Journal of Mathematical Physics

RESEARCH ARTICLE | OCTOBER 31 2022

Regular non-semisimple Dubrovin–Frobenius manifolds

Paolo Lorenzoni ; Sara Perletti

J. Math. Phys. 63, 102301 (2022)
https://doi.org/10.1063/5.0094580

Journal of Mathematical Physics | ARTICLE | scitation.org/journal/jmp

Regular non-semisimple Dubrovin–Frobenius manifolds

Cite as: J. Math. Phys. 63, 102301 (2022); doi: 10.1063/5.0094580
Submitted: 4 April 2022 • Accepted: 5 September 2022 •
Published Online: 31 October 2022

Paolo Lorenzoni[a] and Sara Perletti[b]

AFFILIATIONS

Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Via Roberto Cozzi 55, I-20125 Milano, Italy and INFN Sezione di Milano-Bicocca, Milano, Italy

[a] **E-mail:** paolo.lorenzoni@unimib.it
[b] **Author to whom correspondence should be addressed:** s.perletti1@campus.unimib.it

ABSTRACT

We study regular non-semisimple Dubrovin–Frobenius manifolds in dimensions 2, 3, and 4. Our results rely on the existence of special local coordinates introduced by David and Hertling [Ann. Sc. Norm. Super. Pisa, Cl. Sci. **17**(5), 1121–1152 (2017)] for regular flat F-manifolds endowed with an Euler vector field. In such coordinates, the invariant metric of the Dubrovin–Frobenius manifold takes a special form, which is the starting point of our construction. We give a complete classification in the case where the Jordan canonical form of the operator of multiplication by the Euler vector field has a single Jordan block, and we reduce the classification problem to a third-order ordinary differential equation and to a system of third-order PDEs in the remaining three-dimensional and four-dimensional cases. In all the cases, we provide explicit examples of Dubrovin–Frobenius potentials.

I. INTRODUCTION

Dubrovin–Frobenius manifolds have been introduced by Dubrovin as a coordinate-free reformulation of the so-called Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of two-dimensional topological field theories (see Ref. 1) and play an important role in many areas of mathematics (quantum cohomology, Gromov–Witten theory, singularity theory, integrable PDEs, etc.). Some constructions in the theory of Dubrovin–Frobenius manifolds rely on an additional assumption: the existence of a holonomic frame of idempotents. Dubrovin–Frobenius manifolds having this property are called semisimple or massive since in a physical context they correspond to massive perturbations of two-dimensional topological field theories. Semisimple Dubrovin–Frobenius manifolds are characterized by the existence of a special set of local coordinates, called Dubrovin canonical coordinates or simply canonical coordinates, reducing the structure constants of the product to a constant canonical form. A generalization of canonical coordinates in the non-semisimple regular case was found by David and Hertling in Ref. 2. David–Hertling canonical coordinates depend on the Jordan normal form of the operator of multiplication by the Euler vector field. In this paper, using these coordinates, we construct explicit examples of non-semisimple regular Dubrovin–Frobenius manifolds in the case of a single Jordan block.

This paper is organized as follows. In Sec. II, we recall the definition of the Dubrovin–Frobenius manifold and some known results in the semisimple case. In Sec. III, we introduce David–Hertling canonical coordinates for regular non-semisimple Dubrovin–Frobenius manifolds and some general properties of the invariant metric in such coordinates. In Sec. IV, we focus on the case of a single Jordan block in dimensions 2, 3, and 4, and in Sec. V, we briefly discuss the case of multiple Jordan blocks in both dimensions 3 and 4.

II. DUBROVIN–FROBENIUS MANIFOLDS: THE SEMISIMPLE CASE

Following the work of Dubrovin,[1] we introduce the notion of the Dubrovin–Frobenius manifold.

Definition I.1. A Dubrovin–Frobenius manifold M is a manifold equipped with a metric η, a commutative associative product $\circ$ on the tangent space with unit e, and a second distinguished vector field E called the Euler vector field satisfying the following conditions:

- *Invariance of the metric,*

$$\eta_{il}\, c^l_{jk} = \eta_{jl}\, c^l_{ik}. \tag{1.1}$$

- *Flatness of the metric,*

$$R^m_{ijk} = \partial_j \Gamma^m_{ik} - \partial_i \Gamma^m_{jk} + \Gamma^s_{ik}\Gamma^m_{sj} - \Gamma^s_{jk}\Gamma^m_{is} = 0. \tag{1.2}$$

- *Symmetry of ∇c,*

$$\nabla_i c^l_{jk} = \nabla_j c^l_{ik}. \tag{1.3}$$

- *Constancy of e,*

$$\nabla_i e^k = 0. \tag{1.4}$$

- *Homogeneity conditions,*

$$\mathscr{L}_E c^i_{jk} = c^i_{jk}, \quad \mathscr{L}_E e^i = -e^i, \quad \mathscr{L}_E \eta_{ij} = (2-d)\eta_{ij} \tag{1.5}$$

for some constant d. Here, ∇ denotes the Levi-Cività connection associated with η and $\mathscr{L}_Z$ denotes the Lie derivative along a vector field Z.

From the axioms above, it follows that in flat coordinates for the metric, the structure constants of the product can be written in terms of the third-order partial derivatives of a function F called the prepotential of the Dubrovin–Frobenius manifold,

$$c^i_{jk} = \eta^{il}\partial_l\partial_j\partial_k F.$$

By construction, the function F is a solution of Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations.[3,4]

Remark I.2. The manifold M in the above definition is a real or complex n-dimensional manifold. In the first case, all the geometric data are supposed to be smooth. In the latter case, TM is intended as the holomorphic tangent bundle and all the geometric data are supposed to be holomorphic.

Remark I.3. Since the components of the metric and of the unit vector field are constant in flat coordinates, we clearly have

$$\mathscr{L}_e \eta_{ij} = 0. \tag{1.6}$$

A point $p \in M$ of an n-dimensional Dubrovin–Frobenius manifold is called *semisimple* if T_pM has a basis of idempotents $\pi_1, \dots, \pi_n$ satisfying $\pi_k \circ \pi_l = \delta_{k,l}\pi_k$. Semisimplicity at a point is an open property on M: locally around a semisimple point, one can choose coordinates u^i such that $\frac{\partial}{\partial u^k} \circ \frac{\partial}{\partial u^l} = \delta_{k,l}\frac{\partial}{\partial u^k}$. These coordinates are called *canonical coordinates*.

Due to (1.1), in canonical coordinates, the metric η becomes diagonal: $\eta_{ij} = H_i^2\delta_{ij}$. Let us introduce the *Ricci rotation coefficients*[10] $\beta_{ij} := \frac{\partial_j H_i}{H_j}$, $i \neq j$. In the case of Dubrovin–Frobenius manifolds, the rotation coefficients are symmetric ($\beta_{ij} = \beta_{ji}$), and as a consequence, the metric is potential in canonical coordinates (i.e., $H_i^2 = \partial_i\varphi$ for some function φ). Moreover, it is easy to check that the rotation coefficients satisfy the following overdetermined system of PDEs:

$$\partial_k\beta_{ij} = \beta_{ik}\beta_{kj}, \qquad i \neq j \neq k \neq i, \tag{1.7}$$

$$e(\beta_{ij}) = 0, \qquad i \neq j, \tag{1.8}$$

$$E(\beta_{ij}) = -\beta_{ij}, \qquad i \neq j, \tag{1.9}$$

where

$$e = \sum_{i=1}^{n} \partial_i, \quad E = \sum_{i=1}^{n} u^i \partial_i.$$

Condition (1.8) follows from (1.6). Systems (1.7) and (1.8) are called *Darboux–Egorov system* (see Refs. 5 and 6) and imply the flatness of the metric η. The last condition (1.9) follows from the homogeneity properties. Given a solution of the above system, the Lamé coefficients $(H_1, \ldots, H_n)$ are obtained by solving the overdetermined system of PDEs,

$$\partial_j H_i = \beta_{ij} H_j, \qquad i \neq j, \tag{1.10}$$

$$e(H_i) = 0, \tag{1.11}$$

$$E(H_i) = DH_i, \tag{1.12}$$

where $D = -\frac{d}{2}$ is an eigenvalue of the skew-symmetric matrix $V_{ij} := (u^j - u^i)\beta_{ij}$.[1] In dimension $n = 3$, on the open set $u^1 \neq u^2 \neq u^3 \neq u^1$, the general solution of systems (1.8) and (1.9) is given as

$$\begin{aligned} \beta_{12} &= \frac{1}{u^2 - u^1} F_{12}\left(\frac{u^3 - u^1}{u^2 - u^1}\right), \\ \beta_{23} &= \frac{1}{u^3 - u^2} F_{23}\left(\frac{u^3 - u^1}{u^2 - u^1}\right), \\ \beta_{13} &= \frac{1}{u^3 - u^1} F_{13}\left(\frac{u^3 - u^1}{u^2 - u^1}\right). \end{aligned} \tag{1.13}$$

The remaining conditions (1.7) are equivalent to the following non-autonomous system of ordinary differential equations (ODEs):

$$\begin{aligned} \frac{dF_{12}}{dz} &= \frac{1}{z(z-1)} F_{13} F_{23}, \\ \frac{dF_{13}}{dz} &= -\frac{1}{z-1} F_{12} F_{23}, \\ \frac{dF_{23}}{dz} &= \frac{1}{z} F_{12} F_{13}, \end{aligned} \tag{1.14}$$

where $z := \frac{u^3 - u^1}{u^2 - u^1}$. It is well-known that three-dimensional Dubrovin–Frobenius manifolds are parameterized by solutions of a family of Painlevé VI equations (see Ref. 1). This can be easily proved by also studying system (1.14).

Theorem I.4. *System (1.14) is equivalent to the following sigma form of Painlevé VI equations (see Ref. 7):*

$$z^2(z-1)^2(\sigma'')^2 + 4\left[\sigma'(z\sigma' - \sigma)^2 - (\sigma')^2(z\sigma' - \sigma)\right] = -2R^2(\sigma')^2 + R^4\sigma', \tag{1.15}$$

where the parameter R^2 is the value of the first integral $I = F_{12}^2 + F_{13}^2 + F_{23}^2$.

Proof. First, note that $\frac{dI}{dz} = 0$, as shown by a simple computation. Hence, we set $I = R^2$. Following Ref. 8, it is easy to check that one can write the squares of the functions F_{ij} in terms of a single function $\sigma(z)$,

$$F_{12}^2 = \sigma', \tag{1.16}$$

$$F_{13}^2 = \sigma - z\sigma' + \frac{R^2}{2}, \tag{1.17}$$

$$F_{23}^2 = -\sigma + (z-1)\sigma' + \frac{R^2}{2}. \tag{1.18}$$

From Eqs. (1.16)–(1.18), we immediately have

$$z\frac{d}{dz}(F_{23}^2) = z(z-1)\frac{d}{dz}(F_{12}^2) = -(z-1)\frac{d}{dz}(F_{13}^2) = z(z-1)\sigma''(z). \tag{1.19}$$

On the other hand, due to (1.14), we have

$$z\frac{d}{dz}\left(F_{23}^2\right) = z(z-1)\frac{d}{dz}\left(F_{12}^2\right) = -(z-1)\frac{d}{dz}\left(F_{13}^2\right) = 2F_{12}F_{13}F_{32}. \tag{1.20}$$

By comparing these equations with (1.19) and taking the square, we obtain (1.15). ■

In dimension 4, there is a special class of Dubrovin–Frobenius manifolds that are also related to the Painlevé VI equation.[11] Dropping the assumption of symmetry of the rotation coefficients and allowing for different degrees of homogeneity for the Lamé coefficients, one ends up with the Darboux–Egorov system [(1.7) and (1.8)] with the additional constraint

$$E(\beta_{ij}) = (d_i - d_j - 1)\beta_{ij}, \quad i \neq j. \tag{1.21}$$

In dimension 3, systems (1.7), (1.8), and (1.21) reduce to a system of six ODEs that turned out to be equivalent to the full family of Painlevé VI.[12] The corresponding geometric structure is a generalization of the Dubrovin–Frobenius manifold structure, and it is called bi-flat structure.[8] A similar result can be obtained by studying the following system (see Ref. 13):

$$\partial_k\Gamma^i_{ij} = -\Gamma^i_{ij}\Gamma^i_{ik} + \Gamma^i_{ij}\Gamma^j_{jk} + \Gamma^i_{ik}\Gamma^k_{kj}, \quad i \neq k \neq j \neq i, \tag{1.22}$$

$$e(\Gamma^i_{ij}) = 0, \quad i \neq j, \tag{1.23}$$

$$E(\Gamma^i_{ij}) = -\Gamma^i_{ij}, \quad i \neq j. \tag{1.24}$$

System (1.22) is called Darboux–Tsarev system. Regular non-semisimple bi-flat structures in dimension 3 are also related to Painlevé transcendents. This was proved in Ref. 13 by studying the analog of the Darboux–Tsarev system in the non-semisimple case (see also Ref. 9 and 14 for an alternative approach based on the study of Okubo-type systems).

