Well posedness of the nonlinear Schrödinger equation with isolated singularities

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Abstract

We study the well posedness of the nonlinear Schrödinger (NLS) equation with a point interaction and power nonlinearity in dimension two and three. Behind the autonomous interest of the problem, this is a model of the evolution of so called singular solutions that are well known in the analysis of semilinear elliptic equations. We show that the Cauchy problem for the NLS considered enjoys local existence and uniqueness of strong (operator domain) solutions, and that the solutions depend continuously from initial data. In dimension two well posedness holds for any power nonlinearity and global existence is proved for powers below the cubic. In dimension three local and global well posedness are restricted to low powers.

Keywords: Non-linear Schrödinger equation, Singular solutions, Point interactions.
2020 MSC: 35J10, 35Q55, 35A21

1. Introduction

In the present paper we want to define the dynamics of the nonlinear Schrödinger equation (or briefly NLS equation) in the presence of isolated singularities.
By isolated singularities we mean the solutions of the equation

$$\begin{cases} i\partial_t\psi = -\Delta\psi \pm |\psi|^{p-1}\psi \\ \psi(0) = \psi_0 \end{cases} \tag{1.1}$$

where $\psi \in H^2(\mathbb{R}^n \setminus \{\mathbf{0}\})$. The sign in front of the nonlinearity will be not important in the sequel. Our main result is the local and global well posedness in dimension $n = 2$ or $n = 3$ when the nature of admitted singularities is suitably restricted. To make more clear the premises of our analysis we describe the analogous and well known

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problem in the time independent case. Isolated singularities of semi linear elliptic equations constitute a subject of study flourished at the end of '70s and still prolific, with important ramifications toward quasilinear elliptic and parabolic equations (see for an incomplete but representative bibliography [5, 14, 17, 21, 24, 25, 29, 30] and references therein). An example especially relevant in our context is given by the stationary equation associated to (1.1):

$$-\Delta u \pm |u|^{p-1}u - \omega u = 0. \tag{1.2}$$

Its positive solutions defined and regular on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and vanishing at infinity are called ground states or singular ground states according to their behavior at $\mathbf{0}$, respectively bounded or diverging. Consider for example the equation with the minus sign in (1.1) or (1.2), the so called focusing stationary NLS equation. A typical result is the following. Let $1 < p < \frac{n}{n-2}$ ($p > 1$ if $n = 2$); for any singular ground state u of (1.2) there exist $q \geqslant 0$ depending on u such that

$$\lim_{\mathbf{x}\to\mathbf{0}} u(\mathbf{x})|\mathbf{x}|^{n-2} = q \quad (n \geqslant 3) \quad \text{or} \quad \lim_{\mathbf{x}\to\mathbf{0}} u(\mathbf{x})\frac{1}{\log\frac{1}{|\mathbf{x}|}} = q \quad (n = 2).$$

Moreover u solves as a distribution

$$-\Delta u - |u|^{p-1}u - \omega u = c_n q \delta_{\mathbf{0}}$$

for some positive constant c_n depending on the dimension only and that can be absorbed in the singularity. So the only singularities admitted have the behavior of the fundamental solution of the Laplacian G^0. For $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ the singular ground state still exists but with a different power type singularity u_p depending on p, and for $p \geqslant \frac{n+2}{n-2}$ equation (1.2) has neither a ground state nor a singular ground state (see [24, 25]). Results for the defocusing (plus sign in (1.1) or (1.2)) again show an alternative between a ground state with the singularity of the fundamental solution of the Laplacian G^0, a different singularity u_p and finally no ground states at all. The difference with respect to the focusing case is that the two different singularities can coexist in the range $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. We refer to the already quoted references and the detailed monograph [30] for a complete analysis. In the present work we want to study the time dependent NLS equation in the regime in which the singularities around the origin are of type of the fundamental solution G^0. It is by no means an obvious fact that this behavior, if initially present, could be preserved by the NLS flow for some range of nonlinearities. To analyze this problem and to achieve our main result, we choose to work in a Hilbert space setting. It turns out that a convenient way to rewrite the equation in (1.1) is as the abstract NLS equation

$$i\partial_t \psi = \mathcal{H}_\alpha \psi \pm |\psi|^{p-1}\psi$$

where $\mathcal{H}_\alpha$ is a self-adjoint operator in $L^2(\mathbb{R}^n)$. The operator $\mathcal{H}_\alpha$ is well defined only for $n \leqslant 3$ and it belongs to the class of point interactions (see [2] and references therein, and Section 2.1 below for the essential facts). This well known class of operators is the most suitable linear part for the abstract NLS equation because elements of the domain $\psi \in \mathcal{D}(\mathcal{H}_\alpha)$ behave at the origin exactly as the fundamental solution, and they have

H^2 Sobolev regularity away from the origin. Namely, every element in $\mathcal{D}(\mathcal{H}_\alpha)$ has the structure

$$\psi = \phi^\lambda + qG^\lambda \qquad (\lambda > 0)$$

where G^λ is the Green function of the Laplacian, $(-\Delta + \lambda)G^\lambda = \delta_{\mathbf{0}}$, and $\phi^\lambda \in H^2(\mathbb{R}^n)$ and q is a complex coefficient.

Moreover a precise relation between the coefficient q and the regular part $\phi^\lambda \in H^2$ is needed to have a self-adjoint operator with domain $\mathcal{D}(\mathcal{H}_\alpha)$. In this boundary condition appears a real parameter α and for any such α one has a different self-adjoint operator $\mathcal{H}_\alpha$. Complete definitions will be given in Section 2.1. These facts suggest to treat the nonlinear term as a perturbation in the linear Schrödinger dynamics generated by $\mathcal{H}_\alpha$ and to search for strong solutions of the NLS equation, i.e. solutions of (1.1) with initial data in $\mathcal{D}(\mathcal{H}_\alpha)$. The above considerations lead eventually to the Cauchy problem

$$\begin{cases} i\partial_t\psi = \mathcal{H}_\alpha\psi \pm |\psi|^{p-1}\psi \\ \psi(0) = \psi_0 \in \mathcal{D}(\mathcal{H}_\alpha) \end{cases} \tag{1.3}$$

We are not aware of any result about this evolution problem in dimension $n > 1$. On the contrary, some rigorous literature exists in the much simpler one dimensional case, where the operator $\mathcal{H}_\alpha$ can be interpreted, at least formally, as a Schrödinger operator with a delta potential (see [1] and the treatment in the more general context of quantum graphs given in [8]). We also mention the paper [23], where the Cauchy problem with Hartree nonlinearity is treated. As usual the analysis of (1.3) is reduced via Duhamel formula to the following integral equation:

$$\psi(t) = e^{-it\mathcal{H}_\alpha}\psi_0 - i\int_0^t e^{-i(t-s)\mathcal{H}_\alpha}|\psi(s)|^{p-1}\psi(s)\, ds, \tag{1.4}$$

and afterwards we recovery the solutions of the differential equation, see Section 3. Preliminary to the analysis of well posedness of (1.3) is the construction of some essential technical tools. Namely one has to extend classical interpolation inequalities to a scale of spaces modeled on $\mathcal{D}(\mathcal{H}_\alpha)$ (so including singular behavior), and to prove dispersive and Strichartz estimates. These properties, for the most part new, are discussed and proved in Section 2. The local well posedness is our first main result, the proof of which fills Section 3, including local existence, unconditional uniqueness, continuous dependence and blow-up alternative. More precisely, we prove the following:

Theorem 1 (Local Well-Posedness in $\mathcal{D}(\mathcal{H}_\alpha)$)**.** *Assume $p \geqslant 1$ if $n = 2$ and $1 \leqslant p < 3/2$ if $n = 3$ and let $\psi_0 \in \mathcal{D}(\mathcal{H}_\alpha)$. Then the following properties hold true.*
1) There exists $T \in (0, 1]$ and a strong solution of (1.4) *in $C([0, T]; \mathcal{D}(\mathcal{H}_\alpha)) \cap C^1([0, T]; L^2)$.*
2) The solution $\psi \in W^{1,r}((0, T); L^{p+1})$, where r is such that $(r, p + 1)$ is admissible (as in Def. 2.5).
3) The solution enjoys unconditional uniqueness in $C([0, T]; \mathcal{D}(\mathcal{H}_\alpha))$.
4) There is continuous dependence on initial data, in the following sense. Let $\psi_0^n \to \psi_0$ in $\mathcal{D}$; then denoted as ψ and ψ^n the solutions of (1.4) *corresponding to initial data ψ_0 and ψ_0^n, one has $\psi^n \to \psi$ in $C([0, T]; \mathcal{D}(\mathcal{H}_\alpha))$.*
5) The following blow-up alternative holds. Let the maximal existence time be defined

as

$$T^* = \sup_{T>0} \left\{ \psi \in C([0,T], \mathcal{D}(\mathcal{H}_\alpha)) \cap C^1([0,T], L^2) \ \ solves \ \ (1.4) \right\};$$

then

$$\lim_{t\to T^*} \|\psi(t)\|_{\mathcal{D}(\mathcal{H}_\alpha)} < \infty \implies T^* = \infty.$$

See Section 2 for the definition of the graph norm $\|\psi\|_{\mathcal{D}(\mathcal{H}_\alpha)}$. While the proof of the well posedness is insensitive to the actual value of α (which by this reason and by notational simplicity will be discarded after Section 2.1), the result provides families of singular solutions parametrized by the real α. More detailed analysis of the dynamics can of course depend on it. The proof exploits the framework introduced by Kato (see [18, 19, 20]) to study the standard case of the Laplacian, but with different conclusions. In particular, we stress that the two and three dimensional cases display a rather different behavior with respect to the nonlinearity power. The good news is that, as in the standard case of the Laplacian, local well posedness holds for any power in the two dimensional case. On the contrary, in the three dimensional case the admitted powers are greatly restricted, being in the range $p \in (1, 3/2)$, which means a rather mild nonlinearity. We recall that strong H^2 solutions exist in the standard case for any power p. It seems not possibile to overcome this limitation in the present framework: powers of the logarithm are tame, inverse powers of $|\mathbf{x}|$ are not. We also notice that a consequence of the well posedness for problem (1.3) is that its solutions solve in distributional sense the equation

$$i\partial_t\psi = -\Delta\psi \pm |\psi|^{p-1}\psi - q\delta_{\mathbf{0}} \,.$$

This is a NLS equation with a time dependent delta source (recall that q depends on time for strong solutions of (1.3); see Remark 3.2). This fact gives a further and suggestive interpretation of both (1.3) and the kind of evolution of singular solutions. In Section 4, after showing conservation laws of L^2 mass and energy, the second main result of this paper is global well posedness of the dynamics. Namely, we prove

Theorem 2. *Let $1 < p < 3$ if $n = 2$ and $1 < p < \frac{3}{2}$ if $n = 3$ and consider the maximal solution $\psi \in C([0, T^*); \mathcal{D}(\mathcal{H}_\alpha)) \cap C^1([0, T^*); L^2)$ of the problem (3.1).Then the solution ψ is global, i.e. $T^* = \infty$.*

Again, for the two dimensional case nothing changes with respect to the standard case of the Laplacian: also in the presence of singular solutions, global existence is guaranteed for nonlinearities $p < 3$ for any initial datum. On the other hand, in the three dimensional case global existence holds for the same powers in which local existence is true, $p < 3/2$.
We conclude this introduction describing possible developments and perspectives raised by the present results, not reduced to the always possible extensions and technical refinements of the results (among the latter we include for example well posedness in L^2 or in the form domain of the operator). The problem of singular solutions has an autonomous mathematical interest; however, as recorded in the references cited above, a not secondary physical motivation for their study originated in models of condensed matter, in particular stationary Fermi-Thomas theory and examples of Yang-Mills theories. Due to the considered models it was natural to limit the analysis to the elliptic

stationary case. However, we notice that one could be interested in the dynamics of Bose-Einstein condensates in the presence of defects, conveniently modeled as point interactions. In this case the relevant equation is the time dependent one. In particular vortex solutions could fall within this description. A first step is the analysis of existence of stationary solutions and their stability. The first, and in fact unique, result about existence of stationary solutions in which use has been made of point interactions appeared in [6] and [7]. In those papers a branch of singular ground states bifurcating from the linear eigenstate of $\mathcal{H}$ was proved for the 3D stationary defocusing case in equation (1.2). A more complete classification of standing waves would be desirable, firstly as regards ground states, then including excited (non sign-definite) states and eventually considering the interactions of several singularities (not treated in this paper). Stability of the solutions with respect of the NLS flow is then a natural question. In this respect, we also mention on the physical and modeling side the recent contributions in [26, 27] that seem to have raised an interest on "singular solitons". A second relevant question concerns a detailed analysis of the evolution for the critical ($p = 3$) and supercritical ($p > 3$) nonlinearities in the $n = 2$ case. One expects blow-up and it would be interesting to understand if and how the singularity plays a role. Finally, scattering theory for the NLS in the presence of point defects is also a relevant issue. About all these problems there is no previous analysis, and they seem to deserve some interest.

2. Preliminaries

In this section we fix notations and we prove some technical results used in the following.

We denote by $\mathbf{x}$, $\mathbf{k}$ and so on, points in $\mathbb{R}^n$, $n = 2, 3$. Correspondingly, we use the notation $x \equiv |\mathbf{x}|$, $k \equiv |\mathbf{k}|$.

We denote by $\hat{f}$ the Fourier transform of f, defined to be unitary in $L^2(\mathbb{R}^n)$:

$$\hat{f}(\mathbf{k}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathbf{dx}\, e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \qquad \mathbf{k} \in \mathbb{R}^n.$$

We denote by $\|\cdot\|$ the $L^2(\mathbb{R}^n)$-norm associated with the inner product $\langle\cdot,\cdot\rangle$ and with $\|\cdot\|_p$ the $L^p(\mathbb{R}^n)$-norm while we use $\|\cdot\|_{H^s}$ for the norm in the Sobolev spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$.

