Branching random walks with uncountably many extinction probability vectors

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Abstract. Given a branching random walk on a set X, we study its extinction probability vectors $\mathbf{q}(\cdot, A)$. Their components are the probability that the process goes extinct in a fixed $A \subseteq X$, when starting from a vertex $x \in X$. The set of extinction probability vectors (obtained letting A vary among all subsets of X) is a subset of the set of the fixed points of the generating function of the branching random walk. In particular here we are interested in the cardinality of the set of extinction probability vectors. We prove results which allow to understand whether the probability of extinction in a set A is different from the one of extinction in another set B. In many cases there are only two possible extinction probability vectors and so far, in more complicated examples, only a finite number of distinct extinction probability vectors had been explicitly found. Whether a branching random walk could have an infinite number of distinct extinction probability vectors was not known. We apply our results to construct examples of branching random walks with uncountably many distinct extinction probability vectors.

1 Introduction

The branching random walk (or briefly *BRW*) on an at most countable set X is a process which describes the evolution of a population breeding and dying on X. When X is a singleton, the BRW reduces to the *branching process* (the BRW is also known as *multi-type branching process*). In the long run, for any $A \subseteq X$, a BRW starting with one individual at $x \in X$ can go extinct in A (no individuals alive in A from a certain time on) or survive in A (infinitely many visits to A). If the probability of extinction in A is equal to 1, we say that there is extinction in A, otherwise that there is survival in A.

Letting x vary in X, we get an extinction probability vector, and letting also A vary we have the family of extinction probability vectors. These vectors are of particular interest and can be seen as fixed points of a suitable generating function associated to the process (see Section 2).

It is well known that the generating function of the branching process has at most two fixed points: the extinction probability and 1. When the space X is not a singleton, there are, in principle, more extinction probabilities (see Section 2), thus more fixed points. It remains true that the vector **1** is always a fixed point and the *global extinction* probability vector (that is, the probability of extinction in the whole space X) is always the minimal fixed point.

In order to construct a BRW with a large number of fixed points, a trivial way is to use *reducible* BRWs (see Section 2 for the definition). Roughly speaking, in the reducible case, X is divided into classes and the progeny of particles living in some classes cannot colonize other ones, and it is not difficult to have different extinction probability vectors in each class. The interesting case is when the BRW is *irreducible*, that is, the progeny of a particle has always a positive probability of reaching every point of X. Therefore our study addresses the

question whether an irreducible BRW can have infinitely many distinct extinction probability vectors.

It turns out that for any irreducible BRW on a finite X the situation is the same as in a branching process: there are at most two fixed points. In particular, this means that the value of the probability of extinction in a (nonempty) set A does not depend on A. The interests on fixed points slowed down when in Spataru (1989) the author claimed that the generating function of every irreducible BRW on a finite or countable X has at most two fixed points. In particular, if this were true, in any BRW the probability of extinction in $A \subseteq X$ would be either 1 or equal to the probability of global extinction (this last case is called strong local survival at A, see Bertacchi and Zucca (2014), Machado, Menshikov and Popov (2001)).

In Bertacchi and Zucca (2014), the authors found a gap in the proof of Spataru (1989, Theorem 3) and provided an example of an irreducible BRW with more than two fixed points. Later on, in Bertacchi and Zucca (2017) the authors provided an example of an irreducible BRW with an uncountable number of fixed points which are not extinction probabilities. Since then, some authors have been trying to describe the properties of the space of fixed points and the subspace of extinction probabilities (see Section 4 for further details on the state of the art).

The first example of an irreducible BRW with (at least) two nontrivial extinction probabilities is in Bertacchi and Zucca (2014, Example 4.2) while the first example of an irreducible BRW with (at least) three nontrivial extinction probabilities can be found in Braunsteins and Hautphenne (2018). Both BRWs are inhomogeneous, that is, the reproduction laws differ from site to site. It is natural to wonder whether a BRW can have infinitely many extinction probability vectors and in particular if this is possible without "inhomogeneity". Perhaps the strongest "homogeneity" one can think of is transience, that is when one can map every site to any other site through an automorphism (see Section 2). In this paper, we provide an example of an irreducible and transient BRW on a regular tree with an uncountable number of distinct extinction probabilities (see Section 3.1). The key to the proof is finding an uncountable family $\mathcal{F}$ of unions of subtrees such that the probability that the process starting from the root goes extinct in A, is different for all $A \in \mathcal{F}$. Two members of $\mathcal{F}$ have different "size of their boundaries", so at first one may think that the probability of extinction, starting from the root, only depend on this size, but we prove that this is not the case (again see Section 3.1).

In the example in Section 3.1, there is extinction in all finite sets; this is not necessary, indeed Example 3.4 is an inhomogeneous BRW on the tree with uncountably many extinction probability vectors and survival with positive probability in each finite set. The tree is a graph with a particularly "large" boundary, but even this property is not necessary: in Section 3.2 we generalize our result to a wider class of BRWs with uncountably many extinction probabilities, supported on the comb, which is a subset of $\mathbb{Z}^2$.

In Section 2, we define the process in discrete-time and in continuous-time, survival and extinction and then define the generating function of a BRW (Section 2.2), its fixed points and the extinction probability vectors (Section 2.3). We state and prove two results which tell when two extinction probabilities are equal or not. In particular, Theorem 2.4 compares two BRWs whose reproduction laws are the same outside a set $A \subset X$, while Lemma 2.5 deals with the case when the process survives in a set, without ever visiting another set. In Section 3.1, we prove in detail that on the tree the generating function of a BRW can have uncountably many extinction probabilities. This is not just a property of the tree; indeed, by projecting BRWs (see Definition 3.5) we prove in Section 3.2, that the same holds for a larger class (up to projections) of BRWs. Finally, Section 4 is devoted to a brief description of the state of the art on extinction probabilities and fixed points and contains some open questions.

2 Extinction probabilities: Definitions and properties

Given an at most countable set X, a discrete-time BRW is a process $\{\eta_n\}_{n\in\mathbb{N}}$, where $\eta_n(x)$ is the number of particles alive at $x \in X$ at time n. The dynamics is described as follows: consider the (countable) measurable space $(S_X, 2^{S_X})$ where $S_X := \{f : X \to \mathbb{N}: \sum_y f(y) < \infty\}$ and let $\mu = \{\mu_x\}_{x\in X}$ be a family of probability measures on $(S_X, 2^{S_X})$. A particle of generation n at site $x \in X$ lives one unit of time; after that, a function $f \in S_X$ is chosen at random according to the law μ_x. This function describes the number of children and their positions, that is, the original particle is replaced by $f(y)$ particles at y, for all $y \in X$. The choice of f is independent for all breeding particles. The BRW is denoted by (X, μ) and it is a Markov chain with absorbing state $\mathbf{0}$, the configuration with no particles at all sites.