III. DUBROVIN–FROBENIUS METRIC IN THE GENERAL REGULAR CASE

Let M be a non-semisimple Dubrovin–Frobenius manifold of dimension n, with commutative and associative product $\circ$, metric η, unit vector field e, and Euler vector field E. Let M be *regular* near a point $m \in M$, meaning that each Jordan block of the operator $L = E\,\circ$ is associated with a different eigenvalue.

Let r be the number of Jordan blocks of L, and let $m_1, \ldots, m_r$ be their sizes. Any set of coordinates $u^1, \ldots, u^n$ for M can be re-labelled by means of the following notation: for each $\alpha \in \{2, \ldots, r\}$ and for each $j \in \{1, \ldots, m_\alpha\}$, we write

$$j(\alpha) = m_1 + \cdots + m_{\alpha-1} + j \tag{2.1}$$

[for $\alpha = 1$, we set $j(\alpha) = j$] so that $u^{j(\alpha)}$ denotes the jth coordinate associated with the αth Jordan block. From now on, we will write u^i when seeing the coordinate as running from 1 to the dimension of the manifold, and we will write $u^{i(\alpha)}$ when in need to highlight the Jordan block to which the coordinate refers. According to this notation, ∂_i and $\partial_{i(\alpha)}$ will denote the partial derivative with respect to u^i and $u^{i(\alpha)}$, respectively.

In Ref. 2, David and Hertling provided a generalization of canonical coordinates in the regular case. According to their results, we can assume that the product has the following form:

$$\partial_{i(\alpha)} \circ \partial_{j(\beta)} = \begin{cases} \delta_{\alpha\beta}\,\partial_{(i+j-1)(\alpha)}, & i+j \leq m_\alpha + 1, \\ 0, & i+j \geq m_\alpha + 2 \end{cases} \tag{2.2}$$

for all $i \in \{1, \ldots, m_\alpha\}$, $j \in \{1, \ldots, m_\beta\}$ for each $\alpha, \beta \in \{1, \ldots, r\}$. The unit vector field takes the form

$$e = \sum_{\alpha=1}^{r} \partial_{1(\alpha)}, \tag{2.3}$$

and the Euler vector field becomes

$$E = \sum_{s=1}^{n} u^s\,\partial_s. \tag{2.4}$$

The operator $L = E\,\circ$ is given by

$$L = L^{i(\alpha)}_{j(\beta)}\,\partial_{i(\alpha)} \otimes du^{j(\beta)}, \tag{2.5}$$

where

$$L^{i(\alpha)}_{j(\beta)} = \begin{cases} \delta_{\alpha\beta}\, u^{(i-j+1)(\alpha)}, & i \geq j, \\ 0, & i < j \end{cases} \tag{2.6}$$

for $\alpha, \beta \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, m_\alpha\}, j \in \{1, \ldots, m_\beta\}$.

In fact, given $\alpha, \beta \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, m_\alpha\}, j \in \{1, \ldots, m_\beta\}$, we have

$$\begin{aligned} L^{i(\alpha)}_{j(\beta)} &= \left(E \circ \partial_{j(\beta)}\right)^{i(\alpha)} = u^{k(\gamma)}\left(\partial_{k(\gamma)} \circ \partial_{j(\beta)}\right)^{i(\alpha)} \\ &= \begin{cases} u^{k(\gamma)}\delta_{\beta\gamma}\left(\partial_{(j+k-1)(\beta)}\right)^{i(\alpha)}, & 1 \leq k \leq m_\beta - j + 1, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} u^{k(\beta)}\,\delta_{\alpha\beta}\,\delta^{i}_{j+k-1}, & 1 \leq i - j + 1, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} \delta_{\alpha\beta}\, u^{(i-j+1)(\alpha)}, & i \geq j, \\ 0, & i < j. \end{cases} \end{aligned}$$

Remark 1. Due to the regularity condition, we are implicitly assuming that $u^{2(\alpha)} \neq 0$ and $u^{1(\alpha)} \neq u^{1(\beta)}$ if $\alpha \neq \beta$.

In order for the data $(\eta, \circ, e, E)$ to define an actual Dubrovin–Frobenius manifold, we have to impose all the axioms entering its definition.

In particular, we want to study conditions (1.1)–(1.6) in David–Hertling canonical coordinates. As stated in Ref. 2, the metric η is represented by a block diagonal matrix, each block of which is an upper triangular Hankel matrix [for instance, in the case of a single Jordan block, see (3.3)]. This follows from (1.1). Precisely,

$$\eta = \delta_{\alpha\beta}\,\overline{\eta}_{(i+j-1)(\alpha)}\,du^{i(\alpha)} \otimes du^{j(\beta)} \tag{2.7}$$

for some functions $\left\{\overline{\eta}_{(i)(\alpha)} \,|\, 1 \leq \alpha \leq r,\ 1 \leq i \leq m_\alpha\right\}$ and $\overline{\eta}_{(i)(\alpha)} = 0$ for $i \geq m_\alpha + 1$. Moreover, (1.4) implies the existence of a *metric potential H* such that

$$\overline{\eta}_{i(\alpha)} = \partial_{i(\alpha)} H \tag{2.8}$$

for all $i \in \{1, \ldots, m_\alpha\}$ for each $\alpha \in \{1, \ldots, r\}$.

Since we consider non-semisimple Dubrovin–Frobenius manifolds, there must exist at least one Jordan block of size greater or equal than 2. Without loss of generality, we then assume that the size of the first Jordan block is greater than 1. If one drops this assumption, analogous results will hold, where different coordinates will play the roles here played by u^1, u^2.

If we take into account that the metric must be homogeneous with respect to the Euler vector field and constant with respect to the unity vector field, we are able to get a further expression for the terms $\overline{\eta}_{(i)(\alpha)}$.

Theorem II.1. *The functions $\overline{\eta}_i$ appearing in (2.7) can be written as*

$$\overline{\eta}_i = (u^2)^{-d} F_i, \quad i \in \{1, \ldots, n\}, \tag{2.9}$$

for some functions $F_1, \ldots, F_n$ of the variables

$$z^j = \frac{u^{j+2} - u^1 \sum_{\alpha=2}^{r} \delta^{j+2}_{1(\alpha)}}{u^2}, \quad j \in \{1, \ldots, n-2\}, \tag{2.10}$$

such that

$$F_1 = -\sum_{\alpha=2}^{r} \partial_{z^{1(\alpha)-2}} f + C_1, \tag{2.11}$$

$$F_2 = -z^j\,\partial_{z^j} f - (d-1) f + C_2, \tag{2.12}$$

$$F_j = \partial_{z^{j-2}} f, \quad j \in \{3, \ldots, n\}, \tag{2.13}$$

for some function f of $z^1, \ldots, z^{n-2}$ and constants C_1, C_2. In particular, the quantity

$$\sum_{\alpha=1}^{r} F_{1(\alpha)} = C_1 \tag{2.14}$$

is a constant that vanishes whenever $d \neq 0$.

Proof. By imposing (1.6), we get

$$\sum_{\alpha=1}^{r} \partial_{1(\alpha)} \overline{\eta}_i = \mathscr{L}_e \overline{\eta}_i = 0$$

for $i \in \{1, \ldots, n\}$. It follows that each $\overline{\eta}_i$ can be written as

$$\overline{\eta}_i = \varphi_i \left(u^2, u^3 - u^1 \sum_{\alpha=2}^{r} \delta^3_{1(\alpha)}, \ldots, u^n - u^1 \sum_{\alpha=2}^{r} \delta^n_{1(\alpha)} \right) \tag{2.15}$$

for some function φ_i of $n-1$ variables. By the homogeneity condition (1.5), it can be rewritten as in (2.9) for some function F_i of the variables defined in (2.10).

The flatness of e with respect to ∇ implies that $d(\eta(e, \cdot)) = 0$ (see Ref. 2), that is,

$$\partial_{j(\beta)} \overline{\eta}_{i(\alpha)} \, du^{j(\beta)} \wedge du^{i(\alpha)} = 0.$$

Thus,

$$\partial_{j(\beta)} \overline{\eta}_{i(\alpha)} - \partial_{i(\alpha)} \overline{\eta}_{j(\beta)} = 0 \tag{2.16}$$

for all $i \in \{1, \ldots, m_\alpha\}$, $j \in \{1, \ldots, m_\beta\}$, and $\alpha, \beta \in \{1, \ldots, r\}$. In particular, for $i(\alpha), j(\beta) \in \{3, \ldots, n\}$, we get

$$\partial_{z^{j(\beta)-2}} F_{i(\alpha)} = \partial_{z^{i(\alpha)-2}} F_{j(\beta)}.$$

There must then exist a function f of the variables $z^1, \ldots, z^{n-2}$, realizing (2.13). By fixing $j(\beta) = 2$ and $i(\alpha) \in \{3, \ldots, n\}$ in (2.16), we obtain the relation

$$\partial_{i(\alpha)} \left((u^2)^{-d} F_2 \right) = \partial_2 \left((u^2)^{-d} F_{i(\alpha)} \right),$$

which amounts to

$$(u^2)^{-1} \partial_{z^{i(\alpha)-2}} F_2 = -d \, (u^2)^{-1} F_{i(\alpha)} + \partial_2 F_{i(\alpha)}$$

and, by the chain rule and (2.10),

$$\partial_{z^{i(\alpha)-2}} F_2 = -d \, F_{i(\alpha)} - \sum_{j=1}^{n-2} z^j \, \partial_{z^j} F_{i(\alpha)}.$$

By taking into account (2.13), we get

$$\partial_{z^{i(\alpha)-2}} F_2 = -d \, \partial_{z^{i(\alpha)-2}} f - \sum_{j=1}^{n-2} z^j \, \partial_{z^j} \partial_{z^{i(\alpha)-2}} f.$$

Then, for each $i \in \{1, \ldots, n-2\}$,

$$\partial_{z^i} F_2 = -d \, \partial_{z^i} f - z^j \, \partial_{z^j} \partial_{z^i} f,$$

that is,

$$\partial_{z^i}\left[F_2 - (1-d)f + z^j\,\partial_{z^j} f\right] = 0.$$

Therefore, the quantity $F_2 - (1-d)f + z^j\,\partial_{z^j} f$ equals some constant C_2, proving (2.12).

