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ON THE SUBOPTIMALITY OF THE *P*-VERSION DISCONTINUOUS GALERKIN METHODS FOR FIRST ORDER HYPERBOLIC PROBLEMS

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Abstract. We address the issue of the suboptimality in the *p*-version discontinuous Galerkin (dG) methods for first order hyperbolic problems. The convergence rate is derived for the upwind dG scheme on tensor product meshes in any dimension. The standard proof in seminal work [14] leads to suboptimal convergence in terms of the polynomial degree by $3/2$ order for general convection fields, with the exception of piecewise multi-linear convection fields, which rather yield optimal convergence. Such suboptimality is not observed numerically. Thus, it might be caused by a limitation of the analysis, which we partially overcome: for a special class of convection fields, we shall show that the dG method has a *p*-convergence rate suboptimal by $1/2$ order only.

1 Introduction

Discontinuous Galerkin (dG) finite element methods were introduced in the early 1970s for the numerical solution of first-order hyperbolic problems [17] and the weak imposition of inhomogeneous boundary conditions for elliptic problems [16]. In the past several decades, dG methods have enjoyed considerable success as a standard variational framework for the numerical solution of many classes of problems involving partial differential equations (PDEs); see, e.g., monographs [8, 9, 10] for reviews of some of the main developments of dG methods. The interest in dG methods can be attributed to a number of factors, including the great flexibility in dealing with *hp*-adaptivity and general shaped elements [7, 5, 6], as well as in solving convection- dominated PDEs; see, e.g., early works [3, 2] concerning hyperbolic conservation laws and convection-diffusion problems.

Due to missing tools in the analysis, the convergence rate always contains suboptimality in terms of the polynomial degree *p*. In [13], the first optimal convergence rate of *hp*-dG methods is derived for linear convection problems by using the SUPG stabilisation. However, the authors provide numerical evidence that the hp-optimal convergence rate is achieved even without such stabilisation. In seminal work [14], based on (back then) novel optimal approximation results for the L^2-orthogonal projection, the hp-optimal convergence rate is derived for dG methods applied to hyperbolic problems, under the technical assumption that the convection field is piecewise linear. Moreover, whenever the above assumption is violated, the theoretical analysis in [14] leads to error bounds that are suboptimal in terms of p by $3/2$ order. Such suboptimality is yet not observed in the numerical experiments. Over the last two decades, the above mentioned technical assumption became standard in hp-dG methods for convection-diffusion-reaction and hyperbolic problems; see, e.g., [12, 5, 6, 4]. It is still an open question , whether the p-suboptimality for dG methods by $3/2$ order is true or not in general.

Our contribution represents a further step in shedding light on this issue. Notably, we present the a priori error analysis for hp-dG methods applied to pure hyperbolic problems employing a class of convection field, including nonpolynomial cases. The new error is h-optimal and p-suboptimal by $1/2$ order only.

The rest of the paper is organized as follows: the continuous problem and its dG discretization are addressed in Section 2; the classical analysis from [13] is re-elaborated in Section 3, whereas the improved bounds under suitable assumptions on the convection field are the topic of Section 4; we collect the conclusions in Section 5.

Throughout, we employ a standard notation for Sobolev spaces [1].

2 The continuous problem and its dG formulation

2.1 The continuous problem

Let Ω be a bounded polyhedral domain in $\mathbb{R}^d$, $d \in \mathbb{N}$, with boundary Γ. We denote the unit outward normal vector to Γ at $x \in \Gamma$ by $\mathbf{n}_\Gamma(x)$ and introduce the Fichera function $\mathbf{b}\cdot\mathbf{n}$ on Γ to define

$$\Gamma_- := \{x \in \Gamma : \mathbf{b}(x)\cdot\mathbf{n}(x) < 0\} \quad \text{and} \quad \Gamma_+ := \{x \in \Gamma : \mathbf{b}(x)\cdot\mathbf{n}(x) \geq 0\}. \tag{1}$$

In the following, the sets Γ_- and Γ_+ are referred to as the *inflow* and *outflow* boundary, respectively, and clearly form a nonoverlapping partition of Γ.

