SCUOLA DI DOTTORATO
UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA

Dipartimento di Matematica e Applicazioni

Joint PhD Program in Mathematics Milano Bicocca - Pavia -INdAM

Cycle XXXVI

Curriculum in Mathematics

On a Majorana representation of the group $2 \times PSL(3,2)$

Tameni Marta

Registration number 871619

Tutor: Prof. Clara Franchi

Coordinator: Prof. Pierluigi Colli

ACADEMIC YEAR 2022/23

Declaration of Originality

I, Marta Tameni, declare that this thesis and the research to which it refers are the product of my own work except where acknowledged in accordance with the standard referencing practices of the discipline.

Copyright Declaration

Acknowledgments

I would like to start by expressing my gratitude to Professor Clara Franchi, my supervisor. Her advice and assistance have always been motivating.
I would also like to thank Professor A.A. Ivanov for the months spent in London.
Thanks to my colleagues and friends for the fun and for making life a bit easier.
Last but not least, my family has my deepest gratitude for their patience and support during the course of my studies.

Abstract

The simple group of Fischer and Griess, known as the Monster, is the largest of the sporadic groups. Its order is huge and the only construction we have knowledge of is as an automorphism group of the Griess (or Monster) algebra, a 196.884 dimensional commutative non-associative real algebra, see [23]. In this algebra there are special idempotents, known as axes (see [17]), that are in bijection with the involutions of the class 2A in the Atlas notation which generate the Monster.

In an effort to propose an axiomatic setting for the study of the Monster, A. A. Ivanov presented the notions of Majorana algebra and Majorana representation in 2009 (see [26]), taking inspiration from Sakuma's results in [53].

Finding the subalgebras of the Griess algebra generated by axes is one of the purposes of the study of Majorana algebras. As, by Norton-Sakuma's classification, Majorana algebras generated by two axes are completely known, it then makes sense to look at the classification of algebras generated by three axes or by an axis and a 3-axis. The subalgebras of the Griess algebra generated by an axis and a 3-axis were classified by Norton in [48] and then Ivanov in [36] proposed classifying these in the context of Majorana algebras as an important project. This thesis is a contribution to these two problems.

In Chapter 3, we compute the dimension of some of the algebras in the Norton list (see Tables 3.1 and 3.2) using the GAP Package "Majorana Algebras" [49].

The project of finishing the classification of Majorana algebras that admit the group $PSL(3,2)$ as an automorphism group is examined in Chapter 5. As demonstrated in [31], the group $PSL(3,2)$ has exactly two Majorana representations that satisfy axiom $2A$. Additionally, Table 5 in [41] indicates that there is at

least one Majorana representation of dimension 57 of $PSL(3,2)$ not satisfying axiom $2A$, and there may be more. We analyze the Majorana representation of the group $C_2 \times PSL(3,2)$ with shape $(2B, 3A, 4A)$, as the possibly missing algebra has this shape.

Since $C_2 \times S_4$ is one of the maximal groups in $C_2 \times PSL(3,2)$, in Chapter 4, we first focus on the Majorana representation of the group $C_2 \times S_4$ with shape $(2B, 3A)$.

Contents

List of Tables

Notation

$\mathbb{M}$	the Monster simple group
$V_{\mathbb{M}}$	the Griess or Monster algebra
m^n	the elementary abelian group of order m^n
D_{2n}	the dihedral group of order $2n$
S_n	the symmetric group of degree n
A_n	the alternating group of degree n
$L_n(p^k)$	the projective special linear group of dimension n over the field of order p^k
$GL(V)$	the general linear group of a vector space V

For two groups G and H:

$\langle X \rangle$	the smallest subspace of G containing the set $X \subseteq G$
G^n	the direct product of $n \in \mathbb{N}$ copies of G
$G \times H$	the direct product of G and H
$G.H$	an extension of G by H
$G : H$	the semidirect product of G and H
G wr H	the wreath product of G and H

For a Majorana algebra A:

$\langle X \rangle$	the smallest subspace of A containing the set $X \subseteq A$
$\langle\langle X \rangle\rangle$	the smallest subalgebra of A containing the set $X \subseteq A$
$\tau(a)$	the Majorana involution corresponding to the Majorana axis a
$A_\mu^{(a)}$	the μ-eigenspace of (the adjoint action of) the Majorana axis a

Let G be a finite group.

1	the identity element of G
$o(g)$	the order of $g \in G$
gh	the product of $g \in G$ composed with $h \in G$
$g^h := h^{-1}gh$	the conjugate of $g \in G$ by $h \in G$
g^G	the conjugacy class of g in G

Let X be a non-empty subset of G.

$\langle X \rangle$	the subgroup of G generated by X
$g^X := \{g^x : x \in X\}$	the set of conjugates of g by elements of X
$X^g := \{x^g : x \in X\}$	the set conjugate to X by g

Let Y be another non-empty subset of G.

$X^Y := \{x^y : x \in X \text{ and } y \in Y\}$	the set of elements of X conjugated by elements of Y
$C_G(X)$	the centralizer of X in G
$N_G(X)$	the normalizer of X in G

Let G act on a finite set Ω.

$\lvert\Omega\rvert$	the cardinality of Ω
ω^g	the image of $\omega \in \Omega$ under the action of $g \in G$
ω^G	the orbit of ω
Δ^g	the image of a subset $\Delta \subseteq \Omega$ under g
G_ω	the stabilizer of ω
$orb(G, \Omega)$	the number of orbits of G on Ω

Preliminaries

This chapter is a list of several basic results without any demonstrations. Refer to [38] and [55] for further information.

Let G be a finite group and let Ω be a set.

Theorem 0.1 (Orbit-Stabilizer Theorem). *Let G act on Ω. Then for any $\omega \in \Omega$*

$$|\omega^G| = \frac{|G|}{|G_\omega|}.$$

Theorem 0.2. *Let G act on finite sets Ω_1 and Ω_2. If G acts transitively on Ω_1, then the orbits of G on $\Omega_1 \times \Omega_2$ are in bijective correspondence with the orbits of G_ω on Ω_2 for any fixed $\omega \in \Omega_1$.*

Let V be a vector space over a field $F = \mathbb{R}$ or $\mathbb{C}$ and let $W \subseteq V$.

Let $(,) : V \times V \to F$ be a inner product (i.e. a symmetric positive definite bilinear form) on V.

Theorem 0.3 (Gram Determinant). *Let $S := \{v_i : 1 \leq i \leq n\}$ be a finite set of vectors in V, and let Gr_S be the Gram matrix of S (i.e. the matrix where the (i, j)-entry is the value of the inner product (v_i, v_j)). Then the determinant of Gr_S is non-zero if and only if S is linearly independent.*

If $\cdot : V \times V \to V$ is a bilinear map, then $(V, \cdot)$ is an algebra and V is said to have an algebra product $\cdot$.

We write $U = \langle\langle W \rangle\rangle$ if U is the smallest subalgebra of V containing $W \subseteq V$, i.e. U is the smallest subspace of V which is closed under the algebra product and it contains W.

Motivation, background and main results

One of the great mathematical achievements of the twentieth century was the classification of finite simple groups.
The classification consists of an explicit list of simple groups together with the proof that every finite simple group is isomorphic to some member of the list.
The list is as follows:

- Groups of prime order (abelian simple groups)
- Alternating groups
- Finite groups of Lie type
- Sporadic groups

Every group in the first three classes is included in one or more infinite families of finite simple groups. The fourth class consists of 26 finite simple groups which are not members of any of the previous families. Sporadic groups are by far the most interesting but also the most difficult to construct among the finite simple groups.

Notation	Name	Order
M_{11}	Mathieu	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
M_{12}		$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}		$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
M_{23}		$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{24}		$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
J_1	Janko	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
$J_2 = HJ$	Hall-Janko	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
$J_3 = HJM$	Higman-Janko-McKay	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
J_4	Janko	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
HS	Higman-Sims	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
Mc	McLaughlin	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Sz	Suzuki	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
$Ly = LyS$	Lyons-Sims	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
$He = HHM$	Held-Higman-McKay	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Ru	Rudvalis	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
$O'N = O'NS$	O'Nan-Sims	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
$Co_3 = \cdot 3$	Conway	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
$Co_2 = \cdot 2$		$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
$Co_1 = \cdot 1$		$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
$M(22) = F_{22}$	Fischer	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
$M(23) = F_{23}$		$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
$M(24) = F_{24}$		$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
$F_3 = E$	Thompson	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
$F_5 = D$	Harada	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
$F_2 = B$	Baby Monster	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
$F_1 = \mathbb{M}$	Monster	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

Table 1: Sporadic groups

The largest of the sporadic groups is the simple group of Fischer and Griess, called the Monster, and twenty of the others are its subgroups or quotients of subgroups. The fact that the Monster has connections to other parts of mathematics and to theories in physics (like string theory) shows that there is something very deep to discover and makes the Monster a very mysterious and fascinating object.
The study of the Monster is difficult, first of all because its order is very huge:

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \times 10^{53}$$

and secondly because the only construction we know is as an automorphism group of a 196.884[1] dimensional commutative, non-associative, real algebra known as the Griess or Monster algebra $V_{\mathbb{M}}$, see [23] and [8]. $V_{\mathbb{M}}$ is isomorphic (as a $R[\mathbb{M}]$-module) to the direct sum of the trivial 1-dimensional module $V_{1_{\mathbb{M}}}$ and the smallest non-trivial complex module of the Monster $V_{2_{\mathbb{M}}}$:

$$V_{\mathbb{M}} \cong V_{1_{\mathbb{M}}} \oplus V_{2_{\mathbb{M}}}.$$

Additionally, the Monster algebra has an inner product $(,)_{\mathbb{M}}$ that is $\mathbb{M}$-invariant.

The Monster has 194 conjugacy classes denoted NX where N is the order of elements in the conjugacy class and $X \in \{A, B, C, ...\}$.
If there is more than one conjugacy class of elements of a certain order in $\mathbb{M}$, then the alphabetic order of X, depending on the orders of the centralizers, differentiates the conjugacy classes.
In particular, in [9] we see that $\mathbb{M}$ has two conjugacy classes of involutions denoted $2A$ and $2B$ where $2A$ is the class of involutions whose centralizer is the double cover of the Baby-Monster F_2.
The product of any two involutions in the class $2A$ has order at most 6, more precisely, the conjugacy class of this product is one of the nine classes

$$1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, \text{ and } 6A,$$

in particular the $2A$ involutions are 6-transpositions.
Moreover the $2A$ involutions generate $\mathbb{M}$ making it a 6-transposition group.

[1] The 196883 dimensional module has an algebra structure too but we consider the algebra with the identity element.

J. McKay in 1978 found that the Griess algebra's dimension, 196,884, is equal to the first coefficient of the q-expansion of the elliptic modular function j. This relation suggested the "Monstrous Moonshine" conjecture:
there is an infinite-dimensional graded vector space $V = \bigoplus_{i=0}^{\infty} V_i$ where each V_i carries a finite-dimensional representation of the Monster and the dimensions of these representations relate to the coefficients of the j-function.
Borcherds in [1] demonstrated the monstrous moonshine conjecture by identifying V with the moonshine module $V^\natural$ which is an important example of vertex operator algebra (VOAs) and it has the Monster as its automorphism group.

However, the origins of vertex operators can be found in physics, specifically in string theory, rather than mathematics. Mathematicians became interested in them because of their connections to Monstrous moonshine.
VOAs, on the other hand, do not simplify the study of the Monster because they are extremely complicated.

In both the construction of Conway and the VOA approach to the Monster, a crucial role is played by the $2A$-involutions. Conway showed that to each involution $2A$ in the Monster, there corresponds an idempotent (called axis) in the Griess algebra $V_{\mathbb{M}}$. In 1996 Norton classified the subalgebras of $V_{\mathbb{M}}$ generated by two axes (see [48]) and proved that there are exactly nine such algebras, up to isomorphism, corresponding to the nine conjugacy classes of dihedral subgroups of the Monster generated by $2A$ involutions. These algebras are denoted by the corresponding conjugacy class of the product of the two involutions in $\mathbb{M}$, namely, $1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A$.

In 2007, S. Sakuma, obtained the same classification in the more general context of VOA's. For this reason, the nine dihedral algebras found by Norton are now referred to as Norton-Sakuma algebras.

Inspired by Sakuma's result and methods, in 2009 A. A. Ivanov introduced the concepts of Majorana algebra and Majorana representation, as an attempt to create an axiomatic setting for the study of the Monster, see [26].

In [27], the classification of Norton-Sakuma algebras was obtained within the axiomatics of Majorana algebras, thus showing the power of the new theory.

The concept of Majorana algebras was further extended by Hall, Rehren and Shpectorov, who defined axial algebras [24].

Interest and results in this area increased rapidly. In particular, Majorana representations of several finite groups have been constructed, and consequently various Majorana algebras (see [4] ,[5] ,[7] ,[10], [12], [13], [17], [19], [22], [27], [28], [29], [30], [31], [32], [35], [41], [43], [45], [54], [56], [57], [58]).

It turns out that just a few of the known Majorana algebras exist independently (see [29] and [58]), in fact, in general they show a remarkable tendency to be embedded in the Griess algebras. Indeed, A.A. Ivanov proposed in [32] the following conjecture:

Conjecture 0.4 (Straight Flush Conjecture). *Suppose A is an indecomposable Majorana algebra containing a Norton-Sakuma subalgebra NA, for every $N \in \{1, 2, 3, 4, 5, 6\}$. Then A embeds into the Monster algebra.*

If confirmed, the conjecture would support the belief that Majorana theory is an essential tool for understanding the Monster and the Griess algebra and it would designate the Griess algebra as the universal object in the class of the Majorana algebras.

One of the goals of the study of Majorana algebras is to identify the subalgebras of the Griess algebra generated by axes. Then it makes sense to think about the classification of algebras generated by three axes since Norton-Sakuma's classification of Majorana algebras generated by two axes is complete.

In this direction there are results by Whybrow in [56], by Mamontov, Staroletov, Whybrow in [43], by McInroy and Shpectorov in [45], by Khasraw, McInroy and Shpectorov in [41] and by Ivanov in [35].

Other interesting Majorana algebras are the algebras generated by an axis and a 3-axis (see Chapter 1 for the definition of 3-axis). Norton in [48] classified those contained in the Griess algebra and Ivanov in [36] proposes their study in the context of Majorana algebras as an important project.

This thesis is a contribution to these two problems. Namely, in Chapter 3, using the GAP Package "Majorana Algebras" [49] we determine the dimension of some of the algebras in the Norton list (see Table 3.1 and Table 3.2).

In Chapter 5 we address the problem of completing the classification of Majorana algebras which admit the group $PSL(3,2) \cong PSL(2,7)$ as automorphism group. In fact, in [31] it is shown that the group $PSL(3,2)$ has exactly two Majorana representations satisfying axiom $2A$ (see 2.2 for the definition), and, according to Table 5 in [41], there is at least one faithful Majorana representation of dimension 57 not satisfying axiom $2A$ and there could be one more. Since the possibly missing algebra has shape $(2B, 3A, 4A)$, we study a Majorana representation $\mathcal{R}$ of the group $2 \times PSL(3,2)$ with shape $(2B, 3A, 4A)$.

Note that two computer programs which construct Majorana algebras are available, one for GAP (GAP Package "MajoranaAlgebras" [49]) and one for MAGMA [44]. None of them concludes the computation of the Majorana representation $\mathcal{R}$ in a reasonable time. This clearly suggests that the task of determining $\mathcal{R}$ is not easy at all. On the other hand, a Majorana algebra affording the representation of the group $2 \times PSL(3,2)$ with shape $(2B, 3A, 4A)$ can be found inside the saturated Majorana representation of A_{12} [19] and has dimension 80 [20].

The main result of our work is the following:

Theorem 0.5. *Let A be a Majorana algebra affording a Majorana representation of the group $2 \times PSL(3,2)$ with shape $(2B, 3A, 4A)$. Then A contains a subspace W, containing the 2-closure of A, of dimension 80. Moreover, the inner product on W is uniquely determined.*

To prove our result, in Chapter 4, we first completely describe the Majorana representation of the group $C_2 \times S_4$ with shape $(2B, 3A)$, since it is one of the maximal groups in $C_2 \times PSL(3, 2)$. This representation has dimension 26 and it is well known to exist. Nevertheless, we couldn't find an explicit description of it in literature and moreover, extracting information from the construction obtained with the GAP Package "Majorana Algebras" is quite complicated.

We obtain that the inner product on W is uniquely determined, as stated in the Theorem 0.5, but when we look at the algebra products, things become much more complex. We don't have knowledge of large 2-closed subalgebras of A generated by 2-axes, in fact, the maximal subgroups of $2 \times PSL(3, 2)$ generated by involutions are only those that are isomorphic to $2 \times S_4$, and according to Chapter 4's results, the subalgebras corresponding to these subgroups are not 2-closed.

The next step might be to add vectors for each maximal subgroup of $2 \times PSL(3, 2)$ that is isomorphic to $2 \times S_4$ and use standard techniques to find relations. Unfortunately, there are many unknown products in every equation we found, making it unclear whether the system can be solved or not. A different strategy would be to find every inner product involving every element in the vector space W and each of the new vectors. This would allow us to determine the dimension of the resulting subspace and check whether the new vectors belong to W (as conjectured 5.20) or not. However, this work is expected to require some algebra products, therefore it's not an easy task.

Chapter 1

AXIAL ALGEBRAS

Let $\mathbb{F}$ be a field and let A be a (non-associative) commutative $\mathbb{F}$-algebra.
For $a \in A$, let

$$ad_a : \begin{cases} A \to A \\ v \mapsto av \end{cases}$$

be the adjoint map of a.

For $\lambda \in \mathbb{F}$ let $A_\lambda(a) = \{v \in A : av = \lambda v\}$ be the λ-eigenspace of ad_a.
Note that $A_\lambda(a) = 0$ if λ is not an eigenvalue of ad_a.

For $\Lambda \subseteq \mathbb{F}$ we define $A_\Lambda(a) = \bigoplus_{\lambda \in \Lambda} A_\lambda(a)$.

Remark 1.1. In the sequel, in order to simplify the notation, if $v \in A_\lambda(a)$ we will just say that v is a λ-eigenvector of a, or an a-eigenvector.

***Definition* 1.2.** *A **fusion law** is a pair $(F, *)$ where F is a set and $* : F \times F \to 2^F$ is a symmetric binary operation, where 2^F denotes the power set of F.*

For example, given a group G, define $(F, *)$ as:

$$F = G \text{ and } g * h = \{gh\}.$$

This is what we call a group fusion law.

The elements of F can be used to build a matrix whose rows and columns are indexed by those same elements, and where the entry for row a and column b is $a * b$. This is known as the Pythagorean table of $*$.

***Definition* 1.3.** *Let A be a commutative non-associative algebra over $\mathbb{F}$ and let F be a fusion law with $1 \in F \subseteq \mathbb{F}$. An F-**axis** $a \in A$ is a non-zero idempotent (i.e. $a^2 = a$) such that*

- $A = A_F(a) = \bigoplus_{\lambda \in F} A_\lambda(a)$
- *the above decomposition respects the fusion law F in the sense that* $A_\lambda(a)A_\mu(a) \subseteq A_{\lambda * \mu}(a) = \bigoplus_{\gamma \in \lambda * \mu} A_\gamma(a)$ *for all $\lambda, \mu \in F$.*

Since a is an idempotent, 1 is always an eigenvalue of ad_a, therefore from now on we assume $1 \in F$.

***Definition* 1.4.** *An F-axis $a \in A$ is called **primitive** if $A_1(a) = \mathbb{F}a$.*

***Definition* 1.5.** *Suppose F is a fusion law with $1 \in F \subseteq \mathbb{F}$.*
*An F-**axial algebra** is a pair (A, X), where A is a commutative (non-associative) $\mathbb{F}$-algebra and X is a set of F-axes generating A (i.e. $A = \langle\langle X \rangle\rangle$).*

***Definition* 1.6.** *An F-axial algebra (A, X) is called **primitive** if every element of X is a primitive axis.*

***Definition* 1.7.** *An F-axial algebra (A, X) is called **dihedral** if $|X| = 2$.*

***Definition* 1.8.** *A **Frobenius form** on an axial algebra (A, X) is a (non-zero) bilinear form $(\cdot, \cdot) : A \times A \to \mathbb{F}$ which associates with the algebra product, in the sense that $\forall u, v, w \in A$:*

$$(uv, w) = (u, vw).$$

The usual fact that eigenvectors relative to different eigenvalues are orthogonal holds also in this setting.

Lemma 1.9. *Let A be an F-axial algebra with a Frobenius form $(,)$, let a be an F-axis and let $\lambda, \mu \in \mathbb{F}$. If $u, v \in A$ are eigenvectors for ad_a, relative to the eigenvalues λ and μ respectively with $\lambda \neq \mu$, then $(u, v) = 0$.*

Proof. Since u is a λ-eigenvector of a then $au = \lambda u$ and similarly $av = \mu v$.
Then $\lambda(u, v) = (\lambda u, v) = (au, v) = (u, av) = (u, \mu v) = \mu(u, v)$.
We know that $\lambda \neq \mu$ hence (u, v) must be 0.

□

The fusion law provides much of the structure of axial algebras and play an important role in their construction.

We give now some important examples of axial algebras and their corresponding fusion laws.

First we see a case with only eigenvalues 0 and 1.

1.1. AXIAL ALGEBRAS WITH FUSION LAW $\mathcal{A}$

In this case the axial algebras satisfy the fusion law $\mathcal{A}$:

$*$	1	0
1	1	
0		0

Note that instead of writing the empty set, we just leave a space.

Usually we omit the set brackets in these tables and just write, for example, 1 for the set $\{1\}$ and $1, 0$ for the set $\{1, 0\}$ which means the direct sum of the 1-eigenspace and the 0-eigenspace.

Associative algebras are examples of this kind of algebras. In fact, if A is an associative algebra and $a \in A$ is an idempotent ($a \cdot a = a$), then for every $v \in A$,

$$a \cdot (a \cdot v) = (a \cdot a) \cdot v = a \cdot v.$$

But also, if v is an eigenvector with eigenvalue λ, then

$$\lambda v = a \cdot v = a \cdot (a \cdot v) = a \cdot (\lambda v) = \lambda^2 v.$$

Therefore $\lambda^2 = \lambda$ and so λ can be only 0 or 1.

Now let v, w be 0-eigenvectors of a and let z be a 1-eigenvector of a, then:

$$a \cdot (v \cdot w) = (a \cdot v) \cdot w = (0v) \cdot w = 0$$

$$a \cdot (w \cdot v) = (a \cdot w) \cdot v = (0w) \cdot v = 0$$

Then the product of two 0-eigenvectors of a is again a 0-eigenvector of a.

$$v \cdot z = v \cdot (a \cdot z) = (v \cdot a) \cdot z = 0z = 0$$

$$z \cdot v = (z \cdot a) \cdot v = z \cdot (a \cdot v) = z0 = 0$$

Then the product of a 0-eigenvector and a 1-eigenvector of a is always 0, hence A satisfies the fusion law $\mathcal{A}$.

Indeed, $\mathcal{A}$-axial algebras are all associative algebras and are isomorphic to a direct sum of copies of the field (see [25]).

1.2. AXIAL ALGEBRAS OF JORDAN TYPE η

***Definition* 1.10.** *A primitive axial algebra over* $\mathbb{F}$ *is said to be of **Jordan type** $\eta \in \mathbb{F}$ if it satisfies the fusion law $J(\eta)$ with $\eta \neq 0, 1$.*

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1, 0$

Axial algebras of Jordan type have been defined by Hall, Rehren, and Shpectorov in [24]. The name comes from their link to Jordan algebras, non-associative algebras introduced by Pascual Jordan in [39] to study observables in quantum mechanics with this definition:

***Definition* 1.11.** *A **Jordan algebra** is a commutative non-associative algebra which satisfies $(xy)(xx) = x(y(xx))$ for all x and y.*

It follows at once from the Peirce decomposition of Jordan algebras (see [37]) that Jordan algebras generated by idempotents are examples of axial algebras for the fusion law $J(\eta)$ with $\eta = \frac{1}{2}$.

1.3. AXIAL ALGEBRAS OF MONSTER TYPE (α, β)

Definition* 1.12.** *An axial algebra over $\mathbb{F}$ is said to be of* ***Monster type (α, β) *if it satisfies the fusion law $M(\alpha, \beta)$ with $\alpha, \beta \in \mathbb{F} \backslash \{0, 1\}$ and $\alpha \neq \beta$.*

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1 \quad 0$	β
β	β	β	β	$1 \quad 0 \quad \alpha$

Axial algebras of Monster type (α, β) have been introduced by Rehren in [51].

The Griess algebra is a real axial algebra with fusion law $M(\alpha, \beta)$ with $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{32}$ as demonstrated in [48].