By taking $i(\alpha) = 1(\alpha), j(\beta) \in \{3,\ldots,n\}$ in (2.16) and summing over all $\alpha \in \{1,\ldots,r\}$, we get

$$\sum_{\alpha=1}^{r} \partial_{j(\beta)}\overline{\eta}_{1(\alpha)} = \sum_{\alpha=1}^{r} \partial_{1(\alpha)}\overline{\eta}_{j(\beta)},$$

that is,

$$(u^2)^{-d}\,\partial_{j(\beta)}\left(\sum_{\alpha=1}^{r} F_{1(\alpha)}\right) = \mathscr{L}_e\overline{\eta}_{j(\beta)}.$$

Thus,

$$\partial_{z^{j(\beta)-2}}\left(\sum_{\alpha=1}^{r} F_{1(\alpha)}\right) = 0.$$

This means that

$$\partial_{z^j}\left(\sum_{\alpha=1}^{r} F_{1(\alpha)}\right) = 0$$

for all $j \in \{1,\ldots,n-2\}$, proving that $\sum_{\alpha=1}^{r} F_{1(\alpha)}$ must be equal to some constant C_1. Condition (2.11) follows.

On the other hand, by taking $i(\alpha) = 1(\alpha), j(\beta) = 2$ in (2.16) and summing over all $\alpha \in \{1,\ldots,r\}$, we get

$$\partial_2\left(\sum_{\alpha=1}^{r} (u^2)^{-d} F_{1(\alpha)}\right) = 0,$$

which, since $\sum_{\alpha=1}^{r} F_{1(\alpha)} = C_1$, amounts to

$$\partial_2\left((u^2)^{-d} C_1\right) = 0.$$

This implies $d\,C_1 = 0$, meaning that the constant C_1 must vanish whenever $d \neq 0$. ∎

Proposition II.2. Up to constants, the function f appearing in (2.11)–(2.13) is related to the metric potential H by the following formula:

$$H = (u^2)^{1-d} f + C_2\,\varphi(u^2) + C_1\,u^1, \tag{2.17}$$

where

$$\varphi(u^2) = \begin{cases} \dfrac{(u^2)^{1-d}}{1-d} & \text{if } d \neq 1, \\ \ln u^2 & \text{if } d = 1. \end{cases} \tag{2.18}$$

Proof. By (2.8) and (2.9), we have

$$\partial_i H = (u^2)^{-d} F_i(z^1,\ldots,z^{n-2}) \tag{2.19}$$

for each $i \in \{1,\ldots,n\}$. For $i \geq 3$, we get

$$\partial_i H = (u^2)^{-d}\,\partial_{z^{i-2}} f,$$

that is,

$$\partial_{z^{i-2}} H = (u^2)^{1-d}\,\partial_{z^{i-2}} f$$

or

$$\partial_{z^{i-2}}\Big(H-(u^2)^{1-d}f\Big)=0.$$

It follows that

$$H=(u^2)^{1-d}f+K(u^1,u^2) \tag{2.20}$$

for some function $K(u^1,u^2)$. For $i=2$ in (2.19), we get

$$\partial_2 H=(u^2)^{-d}\left(-z^j\,\partial_{z^j}f-(d-1)f+C_2\right),$$

that is, by the chain rule and (2.10),

$$(u^2)^{-d}\Big((1-d)f-z^j\,\partial_{z^j}f\Big)+\partial_2 K=(u^2)^{-d}\left(-z^j\,\partial_{z^j}f-(d-1)f+C_2\right),$$

yielding

$$\partial_2 K(u^1,u^2)=C_2\,(u^2)^{-d}.$$

Then,

$$K(u^1,u^2)=\begin{cases}C_2\,\dfrac{(u^2)^{1-d}}{1-d}+k(u^1) & \text{if } d\neq 1,\\ C_2\,\ln u^2+k(u^1) & \text{if } d=1\end{cases} \tag{2.21}$$

for some function $k(u^1)$. By putting together (2.20) and (2.21), one gets

$$H=(u^2)^{1-d}f+C_2\,\varphi(u^2)+k(u^1)$$

for

$$\varphi(u^2)=\begin{cases}\dfrac{(u^2)^{1-d}}{1-d} & \text{if } d\neq 1,\\ \ln u^2 & \text{if } d=1.\end{cases} \tag{2.22}$$

For $i=1$ in (2.19), we finally get

$$\partial_1 H=(u^2)^{-d}F_1,$$

that is, by the chain rule and (2.10),

$$-(u^2)^{-d}\sum_{\alpha=2}^{r}\partial_{z^{1(\alpha)-2}}f+\partial_1 k(u^1)=-(u^2)^{-d}\sum_{\alpha=2}^{r}\partial_{z^{1(\alpha)-2}}f+(u^2)^{-d}C_1.$$

Thus,

$$\partial_1 k(u^1)=(u^2)^{-d}C_1=\left.\begin{cases}0 & \text{if } d\neq 0\\ C_1 & \text{if } d=0\end{cases}\right\}=C_1$$

implying

$$k(u^1)=C_1\,u^1+C_3$$

for some constant C_3. We conclude that

$$H = (u^2)^{1-d} f + C_2\,\varphi(u^2) + C_1\,u^1 + C_3$$

for $\varphi(u^2)$ as in (2.22). ■

IV. THE CASE OF A SINGLE JORDAN BLOCK: EXPLICIT RESULTS UP TO DIMENSION 4

In this section, we classify regular non-semisimple Dubrovin–Frobenius manifold structures up to dimension 4 in the case where the operator L has a single Jordan block. Due to the results of Sec. III in the specific case where L has a single Jordan block of size n, the unit vector field becomes $e = \partial_1$, and in canonical coordinates, we have

$$\partial_i \circ \partial_j = \begin{cases} \partial_{i+j-1}, & i+j \le n+1, \\ 0, & i+j \ge n+2 \end{cases} \tag{3.1}$$

for all $i, j \in \{1, \ldots, n\}$ and $u^i = u^{i(1)}$ for each $i \in \{1, \ldots, n\}$. The operator L is described by the following lower triangular Toeplitz matrix:

$$L = \begin{bmatrix} u^1 & 0 & 0 & \cdots & 0 & 0 \\ u^2 & u^1 & 0 & \cdots & 0 & 0 \\ u^3 & u^2 & u^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{n-1} & u^{n-2} & u^{n-3} & \cdots & u^1 & 0 \\ u^n & u^{n-1} & u^{n-2} & \cdots & u^2 & u^1 \end{bmatrix}. \tag{3.2}$$

The metric is represented by an upper triangular Hankel matrix that only depends on the coordinate u^2 and on n functions $F_1, \ldots, F_n$ of the variables,

$$z^i = \frac{u^{i+2}}{u^2}, \qquad i \in \{1, \ldots, n-2\}.$$

It takes the following form:

$$\eta = (u^2)^{-d} \begin{bmatrix} F_1 & F_2 & F_3 & \cdots & F_{n-1} & F_n \\ F_2 & F_3 & F_4 & \cdots & F_n & 0 \\ F_3 & F_4 & F_5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-1} & F_n & 0 & \cdots & 0 & 0 \\ F_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{3.3}$$

In particular, F_1 is equal to a constant C_1 that vanishes whenever $d \neq 0$, and other F_i's are expressed in terms of a function $f(z^1, \ldots, z^{n-2})$ by

$$F_2 = -z^i\,\partial_{z^i} f - (d-1) f + C_2, \tag{3.4}$$

$$F_j = \partial_{z^{j-2}} f \qquad \forall j \in \{3, \ldots, n\} \tag{3.5}$$

for some constant C_2.

A. Dimension $n = 2$

Let M be a two-dimensional Dubrovin–Frobenius manifold with product $\circ$, metric η, unit vector field e, and Euler vector field E. Let us require M to be regular and the operator $L = E\,\circ$ to have a single Jordan block near a point $m \in M$. The unit and the Euler vector fields read, respectively, $e = \partial_1$ and $E = u^1\partial_1 + u^2\partial_2$. It follows directly from (3.3) that the metric has the form

$$\eta = (u^2)^{-d} \begin{bmatrix} C_1 & C_2 \\ C_2 & 0 \end{bmatrix} \tag{3.6}$$

for some constant C_1, which vanishes whenever $d \neq 0$ and for some non-zero constant C_2.

We are able to recover flat coordinates and an explicit expression for the Dubrovin–Frobenius prepotential, as pointed out in the following result.

Theorem III.1. *Flat coordinates coincide with the canonical ones when $d = 0$. Otherwise, they are given by*

$$x^1(u^1, u^2) = u^1,$$
$$x^2(u^1, u^2) = \frac{(u^2)^{1-d}}{1-d}$$

when $d \neq 1$ and by

$$x^1(u^1, u^2) = u^1,$$
$$x^2(u^1, u^2) = \ln u^2$$

when $d = 1$. In all the cases, the prepotential is given by

$$F(x^1, x^2) = \frac{C_1}{6}(x^1)^3 + \frac{C_2}{2}(x^1)^2 x^2 \tag{3.7}$$

up to second-order polynomial terms. In flat coordinates, the unit and the Euler vector fields are, respectively, given by $e = \tilde{\partial}_1$ and

$$E = \begin{cases} x^1 \tilde{\partial}_1 + \tilde{\partial}_2 & \text{if } d = 1, \\ x^1 \tilde{\partial}_1 + x^2(1-d)\tilde{\partial}_2 & \text{if } d \neq 1. \end{cases}$$

Proof. If $d = 0$, then the metric in (3.6) is constant; thus, flat coordinates coincide with the canonical ones. Let us now fix $d \neq 0$. In this case, the flat coordinates are

$$x^1(u^1, u^2) = u^1,$$
$$x^2(u^1, u^2) = \frac{(u^2)^{1-d}}{1-d}$$

when $d \neq 1$ and

$$x^1(u^1, u^2) = u^1,$$
$$x^2(u^1, u^2) = \ln u^2$$

when $d = 1$. In both cases, in flat coordinates, the metric becomes

$$\tilde{\eta} = \begin{bmatrix} C_1 & C_2 \\ C_2 & 0 \end{bmatrix},$$

and the structure constants equal the ones in canonical coordinates,

$$\tilde{c}_{ij}^k = c_{ij}^k, \qquad i, j, k \in \{1, 2\}.$$

It follows that up to second-order polynomial terms, the Dubrovin–Frobenius prepotential F is of the form

$$F(x^1, x^2) = \frac{C_1}{6}(x^1)^3 + \frac{C_2}{2}(x^1)^2 x^2,$$

and it follows that in flat coordinates, the unit and the Euler vector fields are of the form as stated above. ■

B. Dimension $n = 3$

Let M be a three-dimensional Dubrovin–Frobenius manifold with product $\circ$, metric η, unit vector field e, and Euler vector field E. Let us require M to be regular and the operator $L = E\,\circ$ to have a single Jordan block near a point $m \in M$. The unit and the Euler vector fields read, respectively, $e = \partial_1$ and $E = u^1\partial_1 + u^2\partial_2 + u^3\partial_3$. We already know from (3.3) that the metric is of the form

$$\eta = (u^2)^{-d}\begin{bmatrix} F_1\left(\frac{u^3}{u^2}\right) & F_2\left(\frac{u^3}{u^2}\right) & F_3\left(\frac{u^3}{u^2}\right) \\ F_2\left(\frac{u^3}{u^2}\right) & F_3\left(\frac{u^3}{u^2}\right) & 0 \\ F_3\left(\frac{u^3}{u^2}\right) & 0 & 0 \end{bmatrix} \tag{3.8}$$

for some functions F_1, F_2, and F_3 and that F_1 is equal to a constant C_1 that vanishes whenever $d \neq 0$. It turns out from the zero-curvature conditions that the functions F_2, F_3 must be solutions to the following system of ODEs:

$$\begin{cases} F_2' + z\,F_3' + d\,F_3 = 0, \\ 2\,F_3\,F_3'' - 3\,(F_3')^2 = 0. \end{cases} \tag{3.9}$$

In fact, let us introduce the variable $z = \frac{u^3}{u^2}$. We have already seen that there exists a function $f(z)$ such that

$$F_2(z) = -zf'(z) - (d-1)f(z) + C_2,$$
$$F_3(z) = f'(z)$$

for some constant C_2. It follows that

$$F_2' + z\,F_3' + d\,F_3 = 0.$$

Moreover, by requiring that $R^1_{232} = 0$, one obtains the Liouville-type differential equation,

$$2\,F_3\,F_3'' - 3\,(F_3')^2 = 0.$$

This suffices to make all of the conditions in (1.1)–(1.6) hold without imposing more. So far, what we know about the functions F_1, F_2, F_3 is that F_1 equals some constant C_1 and that F_2, F_3 are solutions to system (3.9). Two expressions for the function f appearing in (3.4) and (3.5) are then possible, as shown below.

