UNIQUENESS CONDITIONS:
2 – *REGULAR* CAUCHY PBM.

Notation:

$$r := -V_t + (aV_x)_x \qquad \text{(the } \textit{defect}\text{)}$$

Def. $\mathbb{B}_{ad} := \{\, B \mid B \in L^\infty(D);\ \exists \lim_{x \to x_0^+} B \wedge \lim_{x \to x_0^+} B = 0 \,\}$

Prop. (*uniqueness*)

$$f \in C^0(\bar{T}; H^{-1}(D))\ ;\ u, v \in \mathbb{X}$$

$$\exists\ \tau \in \bar{T} \cdot\ni\cdot \{v_x \neq 0, \text{ a.e. in } D\ ,\ \exists\, u_x(x_0^+,\tau), v_x(x_0^+,\tau)\}$$

$$\exists\ a(u,f) \in \mathbb{A}_{ad}\ ;\ a, b \cdot\ni\cdot B \in \mathbb{B}_{ad}$$

IF $\{ V(\,.\,,\tau) = 0\ \forall x \in \bar{D} \} \wedge \{ V_t(\,.\,,\tau) =_{\text{a.e.}} 0 \text{ in } D \}$

THEN $B =_{\text{a.e.}} 0$ in D .

Applications:

Regular Cauchy data from
$$\begin{cases} a(x_0) \text{ known} \\ \{u, f\}, \{v, g\} \text{ independent} \\ \qquad f, g \not\equiv 0 \\ a'(x_0) = 0 \quad (\textit{zoning}) \end{cases}$$

UNIQUENESS CONDITIONS:
3 – *SINGULAR* CAUCHY PBM.

Def. (set of points where u stationary = critical points)

$$E_u(t) := \text{clos}\,\{\ \xi(t)\,|\ \xi(t) \in \bar{D}\ ,$$

$$u_x(\xi(t),\, t) = 0 \ \vee \lim_{x \to y^{\pm}} u_x(\xi(t),\, t) = 0\ \}$$

Prop. (*uniqueness*)

$$f \in L^2(T\,;\, H^{-1}(D))$$

$$\exists\, a \in \mathbb{A}_{ad} \ \cdot\!\ni\!\cdot\ u(a, f) \in \mathbb{U}_{ad}$$

$$\exists\, b \in \mathbb{A}_{ad} \ \cdot\!\ni\!\cdot\ u(b, f) \in \mathbb{U}_{ad}$$

$$u(a, f) =_{\text{a.e.}} u(b, f) \text{ in } Q$$

IF EITHER

$$\exists\, \tau \in \bar{T} \cdot\!\ni\!\cdot \{E_u(\tau) \neq \emptyset \wedge \text{meas}[E_u(\tau)] = 0\ \} \quad (*)$$

OR $\quad \{\ E_u(t) \neq \emptyset,\ \forall\ t \in T \wedge \text{meas}[\ \bigcap_t E_u(t)\ ] = 0\ \} \quad (**)$

THEN $\qquad b =_{\text{a.e.}} a \text{ in } D$.

Rem.:

Uniqueness relies on properties of E_u (set of critical points)

Kitamura & Nakagiri' s (1977) uniqueness conditions apply to more regular data

UNIQUENESS CONDITIONS:
4 – *SELF–IDENTIFIABILITY*

Def. (admissible data pairs with least possible regularity)

$$\mathbb{P}_{ad} := \{ \{ u, f \} \mid u \in \mathbb{U}_{ad} , f \in C^0(\bar{T}; H^{-1}(D)) ;$$

$$(u_t + f)^{[-1]} := g^{[-1]} \subset C^0(\bar{D} \backslash S_g(t)) \cap L^2(D) \ \forall t \in \bar{T} \}$$

where

$$S_g(t) \cap \{ x_0 ; x_1 \} = \emptyset \ \forall \ t \in \bar{T}, \text{meas}[S_g(t)] = 0, \forall t$$

Thm. (*self–identifiability*)

IF $\{ u, f \} \in \mathbb{P}_{ad}$; $\text{meas}[E_u(\tau)] = 0$

THEN the following are equivalent

i) $\exists \ a \in \mathbb{A}_{ad} , \exists \ \tau \in \bar{T} \cdot\ni\cdot <au_x>(\tau) = 0$

ii) $\{ \exists! \ a \in \mathbb{A}_{ad} \} \wedge \{ a = \left(\dfrac{g_0^{[-1]} - \left\langle g_0^{[-1]} \right\rangle}{u_x}\right)(\tau) \}$

Rem.

non–local condition

physical interpretation: $<au_x>(\tau)$ = avg. flux

$<au_x>(\tau) = k \ (\neq 0)$ also applies with due change

$\exists$ link with singular Cauchy if $g_0^{[-1]} \in C^0(\bar{D})$

counterexample

UNIQUENESS CONDITIONS:
5 – COUNTEREXAMPLE

Features:

$t = \tau$, fixed

self–identifiable solution s.t., singular Cauchy uniqueness Hp. do not apply.

