tag{A.21}$$

so that we can write the Lie derivative in coordinates as

$$\mathcal{L}_{\boldsymbol{v}}\boldsymbol{u} = \left(v^j\frac{\partial u^i}{\partial x^j} - u^j\frac{\partial v^i}{\partial x^j}\right)\partial_i\,. \tag{A.22}$$

Now we can also define the Lie derivative of a function f along a vector field $\boldsymbol{v}$ as

$$\mathcal{L}_{\boldsymbol{v}}(f) = \mathcal{L}_{(v^i\partial_i)}f = v^i\,\partial_i f\,, \tag{A.23}$$

and the Lie derivative of a 1-form by specifying

$$\mathcal{L}_{\boldsymbol{v}}(\,\mathrm{d}x^j) = \mathcal{L}_{v^i\partial_i}(\,\mathrm{d}x^j) = \frac{\partial v^j}{\partial x^i}\,\mathrm{d}x^i\,. \tag{A.24}$$

We can now extend the definition to all tensors just asking the derivative to respect a sort of "Leibniz rule":

$$\mathcal{L}_{\boldsymbol{v}}(T\otimes S) = \mathcal{L}_{\boldsymbol{v}}T\otimes S + T\otimes\mathcal{L}_{\boldsymbol{v}}S\,. \tag{A.25}$$

For example we can take the Lie derivative of the metric tensor and get

$$\mathcal{L}_{\boldsymbol{v}}(g) = \mathcal{L}_{v^i\partial_i}(g_{rs}\,\mathrm{d}x^r\otimes\,\mathrm{d}x^s) = (v^i\,\partial_i g_{rs} + g_{is}\,\partial_r v^i + g_{ri}\,\partial_s v^i)\,\mathrm{d}x^r\otimes\,\mathrm{d}x^s\,. \tag{A.26}$$

Divergence, Laplacian and Hessian

Divergence

We define the divergence of a vector field by checking how the volume form $\mathrm{d}vol = \sqrt{|g|}\,\mathrm{d}x^1 \otimes \ldots \otimes \mathrm{d}x^n$ changes along the flow of the vector field. To do that we compute the Lie derivative of the volume form along the vector field:

$$\mathcal{L}_{\boldsymbol{v}}\,\mathrm{d}vol = \mathrm{div}(\boldsymbol{v})\,\mathrm{d}vol\,. \tag{A.27}$$

In particular, notice that

$$\begin{aligned}\mathcal{L}_{v^i\partial_i}\,\mathrm{d}vol &= \mathcal{L}_{v^i\partial_i}(\sqrt{|g|}\,\mathrm{d}x^1 \otimes \ldots \otimes \mathrm{d}x^n)\\ &= v^i\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\,\mathrm{d}vol + \sqrt{|g|}\sum_{j=1}^{n}\mathrm{d}x^1 \otimes \ldots \otimes \frac{\partial v^j}{\partial x^j}\,\mathrm{d}x^j \otimes \ldots \otimes \mathrm{d}x^n\\ &= \frac{1}{\sqrt{|g|}}\sum_{j=1}^{n}\frac{\partial(\sqrt{|g|}v^j)}{\partial x^j}\,\mathrm{d}vol\,.\end{aligned} \tag{A.28}$$

Hence, omitting the sum on the repeated indices, we can write the divergence of a vector field as

$$\mathrm{div}(\boldsymbol{v}) = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}v^i)\,. \tag{A.29}$$

Notice that, if we multiply the vector field times a scalar function, we get the property

$$\begin{aligned}\mathrm{div}(f\boldsymbol{v}) &= \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}f\,v^i)}{\partial x^i}\\ &= f\frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}v^i)}{\partial x^i} + \frac{1}{\sqrt{|g|}}\frac{\partial f}{\partial x^i}\sqrt{|g|}v^i\\ &= f\mathrm{div}(\boldsymbol{v}) + \mathrm{d}f(\boldsymbol{v})\,.\end{aligned} \tag{A.30}$$

Laplacian

We define the Laplacian of a function f as the divergence of its gradient, so that we get

$$\nabla^2 f = \mathrm{div}(\nabla f) = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right)\,. \tag{A.31}$$

Hessian

We define the Hessian of f as a half of the Lie derivative of the metric tensor along ∇f:

$$\begin{aligned}
\mathrm{Hess} f &= \frac{1}{2}\mathcal{L}_{\nabla f} g \\
&= \frac{1}{2}\left[g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial g_{rs}}{\partial x^i} + g_{is}\frac{\partial}{\partial x^r}\left(g^{ij}\frac{\partial f}{\partial x^j}\right) + g_{ri}\frac{\partial}{\partial x^s}\left(g^{ij}\frac{\partial f}{\partial x^j}\right)\right] \mathrm{d}x^r \otimes \mathrm{d}x^s \\
&= \frac{1}{2}\left[g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial g_{rs}}{\partial x^i} + g_{is}\frac{\partial g^{ij}}{\partial x^r}\frac{\partial f}{\partial x^j} + g_{is}g^{ij}\frac{\partial^2 f}{\partial x^r \partial x^j} + g_{ri}\frac{\partial g^{ij}}{\partial x^s}\frac{\partial f}{\partial x^j} + g_{ri}g^{ij}\frac{\partial^2 f}{\partial x^s \partial x^j}\right] \mathrm{d}x^r \otimes \mathrm{d}x^s \\
&= \left[\frac{\partial^2 f}{\partial x^r \partial x^s} + \frac{1}{2}\left(g^{ij}\frac{\partial g_{rs}}{\partial x^i} + g_{is}\frac{\partial g^{ij}}{\partial x^r} + g_{ri}\frac{\partial g^{ij}}{\partial x^s}\right)\frac{\partial f}{\partial x^j}\right] \mathrm{d}x^r \otimes \mathrm{d}x^s \\
&= (\mathrm{Hess} f)_{rs}\, \mathrm{d}x^r \otimes \mathrm{d}x^s = \nabla_r \nabla_s f \; \mathrm{d}x^r \otimes \mathrm{d}x^s \,.
\end{aligned} \tag{A.32}$$

