Research Article

Amos Uderzo*

On tangential approximations of the solution set of set-valued inclusions

https://doi.org/10.1515/jaa-2021-2049
Received September 23, 2019; revised August 19, 2020; accepted September 2, 2020

Abstract: In the present paper, the problem of estimating the contingent cone to the solution set associated with certain set-valued inclusions is addressed by variational analysis methods and tools. As a main result, inner (resp. outer) approximations, which are expressed in terms of outer (resp. inner) prederivatives of the set-valued term appearing in the inclusion problem, are provided. For the analysis of inner approximations, the evidence arises that the metric increase property for set-valued mappings turns out to play a crucial role. Some of the results obtained in this context are then exploited for formulating necessary optimality conditions for constrained problems, whose feasible region is defined by a set-valued inclusion.

Keywords: Tangential approximation, decrease principle, prederivative, fan, contingent cone, optimality condition

MSC 2010: 49J52, 49J53, 90C30

Dedicated to Diethard Pallaschke (1940–2020)

1 Introduction and problem statement

The present work aims at providing elements for a first-order analysis of the solutions to set-valued inclusions. By set-valued inclusion the following problem is meant: given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a nonempty closed set $S \subseteq \mathbb{R}^n$, and a closed, convex and pointed cone $C \subseteq \mathbb{R}^m$, with $C \neq \{\mathbf{0}\}$,

$$\text{find } x \in S \text{ such that } F(x) \subseteq C. \tag{SVI}$$

The solution set to (SVI) will be denoted throughout the paper by

$$\mathrm{Sol(SVI)} = \{x \in S : F(x) \subseteq C\}.$$

Some motivations for considering such problems as (SVI), mainly coming from the robust approach to uncertain optimization as well as from mathematical economics, are discussed in [25, 26]. It is clear that Sol(SVI) can be nominally expressed in terms of upper inverse image of C through F, i.e. $F^{+1}(C) = \{x \in \mathbb{R}^n : F(x) \subseteq C\}$, resulting in $\mathrm{Sol(SVI)} = F^{+1}(C) \cap S$. In spite of such a simple reformulation, Sol(SVI) may happen to be a rather involved set, as reflecting the complicated nature that set-valued mappings may often exhibit. Therefore, in order to glean information on the local geometry of Sol(SVI), it becomes crucial to undertake a systematic study of first-order approximations of it. In the present paper, such a task will be pursued by focusing on the contingent (a.k.a. Bouligand–Severi tangent) cone to Sol(SVI). Among many local conic approximations of sets currently at disposal in set-valued analysis, this is one of the mostly employed and widely investigated. It plays an essential role in constructing derivatives for set-valued mappings through a graphical approach

***Corresponding author: Amos Uderzo,** Department of Mathematics and Applications, Università di Milano-Bicocca, Via R. Cozzi 53, 20125 Milano, Italy, e-mail: amos.uderzo@unimib.it. https://orcid.org/0000-0001-8120-159X

(see, for instance, [2, 23]) and it emerges as a basic tool in formulating first-order optimality conditions (see, among others, [11, 18, 23]).

The line of thought behind the analysis here proposed is that a workable representation of the contingent cone to $F^{+1}(C) \cap S$ could be obtained by the upper inverse image of C through a first-order approximation of F. In other words, this amounts to considering the interchange of two operations of different nature: namely, on one hand the approximation of sets and mappings and, on the other hand, the operation of taking the upper inverse. It is worth noting that the approach stemming from this line of thought shares the spirit of the celebrated Lyusternik theorem about the representation of the tangent space to a smooth manifold, which is expressed by an equation system (see [16]). In its modern formulation, suitable for problems of the form

$$\text{find } x \in S \text{ such that } f(x) \in C, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a given single-valued mapping, this theorem states that, under proper regularity assumptions valid at a solution $\bar{x} \in f^{-1}(C) \cap S$, the representation

$$\mathrm{T}(f^{-1}(C) \cap S; \bar{x}) = \mathrm{D} f(\bar{x})^{-1}(\mathrm{T}(C; f(\bar{x}))) \cap \mathrm{T}(S; \bar{x}) \tag{1.2}$$

holds, where $\mathrm{D} f(\bar{x})$ stands for the strict derivative of f at $\bar{x}$ (see [23, Theorem 11.4.4]). In a similar manner, the present investigations explore the possibility of exploiting the upper inverse image of first-order approximations of F for providing representations of $\mathrm{T}(F^{+1}(C) \cap S; \bar{x})$. If $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ acts as a first-order approximation of F near $\bar{x}$, one expects that geometric properties of H result in an easy geometry of the set $H^{+1}(C)$. For example, it is well known that if H is positively homogeneous, then $H^{+1}(C)$ is a cone; if H is concave (in the sense of [26, Definition 2.3]), then $H^{+1}(C)$ is a convex set. Such correspondences evidently contribute to a better understanding of the local structure of Sol(SVI).

In developing the above proposed approach of analysis, passing from such problems as (1.1) to (SVI), a methodological question to face is which approximation tool is to be used for the term F. Since the fact that $\bar{x} \in S$ is a solution to (SVI) involves all elements in $F(\bar{x})$, such an approximation tool should not be based on the local behavior of F around a reference element of its graph (as it happens with graphical derivatives and coderivatives [2, 12, 17]), but should take into account the whole set $F(\bar{x})$. For this reason, the present approach utilizes the notion of prederivative (see [13, 14, 20]). The splitting of this notion into an outer and an inner version allows one to study separately the question of inner and outer tangential approximation of Sol(SVI). Prederivatives are not the only set-oriented derivative-like notion for set-valued mappings, that is able to take into account the whole image of F at a reference point. A different construction, which relies on the Rådström embedding theorem, can be found in [5]. An intrinsic limitation of the notion of π-differentiability there proposed consists in referring to mappings with convex and bounded values. Since in the present analysis the set-valued mapping F considered in (SVI) will not be required to satisfy that assumption, a line of research exploiting π-differentiability is left open for future investigations, which will focus on more particular classes of (SVI).

To the best of the author's knowledge, the study of the solution analysis of (SVI) was initiated in [9]. Some advances in this direction, already including representations of the contingent cone to Sol(SVI), have been recently obtained in [26], under assumptions of concavity of F and boundedness of its values. A feature distinguishing the present investigations is the essential employment of the metric C-increase property for set-valued mappings, while avoiding any concavity assumption on F. This property describes a certain behavior of mappings that links the metric structure of the domain with the cone C appearing in (SVI). Roughly speaking, it can be viewed as a counterpart, valid in partially ordered vector spaces, of the so-called decrease principle for scalar functions, in use in variational analysis (see [7, Chapter 3.6] and [21, Chapter 1.6]). It is well known that for traditional equation systems and, to a certain extent, for generalized equations of the form (1.1), open covering (and hence, metric regularity) is the main property for mappings ensuring local solvability and, as such, it became the key concept to achieve tangential approximations of solution sets. In a similar manner, the metric C-increase property turns out to be a key concept in order to establish a proper error bound for (SVI) and, through such kind of estimate, to get the inner tangential approximation of Sol(SVI).

The contents of the paper are arranged in the subsequent sections as follows. In Section 2, the major analysis tools needed to develop the approach analysis summarized above are presented with references.

Essentially, all of them are well-known notions and facts from set-valued analysis and generalized differentiation, with the only exception of the metric C-increase property, to which a specific subsection is devoted. In Section 3, the main contributions of the paper, which concern inner and outer tangential approximations of Sol(SVI), are established and discussed. In Section 4, optimization problems, whose feasible region is defined by set-valued inclusions (SVI), are considered and some of the results achieved in Section 3 are exploited for deriving necessary optimality conditions suitable for problems of that form. Such an application may serve as an evidence of the fact that "the calculus of tangents is one of the main techniques of optimization" (as stated in [15]).

2 Analysis tools

The notation in use throughout the paper is standard: $\mathbb{N}$ and $\mathbb{R}$ denote the natural and the real number set, respectively, $\mathbb{R}^m_+$ denotes the nonnegative orthant in the Euclidean space $\mathbb{R}^m$, whose (Euclidean) norm is indicated by $|\cdot|$. The null vector in any Euclidean space is indicated by $\mathbf{0}$. Given an element x of a metric space (X, d) and a nonnegative real r, $\mathrm{B}(x, r) = \{z \in X : d(z, x) \le r\}$ denotes the closed ball with center x and radius r. In particular, if $X = \mathbb{R}^n$, then $\mathbb{B} = \mathrm{B}(\mathbf{0}, 1)$ and $\mathbb{S} = \{v \in \mathbb{B} : |v| = 1\}$ stand for the unit ball and the unit sphere, respectively. Given a subset S of an Euclidean space, by $\operatorname{int} S$ and $\operatorname{bd} S$ the topological interior and the boundary of S are denoted, respectively, whereas $\operatorname{co} S$ denotes the convex hull of S. By $\operatorname{dist}(x, S) = \inf_{z \in S} d(z, x)$ the distance of x from a subset $S \subseteq \mathbb{R}^n$ is denoted, with the convention that $\operatorname{dist}(x, \varnothing) = +\infty$. The r-enlargement of a set $S \subseteq \mathbb{R}^n$ is indicated by $\mathrm{B}(S, r) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, S) \le r\}$. Given a pair of subsets $S_1, S_2 \subseteq X$, the symbol $\operatorname{exc}(S_1, S_2) = \sup_{s \in S_1} \operatorname{dist}(s, S_2)$ denotes the excess of S_1 over S_2, where the convention $\sup_{x \in \varnothing} = -\infty$ is accepted. The symbol $\operatorname{Haus}(S_1, S_2) = \max\{\operatorname{exc}(S_1, S_2), \operatorname{exc}(S_2, S_1)\}$ indicates the Pompeiu–Hausdorff distance between S_1 and S_2.

Whenever $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued mapping,

$$\operatorname{gph} F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\} \quad \text{and} \quad \operatorname{dom} F = \{x \in \mathbb{R}^n : F(x) \neq \varnothing\}$$

denote the graph and the domain of F, respectively. All set-valued mappings appearing in the paper will be supposed to take closed values, unless otherwise stated. This fact will be implicitly assumed, in particular, with reference to mappings resulting from the sum of set-valued mappings. Moreover, $\mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ indicates the space of all linear mappings acting from $\mathbb{R}^n$ to $\mathbb{R}^m$, endowed with the operator norm $\|\cdot\|$. If $\Lambda \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$, then $\Lambda^\top$ denotes the adjoint operator to Λ. Given a function $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$, where X is a given set, $[\varphi \le 0] = \varphi^{-1}([-\infty, 0])$ and $[\varphi > 0] = \varphi^{-1}((0, +\infty])$ stand for the 0-sublevel and the strict 0-superlevel set of φ, respectively. Other notations will be explained contextually to their use.

Throughout the text, the acronyms l.s.c. and p.h. stand for lower semicontinuous and positively homogeneous, respectively.

2.1 Elements of set-valued and variational analysis

Let us recall that a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be l.s.c. at $\bar{x} \in \mathbb{R}^n$ if for every open set $O \subseteq \mathbb{R}^m$ such that $F(\bar{x}) \cap O \neq \varnothing$ there exists $\delta_O > 0$ such that

$$F(x) \cap O \neq \varnothing \quad \text{for all } x \in \mathrm{B}(\bar{x}, \delta_O).$$

The mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be Lipschitz (continuous) with constant $\kappa \ge 0$ if

$$\operatorname{Haus}(F(x_1), F(x_2)) \le \kappa |x_1 - x_2| \quad \text{for all } x_1, x_2 \in \mathbb{R}^n.$$

A known fact which is relevant to the present analysis is that if a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is l.s.c. at each point of $\mathbb{R}^n$, then $F^{+1}(C)$ is a closed set for every closed set C (see, for instance, [1, Lemma 17.5]). This fact makes it clear that, under the assumptions made on the problem data of (SVI) (namely, S closed and C closed, convex and pointed cone), if F is a l.s.c. set-valued mapping, then the solution set $\mathrm{Sol(SVI)} = S \cap F^{+1}(C)$ is a closed subset (possibly empty) of $\mathbb{R}^n$.

In studying the variational behavior of set-valued mappings, a basic tool of analysis is the excess of a set over another. The following remark gathers several known facts concerning the behavior of the excess, which will be employed in the subsequent analysis (for their proof, whenever not trivial, one can refer to [25]).