The total number of children associated to f is represented by the function $\mathcal{H} : S_X \to \mathbb{N}$ defined by $\mathcal{H}(f) := \sum_{y\in X} f(y)$; the associated law $\rho_x(\cdot) := \mu_x(\mathcal{H}^{-1}(\cdot))$ is the law of the random number of children of a particle living at x. We denote by $m_{xy} := \sum_{f\in S_X} f(y)\mu_x(f)$ the expected number of children that a particle living at x sends to y. It is easy to show that $\sum_{y\in X} m_{xy} = \bar{\rho}_x$ where $\bar{\rho}_x$ is the expected value of the law ρ_x.

In particular, if ρ_x does not depend on $x \in X$, we say that the BRW can be *projected on a branching process* (see Bertacchi and Zucca (2012) for details). More generally, some BRWs can be projected onto BRWs defined on finite sets as explained in Bertacchi and Zucca (2017, Section 2.3) (see also Definition 3.5 for the case of continuous-time BRWs). In the case of the projection on a branching process, the finite set is a singleton. Other examples are the so called *quasi-transitive BRWs* (see Bertacchi and Zucca (2014, Section 2.4, p. 408) for the formal definition) where the action of the group of the automorphisms of the BRW (namely, bijective maps preserving the reproduction laws) has a finite number j of orbits: the finite set onto which we project has cardinality j. When there is just one orbit, then it is called *transitive* (which is thus a particular case of BRW projected on a branching process).

To a generic discrete-time BRW, we associate a graph (X, E_μ) where $(x, y) \in E_\mu$ if and only if $m_{xy} > 0$. We say that there is a path from x to y of length n, and we write $x \xrightarrow{n} y$, if it is possible to find a finite sequence $\{x_i\}_{i=0}^n$ (where $n \in \mathbb{N}$) such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E_\mu$ for all $i = 0, \ldots, n-1$. Clearly $x \xrightarrow{0} x$ for all $x \in X$; if there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} y$ then we write $x \to y$. If the graph (X, E_μ) is *connected* then we say that the BRW is *irreducible*.

We consider initial configurations with only one particle placed at a fixed site x and we denote by $\mathbb{P}^{\delta_x}$ the law of the corresponding process. We now distinguish between the possible long-term behaviours of a BRW.

Definition 2.1.

1. The process *survives locally* in $A \subseteq X$ starting from $x \in X$ if

$$\mathbf{q}(x, A) := 1 - \mathbb{P}^{\delta_x}\left(\limsup_{n\to\infty} \sum_{y\in A} \eta_n(y) > 0\right) < 1.$$

2. The process *survives globally* starting from x if $\bar{\mathbf{q}}(x) := \mathbf{q}(x, X) < 1$.

In the rest of the paper, we use the notation $\mathbf{q}(x, y)$ instead of $\mathbf{q}(x, \{y\})$ for all $x, y \in X$. When there is no survival with positive probability, we say that there is extinction and the fact that extinction occurs almost surely will be tacitly understood. It is worth noting that, in the irreducible case, for every $A \subseteq X$, the inequality $\mathbf{q}(x, A) < 1$ holds for some $x \in X$ if and only if it holds for every $x \in X$. For details and results on survival and extinction, see for instance, Section 2.3 or Bertacchi and Zucca (2012), Zucca (2011).

2.1 Continuous-time branching random walks

In continuous time, each particle has an exponentially distributed random lifetime with parameter 1. The breeding mechanisms can be regulated by putting on each couple (x, y) and for each particle at x, a clock with $Exp(\lambda k_{xy})$-distributed intervals (where $\lambda > 0$), each time the clock rings the particle breeds in y.

If one is only interested in survival and extinction of the process, the continuous-time BRW has a discrete-time counterpart with the same long-term behavior: here is the construction. The initial particles represent the generation 0 of the discrete-time BRW; the generation $n + 1$ (for all $n \geq 0$) is obtained by considering the children of all the particles of generation n (along with their positions).

If X has a graph structure and $K := (k_{xy})_{x, y \in X}$ is the adjacency matrix then we call the process an *edge-breeding BRW*; in this case the graph (X, E_μ) associated to the discrete-time counterpart is the preexisting graph on X. In particular an edge-breeding BRW is quasi-transitive if and only if the underlying graph is.

Given $x \in X$, two critical parameters are associated to the continuous-time BRW: the global survival critical parameter $\lambda_w(x)$ and the local survival one $\lambda_s(x)$. They are defined as

$$
\begin{aligned}
\lambda_w(x) &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_x}(\exists t : \eta_t = \mathbf{0}) < 1\}, \\
\lambda_s(x) &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_x}(\exists \bar{t} : \eta_t(x) = 0, \forall t \geq \bar{t}) < 1\}.
\end{aligned}
\tag{2.1}
$$

In particular when $\lambda < \lambda_w(x)$ (resp. $\lambda > \lambda_w(x)$) then $\bar{\mathbf{q}}(x) = 1$ (resp. $\bar{\mathbf{q}}(x) < 1$); while if $\lambda = \lambda_w(x)$ there can be both global extinction and global survival (see, for instance, Bertacchi and Zucca (2009)). As for the local behavior $\lambda \leq \lambda_s(x)$ if and only if $\mathbf{q}(x, x) = 1$. If the process is irreducible, then the critical parameters do not depend on x. See Bertacchi, Coletti and Zucca (2017), Bertacchi and Zucca (2008, 2009, 2012) for a more detailed discussion on the values of $\lambda_w(x)$ and $\lambda_s(x)$, including their characterizations.

2.2 Infinite-dimensional generating function

To the family $\{\mu_x\}_{x \in X}$, we associate the following generating function $G : [0, 1]^X \to [0, 1]^X$,

$$
G(\mathbf{z}|x) := \sum_{f \in S_X} \mu_x(f) \prod_{y \in X} \mathbf{z}(y)^{f(y)},
$$

where $G(\mathbf{z}|x)$ is the x coordinate of $G(\mathbf{z})$. The family $\{\mu_x\}_{x \in X}$ is uniquely determined by G. G is continuous with respect to the *pointwise convergence topology* of $[0, 1]^X$ and non-decreasing with respect to the usual partial order of $[0, 1]^X$ (see Bertacchi and Zucca (2009, Sections 2 and 3) for further details). Extinction probabilities are fixed points of G and the smallest fixed point is $\bar{\mathbf{q}}$ (see Section 2.3 for details): more generally, given a solution of $G(\mathbf{z}) \leq \mathbf{z}$ then $\mathbf{z} \geq \bar{\mathbf{q}}$.

Consider now the closed sets $F_G := \{\mathbf{z} \in [0, 1]^X : G(\mathbf{z}) = \mathbf{z}\}$, $U_G := \{\mathbf{z} \in [0, 1]^X : G(\mathbf{z}) \leq \mathbf{z}\}$ and $L_G := \{\mathbf{z} \in [0, 1]^X : G(\mathbf{z}) \geq \mathbf{z}\}$; clearly $F_G = U_G \cap L_G$. Moreover, by the monotonicity property, $G(U_G) \subseteq U_G$ and $G(L_G) \subseteq L_G$. The iteration of G produces sequences converging to fixed points.