Given $\mathbf{b} \in [W^{1,\infty}(\Omega)]^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, and $g_D \in L^2(\Gamma_-)$, we consider the convection-reaction problem

$$\begin{aligned} \mathbf{b}\cdot\nabla u + c\,u &= f \qquad \text{in } \Omega,\\ u &= g_D \quad \text{on } \Gamma_-. \end{aligned} \tag{2}$$

Assuming the existence of a positive constant c_s satisfying

$$c_0^2 := c - \frac{1}{2}\nabla\cdot\mathbf{b} \geq c_s \qquad \text{for a.e. } x \in \Omega, \tag{3}$$

the well-posedness of problem (2) follows, e.g., as in [15].

2.2 The dG formulation on quadrilateral/hexahedral meshes

We are interested in discretizing solutions to (2) by means of a dG finite element method. To the aim, consider sequences of meshes $\{\mathcal{T}_n\}_n$ consisting of tensor product elements, which can be defined through an affine mapping Φ_K on the reference d-dimensional cube element $\widehat{K} := (-1,1)^d$. For sake of simplicity, we assume the all elements $K \in \mathcal{T}_n$ are shape regular. We fix a uniform polynomial degree $p \in \mathbb{N}$ and denote the space of tensor polynomials of degree p over $\widehat{K}$ by $\mathbb{Q}_p(\widehat{K})$. Next, we introduce the dG space

$$V_n := \{v_n \in L^2(\Omega) \mid v_{n|K} \circ \Phi_K \in \mathbb{Q}_p(K)\ \forall K \in \mathcal{T}_n\}.$$

Given an element $K \in \mathcal{T}_n$, we set its diameter and outward pointing normal by h_K and $\mathbf{n}_K$, respectively. Given boundary Γ^K of element K, we split it into the *inflow* and *outflow* parts Γ^K_- and Γ^K_+ defined as

$$\Gamma^K_-:=\{x \in \Gamma^K : \mathbf{b}(x) \cdot \mathbf{n}_K(x) < 0\} \quad \text{and} \quad \Gamma^K_+:=\{x \in \Gamma^K : \mathbf{b}(x) \cdot \mathbf{n}_K(x) \geq 0\}. \tag{4}$$

Next, we define the classical *upwind jump* operator. Given an internal face F, let K_1 and K_2 be two elements in $\mathcal{T}_n$ sharing F. Without loss of generality, we assume that K_1 is such that $\mathbf{b} \cdot \mathbf{n}_K(x) < 0$ for almost all x in F. Then, we set

$$\lfloor v \rfloor_F := (v_{|K_1} - v_{|K_2})_{|F} = v^+ - v^- \qquad \forall v \in H^1(\Omega, \mathcal{T}_n). \tag{5}$$

In the rest of this work, when no confusion occurs, we shall write $\mathbf{n}$ instead of $\mathbf{n}_K(x)$.

Following seminal work [14], we consider the upwind dG variational formulation of (2). More precisely, introduce the dG bilinear form

$$\begin{aligned} B_n(u_n, v_n) := \sum_{K \in \mathcal{T}_n} \Big(& (\mathbf{b} \cdot \nabla u_n, v_n)_{0,K} + (c\, u_n, v_n)_{0,K} \\ & - ((\mathbf{b} \cdot \mathbf{n}) \lfloor u_n \rfloor, v_n^+)_{\Gamma^K_- \setminus \Gamma_-} - ((\mathbf{b} \cdot \mathbf{n}) u_n^+, v_n^+)_{\Gamma^K_- \cap \Gamma_-} \Big) \end{aligned} \tag{6}$$

and the discrete right-hand side

$$F_n(v_n) := \sum_{K \in \mathcal{T}_n} \Big((f, v_n)_{0,K} - ((\mathbf{b} \cdot \mathbf{n}) g_D, v_n^+)_{\Gamma^K_- \cap \Gamma_-} \Big) \quad \forall v_n \in V_n. \tag{7}$$

The dG method we consider reads

$$\text{find } u_n \in V_n \quad \text{such that} \quad B_n(u_n, v_n) = F_n(v_n) \quad v_n \in V_n. \tag{8}$$

Furthermore, we introduce the dG norm

$$\begin{aligned} |||v_n|||^2_{\mathrm{dG}} := \sum_{K \in \mathcal{T}_n} \Big(& \|c_0 v_n\|^2_{0,K} + \frac{1}{2} \|\sqrt{|\mathbf{b} \cdot \mathbf{n}|} v_n^+\|^2_{0,\Gamma^K_- \cap \Gamma} + \frac{1}{2} \|\sqrt{|\mathbf{b} \cdot \mathbf{n}|} v_n^+\|^2_{0,\Gamma^K_+ \cap \Gamma} \\ & + \frac{1}{2} \|\sqrt{|\mathbf{b} \cdot \mathbf{n}|} \lfloor v_n \rfloor\|^2_{\Gamma^K_- \setminus \Gamma_-} \Big). \end{aligned} \tag{9}$$

with c_0 defined as in (3).