1.4. MAJORANA ALGEBRAS

Definition* 1.13.** ***Majorana algebras *are primitive real axial algebras of Monster type $M(\frac{1}{4}, \frac{1}{32})$ with a positive definite inner product $(,)$.*

Moreover it is also required that $(,)$ satisfies the Norton inequality:
for every $u, v \in A$

$$(uu, vv) \geq (uv, uv).$$

The Monster fusion law $M(\frac{1}{4}, \frac{1}{32})$ is also called **Ising fusion law** and the idempotents in X generating the algebra are called **Majorana axes** (or simply axes).

Since the Griess algebra has a positive definite Frobenius form which also satisfies Norton inequality, it is a real Majorana algebra.

Definition* 1.14.** *Two Majorana algebras (A_1, X_1) and (A_2, X_2) are* ***isomorphic *if there exists a linear map ϕ from A_1 to A_2 which preserves the inner and* *the algebra products and which induces a bijection from the Majorana axes of A_1 to the Majorana axes of A_2. If $A_1 = A_2$ then ϕ is an **automorphism** of A_1.*

Note that an automorphism of the vector space and of the algebra A defined as above is an isometry of a Majorana algebra A.
Clearly, the set of automorphisms of a Majorana algebra is a group and it will be denoted by $Aut(A)$.

For a Majorana algebra A the Ising fusion law implies that the setting

$$A_+ := A_0 \oplus A_1 \oplus A_{\frac{1}{4}} \text{ and } A_- := A_{\frac{1}{32}}$$

is a $\mathbb{Z}_2$-grading on A.

Miyamoto noticed (in the context of VOA's) that a $\mathbb{Z}_2$-grading of the algebra naturally 'produces' automorphisms of order 2.

The map τ_a, for each axis a, that negates every element of its $1/32$-eigenspace and fixes each element of the other eigenspaces is an involutory isometry of the algebra A.

***Definition* 1.15.** *The automorphism τ_a is called the **Miyamoto involution** associated to the axis a.*

***Definition* 1.16.** *Let $k \in \mathbb{N}$, for $Y \subseteq X$, the subalgebra $\langle\langle Y \rangle\rangle$ is **k-closed** (with respect to Y) if it is the linear span of k-long products, i.e. $\langle\langle Y \rangle\rangle = \langle y_1 \cdot y_2 \cdots y_k | y_i \in Y \rangle$ where $y_1 \cdot y_2 \cdots y_k$ denotes all possible bracketing of k vectors.*

For example, $A = \langle\langle X \rangle\rangle$ is 2-closed if $A = \langle a_i \cdot a_j | a_i, a_j \in X \rangle$.

We point out that a Majorana algebra could have infinite dimension. However, no non-trivial examples of such an algebra are known to exist. In fact, almost all the known examples of Majorana algebras are equal to the linear span of a finite set of elements and they are at most 3-closed.

We close this section with some lemmas which will be often used in the sequel.

Recall that we said that two elements $u, v \in A$ associate if

$$\forall w \in A: \; v \cdot (w \cdot u) = (v \cdot w) \cdot u.$$

Lemma 1.17. *[27, Lemma 1.10] A Majorana axis a associates with every element in $A_0(a)$.*

Proof. Let w be a 0-eigenvector for a and v a μ-eigenvector of a.
On the one hand we have

$$(a \cdot v) \cdot w = (\mu v) \cdot w = \mu(v \cdot w).$$

On the other hand, by the fusion rules, $v \cdot w$ is also a μ-eigenvector for a and so

$$a \cdot (v \cdot w) = \mu(v \cdot w).$$

□

Lemma 1.18. *Let A be an axial algebra of Monster type (α, β) with a Frobenius form $(\ ,\ )$, let a be an axis and $w \in A_+(a)$. Then*

$$\frac{1}{\alpha}(a \cdot w - (a, w)a)$$

is an α-eigenvector for a and

$$w - (a, w)a - \frac{1}{\alpha} a \cdot (w - (a, w)a)$$

is a 0-eigenvector for a.

Proof. This follows from formulas (3.6) and (3.7) in the article [52].

□

The resurrection principle is a significant result of the fusion rules. It gets its name from the fact that the vector v vanishes at the beginning of the proof before reappearing at the conclusion.

Proposition 1.19. *(The Resurrection Principle) [27] Let A be a Majorana algebra and let $a \in A$ be a fixed Majorana axis. Let W be an a-stable subspace of A (i.e. a subspace such that $a \cdot w \in W$ for all $w \in W$). For $v \in A$ suppose that*

$$\alpha_v = v + w_\alpha \in A_0(a) \text{ and } \beta_v = v + w_\beta \in A_{\frac{1}{4}}(a)$$

for some $w_\alpha, w_\beta \in W$. Then

$$v = -[4a \cdot (w_\alpha - w_\beta) + w_\beta]$$

in particular $v \in W$.

1.5. NORTON-SAKUMA ALGEBRAS

Norton in [48] studied the dihedral subalgebras of the Griess algebra and discovered nine different isomorphism classes of these algebras, which are known as Norton-Sakuma algebras.

The isomorphism class of the algebra generated by the axes a and b depends only on the conjugacy class in the Monster of the product $\tau_a\tau_b$ of the two Miyamoto involutions associated to the generating axes a and b.

The same nine algebras were obtained by Sakuma [53] in the context of VOA's and Ivanov, Pasechnik, Seress, and Shpectorov proved in [27] that every dihedral Majorana algebra is isomorphic to one of those nine.

Hence, we use the names of the conjugacy classes in the Monster to identify the isomorphism classes.

Their complete classification is known under the name Norton-Sakuma Theorem:

Theorem 1.20 ([53],[27])**.** *There are nine isomorphism classes of dihedral Majorana algebras labelled:* $1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A$ *and* $6A$.

This result has been generalized to dihedral algebras of Monster type $(\frac{1}{4}, \frac{1}{32})$ over a field of characterisic zero.

Theorem 1.21 ([24],[33])**.** *Let char*$(\mathbb{F}) = 0$ *and* A *be a primitive dihedral axial algebra of Monster type* $(\frac{1}{4}, \frac{1}{32})$*. Then,* A *is isomorphic to one of the nine algebras* $1A$*,* $2A$*,* $2B$*,* $3A$*,* $3C$*,* $4A$*,* $4B$*,* $5A$ *and* $6A$ *given in Table 1.1.*

This result, in particular, confirms the well known fact that the Monster is a 6-transposition group with respect to its 2A-involutions, meaning that the Monster is generated by the 2A conjugacy class and that the product of any two 2A-involutions has a maximum order of 6.

In Table 1.1 we list the nine Norton-Sakuma algebras. For each of them we give, as usual, a basis and the basic algebra and inner products. All the other products can be obtained using the symmetries of the algebra.

The two generating axes are labelled a_0 and a_1, $\rho := \tau_{a_0}\tau_{a_1}$ and for $i \in \mathbb{Z}$, $\varepsilon \in \{0,1\}$,

$$a_{2i+\varepsilon} := (a_\varepsilon)^{\rho^i}.$$

If ρ (resp. ρ^2 or ρ^3) happens to be a Miyamoto involution, its associated axis is a_ρ (resp. a_{ρ^2} or a_{ρ^3}).

In addition, the vectors u_ρ, v_ρ, w_ρ and u_{ρ^2} that appear in algebras of type NA, for $N \in \{3,4,5,6\}$ are particular vectors needed to complete the basis. These vectors are referred to as odd axes, or to be more precise, 3-axis for $N = 6$ and N-axes for $N < 6$.

Note that, in the Griess algebra and the Monster, odd axes depend only on the cyclic subgroup $\langle\rho\rangle$.

Type	Basis	Structure constants	Scalar products
$1A$	a_0	$a_0 \cdot a_0 = a_0$	$(a_0, a_0) = 1$
$2A$	a_0, a_1, a_ρ	$a_0 \cdot a_1 = \frac{1}{2^3}(a_0 + a_1 - a_\rho)$, $a_0 \cdot a_\rho = \frac{1}{2^3}(a_0 + a_\rho - a_1)$ $a_\rho \cdot a_\rho = a_\rho$	$(a_0, a_1) = \frac{1}{2^3}$ $(a_0, a_\rho) = \frac{1}{2^3}$ $(a_1, a_\rho) = \frac{1}{2^3}$
$2B$	a_0, a_1	$a_0 \cdot a_1 = 0$	$(a_0, a_1) = 0$
$3A$	a_{-1}, a_0, a_1, u_ρ	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}} u_\rho$ $a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_\rho$ $u_\rho \cdot u_\rho = u_\rho$	$(a_0, a_1) = \frac{13}{2^8}$, $(a_0, u_\rho) = \frac{1}{4}$, $(u_\rho, u_\rho) = \frac{2^3}{5}$
$3C$	a_{-1}, a_0, a_1	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1})$	$(a_0, a_1) = \frac{1}{2^6}$
$4A$	a_{-1}, a_0, a_1, a_2, v_ρ	$a_0 \cdot a_1 = \frac{1}{2^6}(3a_0 + 3a_1 + a_{-1} + a_2 - 3v_\rho)$ $a_0 \cdot a_2 = 0$ $a_0 \cdot v_\rho = \frac{1}{2^4}(5a_0 - 2a_1 - 2a_{-1} - a_2 + 3v_\rho)$ $v_\rho \cdot v_\rho = v_\rho$	$(a_0, a_1) = \frac{1}{2^5}$ $(a_0, a_2) = 0$ $(a_0, v_\rho) = \frac{3}{2^3}$ $(v_\rho, v_\rho) = 2$
$4B$	a_{-1}, a_0, a_1, a_2 a_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho^2})$ $a_0 \cdot a_2 = \frac{1}{2^3}(a_0 + a_2 - a_{\rho^2})$	$(a_0, a_1) = \frac{1}{2^6}$ $(a_0, a_2) = \frac{1}{2^3}$ $(a_0, a_{\rho^2}) = \frac{1}{2^3}$
$5A$	a_{-2}, a_{-1}, a_0, a_1, a_2, w_ρ	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_{-1} - a_2 - a_{-2}) + w_\rho$ $a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$ $a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_\rho$ $w_\rho \cdot w_\rho = \frac{5^2 \cdot 7}{2^{19}}(a_0 + a_1 + a_{-1} + a_2 + a_{-2})$	$(a_0, a_1) = \frac{3}{2^7}$ $(a_0, w_\rho) = 0$ $(w_\rho, w_\rho) = \frac{5^3 \cdot 7}{2^{19}}$
$6A$	a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, a_{ρ^3}, u_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 - a_{-2} - a_3 + a_{\rho^3}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3})$ $a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5} u_{\rho^2}$ $a_{\rho^3} \cdot u_{\rho^2} = 0$	$(a_0, a_1) = \frac{5}{2^8}$ $(a_0, a_2) = \frac{13}{2^8}$ $(a_0, a_3) = \frac{1}{2^3}$ $(a_{\rho^3}, u_{\rho^2}) = 0$

Table 1.1: Norton-Sakuma algebras.

Lemma 1.22. *Let U be an algebra of type NX that is generated by Majorana axes a_0 and a_1. Then*

- *if U is of type $4A$, $4B$ or $6A$ then the subalgebra generated by a_0 and a_2 is of type $2B$, $2A$ or $3A$ respectively;*
- *if U is of type $6A$ then the subalgebra generated by a_0 and a_3 is of type $2A$.*

We end this chapter by listing in Table 1.2 a basis of eigenvectors of a_0 for each of the Norton-Sakuma algebras. We will extensively make use of such eigenvectors in Chapters 4 and 5.

Type	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
$2A$	$a_1 + a_{\rho(t_0,t_1)} - \frac{1}{2^2}$	$a_1 - a_{\rho(t_0,t_1)}$	
$2B$	a_1		
$3A$	$u_{\rho(t_0,t_1)} + \frac{2\cdot 5}{3^3}a_0 + \frac{2^5}{3^3}(a_1 + a_{-1})$	$u_{\rho(t_0,t_1)} - \frac{2^3}{3^2\cdot 5}a_0$ $- \frac{2^5}{3^2\cdot 5}(a_1 + a_{-1})$	$a_1 - a_{-1}$
$3C$	$a_1 + a_{-1} - \frac{1}{2^5}a_0$		$a_1 - a_{-1}$
$4A$	$v_{\rho(t_0,t_1)} - \frac{1}{2}a_0 + 2(a_1 + a_{-1}), a_2$	$v_{\rho(t_0,t_1)} - \frac{1}{3}a_0$ $- \frac{2}{3}(a_1 + a_{-1}) - \frac{1}{3}a_2$	$a_1 - a_{-1}$
$4B$	$a_1 + a_{-1} - \frac{1}{2^5}a_0 - \frac{1}{2^3}(a_{\rho(t_0,t_2)} - a_2),$ $a_2 + a_{\rho(t_0,t_2)} - \frac{1}{2^2}a_0$	$a_2 - a_{\rho(t_0,t_2)}$	$a_1 - a_{-1}$
$5A$	$w_{\rho(t_0,t_1)} + \frac{3}{2^9}a_0 - \frac{3\cdot 5}{2^7}(a_1 + a_{-1}) - \frac{1}{2^7}(a_2 - a_{-2}),$ $w_{\rho(t_0,t_1)} - \frac{3}{2^9}a_0 + \frac{1}{2^7}(a_1 + a_{-1}) + \frac{3\cdot 5}{2^7}(a_2 + a_{-2})$	$w_{\rho(t_0,t_1)} + \frac{1}{2^7}(a_1 + a_{-1})$ $- \frac{1}{2^7}(a_2 + a_{-2})$	$a_1 - a_{-1},$ $a_2 - a_{-2}$
$6A$	$u_{\rho(t_0,t_2)} + \frac{2}{3^2\cdot 5}a_0 - \frac{2^8}{3^2\cdot 5}(a_1 - a_{-1})$ $- \frac{2^5}{3^2\cdot 5}(a_2 + a_{-2} + a_3 - a_{\rho(t_0,t_3)}),$ $a_3 + a_{\rho(t_0,t_3)} - \frac{1}{2^2}a_0,$ $u_{\rho(t_0,t_2)} - \frac{2\cdot 5}{3^3}a_0 + \frac{2^5}{3^3}(a_2 + a_{-2})$	$u_{\rho(t_0,t_2)} - \frac{2^3}{3^2\cdot 5}a_0$ $- \frac{2^5}{3^2\cdot 5}(a_2 + a_{-2} + a_3)$ $+ \frac{2^5}{3^2\cdot 5}a_{\rho(t_0,t_3)},$ $a_3 - a_{\rho(t_0,t_3)}$	$a_1 - a_{-1},$ $a_2 - a_{-2}$

Table 1.2: The eigenspace decomposition of the dihedral Majorana algebras.

Chapter 2

MAJORANA REPRESENTATIONS

***Definition* 2.1.** *A **Majorana representation** (G,T,A,ψ,ϕ) of G is defined as follows:*

- *G is a group*
- *T is a G-invariant set of involutions generating G (the Majorana set)*
- *A is a Majorana algebra*
- *ψ is an injective map from T to the set of axes of A*
- *$\phi : G \to Aut(A)$ is a group homomorphism*

such that:

1) T^{ψ} generates A as an algebra

2) for every $t \in T$ and every $g \in G$, $(t^g)^{\psi} = (t^{\psi})^{g^{\phi}}$

3) for every $t \in T$, t^{ϕ} is the Miyamoto involution associated to the axis t^{ψ}.

There are also some very natural extra conditions that are known to hold in the Griess algebra, and then they are assumed in almost all papers on Majorana theory that have been published (see [3]).

***Definition* 2.2.** *Let $t_1, t_2, t_3, t_4 \in T$ with corresponding Majorana axes $a_1, a_2, a_3, a_4 \in A$, respectively. If the following conditions hold:*

- *$t_1t_2 \in T$ if and only if $\langle\langle a_1, a_2\rangle\rangle$ has type $2A$;*
- *if $t_1t_2 = t_3$, then a_3 coincides with $a_{t_1t_2}$, i.e. $a_3 = a_1 + a_2 - 8a_1 \cdot a_2$*

then the Majorana representation is said to satisfy the $2A$***-condition****.*

If $\langle t_1 t_2 \rangle = \langle t_3 t_4 \rangle$*, both* $\langle\langle a_1, a_2 \rangle\rangle$ *and* $\langle\langle a_3, a_4 \rangle\rangle$ *are of type* $3A$*,* $4A$ *or* $5A$*, and* $u_{t_1 t_2} = u_{t_3 t_4}$*,* $v_{t_1 t_2} = v_{t_3 t_4}$ *or* $w_{t_1 t_2} = \pm w_{t_3 t_4}$ *respectively, then the Majorana representation is said to satisfy the* $3A$***-,*** $4A$***- or*** $5A$***-condition*** *respectively.*

The Majorana representation of the Monster on the Griess algebra satisfies all conditions of Definition 2.2. On the other hand, A.A. Ivanov and A. Seress in [30] have shown that the 2A-condition cannot be deduced from the other axioms and show an example of Majorana representation that does not satisfy this condition.

Note that, by the Norton-Sakuma theorem, the Majorana set must be a set of 6-transpositions.
In general, there are various options for selecting the Majorana set T if the group G has distinct conjugacy classes of 6-transpositions.
If the corresponding Majorana set is maximal (by inclusion), we refer to a Majorana representation as saturated.
The next result follows immediately from the definition of a Majorana representation (as stated in [19]).

Lemma 2.3. *If* T_0 *is a nonempty subset of* T *such that* T_0 *is* $\langle T_0 \rangle$*-invariant, then* $\phi_{|\langle T_0 \rangle}$ *is again a Majorana representation of* $\langle T_0 \rangle$ *on the subalgebra* $A_{\langle T_0 \rangle}$ *of* A *generated by* T_0^{ψ} *with respect to* $\psi_{|T_0}$ *.*

In general, consider H to be any finite group, T_H to be an H-invariant generating set of involutions of H and ϵ to be an embedding of H in the Monster $\mathbb{M}$ such that T_{H^ϵ} is the intersection of H^ϵ with the set of involutions of type 2A of $\mathbb{M}$. According to Lemma 2.3, the standard action of $\mathbb{M}$ on the Griess algebra therefore induces on H^ϵ (and so on H) a Majorana representation with respect to T_{H^ϵ} (resp. T_H) on the subalgebra of the Griess algebra generated by the set of axes associated to the elements of T_{H^ϵ}. It is said that the Majorana representations of H produced in this manner are based on the embedding ϵ of H in $\mathbb{M}$.

2.1. SHAPE

Definition* 2.4.** *Given a Majorana representation $\mathcal{R}$:=(G,T,A,ψ,ϕ), the* ***shape *of $\mathcal{R}$ is a function sh from the set of the nondiagonal orbitals of G on T to the set $\{2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A\}$ so that*

- *$sh((t,s)^G) = NX$ if and only if ts has order N and the algebra $\langle\langle t^\psi s^\psi\rangle\rangle$ is a Norton–Sakuma algebra of type NX .*
- *sh must respect the embeddings of the algebras:*

$$2A \hookrightarrow 4B,\ 2B \hookrightarrow 4A,\ 2A \hookrightarrow 6A \text{ and } 3A \hookrightarrow 6A.$$

in the sense that, for $t, r_1, r_2 \in T$, if $\langle\langle t^\psi\rangle\rangle < \langle\langle t^\psi, r_1^\psi\rangle\rangle < \langle\langle t^\psi, r_2^\psi\rangle\rangle$, then

$$(sh((t,r_1)^G), sh((t,r_2)^G)) \in (2A,4B),(2A,6A),(2B,4A),(3A,6A).$$

Remark 2.5. The shape is often denoted simply as a list $(N_1X_1, ..., N_nX_n)$, where N_iX_i are the types of Norton-Sakuma algebras that are present.

The following strategy became quite standard in the process of studying Majorana representations (see [57], [42], [36]).

2.2. GENERIC STRATEGY

Let G be a finite group generated by a set of involutions T, such that T is a union of conjugacy classes of G.
We consider the tuple (G, T, A, ψ, ϕ) to be the Majorana representation of the group G that we want to construct.

2.2.1 Step 1. Shape of the representation

The first step is to choose the shape of the Majorana representation (G, T, A, ψ, ϕ). To do this, representatives of the orbitals of G on $T \times T$ must be identified (for all $g \in G$ $\langle\langle a_1, a_2\rangle\rangle \cong \langle\langle a_2, a_1\rangle\rangle \cong \langle\langle a_1^{\phi(g)}, a_2^{\phi(g)}\rangle\rangle$ as $\phi(g) \in Aut(A)$). Then the potential types of dihedral algebras generated by the Majorana axes corresponding to each of these representatives must be determined.

Recall that, $\langle\langle a_1, a_2\rangle\rangle$ has type NX for some X if $|t_1t_2|= N$.

By Norton-Sakuma theorem, $N \leq 6$, and so, in particular, T must be a 6-transposition set. Moreover, the choice of the different types of dihedral algebras must respect the inclusion of algebras required by the definition of shape.

2.2.2 Step 2. Spanning set for A

Based on the shape chosen in step 1, let X be the set of all Majorana axes in A plus the NA-axes ($3 \leq N \leq 5$) identified in the Norton-Sakuma algebras generated by the pairs of Majorana axes in A.

If the algebra product of any two vectors in X is a linear combination of elements of X then X is a spanning set for A and the Majorana representation is 2-closed.

If the Majorana representation is not 2-closed one has to extend the set X of the algebra, adding, for example, the products of one axis with each of the odd axes and try again until all the algebra products of any two vectors in X stays inside the linear span of X.

2.2.3 Step 3. Identify pairs of vectors in X whose products are unknown

From the Norton-Sakuma algebras some of the inner and algebra products between vectors in X are already known.
Furthermore, the products between vectors in the subalgebra $\langle\langle\psi(T \cap H)\rangle\rangle$ are known if G has a subgroup H generated by involutions in $T \cap H$ such that the Majorana representation $(H, T \cap H, \langle\langle(T \cap H)^{\psi}\rangle\rangle, \psi_{|T\cap H}, \phi_{|H})$ has already been constructed and it is only influenced by the shape.

Under the action of $\phi(G)$, the couples with unknown products may be organised into orbits on pairs. Determining the product for one orbit representative is sufficient for deducing the products of the other couples in the orbit because $\phi(G)$ preserves the products.

2.2.4 Step 4. Compute the unknown inner products

Inner products between two Majorana axes are given by the Norton-Sakuma Theorem and Table 1.1. The other unknown inner products can be obtained combining the following two methods:

1. use the associativity of the inner product and the algebra product. If u and v are two axes (Majorana or odd axes) and $v = v_0 + \lambda a_t \cdot a_s$, where v_0 is a linear combination of Majorana axes, then

$$(u, v) = (u, v_0) + \lambda(u, a_t \cdot a_s) = (u, v_0) + \lambda(u \cdot a_t, a_s).$$

If (u, v_0) is known and we know how to express $u \cdot a_t$ as a linear combination of Majorana axes, then we are able to compute (u, v).

2. use the orthogonality between eigenvectors relative to different eigenvalues.

These conditions lead to a system of linear equations that then we solve.

2.2.5 Step 5. Compute the unknown algebra products

Like the unknown inner products, the unknown algebra products can be determined by solving a system of linear equations obtained by applying the resurrection principle or other relations between eigenvectors deduced from the fusion law.

Note that this generic strategy is not always successful as shown by an example given by Whybrow.

Chapter 3

NORTON'S LIST

The key element for creating new Majorana representations has been the classification of algebras generated by pairs of Majorana axes.
Constructing algebras generated by an axis and a 3-axis and possibly classifying them is the logical next step. In [48] Norton did it in the case of the Griess algebra: his result is summarized in Table 3.1. There are 22 of such algebras, each one corresponding to one of the orbits of $(2A, 3A)$-pairs in $\mathbb{M}$. In Table 3.1 we list the representatives for each pair in the first 18 cases, where it can be found inside a subgroup of $\mathbb{M}$ isomorphic to A_{12}. For each pair, the value of the inner product between the corresponding axes is given (note that it is rescaled respect to Norton's computation).

The first two columns of Table 3.1 provide the representatives for $(2A, 3A)$-pairs (t is a 2A-involution and h is a 3A-element).
The isomorphism class of the group generated by t and h is given in the third column $\langle t, h\rangle$.
The conjugacy class of the product th in the Monster is represented by $(th)^{\mathbb{M}}$.
The value of the inner product between the 2A-axis a_t associated with t and the 3A-axis u_h associated with h is indicated in the last column.