As customary, we denote with the same symbol $\langle\cdot,\cdot\rangle$ the duality between Banach spaces or evaluation of distributions. We use sometimes Dirac notation, that is $|u\rangle\langle v|$ stands for the 1-rank operator $f \rightsquigarrow u\langle v, f\rangle$.

For all $\lambda > 0$ we denote by G^λ the L^2 solution of the distributional equation $(-\Delta + \lambda)G^\lambda = \delta_{\mathbf{0}}$, where $\delta_{\mathbf{0}}$ is the Dirac-delta distribution centered in $\mathbf{x} = \mathbf{0}$. Hence, the integral kernel of the resolvent of the Laplacian is given by

$$G^\lambda(\mathbf{x} - \mathbf{y}) = (-\Delta + \lambda)^{-1}(\mathbf{x} - \mathbf{y}) \qquad \lambda \in \mathbb{R}^+.$$

Explicitly we have:

$$G^\lambda(\mathbf{x}) = \begin{cases} \dfrac{1}{2\pi} K_0(\sqrt{\lambda}\, x) & n = 2; \\ \dfrac{e^{-\sqrt{\lambda}\, x}}{4\pi x} & n = 3. \end{cases}$$

Here K_0 is the Macdonald function of order zero. We recall the relation $\frac{1}{2\pi} K_0(\sqrt{\lambda}\, x) = \frac{i}{4} H_0^{(1)}(i\sqrt{\lambda}\, x)$, where $H_0^{(1)}(z)$ is the Hankel function of first kind and order zero (also known as zeroth Bessel function of the third kind), see, e.g., [32] Eq. (8) p. 78).
We use c and C to denote generic positive constants whose dependence on the parameters of the problem is irrelevant, their value may change from line to line.

2.1. Point interactions

We denote by $\mathcal{H}_\alpha$ the self-adjoint operator in $L^2(\mathbb{R}^n)$, $n = 2, 3$, given by the Laplacian with a delta interaction of "strength" α placed in the origin.
We recall that, see [2], both for $n = 2$ and $n = 3$ the structure of the domain of $\mathcal{H}_\alpha$ is the same:

$$D(\mathcal{H}_\alpha) = \left\{ \psi \in L^2(\mathbb{R}^n) |\; \psi = \phi^\lambda + q\, G^\lambda,\; \phi^\lambda \in H^2(\mathbb{R}^n),\;\; q = \Lambda_\alpha^\lambda\, \phi^\lambda(\mathbf{0}) \right\} \tag{2.1}$$

with

$$(\Lambda_\alpha^\lambda)^{-1} = \begin{cases} \dfrac{2\pi}{2\pi\alpha + \gamma + \ln(\sqrt{\lambda}/2)} & n = 2, \\ \dfrac{1}{\alpha + \frac{\sqrt{\lambda}}{4\pi}} & n = 3; \end{cases} \qquad \alpha \in \mathbb{R}.$$

Here λ can be taken in $\mathbb{R}^+$ (possibly excluded one point which we denote by $-E_\alpha$, where E_α is the negative eigenvalue of $\mathcal{H}_\alpha$, see below for the details). For $n = 2$, γ is the Euler-Mascheroni constant. The constant α is real and it parametrizes the family of operators through $q = \Lambda_\alpha^\lambda\, \phi^\lambda(\mathbf{0})$, which plays the role of a boundary condition at the singularity. For both $n = 2$ and 3 the free dynamics is recovered in the limit $\alpha \to +\infty$.
The action of the operator is given by

$$(\mathcal{H}_\alpha + \lambda)\psi = (-\Delta + \lambda)\phi^\lambda \qquad \forall \psi \in D(\mathcal{H}_\alpha). \tag{2.2}$$

The Hamiltonian $\mathcal{H}_\alpha$ has $[0, \infty)$ as continuous spectrum furthermore there is no singular continuous spectrum. For $n = 2$, $\mathcal{H}_\alpha$ has a simple negative eigenvalue $\{E_\alpha\}$ for any $\alpha \in \mathbb{R}$. For $n = 3$, if $\alpha \geqslant 0$ there is no point spectrum, while for $\alpha < 0$ there is a simple negative eigenvalue $\{E_\alpha\}$. Whenever the eigenvalue E_α exists, we denote by ψ_α the corresponding eigenvector. Explicitly one has:

$$E_\alpha = -4e^{-2(2\pi\alpha+\gamma)} \qquad \psi_\alpha(\mathbf{x}) = \frac{1}{2\pi} K_0(2\, e^{-(2\pi\alpha+\gamma)} x) \qquad n = 2;$$

$$E_\alpha = -(4\pi\alpha)^2 \qquad \psi_\alpha(\mathbf{x}) = \frac{e^{4\pi\alpha\, x}}{4\pi x}, \quad \alpha < 0 \qquad n = 3.$$

The resolvent of $\mathcal{H}_\alpha$ is given by the abstract Kreĭn resolvent formula

$$(\mathcal{H}_\alpha + \lambda)^{-1} = (-\Delta + \lambda)^{-1} + (\Lambda_\alpha^\lambda)^{-1} |G_\lambda\rangle\langle G_\lambda| \qquad \lambda \in \mathbb{R}^+ \backslash \{|E_\alpha|\} \tag{2.3}$$

($\lambda \in \mathbb{R}^+$ if $n = 3$ and $\alpha \geqslant 0$ and $\mathbb{R}^+ = (0, +\infty)$).
For $\lambda > |E_\alpha|$ (take $\lambda > 0$ if $n = 3$ and $\alpha \geqslant 0$) we define

$$\mathcal{D}^s_\alpha := D((\mathcal{H}_\alpha + \lambda)^s), \qquad s \in \mathbb{R}$$

where $(\mathcal{H}_\alpha + \lambda)^s$ is a self-adjoint operator on $L^2(\mathbb{R}^n)$ defined through functional calculus. We equip $\mathcal{D}^s_\alpha$ with the norm $\|\psi\|_{\mathcal{D}^s_\alpha} := \|(\mathcal{H}_\alpha + \lambda)^s \psi\|$, equivalent to the graph norm of the operator $(\mathcal{H}_\alpha + \lambda)^s$. Consistently we set

$$\mathcal{D}_\alpha \equiv \mathcal{D}^{s=1}_\alpha = D(\mathcal{H}_\alpha) \tag{2.4}$$

defined in Eq. (2.1).

Remark 2.1. *Given a function $\psi \in \mathcal{D}_\alpha$, we refer to ϕ^λ (defined in Eq.* (2.1)*) as its regular part. By Eq.* (2.2)*, $\|\psi\|_{\mathcal{D}_\alpha} = \|(\mathcal{H}_\alpha + \lambda)\psi\| = \|(-\Delta + \lambda)\phi^\lambda\|$ which is equivalent to $\|\phi^\lambda\|_{H^2}$. Since $\|\cdot\|_{\mathcal{D}_\alpha}$ is also equivalent to the graph norm of $\mathcal{H}_\alpha$, then $\|\cdot\|_{\mathcal{D}_\alpha}$ is equivalent to the H^2-norm of the regular part. From now on we shall always use the notation* (2.4) *for the set $D(\mathcal{H}_\alpha)$, and work with the norm $\|\cdot\|_{\mathcal{D}_\alpha}$.*

From now on, to simplify the notation, and since α is regarded as a fixed parameter, we omit α from the notation for objects that may depend on it and we simply write, for example, $\mathcal{H} \equiv \mathcal{H}_\alpha$, $\Lambda \equiv \Lambda_\alpha$, and $\mathcal{D} \equiv \mathcal{D}_\alpha$.

2.2. Embeddings and interpolation inequalities

We recall the Sobolev embedding (see, e.g., [28, Th. 2.8.1 *b*) and *e*), and Rem. 2 to the theorem]):

$$\begin{aligned} &H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) && 2 \leqslant q < \infty,\ s \geqslant s_c; \\ &H^s(\mathbb{R}^n) \hookrightarrow C_B(\mathbb{R}^n) && s > n/2; \end{aligned} \tag{2.5}$$

where $s_c = n(\frac{1}{2} - \frac{1}{q})$ and $C_B(\mathbb{R}^n)$ denotes the space of bounded, continuous functions on $\mathbb{R}^n$. We will need further embedding properties involving the domains of operators $\mathcal{H} + \lambda$ and the domains of their fractional powers. Recall that, for all $\lambda > 0$,

$$\begin{aligned} &G^\lambda \in L^s(\mathbb{R}^3) && 1 \leqslant s < 3; \\ &G^\lambda \in L^s(\mathbb{R}^2) && 1 \leqslant s < \infty. \end{aligned}$$

Hence, by the definition of the operator domain $\mathcal{D}$, see Eq. (2.1), it follows that

$$\mathcal{D} \hookrightarrow L^q(\mathbb{R}^n) \qquad \text{where } 2 \leqslant q < 3 \text{ if } n = 3, \text{ and } 2 \leqslant q < \infty \text{ if } n = 2. \tag{2.6}$$

A less obvious property is given in the following:

Proposition 2.2. *We have $\mathcal{D}^{s/2} \hookrightarrow H^s$ with continuous embedding when:*

$$\begin{aligned} &0 < s < 1 && \textit{if } n = 2; \\ &0 < s < 1/2 && \textit{if } n = 3. \end{aligned}$$

Proof. Let us start from an abstract result. Recall the integral identity (see, e.g., [4, Ch. 10.4]):

$$x^{s/2} = \frac{\sin(\frac{s}{2}\pi)}{\pi}\int_0^{+\infty} dt\, t^{\frac{s}{2}-1}\frac{x}{t+x} \qquad x \geqslant 0,\ s \in (0,2).$$

The latter, applied to $x = (y+\lambda)^{-1}$, $(y+\lambda) > 0$, an by means of the change of variables $t \to 1/t$, gives

$$(y+\lambda)^{-s/2} = \frac{\sin(\frac{s}{2}\pi)}{\pi}\int_0^{+\infty} \frac{dt}{t^{s/2}}\,(y+\lambda+t)^{-1} \qquad (y+\lambda) > 0,\ s \in (0,2).$$

Hence, by functional calculus and by the Kreĭn resolvent formula (see Eq. (2.3)), one infers (see [13]):

$$(\mathcal{H}+\lambda)^{-s/2} = (-\Delta+\lambda)^{-s/2} + \frac{\sin\frac{s}{2}\pi}{\pi}\int_0^{+\infty}\frac{dt}{t^{s/2}}\,(\Lambda^{\lambda+t})^{-1}|G_{\lambda+t}\rangle\langle G_{\lambda+t}| \qquad \lambda > |E_\alpha|$$

($\lambda > 0$ if $n = 3$ and $\alpha \geqslant 0$).
Starting from the latter identity, for $n = 3$ and $\alpha \geqslant 0$ a stronger result was proven in [13], see Theorem 3.2, namely that $\mathcal{D}^{s/2} = H^s$ for $0 < s < 1/2$ and that the graph norm associated to $(\mathcal{H}+\lambda)^{s/2}$ is equivalent to the standard Sobolev norm.
Being $\psi \in \mathcal{D}^{s/2}$ iff $\psi = (\mathcal{H}+\lambda)^{-s/2} f$ for $f \in L^2$, any function in $\mathcal{D}^{s/2}$ can be written as $\psi = \psi_1 + \psi_2$ with $\psi_1 = (-\Delta+\lambda)^{-s/2} f$ and

$$\psi_2 = \frac{\sin\frac{s}{2}\pi}{\pi}\int_0^{+\infty}\frac{dt}{t^{s/2}}\,(\Lambda^{\lambda+t})^{-1}G_{\lambda+t}\,g(t) \qquad g(t) = \langle G_{\lambda+t}, f\rangle.$$

Since ψ_1 manifestly belongs to $D((-\Delta+\lambda)^{s/2})$ and the graph norm of $(-\Delta+\lambda)^{s/2}$ is equivalent to the H^s-norm, we turn our attention to ψ_2. Taking the Fourier transform we have

$$\hat{\psi}_2(k) = \frac{\sin\frac{s}{2}\pi}{\pi}\int_0^{+\infty}\frac{dt}{t^{s/2}}(\Lambda^{\lambda+t})^{-1}\frac{g(t)}{k^2+\lambda+t}\,, \qquad g(t) = \int_{\mathbb{R}^3}\mathbf{dk}\frac{1}{k^2+\lambda+t}\hat{f}(\mathbf{k}).$$