It is easy to check that

$$B_n(v_n, v_n) = |||v_n|||^2_{\mathrm{dG}} \qquad \forall v_n \in V_n. \tag{10}$$

The well-posedness of method (8) can be found, e.g., in [9, section 2.3].

3 Standard analysis and suboptimality in terms of the polynomial degree

In this section, we recall the convergence analysis for method (8) from [14] and where suboptimal estimates in terms of p appear.

Preliminary, for all $K \in \mathcal{T}_n$, introduce the L^2 projector $\Pi_p : L^2(K) \to \mathbb{Q}_p(K)$ through the affine mapping and recall the standard hp-approximation estimates, see, e.g., [14, 11]: for any function $u \in H^\ell(K)$ on the given element $K \in \mathcal{T}_n$, the follow relations hold

$$\begin{aligned} \|u-\Pi_p u\|_{0,K} &\lesssim \left(\frac{h_K}{p+1}\right)^s |u|_{s,K}, \\ \|u-\Pi_p u\|_{0,\Gamma^K} &\lesssim \left(\frac{h_K}{p+1}\right)^{s-\frac{1}{2}} |u|_{s,K}, \end{aligned} \tag{11}$$

with $s := \min\{p+1, \ell\}$.

It is easy to check that method (8) is consistent, whence the following Galerkin orthogonality follows:

$$B_n(u-u_n, v_n) = 0 \qquad \forall v_n \in V_n. \tag{12}$$

Then, we split error $u-u_n$ into $\eta+\xi$, where

$$\eta := u - \Pi_p u, \qquad \xi := \Pi_p u - u_n. \tag{13}$$

Using Galerkin orthogonality (12) and the properties of orthogonal projector Π_p, we readily have the error equation

$$0 = B_n(u-u_n,\xi) = B_n(\eta,\xi) + B_n(\xi,\xi) \qquad \Longrightarrow \qquad |||\xi|||^2_{\mathrm{dG}} = -B_n(\eta,\xi). \tag{14}$$

Since estimates on term η are standard, error equation (14) allows us to show a bound on term ξ, on which we now focus on.

We begin by computing the following error splitting:

$$\begin{aligned} B_n(\eta,\xi) = \sum_{K\in\mathcal{T}_n} \Big(& (\mathbf{b}\cdot\nabla\eta,\xi)_{0,K} + (c\eta,\xi)_{0,K} \\ & - ((\mathbf{b}\cdot\mathbf{n})\lfloor\eta\rfloor,\xi^+)_{0,\Gamma^K_-\setminus\Gamma_-} - ((\mathbf{b}\cdot\mathbf{n})\eta^+,\xi^+)_{0,\Gamma^K_-\cap\Gamma_-}\Big) \\ = \sum_{K\in\mathcal{T}_n} \Big(& ((c-\nabla\cdot\mathbf{b})\eta,\xi)_{0,K} - (\mathbf{b}\cdot\nabla\xi,\eta)_{0,K} \\ & + ((\mathbf{b}\cdot\mathbf{n})\lfloor\xi\rfloor,\eta^-)_{0,\Gamma^K_-\setminus\Gamma_-} + ((\mathbf{b}\cdot\mathbf{n})\xi^+,\eta^+)_{0,\Gamma^K_+\cap\Gamma_+}\Big) =: T_1+T_2+T_3+T_4. \end{aligned} \tag{15}$$

We show upper bounds for the four terms on the right-hand side of (15) and anticipate that our analysis on term T_2 will lead to suboptimal bounds in terms of p. Under further assumptions on vector $\mathbf{b}$, we shall exhibit improved p-bounds in Section 4 below.