Representatives are shown as components of the A_{12} standard form, i.e. as even permutations of the set $\{1, ..., 12\}$, in the first 18 rows of the table.

t	h	$\langle t, h\rangle$	$(th)^{\mathbb{M}}$	(a_t, u_h)
$(1,2)(4,5)$	$(1,2,3)$	S_3	$2A$	$\frac{1}{4}$
$(4,5)(6,7)$	$(1,2,3)$	6	$6A$	0
$(1,2)(3,4)$	$(1,2,3)$	A_4	$3A$	$\frac{1}{9}$
$(1,4)(5,6)$	$(1,2,3)$	S_4	$4B$	$\frac{1}{36}$
$(1,4)(2,5)$	$(1,2,3)$	A_5	$5A$	$\frac{1}{18}$
$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$	$(1,2,3)$	$2\times A_4$	$6C$	$\frac{1}{45}$
$(1,4)(2,5)(3,6)(7,8)(9,10)(11,12)$	$(1,2,3)$	$3\times S_3$	$6A$	$\frac{1}{20}$
$(1,7)(2,4)$	$(1,2,3)(4,5,6)$	$L_2(7)$	$7A$	$\frac{1}{24}$
$(1,7)(2,8)$	$(1,2,3)(4,5,6)$	$3\times A_5$	$15A$	$\frac{11}{360}$
$(1,7)(4,8)$	$(1,2,3)(4,5,6)$	S_4	$4A$	$\frac{13}{180}$
$(1,7)(8,9)$	$(1,2,3)(4,5,6)$	$3\times S_4$	$12C$	$\frac{1}{36}$
$(1,2)(3,4)(5,7)(6,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$2\times L_2(7)$	$14A$	$\frac{11}{360}$
$(1,2)(3,7)(4,5)(6,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$2\times A_4$	$6A$	$\frac{2}{45}$
$(1,2)(3,7)(4,8)(5,9)(6,10)(11,12)$	$(1,2,3)(4,5,6)$	$S_3\times A_4$	$6B$	$\frac{17}{360}$
$(1,4)(2,7)(3,8)(5,9)(6,10)(11,12)$	$(1,2,3)(4,5,6)$	$2\times A_5$	$10A$	$\frac{1}{30}$
$(1,4)(2,8)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$3^3.A_4$	$9A$	$\frac{13}{360}$
$(1,5)(2,4)(3,7)(6,10)(8,9)(11,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$4^2.S_3$	$8B$	$\frac{1}{36}$
$(1,2)(3,4)(5,7)(6,10)(8,11)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$L_2(11)$	$11A$	$\frac{1}{30}$
†	†	$3\times S_3$	$6D$	$\frac{1}{40}$
†	†	$GL_2(3)$	$8C$	$\frac{1}{36}$
†	†	$SL_2(3):2$	$12A$	$\frac{1}{30}$
†	†	$2.4^2.S_3$	$8A$	$\frac{7}{180}$

Table 3.1: Norton list.

According to Ivanov in [36] a very important project is to describe the algebras $\langle\langle a_t, u_h\rangle\rangle$ both as subalgebras of the Monster algebra and within Majorana theory as subalgebras of 3-generated Majorana algebras.

Some partial results for example can be found in Lim's thesis ([42]).

Using the GAP free package “MajoranaAlgebras” by M. Pfeiffer and M. Whybrow (see [49]), we determined the dimension of some of these algebras.
We list the results in the Table 3.2. Here, H_0 is one of the groups in Table 3.1 generated by the 2A-involution t and a 3-element h (corresponding to a 3-axis). We find two involutions t_1 and t_2 in A_{12}, corresponding to $2A$-involutions in $\mathbb{M}$, such that $t_1t_2 = h$ and set $H = \langle t, t_1, t_2\rangle$. Thus H is a subgroup of A_{12} generated by $2A$-involutions and we can look for the Majorana representation of H with the same shape induced by an embedding of A_{12} into the Monster (see [19]). Such a shape is indicated in the fourth column of Table 3.2.

In the last case considered, when $H_0 = \langle t, h\rangle \cong GL(2, 3) \cong 2S_4$, H_0 is not embedded into A_{12}. One can check that there is a unique group $2S_5$ with a maximal subgroup isomorphic to $GL(2, 3)$ (in GAP it corresponds to SmallGroup(240,90)). Hence, we compute a Majorana representation of this group, using its permutation representation of degree 40 given by GAP.

In order to determine the dimensions of the different A_{H_0}, we first compute the Majorana representation of the corresponding group H with the given shape. Then with the function *MAJORANA_Subalgebra* we determined the dimension of the subalgebra generated by the 2-axis a_t and the 3-axis u_h in the largest algebra.

Some informations, such as the dimensions of the algebras A_H with H one of the following groups:

- S_3
- S_4
- $2 \times S_4$ with shape $(2A, 2B, 3A)$
- S_3^2 with shape $(2A, 3A)$
- $S_3 \times S_4$ with shape $(2A, 2B, 3A)$

are already known and can be found in [42].

H_0	H	Embedding	Shape	(a_t, u_h)	$dim(A_{H_0})$
S_3	S_3	$t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ $t_1 = (1,2)(3,4)(5,6)(7,10)(8,12)(9,11)$ $t_2 = (1,2)(3,4)(5,6)(7,11)(8,9)(10,12)$	(1A,3A)	$\frac{1}{4}$	3
C_6	D_{12}	$t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ $t_1 = (7,9)(8,10)$ $t_2 = (7,11)(8,12)$	(1A,2A,3A,6A)	0	2
A_4	S_4	$t = (1,2)(3,4)$ $t_1 = (3,4)(10,11)$ $t_2 = (2,3)(9,10)$	(1A,2A,3A,4B)	$\frac{1}{9}$	5
S_4	S_4	$t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ $t_1 = (1,2)(3,4)(5,6)(7,12)(8,11)(9,10)$ $t_2 = (1,2)(3,4)(5,6)(7,10)(8,12)(9,11)$	(1A,2A,3A,4B)	$\frac{1}{36}$	9
A_5	A_5	$t = (1,2)(3,4)$ $t_1 = (1,2)(5,6)$ $t_2 = (2,3)(4,6)$	(1A,2A,3A,5A)	$\frac{1}{18}$	17
$2 \times A_4$	$2 \times S_4$	$t = (1,2)(3,4)$ $t_1 = (3,5)(4,6)$ $t_2 = (5,7)(6,8)$	(1A,2A,2B,3A,4B)	$\frac{1}{45}$	7
$3 \times S_3$	$S_3 \times S_3$	$t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ $t_1 = (1,7)(2,8)(3,11)(4,12)(5,10)(6,9)$ $t_2 = (1,7)(2,9)(3,8)(4,12)(5,10)(6,11)$	(1A,2A,3A,6A)	$\frac{1}{20}$	7
$L_2(7) = L_3(2)$	$L_3(2)$	$t = (1,2)(3,4)$ $t_1 = (2,3)(6,7)$ $t_2 = (3,5)(4,7)$	(1A,2A,3A,4B)	$\frac{1}{24}$	5
$3 \times A_5$	$A_5 : S_3$	$t = (1,2)(3,4)$ $t_1 = (3,5)(10,11)$ $t_2 = (2,3)(9,10)$	(1A,2A,3A,4B,5A,6A)	$\frac{11}{360}$	28
S_4	S_4	$t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ $t_1 = (1,2)(3,4)(5,12)(6,10)(7,8)(9,11)$ $t_2 = (1,2)(3,4)(5,12)(6,11)(7,10)(8,9)$	(1A,2B,3A)	$\frac{13}{180}$	9
$3 \times S_4$	$S_3 \times S_4$	$t = (1,2)(3,4)$ $t_1 = (4,5)(9,10)$ $t_2 = (5,6)(10,11)$	(1A,2A,2B,3A,4B,6A)	$\frac{1}{36}$	17

H_0	H	Embedding	Shape	(a_t, u_h)	$dim(A_{H_0})$
$2 \times L_2(7)$	$2 \times (2^3 : L_2(7))$	$t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ $t_1 = (6,7)(10,11)$ $t_2 = (7,9)(11,12)$	(1A,2A,2B,3A,4A,4B,6A)	$\frac{11}{360}$	46
$2 \times A_4$	$2 \times S_4$	$t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ $t_1 = (6,7)(10,11)$ $t_2 = (7,8)(11,12)$	(1A,2B,3A,4A)	$\frac{2}{45}$	10
$2 \times A_5$	$2 \times S_5$	$t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ $t_1 = (3,6)(9,11)$ $t_2 = (6,7)(8,9)$	(1A,2A,2B,3A,4A,5A,6A)	$\frac{1}{30}$	28
$4^2.S_3$	$2^5 : S_5$	$t = (1,5)(2,4)(3,7)(6,10)(8,9)(11,12)$ $t_1 = (1,4)(2,6)(3,5)(7,10)(8,12)(9,11)$ $t_2 = (1,5)(2,4)(3,6)(7,11)(8,10)(9,12)$	(1A,2A,2B,3A,4A,4B,5A,6A)	$\frac{1}{36}$	23
$GL_2(3) \cong$ $2S_4$	$2.S_5$	$t = (1,3)(2,37)(4,9)(5,33)(6,20)(7,27)$ $(10,26)(11,39)(12,13)(14,25)(15,30)$ $(16,18)(17,35)(19,36)(21,38)(22,23)$ $(24,31)(28,40)(29,32)$ $t_1 = (1,11)(2,30)(3,27)(5,15)(6,16)$ $(7,39)(8,31)(9,34)(10,26)(12,40)$ $(13,14)(17,19)(18,21)(20,38)(22,35)$ $(23,36)(25,28)(29,32)(33,37)$ $t_2 = (1,11)(3,17)(4,21)(5,12)(6,13)$ $(7,32)(8,28)(9,15)(10,19)(14,33)$ $(16,37)(20,25)(22,35)(23,39)(24,30)$ $(26,27)(29,36)(31,38)(34,40)$	(1A,2A,3A,4B,6A)	$\frac{1}{36}$	11

Table 3.2: Dimensions of some 3-generated algebras

We need to set the shape of H in order to find the right inner product (a_t, u_h). We will always suppose that the shape of H is the one obtained from a saturated Majorana representation of A_{12}, which results from the embedding of H into A_{12} as indicated in the third column of Table 3.2.

It is important to note that the scalar products are only dependent on the shape given on H by the Majorana representation of A_{12} via the embedding of H in A_{12}, not on the embedding itself.

Chapter 4

A MAJORANA ALGEBRA FOR THE GROUP $C_2 \times S_4$

In this chapter we determine a Majorana representation (G,T,A,ψ,ϕ) of the group $C_2 \times S_4$ where T is the set of involutions not contained in the direct summand S_4 and the shape is $(2B, 3A, 4A)$.

We identify the group $G = C_2 \times S_4$ with the subgroup of $Sym(8)$:

$$G = \langle (3,4)(5,6)(7,8), (1,2)(5,6)(7,8), (1,2)(4,5)(7,8) \rangle.$$

Then G has five classes of involutions:

$(7,8)^G = \{(7,8)\}$, $(1,2)(3,4)^G$, $(3,4)(5,6)^G$, $(1,2)(3,4)(7,8)^G$, and $(3,4)(5,6)(7,8)^G$.

We set $T := (7,8)^G \cup (1,2)(3,4)(7,8)^G \cup (3,4)(5,6)(7,8)^G$.

There is no dihedral subgroup of order greater than 8 in the group $C_2 \times S_4$, while there is a unique class of dihedral subgroups of order 6 and a unique class of dihedral subgroups of order 8 generated by two permutations in $T \setminus (7,8)$.

Let $t_1, t_2 \in T$. If t_1t_2 has order 2, we see that $t_1t_2 \notin T$. Hence, by condition $2A$, $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong 2B$. By the definition of shape it follows that if t_1t_2 has order 4, then $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong 4A$. Finally we choose that if t_1t_2 has order 3, then $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong 3A$. So the chosen shape is $(2B, 3A, 4A)$.

Remark 4.1. The case $(2B, 3C, 4A)$ could also have been chosen. In this case

the algebra is spanned by 2-axes and 4-axes, it is easy to see that it has dimension 12 and we will not write the details here.
Thus, the case considered $(2B, 3A, 4A)$ is the most interesting one and which we therefore report.

The group G has two maximal subgroups isomorphic to S_4:

- the one consisting of all the permutations fixing 7 and 8,
- $H := \langle (1,2)(5,6)(7,8), (1,2)(4,5)(7,8), (1,2)(3,6)(7,8) \rangle$.

This latter subgroup is generated by $T \cap H$ and the Majorana representation induced by ϕ on H is the representation of S_4 of shape $(2B, 3A)$ described in [27].

We begin by considering the following set of vectors, which must be contained in the algebra A. They are:

- the $2A$-axes,
- the $3A$-axes and the $4A$-axes arising from Norton-Sakuma algebras,
- the vectors $\delta_{(ij)(kl)}$, with $(i,j)(k,l)$ contained in the Klein subgroup K of H defined as in [27].

More explicitly we set:

$a_0 := a_{(7,8)}$

$a_1 := a_{(3,4)(5,6)(7,8)}$

$a_2 := a_{(3,5)(4,6)(7,8)}$

$a_3 := a_{(3,6)(4,5)(7,8)}$

$a_4 := a_{(1,2)(5,6)(7,8)}$

$a_5 := a_{(1,2)(4,5)(7,8)}$

$a_6 := a_{(1,2)(4,6)(7,8)}$

$a_7 := a_{(1,2)(3,4)(7,8)}$

$a_8 := a_{(1,2)(3,5)(7,8)}$

$a_9 := a_{(1,2)(3,6)(7,8)}$

$u_1 := u_{(4,5,6)}$

$u_2 := u_{(3,4,5)}$

$u_3 := u_{(3,4,6)}$

$u_4 := u_{(3,5,6)}$

$v_1 := v_{(1,2)(3,4,5,6)}$

$v_2 := v_{(1,2)(3,4,6,5)}$

$v_3 := v_{(1,2)(3,5,4,6)}$

$$\delta_{(34)(56)} := \tfrac{1}{1024}(5a_7 + a_5 + a_6 + a_8 + a_9 - a_4) + \tfrac{135}{65536}(u_1 - u_2 - u_3) - \tfrac{135}{2048} a_7 \cdot u_1$$
$$\delta_{(35)(46)} := \tfrac{1}{1024}(5a_8 + a_4 + a_5 + a_7 + a_9 - a_6) + \tfrac{135}{65536}(u_1 - u_2 - u_4) - \tfrac{135}{2048} a_8 \cdot u_1$$
$$\delta_{(36)(45)} := \tfrac{1}{1024}(5a_9 + a_4 + a_6 + a_7 + a_8 - a_5) + \tfrac{135}{65536}(u_1 - u_3 - u_4) - \tfrac{135}{2048} a_9 \cdot u_1$$

Finally we set:

$w_4 := a_4 \cdot (v_1 + v_2 + v_3)$

$w_5 := a_5 \cdot (v_1 + v_2 + v_3)$

$w_6 := a_6 \cdot (v_1 + v_2 + v_3)$

$w_7 := a_7 \cdot (v_1 + v_2 + v_3)$

$w_8 := a_8 \cdot (v_1 + v_2 + v_3)$

$w_9 := a_9 \cdot (v_1 + v_2 + v_3)$

We shall see that the above 26 vectors are a basis for the algebra A, so we define:

$$\mathcal{B}_0 := \{a_0, .., a_9, u_1, u_2, u_3, u_4, v_1, v_2, v_3\},$$

$$\mathcal{B} := \{a_0, ..., a_9, u_1, u_2, u_3, u_4, v_1, v_2, v_3, \delta_{(34)(56)}, \delta_{(35)(46)}, \delta_{(36)(45)}, w_4, ..., w_9\}.$$

Then we define $B_0 := \langle \mathcal{B}_0 \rangle$ and $B := \langle \mathcal{B} \rangle$.

Nevertheless, in order to determine the algebra products it is convenient to give a name to some vectors, which in the end will show to be linearly dependent from the previous ones in $\mathcal{B}$. We set:

$w_{117} := a_1 \cdot \delta_{(34)(56)}$

$w_{118} := a_1 \cdot \delta_{(35)(46)}$

$w_{119} := a_1 \cdot \delta_{(36)(45)}$

$w_{217} := a_2 \cdot \delta_{(34)(56)}$

$w_{218} := a_2 \cdot \delta_{(35)(46)}$

$w_{219} := a_2 \cdot \delta_{(36)(45)}$

$w_{317} := a_3 \cdot \delta_{(34)(56)}$

$w_{318} := a_3 \cdot \delta_{(35)(46)}$

$w_{319} := a_3 \cdot \delta_{(36)(45)}$

Let us also define $\mathcal{C}$ as the set of all 35 vectors listed above.
And we define $C := \langle \mathcal{C} \rangle$ the vector space generated by the vectors in $\mathcal{C}$.

It is also convenient to introduce the following elements ([27, Lemma 2.3]) so that we can apply formulas as in [27]:

$$s_1 := -\tfrac{27 \cdot 5}{2^{11}} u_1 + \tfrac{1}{32}(a_4 + a_5 + a_6)$$

$$s_2 := -\tfrac{27 \cdot 5}{2^{11}} u_2 + \tfrac{1}{32}(a_5 + a_7 + a_8)$$

$$s_3 := -\tfrac{27 \cdot 5}{2^{11}} u_3 + \tfrac{1}{32}(a_6 + a_7 + a_9)$$

$$s_4 := -\tfrac{27 \cdot 5}{2^{11}} u_4 + \tfrac{1}{32}(a_4 + a_8 + a_9)$$

Recall that some products of the algebra are already known from the table of Norton-Sakuma algebras (Table 1.1) and the paper [27].

In addition, the following products can be obtained immediately from the given vector configuration and will be useful later in order to find some inner products.

Lemma 4.2. *Let $z := (7,8) \in G$. For any $s \in T \setminus \{z\}$ we have $(a_z, a_s) = 0$ and $a_z \cdot a_s = 0$. In particular $A = \overline{A} \oplus \langle\langle z \rangle\rangle$ where $\overline{A}$ is the Majorana algebra for S_4.*

Proof. For each $s \in T \setminus \{z\}$, the product sz is an involution not contained in T. Hence, by the condition $2A$, $\langle\langle a_z, a_s \rangle\rangle \cong 2B$ and by Table 1.1 we get the result. In particular, as the fusion law is Seress, a result on sum decompositions (see [41]) implies that z commutes with all other axes in A, in other words $A = \overline{A} \oplus \langle\langle z \rangle\rangle$.

□

Lemma 4.3. *For each 2-axis a_t with $t \in (3,4)(5,6)(7,8)^G$ and 4-axis v_ρ, the algebra product $a_t \cdot v_\rho \in B$. In particular, if $t\rho = \rho t$, then $a_t \cdot v_\rho = 0$.*

Proof. We may assume $a_t = a_1$. Then $a_1 \cdot v_1$ and $a_1 \cdot v_2$ are in Norton-Sakuma algebras of type $4A$, so they are already known.

For the product $a_1 \cdot v_3$, note that in the algebra $4A$ generated by a_3 and a_4 we know that the 4-axis can be written as $v_3 = a_3 + a_4 + \frac{1}{3}a_2 + \frac{1}{3}a_7 - \frac{64}{3}a_3 \cdot a_4$. Since the involution corresponding to a_1 commutes with all the involutions related to the 2-axes in the expression of v_3, a_1 generates an algebra of type 2B with all of them. In particular, since a_3 and a_4 are both 0-eigenvectors of a_1, $a_3 \cdot a_4$ is also a 0-eigenvector of a_1 according to the fusion law. Thus all the algebra products are 0 and so $a_1 \cdot v_3 = 0$.

□

Lemma 4.4. *For every 2-axis a_t with $t \in (1,2)(3,4)(7,8)^G$ and for every 4-axis v, the algebra product $a_t \cdot v \in B$.*

Proof. We may assume $a_t = a_4$. As the product $a_4 \cdot v_3$ is in an algebra of type $4A$, it is already known.

We can find the algebra product of a_4 with v_1 and v_2 considering that $a_4 \cdot (v_1 + v_2 + v_3) = w_4$ then $a_4 \cdot (v_1 + v_2) = w_4 - a_4 \cdot v_3 \in B$.

In addition τ_{a_4} permutes v_1 and v_2, hence $a_4 \cdot (v_1 - v_2) = \frac{1}{32}(v_1 - v_2)$.

So we can sum and subtract the above two expressions and we find:

$$a_4 \cdot v_1 = \tfrac{1}{2}w_4 + \tfrac{1}{64}(v_1 - v_2 - 6v_3) - \tfrac{1}{32}(5a_4 - 2a_2 - 2a_3 - a_7),$$
$$a_4 \cdot v_2 = \tfrac{1}{2}w_4 + \tfrac{1}{64}(v_2 - v_1 - 6v_3) - \tfrac{1}{32}(5a_4 - 2a_2 - 2a_3 - a_7).$$

Then, in particular, $a_4 \cdot v_1, a_4 \cdot v_2 \in B$.

□

4.1. INNER PRODUCTS

In this section we calculate the inner products on pairs of elements of $\mathcal{B}$.

Let T_3 be the set of all cyclic subgroups of G of order 3, let T_4 be the set of all cyclic subgroups of G of order 4 conjugate to $\langle(1,2)(3,4,5,6)\rangle$, and let $T_\delta := \{(3,4)(5,6), (3,5)(4,6), (3,6)(4,5)\}$. Finally set

$$\mathcal{X} := T \cup T_3 \cup T_4 \cup T_\delta.$$

Clearly G acts on $\mathcal{X}$ by conjugation. We generically denote by $\sum_*$ an orbital of the group G on $\mathcal{X}$ i.e. the orbit of G on the set $\mathcal{X} \times \mathcal{X}$ with respect to the action defined: for every $g \in G$ and $\varepsilon, \sigma \in \mathcal{X}$, by $(\varepsilon, \sigma)^g = (\varepsilon^g, \sigma^g)$.

The inner product in A is invariant under the group action and therefore it is constant on the orbitals so it is sufficient to consider one representative for each orbital.

By Theorem 0.2, such representatives can be found by fixing a permutation ε and considering the pairs (ε, σ) where σ is in the list of orbit representatives of the normalizer of $\langle\varepsilon\rangle$.

Moreover, since the inner product is symmetric, its value is constant on an orbital and its transpose. Therefore for each pair of orbitals $\sum$, $\sum^T$ such that $\sum^T$ is the transpose of $\sum$, we shall ignore $\sum^T$.

In the tables of this chapter we will find:

- in the first column the label of the orbitals

- in the second column one representative of the orbital (for elements in T_3 and T_4 we simply write a generator of the cyclic group)

- in the third column the cycle type of the product of the two permutations in the corresponding representative

- in the fourth column the subgroup generated by the two permutations in the corresponding representative

- in the fifth column, only in the case of the 2-axes, the corresponding shape (Norton-Sakuma)

- in the last column the inner product between the axes corresponding to the two permutations in the representative.

Let us fix the involutions $t := (1,2)(5,6)(7,8)$ and $t^* := (3,4)(5,6)(7,8)$.

Lemma 4.5. *The inner products between two Majorana axes are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Shape*	*Inner Product*
$\sum_1$	(t, t)	1	C_2	$1A$	1
$\sum_2$	$(t, (3,4)(5,6)(7,8))$	2^2	$C_2 \times C_2$	$2B$	0
$\sum_3$	$(t, (3,5)(4,6)(7,8))$	$2 \cdot 4$	D_8	$4A$	$1/2^5$
$\sum_4$	$(t, (1,2)(4,5)(7,8))$	3	S_3	$3A$	$13/2^8$
$\sum_1^*$	(t^*, t^*)	1	C_2	$1A$	1
$\sum_2^*$	$(t^*, (3,5)(4,6)(7,8))$	2^2	$C_2 \times C_2$	$2B$	0
$\sum_3^*$	$(t^*, (1,2)(4,5)(7,8))$	$2 \cdot 4$	D_8	$4A$	$1/2^5$

Proof. The inner products of the above table can be found from Norton-Sakuma algebras of the corresponding shape in Table 1.1.

□

Lemma 4.6. *The inner products between a Majorana axis and a* 3*-axis are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{u,1}$	$(t,(4,5,6))$	2^3	S_3	$1/4$
$\sum_{u,2}$	$(t,(3,4,5))$	$2 \cdot 4 \cdot 2$	S_4	$13/180$
$\sum_{u,1}^{*}$	$(t^*,(4,5,6))$	2^3	$C_2 \times A_4$	$2/45$

Proof. The inner products of pairs in $\sum_{u,1}$ can be found from Norton-Sakuma algebra.

The inner products of pairs in $\sum_{u,2}$ have been found in [27].

For the inner products of pairs in $\sum_{u,1}^{*}$ let us consider the algebra of type 3A generated by a_5 and a_6 and the algebra of type 2B generated by a_1 and a_4. Then we can take the following eigenvectors of a_4 from the Table 1.2:

$u_{40} = u_1 - \frac{10}{27}a_4 + \frac{32}{27}(a_6 + a_5)$ 0-eigenvector
$u_{4al} = u_1 - \frac{8}{45}a_4 - \frac{32}{45}(a_6 + a_5)$ 1/4-eigenvector
a_1 0-eigenvector

Since a_1 and u_{4al} are a_4-eigenvectors with different eigenvalues, they are perpendicular and $(a_1, u_{4al}) = 0$ by lemma 1.9.
When we substitute the expression for u_{4al} in (a_1, u_{4al}) we see that the only unknown product is (a_1, u_1) and solving the equation we get the result.

□

Lemma 4.7. *The inner products $(a_t, \delta_{(ij)(kl)})$ between the Majorana axis a_t and a $\delta_{(ij)(kl)}$ are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{d,1}$	$(t,(3,4)(5,6))$	2^3	$C_2 \times C_2$	$-147/2^{18}$
$\sum_{d,2}$	$(t,(3,5)(4,6))$	$2 \cdot 4 \cdot 2$	D_8	$5/2^{19}$

Proof. These inner products have been computed in [27].