Domain $\quad D = (-1 + \varepsilon , 1 - \varepsilon)$; $\varepsilon > 0$, fixed

data: $\quad u_x(\tau) = \begin{cases} -1 - x , & -1 + \varepsilon \leq x < 0 \\ \frac{1}{2} - \frac{x}{2} , & 0 < x \leq 1 - \varepsilon \end{cases}$

$$g_0^{[-1]}(\tau) = 2\,\theta(x) - 1 - x + \varepsilon \; (\not\in C^0(\bar{D}))$$

prior knowledge: $a \in \mathbb{A}_{ad} \cdot\ni\cdot <au_x> = 0$

work: $\quad <g_0^{[-1]}> = \varepsilon$

$$\text{⁂} \quad a = \frac{g_0^{[-1]} - \left\langle g_0^{[-1]} \right\rangle}{u_x} = 1 + \theta(x)$$

Rem. $\quad \lim_{x \to x_0^+} a u_x(\tau) = -\varepsilon$

Since both $(\frac{1}{u_x})(\tau)$ and $(g_0^{[-1]})(\tau) \in L^4(D)$, then the stability estimate (see below) applies

$$\| B \|_1 \leq \text{const.} \, [\| V \|_{1,4} + \| (V_t)_0^{[-1]} \|_{0,4}](\tau)$$

ASIDE:

FUNCTIONS OF BOUNDED VARIATION (*BV*)

$f \in BV(\bar{D})$ =>

i) f exhibits at most discontinuity points of the first kind;

ii) if J (set of discontinuity points) $\neq \emptyset$ =>

$$\mathrm{card}[J] = k < \infty \text{ or at most } \aleph^0$$

iii) $f' =_{\text{d.w.}} \rho' + \sigma'; \ \rho' \in L^1(D) \ ; \ \sigma' = \sum_{i=1}^{\mathrm{card}[J]} c_i \delta(x - y_i) \ ;$

iv) $\sum_{i=1}^{\mathrm{card}[J]} |c_i| < \infty \ ;$

v) $\exists \ |a'|^{[-1]} \ ; \ |a'|^{[-1]} \subset L^{\infty}(D)$.

Rem.: rules out e.g., $a = 2 + \sin\frac{1}{x}$ in $[-1, +1]$

ASIDE:

A GROWTH ESTIMATE

Lemma 1

IF $\quad B \in \mathbb{B}_{ad}\ ;\ \frac{\phi_{xx}}{\phi_x} \in L^1(D)\ ;$

AND $\left(\frac{\rho}{\phi_x}\right)^{[-1]} \subset L^\infty(D)\ ;\ \exists \lim_{x \to x_0^+} \left(\frac{\rho}{\phi_x}\right)^{[-1]}$

AND $B_x \phi_x + B \phi_{xx} - \rho =_{\text{d.w.}} 0$

THEN

$$\|B\|_{0,\infty} \le \left\| \left(\frac{\rho}{\phi_x}\right)^{[-1]}_0 \right\|_{0,\infty} \exp\left[\, \left\|\frac{\phi_{xx}}{\phi_x}\right\|_{0,1} \right]$$

Proof:

relies on extended form of Gronwall – Bellman inequality

$D = (x_0, x_1)\ ;\ a \in \mathbb{A}_{ad}\ ;$

$g \in \mathbb{G}_{ad} := \{ g \mid g \in L^1(D),\ g \ge_{\text{a.e.}} 0 \}$

$c_+ \ge 0$ (constant)

IF $\quad a \le c_+ + \int_{x_0}^{x} a g \, d\xi$, a.e. in D

THEN

$$\|a\|_{0,\infty} \le c_+ \exp \|g\|_{0,1}$$

Stability Estimates

Uniqueness from Regular Cauchy Problem

The rôle of procedure

$$(Bv_x)_x \underset{d.w.}{=} -r\ , \quad t=\tau$$

$$Bv_x \underset{a.e.}{=} -r_0^{[-1]} \qquad\qquad B' + B\frac{v_{xx}}{v_x} \underset{d.w.}{=} -\frac{r}{v_x}$$

$$\|B\|_{0,\infty} \le C_v(1+2a_H)\|V\|_{\mathcal{X}(\tau)}$$

$$\|B\|_{0,\infty} \le C_v(1+a_H+C_A)\|V\|_{\mathcal{W}(\tau)} \cdot \exp\left[C_v C_S \sqrt{|x_1-x_0|}\right]$$

provided :

$$\left.\begin{array}{l} V_x \in L^\infty(D) \\ V_t \in L^1(D) \\ \exists \lim\limits_{x\to x_0^+} V_x \end{array}\right\} \begin{array}{l}\forall t\in\bar{T} \\ \text{or at} \\ \tau\in\bar{T}\end{array}$$

$a \in \mathcal{A}_{ad}$

$V \in \mathcal{W} :=$

$= \mathcal{C}^0(\bar{T}, H^2(D)) \cap \mathcal{C}^1(\bar{T}, L^2(D))$

$\|v_{xx}\|_{0,2}(\tau) \le C_S$

$a \in \mathcal{A}_{ad} \cap BV(\bar{D})$

$\|\,|a'|_0^{[-1]}\|_{0,\infty} \le C_A$ total variation of a

$\exists -\frac{r}{v_x}(x_0^+,\tau)$

intermediate steps : Gronwall-Bellman ineq. for measurable fcns., to estimate the growth of B.

Note : $\frac{v_{xx}}{v_x} \in L^2 \Rightarrow \frac{1}{v_x} \in AC(\bar{D})$