Remark 2.1. Let $S \subseteq \mathbb{R}^m$ be nonempty and let $C \subseteq \mathbb{R}^m$ be a closed, convex cone.
(i) If $\mathrm{exc}(S, C) > 0$, for any $r > 0$ it holds $\mathrm{exc}(\mathrm{B}(S, r), C) = \mathrm{exc}(S, C) + r$ (additive behavior of the excess with respect to enlargements).
(ii) If $S \subseteq \mathbb{R}^m$, it holds $\mathrm{exc}(S + C, C) = \mathrm{exc}(S, C)$ (invariance of the excess under conic extension).
(iii) Let $r > 0$. It holds $\mathrm{exc}(r\mathbb{B}, C) = \sup_{x\in r\mathbb{B}} \inf_{c\in C} |x - c| \le \sup_{x\in r\mathbb{B}} |x| = r$.

Remark 2.2. (i) Given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, it is known that the aforementioned semicontinuity property of F implies a corresponding semicontinuity property of the excess function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, which is associated with F and C, namely

$$\phi(x) = \mathrm{exc}(F(x), C). \tag{2.1}$$

In other words, if F is l.s.c. at $\bar{x}$, then ϕ is l.s.c. at $\bar{x}$ (the proof can be found in [25, Lemma 2.3]).

(ii) Since C is a closed set, it is clear that $F(x) \subseteq C$ if and only if $\phi(x) \le 0$. Therefore, the solution set to (SVI) can be characterized, via the function ϕ, in the following terms: $\mathrm{Sol(SVI)} = [\phi \le 0] \cap S$.

The main result of this paper will be achieved by means of an error bound estimate for the solution set to (SVI). The technique of proof of the latter one relies on the characterization of error bounds for l.s.c. functions on a complete metric space through the notion of strong slope (see, among others, [3, 4]). Let us recall that, after [10], given a function $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ defined on a metric space (X, d) and $\bar{x} \in \varphi^{-1}(\mathbb{R})$, the strong slope of φ at $\bar{x}$ is defined as the quantity

$$|\nabla\varphi|(\bar{x}) = \begin{cases} 0 & \text{if } \bar{x} \text{ is a local minimizer of } \varphi, \\ \displaystyle\limsup_{x\to\bar{x}} \frac{\varphi(\bar{x}) - \varphi(x)}{d(x, \bar{x})} & \text{otherwise.} \end{cases}$$

Notice that, if as a metric space X one takes a closed subset $S \subseteq \mathbb{R}^m$ containing $\bar{x}$, the above definition becomes

$$|\nabla\varphi|(\bar{x}) = \begin{cases} 0 & \text{if } \bar{x} \text{ is a local minimizer of } \varphi \text{ over } S, \\ \displaystyle\inf_{r>0} \sup_{x\in\mathrm{B}(\bar{x},r)\cap S\setminus\{\bar{x}\}} \frac{\varphi(\bar{x}) - \varphi(x)}{|x - \bar{x}|} & \text{otherwise.} \end{cases}$$

For the purposes of the present work, the following general condition for an error bound, which can be obtained as a special case of [3, Corollary 3.1], will be employed.

Proposition 2.3. *Let (X, d) be a complete metric space, let $\varphi : X \to [0, +\infty]$ be a function l.s.c. on X, and let $\bar{x} \in [\varphi \le 0]$. Suppose that $\sigma > 0$ and $r > 0$ are such that*

$$|\nabla\varphi|(x) \ge \sigma \quad \text{for all } x \in \mathrm{B}(\bar{x}, 2r) \cap [\varphi > 0].$$

Then it holds

$$\mathrm{dist}(x, [\varphi \le 0]) \le \frac{\varphi(x)}{\sigma} \quad \text{for all } x \in \mathrm{B}(\bar{x}, r).$$

2.2 The metric C-increase property

The next definition introduces the main property of set-valued mappings, on which the proposed approach to the solution analysis of (SVI) relies. It postulates a behavior of mappings that links the metric structure of the domain with the partial ordering induced on the range space by the cone C in the standard way (henceforth denoted by $\le_C$), i.e. $y_1 \le_C y_2$ if and only if $y_2 - y_1 \in C$.

Definition 2.4 (Metrically C-increasing mapping). Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set and let $C \subseteq \mathbb{R}^m$ be a closed, convex cone, with $C \neq \{\mathbf{0}\}$. Consider a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$:

(i) F is said to be *metrically C-increasing* around $\bar{x} \in \operatorname{dom} F$ relative to S if there exist $\delta > 0$ and $\alpha > 1$ such that

$$\begin{cases} \text{for all } x \in \mathrm{B}(\bar{x}, \delta) \cap S \text{ and all } r \in (0, \delta) \text{ there exists } z \in \mathrm{B}(x, r) \cap S \\ \text{such that } \mathrm{B}(F(z), \alpha r) \subseteq \mathrm{B}(F(x) + C, r). \end{cases} \tag{2.2}$$

The quantity

$$\operatorname{inc}_C(F; S; \bar{x}) = \sup\{\alpha > 1 : \text{there exists } \delta > 0 \text{ for which the inclusion in (2.2) holds}\}$$

is called *exact bound of metric C-increase* of F around $\bar{x}$, relative to S.

(ii) F is said to be *globally metrically C-increasing* if there exists $\alpha > 1$ such that

$$\text{for all } x \in \mathbb{R}^n \text{ and all } r > 0 \text{ there exists } z \in \mathrm{B}(x, r) \text{ such that } \mathrm{B}(F(z), \alpha r) \subseteq \mathrm{B}(F(x) + C, r). \tag{2.3}$$

The quantity

$$\operatorname{inc}_C(F) = \sup\{\alpha > 1 : \text{the inclusion in (2.3) holds}\}$$

is called *global exact bound of metric C-increase* of F.

As a comment to the above property, let us observe that the behavior that it describes can be regarded as a set-valued version of a phenomenon, which in the case of scalar functions is known as decrease principle of variational analysis. By this term, any condition is denoted which ensures the existence of a constant $\kappa > 0$ such that

$$\inf_{x \in \mathrm{B}(\bar{x}, r)} \varphi(x) \leq \varphi(\bar{x}) - \kappa r,$$

where $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is a l.s.c. and bounded from below function defined on a proper (at least, metric) space, $\bar{x} \in X$ is a reference point and $r > 0$. Often, such a condition finds a formulation in terms of Fréchet subdifferential, provided that X is a Fréchet smooth Banach space (see [7, Theorem 3.6.2]), or, more generally, in terms of strong slope, if X is a complete metric space (see [21]). The decrease principle appeared as a fundamental tool in the analysis of error bounds and solution stability for inequalities and, as such, it plays a key role in establishing implicit multifunction theorems (see [7]). This led the author to employ the term "metric C-increase" in [25].

Remark 2.5. (i) Whenever $\bar{x} \in \operatorname{int} S$, the notion of metric C-increase around $\bar{x}$, relative to S, reduces to the notion of local metric C-increase around $\bar{x}$, as defined in [25].

(ii) An equivalent reformulation of the inclusion (2.2) that will be useful is clearly

$$\begin{aligned} &\text{for all } x \in \mathrm{B}(\bar{x}, \delta) \cap S \text{ and all } r \in (0, \delta) \text{ there exists } z \in \mathrm{B}(x, r) \cap S \\ &\text{such that } F(z) + \alpha r \mathbb{B} \subseteq F(x) + C + r\mathbb{B}. \end{aligned}$$

Example 2.6. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined by

$$F(x) = \{y = (y_1, y_2) \in \mathbb{R}^2 : \min\{y_1, y_2\} = x\}$$

and let $C = \mathbb{R}^2_+$. By a direct check of Definition 2.4 (ii), one can see that the set-valued mapping F is globally metrically $\mathbb{R}^2_+$-increasing, with $\operatorname{inc}_C(F) = 2$.

Other examples of classes of metrically C-increasing set-valued mappings, along with verifiable conditions for detecting such property, will be provided in the next subsection. Further examples can be found in [25].

Below, the aforementioned error bound condition, instrumental to the solution analysis of (SVI), is established. Such a condition can be viewed as a refinement of [25, Theorem 4.3]. It is presented here with its full proof, because it turns out that, by using a more adequate technique of proof, one assumption made in the mentioned theorem can be dropped out and the whole argument gains in clearness.

Lemma 2.7 (Local error bound under metric C-increase). *Suppose that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued mapping, $C \subseteq \mathbb{R}^m$ is a closed convex cone, and S is a closed set defining a problem* (SVI), *with $\bar{x} \in \mathrm{Sol(SVI)}$. Suppose the following conditions:*

(i) *F is l.s.c. in $\mathrm{B}(\bar{x}, \delta) \cap S$ for some $\delta > 0$.*

(ii) *F is metrically C-increasing around $\bar{x}$, relatively to S.*

Then, for every $\alpha \in (1, \mathrm{inc}_C(F; S; \bar{x}))$, there exists $\delta_\alpha > 0$ such that

$$\mathrm{dist}(x, \mathrm{Sol(SVI)}) \le \frac{\mathrm{exc}(F(x), C)}{\alpha - 1} \quad \text{for all } x \in \mathrm{B}(\bar{x}, \delta_\alpha) \cap S. \tag{2.4}$$

Proof. Consider the function $\phi : S \to [0, +\infty]$, defined as in (2.1). According to Remark 2.2 (i), ϕ is l.s.c. on $\mathrm{B}(\bar{x}, \delta) \cap S$ by virtue of hypothesis (i). Take $\alpha \in (1, \mathrm{inc}_C(F; S; \bar{x}))$. Without any loss of generality, it is possible to assume that δ is smaller than the value of δ appearing in inclusion (2.2) of Definition 2.4. Now, fix an arbitrary $x \in \mathrm{B}(\bar{x}, \delta) \cap S \cap [\phi > 0]$. Then, according to hypothesis (ii), taken any $r > 0$ such that

$$r < \min\left\{\delta, \frac{\phi(x)}{\alpha - 1}\right\},$$

there exists $z_r \in \mathrm{B}(x, r) \cap S$ such that

$$\mathrm{B}(F(z_r), \alpha r) \subseteq \mathrm{B}(F(x) + C, r). \tag{2.5}$$

Notice that $z_r \neq x$ must hold. This is evident if $\phi(z_r) = 0$ because $x \in [\phi > 0]$. In the case $\phi(z_r) > 0$, if it were $z_r = x$, on account of inclusion (2.5), by recalling Remark 2.1 (i) and (ii), one would obtain

$$\begin{aligned}\phi(x) + \alpha r &= \mathrm{exc}(F(x), C) + \alpha r \\ &= \mathrm{exc}(\mathrm{B}(F(x), \alpha r), C) \\ &\le \mathrm{exc}(\mathrm{B}(F(x) + C, r), C) \\ &= \mathrm{exc}(F(x), C) + r \\ &= \phi(x) + r,\end{aligned}$$

whence $\alpha \le 1$, in contradiction to the assumption on the value of α. Furthermore, again by Remark 2.1 (i) and (ii), and inclusion (2.5), it is possible to observe that

$$\begin{aligned}\phi(z_r) &= \mathrm{exc}(\mathrm{B}(F(z_r), \alpha r), C) - \alpha r \\ &\le \mathrm{exc}(\mathrm{B}(F(x) + C, r), C) - \alpha r \\ &= \mathrm{exc}(F(x) + C, C) + r - \alpha r \\ &= \phi(x) + (1 - \alpha)r.\end{aligned}$$

The last inequality chain implies

$$\phi(x) - \phi(z_r) \ge (\alpha - 1)r \ge (\alpha - 1)\|x - z_r\|,$$

so x can not be a local minimizer of ϕ over S. By consequence, when calculating the strong slope of ϕ at x in the metric space S, one finds

$$\begin{aligned}|\nabla\phi|(x) &= \limsup_{z \xrightarrow{S} x} \frac{\phi(x) - \phi(z)}{\|x - z\|} \\ &= \inf_{r>0} \sup_{z \in \mathrm{B}(x,r) \cap S \setminus \{x\}} \frac{\phi(x) - \phi(z)}{\|x - z\|} \\ &\ge \inf_{r>0} \frac{\phi(x) - \phi(z_r)}{\|x - z_r\|} \\ &\ge \alpha - 1.\end{aligned}$$

This shows that

$$|\nabla\phi|(x) \ge \alpha - 1 \quad \text{for all } x \in \mathrm{B}(\bar{x}, \delta) \cap S \cap [\phi > 0].$$

Since S, as a closed subset of $\mathbb{R}^n$, is a complete metric space, Proposition 2.3 guarantees that

$$\operatorname{dist}(x, \mathrm{Sol(SVI)}) = \operatorname{dist}(x, [\phi \le 0] \cap S) \le \frac{\phi(x)}{\alpha - 1} \quad \text{for all } x \in \mathrm{B}\Big(\bar{x}, \frac{\delta}{2}\Big) \cap S.$$

Thus, it suffices to set $\delta_\alpha = \frac{\delta}{2}$ to achieve the thesis. □

The following example illustrates the essential role played by the metric C-increase property for the validity of the error bound (2.4).