□

Lemma 4.8. *The inner products* $(a_{t^*}, \delta_{(ij)(kl)})$ *between the Majorana axis* a_{t^*} *and a* $\delta_{(ij)(kl)}$ *are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{d,1}^*$	$(t^*,(3,4)(5,6))$	2	$C_2 \times C_2$	$1/2^{15}$
$\sum_{d,2}^*$	$(t^*,(3,5)(4,6))$	2^3	$C_2 \times C_2$	$-15/2^{16}$

Proof. For the inner products of pairs in $\sum_{d,1}^*$ it is enough to calculate the inner product between a_1 and $\delta_{(34)(56)}$.
In this case by its definition (see Equation (23) in [27]):
$\delta_{(34)(56)} = a_7 \cdot s_1 - \frac{1}{2^5} s_1 + \frac{1}{2^{10}} a_7$, where $s_1 = \frac{1}{2^5}(a_4 + a_5 + a_6) - \frac{27 \cdot 5}{2^{11}} u_1$.
Then, due to the fact that $(,)$ associates with $\cdot$, we obtain:

$$\begin{aligned}(a_1, \delta_{(34)(56)}) &= (a_1, a_7 \cdot s_1) - \tfrac{1}{2^5}(a_1, s_1) + \tfrac{1}{2^{10}}(a_1, a_7) \\ &= (a_1 \cdot a_7, s_1) - \tfrac{1}{2^5}(a_1, \tfrac{1}{2^5}(a_4 + a_5 + a_6) - \tfrac{27 \cdot 5}{2^{11}} u_1) + \tfrac{1}{2^{10}}(a_1, a_7).\end{aligned}$$

Since a_1 and a_7 generate an algebra of type 2B we know that $a_1 \cdot a_7 = 0$ and all the other inner products are known from Norton-Sakuma algebras and from the previous lemmas. Then we obtain $(a_1, \delta_{(34)(56)}) = \frac{1}{2^{15}}$.

For the inner products of pairs in $\sum_{d,2}^*$ it is enough to calculate the product between a_1 and $\delta_{(35)(46)}$.

In this case $\delta_{(35)(46)} = a_8 \cdot s_1 - \frac{1}{2^5}s_1 + \frac{1}{2^{10}}a_8$.

Then, in the same way as before, we obtain $(a_1, \delta_{(35)(46)}) = -\frac{15}{2^{16}}$.

$\square$

Lemma 4.9. *The inner products between two 3-axes are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{u,1}^{u}$	$((4,5,6),(4,5,6))$	3	C_3	8/5
$\sum_{u,2}^{u}$	$((4,5,6),(3,4,5))$	2^2	A_4	56/675

Proof. The inner product of a 3-axis with itself in $\sum_{u,1}^{u}$ can be found in the Norton-Sakuma algebra $3A$.

The inner product of pairs in $\sum_{u,2}^{u}$ have been computed in [27, Table 11].

$\square$

Lemma 4.10. *The inner products between a 3-axis and a $\delta_{(ij)(kl)}$ are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{d,1}^{u}$	$((4,5,6),(3,4)(5,6))$	3	A_4	$-197/(9 \cdot 5 \cdot 2^{12})$

Proof. It is enough to consider the pair u_1 and $\delta_{(34)(56)}$.

In this case $u_1 = \frac{2^6}{27 \cdot 5}(a_4 + a_5 + a_6) - \frac{2^{11}}{27 \cdot 5}s_1$.

When we replace this expression for u_1 in $(u_1, \delta_{(3,4)(5,6)})$ we see that all the inner products of $\delta_{(3,4)(5,6)}$ with a_4, a_5, a_6, and s_1 are known from [27]. Thus we obtain the result.

$\square$

Lemma 4.11. *The inner products between two $\delta_{(ij)(kl)}$ are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{d,1}^{d}$	$((3,4)(5,6),(3,4)(5,6))$	1	C_2	$5 \cdot 3697/2^{29}$
$\sum_{d,2}^{d}$	$((3,4)(5,6),(3,5)(4,6))$	2^2	$C_2 \times C_2$	$3^5 \cdot 29/2^{30}$

Proof. The inner products between two vectors $\delta_{(ij)(kl)}$ in $\sum_{d,1}^{d}$ and $\sum_{d,2}^{d}$ are taken from [27].

□

Lemma 4.12. *The inner products between a Majorana axis and a* 4*-axis* v *are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{v,1}$	$(t,(1,2)(3,5,4,6))$	2^3	D_8	$3/8$
$\sum_{v,2}$	$(t,(1,2)(3,4,5,6))$	$3 \cdot 2$	$C_2 \times S_4$	$5/64$
$\sum_{v,1}^{*}$	$(t^*,(1,2)(3,5,4,6))$	$2 \cdot 4 \cdot 2$	$C_4 \times C_2$	0
$\sum_{v,2}^{*}$	$(t^*,(1,2)(3,4,5,6))$	2^3	D_8	$3/8$

Proof. The inner products of pairs in $\sum_{v,1}$ are given in Norton-Sakuma algebra $4A$ (see Table 1.1).

For the inner product of a_t with v in $\sum_{v,2}$ it is enough to consider the pair a_4 and v_1. In this case $v_1 = a_1 + a_6 + \frac{1}{3}a_3 + \frac{1}{3}a_8 - \frac{64}{3}a_1 \cdot a_6$ as we see inside the algebra $4A$ generated by a_1 and a_6 and due to the fact that $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}(a_4, v_1) &= (a_4, a_1) + (a_4, a_6) + \tfrac{1}{3}(a_4, a_3) + \tfrac{1}{3}(a_4, a_8) - \tfrac{64}{3}(a_4, a_1 \cdot a_6) \\ &= (a_4, a_1) + (a_4, a_6) + \tfrac{1}{3}(a_4, a_3) + \tfrac{1}{3}(a_4, a_8) - \tfrac{64}{3}(a_4 \cdot a_1, a_6).\end{aligned}$$

Since a_1 and a_4 generate an algebra of type 2B we know that $a_1 \cdot a_4 = 0$ and all the other inner products are known from Norton-Sakuma algebras.
Then we obtain $(a_4, v_1) = \frac{5}{64}$.

The inner product of pairs in $\sum_{v,2}^{*}$ is given in the Norton-Sakuma algebra $4A$.

For the inner products of pairs in $\sum_{v,1}^{*}$ it is sufficient to consider the pair a_1 and v_3. In this case $v_3 = a_3 + a_4 + \frac{1}{3}a_2 + \frac{1}{3}a_7 - \frac{64}{3}a_3 \cdot a_4$ as we see inside the algebra $4A$ generated by a_3 and a_4 and due to the fact that $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}(a_1, v_3) &= (a_1, a_3) + (a_1, a_4) + \tfrac{1}{3}(a_1, a_3) + \tfrac{1}{3}(a_1, a_7) - \tfrac{64}{3}(a_1, a_3 \cdot a_4) \\ &= (a_1, a_3) + (a_1, a_4) + \tfrac{1}{3}(a_1, a_3) + \tfrac{1}{3}(a_1, a_7) - \tfrac{64}{3}(a_1 \cdot a_4, a_3).\end{aligned}$$

Since a_1 and a_4 generate an algebra of type 2B we know that $a_1 \cdot a_4 = 0$ and a_1 commutes with all the 2-axes in the expression of v_3. Then all the inner products are 0 and we obtain $(a_1, v_3) = 0$.

□

Lemma 4.13. *The inner products between a* 3*-axis and a* 4*-axis are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{v,1}^{u}$	$((4,5,6),(1,2)(3,4,5,6))$	2^2	S_4	1/9

Proof. It is enough to consider the pair u_1 and v_1.
In this case $v_1 = a_1 + a_6 + \frac{1}{3}a_3 + \frac{1}{3}a_8 - \frac{64}{3}a_1 \cdot a_6$ as we see inside the algebra $4A$ generated by a_1 and a_6 and due to the fact that $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}(u_1, v_1) &= (u_1, a_1 + a_6 + \tfrac{1}{3}a_3 + \tfrac{1}{3}a_8 - \tfrac{64}{3}a_1 \cdot a_6)\\ &= (u_1, a_1) + (u_1, a_6) + \tfrac{1}{3}(u_1, a_3) + \tfrac{1}{3}(u_1, a_8) - \tfrac{64}{3}(u_1, a_1 \cdot a_6)\\ &= (u_1, a_1) + (u_1, a_6) + \tfrac{1}{3}(u_1, a_3) + \tfrac{1}{3}(u_1, a_8) - \tfrac{64}{3}(a_6 \cdot u_1, a_1).\end{aligned}$$

Since a_6 and u_1 are in an algebra of type 3A we know the product $a_6 \cdot u_1$ as a combination of 2-axes and 3-axes, hence all the inner products are known from Norton-Sakuma algebras.
Then we obtain $(u_1, v_1) = \frac{1}{9}$.

□

Lemma 4.14. *The inner products between two* 4*-axes are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{v,1}^{v}$	$((1,2)(3,5,4,6),(1,2)(3,5,4,6))$	2^2	C_4	2
$\sum_{v,2}^{v}$	$((1,2)(3,5,4,6),(1,2)(3,4,5,6))$	3	S_4	11/48

Proof. The inner product of v with itself in $\sum_{v,1}^{v}$ is taken from the Norton-Sakuma algebra $4A$.

For the inner product of pairs in $\sum_{v,2}^{v}$ it is sufficient to consider the pair v_1 and v_3. In this case $v_1 = a_1 + a_6 + \frac{1}{3}a_3 + \frac{1}{3}a_8 - \frac{64}{3}a_1 \cdot a_6$ as we see inside the algebra $4A$ generated by a_1 and a_6 and due to the fact that $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}(v_3, v_1) &= (v_3, a_1) + (v_3, a_6) + \tfrac{1}{3}(v_3, a_3) + \tfrac{1}{3}(v_3, a_8) - \tfrac{64}{3}(v_3, a_1 \cdot a_6)\\ &= (v_3, a_1) + (v_3, a_6) + \tfrac{1}{3}(v_3, a_3) + \tfrac{1}{3}(v_3, a_8) - \tfrac{64}{3}(v_3 \cdot a_1, a_6).\end{aligned}$$

By Lemma 4.3 $v_3 \cdot a_1 = 0$ and all the inner products are known from Norton-Sakuma algebras and from the previous lemmas.

Then we obtain $(v_3, v_1) = \frac{11}{48}$.

$\square$

Lemma 4.15. *The inner products between a 4-axis and a $\delta_{(ij)(kl)}$ are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{d,1}^{v}$	$((1,2)(3,5,4,6),(3,4)(5,6))$	$2 \cdot 4$	C_4	$29/2^{16}$
$\sum_{d,2}^{v}$	$((1,2)(3,5,4,6),(3,5)(4,6))$	2^2	D_8	$21/2^{17}$

Proof. In order to find the inner products of $\delta_{(ij)(kl)}$ with v it is sufficient to replace v in $(v, \delta_{(ij)(kl)})$.

In particular for products of pairs in $\sum_{d,1}^{v}$ we consider the pair v_3 and $\delta_{(34)(56)}$. In this case $v_3 = a_2 + a_4 + \frac{1}{3}a_3 + \frac{1}{3}a_7 - \frac{64}{3}a_2 \cdot a_4$ as we see inside the algebra $4A$ generated by a_2 and a_4 and due to the fact that $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}
(v_3, \delta_{(34)(56)}) &= (a_2, \delta_{(34)(56)}) + (a_4, \delta_{(34)(56)}) + \tfrac{1}{3}(a_3, \delta_{(34)(56)}) \\
&+ \tfrac{1}{3}(a_7, \delta_{(34)(56)}) - \tfrac{64}{3}(a_2 \cdot a_4, \delta_{(34)(56)}) \\
&= (a_2, \delta_{(34)(56)}) + (a_4, \delta_{(34)(56)}) + \tfrac{1}{3}(a_3, \delta_{(34)(56)}) \\
&+ \tfrac{1}{3}(a_7, \delta_{(34)(56)}) - \tfrac{64}{3}(a_2, a_4 \cdot \delta_{(34)(56)}).
\end{aligned}$$

Since a_4 and $\delta_{(34)(56)}$ are in the subalgebra A_H of shape (2B,3A) we know the product $a_4 \cdot \delta_{(34)(56)}$ from [27] as a combination of 2-axes, 3-axes and $\delta_{(ij)(kl)}$, hence all the inner products are known from Norton-Sakuma algebras and from the previous lemmas.
Then we get $(v_3, \delta_{(34)(56)}) = \frac{29}{2^{16}}$.

In the same way we find the inner products of pairs in $\sum_{d,2}^{v}$ considering the pair v_3 and $\delta_{(35)(46)}$.

$\square$

Note that, for $i \in \{4, \dots, 9\}$, the vector w_i is defined as the product between a_i and $v_1 + v_2 + v_3$. Since $v_1 + v_2 + v_3$ is invariant under the group action, the element w_i depends only on the 2-axis a_i and thus the corresponding involution. Then the orbitals for the vectors w_i are indexed on the same sets as those corresponding to 2-axes a_s with s conjugate to t.

Lemma 4.16. *The inner products between a Majorana axis and a vector w_i are given in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum^*_{w,1}$	$(t^*, (1,2)(5,6)(7,8))$	2^2	$C_2 \times C_2$	0
$\sum^*_{w,2}$	$(t^*, (1,2)(4,5)(7,8))$	$2 \cdot 4$	D_8	$-9/256$
$\sum_{w,1}$	$(t, (1,2)(5,6)(7,8))$	1	C_2	$17/32$
$\sum_{w,2}$	$(t, (1,2)(4,5)(7,8))$	3	S_3	$125/2048$
$\sum_{w,3}$	$(t, (1,2)(3,4)(7,8))$	2^2	$C_2 \times C_2$	0

Proof. By the definition of w_i and the associativity of the inner product, for every $s \in T$, we have

$$(a_s, w_i) = (a_s, a_i \cdot (v_1 + v_2 + v_3)) = (a_s \cdot a_i, v_1 + v_2 + v_3). \qquad (4.1)$$

For the inner products of pairs in $\sum^*_{w,1}$ we consider the pair a_1 and w_4.

Since a_1 and a_4 generate an algebra of type 2B we know that $a_1 \cdot a_4 = 0$. Then from Equation 4.1 we get $(a_1, w_4) = 0$.

For the inner products of pairs in $\sum^*_{w,2}$ we consider the pair a_1 and w_5.

Since a_1 and a_5 generate an algebra of type 4A we know the product $a_1 \cdot a_5$ as combination of 2-axes and 4-axis and then, substituting in Equation 4.1, all the inner products are known from Norton-Sakuma algebras and from the previous

lemmas.
Then we obtain $(a_1, w_5) = -\frac{9}{256}$.

For the inner products of pairs in $\sum_{w,1}$ we consider the pair a_4 and w_4.

Since a_4 is an idempotent, $a_4 \cdot a_4 = a_4$ and then all the inner products in Equation 4.1 are known from Norton-Sakuma algebras and from the previous lemmas.
Then we obtain $(a_4, w_4) = \frac{17}{32}$.

For the inner products of pairs in $\sum_{w,2}$ we consider the pair a_4 and w_5.

Since a_4 and a_5 generate an algebra of type 3A we know the product $a_4 \cdot a_5$ as combination of 2-axes and 3-axis and then, substituting in Equation 4.1, all the inner products are known from Norton-Sakuma algebras and from the previous lemmas.
Then we obtain $(a_4, w_5) = \frac{125}{2048}$.

For the inner products of pairs in $\sum_{w,3}$ we consider the pair a_4 and w_7.

Since a_4 and a_7 generate an algebra of type 2B we know that $a_4 \cdot a_7 = 0$.
Then $(a_4, w_7) = 0$.

□

Lemma 4.17. *The inner products between a* 3*-axis and a vector* w *are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{w,1}^{u}$	$((4,5,6),(1,2)(5,6)(7,8))$	2^3	S_3	5/96
$\sum_{w,2}^{u}$	$((4,5,6),(1,2)(3,4)(7,8))$	$2 \cdot 4 \cdot 2$	S_4	59/1440

Proof. For the inner products of pairs in $\sum_{w,1}^{u}$ we consider the pair u_1 and w_4. We have

$$(u_1, w_4) = (u_1, a_4 \cdot (v_1 + v_2 + v_3)) = (u_1 \cdot a_4, v_1 + v_2 + v_3).$$

Since a_4 and u_1 are in an algebra of type 3A we know the product $a_4 \cdot u_1$ as combination of 2-axes and 3-axis and then all the inner products in the above formula are known from Norton-Sakuma algebras and from the previous lemmas. Then we obtain $(u_1, w_4) = \frac{5}{96}$.

For the inner products of pairs in $\sum_{w,2}^{u}$ we consider the pair u_1 and w_7. We have

$$(u_1, w_7) = (u_1, a_7 \cdot (v_1 + v_2 + v_3)) = (u_1 \cdot a_7, v_1 + v_2 + v_3).$$

Since a_7 and u_1 are in the subalgebra A_H of shape (2B,3A) we know the product $a_7 \cdot u_1$ from [27], hence all the inner products in the above formula are known from Norton-Sakuma algebras and from the previous lemmas.
Then we get $(u_1, w_7) = \frac{59}{1440}$.

□

Lemma 4.18. *The inner products between a* 4*-axis and a vector* w *are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{w,1}^{v}$	$((1,2)(3,4,5,6),(1,2)(4,6)(7,8))$	2^3	D_8	13/32
$\sum_{w,2}^{v}$	$((1,2)(3,4,5,6),(1,2)(5,6)(7,8))$	$3 \cdot 2$	$C_2 \times S_4$	179/1536

Proof. For the inner products of pairs in $\sum_{w,1}^{v}$ we consider the pair v_1 and w_6. We have

$$(v_1, w_6) = (v_1, a_6 \cdot (v_1 + v_2 + v_3)) = (v_1 \cdot a_6, v_1 + v_2 + v_3).$$

Since a_6 and v_1 are in an algebra of type 4A we know the product $a_6 \cdot v_1$ as combination of 2-axes and 4-axis and then all the inner products in the above formula are known from Norton-Sakuma algebras and from the previous lemmas.

Then we obtain $(v_1, w_6) = \frac{13}{32}$.

For the inner product of pairs in $\sum_{w,2}^{v}$ let us consider the algebras of type 4A generated by a_1 and a_9 and a_1 and a_6 respectively and the algebra of type 2B generated by a_1 and a_4. Then we can take the following eigenvectors from the Table 1.2:

$x_9 := v_2 - \frac{1}{2}a_1 + 2(a_5 + a_9) + a_2$ 0-eigenvector of a_1
$y_6 := v_1 - \frac{1}{3}a_1 - \frac{2}{3}(a_6 + a_8) - \frac{1}{3}a_3$ 1/4-eigenvector of a_1
a_4 0-eigenvector of a_1

By the fusion law $a_4 \cdot x_9$ is again a 0-eigenvector of a_1.
Since y_6 and $a_4 \cdot x_9$ are eigenvectors of a_1 with different eigenvalues, they are perpendicular by Lemma 1.9: $(y_6, a_4 \cdot x_9) = 0$.

The product $a_4 \cdot x_9$ is expressible as a linear combination of Majorana axes and odd-axes since all the algebra products between a_4 and the summands in x_9 are known thanks to Norton-Sakuma algebras and the Lemma 4.4.

Thus, except for (v_1, w_4), all the inner products are known from Norton-Sakuma algebras and from the previous lemmas and we get:

$$(y_6, a_4 \cdot x_9) = -\frac{179}{3072} + \frac{1}{2}(v_1, w_4).$$

Then $(v_1, w_4) = \frac{179}{1536}$.

□

Lemma 4.19. *The inner products between a $\delta_{(ij)(kl)}$ and a vector w are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{w,1}^{d}$	$((3,4)(5,6),(1,2)(5,6)(7,8))$	2^3	$C_2 \times C_2$	273/2097152
$\sum_{w,2}^{d}$	$((3,4)(5,6),(1,2)(4,5)(7,8))$	$2 \cdot 4 \cdot 2$	D_8	685/4194304

Proof. For the inner products of pairs in $\sum_{w,1}^{d}$ we consider the pair $\delta_{(34)(56)}$ and w_4. We have

$$(\delta_{(34)(56)}, w_4) = (\delta_{(34)(56)}, a_4 \cdot (v_1 + v_2 + v_3)) = (\delta_{(34)(56)} \cdot a_4, v_1 + v_2 + v_3).$$

Since a_4 and $\delta_{(34)(56)}$ are in the subalgebra A_H of shape (2B,3A) we know the product $a_4 \cdot \delta_{(34)(56)}$ from [27], hence all the inner products in the above formula are known from Norton-Sakuma algebras and from the previous lemmas.
Then we get $(\delta_{(34)(56)}, w_4) = \frac{273}{2097152}$.

For the inner products of pairs in $\sum_{w,2}^{d}$ we consider the pair $\delta_{(34)(56)}$ and w_5. We have

$$(\delta_{(34)(56)}, w_5) = (\delta_{(34)(56)}, a_5 \cdot (v_1 + v_2 + v_3)) = (\delta_{(34)(56)} \cdot a_5, v_1 + v_2 + v_3).$$

In the same way as in the previous case, we obtain: $(\delta_{(34)(56)}, w_5) = \frac{685}{4194304}$.

□

Lemma 4.20. *The inner products between two w are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{w,1}^{w}$	$((1,2)(5,6)(7,8),(1,2)(5,6)(7,8))$	1	C_2	4565/12288
$\sum_{w,2}^{w}$	$((1,2)(5,6)(7,8),(1,2)(4,5)(7,8))$	3	S_3	4415/49152
$\sum_{w,3}^{w}$	$((1,2)(5,6)(7,8),(1,2)(3,4)(7,8))$	2^2	$C_2 \times C_2$	629/12288

Proof. For the inner product of pairs in $\sum_{w,1}^{w}$ and in $\sum_{w,3}^{w}$ let us consider the eigenvectors x_9, y_6 defined in the proof of Lemma 4.1.
Let us also consider the algebras of type 2B generated by a_1 and a_4 and a_1 and a_7 respectively. Then we can take the following eigenvectors from the Table 1.2:

a_4 0-eigenvector of a_1
a_7 0-eigenvector of a_1

By the fusion law $a_4 \cdot x_9$ is again a 0-eigenvector of a_1 and $a_4 \cdot y_6$ is again a 1/4-eigenvector of a_1.
Since $a_4 \cdot y_6$ and $a_4 \cdot x_9$ are eigenvectors of a_1 with different eigenvalues, they are perpendicular by Lemma 1.9: $(a_4 \cdot y_6, a_4 \cdot x_9) = 0$.

The products $a_4 \cdot x_9$ and $a_4 \cdot y_6$ are expressible as linear combinations of Majorana axes and odd-axes since all the algebra products between a_4 and the axes in x_9 and y_6 are known from Norton-Sakuma algebras and thanks to the Lemma 4.4.

Thus, except for (w_4, w_4), all the inner products are known from Norton-Sakuma algebras and from the previous lemmas and we get:

$$(a_4 \cdot y_6, a_4 \cdot x_9) = -\frac{4565}{49152} + \frac{1}{4}(w_4, w_4).$$

Then $(w_4, w_4) = \frac{4565}{12288}$.

By the fusion law $a_7 \cdot y_6$ is again a 1/4-eigenvector of a_1.
Since $a_7 \cdot y_6$ and $a_4 \cdot x_9$ are eigenvectors of a_1 with different eigenvalues, they are perpendicular by Lemma 1.9: $(a_7 \cdot y_6, a_4 \cdot x_9) = 0$.

The products $a_4 \cdot x_9$ and $a_7 \cdot y_6$ are expressible as linear combinations of Majorana axes and odd-axes since all the algebra products between a_4 and the axes in x_9 and between a_7 and the axes in y_6 are known from Norton-Sakuma algebras and thanks to the Lemma 4.4.

Thus, except for (w_4, w_7), all the inner products are known from Norton-Sakuma algebras and from the previous lemmas and we have:

$$(a_7 \cdot y_6, a_4 \cdot x_9) = -\frac{629}{49152} + \frac{1}{4}(w_4, w_7).$$

Thus $(w_4, w_7) = \frac{629}{12288}$.

Finally, for the inner product of pairs in $\sum_{w,2}^{w}$ let us consider a_4 that is a 0-eigenvector of a_1 and the following eigenvectors calculated as in Lemma 1.18:

$\widetilde{c}_1 := 4(a_1 \cdot (v_1 + v_2 + v_3) - (a_1, v_1 + v_2 + v_3)a_1)$ 1/4-eigenvector of a_1

$e := w_5 + w_9 - (a_1, w_5 + w_9)a_1 - 4a_1 \cdot (w_5 + w_9 - (w_5 + w_9, a_1)a_1)$ 0-eigenvector of a_1.

By the fusion law $\widetilde{c}_1 \cdot a_4 = 4(a_4 \cdot (a_1 \cdot (v_1 + v_2 + v_3)) - (a_1, v_1 + v_2 + v_3)a_1 \cdot a_4)$ is again a 0-eigenvector of a_1.