In particular we have

$$\nabla^2 f = \mathrm{tr}(\mathrm{Hess} f) = g^{rs}(\mathrm{Hess} f)_{rs} \,. \tag{A.33}$$

Covariant derivative

The Lie derivative is not the only derivative that we can define on a manifold. In particular here we define the *covariant derivative* as a sort of gradient of vector fields:

Definition 2. *The covariant derivative is an operator that takes two vector fields as an input (a direction vector* $\boldsymbol{v}$ *and an input field* $\boldsymbol{u}$*) and gives a new vector field as an output* $\nabla : TM \times TM \to TM,\ (\boldsymbol{v}, \boldsymbol{u}) \mapsto \nabla_{\boldsymbol{v}}\boldsymbol{u}$ *with the following properties:*

- $\nabla_{a\boldsymbol{w}+b\boldsymbol{v}}\boldsymbol{u} = a\,\nabla_{\boldsymbol{w}}\boldsymbol{u} + b\,\nabla_{\boldsymbol{v}}\boldsymbol{u}$ *(linear in the first entry);*
- $\nabla_{\boldsymbol{v}}(\boldsymbol{u} + \boldsymbol{w}) = \nabla_{\boldsymbol{v}}\boldsymbol{u} + \nabla_{\boldsymbol{v}}\boldsymbol{w}$ *(additive in the second entry);*
- $\nabla_{\boldsymbol{v}}(a\boldsymbol{u}) = (\nabla_{\boldsymbol{v}}a)\boldsymbol{u} + a(\nabla_{\boldsymbol{v}}\boldsymbol{u})$ *("Leibniz rule" in the second entry);*
- $\nabla_{\partial_i}(a) = \partial_i a$ *(standard derivative for scalars).*

In particular a covariant derivative is also called a *connection* since it connects different tangent vector spaces, allowing a comparison between vectors defined on different points of the manifold. A covariant derivative is completely determined by its coefficients defined by expanding the output vector field on the basis $\{\partial_i\}$:

$$\nabla_{\partial_i}\partial_j = \Gamma^k{}_{ij}\,\partial_k \,, \tag{A.34}$$

the coefficients ${\Gamma^k}_{ij}$ are known as *Christoffel symbols*.

Levi-Civita connection

From now on, we will focus on a specific connection known as *Levi-Civita connection*:

Definition 3. *A Levi-Civita connection is a connection that is torsion free and compatible with the metric g. In other words it satisfies:*

- $\nabla_{\boldsymbol{v}}\boldsymbol{u} - \nabla_{\boldsymbol{u}}\boldsymbol{v} = [\boldsymbol{v}, \boldsymbol{u}]$ *hence* ${\Gamma^k}_{ij} = {\Gamma^k}_{ji}$*;*
- $\nabla_{\boldsymbol{v}}(\boldsymbol{w} \cdot \boldsymbol{u}) = \nabla_{\boldsymbol{v}}\boldsymbol{w} \cdot \boldsymbol{u} + \boldsymbol{w}\nabla_{\boldsymbol{v}}\boldsymbol{u}$ *hence* $\partial_k g_{ij} = {\Gamma^l}_{ik}\, g_{jl} + {\Gamma^l}_{kj}\, g_{il}$.

Notice that if the connection is torsion free, then the Lie derivative $L_{\boldsymbol{v}}\boldsymbol{u}$ corresponds to the antisymmetrized covariant derivative $\nabla_{\boldsymbol{v}}\boldsymbol{u} - \nabla_{\boldsymbol{u}}\boldsymbol{v}$.

The *fundamental theorem of Riemannian geometry* states that such a connection exists and that it is unique. In particular from these properties we can write an explicit expression for the Christoffel symbols of the Levi-Civita connection in terms of the metric tensor, by simply noticing that

$$\partial_k g_{ij} = {\Gamma^l}_{ik}\, g_{jl} + {\Gamma^l}_{jk}\, g_{il}\,, \quad \partial_j g_{ki} = {\Gamma^l}_{kj}\, g_{il} + {\Gamma^l}_{ij}\, g_{kl}\,, \quad \partial_i g_{jk} = {\Gamma^l}_{ji}\, g_{kl} + {\Gamma^l}_{ki}\, g_{jl}\,, \tag{A.35}$$

hence

$$\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk} = 2{\Gamma^l}_{jk}\, g_{il} \quad \Longrightarrow \quad {\Gamma^m}_{jk} = \frac{1}{2} g^{im}(\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk})\,. \tag{A.36}$$

In particular in coordinates we can write

$$\begin{aligned} \nabla_{(v^i\partial_i)}(u^j\partial_j) &= v^i\nabla_{\partial_i}(u^j\partial_j) = v^i\,\partial_i u^j\,\partial_j + v^i\,u^j\,\nabla_{\partial_i}\partial_j \\ &= v^i\,\partial_i u^j\,\partial_j + v^i\,u^j\,{\Gamma^k}_{ij}\,\partial_k \\ &= v^i\,(\partial_i u^k + u^j\,{\Gamma^k}_{ij})\,\partial_k\,, \end{aligned} \tag{A.37}$$

or, using an equivalent notation,

$$\nabla_{\partial_i}\boldsymbol{u} = \nabla_i\boldsymbol{u} = (\partial_i u^k + u^j\,{\Gamma^k}_{ij})\,\partial_k = {u^k}_{;i}\,\partial_k\,. \tag{A.38}$$