Example 2.8 (Error bound failure). Consider the set-valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) = [-x^2, +\infty),$$

and take $S = \mathbb{R}$, $C = \mathbb{R}_+$ and $\bar{x} = 0$. With these data, the resulting (SVI) evidently admits $\{0\}$ as a solution set. Therefore, one has

$$\operatorname{dist}(x, \mathrm{Sol(SVI)}) = |x| \quad \text{for all } x \in \mathbb{R}.$$

On the other hand, one sees that it is

$$\operatorname{exc}(F(x), \mathbb{R}_+) = x^2 \quad \text{for all } x \in \mathbb{R}.$$

As a consequence, for any $\alpha > 1$, the error bound inequality

$$\operatorname{dist}(x, \mathrm{Sol(SVI)}) = |x| \le \frac{x^2}{\alpha - 1} = \frac{\operatorname{exc}(F(x), \mathbb{R}_+)}{\alpha - 1}$$

fails to hold in any interval $(-\delta_\alpha, \delta_\alpha)$, whatever the value of $\delta_\alpha > 0$ is. Observe that F is l.s.c. in a neighborhood of 0, so hypothesis (i) of Lemma 2.7 is fulfilled. Instead, F is not metrically $\mathbb{R}_+$-increasing around 0, relative to $\mathbb{R}$ (in other terms, locally metrically $\mathbb{R}_+$-increasing around 0).

2.3 Generalized differentiation tools

Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set and let $\bar{x} \in S$. As a first-order approximation of S near $\bar{x}$, the following different cones will be used:

$$\begin{aligned}
\mathrm{T}(S;\bar{x}) &= \{v \in \mathbb{R}^n : \text{there exist } (v_n)_n \text{ with } v_n \to v \text{ and } (t_n)_n \text{ with } t_n \downarrow 0 \text{ such that } \bar{x} + t_n v_n \in S \text{ for all } n \in \mathbb{N}\},\\
\mathrm{I}(S;\bar{x}) &= \{v \in \mathbb{R}^n : \text{there exists } \delta > 0 \text{ such that } \bar{x} + tv \in S \text{ for all } t \in (0,\delta)\},\\
\mathrm{I_w}(S;\bar{x}) &= \{v \in \mathbb{R}^n : \text{for all } \epsilon > 0 \text{ there exists } t_\epsilon \in (0,\epsilon) \text{ such that } \bar{x} + t_\epsilon v \in S\},\\
\mathrm{T_{Cl}}(S;\bar{x}) &= \{v \in \mathbb{R}^n : \text{for all } \epsilon > 0 \text{ there exists } \tau > 0 \text{ such that for all } x \in \mathrm{B}(\bar{x},\tau) \text{ and all } t \in (0,\tau)\\
&\qquad \text{there exists } v' \in \mathrm{B}(v,\epsilon) \text{ such that } x + tv' \in S\}.
\end{aligned}$$

They are called the contingent cone, the feasible direction cone, the weak feasible direction cone and the Clarke tangent cone to S at $\bar{x}$, respectively (see, for instance, [2, 11, 23]). It is to be noted that the above definition of Clarke tangent cone is actually an equivalent reformulation provided in [11, Proposition 2.2] of the original notion. The following relations of inclusion are known to hold in general:

$$\mathrm{I}(S;\bar{x}) \subseteq \mathrm{I_w}(S;\bar{x}) \subseteq \mathrm{T}(S;\bar{x}) \quad \text{and} \quad \mathrm{T_{Cl}}(S;\bar{x}) \subseteq \mathrm{T}(S;\bar{x}).$$

When, in particular, S is locally convex around $\bar{x}$, i.e. there exists $r > 0$ such that $S \cap \mathrm{B}(\bar{x}, r)$ is convex, then

$$\operatorname{cl} \mathrm{I}(S;\bar{x}) = \operatorname{cl} \mathrm{I_w}(S;\bar{x}) = \mathrm{T}(S;\bar{x})$$

(see, for instance, [23, Proposition 11.1.2 (d)]). The Clarke tangent cone is always closed and convex (see [11, Proposition 2.3]). The contingent cone, introduced in [8, 24], will be the main object of study in the present analysis. It follows from its very definition that it is determined only by the geometric shape of a set near the reference point, namely for any $r > 0$ it is

$$\mathrm{T}(S;\bar{x}) = \mathrm{T}(S \cap \mathrm{B}(\bar{x}, r);\bar{x}). \tag{2.6}$$

Of course, whenever S is a closed convex cone, one finds $\mathrm{T}(S;\mathbf{0}) = S$.

Remark 2.9. Given a nonempty $S \subseteq \mathbb{R}^n$ and $\bar{x} \in S$, the following characterization of $\mathrm{T}(S;\bar{x})$ in terms of the Dini lower derivative of the function $x \mapsto \mathrm{dist}(x,S)$ at $\bar{x}$ will be useful:

$$\mathrm{T}(S;\bar{x}) = \Big\{v \in \mathbb{R}^n : \liminf_{t\downarrow 0} \frac{\mathrm{dist}(\bar{x}+tv,S)}{t} = 0\Big\}$$

(see [23, Proposition 11.1.5] and [2], where the above equality actually appears as a definition of the contingent cone to S at $\bar{x}$).

Given a cone $C \subseteq \mathbb{R}^m$, let us recall that the set

$$C^{\ominus} = \{v \in \mathbb{R}^m : \langle v, c\rangle \leq 0 \text{ for all } c \in C\}$$

is called (negative) dual cone of C. Whenever S is locally convex around $\bar{x}$ (and hence $\mathrm{T}(S;\bar{x})$ is convex), such an operator is connected with the normal cone to S at $\bar{x}$ in the sense of convex analysis by the following well-known relation:

$$\mathrm{N}(S;\bar{x}) = \mathrm{T}(S;\bar{x})^{\ominus}.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function which is finite around $\bar{x} \in \mathbb{R}^n$. Following [17], the sets

$$\widehat{\partial}\varphi(\bar{x}) = \Big\{v \in \mathbb{R}^n : \liminf_{x\to\bar{x}} \frac{\varphi(x)-\varphi(\bar{x})-\langle v, x-\bar{x}\rangle}{|x-\bar{x}|} \geq 0\Big\}$$

and

$$\widehat{\partial}^{+}\varphi(\bar{x}) = \Big\{v \in \mathbb{R}^n : \limsup_{x\to\bar{x}} \frac{\varphi(x)-\varphi(\bar{x})-\langle v, x-\bar{x}\rangle}{|x-\bar{x}|} \leq 0\Big\}$$

are called the Fréchet subdifferential of φ at $\bar{x}$ and the Fréchet upper subdifferential of φ at $\bar{x}$, respectively. It is readily seen that, whenever φ is (Fréchet) differentiable at $\bar{x}$, then $\widehat{\partial}\varphi(\bar{x}) = \widehat{\partial}^{+}\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$, whereas whenever $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex (resp. concave), the set $\widehat{\partial}\varphi(\bar{x})$ (resp. $\widehat{\partial}^{+}\varphi(\bar{x})$) reduces to the subdifferential (resp. superdifferential) of φ at $\bar{x}$ in the sense of convex analysis.

Remark 2.10. The following variational description of the Fréchet upper subdifferential of φ at $\bar{x}$ will be exploited in the sequel: for every $v \in \widehat{\partial}^{+}\varphi(\bar{x})$ there exists a function $\sigma : \mathbb{R}^n \to \mathbb{R}$, differentiable at $\bar{x}$ and with $\varphi(\bar{x}) = \sigma(\bar{x})$, such that $\varphi(x) \leq \sigma(x)$ for every $x \in \mathbb{R}^n$ and $\nabla\sigma(\bar{x}) = v$ (see [17, Theorem 1.88]).

While cones are the basic objects for approximating sets, positively homogeneous set-valued mappings are the basic tools for approximating multifunctions. Recall that a set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is positively homogeneous (for short, p.h.) if $\mathbf{0} \in H(\mathbf{0})$ and

$$H(\lambda x) = \lambda H(x) \quad \text{for all } \lambda > 0 \text{ and all } x \in \mathbb{R}^n.$$

Within the class of p.h. set-valued mappings, fans will play a prominent role in the present analysis (see [14]).

Definition 2.11 (Fan). A set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be a *fan* if it fulfils the following conditions:
(i) It is p.h.
(ii) It is convex-valued.
(iii) It holds

$$H(x_1 + x_2) \subseteq H(x_1) + H(x_2) \quad \text{for all } x_1, x_2 \in \mathbb{R}^n.$$

Fans are set-valued mappings with a useful geometric structure, arising in a large variety of contexts. It is clear that the class of all fans acting between $\mathbb{R}^n$ and $\mathbb{R}^m$ includes, as a very special case, the space $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$. An important class of fan, playing a role in the present analysis, is discussed below.

Example 2.12 (Fans generated by linear mappings). Let $\mathcal{G} \subseteq \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ be a nonempty, convex and closed set. The set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$H(x) = \{\Lambda x : \Lambda \in \mathcal{G}\}$$

is known to be a fan (see [14]). In such a circumstance, the set $\mathcal{G}$ will be called a generator for H. In particular, whenever $\mathcal{G}$ is a polytope in $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$, the fan generated by $\mathcal{G}$ will be said to be finitely-generated.

For example, in the case $m = n$, one may take the class of all linear mappings represented by $n \times n$ doubly stochastic matrices. After the Birkhoff–von Neumann theorem, this class is known to be a polytope, resulting from the convex hull of all the permutation matrices, which are its extreme elements (see [6]). Note that any finitely-generated fan takes compact values which are polytopes in the range space $\mathbb{R}^m$. In general, for any fan H generated by linear mappings it must be $H(\mathbf{0}) = \{\mathbf{0}\}$.

Example 2.13. The set-valued mapping $H : \mathbb{R} \rightrightarrows \mathbb{R}$, defined by

$$H(x) = \begin{cases} -\mathbb{R}_+ & \text{if } x < 0, \\ \mathbb{R} & \text{if } x = 0, \\ \mathbb{R}_+ & \text{if } x > 0, \end{cases}$$

is a fan. Since $H(0) = \mathbb{R}$ holds, it is clear that H can not be generated by any set $\mathcal{G} \subseteq \mathcal{L}(\mathbb{R}; \mathbb{R})$.

Further examples of fans are provided in [26].

According to the present approach of analysis, the upper inverse image of C through a given fan will be a key element to express the tangential approximation of Sol(SVI). In this perspective, the next remark gathers some elementary algebraic/topological properties of such a set.

Remark 2.14. (i) It is plain to see that if $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a fan and $C \subseteq \mathbb{R}^m$ is a closed convex cone, then the set $H^{+1}(C)$ is a convex cone (possibly empty). Notice that, in general, $H^{+1}(C)$ may happen to be not closed. For example, if taking $C = \mathbb{R}_+$ and such a fan $H : \mathbb{R} \rightrightarrows \mathbb{R}$ as defined in Example 2.13, one finds $H^{+1}(C) = (0, +\infty)$ (consistently, H fails to be l.s.c. at 0).

(ii) It is worth noting that, in the case of a fan generated by a set $\mathcal{G} \subseteq \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$, it results in

$$H^{+1}(C) = \bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C).$$

As an immediate consequence of the last equality, one deduces that the convex cone $H^{+1}(C)$ is closed whenever H is a fan generated by linear mappings. Furthermore, if a fan H is finitely-generated, i.e.

$$\mathcal{G} = \mathrm{co}\{\Lambda_1, \dots, \Lambda_p\},$$

with $\Lambda_i \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$, for $= 1, \dots, p$, then it results in

$$H^{+1}(C) = \bigcap_{i=1}^{p} \Lambda_i^{-1}(C).$$

In this case, each set $\Lambda_i^{-1}(C)$ turns out to be polyhedral, provided that C is so, and therefore $H^{+1}(C)$ inherits a polyhedral cone structure.

(iii) Whenever $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a fan generated by a bounded set $\mathcal{G} \subseteq \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$, it turns out to be Lipschitz. More precisely, if $l = \sup\{\|\Lambda\| : \Lambda \in \mathcal{G}\} < +\infty$, it holds

$$\mathrm{Haus}(H(x_1), H(x_2)) \le l|x_1 - x_2| \quad \text{for all } x_1, x_2 \in \mathbb{R}^n.$$

Indeed, since for any $y \in \mathbb{R}^m$ it is

$$\mathrm{dist}(y, H(x_2)) = \inf_{\Lambda \in \mathcal{G}} |y - \Lambda x_2|,$$

then, if $y = \widetilde{\Lambda} x_1$ for some $\widetilde{\Lambda} \in \mathcal{G}$, it results in

$$\mathrm{dist}(\widetilde{\Lambda} x_1, H(x_2)) = \inf_{\Lambda \in \mathcal{G}} |\widetilde{\Lambda} x_1 - \Lambda x_2| \le |\widetilde{\Lambda} x_1 - \widetilde{\Lambda} x_2| \le \|\widetilde{\Lambda}\| |x_1 - x_2|.$$

It follows

$$\begin{aligned} \mathrm{exc}(H(x_1), H(x_2)) &= \sup_{y \in H(x_1)} \mathrm{dist}(y, H(x_2)) \\ &= \sup_{\Lambda \in \mathcal{G}} \mathrm{dist}(\Lambda x_1, H(x_2)) \\ &\le \sup_{\Lambda \in \mathcal{G}} \|\Lambda\| |x_1 - x_2| \\ &\le l|x_1 - x_2| \quad \text{for all } x_1, x_2 \in \mathbb{R}^n. \end{aligned}$$

In particular, all finitely-generated fans are Lipschitz continuous and, if $\mathcal{G} = \mathrm{co}\{\Lambda_1, \dots, \Lambda_p\}$, it results in $l = \max_{i=1,\dots,p} \|\Lambda_i\|$.