Proposition 2.2. *Consider a sequence $\{\mathbf{z}_n\}_{n \in \mathbb{N}}$ in $[0, 1]^X$ such that $\mathbf{z}_{n+1} = G(\mathbf{z}_n)$ for all $n \in \mathbb{N}$ and suppose that $\mathbf{z}_n \to \mathbf{z}$ as $n \to +\infty$ for some $\mathbf{z} \in [0, 1]^X$. Then $\mathbf{z} \in F_G$. Moreover, fix $\mathbf{w} \in [0, 1]^X$.*

1. *If $\mathbf{w} \in U_G$, then $\mathbf{w} \geq \mathbf{z}_0$ implies $\mathbf{w} \geq \mathbf{z}$ (the converse holds for $\mathbf{z}_0 \in L_G$).*
2. *If $\mathbf{w} \in L_G$, then $\mathbf{w} \leq \mathbf{z}_0$ implies $\mathbf{w} \leq \mathbf{z}$ (the converse holds for $\mathbf{z}_0 \in U_G$).*

The proof is straightforward (see, for instance, Bertacchi and Zucca (2009)). The convergence of the sequence $\{\mathbf{z}_n\}_{n\in\mathbb{N}}$ defined in the previous proposition holds if $\mathbf{z}_0 \in L_G$ (resp. $\mathbf{z}_0 \in U_G$): in that case $\mathbf{z}_n \uparrow \mathbf{z}$ (resp. $\mathbf{z}_n \downarrow \mathbf{z}$) for some $\mathbf{z} \in F_G$.

The following properties of U_G and L_G allow to identify potentially new fixed points: if we have a collection $\{\mathbf{w}_i\}_{i\in I}$ where $\mathbf{w}_i \in U_G$ (resp. $\mathbf{w}_i \in L_G$) for all $i \in I$ and $\mathbf{z}_0(x) := \inf_{i\in I} \mathbf{w}_i(x)$ then $\mathbf{z}_0 \in U_G$ (resp. if $\mathbf{z}_0(x) := \sup_{i\in I} \mathbf{w}_i(x)$ then $\mathbf{z}_0 \in L_G$); for instance, it is enough to consider a collection $\{\mathbf{w}_i\}_{i\in I}$ of fixed points. In both cases, $\mathbf{z} = \lim_{n\to+\infty} \mathbf{z}_n$ is well defined; moreover, if $\mathbf{z}_0 < \mathbf{w}_i$ (resp. $\mathbf{z}_0 > \mathbf{w}_i$) for all $i \in I$ then $\mathbf{z}$ is a fixed point different from $\mathbf{w}_i$ for any $i \in I$.

2.3 Fixed points and extinction probabilities

Define $\mathbf{q}_n(x, A)$ as the probability of extinction in A before time n starting with one particle at x, namely $\mathbf{q}_n(x, A) = \mathbb{P}^{\delta_x}(\eta_k(y) = 0, \forall k \geq n, \forall y \in A)$. The sequence $\{\mathbf{q}_n(x, A)\}_{n\in\mathbb{N}}$ is nondecreasing and satisfies

$$\begin{cases} \mathbf{q}_n(\cdot, A) = G(\mathbf{q}_{n-1}(\cdot, A)), & \forall n \geq 1, \\ \mathbf{q}_0(x, A) = 0, & \forall x \in A, \\ \mathbf{q}_0(x, A) = G(\mathbf{q}_0(\cdot, A)|x), & \forall x \notin A. \end{cases} \qquad (2.2)$$

Moreover, $\mathbf{q}_n(x, A)$ converges to $\mathbf{q}(x, A)$, which is the probability of local extinction in A starting with one particle at x (see Definition 2.1); more precisely $\mathbf{q}_n(\cdot, A) \in L_G$ for all $n \in \mathbb{N}$. Since G is continuous, by Proposition 2.2 we have that $\mathbf{q}(\cdot, A) = G(\mathbf{q}(\cdot, A))$, hence these extinction probability vectors are fixed points of G. For details on the last equality in equation (2.2) see Remark 2.3. We denote the set of extinction probability vectors by $E_G := \{\mathbf{q}(\cdot, A) : A \subseteq X\} \supseteq \{\bar{\mathbf{q}}, \mathbf{1}\}$, since $\mathbf{q}(\cdot, \varnothing) = \mathbf{1}$ where $\mathbf{1}(x) = 1$ for all $x \in X$.

Clearly $U_G \supseteq F_G \supseteq E_G$ and it is well known that $\bar{\mathbf{q}}$ is the smallest element of each one of these three sets (since $\mathbf{q}_0(\cdot, X) = \mathbf{0}$, it is enough to apply Proposition 2.2) and $\mathbf{1}$ is the largest one. Hence, $\bar{\mathbf{q}} = \mathbf{1}$ (global extinction) if and only if at least one of these sets is a singleton, that is, if and only if they are all singletons.

Note that $A \subseteq B \subseteq X$ implies $\mathbf{q}(\cdot, A) \geq \mathbf{q}(\cdot, B) \geq \bar{\mathbf{q}}$. Since for all finite $A \subseteq X$ we have $\mathbf{q}(x, A) \geq 1 - \sum_{y\in A}(1 - \mathbf{q}(x, y))$ then, for any given finite $A \subseteq X$, $\mathbf{q}(x, A) = 1$ if and only if $\mathbf{q}(x, y) = 1$ for all $y \in A$. If the BRW is irreducible then, for all $A \subseteq X$, $\mathbf{q}(\cdot, A) < \mathbf{1}$ if and only if $\mathbf{q}(x, A) < 1$ for all $x \in X$; moreover for all finite (nonempty) subsets $A, B \subseteq X$ we have $\mathbf{q}(\cdot, A) = \mathbf{q}(\cdot, B)$.

Remark 2.3. We observe that if $d(x, A) := \min\{n \in \mathbb{N}, y \in A : x \overset{n}{\to} y\}$ then $\mathbf{q}_n(x, A) = \mathbf{q}_0(x, A)$ for all x such that $d(x, A) \geq n$. Hence, $\mathbf{q}_1(x, A) = \mathbf{q}_0(x, A)$ for all $x \notin A$ and according to equation (2.2) we have $\mathbf{q}_0(x, A) = G(\mathbf{q}_0|x)$ for all $x \notin A$.

2.4 Extinction probabilities in different sets

We give here a couple of results which allow, in some cases, to know whether $\mathbf{q}(x, A)$ is different from $\mathbf{q}(x, B)$. The first theorem, is a generalization of Bertacchi and Zucca (2014, Theorem 3.3) and Bertacchi and Zucca (2017, Theorems 4.1 and 4.2). We include the proof for the sake of completeness. In the case of global survival, it gives equivalent conditions for strong local survival in terms of extinction probabilities.

Theorem 2.4.