We want to prove that $\psi_2 \in H^s$. For $n = 3$ and $\alpha < 0$ notice that proofs of Lemma 5.1 and Proposition 5.2 of [13] hold true without modifications if one assumes $\lambda > |E_\alpha|$. Lemma 5.1 is actually unrelated to the value of α and Proposition 5.2 uses only the fact that $\sup_{t>0}(\Lambda^{\lambda+t})^{-1} < \infty$, which is indeed the case if $\lambda > |E_\alpha|$. This completes the proof of the three dimensional case.
Let us consider the two dimensional case. First we prove that

$$\int_0^{+\infty} dt\, |g(t)|^2 \leqslant c\|f\|^2. \tag{2.7}$$

We start by noticing that

$$\int_0^{+\infty} dt\left|\int_{\mathbb{R}^2}\mathbf{dk}\frac{1}{k^2+\lambda+t}\hat{f}(\mathbf{k})\right|^2 = \int_0^{+\infty} dt\left|\int_0^{+\infty} dk\frac{\sqrt{k}}{k^2+\lambda+t}\,\sqrt{k}A\hat{f}(k)\right|^2$$

where

$$A\hat{f}(k) = \int_0^{2\pi} d\theta \hat{f}(k,\theta).$$

Notice that $\sqrt{k} A\hat{f} \in L^2(\mathbb{R}^+)$ and that $\| \sqrt{k} A\hat{f}\|_{L^2(\mathbb{R}^+)} \leqslant \sqrt{2\pi}\|f\|$. Then, to complete the proof of Eq. (2.7), it is sufficient to prove that

$$T_1(t,k) = \frac{\sqrt{k}}{k^2+\lambda+t}$$

is the integral kernel of a bounded operator $T_1 : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$. To this aim, let us notice that, by scaling $t \to t/(k^2+\lambda)$ in the integral we have

$$\sup_{k>0} \int_0^{+\infty} dt\, T_1(t,k) \frac{1}{t^{1/4}} \leqslant \sup_{k>0} \frac{\sqrt{k}}{(k^2+\lambda)^{\frac{1}{4}}} \int_0^{+\infty} dt\, \frac{1}{t+1} \frac{1}{t^{1/4}} < \infty$$

and, by scaling $k \to k/\sqrt{t}$ in the integral,

$$\sup_{t>0} t^{1/4} \int_0^{+\infty} dk\, T_1(t,k) < \sup_{t>0} t^{1/4} \int_0^{+\infty} dk\, \frac{\sqrt{k}}{k^2+t} = \int_0^{+\infty} dk\, \frac{\sqrt{k}}{k^2+1} < \infty,$$

then the claim follows from Schur's test, see, e.g., [15].
Next we prove that $\hat{\psi}_2$ is in H^s. Precisely, we are going to prove that $(-\Delta+\lambda)^{s/2}\psi_2 \in L^2$. To this aim we shall show that $\|(-\Delta+\lambda)^{s/2}\psi_2\| \leqslant c\|g\|^2_{L^2(\mathbb{R}^+)}$ and then use the inequality (2.7). Since $\hat{\psi}_2$ is spherically symmetric, this is equivalent to prove that $\sqrt{k}(k^2+\lambda)^{s/2}\hat{\psi}_2$ belongs to $L^2(\mathbb{R}^+)$. Using the above definitions, we have

$$\sqrt{k}(k^2+\lambda)^{s/2}\hat{\psi}_2(k) = \sqrt{k}\, \frac{\sin\frac{s}{2}\pi}{\pi} \int_0^{+\infty} \frac{dt}{t^{s/2}} \frac{2\pi}{2\pi\alpha+\gamma+\ln(\sqrt{\lambda+t}/2)} \frac{(k^2+\lambda)^{s/2}}{k^2+\lambda+t} g(t).$$

Since $t > 0$, for all $\lambda > |E_\alpha|$ there exists a constant c such that

$$0 < \frac{2\pi}{2\pi\alpha+\gamma+\ln(\sqrt{\lambda+t}/2)} < \frac{2\pi}{2\pi\alpha+\gamma+\ln(\sqrt{\lambda}/2)} \leqslant c.$$

Hence, it is sufficient to prove that $T_2 : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ defined by the integral kernel

$$T_2(t,k) = \frac{k^{\frac{1}{2}}(k^2+\lambda)^{s/2}}{t^{s/2}(k^2+\lambda+t)}$$

is a bounded operator. By scaling $t \to t/(k^2+\lambda)$ in the integral we have, on one hand,

$$\sup_{k>0} \sqrt{k} \int_0^{+\infty} dt\, T_2(t,k) \frac{1}{\sqrt{t}} = \sup_{k>0} \frac{k}{\sqrt{k^2+\lambda}} \int_0^{+\infty} dt\, \frac{1}{t^{\frac{s}{2}+\frac{1}{2}}(1+t)} < \infty.$$

On the other hand, by scaling $k \to k/\sqrt{\lambda+t}$ it is easy to see that

$$\sup_{t>0} \sqrt{t} \int_0^{+\infty} dk\, T_2(t,k) \frac{1}{\sqrt{k}} \leqslant c\left(\sup_{t>0} t^{\frac{1}{2}-\frac{s}{2}} \int_0^{+\infty} dk\, \frac{k^s}{k^2+\lambda+t} + \lambda^{s/2} \sup_{t>0} t^{\frac{1}{2}-\frac{s}{2}} \int_0^{+\infty} dk\, \frac{1}{k^2+\lambda+t}\right)$$
$$= c\left(\sup_{t>0}\left(\frac{t}{\lambda+t}\right)^{\frac{1}{2}-\frac{s}{2}} \int_0^{+\infty} dk\, \frac{k^s}{k^2+1} + \lambda^{s/2} \sup_{t>0} \frac{t^{\frac{1}{2}-\frac{s}{2}}}{(t+\lambda)^{\frac{1}{2}}} \int_0^{+\infty} dk\, \frac{1}{k^2+1}\right)$$

here the constant c depends on s. Hence, the claim follows from Schur's test.
Then we have

$$\|(-\Delta+\lambda)^{s/2}\psi_2\|^2 = 2\pi \int_0^{+\infty} dk \left| \sqrt{k}(k^2+\lambda)^{s/2}\hat{\psi}_2(k)\right|^2 \leqslant c\|g\|^2_{L^2(\mathbb{R}^+)} \leqslant c\|f\|^2$$

and the proof is complete. □

Thanks to Sobolev embeddings, see Eq. (2.5), and the results in Proposition 2.2 one has the continuous embeddings

$$\begin{aligned} &\mathcal{D}^{s/2}(\mathbb{R}^2) \hookrightarrow H^s(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2) \qquad 2 \leqslant q < \infty \qquad s \in [s_c, 1); \\ &\mathcal{D}^{s/2}(\mathbb{R}^3) \hookrightarrow H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3) \qquad 2 \leqslant q < 3 \qquad s \in [s_c, 1/2); \end{aligned} \tag{2.8}$$

where $s_c = n(\frac{1}{2} - \frac{1}{q})$, and the corresponding inequalities

$$\begin{aligned} &\text{if } n = 2 \qquad \|\psi\|_{L^q} \leqslant c\|\psi\|_{H^s} \leqslant c_2\|\psi\|_{\mathcal{D}^{s/2}} \qquad 2 \leqslant q < \infty \qquad s \in [s_c, 1); \\ &\text{if } n = 3 \qquad \|\psi\|_{L^q} \leqslant c\|\psi\|_{H^s} \leqslant c_3\|\psi\|_{\mathcal{D}^{s/2}} \qquad 2 \leqslant q < 3 \qquad s \in [s_c, 1/2). \end{aligned} \tag{2.9}$$

On the other hand $\mathcal{H}+\lambda$ is a self-adjoint and positive operator, and the spectral theorem allows to build the scale of Hilbert spaces $\mathcal{D}^{s/2}$ with the inner product $\langle\psi_1,\psi_2\rangle_{\mathcal{D}^{s/2}} := \langle(\mathcal{H}+\lambda)^{s/2}\psi_1, (\mathcal{H}+\lambda)^{s/2}\psi_2\rangle$. This is a family of real interpolation spaces, in particular this means that (see, e.g., Section 4.3.1 in [22])

$$\|\psi\|_{\mathcal{D}^{(1-\theta)a+\theta b}} \leqslant c\|\psi\|^{1-\theta}_{\mathcal{D}^a}\|\psi\|^{\theta}_{\mathcal{D}^b} \qquad a, b \geqslant 0,\ \theta \in (0,1). \tag{2.10}$$

For $a = 0,\ b = 1/2,\ \theta = s$ one in particular obtains the inequality

$$\|\psi\|_{\mathcal{D}^{s/2}} \leqslant c\|\psi\|^{1-s}_{L^2}\|\psi\|^{s}_{\mathcal{D}^{1/2}} \qquad s \in (0,1);$$

and, for $a = 0,\ b = 1,\ \theta = s$, the inequality

$$\|\psi\|_{\mathcal{D}^{s}} \leqslant c\|\psi\|^{1-s}_{L^2}\|\psi\|^{s}_{\mathcal{D}} \qquad s \in (0,1).$$

From (2.10) and (2.9) we finally obtain the Gagliardo-Nirenberg inequalities adapted to the scale of Hilbert spaces $\mathcal{D}^s$:

$$\begin{aligned} &\text{if } n = 2 \qquad \|\psi\|_{L^q} \leqslant c\|\psi\|^{1-s}_{L^2}\|\psi\|^{s}_{\mathcal{D}^{1/2}} \qquad 2 \leqslant q < \infty \qquad s \in [s_c, 1); \\ &\text{if } n = 3 \qquad \|\psi\|_{L^q} \leqslant c\|\psi\|^{1-s}_{L^2}\|\psi\|^{s}_{\mathcal{D}^{1/2}} \qquad 2 \leqslant q < 3 \qquad s \in [s_c, 1/2). \end{aligned} \tag{2.11}$$

2.3. Evolution operators and space-time estimates

Let us now introduce space-time Banach spaces and several properties of the evolution operators generated by $\mathcal{H}$ needed in the sequel.
For any exponent $\rho \in [1,+\infty]$ we denote by $\rho' \in [1,+\infty]$ its Hölder conjugate:

$$\frac{1}{\rho} + \frac{1}{\rho'} = 1.$$

We will denote $L^\rho([0,T];L^\sigma(\mathbb{R}^n))$ by $L^\rho_t L^\sigma_x$ and the corresponding norm by $\|\cdot\|_{L^\rho_t L^\sigma_x}$. The unitary group generated by the operator $\mathcal{H}$ is denoted by:

$$U(t)\phi := (U\phi)(t) = e^{-it\mathcal{H}}\phi. \tag{2.12}$$

The corresponding Duhamel operator is

$$\Gamma u(t) := \int_0^t U(t-s)\, u(s)\, ds. \tag{2.13}$$

By subsection 2.1, the operator $\mathcal{H} = \mathcal{H}_\alpha$, whose domain of definition is $D(\mathcal{H})$, is self-adjoint and by the spectral theorem the space $L^2(\mathbb{R}^n)$ splits as orthogonal sum of the absolutely continuous subspace, the singular continuous subspace (trivial in this case) and the pure point subspace:

$$L^2(\mathbb{R}^n) = \mathscr{H}_{\text{ac}}(\mathcal{H}) \bigoplus \mathscr{H}_{\text{sc}}(\mathcal{H}) \bigoplus \mathscr{H}_{\text{pp}}(\mathcal{H}), \qquad \mathscr{H}_{\text{sc}}(\mathcal{H}) = \{0\}.$$

Let $P_{ac}(\mathcal{H})$ be the spectral projector on the absolutely continuous space $\mathscr{H}_{\text{ac}}(\mathcal{H})$.

Definition 2.3 (Admissible pair)**.** *We say that a pair of (time,space) exponents (ρ,σ) is admissible if*

$$\frac{2}{\rho} + \frac{n}{\sigma} = \frac{n}{2},$$

and

$$\begin{aligned} &\sigma \in [2,+\infty) && \text{if } n = 2; \\ &\sigma \in [2,3) && \text{if } n = 3. \end{aligned}$$

Correspondingly $\rho \in (2,+\infty]$ if $n = 2$ or $\rho \in (4,+\infty]$ if $n = 3$.

Proposition 2.4 (Strichartz estimates for $\mathcal{H}$)**.** *For all the admissible pairs (ρ,σ) and (μ,ν) there exists a positive constant C such that*

$$\|UP_{ac}(\mathcal{H})\phi\|_{L^\rho_t L^\sigma_x} \leqslant C\|\phi\| \tag{2.14}$$

and

$$\|\Gamma P_{ac}(\mathcal{H})u\|_{L^\rho_t L^\sigma_x} \leqslant C\|u\|_{L^{\mu'}_t L^{\nu'}_x} \tag{2.15}$$

for all $T > 0$.

These bounds are a direct consequence of the fundamental bound

$$\|UP_{ac}(\mathcal{H})\phi\|_\sigma \leqslant C|t|^{-n(\frac{1}{2}-\frac{1}{\sigma})}\|\phi\|_{\sigma'} \qquad t \in \mathbb{R}\backslash\{0\}. \tag{2.16}$$

The proof of the bound (2.16) appeared in [12] for $n = 3$ and [10] for $n = 2$ (together with the Strichartz estimates (2.14) and (2.15), with the time interval $[0,T]$ replaced by $\mathbb{R}$), see also [11] and [16]. Notice that the constants appearing in (2.14), (2.15) and (2.16) do not depend on T but depend on ρ, σ, μ, ν.
In the rest of the paper we will find convenient a different parametrization and notation for the admissible pair, obtained changing ρ to r and σ to $p+1$.

Definition 2.5. *For any $p \in [1, +\infty)$ if $n = 2$ or $p \in [1, 2)$ if $n = 3$, we set*

$$r = \frac{4(p+1)}{n(p-1)}$$

so that $(r, p+1)$ is a pair of admissible exponents.

We summarize in the following proposition the properties of the linear dynamics needed in the proof of the main theorems (see [18, 31]).