The aforementioned features motivate the choice of fans as a possible tool for approximating more general and less structured set-valued mappings.

In view of the employment of the metric C-increase property in the present approach, the next proposition provides conditions for a fan to be globally metrically C-increasing. Its proof makes use of a well-known order cancellation law, saying that whenever $A \subseteq \mathbb{R}^m$ is nonempty, $B \subseteq \mathbb{R}^m$ is nonempty convex and bounded, and $C \subseteq \mathbb{R}^m$ is nonempty closed and convex, then the following implication holds (see [19, Theorem 3.2.1]):

$$A + B \subseteq C + B \quad \text{implies} \quad A \subseteq C.$$

Proposition 2.15. *Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a fan. If*

$$\text{there exist } \eta > 0 \text{ and } u \in \mathbb{B} \text{ such that } H(u) + \eta\mathbb{B} \subseteq C, \tag{2.7}$$

then H is globally metrically C-increasing and $\mathrm{inc}_C(H) \geq \eta + 1$. *Conversely, if the fan $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ takes compact values, then condition* (2.7) *is also necessary for H to be globally metrically C-increasing.*

Proof. Take arbitrary $x \in \mathbb{R}^n$ and $r > 0$. Letting $u \in \mathbb{B}$ and $\eta > 0$ as in condition (2.7) and setting $z = x + ru$, one has that $z \in \mathrm{B}(x, r)$ and obtains

$$\begin{aligned} H(z) + (\eta + 1)r\mathbb{B} &\subseteq H(x) + rH(u) + \eta r\mathbb{B} + r\mathbb{B} \\ &= H(x) + r(H(u) + \eta\mathbb{B}) + r\mathbb{B} \\ &\subseteq H(x) + C + r\mathbb{B}. \end{aligned}$$

According to Definition 2.4 (ii) and Remark 2.5, this proves that H is globally metrically C-increasing.

Conversely, observe first of all that if H takes compact values, then it must be $H(\mathbf{0}) = \{\mathbf{0}\}$. Indeed, as H is p.h., one has $\lambda H(\mathbf{0}) = H(\lambda\mathbf{0}) = H(\mathbf{0})$ for any $\lambda > 0$, so $H(\mathbf{0})$ is a cone, but $\{\mathbf{0}\}$ is the only compact cone. Now, if H is globally metrically C-increasing, for some $\alpha \in (1, \mathrm{inc}_C(H))$, taking $x = \mathbf{0}$ and $r = 1$, there exists $v \in \mathbb{B}$ such that

$$H(v) + \alpha\mathbb{B} \subseteq H(\mathbf{0}) + C + \mathbb{B} = C + \mathbb{B}.$$

Since it is $\alpha\mathbb{B} = (\alpha - 1)\mathbb{B} + \mathbb{B}$, by virtue of the order cancellation law, from the last inclusion one obtains

$$H(v) + (\alpha - 1)\mathbb{B} \subseteq H(\mathbf{0}) + C,$$

so condition (2.7) is shown to be satisfied with $\eta = \alpha - 1 > 0$. □

Remark 2.16. (i) Notice that the condition for metric C-increase expressed by (2.7) requires that $\mathrm{int}\, C \neq \varnothing$. As a consequence, whenever working with finitely generated fans, which are supposed to be globally metrically C-increasing, one is forced to assume that $\mathrm{int}\, C \neq \varnothing$.

(ii) Condition (2.7) may be read in terms of strict positivity. Take into account that, with reference to the partial order induced by C, the elements in C are the positive ones. Thus, condition (2.7) postulates the existence of a direction, along which H takes strictly positive values only.

Example 2.17. According to Definition 2.4, the fan H considered in Example 2.13 fails to be metrically $\mathbb{R}_+$-increasing around each point of $\mathbb{R}$, relative to $S = \mathbb{R}$. Observe that, consistently, condition (2.7) is not satisfied.

From condition (2.7) one can derive a sufficient condition for the global metric C-increase property, which is specific for fans generated by regular linear mappings. Recall that if $\Lambda \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ is regular (i.e. onto, or equivalently it is an epimorphism), then there exists $\eta > 0$ such that $\Lambda\mathbb{B} \supseteq \eta\mathbb{B}$. The quantity

$$\mathrm{sur}(\Lambda) = \sup\{\eta > 0 : \Lambda\mathbb{B} \supseteq \eta\mathbb{B}\}$$

is called exact openness bound of Λ and is used to provide a measure of the regularity (openness or covering) of Λ. For more details on the notion of openness of linear mappings, the reader is referred to [17, Section 1.2.3]. In particular, for exact estimates of $\mathrm{sur}(\Lambda)$, see [17, Corollary 1.58].

Corollary 2.18. *Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a fan generated by $\mathcal{G} \subseteq \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$. Suppose that*

$$\inf_{\Lambda \in \mathcal{G}} \operatorname{sur}(\Lambda) > 0.$$

If

$$\operatorname{int}\Big(\bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C) \Big) \neq \varnothing,$$

then H is globally metrically C-increasing.

Proof. By hypothesis, there exist $u \in \mathbb{R}^n$ and $\epsilon > 0$ such that

$$u + \epsilon \mathbb{B} \subseteq \bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C).$$

Notice that it is possible to assume without loss of generality that $u \neq \mathbf{0}$, because if it is

$$\mathbf{0} \in \operatorname{int}\Big(\bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C) \Big),$$

that is,

$$\epsilon \mathbb{B} \subseteq \bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C),$$

then there must exist $x \neq \mathbf{0}$ such that

$$\Lambda x \in C \quad \text{and} \quad \Lambda(-x) \in C \qquad \text{for all } \Lambda \in \mathcal{G}.$$

Since C is a pointed cone, the above inclusions imply $\Lambda x = \mathbf{0}$, so $x \in \Lambda^{-1}(C)$ for every $\Lambda \in \mathcal{G}$, whence

$$x + \epsilon \mathbb{B} \subseteq \bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C).$$

Furthermore, since $\bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C)$ is a cone, it is possible to assume that $u \in \mathbb{B}$. Letting $0 < \eta < \inf_{\Lambda \in \mathcal{G}} \operatorname{sur}(\Lambda)$, since $\operatorname{sur}(\Lambda) > \eta$ for every $\Lambda \in \mathcal{G}$, one has

$$\Lambda(\epsilon \mathbb{B}) \supseteq \epsilon \eta \mathbb{B} \quad \text{for all } \Lambda \in \mathcal{G}.$$

Therefore, it holds

$$\Lambda u + \epsilon \eta \mathbb{B} \subseteq \Lambda(u + \epsilon \mathbb{B}) \subseteq C \quad \text{for all } \Lambda \in \mathcal{G}.$$

According to the definition of H, it follows that $H(u) + \epsilon \eta \mathbb{B} \subseteq C$, so the sufficient condition (2.7) for a fan to be globally metrically C-increasing is satisfied. The thesis follows from Proposition 2.15. □

In order to utilize p.h. set-valued mappings and, in particular, fans as an approximation tool for general multi-valued mappings, a concept of differentiation is needed. Among various proposals extending differential calculus to a set-valued context, motivated by the specific features of the subject under study, the notion of prederivative is employed here, as found in [14]. Such a notion has been recently considered for different purposes in the variational analysis literature also in [13, 20].

Definition 2.19 (Prederivative). Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping and let $\bar{x} \in \operatorname{dom} F$. A p.h. set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be a

(i) *outer prederivative* of F at $\bar{x}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$F(x) \subseteq F(\bar{x}) + H(x - \bar{x}) + \epsilon |x - \bar{x}| \mathbb{B} \quad \text{for all } x \in \mathrm{B}(\bar{x}, \delta);$$

(ii) *inner prederivative* of F at $\bar{x}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$F(\bar{x}) + H(x - \bar{x}) \subseteq F(x) + \epsilon |x - \bar{x}| \mathbb{B} \quad \text{for all } x \in \mathrm{B}(\bar{x}, \delta);$$

(iii) *prederivative* of F at $\bar{x}$ if H is both, an outer and an inner prederivative of F at $\bar{x}$.

It is clear that, whenever a set-valued mapping F happens to be single-valued in a neighborhood of $\bar{x}$ and H is a p.h. mapping, then all cases (i), (ii), and (iii) in Definition 2.19 coincide with the notion of Bouligand derivative (a.k.a. B-derivative), as introduced in [22]. In particular, if $H \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$, then the above three notions collapse to the notion of Fréchet differentiability for mappings. In full analogy with the calculus for single-valued smooth mappings, in the current context a strict variant of the notion of prederivative, which will be employed in the sequel, may be formulated as follows [13, 20].

Definition 2.20 (Strict prederivative). Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping and let $\bar{x} \in \operatorname{dom} F$. A p.h. set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be a *strict prederivative* of F at $\bar{x} \in \operatorname{dom} F$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$F(x_2) \subseteq F(x_1) + H(x_2 - x_1) + \epsilon|x_2 - x_1|\mathbb{B} \quad \text{for all } x_1, x_2 \in \mathrm{B}(\bar{x}, \delta).$$

An articulated discussion on the existence of prederivatives and strict prederivative, on their calculus rules and connections with regularity properties, can be found in [13, 20].

Remark 2.21. The reader should notice that the notion in Definition 2.19 (ii) and, consequently, the one in Definition 2.19 (iii) are different from the notion of inner T-derivative and of T-derivative, respectively, as proposed in [20]. This happens because the term $H(x - \bar{x})$ appears on the left-hand side of the inclusion in Definition 2.19 (ii). Such a choice entails that a strict prederivative in the sense of Definition 2.20 could fail to be a prederivative of the same set-valued mapping. This fact is in contrast with what happens for T-derivative and strict T-derivative, and therefore it causes a shortcoming in the resulting theory. Nevertheless, such a choice seems to be unavoidable in order to obtain the outer tangential approximation of Sol(SVI), where the values of H must be included in $\mathrm{T}(C; \bar{y})$ for $\bar{y} \in F(\bar{x})$ (see the proof of Theorem 3.5). In this regard, it could be relevant to observe that in [14, Definition 9.1] (where F is single-valued), the p.h. term appears on the left-hand side of the inclusion defining the inner prederivative.

The next result shows how local approximations expressed by certain prederivatives can be exploited in order to formulate a condition for the metric C-increase property of set-valued mappings around a reference point, relative to a given set.