Theorem III.2. *The function f realizing (3.4) and (3.5) is either provided by*

$$f(z) = C_3\,z + C_4 \tag{3.10}$$

for some constants C_3, C_4 or by

$$f(z) = -\frac{C_4}{z + C_3} + C_5 \tag{3.11}$$

for some constants C_3, C_4, C_5.

Proof. The first condition in (3.9) amounts to (3.4) and (3.5), while the second one can be rewritten as

$$2f'(z)f'''(z) - 3\,(f''(z))^2 = 0. \tag{3.12}$$

Assuming $f''(z) \neq 0$, the solutions to Eq. (3.12) can be written as (3.11), while (3.10) is recovered by considering solutions corresponding to $f''(z) = 0$. ■

Summarizing, two cases may occur: either

$$\begin{cases} F_1(z) = C_1, \\ F_2(z) = -C_3\, d\, z + C_2, \\ F_3(z) = C_3 \end{cases} \tag{3.13}$$

for some constant C_1 that vanishes for $d \neq 0$ and some constants C_2, C_3 or

$$\begin{cases} F_1(z) = C_1, \\ F_2(z) = \dfrac{C_3\, C_4}{(z + C_3)^2} - \dfrac{(2-d)\, C_4}{z + C_3} + C_2, \\ F_3(z) = \dfrac{C_4}{(z + C_3)^2} \end{cases} \tag{3.14}$$

for some constant C_1 that vanishes for $d \neq 0$ and some constants C_2, C_3, C_4.

Proposition III.3. In the case of (3.13), flat coordinates are given by

$$\begin{aligned} x^1(u^1,u^2,u^3) &= u^1, \\ x^2(u^1,u^2,u^3) &= (u^2)^{-d}\, u^3 + \frac{C_2 (u^2)^{1-d}}{C_3(1-d)}, \\ x^3(u^1,u^2,u^3) &= \frac{2}{2-d} (u^2)^{\frac{2-d}{2}} \end{aligned}$$

when $d \notin \{0, 1, 2\}$, by

$$\begin{aligned} x^1(u^1,u^2,u^3) &= u^1, \\ x^2(u^1,u^2,u^3) &= \frac{u^3}{(u^2)^2} - \frac{C_2}{C_3\, u^2}, \\ x^3(u^1,u^2,u^3) &= \ln u^2 \end{aligned}$$

when $d = 2$, by

$$\begin{aligned} x^1(u^1,u^2,u^3) &= u^1, \\ x^2(u^1,u^2,u^3) &= \frac{u^3}{u^2} + \frac{C_2}{C_3} \ln u^2, \\ x^3(u^1,u^2,u^3) &= 2\sqrt{u^2} \end{aligned}$$

when $d = 1$, and, trivially, by

$$\begin{aligned} x^1(u^1,u^2,u^3) &= u^1, \\ x^2(u^1,u^2,u^3) &= u^2, \\ x^3(u^1,u^2,u^3) &= u^3 \end{aligned}$$

when $d = 0$.

The proof is a straightforward computation.

Proposition III.4. Let x^1, x^2, x^3 denote flat coordinates. Up to second-order polynomial terms, in the case of (3.13), the prepotential is given by

$$F(x^1,x^2,x^3) = \frac{C_3}{2} (x^1)^2\, x^2 + \frac{C_3}{2}\, x^1\, (x^3)^2 \tag{3.15}$$

when $d \neq 0$ and by

$$F(x^1,x^2,x^3) = \frac{C_1}{6} (x^1)^3 + \frac{C_2}{2} (x^1)^2\, x^2 + \frac{C_3}{2} (x^1)^2\, x^3 + \frac{C_3}{2}\, x^1\, (x^2)^2 \tag{3.16}$$

when $d = 0$ *(in this latter case, flat coordinates coincide with the canonical ones). If* $d \neq 0$*, then in flat coordinates, the multiplication is written as*

$$\tilde{\partial}_1 \circ \tilde{\partial}_1 = \tilde{\partial}_1,$$
$$\tilde{\partial}_1 \circ \tilde{\partial}_2 = \tilde{\partial}_2,$$
$$\tilde{\partial}_1 \circ \tilde{\partial}_3 = \tilde{\partial}_3,$$
$$\tilde{\partial}_2 \circ \tilde{\partial}_2 = 0,$$
$$\tilde{\partial}_2 \circ \tilde{\partial}_3 = 0,$$
$$\tilde{\partial}_3 \circ \tilde{\partial}_3 = \tilde{\partial}_2,$$

and the Euler vector field reads

$$E = x^1\, \tilde{\partial}_1 + (1-d)\, x^2\, \tilde{\partial}_2 + \frac{2-d}{2}\, x^3\, \tilde{\partial}_3$$

if $d \notin \{0,1,2\}$*,*

$$E = x^1\, \tilde{\partial}_1 + \frac{C_2}{C_3}\, \tilde{\partial}_2 + \frac{1}{2}\, x^3\, \tilde{\partial}_3$$

if $d = 1$*, and*

$$E = x^1\, \tilde{\partial}_1 - x^2\, \tilde{\partial}_2 + \tilde{\partial}_3$$

if $d = 2$*. In flat coordinates, the unit vector field is* $e = \tilde{\partial}_1$ *for each value of* d*.*

The proof is a straightforward computation.

Analogous results can be achieved for the case of (3.14), as presented below.

Proposition III.5. In the case of (3.14), flat coordinates and the Euler vector field are given by

$$x^1(u^1,u^2,u^3) = u^1 + \frac{C_2 C_3 (u^2)^2 + C_2\, u^2 u^3 - C_4 (u^2)^2}{C_1\,(C_3\, u^2 + u^3)},$$

$$x^2(u^1,u^2,u^3) = \left[\left(-C_2 C_3 \sqrt{C_1 C_4} - C_4 (C_1 - C_4 + C_2 C_3)\right)(u^2)^{\frac{2C_4 - \sqrt{C_1 C_4}}{C_4}} - C_2\, u^3 (u^2)^{\frac{C_4 - \sqrt{C_1 C_4}}{C_4}} \left(C_4 + \sqrt{C_1 C_4}\right)\right] \frac{1}{C_4 (C_1 - C_4)\,(C_3\, u^2 + u^3)},$$

$$x^3(u^1,u^2,u^3) = \left[\left(C_2 C_3 \sqrt{C_1 C_4} - C_4 (C_1 - C_4 + C_2 C_3)\right)(u^2)^{\frac{2C_4 + \sqrt{C_1 C_4}}{C_4}} - C_2\, u^3 (u^2)^{\frac{C_4 + \sqrt{C_1 C_4}}{C_4}} \left(C_4 - \sqrt{C_1 C_4}\right)\right] \frac{1}{C_4 (C_1 - C_4)\,(C_3\, u^2 + u^3)},$$

$$E = x^1\, \tilde{\partial}_1 + b x^2\, \tilde{\partial}_2 + c\, x^3\, \tilde{\partial}_3,$$

where

$$b = \frac{(C_2 C_3 + C_1)(C_4)^{\frac{3}{2}} - C_1 C_2 C_3 \sqrt{C_4} + (C_4)^2 \sqrt{C_1}}{C_4\,(C_2 C_3 \sqrt{C_1} + C_2 C_3 \sqrt{C_4} + C_1 \sqrt{C_4} - C_4 \sqrt{C_4})} + \frac{-(C_1)^{\frac{3}{2}} C_4 - (C_4)^{\frac{5}{2}}}{C_4\,(C_2 C_3 \sqrt{C_1} + C_2 C_3 \sqrt{C_4} + C_1 \sqrt{C_4} - C_4 \sqrt{C_4})},$$

$$c = \frac{\sqrt{C_4}\,(C_1 - C_4)}{\sqrt{C_1} C_4 - (C_4)^{\frac{3}{2}}}$$

when $d = 0$*,* $C_1 \neq 0$*, and* $C_1 \neq C_4$*, by*

$$x^1(u^1,u^2,u^3) = u^1 + \frac{(u^2)^2 (\ln u^2)^2}{2\,(C_3\, u^2 + u^3)} + \frac{C_2\, u^2 \left(2 \ln u^2 - (\ln u^2)^2 - 2\right)}{2 C_4},$$

$$x^2(u^1,u^2,u^3) = -\frac{(u^2)^2 \ln u^2}{C_3\, u^2 + u^3} + \frac{C_2\, u^2 \left(\ln u^2 - 1\right)}{C_4},$$

$$x^3(u^1,u^2,u^3) = -\frac{(u^2)^2}{C_3\, u^2 + u^3} + \frac{C_2\, u^2}{C_4},$$

$$E = (x^1 - x^2)\, \tilde{\partial}_1 + (x^2 + x^3)\, \tilde{\partial}_2 + x^3\, \tilde{\partial}_3$$

when $d = 0$ *and* $C_1 = 0$*, by*

$$
\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 - \frac{(u^2)^2}{C_3\,u^2+u^3} + \frac{C_2\,u^2}{C_4},\\
x^2(u^1,u^2,u^3) &= -\frac{(u^2)^3}{2\,(C_3\,u^2+u^3)} + \frac{C_2\,(u^2)^2}{4C_4},\\
x^3(u^1,u^2,u^3) &= -\frac{u^2}{C_3\,u^2+u^3} + \frac{C_2\,\ln u^2}{C_4},\\
E &= x^1\,\tilde{\partial}_1 + 2\,x^2\,\tilde{\partial}_2 + \frac{C_2}{C_4}\,\tilde{\partial}_3
\end{aligned}
$$

when $d = 0$ *and* $C_1 = C_4$, *by*

$$
\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 + \frac{2(u^2)^2(C_4 - C_2\,C_3)}{C_4(d)^2(C_3\,u^2+u^3)},\\
x^2(u^1,u^2,u^3) &= \frac{(C_2C_3 - C_4\,(1-d))(u^2)^{2-d}}{C_4\,(1-d)(C_3\,u^2+u^3)} + \frac{C_2\,u^3(u^2)^{1-d}}{C_4\,(1-d)(C_3\,u^2+u^3)},\\
x^3(u^1,u^2,u^3) &= \frac{2C_2\,u^3(u^2)^{\frac{2-d}{2}}}{C_4\,(2-d)(C_3\,u^2+u^3)} + \frac{-(C_4(2-d) - 2C_2C_3)(u^2)^{\frac{4-d}{2}}}{C_4\,(2-d)(C_3\,u^2+u^3)},\\
E &= x^1\,\tilde{\partial}_1 - \frac{d-2}{2}\,x^2\,\tilde{\partial}_2 - (d-1)\,x^3\,\tilde{\partial}_3
\end{aligned}
$$