Proposition 2.6. *Let*

$$p \in (1, +\infty) \qquad \text{if } n = 2;$$
$$p \in (1, 2) \qquad \text{if } n = 3.$$

and let $(r, p+1)$ an admissible pair.
Then, the operators U and Γ are defined and bounded between the following spaces with norms uniformly bounded for $T \leqslant 1$:

a) $U : L^2(\mathbb{R}^n) \to L^\infty([0, T]; L^2(\mathbb{R}^n))$
b) $U : L^2(\mathbb{R}^n) \to L^r([0, T]; L^{p+1}(\mathbb{R}^n))$
c) $\Gamma : L^1([0, T]; L^2(\mathbb{R}^n)) \to L^\infty([0, T]; L^2(\mathbb{R}^n))$
d) $\Gamma : L^1([0, T]; L^2(\mathbb{R}^n)) \to L^2 \to L^r([0, T]; L^{p+1}(\mathbb{R}^n))$
e) $\Gamma : L^2 \to L^{r'}([0, T]; L^{1+1/p}(\mathbb{R}^n)) \to L^\infty([0, T]; L^2(\mathbb{R}^n))$
f) $\Gamma : L^{r'}([0, T]; L^{1+1/p}(\mathbb{R}^n)) \to L^r([0, T]; L^{p+1}(\mathbb{R}^n))$

Proof. Properties *a*) and *c*) hold true since U is a unitary operator in L^2. The other properties follow from the spectral theorem together with Strichartz estimates for the continuous part of the spectrum. Let us give few additional details about the proof. Let us consider the two dimensional case and write

$$U = U_1 + U_2 \qquad U_1 = e^{-it\mathcal{H}} P_{ac}(\mathcal{H}) \qquad U_2 = e^{-iE_\alpha t} \frac{|\psi_\alpha\rangle\langle\psi_\alpha|}{\|\psi_\alpha\|^2}. \tag{2.17}$$

We decompose $\Gamma = \Gamma_1 + \Gamma_2$ accordingly. For the two dimensional case, it was proved in [10] that U_1 and Γ_1 satisfy Strichartz estimates and therefore the remaining properties b) and c)–f) are true since the indexes $(r, p+1)$ are admissible. Concerning U_2 and Γ_2, it is sufficient to notice that $\psi_\alpha \in L^\sigma$ for $1 \leqslant \sigma < \infty$ and that we are assuming $T \leqslant 1$; then straightforward calculations using Hölder's inequality give:

$$\text{for } n = 2 \qquad \begin{aligned} \|U_2 f\|_{L^\rho_t L^\sigma_x} &\leqslant T^{1/\rho} \frac{\|\psi_\alpha\|_\sigma \|\psi_\alpha\|_\nu}{\|\psi_\alpha\|^2} \|f\|_{\nu'} \\ \|\Gamma_2 u\|_{L^\rho_t L^\sigma_x} &\leqslant T^{1/\rho + 1/\mu} \frac{\|\psi_\alpha\|_\sigma \|\psi_\alpha\|_\nu}{\|\psi_\alpha\|^2} \|u\|_{\mu',\nu'} \end{aligned} \qquad \sigma, \nu \in [1, +\infty);\ \rho, \mu \in [1, +\infty]. \tag{2.18}$$

Which imply (b), (d), (e), and (f) for U_2 and Γ_2.
In the three dimensional case, for $\alpha < 0$ we argue in the same way using again (2.17). Strichartz estimates for U_1 and Γ_1 have been proved in [12], while for U_2 and Γ_2 it is

important to notice that $\psi_\alpha \in L^\sigma$ only for $1 \leqslant \sigma < 3$. Hence, for $n = 3$ bounds of the form (2.18) still hold true but with the constraint $\sigma, \nu \in [1, 3)$, which causes no problem since to prove (b), (d), (e), and (f) one needs to set $\sigma = 2$ or $\sigma = p + 1$, and similarly for ν. If $\alpha \geqslant 0$ then there is no point spectrum and both U_2 and Γ_2 are absent. □

Remark 2.7. *The statement of Proposition 2.6 holds true if $T \leqslant 1$ is changed in $T \leqslant T_0$, for any positive T_0. Obviously, in this case the norms of the operators U and Γ would depend on T_0. Since in what follows we shall use Proposition 2.6 to study the local well-posedness, there is no loss of generality in restricting to the choice $T_0 = 1$. In the rest of the paper we always make this choice, so that the various norms of U and Γ can be taken uniformly in T.*

Remark 2.8. *Some of the above properties can be strengthened. In particular, in Proposition 2.6 c) the target space is actually $C([0, T], L^2)$. See Proposition 7.3.4 in [9]).*

We end this section introducing four Banach spaces needed in the following, and we reformulate dispersive estimates in these spaces. In what follows we assume that $(r, p+1)$ is an admissible couple, according to Definition 2.5. The first couple is given by:

$$\mathcal{X}_T = L^\infty([0, T]; L^2(\mathbb{R}^n)) \bigcap L^r([0, T]; L^{p+1}(\mathbb{R}^n))$$
$$\tilde{\mathcal{X}}_T = L^1([0, T]; L^2(\mathbb{R}^n)) + L^{r'}([0, T]; L^{1+1/p}(\mathbb{R}^n))$$

with norms defined by

$$\|f\|_{\mathcal{X}_T} = \max\{\|f\|_{L^\infty_t L^2_x}, \|f\|_{L^r_t L^{p+1}_x}\} \qquad \|f\|_{\tilde{\mathcal{X}}_T} = \inf_{g+h=f}\{\|g\|_{L^1_t L^2_x} + \|h\|_{L^{r'}_t L^{1+1/p}_x}\}.$$

Notice that $\mathcal{X}_T$ is the topological dual of $\tilde{\mathcal{X}}_T$, i.e. $\mathcal{X}_T = \tilde{\mathcal{X}}_T{}'$, and that Proposition 2.6 has the following immediate corollary:

Corollary 2.9. *Under the same assumptions of Proposition 2.6 the following holds true:*

$$U : L^2 \to \mathcal{X}_T \qquad \textit{and} \qquad \Gamma : \tilde{\mathcal{X}}_T \to \mathcal{X}_T, \tag{2.19}$$

as bounded operators and the operator norms are uniformly bounded for every finite T.

A second couple of useful spaces is given by:

$$\mathcal{Z}_T = \{v \in \mathcal{X}_T | \partial_t v \in \mathcal{X}_T,\ \mathcal{H}v \in L^\infty([0, T]; L^2(\mathbb{R}^n))\}$$
$$\tilde{\mathcal{Z}}_T = \{v \in L^\infty([0, T]; L^2(\mathbb{R}^n)) | \partial_t v \in \tilde{\mathcal{X}}_T\}$$

with norms given by

$$\|v\|_{\mathcal{Z}_T} = \max\{\|v\|_{\mathcal{X}_T}, \|\partial_t v\|_{\mathcal{X}_T}, \|\mathcal{H}v\|_{L^\infty_t L^2_x}\} \qquad \|v\|_{\tilde{\mathcal{Z}}_T} = \max\{\|v\|_{L^\infty_t L^2_x}, \|\partial_t v\|_{\tilde{\mathcal{X}}_T}\}.$$

In the previous definitions and in the following, the expression $\partial_t v$ is to be interpreted as the distributional derivative of the Y-vector valued distribution $v \in \mathscr{D}'(I, Y) := \mathcal{L}(\mathscr{D}(I), Y)$, where I is an open interval, $\mathscr{D} := C^\infty_0(I)$ and Y is a relevant Banach space (see for example [9], sections 1.4.4 and 1.4.5 for details).

Proposition 2.10. *Assume that $p > 1$ if $n = 2$ and $1 < p < 2$ if $n = 3$. Then, the operators U and Γ are defined and bounded between the following spaces:*

$$U : \mathcal{D} \to \mathcal{Z}_T \qquad \Gamma : \tilde{\mathcal{Z}}_T \to \mathcal{Z}_T$$

with norms uniformly bounded in T:

$$\|U\phi\|_{\mathcal{Z}_T} \leqslant c\|\phi\|_{\mathcal{D}}; \tag{2.20}$$

$$\|\Gamma f\|_{\mathcal{Z}_T} \leqslant c\|f\|_{\tilde{\mathcal{Z}}_T}. \tag{2.21}$$

Proof. By Corollary 2.9 we have $\|U\phi\|_{\mathcal{X}_T} \leqslant c\|\phi\|$, then by spectral theorem and again Corollary 2.9 we have

$$\|\mathcal{H}U\phi\|_{L_t^\infty L_x^2} \leqslant \|\mathcal{H}U\phi\|_{\mathcal{X}_T} = \|\partial_t U\phi\|_{\mathcal{X}_T} = \|U\mathcal{H}\phi\|_{\mathcal{X}_T} \leqslant c\|\mathcal{H}\phi\|,$$

which proves (2.20).
We note that the obvious inclusion $L_t^\infty L_x^2 \subset L_t^1 L_x^2$ implies $\tilde{\mathcal{Z}}_T \subset \tilde{\mathcal{X}}_T$, hence, for $f \in \tilde{\mathcal{Z}}_T$, we have $\Gamma f \in \mathcal{X}_T$ by Corollary 2.9. In particular we have

$$\|\Gamma f\|_{\mathcal{X}_T} \leqslant c\|f\|_{\tilde{\mathcal{X}}_T} \leqslant c\|f\|_{L_t^1 L_x^2} \leqslant c\|f\|_{L_t^\infty L_x^2} \leqslant c\|f\|_{\tilde{\mathcal{Z}}_T}.$$

Notice that

$$\partial_t \Gamma f = \Gamma \partial_t f + Uf(0). \tag{2.22}$$

This identity is justified as an identity in $\mathcal{X}_T$, whenever $f \in \tilde{\mathcal{Z}}_T$ by the following argument. Firstly note that by Sobolev embeddings there holds true $H^1(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$, hence, by duality, $L^{1+1/p}(\mathbb{R}^n) \hookrightarrow H^{-1}(\mathbb{R}^n)$. Which in turn implies $L^{r'}([0,T]; L^{1+1/p}) \hookrightarrow L^1([0,T]; H^{-1})$ (because $L^s([0,T]) \hookrightarrow L^1([0,T])$ for all $s \geqslant 1$). Moreover, trivially, $L^{1,2} \hookrightarrow L^1([0,T]; H^{-1})$. Hence, $\tilde{X} \hookrightarrow L^1([0,T]; H^{-1})$. As a consequence, $\partial_t f \in \tilde{\mathcal{X}}_T \subset L^1([0,T]; H^{-1})$ and then $f \in C([0,T]; H^{-1})$. In particular $f(0)$ is well defined and $f(0) \in L^2$ since $f \in L_t^\infty L_x^2$. Then, again by Corollary 2.9 we have

$$\|\partial_t \Gamma f\|_{\mathcal{X}_T} \leqslant c(\|\partial_t f\|_{\tilde{\mathcal{X}}_T} + \|f(0)\|) \leqslant c\|f\|_{\tilde{\mathcal{Z}}_T}.$$

Notice also that we have

$$\mathcal{H}\Gamma f = i(\partial_t \Gamma f - f), \tag{2.23}$$

hence,

$$\|\mathcal{H}\Gamma f\|_{L_t^\infty L_x^2} \leqslant \|\partial_t \Gamma f\|_{L_t^\infty L_x^2} + \|f\|_{L_t^\infty L_x^2} \leqslant \|f\|_{\tilde{\mathcal{Z}}},$$

and the proof of (2.21) is complete. □

3. Well Posedness

We want to study strong solutions of the Cauchy problem for the abstract NLS equation

$$\begin{cases} i\partial_t \psi(t) = \mathcal{H}\psi(t) + F(\psi)(t) \\ \psi(0) = \psi_0 \in \mathcal{D} \end{cases} \tag{3.1}$$

where $F(\psi) = \pm|\psi|^{p-1}\psi$.
By strong solution of (3.1) we mean a function $\psi \in C([0,T];\mathcal{D}) \cap C^1([0,T];L^2)$ which satisfies the equation and the initial value as L^2 identities.
Through the Duhamel formula we replace the differential equation with its integral version. More explicitly, we formulate the subsequent proposition. The proof straightforwardly follows the lines of the standard situation and we omit it (see section 4.1 in [9] for a detailed analysis in the abstract setting).

Proposition 3.1. *A function $\psi \in C([0,T];\mathcal{D}) \cap C^1([0,T];L^2)$ is a strong solution of* (3.1) *if and only if it solves in $L^2(\mathbb{R}^n)$ for every $t \in [0,T]$ the integral equation*

$$\psi(t) = U(t)\psi_0 - i\int_0^t U(t-s)F(\psi)(s)\,ds.$$

We will often refer to the integral version in the following shortened form

$$\psi = U\psi_0 - i\Gamma F(\psi), \tag{3.2}$$

where U and Γ are defined in (2.12) and (2.13).
The main result we want to prove in this section is **Theorem 1** regarding 3.1 and stated in the Introduction. Before beginning the proof some comments are in order.

Remark 3.2. *Let $\psi \in \mathcal{D}(\mathcal{H})$ be a solution of* (3.1)*, and let $\zeta \in C^\infty((0,T)\times\mathbb{R}^n)$ be a test function. The function ψ can be considered a (regular) distribution. Testing against ζ the distribution*

$$i\partial_t\psi + \Delta\psi \mp |\psi|^{p-1}\psi$$

and recalling that $\psi = \phi^\lambda + q\,G^\lambda$ and $(-\Delta + \lambda)G^\lambda = \delta_0$, we obtain

$$\begin{aligned}
&\langle i\partial_t\psi, \zeta\rangle - \langle\psi, (-\Delta+\lambda)\zeta\rangle + \lambda\langle\psi,\zeta\rangle \mp \langle|\psi|^{p-1}\psi,\zeta\rangle\\
=&\langle i\partial_t\psi, \zeta\rangle - \langle\phi^\lambda + q\,G^\lambda, (-\Delta+\lambda)\zeta\rangle + \lambda\langle\psi,\zeta\rangle \mp \langle|\psi|^{p-1}\psi,\zeta\rangle\\
=&\langle i\partial_t\psi, \zeta\rangle - \langle(-\Delta+\lambda)\phi^\lambda, \zeta\rangle - \langle q\delta_0,\zeta\rangle + \lambda\langle\psi,\zeta\rangle \mp \langle|\psi|^{p-1}\psi,\zeta\rangle\\
=&\langle i\partial_t\psi, \zeta\rangle - \langle(\mathcal{H}+\lambda)\psi, \zeta\rangle - \langle q\delta_0,\zeta\rangle + \lambda\langle\psi,\zeta\rangle \mp \langle|\psi|^{p-1}\psi,\zeta\rangle\\
=&\langle i\partial_t\psi - \mathcal{H}\psi \mp |\psi|^{p-1}\psi, \zeta\rangle - \langle q\delta_0,\zeta\rangle = -\langle q\delta_0,\zeta\rangle\ .
\end{aligned}$$

This means that a strong solution of (3.1) *solves as a distribution the NLS equation with a Dirac delta source*

$$i\partial_t\psi = -\Delta\psi \pm |\psi|^{p-1}\psi - q\delta_0\ .$$

Notice that at this level the special form of $q = q(t)$ given by the boundary condition is not important.