Koszul formula

If ∇ is the Levi-Civita connection, using the two conditions in Definition (3), we can write:

$$\begin{aligned} &X(g(Y,Z)) + Y(g(Z,X)) - Z(g(Y,X)) = \\ &= g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X) \,. \end{aligned} \tag{A.39}$$

By applying the torsion free property

$$\nabla_X Y - \nabla_Y X = [X, Y] \,, \tag{A.40}$$

we can write the right-hand side of (A.39) as

$$2g(\nabla_X Y, Z) - g([X,Y], Z) + g([X,Z], Y) + g([Y,Z], X) \,, \tag{A.41}$$

from which we can deduce the Koszul formula

$$\begin{aligned} g(\nabla_X Y, Z) =& \frac{1}{2} \left[Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \right. \\ & \left. - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \right] . \end{aligned} \tag{A.42}$$

Since Z is arbitrary, this formula uniquely determines $\nabla_X Y$ in order to satisfy the requested properties for the Levi-Civita connection, hence this expression can be thought as the definition of this particular connection and thus demonstrates that it exists and it is unique.

Covariant derivative of tensor fields

Let us extend covariant derivatives to all tensor fields: the operator ∇ takes in input the direction vector field and the tensor field and gives as an output a tensor field of the same type. For example, if we consider 1-forms we can differentiate $\mathrm{d}x^j$ along ∂_i and expressing the resulting 1-form on our basis we get

$$\nabla_i \, \mathrm{d}x^j = -\Gamma^j{}_{ik} \, \mathrm{d}x^k \,, \tag{A.43}$$

hence, given a 1-form $\alpha = \alpha_j \, \mathrm{d}x^j$ we define its covariant derivative by using "Leibniz rule" as

$$\nabla_i(\alpha) = \frac{\partial \alpha_j}{\partial x^i} \mathrm{d}x^j + \alpha_j (\nabla_i \, \mathrm{d}x^j) = \left(\frac{\partial \alpha_k}{\partial x^i} - \alpha_j \, \Gamma^j{}_{ik} \right) \mathrm{d}x^k = \alpha_{k\,;i} \, \mathrm{d}x^k \,. \tag{A.44}$$

With these two definitions in mind we can now define the covariant derivative of any tensor field just thinking of it as a tensor product of vector and covector fields and by applying again "Leibniz rule":

$$\nabla_{\boldsymbol{v}}(T \otimes S) = (\nabla_{\boldsymbol{v}} T) \otimes S + T \otimes (\nabla_{\boldsymbol{v}} S)\,. \tag{A.45}$$

As an example, we can take the covariant derivative of the metric tensor

$$\begin{aligned}
\nabla_i g &= \nabla_i (g_{rs}\, \mathrm{d}x^r \otimes \mathrm{d}x^s) \\
&= \frac{\partial g_{rs}}{\partial x^i}\, \mathrm{d}x^r \otimes \mathrm{d}x^s - g_{rs}\, \Gamma^r{}_{ik}\, \mathrm{d}x^k \otimes \mathrm{d}x^s - g_{rs}\, \mathrm{d}x^r \otimes \Gamma^s{}_{ik}\, \mathrm{d}x^k \\
&= \left(\frac{\partial g_{rs}}{\partial x^i} - g_{ks}\, \Gamma^k{}_{ir} - g_{rk}\, \Gamma^k{}_{is} \right) \mathrm{d}x^r \otimes \mathrm{d}x^s = g_{rs\,;i}\, \mathrm{d}x^r \otimes \mathrm{d}x^s\,.
\end{aligned} \tag{A.46}$$

Using the metric compatibility property in (3) we conclude that

$$\nabla_i g = 0 \quad \forall\, i \quad \implies \quad \nabla g = 0\,. \tag{A.47}$$

To finish we just notice that every $(m, n)-$tensor will have m positive Christoffel terms and n negative Christoffel terms in its covariant derivative expansion.

Hessian, Laplacian and Divergence with connection

By applying (A.4) we can rewrite the Christoffel symbols as

$$\Gamma^m{}_{jk} = \frac{1}{2} g^{im} (\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk}) = -\frac{1}{2} \left(g^{im}\, \partial_i g_{jk} + g_{ik}\, \partial_j g^{im} + g_{ji}\, \partial_k g^{im} \right)\,, \tag{A.48}$$

so that we can write the Hessian of a function in the form

$$\mathrm{Hess} f = \left[\frac{\partial^2 f}{\partial x^r \partial x^s} - \Gamma^j{}_{rs} \frac{\partial f}{\partial x^j} \right] \mathrm{d}x^r \otimes \mathrm{d}x^s = \nabla_r \nabla_s f\, \mathrm{d}x^r \otimes \mathrm{d}x^s = (\mathrm{Hess} f)_{rs}\, \mathrm{d}x^r \otimes \mathrm{d}x^s\,, \tag{A.49}$$

the Laplacian of a function f as

$$\nabla^2 f = \mathrm{tr}(\mathrm{Hess} f) = g^{rs}\, \partial_r \partial_s f - g^{rs}\, \Gamma^j{}_{rs}\, \partial_j f = g^{rs}\, \nabla_r \nabla_s f\,, \tag{A.50}$$

and the divergence of a vector field $\boldsymbol{v}$ as

$$\mathrm{div}(\boldsymbol{v}) = \partial_i v^i + v^j\, \Gamma^i{}_{ij}\,. \tag{A.51}$$

Divergence of a $(0,2)-$tensor

We define the divergence of a $(0,2)-$tensor T to be a $(0,1)-$tensor defined by taking the covariant derivative ∇T, raising the second index by applying the musical isomorphism $\sharp$ and finally taking the trace to get a 1-form.