when $d \notin \{0,1,2\}$ *and* $C_3 \neq 0$, *by*

$$
\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 + \frac{2\big(-C_2\,u^2\,u^3 + C_4(u^2)^2\big)}{C_4\,(d)^2\,u^3},\\
x^2(u^1,u^2,u^3) &= \frac{2C_2\,u^3(u^2)^{\frac{2-d}{2}} - C_4\,(2-d)(u^2)^{\frac{4-d}{2}}}{C_4\,(2-d)\,u^3},\\
x^3(u^1,u^2,u^3) &= \frac{C_2\,u^3(u^2)^{1-d} - C_4\,(1-d)(u^2)^{2-d}}{C_4\,(1-d)\,u^3},\\
E &= x^1\,\tilde{\partial}_1 - \frac{d-2}{2}\,x^2\,\tilde{\partial}_2 - (d-1)\,x^3\,\tilde{\partial}_3
\end{aligned}
$$

when $d \notin \{0,1,2\}$ *and* $C_3 = 0$, *by*

$$
\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 + \frac{2(u^2)^2}{C_3\,u^2+u^3} - \frac{2C_2\,u^2}{C_4},\\
x^2(u^1,u^2,u^3) &= -\frac{u^2}{C_3\,u^2+u^3} + \frac{C_2\,\ln u^2}{C_4},\\
x^3(u^1,u^2,u^3) &= -\frac{(u^2)^{\frac{3}{2}}}{C_3\,u^2+u^3} + \frac{2C_2\,\sqrt{u^2}}{C_4},\\
E &= x^1\,\tilde{\partial}_1 - \frac{d-2}{2}\,x^2\,\tilde{\partial}_2 + \left(\frac{C_2}{C_4} - (d-1)\,x^3\right)\tilde{\partial}_3
\end{aligned}
$$

when $d = 1$, *and by*

$$
\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 + \frac{(u^2)^2}{2(C_3\,u^2+u^3)} - \frac{C_2\,u^2}{2C_4},\\
x^2(u^1,u^2,u^3) &= -\frac{u^2}{C_3\,u^2+u^3} + \frac{C_2\,\ln u^2}{C_4},\\
x^3(u^1,u^2,u^3) &= -\frac{1}{C_3\,u^2+u^3} - \frac{C_2}{C_4\,u^2},\\
E &= x^1\,\tilde{\partial}_1 + \left(\frac{C_2}{C_4} - \frac{d-2}{2}\,x^2\right)\tilde{\partial}_2 - (d-1)\,x^3\,\tilde{\partial}_3
\end{aligned}
$$

when $d = 2$. In each of these cases, the unit vector field reads $e = \tilde{\partial}_1$.

Here are explicit expressions for the Dubrovin–Frobenius potential in some selected cases.

Example III.6. Let us fix $d = 0$, $C_1 = C_4$, and $C_2 = 0$. In flat coordinates, the metric becomes

$$\tilde{\eta} = \begin{bmatrix} C_4 & 0 & 0 \\ 0 & 0 & -C_4 \\ 0 & -C_4 & 0 \end{bmatrix},$$

the multiplication is given by

$$\begin{aligned}
\tilde{\partial}_1 \circ \tilde{\partial}_1 &= \tilde{\partial}_1, \\
\tilde{\partial}_1 \circ \tilde{\partial}_2 &= \tilde{\partial}_2, \\
\tilde{\partial}_1 \circ \tilde{\partial}_3 &= \tilde{\partial}_3, \\
\tilde{\partial}_2 \circ \tilde{\partial}_2 &= -\frac{3\sqrt{2}}{4}\sqrt{\frac{x^3}{x^2}}\tilde{\partial}_2 + \frac{\sqrt{2}}{4}\left(\frac{x^3}{x^2}\right)^{\frac{3}{2}}\tilde{\partial}_3, \\
\tilde{\partial}_2 \circ \tilde{\partial}_3 &= -\tilde{\partial}_1 - \frac{3\sqrt{2}}{4}\sqrt{\frac{x^2}{x^3}}\tilde{\partial}_2 - \frac{3\sqrt{2}}{4}\sqrt{\frac{x^3}{x^2}}\tilde{\partial}_3, \\
\tilde{\partial}_3 \circ \tilde{\partial}_3 &= \frac{\sqrt{2}}{4}\left(\frac{x^2}{x^3}\right)^{\frac{3}{2}}\tilde{\partial}_2 - \frac{3\sqrt{2}}{4}\sqrt{\frac{x^2}{x^3}}\tilde{\partial}_3,
\end{aligned}$$

and the prepotential reads

$$F(x^1, x^2, x^3) = \frac{2\sqrt{2}}{3}C_4\,(x^2)^{\frac{3}{2}}(x^3)^{\frac{3}{2}} + \frac{C_4}{6}(x^1)^3 - C_4\,x^1\,x^2\,x^3 \tag{3.17}$$

up to second-order polynomial terms. In flat coordinates, the unit and the Euler vector fields are, respectively, written as

$$e = \tilde{\partial}_1$$

and

$$E = x^1\,\tilde{\partial}_1 + 2\,x^2\,\tilde{\partial}_2.$$

Example III.7. Let us fix $d = 2$ and $C_2 = 0$. In flat coordinates, the metric becomes

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & C_4 \\ 0 & C_4 & 0 \\ C_4 & 0 & 0 \end{bmatrix},$$

the multiplication is given by

$$\begin{aligned}
\tilde{\partial}_1 \circ \tilde{\partial}_1 &= \tilde{\partial}_1, \\
\tilde{\partial}_1 \circ \tilde{\partial}_2 &= \tilde{\partial}_2, \\
\tilde{\partial}_1 \circ \tilde{\partial}_3 &= \tilde{\partial}_3, \\
\tilde{\partial}_2 \circ \tilde{\partial}_2 &= -\frac{3}{2}\left(\frac{x^2}{x^3}\right)^2\tilde{\partial}_1 + 3\frac{x^2}{x^3}\tilde{\partial}_2 + \tilde{\partial}_3, \\
\tilde{\partial}_2 \circ \tilde{\partial}_3 &= \left(\frac{x^2}{x^3}\right)^3\tilde{\partial}_1 - \frac{3}{2}\left(\frac{x^2}{x^3}\right)^2\tilde{\partial}_2, \\
\tilde{\partial}_3 \circ \tilde{\partial}_3 &= -\frac{3}{4}\left(\frac{x^2}{x^3}\right)^4\tilde{\partial}_1 + \left(\frac{x^2}{x^3}\right)^3\tilde{\partial}_2,
\end{aligned}$$

and the prepotential reads

$$F(x^1,x^2,x^3) = \frac{C_4}{2}(x^1)^2 x^3 + \frac{C_4}{2} x^1 (x^2)^2 + \frac{C_4}{8}\frac{(x^2)^4}{x^3} \tag{3.18}$$

up to second-order polynomial terms. In flat coordinates, the unit and the Euler vector fields are, respectively, written as

$$e = \tilde{\partial}_1$$

and

$$E = x^1\tilde{\partial}_1 - x^3\tilde{\partial}_3.$$

Example III.8. Let us fix $d = 2$ and $C_2 = 1$. In flat coordinates, the metric becomes

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & C_4 \\ 0 & C_4 & 0 \\ C_4 & 0 & 0 \end{bmatrix},$$

the multiplication is given by

$$\begin{aligned}
&\tilde{\partial}_1 \circ \tilde{\partial}_1 = \tilde{\partial}_1,\\
&\tilde{\partial}_1 \circ \tilde{\partial}_2 = \tilde{\partial}_2,\\
&\tilde{\partial}_1 \circ \tilde{\partial}_3 = \tilde{\partial}_3,\\
&\tilde{\partial}_2 \circ \tilde{\partial}_2 = -\frac{3}{2(C_4)^2(x^3)^2} W\Big(C_4 x^3 e^{C_4 x^2 - 1}\Big)^2 \tilde{\partial}_1 + \frac{3}{C_4 x^3} W\Big(C_4 x^3 e^{C_4 x^2-1}\Big)\tilde{\partial}_2 + \tilde{\partial}_3,\\
&\tilde{\partial}_2 \circ \tilde{\partial}_3 = \frac{1}{(C_4)^3(x^3)^3} W\Big(C_4 x^3 e^{C_4 x^2 - 1}\Big)^3 \tilde{\partial}_1 - \frac{3}{2(C_4)^2(x^3)^2} W\Big(C_4 x^3 e^{C_4 x^2-1}\Big)^2\tilde{\partial}_2,\\
&\tilde{\partial}_3 \circ \tilde{\partial}_3 = -\frac{3}{4(C_4)^4(x^3)^4} W\Big(C_4 x^3 e^{C_4 x^2 - 1}\Big)^4 \tilde{\partial}_1 + \frac{1}{(C_4)^3(x^3)^3} W\Big(C_4 x^3 e^{C_4 x^2-1}\Big)^3\tilde{\partial}_2,
\end{aligned}$$

and the prepotential reads

$$\begin{aligned}
F(x^1,x^2,x^3) = {} & \frac{1}{24(C_4)^3 x^3}\Big(3\,W\Big(C_4 x^3 e^{C_4 x^2-1}\Big)^4 + 22\,W\Big(C_4 x^3 e^{C_4 x^2-1}\Big)^3 + 63\,W\Big(C_4 x^3 e^{C_4 x^2-1}\Big)^2 + 72\,W\Big(C_4 x^3 e^{C_4 x^2-1}\Big)\Big)\\
& + \frac{C_4}{2}(x^1)^2 x^3 + \frac{C_4}{2} x^1 (x^2)^2
\end{aligned} \tag{3.19}$$

up to second-order polynomial terms, where W denotes the principal branch of the Lambert W function (see Ref. 15 and references therein). In flat coordinates, the unit and the Euler vector fields are, respectively, written as

$$e = \tilde{\partial}_1$$

and

$$E = x^1\tilde{\partial}_1 + \frac{1}{C_4}\tilde{\partial}_2 - x^3\tilde{\partial}_3.$$

C. Dimension $n = 4$

Let M be a four-dimensional Dubrovin–Frobenius manifold with product $\circ$, metric η, unit vector field e, and Euler vector field E. Let us require M to be regular and the operator $L = E\circ$ to have a single Jordan block near a point $m \in M$. The unit and the Euler vector fields read, respectively, $e = \partial_1$ and $E = u^1\partial_1 + u^2\partial_2 + u^3\partial_3 + u^4\partial_4$. We already know from (3.3) that the metric is of the form

$$\eta = (u^2)^{-d}\begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ F_2 & F_3 & F_4 & 0 \\ F_3 & F_4 & 0 & 0 \\ F_4 & 0 & 0 & 0 \end{bmatrix} \tag{3.20}$$

for some functions F_1, F_2, F_3, and F_4 of the variables $z = \frac{u^3}{u^2}$, $w = \frac{u^4}{u^2}$. In particular, F_1 is equal to a constant C_1, which vanishes whenever $d \neq 0$, and from (3.4) and (3.5), we know that F_2, F_3, and F_4 can be expressed as

$$F_2(z,w) = -z\,\partial_z f(z,w) - w\,\partial_w f(z,w) - (d-1)\,f(z,w) + C_2, \tag{3.21}$$

$$F_3(z,w) = \partial_z f(z,w), \tag{3.22}$$

$$F_4(z,w) = \partial_w f(z,w) \tag{3.23}$$

for some function $f(z,w)$ and some constant C_2. By the flatness conditions, two expressions for f are possible, as shown below. This fully classifies regular four-dimensional Dubrovin–Frobenius manifolds whose operator $L = E\circ$ has a single Jordan block.