(a) *For every subset $A \subseteq X$ and every fixed point $\mathbf{z}$ of G, the following assertions are equivalent.*

(1) $\mathbf{q}(x, A) \leq \mathbf{z}(x)$, *for all $x \in X$;*
(2) $\mathbf{q}_0(x, A) \leq \mathbf{z}(x)$, *for all $x \in X$.*
In particular if $\mathbf{z} = \mathbf{q}(\cdot, B)$, where $A \subseteq B$, then the previous conditions are equivalent to
(3) $\mathbf{q}(x, A) = \mathbf{q}(x, B)$ *for all $x \in X$.*
(b) *Consider two BRWs (X, μ) and (X, ν). Suppose that $A \subseteq X$ is a nonempty set such that $\mu_x = \nu_x$ for all $x \notin A$. Then $\mathbf{q}_0^\mu(\cdot, A) = \mathbf{q}_0^\nu(\cdot, A)$ Moreover, if $A \subseteq B$ then*

$$\mathbf{q}^\mu(x, A) = \mathbf{q}^\mu(x, B), \quad \forall x \in X \quad \Longleftrightarrow \quad \mathbf{q}^\nu(x, A) = \mathbf{q}^\nu(x, B), \quad \forall x \in X.$$

Proof. (a) If $\mathbf{q}_0(\cdot, A) \leq \mathbf{z}(\cdot)$ then, by equation (2.2), $\mathbf{q}_n(\cdot, A) \leq \mathbf{z}(\cdot)$ for all $n \in \mathbb{N}$, whence $\mathbf{q}(\cdot, A) = \lim_{n \to +\infty} \mathbf{q}_n(\cdot, A) \leq \mathbf{z}(\cdot)$. Conversely, if $\mathbf{q}(\cdot, A) \leq \mathbf{z}(\cdot)$ then, by the monotonicity of $\{\mathbf{q}_n(\cdot, A)\}_{n \in \mathbb{N}}$ we have $\mathbf{q}_0(\cdot, A) \leq \mathbf{z}(\cdot)$.

(b) The equality $\mathbf{q}_0^\mu(x, A) = \mathbf{q}_0^\nu(x, A)$ is trivial when $x \in A$ and, when $x \notin A$, it follows from the fact that the behavior of the two BRWs is the same until they first hit the set A. From the previous part of the theorem, by taking $\mathbf{z} = \mathbf{q}^\mu(\cdot, B)$ and $\mathbf{z} = \mathbf{q}^\nu(\cdot, B)$ we have that

$$\mathbf{q}^\mu(x, A) = \mathbf{q}^\mu(x, B), \quad \forall x \in X \quad \Longleftrightarrow \quad \mathbf{q}_0^\mu(x, A) \leq \mathbf{q}^\mu(x, B), \quad \forall x \in X;$$

$$\mathbf{q}^\nu(x, A) = \mathbf{q}^\nu(x, B), \quad \forall x \in X \quad \Longleftrightarrow \quad \mathbf{q}_0^\mu(x, A) \leq \mathbf{q}^\nu(x, B), \quad \forall x \in X.$$

If, for some $x \in X$, $\mathbf{q}^\mu(x, A) \neq \mathbf{q}^\mu(x, B)$ then, since $A \subseteq B$, $\mathbf{q}^\mu(x, A) > \mathbf{q}^\mu(x, B)$; thus, from the previous part of the theorem, with positive probability (X, μ) survives in B without ever visiting A (starting from a suitable $y \in B \setminus A$). Thus, the same holds for (X, ν) (because their behavior is the same until they first hit A), thus, $\mathbf{q}^\nu(x, A) > \mathbf{q}^\nu(x, B)$ for some $x \in X$. By switching, now, the roles of (X, μ) and (X, ν), the equivalence follows. $\qquad\square$

From the previous theorem, we have the following dichotomy: for every sets $A, B \subseteq X$, either $\mathbf{q}(\cdot, A) \leq \mathbf{q}(\cdot, B)$ or there is $x \in B \setminus A$ such that there is a positive probability of local survival in B starting from x without ever visiting A. In particular $\mathbf{q}(x, A) > \mathbf{q}(x, B)$ implies that there is a positive probability of local survival in B and local extinction in A starting from x (if $A \subseteq B$ then also the converse is true). Note that, $\mathbf{q}_0(x, A) > \mathbf{q}(x, B)$ implies $\mathbf{q}(x, A) > \mathbf{q}(x, B)$ but the converse is not true. The second tool that we need is the following lemma.

Lemma 2.5. *Consider a BRW (X, μ) and three subsets $A_1, A_2 \subseteq A \subseteq X$ such that $A_1 \cap A_2 = \varnothing$. If there exists $z \in X$ such that $\mathbb{P}^z(\sum_{x \in A_1} \eta_n(x) > 0 \text{ i.o.}, \lim_{n \to +\infty} \sum_{x \in A_2} \eta_n(x) = 0) > 0$, then $\mathbf{q}(z, A_2) > \mathbf{q}(z, A)$, whence $\mathbf{q}(\cdot, A_2) > \mathbf{q}(\cdot, A)$.*

In particular, if x is such that $x \to z$ then $\mathbf{q}(x, A_2) > \mathbf{q}(x, A)$ provided that $\bar{\mathbf{q}}(y) > 0$ whenever $x \to y$ (for instance, if $\mu_y(\mathbf{0}) > 0$ for all y).

Proof. From the inclusion $A_i \subseteq A$ we have $\mathbf{q}(\cdot, A_i) \geq \mathbf{q}(\cdot, A)$ (for $i = 1, 2$). From the hypotheses, we have

$$\mathbb{P}^z\left(\sum_{x \in A} \eta_i(x) > 0 \text{ i.o.}, \lim_{i \to +\infty} \sum_{x \in A_2} \eta_i(x) = 0\right)$$

$$\geq \mathbb{P}^z\left(\sum_{x \in A_1} \eta_n(x) > 0 \text{ i.o.}, \lim_{n \to +\infty} \sum_{x \in A_2} \eta_n(x) = 0\right) > 0$$

whence $\mathbf{q}(z, A_2) > \mathbf{q}(z, A)$ and this implies $\mathbf{q}(\cdot, A_2) > \mathbf{q}(\cdot, A)$.

If $x \to z$, then there is a positive probability p_0 that the process can reach z and that the progenies of all the particles, except at most one at z, die out; in this case the long term behavior is given by the evolution of the progeny of one particle at z. Thus $\mathbb{P}^x(\sum_{y \in A_1} \eta_i(y) > 0 \text{ i.o.}, \lim_{i \to +\infty} \sum_{y \in A_2} \eta_i(y) = 0) \geq p_0 \mathbb{P}^z(\sum_{y \in A_1} \eta_n(A_1) > 0 \text{ i.o.}, \lim_{n \to +\infty} \sum_{y \in A_2} \eta_n(y) = 0) > 0$. As before, this implies $\mathbf{q}(x, A_2) > \mathbf{q}(x, A)$. $\qquad\square$

3 BRWs with uncountably many extinction probabilities

3.1 A BRW with an uncountable set of extinction probability vectors: The tree

Consider the regular tree $\mathbb{T}_m$ (where $m \geq 3$) and the discrete-time counterpart of a continuous-time BRW where K is the adjacency matrix on $\mathbb{T}_m$; for this BRW it is well known that $\lambda_w = 1/m < 1/2\sqrt{m-1} = \lambda_s$. Denote a vertex by o and call it the root. Given $x \neq o$ we denote by T_x the subtree branching from x, that is, the set of vertices which are disconnected from o by removing x from $\mathbb{T}_m$; moreover, let $T_o := \mathbb{T}_m$. Given any automorphism Ψ of $\mathbb{T}_m$ (that is, a bijective map preserving the edges), one can easily prove that $\mathbf{q}(\cdot, A) = \mathbf{q}(\Psi(\cdot), \Psi(A))$; in particular if $\Psi(o) = o$ and $\Psi(x) = y$ then $\Psi(T_x) = \Psi(T_y)$, thus $\mathbf{q}(\cdot, T_x) = \mathbf{q}(\Psi(\cdot), T_y)$.