Remark 3.3. *The case $p = 1$ corresponds to the linear equation and it is well known. Hence in the forthcoming analysis we shall always assume $p > 1$.*

Remark 3.4. *The presence of $T \in (0,1]$ in part 1) of the statement of the Theorem 1 is only due to avoiding constants depending on the existence time in the many bounds appearing in the proof. This limitation is inessential as regards local existence (see also remark 2.7).*

Remark 3.5. *According to a usual and convenient strategy, we will prove existence and uniqueness of solution of the integral equation* (3.2) *in weaker norms than the ones stated and then the further regularity will follow from the equation.*

We split the proof of the local well posedness for strong solutions in separate subsections.

3.1. Local Existence and conditional uniqueness

The first proposition collects some simple and useful properties of the nonlinearity F used often in the subsequent analysis; the first two are well known, while the third is specific of the present problem.

Proposition 3.6. *Let $q \geqslant p \geqslant 1$ and consider the map $v \mapsto F(v) = \pm|v|^{p-1}v$. Then the following holds true.*
1) $F : L^q \to L^{q/p}$ is continuous and

$$\|F(v)\|_{q/p} \leqslant C\|v\|_q^p$$

2) F is continuously differentiable in the real sense and its derivative at the point $v \in L^q$ is given by

$$F'(v)w = |v|^{p-1}w + (p-1)|v|^{p-3}v\,\mathrm{Re}\,(v\overline{w}) \qquad \forall v, w \in L^q. \tag{3.3}$$

Moreover the derivative map satisfies the bounds

$$\|F'(v)\|_{L^q \to L^{q/p}} \leqslant C\|v\|_q^{p-1} \qquad \text{and} \qquad \|F'(v)w\|_{q/p} \leqslant C\|v\|_q^{p-1}\|w\|_q\,. \tag{3.4}$$

3) Let $p > 1$ if $n = 2$, $1 < p < 3/2$ if $n = 3$ and let $v \in \mathcal{D}$. Then

$$\|F(v)\| \leqslant c\|v\|_{\mathcal{D}}^p. \tag{3.5}$$

Proof. The fact that $F : L^q \to L^{q/p}$ is an easy check, as it is formula (3.3) by using the formula $F'(v)w = \frac{d}{ds}F(v + sw)|_{s=0}$. Continuity and differentiability are well known properties of the Nemitskii operator $v \mapsto F(v)$ (see for example [3], Section 1.3 and [19], Section 4). Concerning 3), notice that, by the definition of $\mathcal{D}$, $v = \phi^\lambda + \phi^\lambda(\mathbf{0})\,G^\lambda$, hence

$$|F(v)| \leqslant c(|\phi^\lambda|^p + |\phi^\lambda(\mathbf{0})|^p\,|G^\lambda|^p).$$

By (2.5), $\|\,|\phi^\lambda|^p\,\| = \|\phi^\lambda\,\|_{2p}^p \leqslant c\|\phi^\lambda\|_{H^2}^p$ and $\|\phi^\lambda\|_\infty \leqslant c\|\phi^\lambda\|_{H^2}$. Moreover $|G^\lambda|^p \in L^2(\mathbb{R}^n)$ for the considered range of p. Hence, recalling that $\|v\|_{\mathcal{D}} = \|\phi^\lambda\|_{H^2}$, (3.5) immediately follows. □

Remark 3.7. *We will often use property* 2) *of Proposition 3.6 in the case $q = p + 1$, obtaining*

$$\|F'(v)\|_{L^{p+1} \to L^{1+1/p}} \leqslant C\|v\|_{p+1}^{p-1} \quad \text{and} \quad \|F'(v)u\|_{1+1/p} \leqslant C\|v\|_{p+1}^{p-1}\|u\|_{p+1}.$$

Remark 3.8. *Validity of Proposition 3.6 is not restricted to the pure power nonlinearity. Let $F : \mathbb{C} \to \mathbb{C}$ be a C^1 function in real sense which satisfies the bounds*

$$|F(z)| \leqslant C|z|^p \quad \textit{and} \quad |F'(z)| \leqslant C|z|^{p-1} \qquad z \in \mathbb{C},$$

then Proposition 3.6 still holds true. The proof is analogous to the one given above for part 1) and for 2) and 3) see [19], Section 4).

In the following Proposition recall that $r = \frac{4(p+1)}{n(p-1)}$ (see Def. 2.5) and that $r > 2$.

Proposition 3.9. *Assume that $p > 1$ if $n = 2$ and $1 < p < 3/2$ if $n = 3$. Set $\beta = \frac{2}{r}$. We have $F : \mathcal{Z}_T \to \tilde{\mathcal{Z}}_T$ and for $T \leqslant 1$ there holds true:*

$$\|F(v) - F(v)(0)\|_{\tilde{\mathcal{Z}}_T} \leqslant cT^{1-\beta}\|v\|^p_{\mathcal{Z}_T} \qquad \forall v \in \mathcal{Z}_T.$$

Proof. We prove first that $\partial_t F(v) \in \tilde{\mathcal{X}}_T$. To this aim we shall prove that

$$F(v) \in W^{1,r'}((0,T); L^{1+1/p}),$$

which implies the claim. Note that $v \in \mathcal{Z}_T$ implies $v \in L^\infty([0,T]; \mathcal{D})$, hence, by the embedding (2.6), there holds $\|v\|_{L^\infty_t L^{p+1}_x} \leqslant c\|v\|_{\mathcal{Z}_T}$. Moreover, $\|F(v)\|_{1+1/p} = \|v\|^p_{p+1}$. If $p < r-1$, by Hölder inequality, one has $\|F(v)\|_{L^{r'}_t L^{1+1/p}_x} \leqslant T^{\frac{r-1-p}{r}}\|v\|^p_{L^r_t L^{p+1}_x}$, while for $p \geqslant r-1$ one obtains $\|F(v)\|_{L^{r'}_t L^{1+1/p}_x} \leqslant c\|v\|^{p-r+1}_{\mathcal{Z}_T}\|v\|^{r-1}_{L^r_t L^{p+1}_x}$. So, $\|F(v)\|_{L^{r'}_t L^{1+1/p}_x} \leqslant c\|v\|^p_{\mathcal{Z}_T}$, and $F(v) \in L^{r'}([0,T]; L^{1+1/p})$. To proceed, note that

$$|F(v)(t',x) - F(v)(t,x)| \leqslant c(|v(t',x)|^{p-1} + |v(t,x)|^{p-1})\,|v(t',x) - v(t,x)|. \tag{3.6}$$

Taking into account that $\|f^{p-1}g\|_{1+1/p} \leqslant \|f\|^{p-1}_{p+1}\|g\|_{p+1}$, we have

$$\left\|F(v)(t') - F(v)(t)\right\|_{1+1/p} \leqslant c(\|v(t')\|^{p-1}_{p+1} + \|v(t)\|^{p-1}_{p+1})\left\|v(t') - v(t)\right\|_{p+1}.$$

Since $\left\|v(t') - v(t)\right\|_{p+1} \leqslant \int_t^{t'} \|\partial_s v(s)\|_{p+1} ds$, setting $\varphi(s) = c\|v\|^{p-1}_{L^\infty_t L^{p+1}_x}\|\partial_s v(s)\|_{p+1}$ (for some constant c large enough) one has

$$\left\|F(v)(t') - F(v)(t)\right\|_{1+1/p} \leqslant \left|\int_t^{t'} \varphi(s) ds\right|$$

for almost all $t, t' \in [0,T]$. By Theorem 1.4.40 in [9] it follows that $F \in W^{1,r'}((0,T); L^{1+1/p})$ and $\|\partial_t F(v)\|_{L^{r'}_t L^{1+1/p}_x} \leqslant \|\varphi\|_{L^{r'}[0,T]}$. Additionally, by Hölder inequality in time,

$$\|\varphi\|_{L^{r'}[0,T]} \leqslant c\|v\|^{p-1}_{L^\infty_t L^{p+1}_x} T^{1-\beta}\|\partial_t v\|_{L^r_t L^{p+1}_x} \leqslant cT^{1-\beta}\|v\|^p_{\mathcal{Z}_T},$$

and

$$\left\|\partial_t(F(v) - F(v(0)))\right\|_{\tilde{\mathcal{X}}_T} = \|\partial_t F(v)\|_{\tilde{\mathcal{X}}_T} \leqslant \|\partial_t F(v)\|_{L^{r'}_t L^{1+1/p}_x} \leqslant cT^{1-\beta}\|v\|^p_{\mathcal{Z}_T}.$$

Note that $\|F(v)\|_{L_t^\infty L_x^2} \leqslant c\|v\|_{\mathcal{Z}_T}^p$ by Prop. 3.6. Hence, $F : \mathcal{Z}_T \to \tilde{\mathcal{Z}}_T$.
Next we prove that

$$\|F(v) - F(v)(0)\|_{L_t^\infty L_x^2} \leqslant cT^{1-\beta}\|v\|_{\mathcal{Z}_T}^p. \tag{3.7}$$

By interpolation (see Eq. (2.10), with $a = 0$, $b = 1$, $\theta = s/2$, $s \in (0,2)$), it follows that

$$\|v(t) - v(t')\|_{\mathcal{D}^{\frac{s}{2}}} \leqslant c\|v(t) - v(t')\|_{\mathcal{D}}^{\frac{s}{2}} \|v(t) - v(t')\|^{1-\frac{s}{2}},$$

for all $s \in (0,2)$. Since $\partial_t v \in L_t^\infty L_x^2$, there holds true $\|v(t) - v(t')\| \leqslant |t - t'|\|\partial_t v\|_{L_t^\infty L_x^2}$ for all $t, t' \in [0,T]$. Moreover, $v \in L^\infty([0,T]; \mathcal{D})$, hence, after a possibile modification on a set of measure zero, v is a bounded mapping of $[0,T] \to \mathcal{D}$ which satisfies the inequality

$$\|v(t) - v(t')\|_{\mathcal{D}^{\frac{s}{2}}} \leqslant c|t - t'|^{1-\frac{s}{2}} \|v\|_{L^\infty([0,T];\mathcal{D})}^{\frac{s}{2}} \|\partial_t v\|_{L_t^\infty L_x^2}^{1-\frac{s}{2}} \leqslant c|t - t'|^{1-\frac{s}{2}} \|v\|_{\mathcal{Z}_T} \qquad s \in (0,2). \tag{3.8}$$

The latter inequality implies that the representative of $v \in \mathcal{Z}_T$ which is a bounded map from $[0,T] \to \mathcal{D}$ is a Hölder continuous map from $[0,T] \to \mathcal{D}^{\frac{s}{2}}$; from now on we denote by $v(0)$ its value in $t = 0$. The function $v(0)$ can be understood as function in $L^\infty([0,T]; \mathcal{D}^{\frac{s}{2}})$, independent on t. Setting $t' = 0$ in Eq. (3.8), and taking the essential supremum we infer

$$\|v - v(0)\|_{L^\infty([0,T];\mathcal{D}^{\frac{s}{2}})} \leqslant cT^{1-\frac{s}{2}} \|v\|_{\mathcal{Z}_T} \qquad s \in (0,2).$$

Next we use the embeddings $\mathcal{D}^{\frac{s}{2}} \hookrightarrow H^s \hookrightarrow L^{2p}$, see Eq. (2.8), with $s = s_c(2p) = \frac{n(p-1)}{2p}$. To proceed, recall the inequality (3.6) and notice that, by Hölder inequality, it follows that

$$\|f^{p-1}g\| \leqslant \|f\|_{2p}^{p-1}\|g\|_{2p}. \tag{3.9}$$

Hence,

$$\begin{aligned}\|F(v) - F(v)(0)\|_{L_t^\infty L_x^2} &\leqslant c(\|v\|_{L_t^\infty L_x^{2p}}^{p-1} + \|v(0)\|_{L_t^\infty L_x^{2p}}^{p-1})\|v - v(0)\|_{L_t^\infty L_x^{2p}} \\ &\leqslant c(\|v\|_{L^\infty([0,T];\mathcal{D}^{\frac{s}{2}})}^{p-1} + \|v(0)\|_{L^\infty([0,T];\mathcal{D}^{\frac{s}{2}})}^{p-1})\|v - v(0)\|_{L^\infty([0,T];\mathcal{D}^{\frac{s}{2}})}\end{aligned} \tag{3.10}$$

where in the latter inequality we used $1 - \frac{s}{2} > 1 - \frac{2}{r}$. This concludes the proof of inequality (3.7). □

Remark 3.10. *As pointed out in the proof of Prop. 3.9, if $v \in \mathcal{Z}_T$ then one has $v \in C^{0,1-\frac{s}{2}}([0,T], \mathcal{D}^{\frac{s}{2}})$, i.e., v is a Hölder continuous map from $[0,T] \to \mathcal{D}^{\frac{s}{2}}$. More precisely, there holds true*

$$v \in \mathrm{Lip}([0,T]; L^2) \cap C^{0,1-\frac{s}{2}}([0,T]; \mathcal{D}^{\frac{s}{2}}) \subset \mathrm{Lip}([0,T]; L^2) \cap C^{0,1-\frac{s}{2}}([0,T]; L^{2p}).$$

As a consequence of inequalities (3.6) *and* (3.9) *one has $F(v) \in C^{0,1-\frac{s}{2}}([0,T]; L^2)$.*

For $v \in \mathcal{Z}_T$ and $\psi_0 \in \mathcal{D}$ let us define the map

$$\Phi(v) = U\psi_0 - i\Gamma F(v). \tag{3.11}$$

and set $B_{\mathcal{Z}_T}(R) = \{v \in \mathcal{Z}_T \,|\, \|v\|_{\mathcal{Z}_T} \leqslant R\}$.