Theorem III.9. *The function f realizing (3.4) and (3.5) is either provided by*

$$f(z,w) = C_3\, w\, e^{C_4 z} + h(z) \tag{3.24}$$

for some constants C_3, C_4 and some function $h(z)$, which is a solution to

$$h'''(z) - 2\,C_4\,h''(z) + C_4^2\,h'(z) + 2\,C_3\,C_4\,e^{C_4 z} = 0, \tag{3.25}$$

or by

$$f(z,w) = C_3 - \frac{A(z)}{2\,B(z) + w} \tag{3.26}$$

for some constant C_3 and solutions $A(z)$, $B(z)$ to the following system of ODEs:

$$A''\,A - (A')^2 + 2\,(C_2 + (1-d)\,C_3)\,A = 0, \tag{3.27}$$

$$A\,B''' - A'\,(B'' + 1) + 2\,(C_2 + (1-d)\,C_3)(B' + z) + C_1 = 0. \tag{3.28}$$

Proof. By requiring that $R^1_{243} = 0$, we get

$$2\,\partial_w f\,\partial_w^3 f - 3(\partial_w^2 f)^2 = 0. \tag{3.29}$$

Let us distinguish two cases: $\partial_w^2 f \neq 0$ and $\partial_w^2 f = 0$. In the first case, we obtain

$$f(z,w) = C(z) - \frac{A(z)}{2B(z) + w} \tag{3.30}$$

for some functions $A(z)$, $B(z)$, and $C(z)$, while in the second one, we obtain

$$f(z,w) = w\,h_1(z) + h_2(z) \tag{3.31}$$

for some functions $h_1(z)$, $h_2(z)$.

If f is as in (3.30), then condition $R^3_{343} = 0$ implies that the function $C(z)$ must be equal to a constant C_3. Conditions $R^3_{234} = 0$ and $R^2_{322} = 0$ yield, respectively,

$$A''\,A - (A')^2 + 2\,(C_2 + (1-d)\,C_3)\,A = 0$$

and

$$A\,B''' - A'\,(B'' + 1) + 2\,(C_2 + (1-d)\,C_3)(B' + z) + C_1 = 0.$$

All the other conditions in (1.1)–(1.6) hold without imposing more.

If, on the other hand, f is as in (3.31), condition $R^3_{234} = 0$ implies that

$$h_1(z)\,h_1''(z) - (h_1'(z))^2 = 0. \tag{3.32}$$

Solutions to (3.32) are given by $h_1(z) = C_3\, e^{C_4 z}$ for some constants C_3 and C_4 so that

$$f(z,w) = C_3\, w\, e^{C_4 z} + h_2(z).$$

By imposing condition $R^2_{322} = 0$, we get

$$h_2'''(z) - 2C_4\, h_2''(z) + C_4^2\, h_2'(z) + 2C_3\, C_4\, e^{C_4 z} = 0$$

that yields

$$h_2(z) = C_7 - \frac{e^{C_4 z}}{C_4^2}\left[C_3\, C_4^2\, z^2 - C_4\,(2\,C_3 + C_5)\, z - C_4\, C_6 + 2C_3 + C_5\right]$$

when $C_4 \neq 0$ and

$$h_2(z) = C_5\, z^2 + C_6\, z + C_7$$

when $C_4 = 0$ for some constants C_5, C_6, and C_7 so that f becomes, respectively,

$$f(z,w) = C_3\, w\, e^{C_4 z} + C_7 - \frac{e^{C_4 z}}{C_4^2}\left[C_3\, C_4^2\, z^2 - C_4\,(2\,C_3 + C_5)\, z - C_4\, C_6 + 2\,C_3 + C_5\right] \tag{3.33}$$

and

$$f(z,w) = C_3\, w\, e^{C_4 z} + C_5\, z^2 + C_6\, z + C_7. \tag{3.34}$$

In both cases, it turns out that all the other conditions in (1.1)–(1.6) hold without imposing more. ∎

Proposition III.10. The functions $A(z)$ and $B(z)$ appearing in (3.27) and (3.28) are expressed via hyperbolic functions and second-order polynomials,

$$A(z) = \frac{C_2 + (1-d)\,C_3}{C_4^2}\,\sinh^2(C_4(z + C_5)),$$
$$B(z) = C_6\,\cosh(2C_4(z + C_5)) + C_7\,\sinh(2C_4(z + C_5)) - \frac{z}{2}\left(\frac{C_1}{C_2 + (1-d)\,C_3} + 4C_4C_7\right) - \frac{z^2}{2} + C_8$$

for some constants C_4, C_5, C_6, C_7, C_8 if $C_2 + (1-d)C_3 \neq 0$ and

$$A(z) = C_5\,(\cosh(C_4\, z) + \sinh(C_4\, z)), \tag{3.35}$$

$$B(z) = \frac{1}{2\,(C_4)^3\, C_5}((2C_6\, C_4\, C_5 + C_1)\cosh(C_4\, z) + (2C_6\, C_4\, C_5 - C_1)\sinh(C_4\, z)) - \frac{z^2}{2} + C_7\, z + C_8 \tag{3.36}$$

for some constants C_4, C_5, C_6, C_7, C_8 if $C_2 + (1-d)C_3 = 0$.

Below, flat coordinates are computed for selected other cases, together with some Dubrovin–Frobenius prepotentials.

Example III.11. Let us consider the case (3.24) with $C_3 = 1$, $C_4 = 0$, and $d \neq 0$. Equation (3.25) becomes $h'''(z) = 0$, yielding $h(z) = az^2 + bz + c$ for some constants a, b, c. In particular, we choose $a = c = 0$ and $b = 1$ so that $h(z) = z$ and $f(z,w) = z + w$. When $d \neq 1$, in the flat coordinates,

$$x^1(u^1,u^2,u^3,u^4) = u^1,$$
$$x^2(u^1,u^2,u^3,u^4) = (u^2)^{-d}(u^3 + u^4),$$
$$x^3(u^1,u^2,u^3,u^4) = \frac{1}{2}u^2 + u^3,$$
$$x^4(u^1,u^2,u^3,u^4) = \frac{1}{1-d}(u^2)^{1-d},$$

we have

$$\tilde{\eta} = \begin{bmatrix} 0 & 1 & 0 & C_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ C_2 & 0 & 1 & 0 \end{bmatrix},$$

$$e = \tilde{\partial}_1,$$
$$E = x^1\,\tilde{\partial}_1 + (1-d)\,x^2\,\tilde{\partial}_2 + x^3\,\tilde{\partial}_3 + (1-d)\,x^4\,\tilde{\partial}_4.$$

Up to second-order polynomial terms, the prepotential is given by

$$F(x^1,x^2,x^3,x^4) = \frac{C_2}{2}(x^1)^2x^4 + x^1x^3x^4 + \frac{1}{2}(x^1)^2x^2 + \frac{(d-1)^{\frac{d-3}{d-1}}\,e^{\frac{2\pi i}{d-1}}\,(x^4)^{\frac{d-3}{d-1}}}{2(d+1)(d-3)}$$

when $d \notin \{-1,0,3\}$, *by*

$$F(x^1,x^2,x^3,x^4) = \frac{1}{4}(x^4)^2\,\ln x^4 + \frac{C_2}{2}(x^1)^2x^4 + x^1x^3x^4 + \frac{1}{2}(x^1)^2x^2$$

when $d = -1$, *by*

$$F(x^1,x^2,x^3,x^4) = \frac{C_2}{2}(x^1)^2x^2 + x^1x^2x^3 + \frac{1}{2}(x^1)^2x^3 + \frac{1}{2}(x^1)^2x^4 + \frac{1}{2}x^1(x^2)^2 + \frac{1}{6}(x^2)^3$$

when $d = 0$, *and by*

$$F(x^1,x^2,x^3,x^4) = \frac{C_2}{2}(x^1)^2x^4 + x^1x^3x^4 + \frac{1}{2}(x^1)^2x^2 - \frac{1}{16}\ln x^4$$

when $d = 3$. *The case where* $d = 1$ *must be treated separately. In the flat coordinates,*

$$x^1(u^1,u^2,u^3,u^4) = u^1,$$
$$x^2(u^1,u^2,u^3,u^4) = \frac{u^3+u^4}{u^2},$$
$$x^3(u^1,u^2,u^3,u^4) = \frac{1}{2}u^2 + u^3,$$
$$x^4(u^1,u^2,u^3,u^4) = \ln u^2,$$

the unit and the Euler vector fields are given by

$$e = \tilde{\partial}_1, \quad E = x^1\,\tilde{\partial}_1 + x^3\,\tilde{\partial}_3 + \tilde{\partial}_4.$$

The metric is as the one for $d \neq 1$, *and up to second-order polynomial terms, the prepotential is*

$$F(x^1,x^2,x^3,x^4) = \frac{1}{8}e^{2x^4} + \frac{C_2}{2}(x^1)^2x^4 + x^1x^3x^4 + \frac{1}{2}(x^1)^2x^2.$$

Example III.12. Let us consider the case (3.24) with $C_3 = C_4 = 1$ *and* $d \neq 0$. *Equation (3.25) becomes* $h'''(z) - 2h''(z) + h'(z) + 2e^z = 0$, *yielding* $h(z) = a - (z^2 + bz + c)e^z$ *for some constants* a, b, c. *In particular, we choose* $a = b = c = 0$ *so that* $h(z) = -z^2e^z$ *and* $f(z,w) = (w - z^2)e^z$. *When* $d \neq 0, 1, 2$, *the flat coordinates are*

$$x^1(u^1,u^2,u^3,u^4) = u^1 + \frac{u^2}{2(1-d)},$$
$$x^2(u^1,u^2,u^3,u^4) = C_2\,\ln u^2 - 2(1-d)\,e^{\frac{u^3}{u^2}} - \frac{(u^3)^2 - u^2u^4}{(u^2)^2}\,e^{\frac{u^3}{u^2}},$$
$$x^3(u^1,u^2,u^3,u^4) = (u^2)^{-d-1}\big(u^2u^4 - (u^3)^2\big)\,e^{\frac{u^3}{u^2}} + \frac{C_2(u^2)^{1-d}}{1-d},$$
$$x^4(u^1,u^2,u^3,u^4) = \frac{(u^2)^{2-d}}{2-d}.$$

In such coordinates, the unit and the Euler vector fields are, respectively, written as $e = \tilde{\partial}_1$ *and*

$$E = x^1\,\tilde{\partial}_1 + C_2\,\tilde{\partial}_2 - (d-1)\,x^3\,\tilde{\partial}_3 - (d-2)\,x^4\,\tilde{\partial}_4.$$

For $d = -1$, *the metric is*

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \end{bmatrix},$$

and up to second-order polynomial terms, the prepotential is given by

$$F(x^1,x^2,x^3,x^4) = -\frac{1}{32}\sqrt[3]{3}C_2(x^4)^{\frac{4}{3}}\,\ln\!\left(3x^4\right) + \frac{15}{128}\sqrt[3]{3}C_2(x^4)^{\frac{4}{3}} + \frac{1}{32}\sqrt[3]{9}(x^4)^{\frac{2}{3}}x^3 + \frac{3}{32}\sqrt[3]{3}(x^4)^{\frac{4}{3}}x^2 + \frac{1}{2}(x^1)^2x^3 - \frac{1}{4}x^1x^2x^4.$$