If $\lambda \leq \lambda_w$ then there is only one fixed point, namely $\mathbf{z} = \mathbf{1}$; if $\lambda > \lambda_s$ then there are just two extinction probability vectors, $\mathbf{q}(\cdot, \mathbb{T}_3)$ and $\mathbf{1}$, indeed in this case $\mathbf{q}(\cdot, A) = \mathbf{q}(\cdot, \mathbb{T}_m)$ for all $A \neq \varnothing$ (see Bertacchi and Zucca (2014, Corollary 3.2) and Bertacchi and Zucca (2017, Example 4.5)). The last case $\lambda \in (\lambda_w, \lambda_s]$ is the most interesting one: $\mathbf{q}(\cdot, \mathbb{T}_m) < \mathbf{1}$ while $\mathbf{q}(\cdot, A) = \mathbf{1}$ for every finite $A \subset \mathbb{T}_3$. We prove now that $x \mapsto \mathbf{q}(\cdot, T_x)$ is injective, thus F_G is at least countable.

Henceforth, for simplicity we consider just the case $m = 3$ although an analogous construction can be done on $\mathbb{T}_m$ for all $m \geq 3$; indeed, a more general example is sketched in Section 3.2 (see Theorem 3.6). Let d be the natural distance on the graph $\mathbb{T}_3$ and consider a sequence $\{y_n\}_{n \in \mathbb{Z}}$ of distinct vertices such that y_n is a neighbor of y_{n+1} (for all $n \in \mathbb{Z}$) and $d(o, y_n) = |n|$ (clearly $y_0 = o$). We denote by x_i the third neighbor of y_{i-1} (outside y_{i-2} and y_i). A graphical representation is depicted in Figure 1. We note that $d(o, x_n) = n$ for all $n \in \mathbb{N}$ and $T_{x_n} \cap T_{x_m} = \varnothing$ whenever $n \neq m$.

Lemma 3.1. *Let $\lambda \in (\lambda_w, \lambda_s]$. For every $n \geq 1$ and $x \in \mathbb{T}_3$.*

$$\mathbf{q}(x, \mathbb{T}_3) = \mathbf{q}(x, T_{y_0}) < \mathbf{q}(x, T_{y_n}) < \mathbf{q}(x, T_{y_{n+1}}) < 1.$$

Proof. Denote by A the subtree T_{y_n}, by A_2 the subtree $T_{y_{n+1}}$ and by A_1 the set $A \setminus (A_2 \cup \{y_n\})$. Since $\mathbf{q}(\cdot, \mathbb{T}_3) < 1$ and $\mathbf{q}(\cdot, \{y_n\}) = 1$ then, by Theorem 2.4 (if we take $B := \{y_n\}$ then (3) fails), there exists $w \in \mathbb{T}_3$ such that the process starting from w survives with positive probability without ever visiting y_n; by rotational symmetry centered in y_n, w can be chosen in A_1. If a process starts in A_1 and never visits y_n it is then confined to A_1 whence, by Lemma 2.5, $\mathbf{q}(\cdot, T_{y_{n+1}}) > \mathbf{q}(\cdot, T_{y_n})$ and the strict inequality holds for every coordinate. □

Observe that, given $x, z \in \mathbb{T}_3$ such that $z \notin T_x$, then $\mathbf{q}(z, T_x)$ depends only on $d(z, x)$. Indeed, there exists an automorphism Ψ such that $\Psi(z) = o$ and $\Psi(x) = y_{d(z,x)}$. Thus, $\Psi(T_x) = T_{y_{d(z,x)}}$, hence $\mathbf{q}(z, T_x) = \mathbf{q}(\Psi(z), \Psi(T_x)) = \mathbf{q}(o, T_{y_{d(z,x)}})$. In particular, if $x, w \in \mathbb{T}_3$ are such that $d(o, x) < d(o, w)$ then $\mathbf{q}(o, T_x) = \mathbf{q}(o, T_{y_{d(o,x)}}) < \mathbf{q}(o, T_{y_{d(o,w)}}) = \mathbf{q}(o, T_w)$ (the case $d(o, x) > d(o, w)$ is analogous). If $d(o, x) = d(o, w)$ then $d(x, w) > 0$ and it is easy to show that $\mathbf{q}(x, T_x) < \mathbf{q}(o, T_{y_{d(x,w)}}) = \mathbf{q}(x, T_w)$.
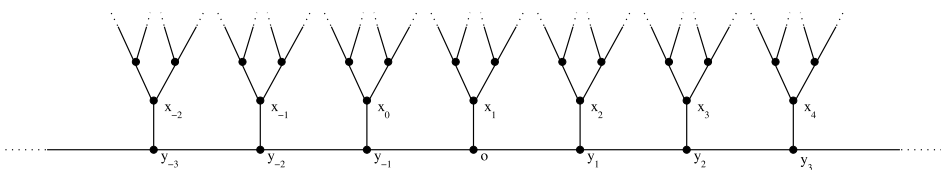

Figure 1 *The tree $\mathbb{T}_3$.*

Now we prove the local extinction on a bi-infinite line.

Lemma 3.2. *Let* $\lambda \in (\lambda_w, \lambda_s]$. *If* γ *is a bi-infinite line in* $\mathbb{T}_3$ *then* $\mathbf{q}(x, \gamma) = 1$.

Proof. It is enough to prove that $\mathbf{q}(x, \gamma) = 1$ when $\gamma := \{y_n\}_{n \in \mathbb{N}}$. Since there is a.s. local extinction then $\mathbf{q}(\cdot, \gamma) \geq \lim_{n \to \infty} \mathbf{q}(\cdot, T_{y_n})$. Now, by using a suitable automorphism, then $\mathbf{q}(o, T_{y_n}) = \mathbf{q}(y_{n-1}, T_{x_1})$ for all $n > 0$. Since $\mathbf{q}(\cdot, T_{x_1}) \neq \mathbf{q}(\cdot, X)$ then a result of Moyal (see Moyal (1962)) and the transitivity of $\mathbb{T}_3$ imply that $\lim_{n \to \infty} \mathbf{q}(y_{n-1}, T_{x_1}) = 1$ whence $\mathbf{q}(o, \gamma) = 1$; the irreducibility yields $\mathbf{q}(x, \gamma) = 1$ for all $x \in \mathbb{T}_3$. $\qquad \square$

There are two interesting consequences of the previous lemma.