Proposition 2.22 (Metric C-increase via strict prederivative). *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping, let $S \subseteq \mathbb{R}^n$ be a closed set and let $\bar{x} \in \operatorname{dom} F \cap S$. Suppose the following conditions:*
(i) *F admits a strict prederivative $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x}$.*
(ii) *There exist $\eta > 0$ and $u \in \mathrm{T}_{\mathrm{Cl}}(S; \bar{x}) \cap \mathbb{B}$ such that $H(u) + \eta\mathbb{B} \subseteq C$.*
(iii) *H is Lipschitz with constant $\kappa \geq 0$.*
Then F is metrically C-increasing around $\bar{x}$ relative to S with

$$\operatorname{inc}_C(F; S; \bar{x}) \geq \frac{\eta}{4(\kappa + 1)} + 1.$$

Proof. Let η and u be as in hypothesis (ii). Notice that, without any loss of generality, it is possible to assume that $\eta \in (0, 1)$. Moreover, by the positive homogeneity of H, one has

$$H\Big(\frac{u}{2}\Big) + \frac{\eta}{2}\mathbb{B} = \frac{1}{2}[H(u) + \eta\mathbb{B}] \subseteq \frac{1}{2}C = C. \tag{2.8}$$

Since $\frac{u}{2} \in \mathrm{T}_{\mathrm{Cl}}(S; \bar{x})$ holds, according to the definition of Clarke tangent cone, corresponding to $\eta/4(\kappa + 1)$ there must exist $\tau > 0$ such that for every $x \in \mathrm{B}(\bar{x}, \tau)$ and for every $t \in (0, \tau)$ there is $v \in \mathrm{B}(\frac{u}{2}, \eta/4(\kappa + 1))$ such that $x + tv \in S$. In view of subsequent estimates, it is useful to observe that, by virtue of the Lipschitz continuity of H, it holds

$$\begin{aligned} H(v) &\subseteq H\Big(\frac{u}{2}\Big) + \kappa\Big|v - \frac{u}{2}\Big|\mathbb{B} \\ &\subseteq H\Big(\frac{u}{2}\Big) + \frac{\kappa\eta}{4(\kappa + 1)}\mathbb{B} \\ &\subseteq H\Big(\frac{u}{2}\Big) + \frac{\eta}{4}\mathbb{B}. \end{aligned} \tag{2.9}$$

Fix $\epsilon > 0$ in such a way that

$$0 < \epsilon < \min\Big\{1, \frac{\eta}{4(\kappa + 1)}\Big\}.$$

According to hypothesis (i), there exists $\delta_\epsilon > 0$ such that

$$F(x_1) \subseteq F(x_2) + H(x_1 - x_2) + \epsilon|x_1 - x_2|\mathbb{B} \quad \text{for all } x_1, x_2 \in \mathrm{B}(\bar{x}, \delta_\epsilon). \tag{2.10}$$

Now, choose $\delta_* \in (0, \min\{\tau,\ \delta_\epsilon/3\})$ and take arbitrary $x \in \mathrm{B}(\bar{x}, \delta_*) \cap S$ and $r \in (0, \delta_*)$.

Since, in particular, $x \in \mathrm{B}(\bar{x}, \tau)$ and $r \in (0, \tau)$, there is $v \in \mathrm{B}(\frac{u}{2}, \eta/4(\kappa + 1))$ such that $x + rv \in S$. Thus, let us define $z = x + rv$. By recalling that $\eta < 1$, it results in

$$\begin{aligned}
|z - \bar{x}| &\le |z - x| + |x - \bar{x}| \\
&\le r|v| + \delta_* < (|v| + 1)\delta_* \\
&\le \Big(\frac{1}{2} + \frac{\eta}{4(\kappa + 1)} + 1\Big)\delta_* \\
&\le \Big(\frac{1}{2} + \frac{1}{4} + 1\Big)\delta_* \\
&< 2\delta_* \\
&< \delta_\epsilon.
\end{aligned}$$

This means that $x, z \in \mathrm{B}(\bar{x}, \delta_\epsilon)$, so it is possible to apply inclusion (2.10), with $x_1 = z$ and $x_2 = x$. Furthermore, it is also useful to remark that

$$r|v| < r\Big(\frac{1}{2} + \frac{\eta}{4(\kappa + 1)}\Big) < r\Big(\frac{1}{2} + \frac{1}{4}\Big) < r. \tag{2.11}$$

Consequently, by taking into account inclusions (2.8), (2.9) and (2.11), one obtains

$$\begin{aligned}
F(z) + \Big(\frac{\eta}{4(\kappa + 1)} + 1 - \epsilon\Big)r\mathbb{B} &\subseteq F(x) + rH(v) + \epsilon r|v|\mathbb{B} + \Big(\frac{\eta}{4(\kappa + 1)} + 1 - \epsilon\Big)r\mathbb{B} \\
&\subseteq F(x) + r\Big[H\Big(\frac{u}{2}\Big) + \frac{\eta}{4}\mathbb{B}\Big] + \frac{\eta r}{4(\kappa + 1)}\mathbb{B} + [\epsilon r + (1 - \epsilon)r]\mathbb{B} \\
&\subseteq F(x) + r\Big[H\Big(\frac{u}{2}\Big) + \frac{\eta}{2}\mathbb{B}\Big] + r\mathbb{B} \\
&\subseteq F(x) + rC + r\mathbb{B} \\
&= F(x) + C + r\mathbb{B}.
\end{aligned}$$

By inequality (2.11), it is true that $z \in \mathrm{B}(x, r) \cap S$. Therefore, since

$$\frac{\eta}{4(\kappa + 1)} + 1 - \epsilon > 1,$$

the last inclusion shows that F is metrically C-increasing around $\bar{x}$ relative to S. The arbitrariness of $\epsilon > 0$ enables one to get the quantitative estimate of $\mathrm{inc}_C(F; S; \bar{x})$ in the thesis. □

The condition appearing in hypothesis (ii) can be regarded as a localization of condition (2.7). This shows that the approximation apparatus based on prederivatives transforms properties of approximations into corresponding properties of the mappings to be approximated, as it happens with classical differential calculus and certain specific properties such as metric regularity (see [14, 17]).

3 Main achievements

The main result of the paper, about a tangential approximation of Sol(SVI) near one of its elements, is established below.

Theorem 3.1 (Inner tangential approximation under C-increase). *With reference to problem* (SVI), *suppose* $\bar{x} \in \mathrm{Sol(SVI)}$. *Suppose the following conditions:*

(i) *F is l.s.c. in a neighborhood of $\bar{x}$.*
(ii) *F is metrically C-increasing around $\bar{x}$ relative to S.*
(iii) *F admits $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ as an outer prederivative at $\bar{x}$.*

Then the following inclusion holds:

$$H^{+1}(C) \cap \mathrm{I_w}(S; \bar{x}) \subseteq \mathrm{T}(\mathrm{Sol(SVI)}; \bar{x}). \tag{3.1}$$

If, in addition,

(iv) *the outer prederivative H of F at $\bar{x}$ is Lipschitz,*

then the following stronger inclusion holds:

$$H^{+1}(C) \cap \mathrm{T}(S; \bar{x}) \subseteq \mathrm{T}(\mathrm{Sol(SVI)}; \bar{x}). \tag{3.2}$$

Proof. Take an arbitrary $v \in H^{+1}(C) \cap \mathrm{I_w}(S; \bar{x})$. If $v = \mathbf{0}$, then it is obviously $v \in \mathrm{T}(\mathrm{Sol(SVI)}; \bar{x})$. So, let us suppose henceforth $v \neq \mathbf{0}$. Observe that, since $H^{+1}(C)$, $\mathrm{I_w}(S; \bar{x})$ and $\mathrm{T}(\mathrm{Sol(SVI)}; \bar{x})$ are all cones (remember Remark 2.14 (i)), it is possible to assume without any loss of generality that $|v| = 1$. According to the characterization of elements in the contingent cone mentioned in Remark 2.9, in order to prove that $v \in \mathrm{T}(\mathrm{Sol(SVI)}; \bar{x})$ it suffices to show that

$$\liminf_{t \downarrow 0} \frac{\mathrm{dist}(\bar{x} + tv, \mathrm{Sol(SVI)})}{t} = \sup_{\tau > 0} \inf_{t \in (0,\tau)} \frac{\mathrm{dist}(\bar{x} + tv, \mathrm{Sol(SVI)})}{t} = 0. \tag{3.3}$$

This means that for every $\tau > 0$ and $\epsilon > 0$ there must exist $t \in (0, \tau)$ such that

$$\frac{\mathrm{dist}(\bar{x} + tv, \mathrm{Sol(SVI)})}{t} \leq \epsilon. \tag{3.4}$$

So, fix positive τ and ϵ. According to Lemma 2.7, by virtue of hypotheses (i) and (ii), a local error bound for (SVI) is valid, so corresponding to $\alpha \in (1, \mathrm{inc}_C(F; S; \bar{x}))$ there exists $\delta_\alpha > 0$ such that inequality (2.4) holds.

On the other hand, by virtue of hypothesis (iii), corresponding to ϵ there exists $\delta_\epsilon > 0$ such that

$$F(x) \subseteq F(\bar{x}) + H(x - \bar{x}) + \epsilon(\alpha - 1)|x - \bar{x}|\mathbb{B} \quad \text{for all } x \in \mathrm{B}(\bar{x}, \delta_\epsilon). \tag{3.5}$$

Now, take δ_* in such a way that

$$0 < \delta_* < \min\{\delta_\alpha,\ \delta_\epsilon,\ \tau\}.$$

Since $v \in \mathrm{I_w}(S; \bar{x})$, there exists $t_* \in (0, \delta_*)$ with the property that $\bar{x} + t_* v \in S \cap \mathrm{B}(\bar{x}, \delta_*)$. As a consequence of inclusion (3.5), taking into account that $v \in H^{+1}(C)$, one finds

$$\begin{aligned} F(\bar{x} + t_* v) &\subseteq F(\bar{x}) + t_* H(v) + \epsilon(\alpha - 1)t_* \mathbb{B} \\ &\subseteq C + t_* C + \epsilon(\alpha - 1)t_* \mathbb{B} \\ &= C + \epsilon(\alpha - 1)t_* \mathbb{B}. \end{aligned}$$

From the last inclusion, on account of what was recalled in Remark 2.1 (iii), it follows that

$$\begin{aligned} \mathrm{exc}(F(\bar{x} + t_* v), C) &\leq \mathrm{exc}(C + \epsilon(\alpha - 1)t_* \mathbb{B}, C) \\ &= \mathrm{exc}(\epsilon(\alpha - 1)t_* \mathbb{B}, C) \\ &\leq \epsilon(\alpha - 1)t_*. \end{aligned}$$

Therefore, by exploiting the error bound inequality (2.4), what is possible to do inasmuch as

$$\bar{x} + t_* v \in \mathrm{B}(\bar{x}, \delta_\alpha) \cap S,$$

one obtains

$$\frac{\mathrm{dist}(\bar{x} + t_* v, \mathrm{Sol(SVI)})}{t_*} \leq \frac{\mathrm{exc}(F(\bar{x} + t_* v), C)}{(\alpha - 1)t_*} \leq \epsilon.$$

As the last inequality shows that condition (3.4) is satisfied for $t = t_* \in (0, \tau)$, inclusion (3.1) is proved.

In order to prove the second inclusion in the thesis, observe first that, since the function

$$x \mapsto \mathrm{dist}(x, \mathrm{Sol(SVI)})$$

is Lipschitz, it holds

$$\liminf_{t \downarrow 0} \frac{\mathrm{dist}(\bar{x} + tv, \mathrm{Sol(SVI)})}{t} = \liminf_{\substack{w \to v \\ t \downarrow 0}} \frac{\mathrm{dist}(\bar{x} + tw, \mathrm{Sol(SVI)})}{t}.$$

By consequence, in order to show that $v \in H^{+1}(C) \cap \mathrm{T}(S;\bar{x})$ implies $v \in \mathrm{T}(\mathrm{Sol(SVI)};\bar{x})$ by means of the characterization in (3.3), it suffices to prove the existence of sequences $(v_n)_n$ with $v_n \to v$ and $(t_n)_n$ with $t_n \downarrow 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\mathrm{dist}(\bar{x}+t_n v_n, \mathrm{Sol(SVI)})}{t_n} = 0. \tag{3.6}$$

Again, one can assume that $|v| = 1$ (the case $v = \mathbf{0}$ being trivial). Since $v \in H^{+1}(C) \cap \mathrm{T}(S;\bar{x})$, there exist $(v_n)_n$ with $v_n \to v$ and $(t_n)_n$ with $t_n \downarrow 0$ such that $\bar{x} + t_n v_n \in S$ for every $n \in \mathbb{N}$. As a consequence of hypothesis (iv), one finds that for some $\kappa > 0$ it must hold

$$H(v_n) \subseteq H(v) + \kappa|v_n - v|\mathbb{B} \quad \text{for all } n \in \mathbb{N}.$$

Fix $\epsilon > 0$. Correspondingly, by hypothesis (iii) there exists $\delta_\epsilon > 0$ such that the following inclusion holds true:

$$F(x) \subseteq F(\bar{x}) + H(x-\bar{x}) + |x-\bar{x}|\Big(\frac{\alpha-1}{\kappa+2}\Big)\epsilon\mathbb{B} \quad \text{for all } x \in \mathrm{B}(\bar{x},\delta_\epsilon). \tag{3.7}$$

Take $\delta_* \in (0, \min\{\delta_\alpha, \delta_\epsilon\})$, where $\delta_\alpha > 0$ and α have the same meaning as in the first part of the proof and do exist by hypotheses (i) and (ii) and by Lemma 2.7. Since $\bar{x} + t_n v_n \to \bar{x}$ as $n \to \infty$, there exists $n_* \in \mathbb{N}$ such that

$$\bar{x} + t_n v_n \in \mathrm{B}(\bar{x}, \delta_*)$$

and

$$|v_n - v| < \frac{(\alpha-1)\epsilon}{\kappa+2}, \quad |v_n| < 2 \qquad \text{for all } n \in \mathbb{N},\ n \geq n_*.$$