We begin with term T_1. Using that $\mathbf{b} \in [W^{1,\infty}(\Omega)]^2,\ c \in L^\infty(\Omega)$, and assumption (3), we obtain

$$T_1 \le \sum_{K\in\mathcal{T}_n} \|c-\nabla\cdot\mathbf{b}\|_{\infty,K}\|\eta\|_{0,K}\|\xi\|_{0,K} \lesssim \|\eta\|_{0,\Omega}\|c_0\xi\|_{0,\Omega} \lesssim \|\eta\|_{0,\Omega}|||\xi|||_{\mathrm{dG}}. \tag{16}$$

As for terms T_3 and T_4, we have

$$T_3 + T_4 \lesssim \Big(\sum_{K\in\mathcal{T}_n} \|(\mathbf{b}\cdot\mathbf{n}_K)^{\frac{1}{2}}\eta\|^2_{0,\Gamma^K}\Big)^{\frac{1}{2}} |||\xi|||_{\mathrm{dG}} \lesssim \Big(\sum_{K\in\mathcal{T}_n} \|\eta\|^2_{0,\Gamma^K}\Big)^{\frac{1}{2}} |||\xi|||_{\mathrm{dG}}. \tag{17}$$

As for term T_2, if we assume that $\mathbf{b}\cdot\nabla\xi \in V_n$ for all $\xi \in V_n$, then $T_2 = 0$. Thus, inserting (16) and (17) in (15) yields

$$|||\xi|||_{\mathrm{dG}} \lesssim \Big(\sum_{K\in\mathcal{T}_n} (\|\eta\|^2_{0,K} + \|\eta\|^2_{0,\Gamma^K})\Big)^{\frac{1}{2}} \lesssim \Big(\frac{h_K}{p+1}\Big)^{s-\frac{1}{2}} |u|_{s,\Omega}, \tag{18}$$

Using a triangle inequality, and combining (11) with (18) leads to a p-optimal error estimate.

Next, we focus on the case of nonzero T_2 to investigate the p-suboptimality. Using the definition of η in (13), and notably the property of orthogonal projection Π_p, we can write

$$T_2 := \sum_{K\in\mathcal{T}_n} \int_K (\mathbf{b}\cdot\nabla\xi)\eta = \sum_{K\in\mathcal{T}_n} \int_K (\mathbf{b}\cdot\nabla\xi - \mathbf{b}_0\cdot\nabla\xi)\eta,$$

where $\mathbf{b}_0$ is the vector average over every K of $\mathbf{b}$. We deduce

$$T_2 \le \sum_{K\in\mathcal{T}_n} \|\mathbf{b}-\mathbf{b}_0\|_{\infty,K} |\xi|_{1,K} \|\eta\|_{0,K}.$$

On each element $K \in \mathcal{T}_n$, we have the following approximation property and hp-polynomial inverse inequality:

$$\|\mathbf{b}-\mathbf{b}_0\|_{\infty,K} \lesssim h_K |\mathbf{b}|_{W^{1,\infty}(K)}, \qquad |\xi|_{1,K} \lesssim \frac{p^2}{h_K} \|\xi\|_{0,K}.$$

In the light of the two above bounds and (3), we have the following bound on term T_2:

$$T_2 \lesssim p^2 \|\eta\|_{0,\Omega} |||\xi|||_{\mathrm{dG}}. \tag{19}$$

Inserting (16), (19), and (17) in (15), and using (11) yield

$$|||\xi|||_{\mathrm{dG}} \lesssim (1+p^2)\|\eta\|_{0,\Omega} + \Big(\sum_{K\in\mathcal{T}_n} \|\eta\|^2_{0,\Gamma^K}\Big)^{\frac{1}{2}} \lesssim \left(\frac{h^s}{(p+1)^{s-2}} + \frac{h^{s-\frac{1}{2}}}{(p+1)^{s-\frac{1}{2}}}\right) |u|_{s,\Omega}. \tag{20}$$

Using a triangle inequality, and combining (11) with (20) leads to the following p-suboptimal error estimate:

$$|||u-u_n|||_{\mathrm{dG}} \lesssim \left(\frac{h^{s-\frac{1}{2}}}{(p+1)^{s-2}}\right) |u|_{s,\Omega}. \tag{21}$$

The above error bound is optimal in h but suboptimal in terms of p by $3/2$ order, which is in accordance with [14, Remark 3.13]. Notwithstanding, such suboptimality is not observed in practice; see, e.g., [14, Numerical Example 1].

This motivates Section 4, where we shall exhibit improved estimates in terms of p, under further assumptions on convection field $\mathbf{b}$.