1. Any surviving population leaves a.s. every bi-infinite line γ.
2. Since $\mathbf{q}(\cdot, \gamma) = \mathbf{1}$ then the population visits a.s. a finite number of vertices $\{y_n\}_{n \in \mathbb{N}}$, hence survival occurs in a finite number of subtrees $\{T_{x_n}\}_{n \in \mathbb{Z}}$ (this argument can be repeated inside each subtree and so on). Thus, for all $I \subseteq \mathbb{N} \setminus \{0\}$ we have $\mathbf{q}(\cdot, \bigcup_{i \in I : i \leq n} T_{x_i}) \downarrow \mathbf{q}(\cdot, \bigcup_{i \in I} T_{x_i})$ as $n \to +\infty$.

By Lemma 3.1 we have at least a countable collection of distinct extinction probability vectors. The following theorem proves the existence of an uncountable collection.

Theorem 3.3. *Let* $\lambda \in (\lambda_w, \lambda_s]$. *If* $I_1, I_2 \subseteq \mathbb{N} \setminus \{0\}$ *such that* $\sum_{n \in I_1} 2^{-i} \neq \sum_{n \in I_2} 2^{-i}$ *then* $\mathbf{q}(\cdot, \bigcup_{i \in I_1} T_{x_i}) \neq \mathbf{q}(\cdot, \bigcup_{i \in I_2} T_{x_i})$.

Proof. We observe that $\mathbf{q}(o, T_{x_n}) = \mathbf{q}(o, T_{y_n})$ for all $n \geq 1$. We start by proving that for all finite $I \subseteq \mathbb{N} \setminus \{0\}$, if $i_0 := i_0(I) = \max I$,

$$\mathbf{q}\left(o, \bigcup_{i \in I} T_{x_i}\right) = \mathbf{q}\left(o, \bigcup_{i \in \bar{I}} T_{x_i}\right), \tag{3.1}$$

where $\bar{I} := \{i \in I : i < i_0\} \cup \{i > i_0\}$. Indeed, by a simple automorphism argument (choose an automorphism Ψ such that $\Psi(o) = o$ and $\Psi(x_{i_0}) = \Psi(y_{i_0})$), $\mathbf{q}(o, \bigcup_{i \in I} T_{x_i}) = \mathbf{q}(o, \bigcup_{i \in I : i < i_0} T_{x_i} \cup T_{y_{i_0}})$. Since there is a.s. extinction in every infinite line, then survival in $T_{y_{i_0}}$ is equivalent to survival in $\bigcup_{i > i_0} T_{x_i}$ and this yields equation (3.1).

From Lemma 2.5 as in Lemma 3.1, we have that for all $I, J \subseteq \mathbb{N} \setminus \{0\}$

$$I \subsetneq J \implies \mathbf{q}\left(o, \bigcup_{i \in I} T_{x_i}\right) > \mathbf{q}\left(o, \bigcup_{i \in J} T_{x_i}\right). \tag{3.2}$$

Since $I_1 \neq I_2$ we can define $j_0 := j_0(I_1, I_2) = \min I_1 \triangle I_2$; suppose, without loss of generality, that $j_0 \in I_1 \setminus I_2$. Define $I_3 := \{i \in I_1 : i \leq j_0\}$ and $I_4 := \{i \in I_3 : i < j_0\} \cup \{i > j_0\}$. Note that by equation (3.1),

$$\mathbf{q}\left(o, \bigcup_{i \in I_3} T_{x_i}\right) = \mathbf{q}\left(o, \bigcup_{i \in I_4} T_{x_i}\right). \tag{3.3}$$

Moreover, $I_3 \subseteq I_1$ and $I_2 \subseteq I_4$. Since $\sum_{n \in I_1} 2^{-i} \geq \sum_{n \in I_3} 2^{-i} = \sum_{n \in I_4} 2^{-i} \geq \sum_{n \in I_2} 2^{-i}$ but $\sum_{n \in I_1} 2^{-i} > \sum_{n \in I_2} 2^{-i}$ (remember that $j_0 \in I_1 \setminus I_2$) then we have just two possible cases.

- $I_2 \subsetneq I_4$, $I_3 \subseteq I_1$. In this case, by equation (3.2),

$$\mathbf{q}\left(o, \bigcup_{i \in I_1} T_{x_i}\right) \leq \mathbf{q}\left(o, \bigcup_{i \in I_3} T_{x_i}\right) = \mathbf{q}\left(o, \bigcup_{i \in I_4} T_{x_i}\right) < \mathbf{q}\left(o, \bigcup_{i \in I_2} T_{x_i}\right). \tag{3.4}$$

- $I_2 = I_4$, $I_3 \subsetneq I_1$.In this case, again by equation (3.2),

$$\mathbf{q}\left(o, \bigcup_{i \in I_1} T_{x_i}\right) < \mathbf{q}\left(o, \bigcup_{i \in I_3} T_{x_i}\right) = \mathbf{q}\left(o, \bigcup_{i \in I_4} T_{x_i}\right) = \mathbf{q}\left(o, \bigcup_{i \in I_2} T_{x_i}\right). \qquad (3.5)$$
$\square$

Note that the previous theorem contradicts what had been written in Bertacchi and Zucca (2017, p. 244), namely it is not true that on quasi-transitive irreducible BRWs $\{\bar{\mathbf{q}}, \mathbf{1}\} = E_G$ (unless $\mathbf{q}(x, x) = \bar{\mathbf{q}}(x)$ for some $x \in X$). Indeed, Bertacchi and Zucca (2014, Corollary 3.2) implies that either $q(\cdot, A) = \mathbf{1}$ for all finite A or $q(\cdot, A) = \bar{\mathbf{q}}(\cdot)$ for all nonempty subsets A (in particular if there is local survival at x, then there is strong local survival at each $y \in X$). This implies that when $\bar{\mathbf{q}} < \mathbf{1} = \mathbf{q}(\cdot, A)$ for all finite subsets A then in general nothing can be said about $\mathbf{q}(\cdot, A)$ when A is infinite: in Bertacchi and Zucca (2017, Example 3.6) $\mathbf{q}(\cdot, A) = \bar{\mathbf{q}}(\cdot)$ for every infinite A, in Braunsteins and Hautphenne (2018, Examples 1 and 2) there are BRWs with a finite number of extinction probability vectors corresponding to different choices of the infinite set A, while in the above BRW on the tree there are uncountably many different extinction probability vectors.

An uncountable set of of extinction probability vectors can also be found in BRWs where there is local survival as the following example shows.