$$
\begin{aligned}
&T = T_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \implies \\
&\nabla_k T = (\partial_k T_{ij} - T_{lj}\Gamma^l{}_{ki} - T_{il}\Gamma^l{}_{kj})\,\mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j = \nabla_k T_{ij}\,\mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j \implies \\
&\sharp\nabla_k T = (g^{ri}\partial_k T_{ij} - g^{ri}T_{lj}\Gamma^l{}_{ki} - g^{ri}T_{il}\Gamma^l{}_{kj})\,\mathrm{d}x^k \otimes \partial_r \otimes \mathrm{d}x^j = g^{ri}\nabla_k T_{ij}\,\mathrm{d}x^k \otimes \partial_r \otimes \mathrm{d}x^j \implies \\
&\mathrm{tr}(\sharp\nabla_k T) = (g^{ki}\,\partial_k T_{ij} - g^{ki}\,T_{lj}\,\Gamma^l{}_{ki} - g^{ki}\,T_{il}\,\Gamma^l{}_{kj})\,\mathrm{d}x^j = g^{ki}\,\nabla_k T_{ij}\,\mathrm{d}x^j\,,
\end{aligned}
\tag{A.52}
$$

thus

$$
\mathrm{div}\,T = g^{ik}\,\nabla_k T_{ij}\,\mathrm{d}x^j = g^{ik}(\partial_k T_{ij} - T_{lj}\,\Gamma^l{}_{ki} - T_{il}\,\Gamma^l{}_{kj})\,\mathrm{d}x^j\,. \tag{A.53}
$$

Riemann, Ricci and Einstein tensors

Definition 4. *We define Riemann curvature tensor by*

$$
R(\boldsymbol{u},\boldsymbol{v}) = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\,, \tag{A.54}
$$

hence it is a linear $(1,3)-$tensor that acts on vectors as

$$
R(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w} = u^i v^j w^k R(\partial_i,\partial_j)\partial_k = u^i v^j w^k R^l{}_{kij}\partial_l\,, \tag{A.55}
$$

where we define the Riemann curvature tensor coefficients as

$$
R(\partial_i,\partial_j)\partial_k = R^l{}_{kij}\,\partial_l\,. \tag{A.56}
$$

Explicitly we have

$$
\begin{aligned}
R(\partial_i,\partial_j)\partial_k &= \nabla_{\partial_i}(\nabla_{\partial_j}\partial_k) - \nabla_{\partial_j}(\nabla_{\partial_i}\partial_k) - \nabla_{\cancel{[\partial_i,\partial_j]}}\partial_k \\
&= \nabla_{\partial_i}(\Gamma^a{}_{jk}\partial_a) - \nabla_{\partial_j}(\Gamma^b{}_{ik}\partial_b) \\
&= (\partial_i\Gamma^a{}_{jk})\partial_a + \Gamma^a{}_{jk}\Gamma^l{}_{ia}\partial_l - (\partial_j\Gamma^b{}_{ik})\partial_b - \Gamma^b{}_{ik}\Gamma^l{}_{jb}\partial_l \\
&= \left[\partial_i\Gamma^l{}_{jk} - \partial_j\Gamma^l{}_{ik} + \Gamma^a{}_{jk}\Gamma^l{}_{ia} - \Gamma^b{}_{ik}\Gamma^l{}_{jb}\right]\partial_l\,,
\end{aligned}
\tag{A.57}
$$

thus

$$R^l{}_{kij} = \left[\partial_i \Gamma^l{}_{jk} - \partial_j \Gamma^l{}_{ik} + \Gamma^a{}_{jk}\,\Gamma^l{}_{ia} - \Gamma^b{}_{ik}\,\Gamma^l{}_{jb}\right] , \tag{A.58}$$

and we can lower indices by

$$R^l{}_{kij} = g^{lm} R_{mkij} \,. \tag{A.59}$$

Theorem 6. *Properties of the Riemann tensor*

- $R^l{}_{kij} = -R^l{}_{kji}$
- $R^l{}_{kij} + R^l{}_{jki} + R^l{}_{ijk} = 0$ *First (algebraic) Bianchi identity (Torsion-free condition)*
- $R_{mkij} = -R_{kmij}$ *(Metric compatibility condition)*
- $R_{mkij} = R_{ijmk}$

Definition 5. *We define Ricci curvature tensor by contracting two indices*

$$R_{kj} = R^i{}_{kij} = \sum_i R(\partial_i, \partial_j)\partial_k \cdot \partial_i \,. \tag{A.60}$$

In particular we can also write it as $R_{kj} = R^i{}_{kij} = -R^i{}_{kji}$ and it happens that the tensor is symmetric $R_{kj} = R_{jk}$. In components we have

$$R_{kj} = \partial_i \Gamma^i{}_{jk} - \partial_j \Gamma^i{}_{ik} + \Gamma^a{}_{jk}\,\Gamma^i{}_{ia} - \Gamma^b{}_{ik}\,\Gamma^i{}_{jb} \,. \tag{A.61}$$

Definition 6. *We define Ricci scalar by contraction of Ricci tensor with the metric*

$$R = R^k{}_k = g^{kj} R_{kj} \,. \tag{A.62}$$

hence in components we have

$$R = R^k{}_k = g^{kj}\left(\partial_i \Gamma^i{}_{jk} - \partial_j \Gamma^i{}_{ik} + \Gamma^a{}_{jk}\,\Gamma^i{}_{ia} - \Gamma^b{}_{ik}\,\Gamma^i{}_{jb}\right) . \tag{A.63}$$

If we take the covariant derivative of the Riemann tensor we get

$$\begin{aligned}
\nabla_a R &= \nabla_a (R^l{}_{kij}\ \partial_l \otimes \mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j) \\
&= (\partial_a R^l{}_{kij} + R^x{}_{kij}\Gamma^l{}_{ax} - R^l{}_{xij}\Gamma^x{}_{ak} - R^l{}_{kxj}\Gamma^x{}_{ai} - R^l_{kix}\Gamma^x{}_{aj})\ \partial_l \otimes \mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j \\
&= R^l{}_{kij\,;a}\ \partial_l \otimes \mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j \,.
\end{aligned} \tag{A.64}$$