Proposition 3.11. *Let $\psi_0 \in \mathcal{D}$ and define*

$$\mathcal{E} = \{v \in B_{\mathcal{Z}_T}(R) | \, v(0) = \psi_0\}.$$

Then:
1) $\mathcal{E}$ is a complete metric space with respect to the metric induced by the $\mathcal{X}_T$-norm.
2) Assume that $p > 1$ if $n = 2$ and $1 < p < 3/2$ if $n = 3$. Then there exist R big enough and T sufficiently small such that $\Phi : \mathcal{E} \to \mathcal{E}$.

Proof. Let us prove that $\mathcal{E}$, with the metric induced by the $\mathcal{X}_T$-norm, is a complete metric space. One has $\mathcal{E} \subset \mathcal{Z}_T \subset \mathcal{X}_T$. $\mathcal{X}_T$ is a Banach space, hence, to prove that it $\mathcal{E}$ is complete it is enough to prove that it is a closed subset in $\mathcal{X}_T$. Let u be a limit point of $\mathcal{E}$ so that there exists $\{u_n\}$ with $u_n \in \mathcal{E}$ and $\|u_n - u\|_{\mathcal{X}_T} \to 0$; we want to prove that $u \in \mathcal{E}$. Obviously, $u \in \mathcal{X}_T$ and $\|u\|_{\mathcal{X}_T} \leqslant R$. We are left to prove that $\|\partial_t u\|_{\mathcal{X}_T} \leqslant R$, $\|\mathcal{H}u\|_{L_t^\infty L_x^2} \leqslant R$, and $u(0) = \Psi_0$. We recall that $\partial_t u$ is defined as a distribution on the test functions $\varphi \in C_0^\infty((0,T))$ by $\langle \varphi, \partial_t u\rangle = -\langle \partial_t \varphi, u\rangle$. Notice that by hypothesis we have $u_n \in W^{1,r}((0,T), L^{p+1})$ and $\|\partial_t u_n\|_{L_t^r L_x^{p+1}} \leqslant R$. By well known properties of vector valued Sobolev spaces (see for example Corollary 1.4.42 in [9]) one concludes that there exists a subsequence $u_{n_k} \rightharpoonup v \in W^{1,r}((0,T), L^{p+1})$ and $\|\partial_t v\|_{L_t^r L_x^{p+1}} \leqslant \liminf \|u_{n_k}\| \leqslant R$. Finally by uniqueness $u = v$. In order to prove that $\partial_t u \in L_t^\infty L_x^2$ and $\|\partial_t u\|_{L_t^\infty L_x^2} \leqslant R$, the same reasoning works replacing weak convergence with weak-* convergence and invoking again Corollary 1.4.42 in [9]. Now we prove that $\|\mathcal{H}u\|_{L_t^\infty L_x^2} \leqslant R$. Since $\|\mathcal{H}u_n(t)\| \leqslant R$ for a.e. $t \in [0,T]$ then there exists $v(t) \in L^2$ and a subsequence that we denote with $\mathcal{H}u_{n_k}(t)$ such that $\mathcal{H}u_{n_k}(t) \rightharpoonup v(t)$ a.e. in $[0,T]$ and $\|v(t)\| \leqslant R$. Recall that $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ is a dense subset in $L^2(\mathbb{R}^n)$ $n = 2, 3$ and let $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. We have $\langle \varphi, \mathcal{H}u_{n_k}(t)\rangle \to \langle \varphi, v(t)\rangle$, where now $\langle \, , \rangle$ is the L^2 scalar product. Moreover

$$\langle \varphi, \mathcal{H}u_n(t)\rangle = \langle \mathcal{H}\varphi, u_n(t)\rangle \to \langle \mathcal{H}\varphi, u(t)\rangle = \langle \varphi, \mathcal{H}u(t)\rangle$$

since $u(t) \in \mathcal{D}$ a.e. in $[0,T]$. Then $\mathcal{H}u(t) = v(t)$ and $\|\mathcal{H}u(t)\| \leqslant R$ a.e. in $[0,T]$.
We are left to prove that $u(0) = \psi_0$ as an L^2 identity. We know that $u_n(t) \in W^{1,\infty}((0,T); L^2)$ and $u_n(0) = \psi_0$. Being $W^{1,\infty}((0,T); L^2) \hookrightarrow C([0,T]; L^2)$ one has that u_n converges in $C([0,T]; L^2)$ and $u(0) = \psi_0$ as an identity in L^2.
Now we prove that $\Phi : \mathcal{E} \to \mathcal{E}$. Notice that, by Proposition 2.10, $U\psi_0 \in \mathcal{Z}_T$. Moreover since $F(\psi_0) \in L^2$ (by Proposition 3.6), we can say $F(\psi_0) \in \tilde{\mathcal{Z}}_T$ since it depends on t in a trivial way; therefore $\Gamma F(\psi_0) \in \mathcal{Z}_T$ by Proposition 2.10. We choose $R > \|U\psi_0\|_{\mathcal{Z}_T} + \|\Gamma F(\psi_0)\|_{\mathcal{Z}_T}$ such that $\mathcal{E}$ is not empty since $U\psi_0 \in \mathcal{E}$. Adding and subtracting $\Gamma F(\psi_0) = \Gamma F(v)(0)$ to the r.h.s. of Eq. (3.11), we have by Proposition 3.9

$$\begin{aligned}
\|\Phi(v)\|_{\mathcal{Z}_T} &\leqslant \|U\psi_0\|_{\mathcal{Z}_T} + \|\Gamma F(\psi_0)\|_{\mathcal{Z}_T} + \|\Gamma(F(v) - F(v)(0))\|_{\mathcal{Z}_T} \\
&\leqslant \|U\psi_0\|_{\mathcal{Z}_T} + \|\Gamma F(\psi_0)\|_{\mathcal{Z}_T} + \|F(v) - F(v)(0)\|_{\tilde{\mathcal{Z}}_T} \\
&\leqslant \|U\psi_0\|_{\mathcal{Z}_T} + \|\Gamma F(\psi_0)\|_{\mathcal{Z}_T} + cT^{1-\beta}R^p
\end{aligned}$$

Then $\Phi : \mathcal{E} \to \mathcal{E}$ if T is sufficiently small. □

Now we can prove part 1) and 2) of Theorem 1.

Proof of Theorem 1. Parts 1) **and** 2)**.** For sufficiently small T, Φ is a contraction in the $\mathcal{X}_T$-norm. Indeed by (2.19), we have

$$\|\Phi(u)-\Phi(v)\|_{\mathcal{X}_T} = \|\Gamma(F(u)-F(v))\|_{\mathcal{X}_T} \leqslant c\|F(u)-F(v)\|_{\tilde{\mathcal{X}}_T}.$$

Notice that

$$|(F(u)-F(v))(t,x)| \leqslant c(|u(t,x)|^{p-1}+|v(t,x)|^{p-1})\,|u(t,x)-v(t,x)|.$$

Using the inequality $\|f^{p-1}g\|_{1+1/p} \leqslant \|f\|_{p+1}^{p-1}\|g\|_{p+1}$, we have

$$\|(F(u)-F(v))(t)\|_{1+1/p} \leqslant c(\|u(t)\|_{p+1}^{p-1}+\|v(t)\|_{p+1}^{p-1})\,\|u(t)-v(t)\|_{p+1}.$$

Therefore we have

$$\|F(u)-F(v)\|_{L_t^{r'}L_x^{1+1/p}} \leqslant T^{1-\beta}\|F(u)-F(v)\|_{L_t^{r}L_x^{1+1/p}} \leqslant cT^{1-\beta}(\|u\|_{L_t^\infty L_x^{p+1}}^{p-1}+\|v\|_{L_t^\infty L_x^{p+1}}^{p-1})\,\|u-v\|_{L_t^r L_x^{p+1}} \tag{3.12}$$

with $\beta = 2/r$ (as in Proposition 3.9). Hence, by the embedding (2.6), for $u, v \in \mathcal{E}$ we obtain

$$\|\Phi(u)-\Phi(v)\|_{\mathcal{X}_T} \leqslant c\,T^{1-\beta}R^{p-1}\|u-v\|_{\mathcal{X}_T}. \tag{3.13}$$

For sufficiently small T, Φ is a contraction in the $\mathcal{X}_T$-norm. Since $\mathcal{E}$ is complete with respect to the metric induced by the $\mathcal{X}_T$-norm then the fixed point equation

$$u = \Phi(u)$$

admits a solution $\psi \in \mathcal{E}$. In particular, $\psi \in L^\infty([0,T];\mathcal{D})$, $\partial_t\psi \in L^\infty([0,T];L^2)$, $\psi \in W^{1,r}((0,T);L^{p+1})$ and ψ satisfies the identity $\psi = U\psi_0 - i\Gamma F(\psi)$. We are left to prove that $\psi \in C([0,T];\mathcal{D}) \cap C^1([0,T];L^2)$. Obviously, $U\psi_0$ has the required properties, since $\psi_0 \in \mathcal{D}$ and thanks to the properties of the linear evolution. Concerning $\Gamma F(\psi)$, we start by noticing that $F(\psi)(t) \in L^2$ (a.e. in $[0,T]$), by Proposition 3.6, hence $F(\psi) \in L_t^\infty L_x^2$ and finally $U(t-\cdot)F(\psi)(\cdot) \in L^1([0,T];L^2)$. So, by absolute continuity of the integral, one concludes $\psi \in C([0,T];L^2)$. From the Duhamel formula it is also immediate that $\Gamma F(\psi) \in C([0,T];L^2)$. By the identity (see also (2.22))

$$\partial_t\Gamma F(\psi)(t) = UF(\psi)(0) + \int_0^t U(t-s)\partial_s F(\psi)(s)\,ds.$$

using Proposition 2.6 e) and taking into account that $\partial_s F(\psi)(s) \in L^{r',1+1/p}$ we have that $\partial_t\Gamma F(\psi)(t) \in L_t^\infty L_x^2$. On the other hand it is well known that it actually holds the stronger result $\Gamma v \in C([0,T],L^2)$ for $v \in L_t^{r'}L_x^{1+1/p}$ with $(r, p+1)$ admissible (see Remark 2.8). Finally, exploiting the fact that $\mathcal{H}$ is the infinitesimal generator of $U(t)$ (see also (2.23)) we have the identity

$$\mathcal{H}\Gamma F(\psi)(t) = i\partial_t\Gamma F(\psi) - iF(\psi)\,.$$

The r.h.s. belongs to $C([0,T];L^2)$, or equivalently $\Gamma F(\psi) \in C([0,T];\mathcal{D})$. □

Corollary 3.12 (Local well-posedness for strong solutions)**.** *Let $p > 1$ if $n = 2$ and $1 < p < 3/2$ if $n = 3$. For any $\psi_0 \in D(\mathcal{H})$ there exists $T \in (0, +\infty)$ s.t. the initial value problem* (3.1) *has a unique solution $\psi \in C([0, T); \mathcal{D}) \cap C^1([0, T); L^2(\mathbb{R}))$.*

Remark 3.13. *Notice that for a strong solution of the equation $\psi \in C([0, T); \mathcal{D}) \cap C^1([0, T); L^2)$, the existence time given in the local well-posedness Theorem actually depends only on $\|\psi_0\|_{\mathcal{D}}$. In fact, the L^2-norm of $\partial_t\psi$ can be bounded in terms of the graph norm of ψ just taking into account that equation in* (3.1) *holds as an L^2 identity and using the estimate* (3.5)*.*

3.2. Unconditional Uniqueness.

Proof of Theorem 1. Part 3). In the proof of local existence, the fixed point technique guarantees uniqueness only for those solutions $\psi \in C([0, T]; \mathcal{D})$ that belong to the auxiliary space $L^{r,p+1}$. In this paragraph we show that actually the latter condition is not needed.