Example 3.4. Consider the BRW on the tree obtained by adding a loop at o. If the loop has a sufficiently large reproduction rate, the BRW has local survival at every vertex (see Bertacchi and Zucca (2017, Example 4.5)). It also has an uncountable number of extinction probability vectors. Indeed, in order to apply Lemma 2.5 to obtain Lemma 3.1, we just need to prove that there is a positive probability of surviving in $T_{y_{n+1}}$ without ever visiting y_n: this is equivalent to surviving in $B := T_{y_{n+1}} \cup \{o\}$ without ever visiting $A := \{y_n, o\}$ and this follows from Theorem 2.4. Moreover, in case Lemma 3.2 does not hold, the equality in equation (3.1) becomes an inequality which implies $\mathbf{q}(o, \bigcup_{i \in I_3} T_{x_i}) \leq \mathbf{q}(o, \bigcup_{i \in I_4} T_{x_i})$ instead of the equality in equation (3.3), but this does not change the conclusions in equations (3.4) and (3.5).

Theorem 3.3 implies that the relation $\{(\sum_{i \in I} 2^{-i}, \mathbf{q}(o, \bigcup_{i \in I} T_{x_i}))\}_{I \subseteq \mathbb{N} \setminus \{0\}}$ is a well-defined, strictly decreasing map. One may conjecture that $q(x, A)$ only depends on "how large A is at infinity". To be more precise, consider the simple random walk and the branching random walk (with rates 1 on each edge) on the regular tree $\mathbb{T}_m$. Denote by $(\mathcal{M}, \nu_x)$ the measure space where $\mathcal{M}$ is the Martin boundary and ν_x is the harmonic measure related to the random walk starting from $x \in \mathbb{T}_m$. For any $A \subseteq \mathbb{T}_m$ there is a well-defined (possibly empty) boundary $\partial A \in \mathcal{M}$ (see Woess (2000) for details on the Martin boundary of a random walk and the associated measure). Is the relation $\{(\nu_x(\partial A), \mathbf{q}(x, A))\}_{A \subseteq \mathbf{T}_m}$ a well-defined map?

The answer is no, by the following argument. It is enough to prove that $q(o, A) \neq q(o, B)$ for A and B such that $\nu_o(\partial A) = \nu_o(\partial B)$. Let $A = T_{x_1}$ and let s and v be the two neighbors of x_1 which are in A. Since there is local extinction, $q(o, A) = q(o, A_1 \cup A_2)$ where $A_1 = T_s$ and $A_2 = T_v$. Let $B = A_2 \cup T_{x_2}$. We focus on survival in A and in B: it suffices to prove that the probability of survival in A is different from the probability of survival in B. If the process survives in A_2 then there is survival both in A and in B. We prove that the probability of the event $\mathcal{A}^* = $ surviving in A_1 but not in A_2 is different from the probability of the event $\mathcal{B}^* = $ surviving in T_{x_2} and not in A_2. Let C_n^z be the event that the original particle at o has exactly n descendants at z, whose reproduction trail hits z for the first time (roughly speaking, the reproduction trail is the path which traces the lineage, see Pemantle and Stacey (2001) for a formal definition). By symmetry, $\mathbb{P}(C_n^{x_1}) = \mathbb{P}(C_n^{y_1})$, moreover $\mathcal{A}^* \subseteq \bigcup_{n \geq 1} C_n^{x_1}$ and $\mathcal{B}^* \subseteq \bigcup_{n \geq 1} C_n^{y_1}$. Again by symmetry, $\mathcal{A}^* \cap C_n^{x_1}$ is the event where none of the n children has an infinite progenies in A_2, while at least one of them has an infinite progenies in A_1 and

its probability is equal to the probability that, starting with n particles at o, there is extinction in T_{y_1} and survival in A. Similarly, $\mathcal{B}^* \cap C_n^{y_1}$ is the event where none of the n children has an infinite progenies in A_2, while at least one of them has an infinite progenies in T_{x_2} and its probability is equal to the probability that, starting with n particles at o, there is extinction in T_{y_3} and survival in A. By Lemma 3.1, $q(o, T_{y_1}) < q(o, T_{y_3})$ and the same inequality holds for the process starting with n particles. Since the event "extinction in T_{y_1}" is a subset of "extinction in T_{y_3}", it is enough to note that the event "extinction in T_{y_3} with survival both in A and in T_{y_1}" has a positive probability.

3.2 A BRW with an uncountable set of extinction probability vectors: The comb

In this section, we sketch the proof of a generalization of Theorem 3.3. To this aim, consider the BRW in Figure 2 on the 2-dimensional comb C_2, that is, the graph on $\{(x, y) \in \mathbb{Z}^2 : y \geq 0\}$ where (x, y) and (x_1, y_1) are neighbors if and only if either "$x = x_1$ and $|y - y_1| = 1$" or "$y = y_1 = 0$ and $|x - x_1| = 1$". Let $\alpha \geq 1$ and consider the rates k_{xy} as in Figure 2, that is 1 on the horizontal neighbors, 1 downward and $\alpha + 1$ upward except when leaving the horizontal axis where the rate is α. We denote by V_i the vertical line from y_i: when $i \geq 1$ this is $T_{y_i} \setminus T_{y_{i+1}}$.

The following definition of *projection of a BRW* first appeared in Bertacchi and Zucca (2008) for multigraphs, in Bertacchi and Zucca (2009) for continuous-time BRWs and Zucca (2011) for generic discrete-time BRWs (in these papers it was called *local isomorphism*). We just need it in the case of a continuous-time process.

Definition 3.5. A projection of a BRW (X, K) onto $(Y, \widetilde{K})$ is a surjective map $g : X \to Y$, such that $\sum_{z \in g^{-1}(y)} k_{xz} = \widetilde{k}_{g(x)y}$ for all $x \in X$ and $y \in Y$.

If $\{\eta_t\}_{t \geq 0}$ is a realization of the BRW (X, K), then $\{\sum_{z \in g^{-1}(\cdot)} \eta_t(z)\}_{t \geq 0}$ is a realization of the BRW $(Y, \widetilde{K})$. In particular, it is easy to prove that $\mathbf{q}(x, g^{-1}(A)) = \widetilde{\mathbf{q}}(g(x), A)$ for all $x \in X$ and $A \subseteq Y$.

To give an explicit example, consider the BRW on the comb: this can be projected on a continuous-time branching process, that is a BRW on a singleton with rate $\alpha + 2$. This implies that $\lambda_w = 1/(\alpha + 2)$ while, by applying Bertacchi and Zucca (2012, Proposition 4.33), $\lambda_s = 1/(s\sqrt{\alpha + 1})$. We note that the edge-breeding BRW on $\mathbb{T}_m$ can be projected on the BRW on the comb with $\alpha = m - 2$ (one can easily understand it by comparing Figures 1 and 2). One last example, which will be useful in the main result of the section is the BRW on the set B in Figure 3: this BRW can be projected on the BRW on V_i by a map $g : B \to V_i$ where $d(y_i, g(x)) = d(y_i', x)$.