4 Improved bounds for special convection fields

In this section, we show improved p-error estimates, under the following assumption on convection field $\mathbf{b}$:

$$\mathbf{b} \in [W^{2,\infty}(K)]^d \quad \forall K \in \mathcal{T}_n, \qquad \mathbf{b} = [b_1(x_1), b_2(x_2), \dots, b_d(x_d)]^T. \tag{22}$$

Since the j-th, $j = 1, \dots, d$, component of $\mathbf{b}$ is assumed to be single-valued in the x_j variable, without loss of generality, we can assume that element K is the Cartesian product of intervals $I_j := (\alpha_j, \beta_j)$, $j = 1, 2, \dots, d$.

We can re-write term T_2 in (15) as

$$T_2 = \sum_{j=1}^{d} \int_K b_j(x_j) \partial_j \xi \, \eta.$$

Fix $j = 1, \dots, d$. The j-th partial derivative of ξ is a tensor polynomial of degree $p-1$ along direction x_j and p along the others. Define $I_j b_j$ as the linear interpolant of b_j at the end-points of interval I_j :

$$\Big(b_j - I_j b_j\Big)(\alpha_j) = \Big(b_j - I_j b_j\Big)(\beta_j) = 0. \tag{23}$$

Then, we clearly have that $I_j b_j \, \partial_i \xi \in \mathbb{P}_p(I_j)$ and $I_j b_j \, \partial_i \xi \in V_n$. Consequently, the definition of orthogonal projection Π_p allows us to write

$$\int_K b_j \partial_j \xi \, \eta = \int_K (b_j - I_j b_j) \partial_j \xi \, \eta.$$

As α_j and β_j denote the endpoints of interval I_j, standard properties of one dimensional linear interpolation operators guarantee the existence of $\widetilde{x}_j$ such that

$$b_j(x_j) - I_j b_j(x_j) = \frac{b_j^{(2)}(\widetilde{x}_j)}{2}(x - a_j)(x - b_j) \qquad \forall x_j \in I_j.$$

Defining the standard quadratic bubble function on I_j

$$w_j := -(x - a_j)(x - b_j),$$

we can thus write

$$\int_K (b_j - I_j b_j) \partial_j \xi \, \eta = \int_K \Big(\frac{b_j - I_j b_j}{\sqrt{w_j}}\Big)\big(\sqrt{w_j}\partial_j \xi\big) \, \eta \leq \left\| \frac{b_j - I_j b_j}{\sqrt{w_j}} \right\|_{\infty,K} \|\sqrt{w_j}\partial_j \xi\|_{0,K} \|\eta\|_{0,K}.$$

Thanks to assumption (22) and an hp-polynomial inverse estimate involving bubbles, see, e.g., [18, Lemma 3.42], we deduce

$$\left\| \frac{b_j - I_j b_j}{\sqrt{w_j}} \right\|_{\infty,K} \lesssim h_K |b_j|_{W^{2,\infty}(K)}, \qquad \|\sqrt{w_j}\partial_j \xi\|_{0,K} \lesssim \sqrt{p(p+1)} \|\xi\|_{0,K}.$$

Collecting all the above estimates yields the following improved bound on term T_2:

$$T_2 \lesssim hp \|\eta\|_{0,\Omega} |||\xi|||_{\mathrm{dG}}. \tag{24}$$

Inserting (16), (24), and (17) in (15) gives the bound

$$|||\xi|||_{\mathrm{dG}} \lesssim \frac{h^{s-\frac{1}{2}}}{(p+1)^{s-1}} |u|_{s,\Omega},$$

which, combined with a triangle inequality and (11) , eventually entails the error estimate

$$|||u - u_n|||_{\mathrm{dG}} \lesssim \frac{h^{s-\frac{1}{2}}}{(p+1)^{s-1}} |u|_{s,\Omega}.$$

Compared with error estimate (21), the p-suboptimality improved by one order.

5 Conclusion

Employing a special class of convection fields, we derived an improved hp-bound for dG methods discretising linear hyperbolic problems. The new error bound is suboptimal by $1/2$ order in p only, which improves the $3/2$ suboptimal order presented in [14]. Needless to say, the results in this work do not provide full answers to the open questions in [14], notably on the mismatch between the theoretical and numerical results on p-convergence. However, it sheds some additional light on such issues and shows the possibility of deriving sharper hp-error bounds for dG methods.

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