Theorem 7. *Second (differential) Bianchi identity states that*

$$(\nabla_{\boldsymbol{z}}R)(\boldsymbol{u},\boldsymbol{v}) + (\nabla_{\boldsymbol{v}}R)(\boldsymbol{z},\boldsymbol{u}) + (\nabla_{\boldsymbol{u}}R)(\boldsymbol{v},\boldsymbol{z}) = 0\,, \tag{A.65}$$

or in component form

$$R^{l}{}_{kij\,;a} + R^{l}{}_{kai\,;j} + R^{l}{}_{kja\,;i} = 0\,. \tag{A.66}$$

From this expression we can get the *contracted Bianchi identity* on Ricci tensor by simply noticing that, using the metric compatibility ($\nabla g = 0$), we can commute the covariant derivative and the metric tensor

$$\nabla_a(g(R)) = (\nabla_a g)(R) + g(\nabla_a R) = g(\nabla_a R)\,. \tag{A.67}$$

Hence, contracting the metric with Riemann and then taking the covariant derivative is the same thing as taking the covariant derivative of Riemann and then contracting with the metric tensor. In coordinates we have

$$(g^{lm}R_{lkij})_{\,;a} = g^{lm}(R_{lkij\,;a}) = (R^{m}{}_{kij})_{\,;a}\,, \tag{A.68}$$

from which we can prove the following property:

Theorem 8. *Twice contracted Bianchi identity:*

$$(R^{mn})_{\,;n} - \frac{1}{2}g^{mn}R_{\,;n} = 0\,, \tag{A.69}$$

or equivalently

$$\nabla^m R_{mn} = \frac{1}{2}\nabla_n R\,. \tag{A.70}$$

Definition 7. *We define the Einstein tensor as*

$$G_{mn} = R_{mn} - \frac{1}{2}g_{mn}R\,. \tag{A.71}$$

The contracted Bianchi identity (A.70) states that Einstein's tensor G_{mn} has zero covariant derivative when the index is matched:

$$\nabla^m(G_{mn}) = 0\,. \tag{A.72}$$

Theorem 9. *Let T_{mn} denote the energy-momentum tensor. The laws of conservation of energy and momentum can be written in a compact way as*

$$\nabla^n T_{mn} = 0\,. \tag{A.73}$$

Einstein Field equation states that

$$G_{mn} = R_{mn} - \frac{1}{2} g_{mn} R = \frac{8\pi G_N}{c^4}\, T_{mn}\,, \tag{A.74}$$

where G_N is Newton's gravitational constant, c is speed of light and R is Ricci scalar.

Moreover, we know that, because of metric compatibility property, the metric tensor has zero covariant derivative in any direction, hence we can modify the equation by adding an extra term proportional to the metric tensor, obtaining

$$R_{mn} - \frac{1}{2} g_{mn} R + \Lambda g_{mn} = \frac{8\pi G_N}{c^4}\, T_{mn}\,, \tag{A.75}$$

where Λ is called the *cosmological constant* and is related to the expansion of the universe. This equation can be easily modified into the trace subtracted Einstein equation, obtaining

$$R_{mn} - \Lambda g_{mn} = \frac{8\pi G_N}{c^4} (T_{mn} - \frac{1}{2} g_{mn} T)\,, \tag{A.76}$$

where T denotes the trace of the energy-momentum tensor.

Appendix B

Non-dimensional Gross-Pitaevskii equation

Using the definition of healing length (1.13), we can rewrite the flat GPE (1.1) in non-dimensional form and recover all the hydrodynamic results. In order to do that, let us start by writing the explicit dependence on the chemical potential μ by decomposing the wave function

$$\Psi(\mathbf{r},t) = \psi(\mathbf{r},t)e^{-i\mu t/\hbar}\,. \tag{B.1}$$

Substituting into (1.1) we find an equation for ψ

$$i\hbar\partial_t\psi(\mathbf{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + \mathfrak{g}|\psi(\mathbf{r},t)|^2 - \mu\right)\psi(\mathbf{r},t)\,. \tag{B.2}$$

Considering the external potential to be zero ($V = 0$) first notice that the constant solution satisfies

$$|\psi_0|^2 = \frac{\mu}{\mathfrak{g}} = \rho_0\,, \tag{B.3}$$

hence we have that the healing length can be written into two equivalent forms

$$\xi = \frac{\hbar}{\sqrt{2m\mathfrak{g}\rho_0}} = \frac{\hbar}{\sqrt{2m\mu}}\,. \tag{B.4}$$

The non-dimensional GPE can be obtained by simply performing a change of coordinates

$$\mathbf{r} \mapsto \xi\,\mathbf{r}', \qquad t \mapsto \frac{\hbar}{2\mu}\,t', \qquad \psi \mapsto \sqrt{\frac{\mu}{\mathfrak{g}}}\,\psi'\,, \tag{B.5}$$

and equation (B.2) becomes

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + \frac{1}{2}(|\psi|^2 - 1)\psi\,, \tag{B.6}$$

which we will refer to as non-dimensional GPE.