Since e and $\widetilde{c}_1 \cdot a_4$ are eigenvectors of a_1 with different eigenvalues, they are perpendicular by Lemma 1.9: $(e, \widetilde{c}_1 \cdot a_4) = 0$.

The product $a_1 \cdot (v_1 + v_2 + v_3)$ in $\widetilde{c}_1$ is expressible as a combination of 2-axes and 4-axes thanks to Norton-Sakuma algebras and the Lemma 4.3.
Moreover, we can also calculate $\widetilde{c}_1 \cdot a_4$, since all the algebra products between a_4 and the axes in $\widetilde{c}_1$ are known from Norton-Sakuma algebras and thanks to the Lemma 4.4.

Thus, we can write $(e, \widetilde{c}_1 \cdot a_4)$ as a linear combination of inner products, where, except for $(w_4, w_5), (w_4, w_6), (w_4, w_8), (w_4, w_9)$, all the inner products are known from Norton-Sakuma algebras and from the previous lemmas and we get:

$$(e, \widetilde{c}_1 \cdot a_4) = -\tfrac{4415}{32768} + \tfrac{3}{8}(w_4, w_5) + \tfrac{3}{8}(w_4, w_6) + \tfrac{3}{8}(w_4, w_8) + \tfrac{3}{8}(w_4, w_9).$$

In addition, we have $(w_4, w_5) = (w_4, w_6) = (w_4, w_8) = (w_4, w_9)$ since the pairs of representatives are all in the same orbital $\sum_{w,2}^{w}$.

Thus we obtain $(w_4, w_5) = \frac{4415}{49152}$.

$\square$

In this way, all the inner products between the vectors of the set $\mathcal{B}$ have been found.

Proposition 4.21. *The set $\mathcal{B}$ is linear independent.*

Proof. We computed the determinant of the Gram matrix which is different from 0 so the vectors of $\mathcal{B}$ are all linearly independent.

$\square$

In this section we will assume that the dimension of the algebra A is 26, in fact, this representation is well known to exist.

Since the algebra product is commutative and all other products can be obtained using the action of G, once the expression for the product of two vectors corresponding to each orbital used in the previous section is known, the others can be derived.

Recall from Section 4.1 that $B = \langle \mathcal{B} \rangle$.

Lemma 4.22. *Let $\eta, \varepsilon \in G$ and let x_η and x_ε be the corresponding axes or δ-elements. If the pair (η, ε) is contained in one of the following orbitals:*

$\sum_1$, $\sum_2$, $\sum_3$, $\sum_4$, $\sum_1^*$, $\sum_2^*$, $\sum_3^*$, $\sum_{u,1}$, $\sum_{u,2}$, $\sum_{d,1}$, $\sum_{d,2}$, $\sum_{u,1}^u$, $\sum_{u,2}^u$, $\sum_{d,1}^u$, $\sum_{d,1}^d$, $\sum_{d,2}^d$, $\sum_{v,1}$, $\sum_{v,2}^*$, $\sum_{v,1}^v$

then the product $x_\eta \cdot x_\varepsilon \in B$.

Proof. The algebra products in the orbitals

$$\sum_1, \sum_2, \sum_3, \sum_4, \sum_1^*, \sum_2^*, \sum_3^*, \sum_{u,1}, \sum_{u,1}^u, \sum_{v,1}, \sum_{v,2}^*, \sum_{v,1}^v$$

can be found in Norton-Sakuma algebras (Table 1.1).
The algebra products in the orbitals

$$\sum_{u,2}, \sum_{d,1}, \sum_{d,2}, \sum_{u,2}^u, \sum_{d,1}^u, \sum_{d,1}^d, \sum_{d,2}^d$$

can be obtained with the formulas in [27].

□

Remark 4.23. Recall that the Miyamoto involutions τ_{a_i} associated to the 2-axes defined in 1.15 switch some vectors with others.
For example τ_{a_5} switches:

a_1 and a_2

a_4 and a_6

a_7 and a_8

u_3 and u_4

v_1 and v_3

$\delta_{(34)(56)}$ and $\delta_{(35)(46)}$

Lemma 4.24. *If* $(\eta, \varepsilon) \in \sum_{v,1}^{*}$*, then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. This is part of Lemma 4.3.

□

Lemma 4.25. *If* $(\eta, \varepsilon) \in \sum_{v,2}$*, then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. This is Lemma 4.4.

□

Lemma 4.26. $a_i \cdot w_j = a_j \cdot w_i$ *for every* $i, j \in \{4, \dots, 9\}$ *such that* $a_i \cdot a_j = 0$.

Proof. Since $a_i \cdot a_j = 0$, a_i is a 0-eigenvector of a_j. Then, by Lemma 1.17 we have

$$a_i \cdot w_j = a_i \cdot (a_j \cdot (v_1 + v_2 + v_3)) = a_j \cdot (a_i \cdot (v_1 + v_2 + v_3)) = a_j \cdot w_i.$$

□

Lemma 4.27. *If* $(\eta, \varepsilon) \in \sum_{w,1}^{*}$*, then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. It is enough to compute $a_1 \cdot w_4$.

Since a_4 is a 0-eigenvector for a_1, by Lemma 1.17 we have

$$a_1 \cdot w_4 = a_1 \cdot (a_4 \cdot (v_1 + v_2 + v_3)) = a_4 \cdot (a_1 \cdot (v_1 + v_2 + v_3)).$$

By Lemma 4.3 $a_1 \cdot (v_1 + v_2 + v_3)$ is a linear combination of 2-axes and 4-axes. Hence, by Lemma 4.4, $a_4 \cdot (a_1 \cdot (v_1 + v_2 + v_3)) \in B$ and we get the result.

$\square$

Lemma 4.28. *If* $(\eta, \varepsilon) \in \sum^{*}_{u,1}$*, then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. It is enough to calculate $a_1 \cdot u_1$.

Let us consider the algebras of type 4A generated by a_1, a_9 and a_1, a_6 respectively and the algebra of type 2B generated by a_1 and a_4. Then we can take the following eigenvectors from the Table 1.2:

$x_9 = v_2 - \frac{1}{2}a_1 + 2(a_5 + a_9) + a_2$ 0-eigenvector of a_1
$y_9 = v_2 - \frac{1}{3}a_1 - \frac{2}{3}(a_5 + a_9) - \frac{1}{3}a_2$ 1/4-eigenvector of a_1

$x_6 = v_1 - \frac{1}{2}a_1 + 2(a_6 + a_8) + a_3$ 0-eigenvector of a_1
$y_6 = v_1 - \frac{1}{3}a_1 - \frac{2}{3}(a_6 + a_8) - \frac{1}{3}a_3$ 1/4-eigenvector of a_1

a_4 0-eigenvector of a_1

By the fusion law $a_4 \cdot x_9$ is again a 0-eigenvector of a_1.
Then $a_1 \cdot (a_4 \cdot x_9) = 0$.

We know all the products between a_4 and the axes in the formula of x_9 then in $a_1 \cdot (a_4 \cdot x_9)$ the only missing products are $a_1 \cdot u_1$ and $a_1 \cdot u_4$ and so we get $a_1 \cdot (u_1 + u_4) \in B$.

In addition τ_{a_1} permutes u_1 and u_4, hence $a_1 \cdot (u_1 - u_4) = \frac{1}{32}(u_1 - u_4)$.

Hence we have

$$a_1 \cdot u_1 = \frac{1}{2}(a_1 \cdot (u_1 + u_4) + a_1 \cdot (u_1 - u_4)) \in B.$$

$\square$

Lemma 4.29. *If* $(\eta, \varepsilon) \in \sum_{w,1}$*, then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. In this case it is enough to compute $a_4 \cdot w_4$.

Let us consider the eigenvectors x_9 and y_9 of a_1 defined in the proof of Lemma 4.27. By the fusion law $a_4 \cdot (a_4 \cdot x_9)$ and $a_4 \cdot (a_4 \cdot y_9)$ are again 0 and 1/4-eigenvectors of a_1, then:

$$a_1 \cdot (a_4 \cdot (a_4 \cdot (y_9 - x_9))) = \tfrac{1}{4} a_4 \cdot (a_4 \cdot y_9).$$

The only missing product in $a_4 \cdot (a_4 \cdot y_9)$ is $a_4 \cdot w_4$ while all the algebra products in $a_1 \cdot (a_4 \cdot (a_4 \cdot (x_9 - y_9)))$ are in B from the previous lemmas.
Then $a_4 \cdot w_4 \in B$.

□

Lemma 4.30. *A basis for the* $\frac{1}{32}$*-eigenspace of* a_4 *is given by the set*

$$a_5 - a_6, a_8 - a_9, u_2 - u_3, \delta_{(35)(46)} - \delta_{(36)(45)}, a_2 - a_3, v_1 - v_2, w_5 - w_6, w_8 - w_9.$$

Proof. The given vectors are negated by τ_{a_4} and so they are $\frac{1}{32}$-eigenvectors of a_4. Moreover, they are linearly independent, as the set $\mathcal{B}$ is by Proposition 4.21. To see that they generate the $\frac{1}{32}$-eigenspace of a_4, we note that the vectors a_0, a_1, a_4, a_7, $a_5 + a_6$, $a_8 + a_9$, u_1, $u_2 + u_3$, $\delta_{(34)(56)}$, $\delta_{(35)(46)} - \delta_{(36)(45)}$, w_4, $w_5 + w_6$, w_7, $w_8 + w_9$ are fiexd by τ_{a_4} and so they lie in $A_{1,0,\frac{1}{4}}(a_4)$.
Since they are also linearly independent, under our assumption that A has dimension 26, we can deduce that the $\frac{1}{32}$-eigenspace has dimension 8 and the result follows.

□

Lemma 4.31. *If* $(\eta, \varepsilon) \in \sum^*_{w,2}$*, then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. It is enough to compute $a_1 \cdot w_5$.

Let us consider the eigenvectors x_6, x_9, y_6, y_9 for a_1 already used in the proofs of previous lemmas.

Define $s := x_9 \cdot x_6 - y_9 \cdot y_6$. By the fusion law s is the sum of $(s, a_1)a_1$ with a 0-eigenvector. A direct check shows that $(s, a_1) = 0$.
Hence s is a 0-eigenvector of a_1. An explicit computation of s and of the product $a_1 \cdot s$ shows that the only missing products are $a_1 \cdot w_5$, $a_1 \cdot w_6$, $a_1 \cdot w_8$ and $a_1 \cdot w_9$ because all the other algebra products are known from the previous lemmas.
Thus we get $a_1 \cdot (w_5 + w_6 + w_8 + w_9) \in B$.

Since τ_{a_1} swaps $w_5 + w_6$ and $w_8 + w_9$, we have

$$a_1 \cdot (w_5 + w_6 - w_8 - w_9) = \frac{1}{32}(w_5 + w_6 - w_8 - w_9).$$

So $a_1 \cdot (w_5 + w_6) = \frac{1}{2}(a_1 \cdot (w_5 + w_6 + w_8 + w_9) + a_1 \cdot (w_5 + w_6 - w_8 - w_9))$ belongs to B.

Now $w_5 - w_6$ is a $\frac{1}{32}$-eigenvector of a_4 and so, by the fusion law, $a_1 \cdot (w_5 - w_6)$ is again a $\frac{1}{32}$-eigenvector of a_4. Hence, by Lemma 4.30, there exist real numbers $z_1, ..., z_8$ such that

$$\begin{aligned} a_1 \cdot (w_5 - w_6) = & z_1(a_5 - a_6) + z_2(a_8 - a_9) + z_3(u_2 - u_3) + z_4(\delta_{(35)(46)} - \delta_{(36)(45)}) \\ & + z_5(a_2 - a_3) + z_6(v_1 - v_2) + z_7(w_5 - w_6) + z_8(w_8 - w_9). \end{aligned}$$

Since

$$a_1 \cdot w_5 = \frac{1}{2}(a_1 \cdot (w_5 + w_6) + a_1 \cdot (w_5 - w_6))$$
$$a_1 \cdot w_6 = \frac{1}{2}(a_1 \cdot (w_5 + w_6) - a_1 \cdot (w_5 - w_6))$$

we can express these two products as linear combinations of the elements of $\mathcal{B}$ with coefficients that are functions in the unknowns $z_1, ...z_8$.

In the same way we can find the other products as a function of the unknowns.

Now let us consider the following eigenvectors obtained as in Lemma 1.18:

$b := u_2 + u_3 - (u_2 + u_3, a_1)a_1 - 4(a_1 \cdot (u_2 + u_3) - (u_2 + u_3, a_1)a_1)$ 0-eigenvector of a_1
$c := 4(a_1 \cdot (u_2 + u_3) - (u_2 + u_3, a_1)a_1)$ 1/4-eigenvector of a_1

$b_1 := u_1 + u_4 - (u_1 + u_4, a_1)a_1 - 4(a_1 \cdot (u_1 + u_4) - (u_1 + u_4, a_1)a_1)$ 0-eigenvector of a_1

$c_1 := 4(a_1 \cdot (u_1 + u_4) - (u_1 + u_4, a_1)a_1)$ 1/4-eigenvector of a_1

a_2 0-eigenvector of a_1.

By the fusion law $a_2 \cdot (b_1 - b)$ is again a 0-eigenvector.

By the previous lemmas, we can compute explicitly the product $a_1 \cdot (a_2 \cdot (b_1 - b_2)))$ as a linear combination of the elements of $\mathcal{B}$ and coefficients in terms of the unknowns. From the equality $a_1 \cdot (a_2 \cdot (b_1 - b_2))) = 0$, using the fact that $\mathcal{B}$ is a linearly independent set (Propostition 4.21) we get:

$z_2 = -z_1$,

$z_3 = 0$,

$z_8 = \frac{1}{32} - z_7$.

By the fusion law, $a_2 \cdot b$ is again a 0-eigenvector and $a_2 \cdot c$ is a 1/4-eigenvector of a_1.

Then $a_1 \cdot (a_2 \cdot b) + a_1 \cdot (a_2 \cdot c) - \frac{1}{4} a_2 \cdot c = 0$ and similarly as above we get:

$z_4 = 0$,

$z_5 = \frac{11}{512}$,

$z_1 = -\frac{13}{256}$,

$z_6 = \frac{13}{512}$,

$z_7 = \frac{9}{64}$,

$z_8 = -\frac{7}{64}$.

Hence we have written $a_1 \cdot w_5$ as a linear combination of the vectors $a_5 - a_6$, $a_8 - a_9, u_2 - u_3, \delta_{(35)(46)} - \delta_{(36)(45)}, a_2 - a_3, v_1 - v_2, w_5 - w_6, w_8 - w_9$ with coefficients $z_1, ..., z_8$ and so $a_1 \cdot w_5 \in B$.

□

Lemma 4.32. *If $(\eta, \varepsilon) \in \sum_{v,2}^{v}$, then $x_\eta \cdot x_\varepsilon \in B$.*

Proof. In this case it is enough to compute $v_2 \cdot v_3$.

Let us consider the eigenvectors v_3, x_9 and y_9 for a_1 already used in the proofs of previous lemmas.
By the fusion law $v_3 \cdot x_9$ and $v_3 \cdot y_9$ are again 0 and 1/4-eigenvectors of a_1 respectively, then:

$$a_1 \cdot (v_3 \cdot (y_9 - x_9)) = \frac{1}{4} v_3 \cdot y_9$$

The only missing product in the above equation is $v_2 \cdot v_3$.
Hence $v_2 \cdot v_3 \in B$.

□

Lemma 4.33. *If $(\eta, \varepsilon) \in \sum_{w,2}$, then $x_\eta \cdot x_\varepsilon \in B$.*

Proof. It is enough to calculate $a_5 \cdot w_4$.

Let us consider the algebra of type 4A generated by a_1 and a_6 and the algebra of type 3A generated by a_4 and a_6. Then we can take the following eigenvectors from the Table 1.2:

$v_0 := v_1 - \frac{1}{2}a_6 + 2(a_1 + a_3)$ 0-eigenvector of a_6
$v_{al} := v_1 - \frac{1}{3}a_6 - \frac{2}{3}(a_1 + a_3) - \frac{1}{3}a_8$ 1/4-eigenvector of a_6

$u_0 := u_1 - \frac{10}{27}a_6 + \frac{32}{27}(a_4 + a_5)$ 0-eigenvector of a_6
$u_{al} := u_1 - \frac{8}{45}a_6 - \frac{32}{45}(a_4 + a_5)$ 1/4-eigenvector of a_6

By the fusion law $u_{al} \cdot v_{al} - (v_{al}, u_{al} \cdot a_6)a_6$ and $u_0 \cdot v_0$ are 0-eigenvectors.

When computing the product $a_6 \cdot (u_{al} \cdot v_{al} - u_0 \cdot v_0 - (v_{al}, u_{al} \cdot a_6)a_6)$ we see that the only unkown term is $a_6 \cdot (w_4 + w_5)$.
Since $a_6 \cdot (u_{al} \cdot v_{al} - u_0 \cdot v_0 - (v_{al}, u_{al} \cdot a_6)a_6)$ must be zero, we get $a_6 \cdot (w_4 + w_5) \in B$.

In addition, since τ_{a_6} permutes w_4 and w_5, $a_6 \cdot (w_4 - w_5) = \frac{1}{32}(w_4 - w_5)$.

So as above it follows that $a_6 \cdot w_4 \in B$.

□

Lemma 4.34. *If* $(\eta, \varepsilon) \in \sum_{v,1}^{u}$, *then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. It is enough to calculate $u_1 \cdot v_1$.

Let us consider the eigenvectors u_0, v_0 and v_{al} for a_6 already used in the proofs of previous lemmas.
By the fusion law $u_0 \cdot v_{al}$ is a 1/4-eigenvector and $u_0 \cdot v_0$ is a 0-eigenvector.

The result follows applying the resurrection principle (Lemma 1.19) to eigenvectors $u_0 \cdot v_0$ and $u_0 \cdot v_{al}$ as in the proof of Lemma 4.32.

□

Lemma 4.35. *If* $(\eta, \varepsilon) \in \sum_{w,3}$, *then* $x_\eta \cdot x_\varepsilon \in C$.

Proof. It is enough to calculate $a_4 \cdot w_7$.

Let us consider the eigenvectors b, c and a_4 for a_1 already used in the proofs of previous lemmas.

By the fusion law $a_4 \cdot c$ is a 1/4-eigenvector and $a_4 \cdot b$ is a 0-eigenvector, then:

$$a_1 \cdot (a_4 \cdot (b + c)) = \frac{1}{4} a_4 \cdot c.$$

The only missing product in $a_4 \cdot c$ is $a_4 \cdot w_7$ and $a_1 \cdot (a_4 \cdot (b+c))$ can be calculate since all the other algebra products are known from the previous lemmas but in this case there is also the vector w_{117} which is the product $a_1 \cdot \delta_{(34)(56)}$.
Then $a_4 \cdot w_7 \in C$.

□

Lemma 4.36. *If* $(\eta, \varepsilon) \in \sum_{w,1}^{v}$, *then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. It is enough to calculate $v_3 \cdot w_4$.

Let us consider the eigenvectors b_1 and c_1 for a_1 already used in the proofs of previous lemmas and let us consider the algebra of type 4A generated by a_2 and a_4. Then we can take the following eigenvectors from the table 1.2:

$v_{40} := v_3 - \frac{1}{2}a_4 + 2(a_2 + a_3)$ 0-eigenvector of a_1 and a_4
$v_{4al} := v_3 - \frac{1}{3}a_4 - \frac{2}{3}(a_2 + a_3) - \frac{1}{3}a_7$ 0-eigenvector of a_1 and 1/4-eigenvector of a_4

By the fusion law $c_1 \cdot v_{40}$ is a 1/4-eigenvector and $b_1 \cdot v_{40}$ is a 0-eigenvector a_1.

The result follows applying the resurrection principle (Lemma 1.19) to eigenvectors $b_1 \cdot v_{40}$ and $c_1 \cdot v_{40}$ as in the proof of Lemma 4.32.

□

Lemma 4.37. *The algebra product $a_4 \cdot w_{117}$ is in C.*

Proof. By Lemma 1.17, since a_4 is a 0-eigenvector of a_1, we have:

$$a_4 \cdot w_{117} = a_4 \cdot (a_1 \cdot \delta_{(34)(56)}) = a_1 \cdot (a_4 \cdot \delta_{(34)(56)}).$$

Now $a_4 \cdot \delta_{(34)(56)}$ is contained in the subalgebra A_H, so by [27, Lemma 4.8] it is a linear combination of elements of $\mathcal{B}$ dinstinct from $w_4, ..., w_9$.
When we multiply this linear combination by a_1, all products are either in B by the previous lemmas, or equal to one of $w_{117}, w_{118}, w_{119}$. Hence $a_4 \cdot w_{117} \in C$.

□

Lemma 4.38. *If $(\eta, \varepsilon) \in \sum_{d,1}^{*}$, then $x_\eta \cdot x_\varepsilon \in B$.*

Proof. It is enough to calculate $a_1 \cdot \delta_{(34)(56)}$.

Now let us consider the following eigenvectors of a_4 obtained as in the Lemma 1.18:

$b_{56} := w_5 + w_6 - (w_5 + w_6, a_4)a_4 - 4(a_4 \cdot (w_5 + w_6) - (w_5 + w_6, a_4)a_4)$ 0-eigenvector of a_4

$c_{56} := 4(a_4 \cdot (w_5 + w_6) - (w_5 + w_6, a_4)a_4)$ 1/4-eigenvector of a_4

By the fusion law, $a_1 \cdot b_{56}$ is again a 0-eigenvector for a_4 then $a_4 \cdot (a_1 \cdot b_{56})$ must be 0 but we can compute it and we see that it is equal to $l - 4w_{117}$, where $l \in B$.

So $a_1 \cdot \delta_{(34)(56)} = w_{117} = \frac{1}{4}l \in B$.

□

Lemma 4.39. *If* $(\eta, \varepsilon) \in \sum_{w,2}^{u}$*, then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. It is enough to calculate $u_1 \cdot w_7$.

Let us consider the algebra of type 3A generated by a_4 and a_6. Then we can take the following eigenvectors from the Table 1.2:

$u_{40} := u_1 - \frac{10}{27}a_4 + \frac{32}{27}(a_6 + a_5)$ 0-eigenvector of a_4
$u_{4al} := u_1 - \frac{8}{45}a_4 - \frac{32}{45}(a_6 + a_5)$ 1/4-eigenvector of a_4

Now let us consider the following eigenvector obtained as in the Lemma 1.18:

$b_7 := w_7 - (w_7, a_4)a_4 - 4(a_4 \cdot w_7 - (w_7, a_4)a_4)$ 0-eigenvector of a_4

By the fusion law $u_{4al} \cdot b_7$ is a 1/4-eigenvector and $u_{40} \cdot b_7$ is a 0-eigenvector, then:

$$a_4 \cdot ((u_{4al} - u_{40}) \cdot b_7) = \frac{1}{4}u_{4al} \cdot b_7.$$

By proceeding as in the previous cases, we get $u_1 \cdot w_7 \in B$.

□

Lemma 4.40. *If* $(\eta, \varepsilon) \in \sum_{w,1}^{u}$*, then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. It is enough to calculate $u_2 \cdot w_8$.

Let us consider the algebra of type $3A$ generated by a_7 and a_8. Then we can take the following eigenvectors from Table 1.2:

$u_{80} := u_2 - \frac{10}{27}a_8 + \frac{32}{27}(a_7 + a_5)$ 0-eigenvector of a_8
$u_{8al} := u_2 - \frac{8}{45}a_8 - \frac{32}{45}(a_7 + a_5)$ 1/4-eigenvector of a_8

And let us consider the following eigenvector obtained as in Lemma 1.18:

$b_{49} := w_4 + w_9 - (w_4 + w_9, a_8)a_8 - 4(a_8 \cdot (w_4 + w_9) - (w_4 + w_9, a_8)a_8)$ 0-eigenvector of a_8

By the fusion law $u_{8al} \cdot b_{49}$ is a 1/4-eigenvector and $u_{80} \cdot b_{49}$ is a 0-eigenvector, then:

$$a_8 \cdot ((u_{8al} - u_{80}) \cdot b_{49}) = \frac{1}{4}u_{8al} \cdot b_{49}.$$

By proceeding as in the previous cases, we get $u_2 \cdot w_8 \in B$.

□

Lemma 4.41. *If $(\eta, \varepsilon) \in \sum_{w,2}^{v}$, then $x_\eta \cdot x_\varepsilon \in C$.*

Proof. It is enough to calculate $v_1 \cdot w_4$.

Let us consider the eigenvectors x_6, y_6 and b_1 for a_1.
By the fusion law $y_6 \cdot b_1$ is a 1/4-eigenvector and $x_6 \cdot b_1$ is a 0-eigenvector, then:

$$a_1 \cdot ((y_6 - x_6) \cdot b_1) = \frac{1}{4}y_6 \cdot b_1.$$

By proceeding as in the previous cases, we get $v_1 \cdot w_4 \in C$.