Thus, by recalling that $v \in H^{+1}(C)$, in the light of inclusion (3.7), which can be used because

$$\bar{x} + t_n v_n \in \mathrm{B}(\bar{x}, \delta_\epsilon) \quad \text{for every } n \geq n_*,$$

one obtains

$$\begin{aligned} F(\bar{x}+t_n v_n) &\subseteq F(\bar{x}) + t_n H(v_n) + t_n|v_n|\Big(\frac{\alpha-1}{\kappa+2}\Big)\epsilon\mathbb{B} \\ &\subseteq C + t_n[H(v) + \kappa|v_n - v|\mathbb{B}] + t_n|v_n|\Big(\frac{\alpha-1}{\kappa+2}\Big)\epsilon\mathbb{B} \\ &\subseteq C + t_n C + t_n\frac{\kappa}{\kappa+2}(\alpha-1)\epsilon\mathbb{B} + t_n\frac{2}{\kappa+2}(\alpha-1)\epsilon\mathbb{B} \\ &= C + t_n\Big(\frac{\kappa}{\kappa+2} + \frac{2}{\kappa+2}\Big)(\alpha-1)\epsilon\mathbb{B}, \\ &= C + t_n(\alpha-1)\epsilon\mathbb{B} \quad \text{for all } n \in \mathbb{N},\ n \geq n_*. \end{aligned}$$

Now, by passing to the excess function, on account of Remark 2.1 (ii) and (iii) from the last inclusions, one deduces

$$\begin{aligned} \mathrm{exc}(F(\bar{x}+t_n v_n), C) &\leq \mathrm{exc}(C + t_n(\alpha-1)\epsilon\mathbb{B}, C) \\ &\leq \mathrm{exc}(t_n(\alpha-1)\epsilon\mathbb{B}, C) \\ &\leq t_n(\alpha-1)\epsilon \quad \text{for all } n \in \mathbb{N},\ n \geq n_*. \end{aligned}$$

Since $\bar{x} + t_n v_n \in \mathrm{B}(\bar{x}, \delta_*) \cap S$ for every $n \in \mathbb{N}$ with $n \geq n_*$, by virtue of the error bound inequality valid in $\mathrm{B}(\bar{x}, \delta_\alpha) \cap S$, it results in

$$\frac{\mathrm{dist}(\bar{x}+t_n v_n, \mathrm{Sol(SVI)})}{t_n} \leq \frac{\mathrm{exc}(F(\bar{x}+t_n v_n), C)}{(\alpha-1)t_n} \leq \epsilon \quad \text{for all } n \in \mathbb{N},\ n \geq n_*.$$

The last inequality, by arbitrariness of ϵ, allows one to conclude that equality (3.6) holds true, thereby completing the proof. □

Inclusions (3.1) and (3.2) provide a convenient description of (in the general case) some elements in $\mathrm{T}(\mathrm{Sol(SVI)};\bar{x})$. Theorem 3.1 ensures that, as far as working with solutions of the approximated (actually, homogenized) set-valued inclusion

$$\text{find } x \in \mathrm{I_w}(S;\bar{x}) \text{ such that } H(x) \subseteq C,$$

one keeps within the conic (contingent) approximation of Sol(SVI) near $\bar{x}$. The reader should notice that, very often, finding all solutions of problem (SVI) turns out to be a hard problem. Consequently, the set $\mathrm{T}(\mathrm{Sol(SVI)}; \bar{x})$ can not be calculated explicitly starting from Sol(SVI). On the other hand, since S and C are problem data, while the structure of H is supposed to be simpler than the one of F, cones $H^{+1}(C)$ and $\mathrm{I_w}(S; \bar{x})$, or $\mathrm{T}(S; \bar{x})$, can be calculated more easily. This fact is more evident when H is a fan generated by linear mappings and, in particular, is finitely generated (remember indeed Remark 2.14 (ii)). With such a reading, Theorem 3.1 can be considered as a modern version of an implicit function theorem.

Since outer prederivatives are only one-side approximation tools, one can not expect that any inclusion achieved through them, such as (3.1), could be reverted to get an equality. A simple counterexample is discussed below.

Example 3.2 (Strict inclusion may hold). Let us consider the set-valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ introduced in Example 2.6. Take $S = \mathbb{R}$, $C = \mathbb{R}^2_+$ and $\bar{x} = 0$. As F is globally metrically $\mathbb{R}^2_+$-increasing, it is metrically $\mathbb{R}^2_+$-increasing relative to $\mathbb{R}$ around 0. It is plain to check that F is l.s.c. on $\mathbb{R}$. From Definition 2.19 (i) it follows that the constant mapping $H : \mathbb{R} \rightrightarrows \mathbb{R}^2$, defined by $H(x) = \mathbb{R}^2$ for every $x \in \mathbb{R}$, is an outer prederivative of F at 0. As one readily sees, it holds $\mathrm{Sol(SVI)} = F^{+1}(\mathbb{R}^2_+) = [0, +\infty)$. Thus, it results in $\mathrm{T}(\mathrm{Sol(SVI)}; 0) = [0, +\infty)$. On the other hand, it is clear that $H^{+1}(\mathbb{R}^2_+) = \varnothing$. So, in the current case it happens that

$$H^{+1}(\mathbb{R}^2_+) \cap \mathrm{I_w}(S; 0) = \varnothing \neq [0, +\infty) = \mathrm{T}(\mathrm{Sol(SVI)}; 0).$$

Now, to work with a more reasonable approximation of F at 0, one may consider the set-valued mapping $H : \mathbb{R} \rightrightarrows \mathbb{R}^2$, defined by

$$H(x) = (x, x) + O,$$

where $O = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 y_2 = 0\}$. Clearly, H is p.h. because, as O is a cone, it holds

$$\begin{aligned} H(\lambda x) &= (\lambda x, \lambda x) + O \\ &= \lambda(x, x) + \lambda O \\ &= \lambda[(x, x) + O] \\ &= \lambda H(x) \quad \text{for all } \lambda > 0 \text{ and all } x \in \mathbb{R}. \end{aligned}$$

Moreover, since it is

$$F(x) \subseteq H(x) \quad \text{for all } x \in \mathbb{R} \text{ and } (0, 0) \in F(0),$$

for every $\epsilon > 0$ one has

$$F(x) \subseteq F(0) + H(x) \subseteq F(0) + H(x) + \epsilon|x|\mathbb{B} \quad \text{for all } x \in \mathbb{R}.$$

Consequently, H is an outer prederivative of F at 0, so all the hypotheses of Theorem 3.1 are fulfilled. Since

$$H(x) \not\subseteq \mathbb{R}^2_+ \quad \text{for all } x \in \mathbb{R},$$

it happens that $H^{+1}(\mathbb{R}^2_+) = \varnothing$. So, again one has

$$H^{+1}(\mathbb{R}^2_+) \cap \mathrm{I_w}(S; 0) = \varnothing \neq [0, +\infty) = \mathrm{T}(\mathrm{Sol(SVI)}; 0).$$

In the particular case where F admits a strict prederivative at $\bar{x}$, from Theorem 3.1 the following inner approximation of Sol(SVI) can be derived.

Corollary 3.3. *With reference to problem* (SVI), *let* $\bar{x} \in \mathrm{Sol(SVI)}$. *Suppose the following conditions:*
(i) *F is l.s.c. in a neighborhood of $\bar{x}$.*
(ii) *F admits a strict prederivative $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x}$.*
(iii) *There exist $\eta > 0$ and $u \in \mathrm{T_{Cl}}(S; \bar{x}) \cap \mathbb{B}$ such that $H(u) + \eta\mathbb{B} \subseteq C$.*
(iv) *H is Lipschitz.*
Then inclusion (3.2) *holds true.*

Proof. In the light of Proposition 2.22, under hypotheses (ii), (iii) and (iv), F turns out to be metrically C-increasing around $\bar{x}$, relative to S. Then it suffices to apply Theorem 3.1. □

Besides an inner tangential approximation of the solution set to (SVI), which is already useful in applications to optimization (see Section 4), it seems to be worthwhile to consider also an outer tangential approximation of this set. In doing so, the following remark is relevant.

Remark 3.4. Under the assumption that F is Hausdorff C-u.s.c. at $\bar{x}$, which seems to be reasonable for the problem at the issue, if there exists $\eta > 0$ such that $F(\bar{x}) + \eta\mathbb{B} \subseteq C$ (strong satisfaction of the set-valued inclusion), then one has $\bar{x} \in \operatorname{int} F^{+1}(C)$. Indeed, corresponding with η, there exists $\delta > 0$ such that

$$F(x) \subseteq F(\bar{x}) + C + \eta\mathbb{B} \subseteq C + C = C \quad \text{for all } x \in \mathrm{B}(\bar{x}, \delta).$$

Therefore, whenever it happens that $\bar{x} \in \operatorname{int} S$, one obtains

$$\bar{x} \in \operatorname{int} F^{+1}(C) \cap \operatorname{int} S = \operatorname{int} \mathrm{Sol}(\mathrm{SVI}).$$

Consequently, it results in

$$\mathrm{T}(\mathrm{Sol}(\mathrm{SVI}); \bar{x}) = \mathbb{R}^n.$$

In such a circumstance, it is clear that an outer description of the contingent cone is no longer interesting.

In the light of Remark 3.4, the below analysis is focussed on the case $F(\bar{x}) \cap \operatorname{bd} C \neq \varnothing$. Such a choice clearly excludes the case $F(\bar{x}) \subseteq \operatorname{int} C$. It is worth noting that, whenever $F(\bar{x})$ fails to be a compact set, the latter case is more general than the one considered in Remark 3.4 (strong satisfaction of the set-valued inclusion).

Theorem 3.5 (Outer tangential approximation by fans). *With reference to problem* (SVI), *let* $\bar{x} \in \mathrm{Sol}(\mathrm{SVI})$. *Suppose the following conditions:*
(i) $F(\bar{x}) \cap \operatorname{bd} C \neq \varnothing$.
(ii) *F admits an inner prederivative $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x}$.*
(iii) *H is a fan generated by a bounded set $\mathcal{G} \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.*
Then it holds

$$\mathrm{T}(\mathrm{Sol}(\mathrm{SVI}); \bar{x}) \subseteq \Big[\bigcap_{y \in F(\bar{x}) \cap \operatorname{bd} C} H^{+1}(\mathrm{T}(C; y)) \Big] \cap \mathrm{T}(S; \bar{x}).$$

Proof. It is clear that

$$\mathbf{0} \in \Big[\bigcap_{y \in F(\bar{x}) \cap \operatorname{bd} C} H^{+1}(\mathrm{T}(C; y)) \Big] \cap \mathrm{T}(S; \bar{x}).$$

Indeed, $\mathbf{0} \in \mathrm{T}(S; \bar{x})$ and, since H is generated by linear mappings, $H(\mathbf{0}) = \{\mathbf{0}\}$, with the consequence that $\mathbf{0} \in H^{+1}(\mathrm{T}(C; y))$ for every $y \in F(\bar{x}) \cap \operatorname{bd} C$. Now, let $v \neq \mathbf{0}$ be an arbitrary element of $\mathrm{T}(\mathrm{Sol}(\mathrm{SVI}); \bar{x})$. As already done above, in consideration of the conical nature of all involved sets, it is possible to assume that $|v| = 1$. Then there exist $(v_n)_n$ with $v_n \to v$ and $(t_n)_n$ with $t_n \downarrow 0$ as $n \to \infty$ such that

$$\bar{x} + t_n v_n \in \mathrm{Sol}(\mathrm{SVI}) = F^{+1}(C) \cap S \quad \text{for every } n \in \mathbb{N}.$$

This fact immediately implies that $v \in \mathrm{T}(S; \bar{x})$. By virtue of hypothesis (ii), for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$F(\bar{x}) + H(x - \bar{x}) \subseteq F(x) + \epsilon|x - \bar{x}|\mathbb{B} \quad \text{for all } x \in \mathrm{B}(\bar{x}, \delta_\epsilon). \tag{3.8}$$

According to what was noted in Remark 2.14 (iii), since H is generated by a bounded set, it is Lipschitz, so there exists $\kappa > 0$ such that

$$H(x_1) \subseteq H(x_2) + \kappa|x_1 - x_2|\mathbb{B} \quad \text{for all } x_1, x_2 \in \mathbb{R}^n. \tag{3.9}$$