It is still a Schrödinger equation with Hamiltonian given by

$$H\psi = \left(-\frac{1}{2}\nabla^2 - \frac{1}{2}(1 - |\psi|^2)\right)\psi\,, \tag{B.7}$$

so that we can write the energy as

$$\langle H\rangle = \langle\psi|H\psi\rangle = \int_{\mathcal{H}} \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}|\psi|^2 + \frac{1}{4}|\psi|^4\,. \tag{B.8}$$

Obviously, all the hydrodynamic results can be recovered also from the non-dimensional GPE (B.6) by performing again a Madelung transform ($\psi = \sqrt{\rho}e^{i\theta}$). Indeed, we can still recover the continuity equation for the density ρ

$$\partial_t\rho = -\mathrm{div}(\rho\nabla\theta)\,, \tag{B.9}$$

and the Bernoulli type equation for the phase θ in non-dimensional form

$$\partial_t\theta = \frac{1}{2}(1-\rho) - \frac{1}{2}|\nabla\theta|^2 + \frac{1}{2\sqrt{\rho}}\nabla^2\sqrt{\rho}\,. \tag{B.10}$$

If we consider θ as the scalar potential for the velocity field $\boldsymbol{u} = \nabla\theta$, we can still write an Euler equation (1.39), which, in our non-dimensional setup, takes the form

$$\rho(\partial_t\boldsymbol{u} + \nabla_{\boldsymbol{u}}\boldsymbol{u}) = -\nabla\mathrm{p} + \rho\nabla Q\,, \tag{B.11}$$

where $\mathrm{p} = \dfrac{\rho^2}{4}$ is the non-dimensional pressure and $Q = \dfrac{1}{2\sqrt{\rho}}\nabla^2\sqrt{\rho}$ is the non-dimensional quantum potential. Using these definitions we obtain the speed of sound to be

$$c^2 = \frac{\mathrm{dp}}{\mathrm{d}\rho} = \frac{\rho}{2} \quad\Longrightarrow\quad c = \sqrt{\frac{\rho}{2}}\,. \tag{B.12}$$

The Navier-Stokes equation can be recovered as in (1.32), taking the form

$$\rho(\partial_t\boldsymbol{u} + \nabla_{\boldsymbol{u}}\boldsymbol{u}) = -\nabla\mathrm{p} + \mathrm{div}\,\tau + \mathcal{E}\,, \tag{B.13}$$

where $\tau = \dfrac{1}{4}\rho\mathrm{Hess}(\ln\rho)$ is the non-dimensional stress tensor and $\mathcal{E}^j = -\dfrac{1}{4}R^{jk}\partial_k\rho$ is the non-dimensional curvature density vector.

Finally, we can also find the non-dimensional momentum equation (1.57), in the form

$$\partial_t(\rho u_j) = -g^{ik}\nabla_k\left(\rho u_i u_j + \frac{\rho^2}{4}g_{ij} - \frac{1}{4}\rho \mathrm{Hess}_{ij}(\ln\rho) + \frac{1}{4}\rho G_{ij}\right) - \frac{1}{8}R\partial_j\rho\,, \tag{B.14}$$

from which, by imposing the same steady state conditions as in Section 1.5.3, we obtain

$$\begin{cases} G_{ij} = \mathrm{Hess}_{ij}(\ln\rho) - 4u_i u_j \\ R = -2\rho = -4c^2 \end{cases} \tag{B.15}$$

Appendix C

Dini surface

A Dini Surface is a constant negative curvature surface obtained by twisting a Pseudosphere around its central axis in a helicoidal motion. In order to write it in the standard form we start from the the unit speed tractrix (2.4) and set $f(w) = ae^{-w/a} = a\sin\varphi$ with $\varphi \in (0, \pi/2]$, so that we get

$$g(w(\varphi)) = \mp a\left(\ln\left(\frac{1+\cos\varphi}{\sin\varphi}\right) - \cos\varphi\right) = \pm a\left(\cos\varphi + \ln\left(\tan\frac{\varphi}{2}\right)\right) \tag{C.1}$$

where we used the fact that

$$\frac{1+\cos\varphi}{\sin\varphi} = \frac{\sin\varphi}{1-\cos\varphi} = \left(\tan\frac{\varphi}{2}\right)^{-1} .$$

Hence the tractrix curve can be also written in parametric form as

$$\gamma(w(\varphi)) = \left(a\sin\varphi, 0, a\left(\cos\varphi + \ln\left(\tan\frac{\varphi}{2}\right)\right)\right) \qquad \text{with } \varphi \in \left(0, \frac{\pi}{2}\right] , \tag{C.2}$$

from which we obtain another parametric expression for the pseudosphere given by

$$\sigma_3(\varphi, \theta) = \begin{pmatrix} a\sin\varphi\cos\theta \\ a\sin\varphi\sin\theta \\ a(\cos\varphi + \ln(\tan(\frac{\varphi}{2}))) \end{pmatrix} \quad \varphi \in \left(0, \frac{\pi}{2}\right] ,\ \theta \in [0, 2\pi] , \tag{C.3}$$

whose metric is

$$\mathrm{d}s^2 = a^2\left(\cot^2\varphi\,\mathrm{d}\varphi^2 + \sin^2\varphi\,\mathrm{d}\theta^2\right) . \tag{C.4}$$

Starting from (C.3) we can easily describe a Dini surface by imposing a translation in the $z-$axis, obtaining the parametric equation

$$\sigma(\varphi,\theta) = \begin{pmatrix} a\sin\varphi\cos\theta \\ a\sin\varphi\sin\theta \\ a(\cos\varphi + \ln(\tan\left(\frac{1}{2}\varphi\right)) + b\,\theta \end{pmatrix} \tag{C.5}$$

with a and b positive constants, $\theta \in \mathbb{R}$ and $\varphi \in (0, \pi/2]$. In particular, φ represents the angle that the helicoidal axis generates with the tangent to the tractrix.

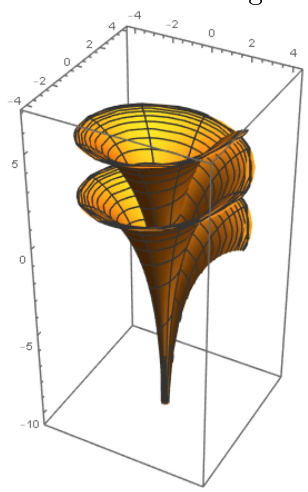

Figure C.1: Dini Surface with $a = 4$ and $b = 0.5$, $\theta \in [0, 4\pi]$ and $\varphi \in (0, \pi/2]$.