□

Lemma 4.42. *If $(\eta, \varepsilon) \in \sum_{w,1}^{w}$, then $x_\eta \cdot x_\varepsilon \in C$.*

Proof. It is enough to calculate $w_4 \cdot w_4$.

Let us consider the eigenvectors b_1 and c_1 for a_1.
By the fusion law $b_1 \cdot c_1$ is a 1/4-eigenvector and $b_1 \cdot b_1$ is a 0-eigenvector, then:

$$a_1 \cdot (b_1 \cdot (c_1 + b_1)) = \frac{1}{4} b_1 \cdot c_1.$$

By proceeding as in the previous cases, we get $w_4 \cdot w_4 \in C$.

□

Lemma 4.43. *If $(\eta, \varepsilon) \in \sum_{w,3}^{w}$, then $x_\eta \cdot x_\varepsilon \in C$.*

Proof. It is enough to calculate $w_4 \cdot w_7$.

Let us consider the eigenvectors b, b_1 and c_1 for a_1.
By the fusion law $b \cdot c_1$ is a 1/4-eigenvector and $b \cdot b_1$ is a 0-eigenvector, then:

$$a_1 \cdot (b \cdot (c_1 + b_1)) = \frac{1}{4} b \cdot c_1.$$

By proceeding as in the previous cases, we get $w_4 \cdot w_7 \in C$.

□

Lemma 4.44. *We have $w_{118} + w_{119} \in B$.*

Proof. We consider the eigenvectors b and c of a_1 defined in the proof of 4.31.
By the fusion law the product $a_1 \cdot (b \cdot b - c \cdot c + \frac{1}{4}(c, c)a_1)$ must be zero.
Computing explicitely the above product we get the result.

□

Lemma 4.45. *If $(\eta, \varepsilon) \in \sum_{d,1}^{v}$, then $x_\eta \cdot x_\varepsilon \in C$.*

Proof. It is enough to calculate $v_3 \cdot \delta_{(34)(56)}$.

Let us consider the eigenvectors p and q in S_4 as in [27, Table 6]:

$p = \delta_{(34)(56)} - \frac{7}{2^6}(s_2 + s_3) + \frac{7}{2^{15}} a_4$ 0-eigenvector of a_4
$q = \delta_{(34)(56)} + \frac{1}{2^6}(s_2 + s_3) + \frac{5}{2^{13}} a_4$ 1/4-eigenvector of a_4

Now let us consider the following eigenvector constructed as in the Lemma 1.18:

$a_{23} = a_2 + a_3 - (a_2 + a_3, a_4)a_4 - 4(a_4 \cdot (a_2 + a_3) - (a_2 + a_3, a_4)a_4)$ 0-eigenvector of a_4

By the fusion law $a_{23} \cdot q$ is a 1/4-eigenvector and $a_{23} \cdot p$ is a 0-eigenvector, then:

$$a_4 \cdot (a_{23} \cdot (q - p)) = \tfrac{1}{4} a_{23} \cdot q.$$

By proceeding as in the previous cases, we get $v_3 \cdot \delta_{(34)(56)} \in C$.

□

Lemma 4.46. *If $(\eta, \varepsilon) \in \sum_{w,1}^{d}$, then $x_\eta \cdot x_\varepsilon \in B$.*

Proof. It is enough to calculate $w_4 \cdot \delta_{(34)(56)}$.

Let us consider the eigenvectors p and q defined in the proof of the previous lemma and let us consider the following eigenvector constructed as in Lemma 1.18:

$\widetilde{c}_4 := 4(a_4 \cdot (v_1 + v_2 + v_3) - (v_1 + v_2 + v_3, a_4)a_4)$ 1/4-eigenvector of a_4

By the fusion law $\widetilde{c}_4 \cdot p$ is a 1/4-eigenvector and $\widetilde{c}_4 \cdot q$ is the sum of a 0-eigenvector and a 1-eigenvector, then:

$$a4 \cdot (\widetilde{c}_4 \cdot (p - q)) = \tfrac{1}{4} \widetilde{c}_4 \cdot p - \tfrac{1}{4}(\widetilde{c}_4, q)a_4.$$

By proceeding as in the previous cases, we get $\delta_{(34)(56)} \cdot w_4 \in B$.

□

Lemma 4.47. *The algebra product $a_1 \cdot w_{317}$ is in B.*

Proof. Since a_1 is a 0-eigenvector of a_3, by Lemma 1.17 we have

$$a_1 \cdot w_{317} = a_1 \cdot (a_3 \cdot \delta_{(34)(56)}) = a_3 \cdot (a_1 \cdot \delta_{(34)(56)}).$$

The last product can be calculate since all the algebra products involved are known from the previous lemmas.
Hence $a_1 \cdot w_{317} \in B$.

$\square$

Lemma 4.48. *If* $(\eta, \varepsilon) \in \sum_{d,2}^{v}$*, then* $x_\eta \cdot x_\varepsilon \in C$*.*

Proof. It is enough to calculate $v_1 \cdot \delta_{(34)(56)}$.

Let us consider the eigenvectors x_6, y_6 and b for a_1.
By the fusion law $b \cdot a_4$ is again a 0-eigenvector of a_1, so $x_6 \cdot (b \cdot a_4)$ is a 0-eigenvector and $y_6 \cdot (b \cdot a_4)$ is a 1/4-eigenvector, then:

$$a_1 \cdot ((y_6 - x_6) \cdot (b \cdot a_4)) = \frac{1}{4} y_6 \cdot (b \cdot a_4).$$

By proceeding as in the previous cases, we get $v_1 \cdot \delta_{(34)(56)} \in C$.

$\square$

Lemma 4.49. *We have* $w_{217} + w_{317} \in B$*.*

Proof. Now we want to find the sum of the vectors $w_{217} + w_{317}$. In order to do that let us consider the eigenvectors p and q of a_4 defined in 4.45 and the following eigenvector constructed as in Lemma 1.18:

$$\widetilde{b}_4 := v_1 + v_2 + v_3 - (v_1 + v_2 + v_3, a_4)a_4 - 4(a_4 \cdot (v_1 + v_2 + v_3) - (v_1 + v_2 + v_3, a_4)a_4)$$
0-eigenvector of a_4

By the fusion law, $\widetilde{b}_4 \cdot q$ is again 1/4-eigenvector and $\widetilde{b}_4 \cdot p$ is again 0-eigenvector of a_4 then $a_4 \cdot (\widetilde{b}_4 \cdot (q - p)) - \frac{1}{4}\widetilde{b}_4 \cdot q$ must be 0.
We can express this vector explicitely as a linear combination of elements of

$\mathcal{B} \cup \{w_{217}, w_{317}\}$. So we find $w_{217} + w_{317} \in B$.

□

And then we can also find $w_{118}+w_{318}$, $w_{119}+w_{219}$, $w_{217}+w_{219}$ and $w_{317}+w_{318}$ as combination of vectors in $\mathcal{B}$ applying τ_{a_5} and τ_{a_6} to $w_{217} + w_{317}$.

Lemma 4.50. *We have $C = B + \langle w_{118} \rangle$.*

Proof. By Lemma 4.38 we already know that w_{117}, w_{218}, and w_{319} are in B.
From Lemma 4.49 we get

$$\begin{aligned}
w_{119} =&(w_{118} + w_{119}) - w_{118} \in B + \langle w_{118} \rangle \\
w_{217} =&(w_{217} + w_{317}) - (w_{317} + w_{318}) + (w_{118} + w_{318}) - w_{118} \in B + \langle w_{118} \rangle \\
w_{219} =&(w_{217} + w_{219}) - (w_{217} + w_{317}) + (w_{317} + w_{318}) - (w_{118} + w_{318}) + w_{118} \in B + \langle w_{118} \rangle \\
w_{317} =&(w_{317} + w_{318}) - (w_{118} + w_{318}) + w_{118} \in B + \langle w_{118} \rangle \\
w_{318} =&(w_{118} + w_{318}) - w_{118} \in B + \langle w_{118} \rangle
\end{aligned}$$

□

So now it remains to find only the vector w_{118} as a combination of the vectors in $\mathcal{B}$.

Lemma 4.51. *The algebra product $a_4 \cdot w_{118}$ is in B.*

Proof. Since a_4 is a 0-eigenvector of a_1, by Lemma 1.17 we have

$$a_4 \cdot w_{118} = a_4 \cdot (a_1 \cdot \delta_{(35)(46)}) = a_1 \cdot (a_4 \cdot \delta_{(35)(46)}).$$

The last product can be calculate since all the algebra products involved are known from the previous lemmas.
Hence $a_4 \cdot w_{118} \in B$.

□

Lemma 4.52. *The algebra product $a_1 \cdot w_{118}$ is in C.*

Proof. Let us consider the following eigenvector constructed as in Lemma 1.18:

$a_1 \cdot \delta_{(35)(46)} - (\delta_{(35)(46)}, a_1)a_1$ 1/4-eigenvector of a_1.

Then $a_1 \cdot (a_1 \cdot \delta_{(35)(46)} - (\delta_{(35)(46)}, a_1)a_1) = \frac{1}{4}(a_1 \cdot \delta_{(35)(46)} - (\delta_{(35)(46)}, a_1)a_1)$.

Note that $a_1 \cdot \delta_{(35)(46)}$ is the vector w_{118}, then $a_1 \cdot w_{118} = \frac{3}{4}(\delta_{(35)(46)}, a_1)a_1 + \frac{1}{4}w_{118}$.

□

Lemma 4.53. *If* $(\eta, \varepsilon) \in \sum_{d,2}^*$, *then* $x_\eta \cdot x_\varepsilon \in B$.

Proof. It is enough to calculate $a_1 \cdot \delta_{(35)(46)}$.

Let us consider the following eigenvector constructed as in Lemma 1.18:

$k_{10} := \delta_{(34)(56)} - (\delta_{(34)(56)}, a_1)a_1 - 4w_{117} + 4(\delta_{(34)(56)}, a_1)a_1$ 0-eigenvector of a_1

And let us consider the following eigenvectors:

$a_2 - a_3$ 0-eigenvector of a_1
a_4 0-eigenvector of a_1

By the fusion law $a_4 \cdot ((a_2 - a_3) \cdot k_{10})$ is again a 0-eigenvector for a_1 then $a_1 \cdot (a_4 \cdot ((a_2 - a_3) \cdot k_{10}))$ must be 0 but we can compute it and we see that it is equal to $l - \frac{1}{64}w_{118}$, where $l \in B$.

So $a_1 \cdot \delta_{(35)(46)} = w_{118} = \frac{1}{64}l \in B$.

□

Corollary 4.54. *We have* $C = B$.

Proof. In the previous lemma we found that the vector w_{118} is a combination of the vectors in $\mathcal{B}$.

□

Lemma 4.55. *If* $(\eta, \varepsilon) \in \sum_{w,2}^{w}$*, then* $x_\eta \cdot x_\varepsilon \in B$*.*

Proof. It is enough to calculate $w_4 \cdot w_5$.

Let us consider the eigenvectors $\widetilde{b}_4, \widetilde{c}_4$ and b_{56} for a_4.

By the fusion law $\widetilde{b}_4 \cdot b_{56}$ is a 0-eigenvector and $\widetilde{c}_4 \cdot b_{56}$ is a 1/4-eigenvector, then:

$a_4 \cdot ((\widetilde{c}_4 + \widetilde{b}_4) \cdot b_{56}) = \frac{1}{4}\widetilde{c}_4 \cdot b_{56}$.

The only missing products in $\widetilde{c}_4 \cdot b_{56}$ are $w_4 \cdot w_5$, $w_4 \cdot w_6$ and $a_4 \cdot ((\widetilde{c}_4 + \widetilde{b}_4) \cdot b_{56})$ can be calculated since all the other algebra products are known from the previous lemmas.

Then $w_4 \cdot (w_5 + w_6) \in B$.

In addition applying the Miyamoto involution τ_{a_6} to $w_4 \cdot (w_5 + w_6)$ we can find also that $w_4 \cdot (w_5 - w_6) \in B$.

Then $w_4 \cdot w_5 = \frac{1}{2}(w_4 \cdot (w_5 + w_6) + w_4 \cdot (w_5 + w_6)) \in B$.

□

Lemma 4.56. *If* $(\eta, \varepsilon) \in \sum_{w,2}^{d}$*, then* $x_\eta \cdot x_\varepsilon \in B$*.*

Proof. It is enough to calculate $w_4 \cdot \delta_{(35)(46)}$.

Let us consider the following eigenvector constructed as in the Lemma 1.18:

$k_{18} := \delta_{(35)(46)} - (\delta_{(35)(46)}, a_1)a_1 - 4w_{118} + 4(\delta_{(35)(46)}, a_1)a_1$ 0-eigenvector of a_1
$\overline{b}_1 := \widetilde{b}_4 - (\widetilde{b}_4, a_1)a_1 - 4(a_1 \cdot \widetilde{b}_4 - (\widetilde{b}_4, a_1)a_1)$ 0-eigenvector of a_1
$\overline{c}_1 := \frac{1}{3}(4(a_1 \cdot \widetilde{b}_4 - (\widetilde{b}_4, a_1)a_1))$ 1/4-eigenvector of a_1

By the fusion law $\widetilde{b}_4 \cdot k_{18}$ is a 0-eigenvector and $\widetilde{c}_4 \cdot k_{18}$ is a 1/4-eigenvector, then:

$a_1 \cdot (k_{18} \cdot (\bar{c}_1 - \bar{b}_1)) = \frac{1}{4} k_{18} \cdot \bar{c}_1.$

Proceeding as in the previous cases we get $w_4 \cdot \delta_{(35)(46)} \in B$.

□

Proposition 4.57. *$\mathcal{B}$ is a basis for the algebra A.*

Proof. By the lemmas in this subsection we know that the product of two elements in $\mathcal{B}$ is contained in C and by Corollary 4.54, $C = B$.
Since A is generated as algebra by the elements in $\mathcal{B}$, it follows that $A = B$.
By Proposition 4.21 $\mathcal{B}$ is a basis for A.

□

4.3. NORTON BASIS FOR $C_2 \times S_4$

Note that as written in [27], instead of the vectors $\delta_{(ij)(kl)}$ one could use different vectors $\overline{v}$ as in the Norton basis. These vectors are idempotents with $(\overline{v}, \overline{v}) = 2$ and are in a bijective correspondence with the cyclic subgroups of order 4 in the subgroup H isomorphic to S_4 defined at the beginning of this chapter.
So we have

$$\overline{v}_1 = \overline{v}_{(1,2)(3,5,4,6)(7,8)}$$
$$\overline{v}_2 = \overline{v}_{(1,2)(3,4,5,6)(7,8)}$$
$$\overline{v}_3 = \overline{v}_{(1,2)(3,4,6,5)(7,8)}$$

Remark 4.58. Note that the vectors $\overline{v}_i$ in the Griess algebra are actually real 4-axes but in general this is not always true for subalgebras.
In our algebra A for example these vectors are not real 4-axes arising from Norton-Sakuma algebras of type 4A because the permutations associated to these vectors are not the products of any two Majorana involutions in T.
We will call this vectors *fake* 4*-axes*.

Since in the next chapter we will use this basis with the $\overline{v}_i$ instead of $\delta_{(ij)(kl)}$, we will need the scalar products between these vectors $\overline{v}_i$ and the axes in $\mathcal{B}$.

Some of them are already known from Majorana's representation of S_4 with shape (2B,3A) and Norton's basis in [27].
We report the missing inner products in the following lemmas.

Lemma 4.59. *The inner products between an axis a_i and a vector $\overline{v}_j$ with $i,j \in \{1,2,3\}$ are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{\overline{v},1}^{*}$	$(t^*, (1,2)(3,5,4,6)(7,8))$	$2 \cdot 4$	$C_4 \times C_2$	0
$\sum_{\overline{v},2}^{*}$	$(t^*, (1,2)(3,4,5,6)(7,8))$	2^2	D_8	1/24

Lemma 4.60. *The inner products $(v, \overline{v})$ between a 4-axis and a vector $\overline{v}$ are listed in the following table.*

Orbitals	*Representative*	*Cycle type*	*Group*	*Inner Product*
$\sum_{\overline{v},1}^{v}$	$((1,2)(3,5,4,6), (1,2)(3,5,4,6)(7,8))$	2^3	$C_4 \times C_2$	0
$\sum_{\overline{v},2}^{v}$	$((1,2)(3,5,4,6), (1,2)(3,4,5,6)(7,8))$	$3 \cdot 2$	$C_2 \times S_4$	11/48

It is straightforward to check the scalar products of the tables by using the results of Section 4.1 and expressing the vectors $\delta_{(ij)(kl)}$ in terms of the Norton basis.
For the sake of completeness, we give the formulas for converting vectors $\delta_{(ij)(kl)}$ to $\overline{v}$ and vice versa using the notation of [27]:

$\delta_{(ij)(kl)} = -\frac{1}{2048}(a_{ij} + a_{kl}) + \frac{3}{4096}(a_{jl} + a_{jk} + a_{ik} + a_{il}) - \frac{225}{131072}(u_i + u_j + u_k + u_l) - \frac{3}{2048}(v_j - 2v_k - 2v_l)$

$v_j = -\frac{7}{27}(a_{ij} + a_{kl}) + \frac{1}{54}(a_{jl} + a_{jk} + a_{ik} + a_{il}) + \frac{25}{64}(u_i + u_j + u_k + u_l) - \frac{2048}{27}(\delta_{(ij)(kl)} - 2\delta_{(ik)(jl)} - 2\delta_{(il)(jk)})$.

Chapter 5

A MAJORANA ALGEBRA FOR THE GROUP $C_2 \times PSL(3,2)$

In this chapter we study a Majorana representation ($\overline{G}$,T,A,ψ,ϕ) of the group $C_2 \times PSL(3,2)$ with shape (2B,3A,4A).

Let $\overline{G} = C_2 \times PSL(3,2)$ and $G = PSL(3,2)$.

We identify the group $\overline{G} = C_2 \times PSL(3,2)$ with the subgroup of $Sym(9)$ generated by $(4,6)(5,7)(8,9), (2,4)(3,5)(8,9), (1,2)(5,6)(8,9)$.
Then $\overline{G}$ has three classes of involutions:

$$(8,9)^{\overline{G}} = \{(8,9)\}, (4,5)(6,7)^{\overline{G}}, (4,5)(6,7)(8,9)^{\overline{G}}.$$

We set $T := (8,9)^{\overline{G}} \cup (4,5)(6,7)(8,9)^{\overline{G}}$.

There is no dihedral subgroup of order greater than 8 in the group $\overline{G}$, while there is a unique class of dihedral subgroups of order 6 and a unique class of dihedral subgroups of order 8 generated by two involutions in $T \setminus (8,9)$.

Since for every $s,t \in T$ we have $st \notin T$, so by the $2A$-condition, every dihedral subalgebra of type $2X$ is $2B$ and hence, by Lemma 1.22, every dihedral algebra of type $4X$ is always $4A$. Finally, we choose that every dihedral algebra of type $3X$ is of type $3A$.

Let $z := (8,9) \in \overline{G}$, then, as in chapter 4, we have $A = \overline{A} \oplus \langle\langle z \rangle\rangle$ where $\overline{A}$ is the Majorana algebra for G.

Let us list the maximal subgroups of $\overline{G}$:

- the group $G = PSL(3, 2)$,
- fourteen subgroups isomorphic to $C_2 \times S_4$,
- eight subgroups isomorphic to $C_2 \times (C_7 : C_3)$.

Every subgroup isomorphic to $C_2 \times S_4$ has two maximal subgroups isomorphic to S_4, of which only one is generated by involutions in T.
Consequently, there exist 14 subalgebras corresponding to the subgroups S_4 of this type (split into two conjugacy classes).

Then some inner and algebra products are already known from the table of Norton-Sakuma algebras (Table 1.1) and [27].
We also have all the inner products of the subalgebras corresponding to subgroups $C_2 \times S_4$ that we computed in the previous chapter.

We begin by considering the following set of vectors $\mathcal{D}_0$, which must be contained in the algebra A. They are :

- 22 2-axes a_t, $t \in T$,

- 28 3-axes u_h, such that $\langle h \rangle$ is a subgroup of order 3 of G,

- 21 4-axes v_g, where $\langle g \rangle$ is a subgroup of order 4 in G.

We need then to consider fake 4-axes coming from the Norton basis of the subalgebras corresponding to the fourteen subgroups of $\overline{G}$ which are isomorphic to S_4.

For each element g of order 4 in G, we can consider the element of order 4 $g(8, 9)$ that is in $\overline{G}$ but not in G.
Each of these elements $g(8, 9)$ is contained in two different subgroups, H_1 and

H_2, isomorphic to S_4 and not conjugate.
Since each of these elements belongs to two distinct subgroups isomorphic to S_4, they can in principle correspond to two distinct fake 4-axes $\overline{v}_1$ and $\overline{v}_2$.

By analogy with condition 4A of the Definition 2.2 for the 4-axes, we assume

$4Af$**-condition**: Suppose g_1, g_2 are two elements of order 4 in G and $\overline{v}_{g_1(8,9)}$ and $\overline{v}_{g_2(8,9)}$ are two fake 4-axes in a Majorana representation of $\overline{G}$.
If $\langle g_1 \rangle = \langle g_2 \rangle$, then $\overline{v}_{g_1(8,9)} = \overline{v}_{g_2(8,9)}$.

Let $\mathcal{D}$ be the union of the set $\mathcal{D}_0$ together with the 21 fake 4-axes $\overline{v}_{g(8,9)}$, with $\langle g \rangle$ a subgroup of order 4 in G coming from the Norton basis of the subalgebras corresponding to the fourteen subgroups of $\overline{G}$ isomorphic to S_4 and generated by elements of T. Moreover, set $D := \langle \mathcal{D} \rangle$.

We will compute the dimension of D.

Since both the inner and the algebra products and the spanning set $\mathcal{D}$ are $\overline{G}$-invariant, while the products are symmetric, it is sufficient to calculate the values of the products (x, y) and $x \cdot y$ with $x, y \in \mathcal{D}$ for the representatives of the orbits of $\overline{G}$ on the set of unordered pairs of vectors from $\mathcal{D}$.

5.1. INNER PRODUCTS

All inner products between two 2-axes are given by Norton-Sakuma algebras.

Inner products (a_t, u_h), where $t \in T$ and h is an element of order 3 in G, can be computed in a subalgebra A_L, where L is a maximal subgroup isomorphic to $2 \times S_4$ (and so it is known by the results of Chapter 4) unless $\langle t, h \rangle = \overline{G}$.
All pairs $(t, \langle h \rangle)$ with this property are in a unique $\overline{G}$-orbit and so the inner products (a_t, u_h) for all pairs $(t, \langle h \rangle)$ such that $\langle t, h \rangle = \overline{G}$ are all equal. We set

$$x_1 := (a_t, u_h) \text{ if } \langle t, h \rangle = \overline{G}. \tag{5.1}$$

Inner products (a_t, v_g), where $t \in T$ and g is an element of order 4 in G, can be computed in a subalgebra A_L, where L is a maximal subgroup isomorphic to $2 \times S_4$ (and so it is known by the results of Chapter 4) unless $\langle t, g \rangle = \overline{G}$.
All pairs $(t, \langle g \rangle)$ with this property are in a unique $\overline{G}$-orbit and so the inner products (a_t, v_g) for all pairs $(t, \langle g \rangle)$ such that $\langle t, g \rangle = \overline{G}$ are all equal.

Lemma 5.1. *Let $t \in T$ and let g be an element of order 4 in G such that $\langle t, g \rangle = \overline{G}$. Then $(a_t, v_g) = \frac{45}{32}x_1 - \frac{1}{128}$.*

Proof. It is enough to consider the pair $a_{(4,5)(6,7)(8,9)}, v_{(1,2,7,4)(3,5)}$.

In this case $v_{(1,2,7,4)(3,5)} = a_{(2,4)(3,5)(8,9)} + a_{(1,4)(2,7)(8,9)} + \frac{1}{3}a_{(1,2)(4,7)(8,9)} + \frac{1}{3}a_{(1,7)(3,5)(8,9)} - \frac{64}{3}a_{(2,4)(3,5)(8,9)} \cdot a_{(1,4)(2,7)(8,9)}$ as we see inside the algebra $4A$ generated by $a_{(2,4)(3,5)(8,9)}$ and $a_{(1,4)(2,7)(8,9)}$. Since $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}
(a_{(4,5)(6,7)(8,9)}, v_{(1,2,7,4)(3,5)}) &= (a_{(4,5)(6,7)(8,9)}, a_{(2,4)(3,5)(8,9)} + a_{(1,4)(2,7)(8,9)} \\
&\quad + \tfrac{1}{3}a_{(1,2)(4,7)(8,9)} + \tfrac{1}{3}a_{(1,7)(3,5)(8,9)}) \\
&\quad - \tfrac{64}{3}(a_{(4,5)(6,7)(8,9)}, a_{(2,4)(3,5)(8,9)} \cdot a_{(1,4)(2,7)(8,9)}) \\
&= (a_{(4,5)(6,7)(8,9)}, a_{(2,4)(3,5)(8,9)} + a_{(1,4)(2,7)(8,9)} \\
&\quad + \tfrac{1}{3}a_{(1,2)(4,7)(8,9)} + \tfrac{1}{3}a_{(1,7)(3,5)(8,9)}) \\
&\quad - \tfrac{64}{3}(a_{(2,4)(3,5)(8,9)}, a_{(4,5)(6,7)(8,9)} \cdot a_{(1,4)(2,7)(8,9)}).
\end{aligned}$$

Now $a_{(4,5)(6,7)(8,9)}$ and $a_{(1,4)(2,7)(8,9)}$ generate an algebra of type $3A$ with 3-axis $u_h = u_{(1,4,5)(2,6,7)}$ such that $\langle (2,4)(3,5)(8,9), h \rangle = \overline{G}$. Hence the result follows when we substitute the product $a_{(4,5)(6,7)(8,9)} \cdot a_{(1,4)(2,7)(8,9)}$ by its expression as a linear combination of axes in the subalgebra $\langle\langle a_{(4,5)(6,7)(8,9)}, a_{(1,4)(2,7)(8,9)} \rangle\rangle$.