Fix an element $\bar{y} \in F(\bar{x}) \cap \operatorname{bd} C$ and $\epsilon > 0$. Without any loss of generality, it is possible to assume that

$$\delta_\epsilon < \frac{\epsilon}{\kappa}.$$

Since the sequence $(\bar{x} + t_n v_n)_n$ converges to $\bar{x}$, starting with a proper $n_* \in \mathbb{N}$ it must be $\bar{x} + t_n v_n \in \mathrm{B}(\bar{x}, \delta_\epsilon)$ for every $n \geq n_*$. Thus, from inclusion (3.8) it follows

$$\bar{y} + t_n H(v_n) \subseteq F(\bar{x} + t_n v_n) + \epsilon t_n |v_n|\mathbb{B} \quad \text{for all } n \geq n_*,$$

whence, by recalling that $\bar{x} + t_n v_n \in \mathrm{Sol(SVI)}$, one gets

$$t_n H(v_n) \subseteq F(\bar{x} + t_n v_n) - \bar{y} + \epsilon t_n |v_n| \mathbb{B} \subseteq C - \bar{y} + \epsilon t_n |v_n| \mathbb{B} \quad \text{for all } n \geq n_*.$$

Since it is $v_n \to v$, by increasing, if needed, the value of n_*, it is possible to assume that

$$|v_n| < 2 \quad \text{and} \quad |v - v_n| < \frac{\epsilon}{\kappa} \qquad \text{for all } n \geq n_*.$$

From the last inclusion, one obtains

$$H(v_n) \subseteq \frac{C - \bar{y}}{t_n} + 2\epsilon \mathbb{B} \quad \text{for all } n \geq n_*. \tag{3.10}$$

By recalling inclusion (3.9), one deduces

$$H(v) \subseteq H(v_n) + \kappa |v - v_n| \mathbb{B} \subseteq H(v_n) + \epsilon \mathbb{B} \quad \text{for all } n \geq n_*.$$

On account of (3.10), the last obtained inclusion gives

$$H(v) \subseteq \frac{C - \bar{y}}{t_n} + 3\epsilon \mathbb{B} \quad \text{for all } n \geq n_*.$$

According to this, for each $w \in H(v)$ there exist sequences $(c_n)_n$ with $c_n \in C$ and $(b_n)_n$ with $b_n \in \mathbb{B}$ such that

$$w = \frac{c_n - \bar{y}}{t_n} + 3\epsilon b_n \quad \text{for all } n \geq n_*.$$

Since, up to a sequence relabeling, it is $b_n \to b \in \mathbb{B}$ for some $b \in \mathbb{B}$ as $n \to \infty$, $\mathbb{B}$ being compact, it must result in

$$z_n = \frac{c_n - \bar{y}}{t_n} \to z \in \mathrm{T}(C; \bar{y}) \quad \text{as } n \to \infty.$$

As it is $\bar{y} + t_n z_n \in C$ for every $n \geq n_*$, this means that $w \in \mathrm{T}(C; \bar{y}) + 3\epsilon \mathbb{B}$. By arbitrariness of $w \in H(v)$, the above argument shows that

$$H(v) \subseteq \mathrm{T}(C; \bar{y}) + 3\epsilon \mathbb{B}. \tag{3.11}$$

Since $H(v)$ is a closed set, $\mathrm{T}(C; \bar{y})$ is a closed cone and inclusion (3.11) remains true for every $\epsilon > 0$ (notice, indeed, that v has been fixed before fixing ϵ), it is possible to assert that $H(v) \subseteq \mathrm{T}(C; \bar{y})$, or, equivalently, $v \in H^{+1}(\mathrm{T}(C; \bar{y}))$. By arbitrariness of $\bar{y} \in F(\bar{x}) \cap \operatorname{bd} C$, the above argument shows that

$$v \in \bigcap_{y \in F(\bar{x}) \cap \operatorname{bd} C} H^{+1}(\mathrm{T}(C; y)),$$

and thus allows one to conclude that

$$\mathrm{T}(\mathrm{Sol(SVI)}; \bar{x}) \subseteq \bigcap_{y \in F(\bar{x}) \cap \operatorname{bd} C} H^{+1}(\mathrm{T}(C; y)).$$

The proof is complete. □

In the special case in which $\mathbf{0} \in F(\bar{x})$, by exploiting the bilateral approximation of a set-valued mapping provided by prederivatives, one can achieve the following characterization on the contingent cone to the solution set of an (SVI).

Theorem 3.6 (Tangential approximation of Sol(SVI)). *With reference to problem* (SVI), *let* $\bar{x} \in \mathrm{Sol(SVI)}$. *Suppose the following conditions:*
(i) *F is l.s.c. in a neighborhood of $\bar{x}$.*
(ii) *F is metrically C-increasing around $\bar{x}$ relative to S.*
(iii) *F admits a prederivative $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x}$.*
(iv) *H is a fan generated by a bounded subset of $\mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$.*
(v) $\mathbf{0} \in F(\bar{x})$.

Then the following equality holds:

$$\mathrm{T}(\mathrm{Sol(SVI)}; \bar{x}) = H^{+1}(C) \cap \mathrm{T}(S; \bar{x}). \tag{3.12}$$

Proof. Under the above hypotheses one can invoke Theorem 3.1. In doing so, as H is generated by a bounded set of linear mappings, it is a Lipschitz outer prederivative of F at $\bar{x}$. Consequently, inclusion (3.2) must hold true.

On the other hand, the hypotheses in force allows one to apply Theorem 3.5. Thus, since $\mathbf{0} \in F(\bar{x}) \cap \operatorname{bd} C$ and $\mathrm{T}(C; \mathbf{0}) = C$, one finds

$$\mathrm{T}(\mathrm{Sol(SVI)}; \bar{x}) \subseteq H^{+1}(\mathrm{T}(C; \mathbf{0})) \cap \mathrm{T}(S; \bar{x}) = H^{+1}(C) \cap \mathrm{T}(S; \bar{x}).$$

The last inclusion, along with (3.2), certifies that the equality in the assertion is true. □

It is reasonable to expect that, owing to the local nature of the contingent tangential approximation, in the case $\bar{x} \in \operatorname{int} S$ the presence of S does not affect the representation of $\mathrm{T}(\mathrm{Sol(SVI)}; \bar{x})$. This fact is established below.

Corollary 3.7. *Under the hypotheses of Theorem 3.6, suppose that $\bar{x} \in \operatorname{int} S$. Then it holds*

$$\mathrm{T}(F^{+1}(C); \bar{x}) = \mathrm{T}(\mathrm{Sol(SVI)}; \bar{x}) = H^{+1}(C). \tag{3.13}$$

Proof. Since $\bar{x}$ is an interior point of S, there exists $\delta_0 > 0$ such that $\mathrm{B}(\bar{x}, \delta_0) \subseteq S$. By taking into account equality (2.6), one obtains

$$\begin{aligned}
\mathrm{T}(\mathrm{Sol(SVI)}; \bar{x}) &= \mathrm{T}(S \cap F^{+1}(C); \bar{x}) \\
&= \mathrm{T}((S \cap F^{+1}(C)) \cap \mathrm{B}(\bar{x}, \delta_0); \bar{x}) \\
&= \mathrm{T}(F^{+1}(C) \cap \mathrm{B}(\bar{x}, \delta_0); \bar{x}) \\
&= \mathrm{T}(F^{+1}(C); \bar{x}).
\end{aligned}$$

On the other hand, again by the fact that $\bar{x} \in \operatorname{int} S$, we have $\mathrm{T}(S; \bar{x}) = \mathbb{R}^n$. Thus, in the current case (3.12) becomes (3.13). □

Even though the present approach has been conceived for problem (SVI) involving multi-valued mappings, its impact in the case in which F happens to be single-valued is worth being considered too. In such a circumstance, it is known that metrically regular mappings around a reference point $\bar{x}$ are, in particular, $\mathbb{R}^m_+$-increasing around the same point (see [25, Example 3.4]). Moreover, the notion of prederivative forces H to be single-valued and collapses to the notion of B-derivative in the sense of Robinson (see [22]). If, in addition, H is supposed to be generated by linear mappings, prederivatives turns out to be the classic (Fréchet) derivative. Thus, it is possible to see that Theorem 3.6 extends to a set-valued setting the well-known tangential representation (1.2) of the solution set to problem (1.1) in the case $C = \mathbb{R}^m_+$. In this regard, notice that strict differentiability is not required for such an equality to hold because $\operatorname{int} \mathbb{R}^m_+ \neq \varnothing$ (no equality constraint is involved in (1.1)).

4 An application to constrained optimization

In the present section, the tangential analysis of the solution set to set-valued inclusions (SVI) is exploited for deriving necessary optimality conditions. Let us focus on constrained scalar optimization problems that can be formalized as

$$\min_{x \in S} \varphi(x) \quad \text{subject to} \quad F(x) \subseteq C, \tag{P}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ denotes the objective (or cost) function, while the sets $S \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^m$, and the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, define an (SVI) problem. With this format, the feasible region of the problem is therefore $\mathcal{R} = \mathrm{Sol(SVI)} = F^{+1}(C) \cap S$. As in the previous sections, S is assumed to be a nonempty closed set, whereas C a closed, convex and pointed cone with $C \neq \{\mathbf{0}\}$.

Proposition 4.1 (Necessary optimality condition). *Let $\bar{x} \in \mathcal{R}$ be a local solution to problem* (P). *Suppose the following conditions:*
(i) *F is l.s.c. in a neighborhood of $\bar{x}$.*
(ii) *F is metrically C-increasing around $\bar{x}$ relative to S.*
(iii) *F admits $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ as an outer prederivative at $\bar{x}$.*
Then the following inclusion holds:

$$-\widehat{\partial}^+\varphi(\bar{x}) \subseteq \left[H^{+1}(C) \cap \mathrm{I_w}(S; \bar{x})\right]^{\ominus}. \tag{4.1}$$

If, in particular, φ is differentiable at $\bar{x}$, condition (4.1) *becomes*

$$\mathbf{0} \in \nabla\varphi(\bar{x}) + \left[H^{+1}(C) \cap \mathrm{I_w}(S; \bar{x})\right]^{\ominus}.$$

Proof. By the local optimality of $\bar{x}$, there exists $\delta > 0$ such that

$$\varphi(\bar{x}) \le \varphi(x) \quad \text{for all } x \in \mathcal{R} \cap \mathrm{B}(\bar{x}, \delta).$$

Take an arbitrary $w \in \widehat{\partial}^+\varphi(\bar{x})$. According to what was recalled in Remark 2.10, there exists $\sigma : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sigma(\bar{x}) = \varphi(\bar{x}) \le \varphi(x) \le \sigma(x) \quad \text{for all } x \in \mathcal{R} \cap \mathrm{B}(\bar{x}, \delta), \tag{4.2}$$

with $\nabla\sigma(\bar{x}) = w$. Take $v \in (H^{+1}(C) \cap \mathrm{I_w}(S; \bar{x})) \setminus \{\mathbf{0}\}$. By virtue of inclusion (3.1), which holds true because all hypotheses of Theorem 3.1 are in force, there must exist sequences $(v_n)_n$ with $v_n \to v$ and $(t_n)_n$ with $t_n \downarrow 0$ such that $\bar{x} + t_n v_n \in \mathcal{R}$ for every $n \in \mathbb{N}$. Since $\bar{x} + t_n v_n \to \bar{x}$ as $n \to \infty$, there exists a proper $n_* \in \mathbb{N}$ such that

$$\bar{x} + t_n v_n \in \mathcal{R} \cap \mathrm{B}(\bar{x}, \delta) \quad \text{for all } n \ge n_*, \ n \in \mathbb{N}.$$

Thus, from inequality (4.2), by using the differentiability of σ at $\bar{x}$, one obtains

$$0 \le \frac{\sigma(\bar{x} + t_n v_n) - \sigma(\bar{x})}{t_n} = \langle w, v_n \rangle + \frac{o(|t_n v_n|)}{t_n |v_n|} \cdot |v_n| \quad \text{for all } n \ge n_*, \ n \in \mathbb{N}.$$

Take into account that, as a converging sequence, $(v_n)_n$ is bounded. So, passing to the limit as $n \to \infty$ in the last inequality, one finds

$$\langle w, v \rangle \ge 0.$$

As this is true for every $v \in H^{+1}(C) \cap \mathrm{I_w}(S; \bar{x})$ (the case $v = \mathbf{0}$ being trivial), one can deduce that

$$-w \in \left[H^{+1}(C) \cap \mathrm{I_w}(S; \bar{x})\right]^{\ominus}.$$

By arbitrariness of $w \in \widehat{\partial}^+\varphi(\bar{x})$, the last inclusion gives (4.1). The second assertion in the thesis follows at once. □

Inclusion (4.1) is a counterpart for problem (P) to the upper subdifferential optimality condition valid for cone constrained problems (see, for instance, [18, Theorem 5.7]).

Remark 4.2. It is to be noted that, whenever $\bar{x} \in \operatorname{int} S$ satisfies the constraint system in a "strict" way, i.e. the constraint system is strongly satisfied at $\bar{x}$ in the sense of Remark 3.4, then under a Hausdorff upper semicontinuity assumption on F one has $\bar{x} \in \operatorname{int} \mathcal{R}$. In such a circumstance, the local optimality of $\bar{x}$ clearly implies $\mathbf{0} \in \widehat{\partial}\varphi(\bar{x})$.