Its first fundamental form is given by

$$\mathrm{d}s^2 = (a^2\sin^2\varphi + b^2)\,\mathrm{d}\theta^2 + 2(ab\cos\varphi\cot\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi + (a^2\cot^2\varphi)\,\mathrm{d}\varphi^2\,, \tag{C.6}$$

hence the metric matrix is

$$g_{ij} = \begin{bmatrix} (a^2\sin^2\varphi + b^2) & ab\cos\varphi\cot\varphi \\ ab\cos\varphi\cot\varphi & a^2\cot^2\varphi \end{bmatrix}, \tag{C.7}$$

whose determinant is

$$\det g = |g| = a^2\left(a^2 + b^2\right)\cos^2\varphi\,, \tag{C.8}$$

giving an inverse metric matrix

$$g^{ij} = \begin{bmatrix} \dfrac{1}{\sin^2\varphi(a^2+b^2)} & -\dfrac{b}{a\sin\varphi(a^2+b^2)} \\ -\dfrac{b}{a\sin\varphi(a^2+b^2)} & \dfrac{1}{a^2\cos^2\varphi} - \dfrac{1}{a^2+b^2} \end{bmatrix}. \tag{C.9}$$

We now calculate the second fundamental form by defining the normal vector

$$\hat{\mathbf{n}} = \frac{\sigma_\theta \times \sigma_\varphi}{||\sigma_\theta \times \sigma_\varphi||} = \begin{pmatrix} (a\cos\theta\cos\varphi - b\sin\theta)/\sqrt{a^2+b^2} \\ (b\cos\theta + a\cos\varphi\sin\theta)/\sqrt{a^2+b^2} \\ -a\sin\varphi/\sqrt{a^2+b^2} \end{pmatrix} \tag{C.10}$$

and computing the second derivatives of (C.5), obtaining

$$h_{ij} = \frac{a\cos\varphi}{\sqrt{a^2+b^2}} \begin{bmatrix} -a\sin\varphi & b \\ b & a\csc\varphi \end{bmatrix} . \tag{C.11}$$

Using these expressions, we easily get the mean and Gaussian curvatures

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = -\frac{\cot 2\varphi}{\sqrt{a^2+b^2}} , \tag{C.12}$$

$$K = \frac{eg - f^2}{EG - F^2} = -\frac{1}{a^2+b^2} , \tag{C.13}$$

from which we notice that the Dini surface is another example of constant negative Gaussian curvature surface.

Christoffel symbols, Ricci tensor, Ricci scalar and Einstein tensor

We first compute the Christoffel symbols for the upper θ component:

$$\begin{aligned} \Gamma^\theta{}_{\theta\theta} &= \frac{1}{2} g^{\theta\theta}(\partial_\theta g_{\theta\theta}) + \frac{1}{2}(2\partial_\theta g_{\varphi\theta} - \partial_\varphi g_{\theta\theta}) = -\frac{1}{2} g^{\varphi\theta}\partial_\varphi g_{\theta\theta} \\ &= \frac{ab}{a^2+b^2}\cos\varphi , \\ \Gamma^\theta{}_{\theta\varphi} &= \frac{1}{2} g^{\theta\theta}(\partial_\varphi g_{\theta\theta}) + \frac{1}{2} g^{\varphi\theta}(\partial_\theta g_{\varphi\varphi}) = \frac{1}{2} g^{\theta\theta}(\partial_\varphi g_{\theta\theta}) \\ &= \frac{a^2}{a^2+b^2}\cot\varphi , \\ \Gamma^\theta{}_{\varphi\varphi} &= \frac{1}{2} g^{\theta\theta}(2\partial_\varphi g_{\theta\varphi} - \partial_\theta g_{\varphi\varphi}) + \frac{1}{2} g^{\varphi\theta}(\partial_\varphi g_{\varphi\varphi}) \\ &= -\frac{ab}{a^2+b^2}\frac{\cos\varphi}{\sin^2\varphi} . \end{aligned} \tag{C.14}$$

Hence we can compactly write the matrix

$$\Gamma^\theta = \begin{bmatrix} \frac{ab}{a^2+b^2}\cos\varphi & \frac{a^2}{a^2+b^2}\cot\varphi \\ \frac{a^2}{a^2+b^2}\cot\varphi & -\frac{ab}{a^2+b^2}\frac{\cos\varphi}{\sin^2\varphi} \end{bmatrix} . \tag{C.15}$$

For the upper φ component we have:

$$\begin{aligned}
\Gamma^{\varphi}{}_{\theta\theta} &= \frac{1}{2}g^{\theta\varphi}(\partial_\theta g_{\theta\theta}) + \frac{1}{2}g^{\varphi\varphi}(2\partial_\theta g_{\theta\varphi} - \partial_\varphi g_{\theta\theta}) = -\frac{1}{2}g^{\varphi\varphi}\partial_\varphi g_{\theta\theta} \\
&= -\tan\varphi + \frac{a^2}{a^2+b^2}\sin\varphi\cos\varphi\,, \\
\Gamma^{\varphi}{}_{\theta\varphi} &= \frac{1}{2}g^{\theta\varphi}(\partial_\varphi g_{\theta\theta}) + \frac{1}{2}g^{\varphi\varphi}(\partial_\theta g_{\varphi\varphi}) = \frac{1}{2}g^{\theta\varphi}(\partial_\varphi g_{\theta\theta}) \\
&= -\frac{ab}{a^2+b^2}\cos\varphi\,, \\
\Gamma^{\varphi}{}_{\varphi\varphi} &= \frac{1}{2}g^{\theta\varphi}(2\partial_\varphi g_{\varphi\theta} - \partial_\theta g_{\varphi\varphi}) + \frac{1}{2}g^{\varphi\varphi}(\partial_\varphi g_{\varphi\varphi}) \\
&= \frac{b^2}{a^2+b^2}\cot\varphi - \frac{1}{\cos\varphi\sin\varphi}\,.
\end{aligned} \tag{C.16}$$