□

Inner products $(a_t, \overline{v}_{g(8,9)})$, where $t \in T$ and g is an element of order 4 in G, can be computed in a subalgebra A_L, where L is a maximal subgroup isomorphic to $2 \times S_4$ (and so it is known by the results of Chapter 4) unless $\langle t, g(8,9) \rangle = \overline{G}$.

All pairs $(t, \langle g(8,9)\rangle)$ with this property are in a unique $\overline{G}$-orbit and so the inner products $(a_t, \overline{v}_{g(8,9)})$ for all pairs $(t, \langle g(8,9)\rangle)$ such that $\langle t, g(8,9)\rangle = \overline{G}$ are all equal.

Lemma 5.2. *Let $t \in T$ and let g be an element of order 4 in G such that $\langle t, g(8,9)\rangle = \overline{G}$. Then $(a_t, \overline{v}_{g(8,9)}) = \frac{15}{16}x_1 + \frac{29}{768}$.*

Proof. It is enough to consider the pair $a_{(2,7)(3,6)(8,9)}, \overline{v}_{(1,3)(4,7,6,5)(8,9)}$.
Let us consider the following 1/4-eigenvector of $a_{(2,3)(6,7)(8,9)}$ constructed as in the Lemma 1.18:

$$\begin{aligned} c =& 4(a_{(2,3)(6,7)(8,9)} \cdot (\overline{v}_{(1,3)(4,7,6,5)(8,9)} + \overline{v}_{(1,2)(4,6,7,5)(8,9)}) - \\ & (\overline{v}_{(1,3)(4,7,6,5)(8,9)} + \overline{v}_{(1,2)(4,6,7,5)(8,9)}, a_{(2,3)(6,7)(8,9)})a_{(2,3)(6,7)(8,9)}). \end{aligned}$$

Since $(2,7)(3,6)(8,9)$ commutes with $(2,3)(6,7)(8,9)$, $a_{(2,7)(3,6)(8,9)}$ is a 0-eigenvector of $a_{(2,3)(6,7)(8,9)}$.

Since c and $a_{(2,7)(3,6)(8,9)}$ are eigenvectors of $a_{(2,3)(6,7)(8,9)}$ with different eigenvalues, they are perpendicular and $(a_{(2,7)(3,6)(8,9)}, c) = 0$ by Lemma 1.9.
When we substitute the expression for c in $(a_{(2,7)(3,6)(8,9)}, c)$ we see that the only unknown products are

$$(a_{(2,7)(3,6)(8,9)}, \overline{v}_{(1,3)(4,7,6,5)(8,9)}) \text{ and } (a_{(2,7)(3,6)(8,9)}, \overline{v}_{(1,2)(4,6,7,5)(8,9)}).$$

Since $(2,7)(3,6)(8,9)$ generates $\overline{G}$ with both $(1,3)(4,7,6,5)(8,9)$ and $(1,2)(4,6,7,5)(8,9)$, the inner products of $a_{(2,7)(3,6)(8,9)}$ with $\overline{v}_{(1,3)(4,7,6,5)(8,9)}$ and $\overline{v}_{(1,2)(4,6,7,5)(8,9)}$ are equal, and solving the equation we get the result.

□

Inner products (u_h, u_g), where h and g are elements of order 3 in G, can be computed in a subalgebra A_L, where L is a maximal subgroup isomorphic to $2 \times S_4$ (and so it is known by the results of Chapter 4) unless $\langle h, g\rangle \cong C_7 : C_3$ or $\langle h, g\rangle = G$.

The inner products for all pairs $(\langle h\rangle, \langle g\rangle)$ such that $\langle h, g\rangle \cong C_7 : C_3$ are all equal, since there are two orbitals that are the transpose of each other and the scalar product is symmetric. Hence we only consider one of them.

Lemma 5.3. *Let h and g be two elements of order 3 in G such that $\langle h, g\rangle \cong C_7 : C_3$. Then $(u_h, u_g) = \frac{64}{135}x_1 - \frac{812}{6075}$.*

Proof. We may assume $h = (1,4,5)(3,6,7)$ and $g = (1,2,3)(4,5,7)$.

In this case $u_{(1,4,5)(3,6,7)} = \frac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}) - \frac{2048}{135}a_{(4,5)(6,7)(8,9)} \cdot a_{(1,5)(3,7)(8,9)}$ as we see inside the algebra $3A$ generated by $a_{(4,5)(6,7)(8,9)}$ and $a_{(1,5)(3,7)(8,9)}$ and since $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}
(u_h, u_g) &= \tfrac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}, u_{(1,2,3)(4,5,7)}) \\
&\quad - \tfrac{2048}{135}(a_{(4,5)(6,7)(8,9)} \cdot a_{(1,5)(3,7)(8,9)}, u_{(1,2,3)(4,5,7)}) \\
&= \tfrac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}, u_{(1,2,3)(4,5,7)}) \\
&\quad - \tfrac{2048}{135}(a_{(4,5)(6,7)(8,9)}, a_{(1,5)(3,7)(8,9)} \cdot u_{(1,2,3)(4,5,7)}).
\end{aligned}$$

Now the product $a_{(1,5)(3,7)(8,9)} \cdot u_{(1,2,3)(4,5,7)}$ can be written as a combination of vectors in $\mathcal{D}$ because it is contained in a subalgebra S_4 and hence we know all the inner products between them and the axis $a_{(4,5)(6,7)(8,9)}$ in terms of x_1.

Then we obtain $(u_{(1,4,5)(3,6,7)}, u_{(1,2,3)(4,5,7)}) = \frac{64}{135}x_1 - \frac{812}{6075}$.

□

Now let h, g be two elements of order 3 in G such that $\langle h, g\rangle = G$.
Pairs $(\langle h\rangle, \langle g\rangle)$ of this type fall into two different $\overline{G}$-orbitals:
- in one of them the orders of the products are $\{o(hg), o(hg^{-1})\} = \{4, 7\}$,
- in the other one the orders are both 4.

We set

$$x_2 := (u_h, u_g), \text{ if } \langle h, g\rangle = G \text{ and } \{o(hg), o(hg^{-1})\} = \{4, 7\} \tag{5.2}$$

Lemma 5.4. *Let h and g be two elements of order 3 in G such that $\langle h, g\rangle = G$ and $o(hg) = o(hg^{-1}) = 4$. Then $(u_h, u_g) = -\frac{128}{135}x_1 + \frac{208}{1215}$.*

Proof. We may assume $h = (1,4,5)(3,6,7)$ and $g = (1,4,2)(3,5,6)$.

In this case $u_{(1,4,2)(3,5,6)} = \frac{64}{135}(2a_{(2,4)(3,5)(8,9)} + 2a_{(1,4)(3,6)(8,9)} + a_{(1,2)(5,6)(8,9)}) - \frac{2048}{135}a_{(2,4)(3,5)(8,9)} \cdot a_{(1,4)(3,6)(8,9)}$ as we see inside the algebra $3A$ generated by $a_{(2,4)(3,5)(8,9)}$ and $a_{(1,4)(3,6)(8,9)}$ and since $(,)$ associates with $\cdot$ we obtain:

$$
\begin{aligned}
(u_h, u_g) &= \tfrac{64}{135}(2a_{(2,4)(3,5)(8,9)} + 2a_{(1,4)(3,6)(8,9)} + a_{(1,2)(5,6)(8,9)}, u_{(1,4,5)(3,6,7)}) \\
&- \tfrac{2048}{135}(a_{(2,4)(3,5)(8,9)} \cdot a_{(1,4)(3,6)(8,9)}, u_{(1,4,5)(3,6,7)}) \\
&= \tfrac{64}{135}(2a_{(2,4)(3,5)(8,9)} + 2a_{(1,4)(3,6)(8,9)} + a_{(1,2)(5,6)(8,9)}, u_{(1,4,5)(3,6,7)}) \\
&- \tfrac{2048}{135}(a_{(2,4)(3,5)(8,9)}, a_{(1,4)(3,6)(8,9)} \cdot u_{(1,4,5)(3,6,7)}).
\end{aligned}
$$

Now the product $a_{(1,4)(3,6)(8,9)} \cdot u_{(1,4,5)(3,6,7)}$ can be written as a combination of vectors in $\mathcal{D}$ because it is contained in the subalgebra $3A$ and hence we know all the inner products between them and the axis $a_{(2,4)(3,5)(8,9)}$ in terms of x_1.

Then we obtain $(u_{(1,4,5)(3,6,7)}, u_{(1,4,2)(3,5,6)}) = -\frac{128}{135}x_1 + \frac{208}{1215}$.

□

Inner products (u_h, v_g), where h is an element of order 3 and g is an element of order 4 in G, can be computed in a subalgebra A_L, where L is a maximal subgroup isomorphic to $2 \times S_4$ (and so it is known by the results of Chapter 4) unless $\langle g, h\rangle = G$.

Pairs $(\langle h\rangle, \langle g\rangle)$ of this type such that $\langle h, g\rangle = G$, fall into four different $\overline{G}$-orbitals:

- in the first two of them the orders of the products are $\{o(hg), o(hg^{-1})\} = \{3, 4\}$, these are two cases but we will see that the values of the inner product on these two orbitals are equal,
- in the third one the orders are both 7,

- in the fourth one the orders are $\{o(hg), o(hg^{-1})\} = \{3, 7\}$.

Lemma 5.5. *Let h be an element of order 3 and g an element of order 4 in G such that $\langle h, g\rangle = G$, $o(hg) = 4$ and $o(hg^{-1}) = 3$.*
Then $(u_h, v_g) = -2x_1 - \frac{14}{45}$.

Proof. Since there are two orbitals it is enough to compute the inner products (u_h, v_g) and (u_h, v_{g_1}) with $h = (1,4,5)(3,6,7)$, $g = (2,4,3,5)(6,7)$ and $g_1 = (1,2)(4,6,7,5)$.

In this case $u_{(1,4,5)(3,6,7)} = \frac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}) - \frac{2048}{135} a_{(4,5)(6,7)(8,9)} \cdot a_{(1,5)(3,7)(8,9)}$ as we see inside the algebra $3A$ generated by $a_{(4,5)(6,7)(8,9)}$ and $a_{(1,5)(3,7)(8,9)}$ and since $(,)$ associates with $\cdot$ we obtain:

$$
\begin{aligned}
(u_h, v_g) &= \tfrac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}, v_{(2,4,3,5)(6,7)}) \\
&- \tfrac{2048}{135}(a_{(4,5)(6,7)(8,9)} \cdot a_{(1,5)(3,7)(8,9)}, v_{(2,4,3,5)(6,7)}) \\
&= \tfrac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}, v_{(2,4,3,5)(6,7)}) \\
&- \tfrac{2048}{135}(a_{(1,5)(3,7)(8,9)}, a_{(4,5)(6,7)(8,9)} \cdot v_{(2,4,3,5)(6,7)}).
\end{aligned}
$$

Now the product $a_{(4,5)(6,7)(8,9)} \cdot v_{(2,4,3,5)(6,7)}$ can be written as a combination of vectors in $\mathcal{D}$ because it is contained in the subalgebra $4A$ and hence we know all the inner products between them and the axis $a_{(1,5)(3,7)(8,9)}$ in terms of x_1.

Then we obtain $(u_{(1,4,5)(3,6,7)}, v_{(2,4,3,5)(6,7)}) = -2x_1 - \frac{14}{45}$.

Computation of (u_h, v_{g_1}) is analogous.

$\square$

Lemma 5.6. *Let h be an element of order 3 and g an element of order 4 in G such that $\langle h, g\rangle = G$ and $o(hg) = o(hg^{-1}) = 7$. Then $(u_h, v_g) = 2x_1 - \frac{1}{90}$.*

Proof. It is enough to consider the pair $u_{(2,4,6)(3,5,7)}, v_{(1,3)(4,7,6,5)}$.

In this case $v_{(1,3)(4,7,6,5)} = a_{(4,5)(6,7)(8,9)} + a_{(1,3)(4,6)(8,9)} + \frac{1}{3}a_{(1,3)(5,7)(8,9)} + \frac{1}{3}a_{(4,7)(5,6)(8,9)} - \frac{64}{3}a_{(4,5)(6,7)(8,9)} \cdot a_{(1,3)(4,6)(8,9)}$ as we see inside the algebra $4A$ generated by $a_{(4,5)(6,7)(8,9)}$ and $a_{(1,3)(4,6)(8,9)}$ and since $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}
(u_{(2,4,6)(3,5,7)}, v_{(1,3)(4,7,6,5)}) &= (u_{(2,4,6)(3,5,7)}, a_{(4,5)(6,7)(8,9)} + a_{(1,3)(4,6)(8,9)} \\
&+ \tfrac{1}{3}a_{(1,3)(5,7)(8,9)} + \tfrac{1}{3}a_{(4,7)(5,6)(8,9)}) \\
&- \tfrac{64}{3}(u_{(2,4,6)(3,5,7)}, a_{(4,5)(6,7)(8,9)} \cdot a_{(1,3)(4,6)(8,9)}) \\
&= (u_{(2,4,6)(3,5,7)}, a_{(4,5)(6,7)(8,9)} + a_{(1,3)(4,6)(8,9)} \\
&+ \tfrac{1}{3}a_{(1,3)(5,7)(8,9)} + \tfrac{1}{3}a_{(4,7)(5,6)(8,9)}) \\
&- \tfrac{64}{3}(a_{(4,5)(6,7)(8,9)}, u_{(2,4,6)(3,5,7)} \cdot a_{(1,3)(4,6)(8,9)}).
\end{aligned}$$

Now the product $a_{(1,3)(4,6)(8,9)} \cdot u_{(2,4,6)(3,5,7)}$ can be written as a combination of vectors in $\mathcal{D}$ because it is contained in a subalgebra S_4 and hence we know all the inner products between them and the axis $a_{(4,5)(6,7)(8,9)}$ in terms of x_1.

Then we obtain $(u_{(2,4,6)(3,5,7)}, v_{(1,3)(4,7,6,5)}) = 2x_1 - \frac{1}{90}$.

□

Lemma 5.7. *Let h be an element of order 3 and g an element of order 4 in G such that $\langle h, g\rangle = G$, $o(hg) = 7$ and $o(hg^{-1}) = 3$.*
Then $(u_h, v_g) = \frac{2}{3}x_1 - \frac{191}{1080}$.

Proof. It is enough to consider the pair $u_{(1,4,5)(2,3,7)}, v_{(1,3)(4,7,6,5)}$.

In this case $v_{(1,3)(4,7,6,5)} = a_{(4,5)(6,7)(8,9)} + a_{(1,3)(5,7)(8,9)} + \frac{1}{3}a_{(1,3)(4,6)(8,9)} + \frac{1}{3}a_{(4,7)(5,6)(8,9)} - \frac{64}{3}a_{(4,5)(6,7)(8,9)} \cdot a_{(1,3)(5,7)(8,9)}$ as we see inside the algebra $4A$ generated by $a_{(4,5)(6,7)(8,9)}$ and $a_{(1,3)(5,7)(8,9)}$ and since $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}(u_{(1,4,5)(2,3,7)}, v_{(1,3)(4,7,6,5)}) &= (u_{(1,4,5)(2,3,7)}, a_{(4,5)(6,7)(8,9)} + a_{(1,3)(5,7)(8,9)} \\ &+ \tfrac{1}{3}a_{(1,3)(4,6)(8,9)} + \tfrac{1}{3}a_{(4,7)(5,6)(8,9)}) \\ &- \tfrac{64}{3}(u_{(1,4,5)(2,3,7)}, a_{(4,5)(6,7)(8,9)} \cdot a_{(1,3)(5,7)(8,9)}) \\ &= (u_{(1,4,5)(2,3,7)}, a_{(4,5)(6,7)(8,9)} + a_{(1,3)(5,7)(8,9)} \\ &+ \tfrac{1}{3}a_{(1,3)(4,6)(8,9)} + \tfrac{1}{3}a_{(4,7)(5,6)(8,9)}) \\ &- \tfrac{64}{3}(a_{(4,5)(6,7)(8,9)}, u_{(1,4,5)(2,3,7)} \cdot a_{(1,3)(5,7)(8,9)}).\end{aligned}$$

Now the product $a_{(1,3)(5,7)(8,9)} \cdot u_{(1,4,5)(2,3,7)}$ can be written as a combination of vectors in $\mathcal{D}$ because it is contained in a subalgebra S_4 and hence we know all the inner products between them and the axis $a_{(4,5)(6,7)(8,9)}$ in terms of x_1.

Then we obtain $(u_{(1,4,5)(2,3,7)}, v_{(1,3)(4,7,6,5)}) = \frac{2}{3}x_1 - \frac{191}{1080}$.

□

Inner products $(u_h, \overline{v}_{g(8,9)})$, where h is an element of order 3 and g is an element of order 4 in G, can be computed in a subalgebra A_L, where L is a maximal subgroup isomorphic to $2 \times S_4$ (and so it is known by the results of Chapter 4) unless $\langle h, g(8,9)\rangle = \overline{G}$.
Pairs $(\langle h\rangle, \langle g(8,9)\rangle)$ of this type such that $\langle h, g(8,9)\rangle = \overline{G}$, fall into four different $\overline{G}$-orbitals:
- in the first two the orders of the products are $\{o(hg(8,9)), o(hg^{-1}(8,9))\} = \{4, 6\}$, these are two cases but we will see that the values of the inner products are the same,
- in the third one the orders are both 14,
- in the fourth one the orders are $\{o(hg(8,9)), o(hg^{-1}(8,9))\} = \{6, 14\}$.

We set

$$x_3 := (u_h, \overline{v}_{g(8,9)}), \text{ if } \langle h, g(8,9)\rangle = \overline{G} \text{ and } \{o(hg(8,9)), o(hg^{-1}(8,9))\} = \{14\} \tag{5.3}$$

$$x_4 := (u_h, \overline{v}_{g(8,9)}), \text{ if } \langle h, g(8,9)\rangle = \overline{G} \text{ and } \{o(hg(8,9)), o(hg^{-1}(8,9))\} = \{6, 14\} \tag{5.4}$$

$$x_5 := (u_h, \overline{v}_{g(8,9)}), \text{ if } \langle h, g(8,9)\rangle = \overline{G} \text{ and } \{o(hg(8,9)), o(hg^{-1}(8,9))\} = \{4,6\} \tag{5.5}$$

for the first orbital.

$$x_6 := (u_h, \overline{v}_{g(8,9)}), \text{ if } \langle h, g(8,9)\rangle = \overline{G} \text{ and } \{o(hg(8,9)), o(hg^{-1}(8,9))\} = \{4,6\} \tag{5.6}$$

for the second orbital.

Inner products (v_h, v_g), where h and g are elements of order 4 in G, can be computed in a subalgebra A_L, where L is a maximal subgroup isomorphic to $2 \times S_4$ (and so it is known by the results of Chapter 4) unless $\langle h, g\rangle = G$.

Pairs $(\langle h\rangle, \langle g\rangle)$ of this type such that $\langle h, g\rangle = G$, fall into three different $\overline{G}$-orbitals:

- in the first two the orders of the products $\{o(hg), o(hg^{-1})\} = \{3,4\}$, these are two cases but the two orbitals are one the transposed of the other and the inner product is symmetric, so the values of the inner products are equal, hence we only consider one of them,

- in the third one the orders are $\{o(hg), o(hg^{-1})\} = \{4,7\}$.

Lemma 5.8. *Let h and g be two elements of order 4 in G such that $\langle h, g\rangle = G$, $o(hg) = 4$ and $o(hg^{-1}) = 3$. Then $(v_h, v_g) = -\frac{15}{4}x_1 + \frac{1}{2}$.*

Proof. It is enough to consider the pair $v_{(2,4,3,5)(6,7)}, v_{(1,2,7,4)(3,5)}$.

In this case $v_{(1,2,7,4)(3,5)} = a_{(2,4)(3,5)(8,9)} + a_{(1,4)(2,7)(8,9)} + \frac{1}{3}a_{(1,2)(4,7)(8,9)} + \frac{1}{3}a_{(1,7)(3,5)(8,9)} - \frac{64}{3}a_{(2,4)(3,5)(8,9)} \cdot a_{(1,4)(2,7)(8,9)}$ as we see inside the algebra $4A$ generated by $a_{(2,4)(3,5)(8,9)}$ and $a_{(1,4)(2,7)(8,9)}$ and since $(,)$ associates with $\cdot$ we obtain:

$$
\begin{aligned}
(v_{(2,4,3,5)(6,7)}, v_{(1,2,7,4)(3,5)}) &= (v_{(2,4,3,5)(6,7)}, a_{(2,4)(3,5)(8,9)} + a_{(1,4)(2,7)(8,9)} \\
&+ \tfrac{1}{3}a_{(1,2)(4,7)(8,9)} + \tfrac{1}{3}a_{(1,7)(3,5)(8,9)}) \\
&- \tfrac{64}{3}(v_{(2,4,3,5)(6,7)}, a_{(2,4)(3,5)(8,9)} \cdot a_{(1,4)(2,7)(8,9)}) \\
&= (v_{(2,4,3,5)(6,7)}, a_{(2,4)(3,5)(8,9)} + a_{(1,4)(2,7)(8,9)} \\
&+ \tfrac{1}{3}a_{(1,2)(4,7)(8,9)} + \tfrac{1}{3}a_{(1,7)(3,5)(8,9)}) \\
&- \tfrac{64}{3}(a_{(2,4)(3,5)(8,9)}, v_{(2,4,3,5)(6,7)} \cdot a_{(1,4)(2,7)(8,9)}).
\end{aligned}
$$

Now the product $a_{(2,4)(3,5)(8,9)} \cdot v_{(2,4,3,5)(6,7)}$ can be written as a combination of vectors in $\mathcal{D}$ because it is contained in the subalgebra $4A$ and hence we know all the inner products between them and the axis $a_{(1,4)(2,7)(8,9)}$ in terms of x_1.

Then we obtain $(v_{(2,4,3,5)(6,7)}, v_{(1,2,7,4)(3,5)}) = -\frac{15}{4}x_1 + \frac{1}{2}$.

□

Lemma 5.9. *Let h and g be two elements of order 4 in G such that $\langle h, g\rangle = G$, $o(hg) = 4$ and $o(hg^{-1}) = 7$. Then $(v_h, v_g) = \frac{2025}{1024}x_2 + \frac{1}{64}$.*

Proof. We may assume $h = (2,4,3,5)(6,7)$ and $g = (1,6,3,4)(5,7)$.

In this case $v_{(2,4,3,5)(6,7)} = a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} + \frac{1}{3}a_{(2,3)(6,7)(8,9)} + \frac{1}{3}a_{(2,5)(3,4)(8,9)} - \frac{64}{3}a_{(4,5)(6,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}$ as we see inside the algebra $4A$ generated by $a_{(4,5)(6,7)(8,9)}$ and $a_{(2,4)(3,5)(8,9)}$ and $v_{(1,6,3,4)(5,7)} = a_{(4,6)(5,7)(8,9)} + a_{(1,4)(3,6)(8,9)} + \frac{1}{3}a_{(1,3)(5,7)(8,9)} + \frac{1}{3}a_{(1,6)(3,4)(8,9)} - \frac{64}{3}a_{(4,6)(5,7)(8,9)} \cdot a_{(1,4)(3,6)(8,9)}$ as we see inside the algebra $4A$ generated by $a_{(4,6)(5,7)(8,9)}$ and $a_{(1,4)(3,6)(8,9)}$.