The following theorem can be considered a generalization of Theorem 3.3 in the sense that every BRW which can be projected on the comb, including the BRW on $\mathbb{T}_m$, satisfies the same property below (by using $g^{-1}(V_i)$ instead of V_i, where g is the projection).
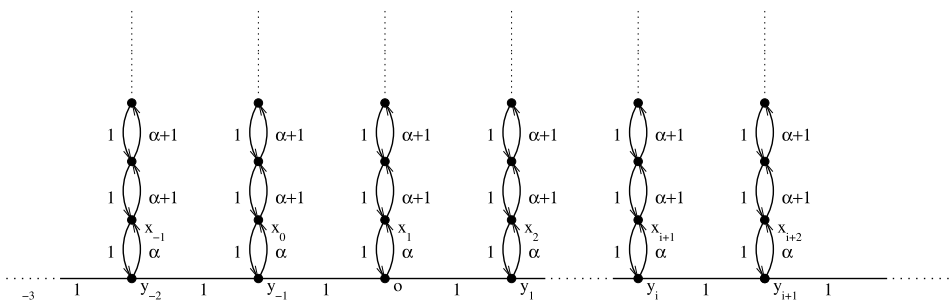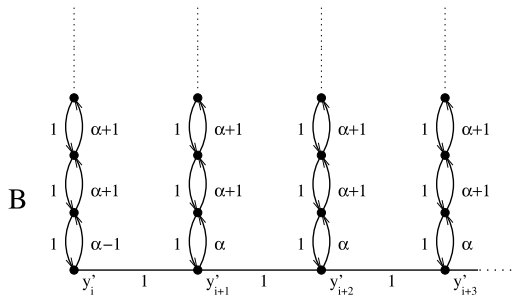

Figure 2 *The BRW on the comb.*

Figure 3 *The BRW on B.*

Theorem 3.6. *Let* $\lambda \in (\lambda_w, \lambda_s]$. *If* $I_1, I_2 \subseteq \mathbb{N} \setminus \{0\}$ *such that* $\sum_{n \in I_1} 2^{-i} \neq \sum_{n \in I_2} 2^{-i}$ *then* $\mathbf{q}(\cdot, \bigcup_{i \in I_1} V_i) \neq \mathbf{q}(\cdot, \bigcup_{i \in I_2} V_i)$.

Proof. We quickly sketch the main steps of the proof. We start by proving the analogous of Lemma 3.1 with V_i instead of T_{y_i}. To this aim consider the BRW obtained from the BRW on C_2 by replacing V_i with B (we identify y_i with y_i'): we call this BRW C_2'. Clearly the BRW on C_2' can be projected on the BRW on C_2 by simply extending the function g (defined above on B) with the identity map on $C_2' \setminus B \equiv C_2 \setminus V_i$. Clearly $\mathbf{q}'(o, B) = \mathbf{q}'(o, g^{-1}(V_i)) = \mathbf{q}(o, V_i)$. But in C_2' there are both V_{i+1} and a copy $V_{i+1}' = T_{y_{i+2}'} \setminus T_{y_{i+1}'} \subseteq B$ (the vertical line from y_{i+1}'). By using Lemma 2.5, we have

$$\mathbf{q}(o, V_{i+1}) = \widetilde{\mathbf{q}}(o, V_{i+1}) = \widetilde{\mathbf{q}}(o, V_{i+1}') > \mathbf{q}'(o, B) = \mathbf{q}(o, V_i)$$

and from this $\mathbf{q}(x, V_{i+1}) > \mathbf{q}(x, V_i)$ for all $x \in C_2$.

The last step, as in Example 3.4, is to prove

$$\mathbf{q}\left(o, \bigcup_{j \in I} V_j\right) \leq \mathbf{q}\left(o, \bigcup_{j \in \bar{I}} V_j\right),$$

where $I \subseteq \mathbb{N} \setminus \{0\}$ is finite and $\bar{I} := \{j \in I : \ < i\} \cup \{j > i\}$ ($i = \max I$). By using the projection on the BRW on C_2' we have that

$$\mathbf{q}\left(o, \bigcup_{j \in \bar{I}} V_j\right) = \widetilde{\mathbf{q}}\left(o, \bigcup_{j \in I:\, j < i} V_j \cup \bigcup_{j > i} V_j'\right)$$

$$\geq \widetilde{\mathbf{q}}\left(o, \bigcup_{j \in I:\, j < i} V_j \cup B\right) = \mathbf{q}\left(o, \bigcup_{j \in I} V_i\right).$$

Now we can use it, as in Example 3.4, instead of equation (3.1) to prove an analogous inequality instead of the equality (3.3). The claim then follows easily. $\square$

4 State of the art and open questions

Let us summarize the (main) known relations between $\{\bar{\mathbf{q}}, \mathbf{1}\}$, E_G and F_G in the irreducible case. In between, we list some interesting questions that, up to our knowledge, are still open. Henceforth, we denote the cardinality of a set by $|\cdot|$.

- As we already noted, $F_G \supseteq E_G \supseteq \{\bar{\mathbf{q}}, \mathbf{1}\}$; moreover $\bar{\mathbf{q}} = \mathbf{1} \iff |F_G| = 1 \iff |E_G| = 1$. This is equivalent to global extinction.
- X finite $\implies F_G = E_G = \{\bar{\mathbf{q}}, \mathbf{1}\}$ (see, for instance, Spataru (1989) or Bertacchi and Zucca (2014, Corollary 3.1)).

- X infinite, (X, μ) quasi-transitive and $\mathbf{q}(\cdot, x) < \mathbf{1}$ for some $x \in X \Longrightarrow E_G = \{\bar{\mathbf{q}}, \mathbf{1}\}$ (here local survival implies strong local survival). Indeed, in this case, $\mathbf{q}(\cdot, A) = \bar{\mathbf{q}}$ for all $A \neq \varnothing$. Whether the cardinality $|F_G \setminus E_G|$ can be positive (finite, countable or uncountable) is an open question.

- X infinite, (X, μ) quasi-transitive and $\mathbf{q}(\cdot, x) = \mathbf{1}$ for all $x \in X$: our example in Section 3.1 shows that E_G can be uncountable. The cardinality $|F_G \setminus E_G|$ is unknown. We conjecture that it can be uncountable, at least when E_G is finite; indeed, we believe that Bertacchi and Zucca (2017, Example 3.6) can be extended as explained in Bertacchi and Zucca (2017, Remark 3.7).

- X infinite and $\mathbf{q}(\cdot, x) < \mathbf{1}$: Example 3.4 shows that E_G can be uncountable. The cardinality $|F_G \setminus E_G|$ is unknown.

- X infinite, projected on a branching process: Bertacchi and Zucca (2017, Example 3.6) shows that $F_G \setminus E_G$ can be uncountable ($E_G = \{\bar{\mathbf{q}}, \mathbf{1}\}$ in this case).

- In Braunsteins and Hautphenne (2017, 2018), there are examples of BRWs where either $|E_G| = |F_G| = 2$ or E_G is finite (larger than 2) and F_G is uncountable.

Other interesting open questions on $|E_G|$ and F_G are the following.

- Is it possible that $|E_G| < |F_G| < +\infty$?
- Is it possible that E_G and $F_G \setminus E_G$ are both infinite?

In particular we conjecture that E_G (resp. F_G) is either finite or uncountable.

Acknowledgments

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