Proposition 3.14. *Assume that $p > 1$ if $n = 2$ or $1 < p < \frac{3}{2}$ if $n = 3$. Take $\psi_0 \in \mathcal{D}$. If ψ_1 and ψ_2 are in $L^\infty([0, T]; \mathcal{D})$ for some $T > 0$ and are two solutions of Eq.* (3.2)*, then $\psi_1 = \psi_2$.*

Proof. Let τ be any time in $(0, T]$. Reasoning as in the derivation of (3.12) we obtain

$$\left\|F(\psi_1) - F(\psi_2)\right\|_{L^{r'}([0,\tau];L^{1+\frac{1}{p}})} \leqslant C(\|\psi_1\|^{p-1}_{L^\infty([0,\tau];L^{p+1})} + \|\psi_2\|^{p-1}_{L^\infty([0,\tau];L^{p+1})})\|\psi_1 - \psi_2\|_{L^{r'}([0,\tau];L^{p+1})}$$

with $r = \frac{4(p+1)}{n(p-1)}$ as in Definition 2.5 so that we can apply Proposition 2.6 (for any $T > 0$, see Remark 2.7).
By Eq. (3.2),

$$|\psi_1 - \psi_2| = \left|\Gamma(F(\psi_1) - F(\psi_2))\right|.$$

Hence, using Prop. 2.6.f) and the inequality above, we infer

$$\begin{aligned}\|\psi_1 - \psi_2\|_{L^r([0,\tau];L^{p+1})} \leqslant& C\|F(\psi_1) - F(\psi_2)\|_{L^{r'}([0,\tau];L^{1+\frac{1}{p}})} \\ \leqslant& C(\|\psi_1\|^{p-1}_{L^\infty([0,\tau];L^{p+1})} + \|\psi_2\|^{p-1}_{L^\infty([0,\tau];L^{p+1})})\|\psi_1 - \psi_2\|_{L^{r'}([0,\tau];L^{p+1})}.\end{aligned} \tag{3.14}$$

Since, by assumption, $\psi_1, \psi_2 \in L^\infty([0, \tau]; \mathcal{D})$, the embedding (2.6), together with the inequality (3.14), give

$$\|\psi_1 - \psi_2\|_{L^r([0,\tau];L^{p+1})} \leqslant C\|\psi_1 - \psi_2\|_{L^{r'}([0,\tau];L^{p+1})} \qquad \forall\tau \in (0, T]. \tag{3.15}$$

Let $\phi(t) := \|\psi_1(t) - \psi_2(t)\|_{L^{p+1}}$. By the inequality above, together with Hölder's inequality, we infer

$$\|\phi\|_{L^r([0,\tau_*])} \leqslant C\tau_*^{1-\frac{2}{r}}\|\phi\|_{L^r([0,\tau_*])}.$$

Since $r > 2$, for τ_* small enough (such that $C\tau_*^{1-\frac{2}{r}} < 1$), the latter inequality implies $\phi(t) = 0$ a.e. in $[0, \tau_*]$. Next, assume that $\phi(t) = 0$ a.e. in $[0, k\tau_*]$ for some positive integer k, then, inequality (3.15) (applied for $\tau = (k + 1)\tau_*$) is equivalent to

$$\|\phi\|_{L^r([k\tau_*,(k+1)\tau_*])} \leqslant C\|\phi\|_{L^{r'}([k\tau_*,(k+1)\tau_*])}.$$

Hence, by using again Hölder's inequality, we infer $\phi(t) = 0$ a.e. in $[0,(k+1)\tau_*]$. We proceed in this way, by induction, until $(k_*+1)\tau_* \geqslant T$, for some positive integer k_*. In the final step we address the interval $[k_*\tau_*, T]$. In this way we prove $\phi(t) = 0$ a.e. in $[0,T]$, henceforth $\psi_1 = \psi_2$ a.e. This concludes the proof of the proposition. □

3.3. Continuous dependence on initial data

Proof of Theorem 1. Part 4)**.** Assume that $\|\psi_0 - \psi_0^n\|_{\mathcal{D}} \to 0$; let ψ be the solution corresponding to the initial datum ψ_0 and ψ^n the solution corresponding to ψ_0^n according to the local existence result proved in the previous section. Notice preliminarily that by hypothesis we have $\|\psi_0^n\|_{\mathcal{D}} \leqslant 2\|\psi_0\|_{\mathcal{D}}$ and from the local existence we obtain that there exists a time $T = T(\|\psi_0\|_{\mathcal{D}})$ and n_0 such that both ψ and ψ_n are defined in $[0,T]$ for $n \geqslant n_0$; moreover the following uniform bound holds

$$\|\psi\|_{L^\infty([0,T];\mathcal{D})} + \|\psi^n\|_{L^\infty([0,T];\mathcal{D})} \leqslant C\|\psi_0\|_{\mathcal{D}} \,. \tag{3.16}$$

From (3.2) and the analogous

$$\psi^n = U\psi_0^n - i\Gamma F(\psi^n)$$

we obtain

$$\psi - \psi^n = U(\psi_0 - \psi_0^n) - i(\Gamma F(\psi) - \Gamma F(\psi^n)).$$

From Strichartz estimates and contractivity in the $\mathcal{X}_T$- norm of $\psi \mapsto \Gamma F(\psi)$ given in (3.13) it follows that, choosing possibly a $T' < T$,

$$\|\psi - \psi^n\|_{\mathcal{X}_T} \leqslant C\|\psi_0 - \psi_0^n\| + \frac{1}{2}\|\psi - \psi^n\|_{\mathcal{X}_T} \leqslant C\|\psi_0 - \psi_0^n\|_{\mathcal{D}} + \frac{1}{2}\|\psi - \psi^n\|_{\mathcal{X}_T}$$

and hence

$$\|\psi - \psi^n\|_{\mathcal{X}_T} \leqslant 2C\|\psi_0 - \psi_0^n\|_{\mathcal{D}}.$$

This gives continuity of the solution map in $\mathcal{X}_T$ and in particular in $L^r([0,T'];L^{p+1})$. Let us show that we also have $\|\partial_t\psi - \partial_t\psi^n\|_{r,p+1} \leqslant C\|\psi - \psi_0^n\|_{\mathcal{D}}$, so that the solution map $\psi_0 \to \psi(t,\psi_0)$ is continuous as a map from $\mathcal{D}$ to $W^{1,r}((0,T');L^{p+1})$ for suitable $T' \leqslant T$. Taking the time derivative of the integral equation both for ψ and ψ^n, subtracting and rearranging we obtain

$$\partial_t(\psi - \psi^n) = -i\Gamma F'(\psi^n)\partial_t(\psi - \psi^n) + \mathcal{R}_1 + \mathcal{R}_2 \tag{3.17}$$

where

$$\begin{aligned}
&\mathcal{R}_1 = -iU(\mathcal{H}(\psi_0 - \psi_0^n)) - iU(F(\psi_0) - F(\psi_0^n))\\
&\mathcal{R}_2 = -i\Gamma(F'(\psi) - F'(\psi^n))\partial_t\psi.
\end{aligned}$$

By means of dispersive estimates 2.6-*b*) and 2.6-*f*) on $(0,T')$ we obtain

$$\begin{aligned}
\|\partial_t(\psi - \psi^n)\|_{L^r([0,T'];L^{p+1})} \leqslant & C(T')\|F'(\psi^n)\partial_t(\psi - \psi^n)\|_{L^{r'}([0,T'];L^{1+1/p})}\\
&+ C(T')(\|\mathcal{H}(\psi_0 - \psi_0^n)\| + \|F(\psi_0) - F(\psi_0^n)\|)\\
&+ C(T')\|(F'(\psi) - F'(\psi^n))\partial_t\psi\|_{L^{r'}([0,T'];L^{1+1/p})}
\end{aligned}$$

where $C(T')$ is a constant which is uniformly bounded for $T' \in (0, 1]$. Let us consider the first addendum in the previous inequality. By the bound in Eq. (3.4) we have

$$\|F'(\psi^n(t))\partial_t(\psi - \psi^n)(t)\|_{1+1/p} \leqslant C\|\psi^n(t)\|_{p+1}^{p-1}\|\partial_t(\psi - \psi^n)(t)\|_{p+1}.$$

Hence, by Hölder inequality in time,

$$\|F'(\psi^n)\partial_t(\psi - \psi^n)\|_{L^{r'}([0,T'];L^{1+1/p})} \leqslant CT'^{1-\frac{2}{r}}\|\partial_t(\psi - \psi^n)\|_{L^r([0,T'];L^{p+1})}$$

where the bound is uniform in n thanks to (3.16) and embedding (2.6). Taking a smaller T' if needed, one gets

$$\begin{aligned}\|\partial_t(\psi - \psi^n)\|_{L^r([0,T'];L^{p+1})} \leqslant &\frac{1}{2}\|\partial_t(\psi - \psi^n)\|_{L^r([0,T'];L^{p+1})} \\ &+ C(T')(\|\mathcal{H}(\psi_0 - \psi_0^n)\| + \|F(\psi_0) - F(\psi_0^n)\|) \\ &+ C(T')\|(F'(\psi) - F'(\psi^n))\partial_t\psi\|_{L^{r'}([0,T'];L^{1+1/p})}\end{aligned}$$

and hence

$$\begin{aligned}\|\partial_t(\psi - \psi^n)\|_{L^r([0,T'];L^{p+1})} \leqslant &2C(T')(\|\mathcal{H}(\psi_0 - \psi_0^n)\| + \|F(\psi_0) - F(\psi_0^n)\|) \\ &+ 2C(T')\|(F'(\psi) - F'(\psi^n))\partial_t\psi\|_{L^{r'}([0,T'];L^{1+1/p})}.\end{aligned}$$

We have to show that the three terms on the r.h.s vanish when $\|\psi - \psi^n\|_{\mathcal{D}} \to 0$. For the first term this is obvious. For the second term we have

$$\|F(\psi_0^n) - F(\psi_0)\| \leqslant C\|\psi_0^n - \psi_0\|_{2p} \leqslant C\|\psi_0^n - \psi_0\|_{\mathcal{D}}$$

where the first inequality is obtained as in (3.10) and the last inequality follows from (2.6). One concludes that $\|F(\psi_0^n) - F(\psi_0)\| \to 0$ as $\|\psi_0^n - \psi_0\|_{\mathcal{D}} \to 0$. For the last term, exploiting again Proposition 3.6-2), in particular the continuity of F', one has that $\|(F'(\psi) - F'(\psi^n))\partial_t\psi\|_{1+1/p} \to 0$ point-wise a.e. in time. Moreover, notice that

$$\|(F'(\psi) - F'(\psi^n))\partial_t\psi\|_{1+1/p} \leqslant C(\|\psi\|_{p+1}^{p-1} + \|\psi^n\|_{p+1}^{p-1})\|\partial_t\psi\|_{p+1} \leqslant C\|\partial_t\psi\|_{p+1}$$

where the latter bound is uniform in n thanks again to (2.6) and (3.16). Now we know from local existence part that $\partial_t\psi \in L_t^r L_x^{p+1}$, and being $r' < r$, it also holds $\partial_t\psi \in L_t^{r'} L_x^{p+1}$, so that by dominated convergence theorem (on the time integral) $\|(F'(\psi) - F'(\psi^n))\partial_t\psi\|_{L_t^{r'} L_x^{1+1/p}} \to 0$ as $n \to \infty$. We conclude that $\|\psi - \psi^n\|_{W^{1,r}((0,T');L^{p+1})} \to 0$ as $\|\psi - \psi^n\|_{\mathcal{D}} \to 0$. An almost identical analysis, starting again from (3.17), but this time making use of 2.6-*a*) and 2.6-*e*) shows that $\|\partial_t\psi - \partial_t\psi^n\|_{L_t^\infty L_x^2} \to 0$ as $\|\psi_0 - \psi_0^n\|_{\mathcal{D}} \to 0$. From this last property and again exploiting the equation, we want to deduce finally that $\|\psi - \psi^n\|_{\mathcal{D}} \to 0$ as $\|\psi_0 - \psi_0^n\|_{\mathcal{D}} \to 0$. To this end, notice that we have

$$\|i\partial_t\psi - i\partial_t\psi^n\|_{L_t^\infty L_x^2} = \|\mathcal{H}\psi + F(\psi) - \mathcal{H}\psi^n - F(\psi^n)\|_{L_t^\infty L_x^2} \to 0 \tag{3.18}$$

Notice that from (2.8) we have $\mathcal{D}^s \hookrightarrow L^{2p}$ with s at least equal to $\frac{n}{4}(1 - 1/p)$ (and at most $1/2$ or $1/4$ according to dimension 2 or 3). From this and 3.6-1)) we obtain continuity of $F : \mathcal{D}^s \to L^2$. On the other hand (see the following Remark 3.15) from continuity of solution with respect to data in $\mathcal{X}_T$, already shown, it follows continuity in $\mathcal{D}^s$, so that from $\|\psi_0^n - \psi_0\|_{\mathcal{D}^s} \to 0$ we get $\|F(\psi^n) - F(\psi)\|_{L_t^\infty L_x^2} \to 0$ and from (3.18) we conclude. Finally, to cover the whole interval $[0, T]$ we iterate the argument a finite number of times, possibly extracting a different subsequence from $\{\psi_0^n\}$. $\square$

Remark 3.15. *Notice that from the sole continuity of the solution map in $\mathcal{X}_T$, by interpolation and use of the uniform bound* (3.16) *we obtain*

$$\|\psi - \psi^n\|_{\mathcal{D}^s} \leqslant \|\psi - \psi^n\|_{\mathcal{D}}^s \|\psi - \psi^n\|^{1-s} \leqslant C\|\psi_0 - \psi_0^n\|^{1-s} \qquad s \in (0,1)$$

which assures continuity of the solution map with values in $\mathcal{D}^s$. This does not use the differentiability of the nonlinearity F, needed to derive continuity in $\mathcal{D}$.

Remark 3.16. *Concerning continuity with respect to initial data several results are possible, depending on the functional spaces where continuity is desired. In the previous proof we actually proved that $\|\psi^n - \psi\|_{\mathcal{Z}_T} \to 0$ as $\|\psi_0^n - \psi_0\|_{\mathcal{D}} \to 0$, which more than stated in Theorem 1.*

3.4. Blow-up alternative

Proof of Theorem 1. Part 5)**.** Let us define

$$M^* := \sup_{t \in [0,T^*)} \|\psi(t)\|_{\mathcal{D}}$$

and suppose that $M^* < \infty$.
It follows that there exists a sequence of times $\{t_n\} \subset \mathbb{R}^+$, $t_n \to T^*$, such that

$$\lim_{n\to\infty} \|\psi(t_n)\|_{\mathcal{D}} = M \leqslant M^*.$$

Let $T^* < +\infty$. The local well-posedness proof does not depend on the initial time t_0, but only on the fact that $\psi(t_0) \in \mathcal{D}$. In particular, one sees that from the definition of M^*, $\|\psi(t_0)\|_{\mathcal{D}} \leqslant M^*$ for every $t_0 \in (0, T^*)$. As a consequence the existence time $T(t_0)$ obtained starting from any $t_0 \in (0, T^*)$ satisfies

$$T(t_0) \geqslant C(M^*) > 0,$$

for some suitable constant $C(M^*)$ depending only on M^*. Setting now $t_0 := t_{n_0}$ with $t_{n_0} > T^* - C(M^*)$, one concludes that the solution exists beyond T^*, which contradicts its definition. $\square$

Remark 3.17. *According to the definition of T^*, there exists a unique function $\psi \in C([0,T^*);\mathcal{D}) \cap C^1([0,T^*);L^2)$ coinciding for every $T < T^*$ with the solution $\psi \in C([0,T];\mathcal{D}) \cap C^1([0,T];L^2)$ of* (3.2) *as defined by the local existence theorem. The function ψ so defined on* $[0, T^*)$ *is called the maximal solution of* (3.2)*.*

4. Conservation laws and global well posedness

In this section we prove **Theorem 2**. We preliminarily show conservation laws for the model.