For $d = -2$, *the metric is*

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{6} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 & 0 \end{bmatrix},$$

and up to second-order polynomial terms, the prepotential is given by

$$F(x^1,x^2,x^3,x^4) = -\frac{1}{45}\sqrt{2}C_2(x^4)^{\frac{5}{4}}\,\ln\!\left(2\sqrt{x^4}\right) + \frac{4}{75}\sqrt{2}C_2(x^4)^{\frac{5}{4}} + \frac{2}{45}\sqrt{2}(x^4)^{\frac{5}{4}}x^2 + \frac{1}{72}\sqrt{x^4}\,x^3 + \frac{1}{2}(x^1)^2x^3 - \frac{1}{6}x^1x^2x^4.$$

The case $d = 2$ *must be treated separately. In the flat coordinates,*

$$\begin{aligned} x^1(u^1,u^2,u^3,u^4) &= u^1 - \frac{u^2}{2}, \\ x^2(u^1,u^2,u^3,u^4) &= \frac{2(u^2)^2 + u^2u^4 - (u^3)^2}{(u^2)^2}\, e^{\frac{u^3}{u^2}}, \\ x^3(u^1,u^2,u^3,u^4) &= \frac{u^2u^4 - (u^3)^2}{(u^2)^3}\, e^{\frac{u^3}{u^2}} - \frac{C_2}{u^2}, \\ x^4(u^1,u^2,u^3,u^4) &= \ln u^2, \end{aligned}$$

we have

$$e = \tilde{\partial}_1,\;\; E = x^1\,\tilde{\partial}_1 - x^3\,\tilde{\partial}_3 + \tilde{\partial}_4,\;\; \tilde{\eta} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & C_2 \end{bmatrix}.$$

Up to second-order polynomial terms, the prepotential is

$$F(x^1,x^2,x^3,x^4) = -\frac{1}{16}x^3\,e^{2x^4} + \left(C_2 + \frac{x^2}{2}\right)e^{x^4} + \frac{C_2}{2}x^1(x^4)^2 + \frac{1}{2}x^1x^2x^4 + \frac{1}{2}(x^1)^2x^3.$$

In the case where $d = 1$, *which must be handled separately as well, flat coordinates are given by*

$$x^1(u^1,u^2,u^3,u^4) = u^1 + \frac{u^2}{2} - \frac{u^2}{2}\ln u^2,$$
$$x^2(u^1,u^2,u^3,u^4) = \frac{u^2u^4-(u^3)^2}{(u^2)^2}\, e^{\frac{u^3}{u^2}} + C_2\ln u^2,$$
$$x^3(u^1,u^2,u^3,u^4) = \left(\frac{u^2u^4-(u^3)^2}{(u^2)^2}\ln u^2 + 2\right) e^{\frac{u^3}{u^2}} + \frac{C_2}{2}\left(\ln u^2\right)^2,$$
$$x^4(u^1,u^2,u^3,u^4) = u^2$$

and

$$e = \tilde{\partial}_1,\ E = \left(x^1 - \frac{x^4}{2}\right)\tilde{\partial}_1 + C_2\,\tilde{\partial}_2 + x^2\,\tilde{\partial}_3 + x^4\,\tilde{\partial}_4,$$
$$\tilde{\eta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Up to second-order polynomial terms, the prepotential is

$$F(x^1,x^2,x^3,x^4) = \frac{C_2}{24}(x^4)^2\left(\ln x^4\right)^3 - \frac{3}{16}\left(C_2 + \frac{2}{3}x^2\right)(x^4)^2\left(\ln x^4\right)^2 + \frac{7}{16}\left(C_2 + \frac{6}{7}x^2 + \frac{4}{7}x^3\right)(x^4)^2\ln x^4 + \frac{1}{2}x^1x^3x^4 + \frac{1}{2}(x^1)^2x^2 - \frac{7x^2+6x^3}{16}(x^4)^2.$$

The last case to be considered separately is the one for $d = 0$, as C_1 may not vanish. Here, the flat coordinates are

$$x^1(u^1,u^2,u^3,u^4) = u^1 + \frac{u^2}{2},$$
$$x^2(u^1,u^2,u^3,u^4) = \frac{u^2u^4-(u^3)^2}{u^2}\, e^{\frac{u^3}{u^2}} - \frac{C_1-2C_2}{2}\,u^2,$$
$$x^3(u^1,u^2,u^3,u^4) = \frac{u^2u^4-(u^3)^2-2(u^2)^2}{(u^2)^2}\, e^{\frac{u^3}{u^2}} - \frac{C_1-4C_2}{4}\ln u^2,$$
$$x^4(u^1,u^2,u^3,u^4) = \frac{(u^2)^2}{2}.$$

In such coordinates, the unit and the Euler vector fields are, respectively, $e = \tilde{\partial}_1$ and

$$E = x^1\,\tilde{\partial}_1 + x^2\,\tilde{\partial}_2 - \left(C_2 - \frac{C_1}{4}\right)\tilde{\partial}_3 - 2x^4\,\tilde{\partial}_4,$$

and up to second-order polynomial terms, the prepotential reads

$$F(x^1,x^2,x^3,x^4) = \frac{3(C_1-4C_2)\ln(2x^4) - 8C_1}{72}\sqrt{2}(x^4)^{\frac{3}{2}} + \frac{32C_2+24x^3}{72}\sqrt{2}(x^4)^{\frac{3}{2}} - \frac{1}{8}x^2x^4\ln(x^4) + \frac{C_1}{6}(x^1)^3 + \frac{1}{2}(x^1)^2x^2 - \frac{1}{2}x^1x^3x^4.$$

V. A MENTION OF THE MULTIPLE-BLOCKS CASES

As seen in Sec. III, an expression for the Dubrovin–Frobenius metric in terms of a function f realizing (2.11)–(2.13) can be achieved in the case where the operator $L = E\circ$ has multiple Jordan blocks as well. This section is devoted to show how, in this case, it is possible to reduce the conditions defining a Frobenius manifold to a single ODE in dimension 3 and to a system of PDEs in dimension 4.

A. The three-dimensional case

In dimension 3, the only regular non-semisimple case with multiple Jordan blocks is the one of two blocks of dimensions 2 and 1, respectively.

We already know that there exists a function f of the variable

$$z = \frac{u^3 - u^1}{u^2}$$

such that the metric can be written as

$$\eta = (u^2)^{-d} \begin{bmatrix} F_1 & F_2 & 0 \\ F_2 & 0 & 0 \\ 0 & 0 & F_3 \end{bmatrix}$$

for

$$F_1(z) = -f'(z) + C_1,$$
$$F_2(z) = -zf'(z) - (d-1)f(z) + C_2,$$
$$F_3(z) = f'(z),$$

where C_1, C_2 are constants. In particular, the quantity $F_1 + F_3 = C_1$ must vanish whenever $d \neq 0$. The flatness condition amounts to the following equation:

$$z^2(-zf' - (d-1)f + C_2)f'f''' = -z^2(f'')^2\Big(zf' - \frac{1}{2}(-zf' - (d-1)f + C_2)\Big) + 2\big(-dzf' - (-zf' - (d-1)f + C_2)\big)zf'f'' + d(f')^2\Big(-dzf' + \frac{d-2}{2}(-zf' - (d-1)f + C_2)\Big). \quad (4.1)$$

By solving (4.1), one can determine explicitly the function f, which turns out to be expressed in terms of hyperbolic functions.

B. The four-dimensional case

In dimension 4, three rearrangements in Jordan blocks are possible. Let us briefly illustrate the results of computations:

- In the case of two blocks of sizes 3 and 1, respectively, we have

$$\eta = (u^2)^{-d} \begin{bmatrix} F_1 & F_2 & F_3 & 0 \\ F_2 & F_3 & 0 & 0 \\ F_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_4 \end{bmatrix},$$

where

$$F_1(z,w) = -\partial_w f(z,w) + C_1,$$
$$F_2(z,w) = -z\partial_z f(z,w) - w\,\partial_w f(z,w) - (d-1)f(z,w) + C_2,$$
$$F_3(z,w) = \partial_z f(z,w),$$
$$F_4(z,w) = \partial_w f(z,w)$$

for some constants C_1, C_2 and a function f of the variables

$$z = \frac{u^3}{u^2}, \qquad w = \frac{u^4 - u^1}{u^2}.$$

In particular, the quantity $F_1 + F_4$ is a constant that must vanish whenever $d \neq 0$.

- In the case of two blocks, both of size 2, we have

$$\eta = (u^2)^{-d}\begin{bmatrix} F_1 & F_2 & 0 & 0 \\ F_2 & 0 & 0 & 0 \\ 0 & 0 & F_3 & F_4 \\ 0 & 0 & F_4 & 0 \end{bmatrix},$$

where

$$\begin{aligned} F_1(z,w) &= -\partial_z f(z,w) + C_1, \\ F_2(z,w) &= -z\partial_z f(z,w) - w\,\partial_w f(z,w) - (d-1)f(z,w) + C_2, \\ F_3(z,w) &= \partial_z f(z,w), \\ F_4(z,w) &= \partial_w f(z,w) \end{aligned}$$

for some constants C_1, C_2 and a function f if the variables

$$z = \frac{u^3 - u^1}{u^2}, \qquad w = \frac{u^4}{u^2}.$$

In particular, the quantity $F_1 + F_3$ is a constant that must vanish whenever $d \neq 0$.

- In the case of three blocks, of sizes 2, 1, and 1, respectively, we have

$$\eta = (u^2)^{-d}\begin{bmatrix} F_1 & F_2 & 0 & 0 \\ F_2 & 0 & 0 & 0 \\ 0 & 0 & F_3 & 0 \\ 0 & 0 & 0 & F_4 \end{bmatrix},$$

where

$$\begin{aligned} F_1(z,w) &= -\partial_z f(z,w) - \partial_w f(z,w) + C_1, \\ F_2(z,w) &= -z\,\partial_z f(z,w) - w\,\partial_w f(z,w) - (d-1)f(z,w) + C_2, \\ F_3(z,w) &= \partial_z f(z,w), \\ F_4(z,w) &= \partial_w f(z,w) \end{aligned}$$

for some constants C_1, C_2 and a function f of the variables

$$z = \frac{u^3 - u^1}{u^2}, \qquad w = \frac{u^4 - u^1}{u^2}.$$

In particular, the quantity $F_1 + F_3 + F_4$ is a constant that must vanish whenever $d \neq 0$.