Hence we can compactly write the matrix

$$\Gamma^{\varphi} = \begin{bmatrix} -\tan\varphi + \dfrac{a^2}{a^2+b^2}\sin\varphi\cos\varphi & -\dfrac{ab}{a^2+b^2}\cos\varphi \\ -\dfrac{ab}{a^2+b^2}\cos\varphi & \dfrac{b^2}{a^2+b^2}\cot\varphi - \dfrac{1}{\cos\varphi\sin\varphi} \end{bmatrix}. \tag{C.17}$$

Using the Christoffel symbols we can calculate Ricci's tensor obtaining

$$\begin{aligned}
R_{\theta\theta} &= R^{\theta}{}_{\theta\theta\theta} + R^{\varphi}{}_{\theta\varphi\theta} = \partial_\varphi\Gamma^{\varphi}{}_{\theta\theta} + \Gamma^{\theta}{}_{\theta\theta}\Gamma^{\varphi}{}_{\varphi\theta} + \Gamma^{\varphi}{}_{\theta\theta}\Gamma^{\varphi}{}_{\varphi\varphi} - \Gamma^{\theta}{}_{\varphi\theta}\Gamma^{\varphi}{}_{\theta\theta} - \Gamma^{\varphi}{}_{\varphi\theta}\Gamma^{\varphi}{}_{\theta\varphi} \\
&= \frac{a^2}{(a^2+b^2)}\cos^2\varphi - 1\,, \\
R_{\theta\varphi} &= R^{\theta}{}_{\theta\theta\varphi} + R^{\varphi}{}_{\theta\varphi\varphi} = -\partial_\varphi\Gamma^{\theta}{}_{\theta\theta} + \Gamma^{\varphi}{}_{\varphi\theta}\Gamma^{\theta}{}_{\theta\varphi} - \Gamma^{\varphi}{}_{\theta\theta}\Gamma^{\theta}{}_{\varphi\varphi} \\
&= -\frac{ab}{a^2+b^2}\frac{\cos^2\varphi}{\sin\varphi}\,, \\
R_{\varphi\varphi} &= R^{\theta}{}_{\varphi\theta\varphi} + R^{\varphi}{}_{\varphi\varphi\varphi} = -\partial_\varphi\Gamma^{\theta}{}_{\theta\varphi} + \Gamma^{\theta}{}_{\varphi\varphi}\Gamma^{\theta}{}_{\theta\varphi} + \Gamma^{\varphi}{}_{\varphi\varphi}\Gamma^{\theta}{}_{\theta\varphi} - \Gamma^{\theta}{}_{\theta\varphi}\Gamma^{\theta}{}_{\varphi\theta} - \Gamma^{\varphi}{}_{\theta\varphi}\Gamma^{\theta}{}_{\varphi\varphi} \\
&= -\frac{a^2}{(a^2+b^2)}\cot^2\varphi\,.
\end{aligned} \tag{C.18}$$

We now compute Ricci's scalar

$$\begin{aligned}
R &= g^{ij}R_{ij} = g^{\theta\theta}R_{\theta\theta} + 2g^{\theta\varphi}R_{\theta\varphi} + g^{\varphi\varphi}R_{\varphi\varphi} \\
&= \frac{2\cos^2\varphi}{(a^2+b^2)\sin^2\varphi} - \frac{2}{(a^2+b^2)\sin^2\varphi} = -\frac{2}{a^2+b^2}\,,
\end{aligned} \tag{C.19}$$

from which we deduce that our computation for the Gaussian curvature was correct, since

$$K = \frac{R}{2} = -\frac{1}{a^2 + b^2} \,. \tag{C.20}$$

We can finally verify that all components of the Einstein tensor are zero.

Frenet-Serret

In order to write the Frenet-Serret equations, let $\mathbf{T} = \sigma_\theta / ||\sigma_\theta||$, namely

$$\mathbf{T} = \frac{1}{\sqrt{a^2 \sin^2 \varphi + b^2}} \begin{pmatrix} -a \sin\varphi \sin\theta \\ a \sin\varphi \cos\theta \\ b \end{pmatrix} \,, \tag{C.21}$$

and compute its derivative to find the curvature

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{T} = \frac{a \sin\varphi}{\sqrt{a^2 \sin^2 \varphi + b^2}} \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} = \kappa \, \mathbf{U} \,. \tag{C.22}$$

Now, let us compute the binormal

$$\mathbf{B} = \mathbf{T} \times \mathbf{U} = \frac{1}{\sqrt{a^2 \sin^2 \varphi + b^2}} \begin{pmatrix} b \sin\theta \\ -b \cos\theta \\ a \sin\varphi \end{pmatrix} \,, \tag{C.23}$$

and its derivative in order to get the torsion

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{B} = \frac{b}{\sqrt{a^2 \sin^2 \varphi + b^2}} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = -\tau \, \mathbf{U} \,. \tag{C.24}$$

Finally, we can write the Frenet-Serret equations as

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \begin{pmatrix} \mathbf{T} \\ \mathbf{U} \\ \mathbf{B} \end{pmatrix} = \frac{1}{\sqrt{a^2 \sin^2 \varphi + b^2}} \begin{pmatrix} 0 & a \sin\varphi & 0 \\ -a \sin\varphi & 0 & b \\ 0 & -b & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{U} \\ \mathbf{B} \end{pmatrix} \,. \tag{C.25}$$

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