4.1. Mass and Energy conservation

Proposition 4.1. *(Conservation of Mass and Energy) In the hypotheses of Theorem 1 we have:*

1. *L^2- mass is conserved along the evolution:* $\|\psi(t)\|^2 = \|\psi_0\|^2 \qquad \forall t \in [0, T^*)$;
2. *Energy is conserved along the evolution:* $E(\psi(t)) = E(\psi_0) \qquad \forall t \in [0, T^*)$ *where*
$$E(\psi) = \frac{1}{2}\langle\psi, \mathcal{H}\psi\rangle \pm \frac{1}{p+1}\|\psi\|_{p+1}^{p+1} \qquad \psi \in \mathcal{D}.$$

Proof. Thanks to Corollary 3.12, the equation $i\partial_t\psi = \mathcal{H}\psi \pm |\psi|^{p-1}\psi$ holds as an identity in L^2. After taking the inner product with ψ and then the imaginary part of the resulting equation one gets mass conservation.
Consider now the energy. Recall that $\mathcal{H}$ is self-adjoint on $\mathcal{D}$, the corresponding quadratic form
$$E_{lin} : \mathcal{D} \to \mathbb{R}, \qquad \psi \mapsto E_{lin}(\psi) := \frac{1}{2}\langle\psi, \mathcal{H}\psi\rangle$$
is continuous with respect to the graph norm and differentiable with respect to the L^2 norm, with gradient given by $\mathcal{H}\psi$.
Being $p > 1$, the same holds true for the nonlinear functional
$$E_{nl} : \mathcal{D} \to \mathbb{R}, \qquad \psi \mapsto E_{nl}(\psi) := \pm\frac{1}{p+1}\|\psi\|_{p+1}^{p+1}$$
with gradient given by $\pm|\psi|^{p-1}\psi$.
We can now differentiate with respect to time the total energy along a solution $\psi(t)$ of (3.1) and we get
$$\begin{aligned}\frac{d}{dt}E(\psi(t)) &= \mathrm{Re}\,\{\langle\partial_t\psi(t), \mathcal{H}\psi(t) \pm |\psi(t)|^{p-1}\psi(t)\rangle\} \\ &= \mathrm{Re}\,\{\langle\partial_t\psi(t), i\partial_t\psi(t)\rangle\} = 0 \qquad \forall t \in (0, T^*)\ .\end{aligned}$$

□

4.2. Energy bound

Let us consider the focusing case. Replacing in (2.11) $q = p + 1$ we obtain
$$\begin{aligned}&\|\psi\|_{L^{p+1}}^{p+1} \leqslant c\|\psi\|_{L^2}^{(1-s)(p+1)}\|\psi\|_{\mathcal{D}^{1/2}}^{s(p+1)} \qquad s \in \Big(\frac{p-1}{p+1}, 1\Big) \qquad n = 2;\\ &\|\psi\|_{L^{p+1}}^{p+1} \leqslant c\|\psi\|_{L^2}^{(1-s)(p+1)}\|\psi\|_{\mathcal{D}^{1/2}}^{s(p+1)} \qquad s \in \Big(\frac{3p-3}{2p+2}, 1/2\Big) \qquad n = 3.\end{aligned} \tag{4.1}$$

From mass and energy conservation and inequalities (4.1) we conclude that both the linear energy $\langle\psi, \mathcal{H}\psi\rangle := \|\psi\|_{\mathcal{D}^{1/2}}^2 - \lambda\|\psi\|^2$ and the nonlinear term $\|\psi\|_{L^{p+1}}^{p+1}$, are uniformly bounded in terms of the mass and energy of the initial datum if the quantity $s(p+1) < 2$. From the limitation on s this occurs in the $n = 2$ case for $p < 3$ and in the $n = 3$ case for $p < 7/3$. These limitations coincide with the ones of the standard NLS equation. Notice however that in the $n = 3$ case well posedness in $\mathcal{D}$ prevents $p \geqslant 3/2$.

4.3. Global existence.

Proof of Theorem 2. From Cor. 3.12 we know that the solution ψ is in $C([0,T^*),\mathcal{D}) \cap C^1([0,T^*),L^2(\mathbb{R}))$, here T^* is the maximal time of existence of the solution. By rephrasing the blow-up alternative, we know that if $T^* < \infty$ it must be $\lim_{t\to T^*} \|\psi(t)\|_{\mathcal{D}} = \infty$.
To prove that the solution is global we reason by absurd: we show that if $T^* < \infty$ it must hold $\lim_{t\to T^*} \|\psi(t)\|_{\mathcal{D}} < \infty$; but this contradicts the blow-up alternative and so $T^* = \infty$.
Let us assume that $T^* < \infty$. The first observation is that conservation laws imply that

$$\|\psi\|_{L^\infty((0,T^*);L^2)} < \infty \qquad \text{and} \qquad \|\psi\|_{L^\infty((0,T^*);\mathcal{D}^{1/2})} < \infty. \tag{4.2}$$

Hence, to prove that $\lim_{t\to T^*} \|\psi(t)\|_{\mathcal{D}} < \infty$ it is enough to show that $\|\mathcal{H}\psi(t)\|$ does not blow-up in finite time. In particular, we are going to show that

$$\|\mathcal{H}\psi\|_{L^\infty((0,T^*);L^2)} < \infty,$$

then, by continuity of $\|\mathcal{H}\psi(t)\|$, this guarantees that $\lim_{t\to T^*} \|\mathcal{H}\psi(t)\| < \infty$.
Since ψ is a strong solution we have

$$\mathcal{H}\psi(t) = i\partial_t\psi(t) - F(\psi(t)).$$

Hence,

$$\|\mathcal{H}\psi(t)\| \leqslant \|\partial_t\psi(t)\| + \|\psi(t)\|_{2p}^p \qquad \forall t \in (0,T^*),$$

and,

$$\|\mathcal{H}\psi\|_{L^\infty((0,T^*);L^2)} \leqslant \|\partial_t\psi\|_{L^\infty((0,T^*);L^2)} + \|\psi\|^p_{L^\infty((0,T^*);L^{2p})}.$$

By the Gagliardo-Nirenberg inequalities in Eq. (2.11) we infer

$$\|\psi(t)\|_{2p} \leqslant c\|\psi(t)\|_{\mathcal{D}^{1/2}} \qquad \forall t \in (0,T^*),$$

hence, $\|\psi\|_{L^\infty((0,T^*);L^{2p})} < \infty$ and we are left to prove that $\partial_t\psi \in L^\infty((0,T^*);L^2)$. Preliminarily we prove that $\partial_t\psi \in L^r((0,T^*);L^{p+1})$. We start by noticing that from part 2) of Th. 1 it follows that

$$\partial_t\psi \in L^r((0,\tau);L^{p+1}) \qquad \forall 0 < \tau < T^*.$$

Hence, by Hölder inequality in time,

$$\|\partial_t\psi\|_{L^{r'}((0,\tau);L^{p+1})} \leqslant {T^*}^{1-\frac{2}{r}} \|\partial_t\psi\|_{L^r((0,\tau);L^{p+1})} < \infty,$$

and

$$\partial_t\psi \in L^{r'}((0,\tau);L^{p+1}) \qquad \forall 0 < \tau < T^*. \tag{4.3}$$

Then, by (2.22), (2.23) and (3.2) we infer

$$\partial_t\psi(t) = -iU(t)\mathcal{H}\psi_0 - iU(t)F(\psi_0) - i(\Gamma\partial_t F(\psi))(t). \tag{4.4}$$

Next we apply Prop. 2.6-*b*) and 2.6-*f*) on $(0,\tau)$ to obtain

$$\|\partial_t\psi\|_{L^r((0,\tau);L^{p+1})} \leqslant C(\tau)(\|\mathcal{H}\psi_0\| + \|F(\psi_0)\| + \|\partial_t F(\psi)\|_{L^{r'}((0,\tau);L^{1+1/p})}).$$

where $C(\tau)$ is bounded by a constant that depends on T^*. Since $\psi_0 \in \mathcal{D}$, also $\|\mathcal{H}\psi_0\|$ and $\|F(\psi_0)\|$ are bounded and can be absorbed in the constant. Hence, we have the bound

$$\|\partial_t \psi\|_{L^r((0,\tau);L^{p+1})} \leqslant C\big(1 + \|\partial_t F(\psi)\|_{L^{r'}((0,\tau);L^{1+1/p})}\big) \tag{4.5}$$

where the constant C depends on T^* and ψ_0 but not on τ.
By the inequality already used several times before,

$$\|F(\psi)(t') - F(\psi)(t)\|_{1+1/p} \leqslant C(\|\psi(t')\|_{p+1}^{p-1} + \|\psi(t)\|_{p+1}^{p-1})\, \|\psi(t') - \psi(t)\|_{p+1},$$

we obtain

$$\|\partial_t F(\psi)(t)\|_{1+1/p} \leqslant C\|\psi(t)\|_{p+1}^{p-1}\|\partial_t \psi(t)\|_{p+1} \qquad \forall t \in (0, T^*).$$

Hence, the bound (4.1) and the a-priori bounds (4.2) give

$$\|\partial_t F(\psi)\|_{L^{r'}((0,\tau);L^{1+1/p})} \leqslant C\|\partial_t \psi\|_{L^{r'}((0,\tau);L^{p+1})}, \tag{4.6}$$

so that, by inequality (4.5) we infer

$$\|\partial_t \psi\|_{L^r((0,\tau);L^{p+1})} \leqslant C\big(1 + \|\partial_t \psi\|_{L^{r'}((0,\tau);L^{p+1})}\big). \tag{4.7}$$

Fix $0 < \varepsilon < \tau < T^*$, and notice that

$$\begin{aligned}
\|\partial_t \psi\|_{L^{r'}((0,\tau);L^{p+1})} &\leqslant C\big(\|\partial_t \psi\|_{L^{r'}((0,\tau-\varepsilon);L^{p+1})} + \|\partial_t \psi\|_{L^{r'}((\tau-\varepsilon,\tau);L^{p+1})}\big) \\
&\leqslant C\big(\|\partial_t \psi\|_{L^{r'}((0,T^*-\varepsilon);L^{p+1})} + \varepsilon^{1-\frac{2}{r}}\|\partial_t \psi\|_{L^{r}((\tau-\varepsilon,\tau);L^{p+1})}\big) \\
&\leqslant C\big(\|\partial_t \psi\|_{L^{r'}((0,T^*-\varepsilon);L^{p+1})} + \varepsilon^{1-\frac{2}{r}}\|\partial_t \psi\|_{L^{r}((0,\tau);L^{p+1})}\big)
\end{aligned}$$

Inserting the latter bound in Eq. (4.7), we obtain the inequality

$$\|\partial_t \psi\|_{L^r((0,\tau);L^{p+1})} \leqslant C\big(1 + \|\partial_t \psi\|_{L^{r'}((0,T^*-\varepsilon);L^{p+1})} + \varepsilon^{1-\frac{2}{r}}\|\partial_t \psi\|_{L^{r}((0,\tau);L^{p+1})}\big).$$

For ε small enough the latter term at the r.h.s. can be absorbed in the l.h.s. to obtain

$$\|\partial_t \psi\|_{L^r((0,\tau);L^{p+1})} \leqslant C\big(1 + \|\partial_t \psi\|_{L^{r'}((0,T^*-\varepsilon);L^{p+1})}\big).$$

Since $\partial_t \psi \in L^{r'}((0, T^* - \varepsilon); L^{p+1})$ by Eq. (4.3), and C does not depend on τ, taking the limit $\tau \to T^*$ we obtain the desired claim $\partial_t \psi \in L^r((0, T^*); L^{p+1})$.
To conclude we go back to Eq. (4.4) and use Prop. 2.6-*a*) and *e*) to obtain

$$\|\partial_t \psi\|_{L^\infty((0,T^*);L^2)} \leqslant C\big(\|\mathcal{H}\psi_0\| + \|F(\psi_0)\| + \|\partial_t F(\psi)\|_{L^{r'}((0,T^*);L^{1+1/p})}\big)$$

As before, see Eq. (4.6), the bound (4.1) and the formula above give

$$\|\partial_t F(\psi)\|_{L^{r'}((0,T^*);L^{1+1/p})} \leqslant C\|\partial_t \psi\|_{L^{r'}((0,T^*);L^{p+1})} \leqslant C{T^*}^{1-\frac{2}{r}}\|\partial_t \psi\|_{L^{r}((0,T^*);L^{p+1})},$$

hence, $\partial_t F(\psi) \in L^{r'}((0, T^*); L^{1+1/p})$, which in turn implies $\partial_t \psi \in L^\infty((0, T^*); L^2)$ and concludes the proof.

Acknowledgments.

D. Noja acknowledges for funding the EC grant IPaDEGAN (MSCA-RISE-778010). The authors acknowledge the support of the Gruppo Nazionale di Fisica Matematica (GNFM-INdAM).

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