Let us consider the flatness conditions in the first case. They amount to the following system of PDEs for the third derivatives of f:

$$\partial_z^3 f = \frac{3(\partial_z^2 f)^2}{2\partial_z f}, \tag{4.2}$$

$$\partial_z^2\partial_w f = \frac{\partial_z\partial_w f\left(\partial_z f\,\partial_z\partial_w f + 2\,\partial_z^2 f\,\partial_w f\right)}{2\partial_z f\,\partial_w f}, \tag{4.3}$$

$$\begin{aligned} \partial_z\partial_w^2 f = \frac{1}{2w(\partial_z f)^2\,\partial_w f}&\Big(2w\,\partial_z f\,\partial_w f(\partial_z\partial_w f)^2 + \big(-\partial_w f\,\partial_z^2 f\,(w\,\partial_w f + (d-1)f - C_2) \\ &+ (\partial_z f)^2\big((d-2)\,\partial_w f + w\,\partial_w^2 f\big)\big)\,\partial_z\partial_w f + \partial_z f\,\partial_z^2 f\,\partial_w f\left(w\,\partial_w^2 f + d\,\partial_w f\right)\Big), \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
\partial_w^3 f = {} & \frac{1}{2\,w^3\,\partial_w f\,(\partial_z f)^3}\big(-\partial_z f\,\partial_w f\,w^2\,(w\,\partial_w f + (d-1)f - C_2)\,(\partial_z\partial_w f)^2 \\
& - \big((w^3\,(\partial_w f)^2 - 2\,(-(d-1)f + C_2)\,w^2\,\partial_w f + (-(d-1)f - w\,C_1 + C_2)\,\partial_z f + (-(d-1)f + C_2)^2\,w)\,\partial_z^2 f \\
& - 3\,(\partial_z f)^2\,(w^3\,\partial_w^2 f - (-4\,w^2\,d\,\partial_w f)\frac{1}{3} - \frac{d-2}{3}\,\partial_z f + (-\frac{d}{3}\,(-(d-1)f + C_2)\,w))\big)\,\partial_w f\,\partial_z\partial_w f \\
& + \partial_z f\,\big((w^2\,(w\,\partial_w f + (d-1)f - C_2)\,\partial_w^2 f + (w^2\,\partial_w f - \partial_z f + (d-1)\,wf - w\,C_2)\,d\,\partial_w f)\,\partial_w f\,\partial_z^2 f \\
& + (\partial_z f)^2\,(w^2\,(\partial_w^2 f)^2 - w\,\partial_w f\,(d+4)\,\partial_w^2 f - 2\,(\partial_w f)^2\,d)\,w)\big).
\end{aligned}
\tag{4.5}
$$

In the remaining cases, flatness conditions are given by similar but much more cumbersome systems of third-order PDEs. We skip details of these computations.

We conclude this section by providing two examples of solutions to systems (4.2)–(4.5).

Example IV.1. If $d = 0$, then the function

$$f(z,w) = a\,z + b\,w + c$$

(where a, b, and c are constants) is a solution to systems (4.2)–(4.5). With this choice of f, the Dubrovin–Frobenius metric turns out to be constant in canonical coordinates. It reads

$$\eta = \begin{bmatrix} C_1 - b & C_2 + c & a & 0 \\ C_2 + c & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{bmatrix},$$

and up to second-order polynomial terms, the Dubrovin–Frobenius potential is

$$F(u^1,u^2,u^3,u^4) = \frac{C_1 - b}{6}\,(u^1)^3 + \frac{C_2 + c}{2}\,(u^1)^2\,u^2 + \frac{a}{2}\,(u^1)^2\,u^3 + \frac{a}{2}\,u^1\,(u^2)^2 + \frac{b}{6}\,(u^4)^3. \tag{4.6}$$

Example IV.2. Let $L = E\,\circ$ have two Jordan blocks of sizes 3 and 1. When looking for a function f of the form

$$f(z,w) = a\,z + g(w)$$

for some function $g(w)$, systems (4.2)–(4.5) come down to a single ODE for $g(w)$,

$$2\,w^2\,g'(w)\,g'''(w) - w^2\,\big(g''(w)\big)^2 + (d+4)\,w\,g'(w)\,g''(w) + 2d\,\big(g'(w)\big)^2 = 0. \tag{4.7}$$

This yields

$$g(w) = a_1 + \frac{d^2\,(a_2)^2}{16a_3\,(d-1)\,w} + a_2\,w^{-\frac{d}{2}} + a_3\,w^{1-d}$$

when $d \neq 1$ and

$$g(w) = a_1 - \frac{(a_2)^2\,\ln w}{16a_3} + \frac{a_2}{\sqrt{w}} + \frac{a_3}{w}$$

when $d = 1$ for some constants a_1, a_2, and a_3. For instance, when $d = 2$ in the flat coordinates,

$$
\begin{aligned}
x^1(u^1,u^2,u^3,u^4) &= \frac{u^3}{(u^2)^2} + \frac{a - C_2}{au^2} + \frac{(a_2)^2}{4a\,a_3\,(u^4 - u^1)} + \frac{a_2 + a_3}{a\,(u^4 - u^1)}, \\
x^2(u^1,u^2,u^3,u^4) &= \ln u^2, \\
x^3(u^1,u^2,u^3,u^4) &= \ln(u^4 - u^1), \\
x^4(u^1,u^2,u^3,u^4) &= u^1,
\end{aligned}
$$

the metric becomes

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & -\dfrac{(2a_3 + a_2)^2}{4\,a_3} & 0 \\ a & 0 & 0 & 0 \end{bmatrix},$$

and up to second-order polynomial terms, the Dubrovin–Frobenius potential is

$$F(x^1, x^2, x^3, x^4) = -\frac{(2a_3 + a_2)^2}{8a_3}\left(2e^{x^3} + (x^3)^2 x^4\right) + \frac{a}{2}\left(x^1 (x^4)^2 + (x^2)^2 x^4\right). \tag{4.8}$$

In flat coordinates, the unit and the Euler vector fields are, respectively, written as

$$\tilde{e} = \tilde{\partial}_4$$

and

$$\tilde{E} = -x^1\,\tilde{\partial}_1 + \tilde{\partial}_2 + \tilde{\partial}_3 + x^4\,\tilde{\partial}_4.$$

VI. CONCLUSIONS

In this paper, we have studied regular Dubrovin–Frobenius manifold structures $(\eta, \circ, e, E)$. Assuming that the Jordan form of the operator of multiplication by the Euler vector field L contains a single Jordan block and using a special set of coordinates introduced by David and Hertling where the operator L and the Dubrovin–Frobenius metric take the form

$$L = \begin{bmatrix} u^1 & 0 & 0 & \cdots & 0 & 0 \\ u^2 & u^1 & 0 & \cdots & 0 & 0 \\ u^3 & u^2 & u^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{n-1} & u^{n-2} & u^{n-3} & \cdots & u^1 & 0 \\ u^n & u^{n-1} & u^{n-2} & \cdots & u^2 & u^1 \end{bmatrix},$$

$$\eta = \begin{bmatrix} \partial_1 H & \partial_2 H & \partial_3 H & \cdots & \partial_{n-1} H & \partial_n H \\ \partial_2 H & \partial_3 H & \partial_4 H & \cdots & \partial_n H & 0 \\ \partial_3 H & \partial_4 H & \partial_5 H & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial_{n-1} H & \partial_n H & 0 & \cdots & 0 & 0 \\ \partial_n H & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

we have shown that, up to constants, the metric potential H has the form

$$H = (u^2)^{1-d} f + C_2\,\varphi(u^2) + C_1\,u^1, \tag{5.1}$$

where

$$\varphi(u^2) = \begin{cases} \dfrac{(u^2)^{1-d}}{1-d} & \text{if } d \neq 1, \\ \ln u^2 & \text{if } d = 1 \end{cases} \tag{5.2}$$

and $f = f(z^1, \ldots, z^{n-2})$ with $z^i = \frac{u^{i+2}}{u^2}$ for each $i = 1, \ldots, n-2$. In dimension $n = 2, 3, 4$, we have obtained explicit formulas for the metric potential H (see Table I) and, in some special cases, we have computed the flat coordinates and the corresponding prepotential.

TABLE I. Metric potential.

$n = 2$	$f = C_3$
$n = 3$	$f = C_3 z + C_4$ or $f = \frac{C_4}{z+C_3} + C_5$.
$n = 4$	$f = C_3 w e^{C_4 z} + C_7 - \frac{e^{C_4 z}}{C_4^2}\left[C_3 C_4^2 z^2 - C_4(2C_3 + C_5)z - C_4 C_6 + 2C_3 + C_5\right]$ or $f = C_3 w e^{C_4 z} + C_5 z^2 + C_6 z + C_7$ or $f = C_3 - \frac{A(z)}{2B(z)+w}$ with $z = \frac{u^3}{u^2}$, $w = \frac{u^4}{u^2}$, and $A(z)$ and $B(z)$ defined as in Proposition III.10

We also considered the case of multiple Jordan blocks in dimensions 3 and 4, showing that the flatness conditions reduce to a third-order ODE in the first case and to a system of third-order PDEs in the remaining cases, computing the flat coordinates and the corresponding prepotential in some selected cases.

According to the general theory the metrics η^{-1} and $L\eta^{-1}$, define a flat pencil of metrics. Therefore, a by-product of our results is a list of non-semisimple flat pencils of metrics that define the bi-Hamiltonian structures of the principal hierarchies of the associated Dubrovin–Frobenius manifolds. The study of this class of bi-Hamiltonian structures and of their bi-Hamiltonian deformations in the non-semisimple case is at a preliminary stage, and only a few results are available so far (see, for instance, Ref. 16). The explicit examples obtained in this work might be the starting point of future investigations.

ACKNOWLEDGMENTS

The authors were supported by funds of INFN (Istituto Nazionale di Fisica Nucleare) by IS-CSN4 Mathematical Methods of Nonlinear Physics. P.L. was supported by funds of H2020-MSCA-RISE-2017 under Project No. 778010 IPaDEGAN. The authors acknowledge GNFM-INdAM for supporting activities that contributed to the research reported in this paper.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Paolo Lorenzoni: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Sara Perletti**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] B. Dubrovin, "Geometry of 2D topological field theory," in *Integrable Systems and Quantum Groups*, edited by M. Francaviglia and S. Greco, Lecture Notes in Mathematics Vol. 1260 (Springer, 1996), pp. 120–348.
[2] L. David and C. Hertling, "Regular F-manifolds: Initial conditions and Frobenius metrics," Ann. Sc. Norm. Super. Pisa, Cl. Sci. **17**(5), 1121–1152 (2017).
[3] R. Dijkgraaf, H. Verlinde, and E. Verlinde, "Topological strings in $d < 1$," Nucl. Phys. B **352**(1), 59–86 (1991).
[4] E. Witten, "On the structure of the topological phase of two-dimensional gravity," Nucl. Phys. B **340**(2–3), 281–332 (1990).
[5] G. Darboux, *Leçons sur les systémes orthogonaux et les coordonnées curvilignes* (Gauthier Villars, Paris, 1910).
[6] D. T. Egorov, *Collected Papers on Differential Geometry* (Nauka, Moscow, 1970) (in Russian).
[7] K. Okamoto, "Studies on the Painlevé equations," Ann. Mat. Pura Appl. **146**, 337–381 (1986).
[8] A. Arsie and P. Lorenzoni, "From the Darboux–Egorov system to bi-flat F-manifolds," J. Geom. Phys. **70**, 98 (2013).

[9] M. Kato, T. Mano, and J. Sekiguchi, “Flat structure on the space of isomonodromic deformations,” SIGMA **16**, 110 (2020).
[10] G. Ricci, “Dei sistemi di congruenze ortogonali in una varietà qualunque,” Mem. R. Accad. Naz. Lincei, Cl. Sci. **2**(5), 276 (1896).
[11] S. Romano, “4-dimensional Frobenius manifolds and Painlevé VI,” Math. Ann. **360**(3), 715–751 (2014).
[12] P. Lorenzoni, “Darboux–Egorov system, bi-flat F-manifolds and Painlevé VI,” Int. Math. Res. Not. **2014**(12), 3279–3302.
[13] A. Arsie and P. Lorenzoni, “F-manifolds, multi-flat structures and Painlevé trascendents,” Asian J. Math. **23**(5), 877–904 (2019).
[14] H. Kawakami and T. Mano, “Regular flat structure and generalized Okubo system,” Commun. Math. Phys. **369**(2), 403–431 (2019).
[15] R. M. Coreless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, “On the Lambert W function,” Adv. Comput. Math. **5**, 329–359 (1996).
[16] A. Della Vedova, P. Lorenzoni, and A. Savoldi, “Deformations of non-semisimple Poisson pencils of hydrodynamic type,” Nonlinearity **29**(9), 2715–2754 (2016).