The optimality condition formulated in Proposition 4.1 requests F to admit an outer prederivative H, but does not impose specific requirements on H (all hypotheses refer indeed to F). As one expects, by adding proper assumptions on the geometric structure of H, along with adequate qualification conditions, it is possible to achieve finer optimality conditions, having a stronger computational impact. This is done in the next result. Since in what follows calculus rules for the normal cone are employed, for the reader's convenience they are recalled along with their qualification condition in a technical remark.

Remark 4.3. Given $\Lambda \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ and a pair of closed convex cones $Q \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^m$, the following useful calculus rule holds (see [23, Lemma 2.4.1]):

$$\left(Q \cap \Lambda^{-1}(C)\right)^{\ominus} = \mathrm{cl}(Q^{\ominus} + \Lambda^{\top}(C^{\ominus})).$$

Notice that the equalities

$$(Q_1 \cap Q_2)^{\ominus} = \mathrm{cl}(Q_1^{\ominus} + Q_2^{\ominus}) \tag{4.3}$$

and

$$(\Lambda^{-1}(C))^{\ominus} = \mathrm{cl}\,\Lambda^{\top}(C^{\ominus}) \tag{4.4}$$

are special cases of the above formula. If, in particular, the qualification condition $\mathrm{int}\,Q_1 \cap \mathrm{int}\,Q_2 \neq \varnothing$ happens to be satisfied, then formula (4.3) takes the simpler form

$$(Q_1 \cap Q_2)^{\ominus} = Q_1^{\ominus} + Q_2^{\ominus}, \tag{4.5}$$

whereas the closure operation in (4.4) can be omitted, provided that there exists some x such that $\Lambda x \in \mathrm{int}\,C$.

Theorem 4.4. *Let $\bar{x} \in \mathcal{R}$ be a local solution to problem* (P). *Suppose the following conditions:*
(i) *F is l.s.c. in a neighborhood of $\bar{x}$.*
(ii) *F admits a strict prederivative $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x}$.*
(iii) *H is a fan generated by a bounded set $\mathcal{G} \subseteq \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$.*
(iv) *There exist $\eta > 0$ and $u \in \mathrm{T_{Cl}}(S; \bar{x}) \cap \mathbb{B}$ such that $H(u) + \eta\mathbb{B} \subseteq C$.*
(v) *S is locally convex around $\bar{x}$ and it holds*

$$\mathrm{int}\,\mathrm{T}(S; \bar{x}) \cap \mathrm{int}\Big(\bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C)\Big) \neq \varnothing.$$

Then it holds

$$-\widehat{\partial}^{+}\varphi(\bar{x}) \subseteq \Big(\bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C)\Big)^{\ominus} + \mathrm{N}(S; \bar{x}). \tag{4.6}$$

If, in particular, it holds
(vi) *$\mathcal{G} = \mathrm{co}\{\Lambda_1, \dots, \Lambda_p\}$ and there exists x_0 such that $\Lambda_i x_0 \in \mathrm{int}\,C$ for all $i = 1, \dots, p$,*
then inclusion (4.6) *becomes*

$$-\widehat{\partial}^{+}\varphi(\bar{x}) \subseteq \sum_{i=1}^{p} \Lambda_i^{\top}(C^{\ominus}) + \mathrm{N}(S; \bar{x}). \tag{4.7}$$

Proof. Observe first that, by virtue of hypothesis (iii), the fan H is Lipschitz (remember Remark 2.14 (iii)). By consequence, upon hypotheses (i), (ii) and (iv), it is possible to apply Corollary 3.3, which ensures that the inclusion

$$H^{+1}(C) \cap \mathrm{T}(S; \bar{x}) \subseteq \mathrm{T}(\mathcal{R}; \bar{x})$$

holds true. By reasoning exactly as in the proof of Proposition 4.1, one finds

$$-\widehat{\partial}^{+}\varphi(\bar{x}) \subseteq [H^{+1}(C) \cap \mathrm{T}(S; \bar{x})]^{\ominus}.$$

Now, since $H^{+1}(C) = \bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C)$ and $\mathrm{T}(S; \bar{x})$ are closed cones satisfying the qualification condition in hypothesis (v), by what was recalled in Remark 4.3 (see, in particular, formula (4.5)), one obtains

$$-\widehat{\partial}^{+}\varphi(\bar{x}) \subseteq \Big(\bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C)\Big)^{\ominus} + (\mathrm{T}(S; \bar{x}))^{\ominus},$$

which gives inclusion (4.6) on account of the relation between the (negative) dual cone of the contingent cone and the normal cone.

Under the additional hypothesis (vi), the further qualification condition allows one to exploit once again formula (4.5), which, along with formula (4.4), leads to obtain

$$\Big(\bigcap_{\Lambda \in \mathcal{G}} \Lambda^{-1}(C)\Big)^{\ominus} = \Big(\bigcap_{i=1}^{p} \Lambda_i^{-1}(C)\Big)^{\ominus} = \sum_{i=1}^{p} (\Lambda_i^{-1}(C))^{\ominus} = \sum_{i=1}^{p} \Lambda_i^{\top}(C^{\ominus}).$$

Indeed, in this case $\Lambda_i x_0 \in \mathrm{int}\,C$ implies $x_0 \in \Lambda_i^{-1}(\mathrm{int}\,C) \subset \mathrm{int}(\Lambda_i^{-1}(C))$, so $x_0 \in \bigcap_{i=1}^{p} \mathrm{int}\,\Lambda_i^{-1}(C)$.

This completes the proof. □

Corollary 4.5. *Let $\bar{x} \in \mathcal{R}$ be a local solution to problem* (P). *Suppose the following conditions:*

(i) *F is l.s.c. in a neighborhood of $\bar{x}$.*
(ii) *F admits a strict prederivative $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x}$.*
(iii) *H is a fan finitely-generated by a set $\mathcal{G} = \mathrm{co}\{\Lambda_1, \dots, \Lambda_p\}$, with the property $\bigcap_{i=1}^p \mathrm{int}\, \Lambda_i^{-1}(C) \neq \varnothing$.*
(iv) *There exist $\eta > 0$ and $u \in \mathbb{B}$ such that $H(u) + \eta\mathbb{B} \subseteq C$.*
(v) *$\bar{x} \in \mathrm{int}\, S$.*
(vi) *φ is differentiable at $\bar{x}$.*

Then there exist $y_1, \dots, y_p \in \mathbb{R}^m$ such that

$$y_i \in C^{\ominus} \quad \textit{for all } i = 1, \dots, p$$

and

$$\nabla\varphi(\bar{x}) + \sum_{i=1}^{p} \Lambda_i^{\top}(y_i) = \mathbf{0}. \tag{4.8}$$

Proof. Since by hypothesis (v) it is $\bar{x} \in \mathrm{int}\, S$, then it holds $\mathrm{T_{Cl}}(S; \bar{x}) = \mathbb{R}^n$ (see, for instance, [2, Chapter 4.1.3]). Thus, the current hypothesis (iv) ensures that hypothesis (iv) in Theorem 4.4 is actually satisfied. Then the thesis follows immediately from Theorem 4.4 by taking into account that, in the present case, it is $\mathrm{N}(S; \bar{x}) = \{\mathbf{0}\}$ and $\widehat{\partial}^{+}\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$. □

The necessary optimality condition formulated in Corollary 4.5 might remind of a multiplier rule, with elements y_i, $i = 1, \dots, p$, playing the role of multipliers. Nevertheless, in comparison with classical Lagrangian-type optimality conditions, some substantial differences evidently emerge. Notice indeed that each y_i is a vector of $\mathbb{R}^m$, not a scalar. Besides, all terms y_i refer to the same constraint $F(x) \subseteq C$. Their number is given by the number of linear mappings needed to represent the strict outer prederivative of F at $\bar{x}$. So, it depends on the tool utilized for approximating F near $\bar{x}$, it is not an intrinsic constant of the constraint system (and hence of the problem). On the other hand, the conditions $y_i \in C^{\ominus}$, $i = 1, \dots, p$, can be regarded as a vector counterpart of a sign condition, which is typical of optimality conditions for problems with side-constraints (inequality systems and their generalizations).

As a further comment referring both, conditions (4.7) and (4.8), let us point out the computational appeal that these conditions display: such a nontrivial constraint system as (SVI) turns out to be treated, under proper assumptions, by means of linear algebra tools. This holds a fortiori whenever C is polyhedral.

Acknowledgment: The author would like to thank an anonymous referee for several relevant remarks, in particular the one concerning Proposition 2.22, which led to improve the quality of the paper.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis. A Hitchhiker's Guide*, 3rd ed., Springer, Berlin, 2006.
[2] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Mod. Birkhäuser Class., Birkhäuser, Boston, 2009.
[3] D. Azé and J.-N. Corvellec, Nonlinear local error bounds via a change of metric, *J. Fixed Point Theory Appl.* **16** (2014), no. 1–2, 351–372.
[4] D. Azé, J.-N. Corvellec and R. E. Lucchetti, Variational pairs and applications to stability in nonsmooth analysis, *Nonlinear Anal.* **49** (2002), no. 5, 643–670.
[5] H. T. Banks and M. Q. Jacobs, A differential calculus for multifunctions, *J. Math. Anal. Appl.* **29** (1970), 246–272.
[6] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization. Theory and Examples*, CMS Books Math./Ouvrages Math. SMC 3, Springer, New York, 2000.
[7] J. M. Borwein and Q. J. Zhu, *Techniques of Variational Analysis*, CMS Books Math./Ouvrages Math. SMC 20, Springer, New York, 2005.
[8] G. Bouligand, Sur les surfaces dépourvues de points hyperlimites, *Ann. Soc. Polon. Math.* **9** (1930), 32–41.
[9] M. Castellani, Error bounds for set-valued maps, *Generalized Convexity and Optimization for Economic and Financial Decisions* (Verona 1998), Pitagora, Bologna (1999), 121–135.
[10] E. De Giorgi, A. Marino and M. Tosques, Problems of evolution in metric spaces and maximal decreasing curve, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* **68** (1980), no. 3, 180–187.

[11] V. F. Demyanov and A. M. Rubinov, *Constructive Nonsmooth Analysis*, Approx. Optim. 7, Peter Lang, Frankfurt am Main, 1995.
[12] A. L. Dontchev and R. T. Rockafellar, *Implicit Functions and Solution Mappings*, 2nd ed., Springer Ser. Oper. Res. Financ. Eng., Springer, New York, 2014.
[13] M. Gaydu, M. H. Geoffroy and Y. Marcelin, Prederivatives of convex set-valued maps and applications to set optimization problems, *J. Global Optim.* **64** (2016), no. 1, 141–158.
[14] A. D. Ioffe, Nonsmooth analysis: Differential calculus of nondifferentiable mappings, *Trans. Amer. Math. Soc.* **266** (1981), no. 1, 1–56.
[15] A. G. Kusraev and S. S. Kutateladze, Calculus of tangents and beyond, *Vladikavkaz. Mat. Zh.* **19** (2017), no. 4, 27–34.
[16] L. A. Lyusternik, On the conditional extrema of functionals (in Russian), *Mat. Sb.* **41** (1934), no. 3, 390–401.
[17] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. I. Basic Theory*, Grundlehren Math. Wiss. 330, Springer, Berlin, 2006.
[18] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. II. Applications*, Grundlehren Math. Wiss. 331, Springer, Berlin, 2006.
[19] D. Pallaschke and R. Urbański, *Pairs of Compact Convex Sets. Fractional Arithmetic with Convex Sets*, Math. Appl. 548, Kluwer Academic, Dordrecht, 2002.
[20] C. H. J. Pang, Generalized differentiation with positively homogeneous maps: Applications in set-valued analysis and metric regularity, *Math. Oper. Res.* **36** (2011), no. 3, 377–397.
[21] J.-P. Penot, *Calculus Without Derivatives*, Grad. Texts in Math. 266, Springer, New York, 2013.
[22] S. M. Robinson, An implicit-function theorem for a class of nonsmooth functions, *Math. Oper. Res.* **16** (1991), no. 2, 292–309.
[23] W. Schirotzek, *Nonsmooth Analysis*, Universitext, Springer, Berlin, 2007.
[24] F. Severi, Su alcune questioni di topologia infinitesimale, *Ann. Soc. Polon. Math.* **9** (1930), 97–108.
[25] A. Uderzo, On some generalized equations with metrically *C*-increasing mappings: Solvability and error bounds with applications to optimization, *Optimization* **68** (2019), no. 1, 227–253.
[26] A. Uderzo, Solution analysis for a class of set-inclusive generalized equations: A convex analysis approach, *Pure Appl. Funct. Anal.* **5** (2020), no. 3, 769–790.