Then due to the fact that $(,)$ associates with $\cdot$ we obtain:

$$
\begin{aligned}
(v_h, v_g) = {}& \big(a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} + \tfrac{1}{3}a_{(2,3)(6,7)(8,9)} \\
& + \tfrac{1}{3}a_{(2,5)(3,4)(8,9)} - \tfrac{64}{3}a_{(4,5)(6,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}, \\
& \quad a_{(4,6)(5,7)(8,9)} + a_{(1,4)(3,6)(8,9)} + \tfrac{1}{3}a_{(1,3)(5,7)(8,9)} \\
& + \tfrac{1}{3}a_{(1,6)(3,4)(8,9)} - \tfrac{64}{3}a_{(4,6)(5,7)(8,9)} \cdot a_{(1,4)(3,6)(8,9)}\big) \\
& = \tfrac{64\cdot 64}{3\cdot 3}\big(a_{(4,6)(5,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}, a_{(1,4)(3,6)(8,9)} \cdot a_{(4,5)(6,7)(8,9)}\big) \\
& - \tfrac{64}{3}\big(a_{(4,6)(5,7)(8,9)}, a_{(1,4)(3,6)(8,9)} \cdot 3a_{(4,5)(6,7)(8,9)}\big) \\
& - \tfrac{64}{3}\big(a_{(4,6)(5,7)(8,9)}, a_{(1,4)(3,6)(8,9)} \cdot 3a_{(2,4)(3,5)(8,9)}\big) \\
& - \tfrac{64}{3}\big(a_{(4,6)(5,7)(8,9)}, a_{(1,4)(3,6)(8,9)} \cdot a_{(2,3)(6,7)(8,9)}\big) \\
& - \tfrac{64}{3}\big(a_{(4,6)(5,7)(8,9)}, a_{(1,4)(3,6)(8,9)} \cdot a_{(2,5)(3,4)(8,9)}\big) \\
& + 3\big(a_{(4,6)(5,7)(8,9)}, v_{(2,4,3,5)(6,7)}\big) + 3\big(a_{(1,4)(3,6)(8,9)}, v_{(2,4,3,5)(6,7)}\big) \\
& + \big(a_{(1,3)(5,7)(8,9)}, v_{(2,4,3,5)(6,7)}\big) + \big(a_{(1,6)(3,4)(8,9)}, v_{(2,4,3,5)(6,7)}\big).
\end{aligned}
$$

Now the algebra products $a_{(4,6)(5,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}$, $a_{(1,4)(3,6)(8,9)} \cdot a_{(4,5)(6,7)(8,9)}$, $a_{(1,4)(3,6)(8,9)} \cdot a_{(4,5)(6,7)(8,9)}$, $a_{(1,4)(3,6)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}$, $a_{(1,4)(3,6)(8,9)} \cdot a_{(2,3)(6,7)(8,9)}$, $a_{(1,4)(3,6)(8,9)} \cdot a_{(2,5)(3,4)(8,9)}$ can be written as combinations of 2-axes, 3-axes and 4-axes and hence we know all the inner products between them and the other two axes in terms of x_2.

Then we obtain $(v_{(2,4,3,5)(6,7)}, v_{(1,6,3,4)(5,7)}) = \frac{2025}{1024}x_2 + \frac{1}{64}$.

□

Inner products $(v_h, \overline{v}_{g(8,9)})$, where h and g are elements of order 4 in G, can be computed in a subalgebra A_L, where L is a maximal subgroup isomorphic to $2 \times S_4$ (and so it is known by the results of Chapter 4) unless $\langle h, g(8,9)\rangle = \overline{G}$. Pairs $(\langle h\rangle, \langle g(8,9)\rangle)$ of this type such that $\langle h, g(8,9)\rangle = \overline{G}$, fall into three different $\overline{G}$-orbitals:
- in the first one the orders are $\{o(hg(8,9)), o(hg^{-1}(8,9))\} = \{4, 14\}$,
- in the last two the orders are $\{o(hg(8,9)), o(hg^{-1}(8,9))\} = \{4, 6\}$, these are two cases but we will see that the values of the inner products on the two orbitals are equal.

Lemma 5.10. *Let h and g be two elements of order 4 in G such that $\langle h, g(8,9)\rangle = \overline{G}$, $o(hg(8,9)) = 4$ and $o(hg^{-1}(8,9)) = 14$. Then $(v_h, \overline{v}_{g(8,9)}) = \frac{25}{16}x_1 + \frac{55}{2304}$.*

Proof. It is enough to consider the pair $v_{(2,4,3,5)(6,7)}, \overline{v}_{(1,5,2,6)(4,7)(8,9)}$.

In this case $v_{(2,4,3,5)(6,7)} = a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} + \frac{1}{3}a_{(2,3)(6,7)(8,9)} + \frac{1}{3}a_{(2,5)(3,4)(8,9)} - \frac{64}{3}a_{(4,5)(6,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}$ as we see inside the algebra $4A$ generated by $a_{(4,5)(6,7)(8,9)}$ and $a_{(2,4)(3,5)(8,9)}$.

$$
\begin{aligned}
(v_{(2,4,3,5)(6,7)}, \overline{v}_{(1,5,2,6)(4,7)(8,9)}) &= (\overline{v}_{(1,5,2,6)(4,7)(8,9)}, a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} \\
&\quad + \tfrac{1}{3}a_{(2,3)(6,7)(8,9)} + \tfrac{1}{3}a_{(2,5)(3,4)(8,9)}) \\
&\quad - \tfrac{64}{3}(\overline{v}_{(1,5,2,6)(4,7)(8,9)}, a_{(4,5)(6,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}) \\
&= (\overline{v}_{(1,5,2,6)(4,7)(8,9)}, a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} \\
&\quad + \tfrac{1}{3}a_{(2,3)(6,7)(8,9)} + \tfrac{1}{3}a_{(2,5)(3,4)(8,9)}) \\
&\quad - \tfrac{64}{3}(\overline{v}_{(1,5,2,6)(4,7)(8,9)} \cdot a_{(4,5)(6,7)(8,9)}, a_{(2,4)(3,5)(8,9)}).
\end{aligned}
$$

Now the product $a_{(4,5)(6,7)(8,9)} \cdot \overline{v}_{(1,5,2,6)(4,7)(8,9)}$ can be written as a combination of vectors in $\mathcal{D}$ from [27] and hence we know all the inner products between them and the axis $a_{(2,4)(3,5)(8,9)}$ in terms of x_1.

Then we obtain $v_{(2,4,3,5)(6,7)}, \overline{v}_{(1,5,2,6)(4,7)(8,9)} = \frac{25}{16}x_1 + \frac{55}{2304}$.

□

Lemma 5.11. *Let h and g be two elements of order 4 in G such that $\langle h, g(8,9)\rangle = \overline{G}$, $o(hg(8,9)) = 4$ and $o(hg^{-1}(8,9)) = 6$. Then $(v_h, \overline{v}_{g(8,9)}) = \frac{25}{96}$.*

Proof. Since there are two orbitals it is enough to compute the inner products $(v_h, \overline{v}_{g(8,9)})$ and $(v_h, \overline{v}_{g_1(8,9)})$ with $h = (2,4,3,5)(6,7)$, $g = (1,2,7,4)(3,5)$ and $g_1 = (1,3)(4,7,6,5)$.

In this case $v_{(2,4,3,5)(6,7)} = a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} + \frac{1}{3}a_{(2,3)(6,7)(8,9)} + \frac{1}{3}a_{(2,5)(3,4)(8,9)} - \frac{64}{3}a_{(4,5)(6,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}$ as we see inside the algebra $4A$ generated by $a_{(4,5)(6,7)(8,9)}$ and $a_{(2,4)(3,5)(8,9)}$.

$$
\begin{aligned}
(v_{(2,4,3,5)(6,7)}, \overline{v}_{(1,2,7,4)(3,5)(8,9)}) = {} & (\overline{v}_{(1,2,7,4)(3,5)(8,9)}, a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} \\
& + \tfrac{1}{3}a_{(2,3)(6,7)(8,9)} + \tfrac{1}{3}a_{(2,5)(3,4)(8,9)}) \\
& - \tfrac{64}{3}(\overline{v}_{(1,2,7,4)(3,5)(8,9)}, a_{(4,5)(6,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}) \\
= {} & (\overline{v}_{(1,2,7,4)(3,5)(8,9)}, a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} \\
& + \tfrac{1}{3}a_{(2,3)(6,7)(8,9)} + \tfrac{1}{3}a_{(2,5)(3,4)(8,9)}) \\
& - \tfrac{64}{3}(\overline{v}_{(1,2,7,4)(3,5)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}, a_{(4,5)(6,7)(8,9)}).
\end{aligned}
$$

Now the product $a_{(2,4)(3,5)(8,9)} \cdot \overline{v}_{(1,2,7,4)(3,5)(8,9)}$ can be written as a combination of vectors in $\mathcal{B}$ from [27] and hence we know all the inner products between them and the axis $a_{(4,5)(6,7)(8,9)}$.

Then we obtain $(v_{(2,4,3,5)(6,7)}, \overline{v}_{(1,2,7,4)(3,5)(8,9)}) = \frac{25}{96}$.

The computation of $(v_h, \overline{v}_{g_1(8,9)})$ is analogous.

□

Inner products $(\overline{v}_{h(8,9)}, \overline{v}_{g(8,9)})$, where h and g are elements of order 4 in G, can be computed in a subalgebra A_L, where L is a maximal subgroup isomorphic to $2 \times S_4$ (and so it is known by the results of Chapter 4) unless $\langle h(8,9), g(8,9) \rangle = \overline{G}$. Pairs $(\langle h(8,9) \rangle, \langle g(8,9) \rangle)$ of this type such that $\langle h(8,9), g(8,9) \rangle = \overline{G}$, fall into three different $\overline{G}$-orbitals:

- in the first one the orders of the products are $\{o(hg), o(hg^{-1})\} = \{4, 7\}$,
- in the last two the orders are $\{o(hg), o(hg^{-1})\} = \{3, 4\}$, these are two cases but the two orbitals are one the transposed of the other and the inner product is symmetric, so the values of the inner products are equal. Hence we only consider one of them.

We set

$$x_7 := (\overline{v}_{h(8,9)}, \overline{v}_{g(8,9)}), \text{ if } \langle h(8,9), g(8,9)\rangle = \overline{G} \text{ and } \{o(hg), o(hg^{-1})\} = \{4,7\} \tag{5.7}$$

$$x_8 := (\overline{v}_{h(8,9)}, \overline{v}_{g(8,9)}), \text{ if } \langle h(8,9), g(8,9)\rangle = \overline{G} \text{ and } \{o(hg), o(hg^{-1})\} = \{3,4\} \tag{5.8}$$

This finishes the list of different inner products between two elements of $\mathcal{D}$. In the following lemmas we find relations between them and determine the values of $x_1, ..., x_8$.

Lemma 5.12. *Let $t \in T$ and let h be an element of order 3 in G such that $\langle t, h\rangle = \overline{G}$. Then $(a_t, u_h) = \frac{11}{360}$, that is $x_1 = \frac{11}{360}$.*

Proof. Let us consider $h = (2,4,3,5)(6,7)$ and $g = (1,5,6,2)(3,4)$.
Then $o(hg(8,9)) = 4$ and $o(hg^{-1}(8,9)) = 6$ and so, by Lemma 5.11, we already know $(v_h, \overline{v}_{g(8,9)}) = \frac{25}{96}$.

In this case we also have $v_{(2,4,3,5)(6,7)} = a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} + \frac{1}{3}a_{(2,3)(6,7)(8,9)} + \frac{1}{3}a_{(2,5)(3,4)(8,9)} - \frac{64}{3}a_{(4,5)(6,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}$ as we see inside the algebra $4A$ generated by $a_{(4,5)(6,7)(8,9)}$ and $a_{(2,4)(3,5)(8,9)}$.

$$\begin{aligned}
(v_{(2,4,3,5)(6,7)}, \overline{v}_{(1,5,6,2)(3,4)(8,9)}) &= (\overline{v}_{(1,5,6,2)(3,4)(8,9)}, a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} \\
&+ \tfrac{1}{3}a_{(2,3)(6,7)(8,9)} + \tfrac{1}{3}a_{(2,5)(3,4)(8,9)}) \\
&- \tfrac{64}{3}(\overline{v}_{(1,5,6,2)(3,4)(8,9)}, a_{(4,5)(6,7)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}) \\
&= (\overline{v}_{(1,5,6,2)(3,4)(8,9)}, a_{(4,5)(6,7)(8,9)} + a_{(2,4)(3,5)(8,9)} \\
&+ \tfrac{1}{3}a_{(2,3)(6,7)(8,9)} + \tfrac{1}{3}a_{(2,5)(3,4)(8,9)}) \\
&- \tfrac{64}{3}(\overline{v}_{(1,5,6,2)(3,4)(8,9)} \cdot a_{(2,4)(3,5)(8,9)}, a_{(4,5)(6,7)(8,9)}).
\end{aligned}$$

Now the product $a_{(2,4)(3,5)(8,9)} \cdot \overline{v}_{(1,5,6,2)(3,4)(8,9)}$ can be written as a combination of vectors in $\mathcal{D}$ from [27] and hence we know all the inner products between them and the axis $a_{(4,5)(6,7)(8,9)}$ in terms of x_1.

Then we obtain $(v_{(2,4,3,5)(6,7)}, \overline{v}_{(1,5,2,6)(4,7)(8,9)}) = -\frac{15}{8}x_1 + \frac{61}{192}$.
So $-\frac{15}{8}x_1 + \frac{61}{192} = \frac{25}{96}$ and thus $x_1 = \frac{11}{360}$.

□

Lemma 5.13. *We have:*

$$x_4 = \tfrac{1}{4}x_6 + \tfrac{1}{80},$$

$$x_5 = \tfrac{15}{4}x_2 + \tfrac{1}{2}x_3 - \tfrac{32}{15}x_7 + \tfrac{16}{15}x_8 - \tfrac{107}{270}.$$

Proof. Let us define the following vectors:

$$\widetilde{u} := u_{(1,4,5)(3,6,7)} + u_{(1,6,7)(3,4,5)} + u_{(1,4,7)(3,6,5)} + u_{(1,6,5)(3,4,7)},$$

$$\widetilde{u}' := u_{(1,4,5)(3,6,7)} + u_{(1,6,7)(3,4,5)},$$

$$\widehat{v} := \overline{v}_{(2,4,3,5)(6,7)(8,9)} + \overline{v}_{(2,3)(4,7,5,6)(8,9)} + \overline{v}_{(2,6,3,7)(4,5)(8,9)}.$$

Now let us consider the 0-eigenvectors $\widetilde{b}_{34}$ and $\overline{b}_{34}$ and the $\frac{1}{4}$-eigenvector c_{24} of $a_{(4,7)(5,6)(8,9)}$ defined as follows (see Lemma 1.18) :

$$\begin{aligned}\widetilde{b}_{34} = \widetilde{u} &- (\widetilde{u}, a_{(4,7)(5,6)(8,9)})a_{(4,7)(5,6)(8,9)} \\ &- 4(a_{(4,7)(5,6)(8,9)} \cdot \widetilde{u} - (\widetilde{u}, a_{(4,7)(5,6)(8,9)})a_{(4,7)(5,6)(8,9)})\end{aligned}$$

$$\begin{aligned}\overline{b}_{34} = \widetilde{u}' &- (\widetilde{u}', a_{(4,7)(5,6)(8,9)})a_{(4,7)(5,6)(8,9)} \\ &- 4(a_{(4,7)(5,6)(8,9)} \cdot \widetilde{u}' - (\widetilde{u}', a_{(4,7)(5,6)(8,9)})a_{(4,7)(5,6)(8,9)})\end{aligned}$$

$$c_{24} = 4(a_{(4,7)(5,6)(8,9)} \cdot \widehat{v} - (\widehat{v}, a_{(4,7)(5,6)(8,9)})a_{(4,7)(5,6)(8,9)})$$

Since $\widetilde{b}_{34}$ and c_{24} are $a_{(4,7)(5,6)(8,9)}$-eigenvectors with different eigenvalues, they are perpendicular and $(\widetilde{b}_{34}, c_{24}) = 0$ by Lemma 1.9.
The explicit expressions of $\widetilde{b}_{34}$ and c_{34} as linear combinations of elements of $\mathcal{D}$ can be computed in subalgebras of A corresponding to maximal subgroups isomorphic to $2 \times S_4$ using the results of Chapter 4. Substituting these expressions

in $(\widetilde{b}_{34}, c_{24}) = 0$ we get :

$$-\tfrac{25}{24}x_2 - \tfrac{5}{36}x_3 + \tfrac{35}{36}x_4 + \tfrac{5}{18}x_5 - \tfrac{35}{144}x_6 + \tfrac{16}{27}x_7 - \tfrac{8}{27}x_8 + \tfrac{1523}{15552} = 0.$$

Similarly, from $(\overline{b}_{34}, c_{24}) = 0$ we get:

$$-\tfrac{25}{24}x_2 - \tfrac{5}{36}x_3 + \tfrac{25}{72}x_4 + \tfrac{5}{18}x_5 - \tfrac{25}{288}x_6 + \tfrac{16}{27}x_7 - \tfrac{8}{27}x_8 + \tfrac{3289}{31104} = 0.$$

We subtract the previous two equations to get $\frac{5}{8}x_4 - \frac{5}{32}x_6 - \frac{1}{128} = 0$.
Then $x_4 = \frac{1}{4}x_6 + \frac{1}{80}$.

So we can also obtain x_5 replacing x_4 in $(\widetilde{b}_{34}, c_{24}) = 0$:

$$x_5 = \tfrac{15}{4}x_2 + \tfrac{1}{2}x_3 - \tfrac{32}{15}x_7 + \tfrac{16}{15}x_8 - \tfrac{107}{270}.$$

$\square$

Lemma 5.14. $x_6 = \frac{15}{8}x_2 + 2x_3 - \frac{32}{15}x_7 - \frac{16}{15}x_8 - \frac{343}{1080}$.

Proof. Let us consider the vector $\widehat{v}$ defined in the proof of the previous lemma and let us define the vector:

$$\widetilde{v} := \overline{v}_{(1,6,3,4)(5,7)(8,9)} + \overline{v}_{(1,5,3,7)(4,6)(8,9)} + \overline{v}_{(1,3)(4,7,6,5)(8,9)}$$

Now let us consider the 0-eigenvector b_{24} and the $\frac{1}{4}$-eigenvector c_{34} of $a_{(4,7)(5,6)(8,9)}$ defined as follows (see Lemma 1.18) :

$$\begin{aligned} b_{24} &= \widehat{v} - (\widehat{v}, a_{(4,7)(5,6)(8,9)})a_{(4,7)(5,6)(8,9)} \\ &\quad - 4(a_{(4,7)(5,6)(8,9)} \cdot \widehat{v} - (\widehat{v}, a_{(4,7)(5,6)(8,9)})a_{(4,7)(5,6)(8,9)}) \\ c_{34} &= 4(a_{(4,7)(5,6)(8,9)} \cdot \widetilde{v} - (\widetilde{v}, a_{(4,7)(5,6)(8,9)})a_{(4,7)(5,6)(8,9)}) \end{aligned}$$

Since b_{24} and c_{34} are $a_{(4,7)(5,6)(8,9)}$-eigenvectors with different eigenvalues, they are perpendicular and $(b_{24}, c_{34}) = 0$ by Lemma 1.9. So proceeding similarly as in the proof of Lemma 5.13 we can obtain x_6.

$\square$

Lemma 5.15. *Let h be an element of order 3 and g an element of order 4 in G such that $\langle h, g(8,9)\rangle = \overline{G}$, $o(hg(8,9)) = 4$ and $o(hg^{-1}(8,9)) = 6$.*
Then $(u_h, \overline{v}_{g(8,9)}) = \frac{19}{108}$, that is $x_5 = x_6 = \frac{19}{108}$.

Proof. We need to compute x_5 and x_6. Let $h = (1,4,5)(3,6,7)$, $g_1 = (2,4,3,5)(6,7)$, and $g_2 = (1,2)(4,6,7,5)$. Then $x_5 = (u_h, \overline{v}_{g_1(8,9)})$ and $x_6 = (u_h, \overline{v}_{g_2(8,9)})$.

In both cases $u_{(1,4,5)(3,6,7)} = \frac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}) - \frac{2048}{135}a_{(4,5)(6,7)(8,9)} \cdot a_{(1,5)(3,7)(8,9)}$ as we see inside the algebra $3A$ generated by $a_{(4,5)(6,7)(8,9)}$ and $a_{(1,5)(3,7)(8,9)}$. Since $(,)$ associates with $\cdot$ we obtain:

$$\begin{aligned}(u_h, \overline{v}_{g_1(8,9)}) &= \tfrac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}, \overline{v}_{(2,4,3,5)(6,7)(8,9)}) \\ &\quad - \tfrac{2048}{135}(a_{(4,5)(6,7)(8,9)} \cdot a_{(1,5)(3,7)(8,9)}, \overline{v}_{(2,4,3,5)(6,7)(8,9)}) \\ &= \tfrac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}, \overline{v}_{(2,4,3,5)(6,7)(8,9)}) \\ &\quad - \tfrac{2048}{135}(a_{(1,5)(3,7)(8,9)}, a_{(4,5)(6,7)(8,9)} \cdot \overline{v}_{(2,4,3,5)(6,7)(8,9)}).\end{aligned}$$

$$\begin{aligned}(u_h, \overline{v}_{g_2(8,9)}) &= \tfrac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}, \overline{v}_{(1,2)(4,6,7,5)(8,9)}) \\ &\quad - \tfrac{2048}{135}(a_{(4,5)(6,7)(8,9)} \cdot a_{(1,5)(3,7)(8,9)}, \overline{v}_{(1,2)(4,6,7,5)(8,9)}) \\ &= \tfrac{64}{135}(2a_{(4,5)(6,7)(8,9)} + 2a_{(1,5)(3,7)(8,9)} + a_{(1,4)(3,6)(8,9)}, \overline{v}_{(1,2)(4,6,7,5)(8,9)}) \\ &\quad - \tfrac{2048}{135}(a_{(1,5)(3,7)(8,9)}, a_{(4,5)(6,7)(8,9)} \cdot \overline{v}_{(1,2)(4,6,7,5)(8,9)}).\end{aligned}$$

Now the products $a_{(4,5)(6,7)(8,9)} \cdot \overline{v}_{(2,4,3,5)(6,7)(8,9)}$ and $a_{(4,5)(6,7)(8,9)} \cdot \overline{v}_{(1,2)(4,6,7,5)(8,9)}$ can be computed in a subalgebra corresponding to a subgroup S_4 of $\overline{G}$ and so, by [27], they can be written as a linear combination of vectors in $\mathcal{D}$.
Hence we know all the inner products between them and the axis $a_{(1,5)(3,7)(8,9)}$.

Then we obtain $(u_{(1,4,5)(3,6,7)}, \overline{v}_{(2,4,3,5)(6,7)(8,9)}) = (u_{(1,4,5)(3,6,7)}, \overline{v}_{(1,2)(4,6,7,5)(8,9)}) = x_5 = x_6 = \frac{19}{108}$.

$\square$

Lemma 5.16. $x_2 = \frac{256}{525}x_7 - \frac{128}{315}x_8 + \frac{5}{27}$ *and* $x_3 = \frac{64}{105}x_7 + \frac{32}{35}x_8 - \frac{11}{45}$.

Proof. By Lemma 5.15, $x_6 = \frac{19}{108}$ and thus, by Lemma 5.13, $x_4 = \frac{1}{4}x_6 + \frac{1}{80} = \frac{61}{1080}$.

By Lemma 5.14 and Lemma 5.13, $x_6 = \frac{15}{8}x_2 + 2x_3 - \frac{32}{15}x_7 - \frac{16}{15}x_8 - \frac{343}{1080}$ and $x_4 = \frac{1}{4}x_6 + \frac{1}{80}$, then $x_4 = \frac{15}{32}x_2 + \frac{1}{2}x_3 - \frac{8}{15}x_7 - \frac{4}{15}x_8 + \frac{397}{4320}$.

Now we can replace $x_6 = \frac{19}{108}$ in the first expression and we can obtain $x_3 = -\frac{15}{2}x_2 + \frac{64}{15}x_7 - \frac{32}{15}x_8 + \frac{103}{90}$.

In the second expression we can replace $x_4 = \frac{61}{1080}$ and the expression of x_3 just found and we obtain $x_2 = \frac{256}{525}x_7 - \frac{128}{315}x_8 + \frac{5}{27}$.

Substituting the expression of x_2 just found into the expression of x_3, we obtain $x_3 = \frac{64}{105}x_7 + \frac{32}{35}x_8 - \frac{11}{45}$.

□

Lemma 5.17. $x_7 = -5x_8 + \frac{413}{256}$.

Proof. The result follows from the orthogonality of c_{34} (defined above) and $\overline{v}_{(1,2)(4,6,7,5)(8,9)}$, that are a $1/4$ and a 0-eigenvector for $a_{(4,7)(5,6)(8,9)}$ respectively.

□

Lemma 5.18. *Let h, g be two elements of order 4 in G such that $\langle h(8,9), g(8,9)\rangle = \overline{G}$ and $o(hg) = 4$ and $o(hg^{-1}) = 3$. Then $(\overline{v}_{h(8,9)}, \overline{v}_{g(8,9)}) = \frac{29}{96}$, that is $x_8 = \frac{29}{96}$.*

Proof. Let us consider the vector $\widetilde{u}$ defined in the proof of the lemma 5.13 and let us consider $\overline{v}_{(1,2)(4,6,7,5)(8,9)}$ which is a 0-eigenvector of $a_{(4,7)(5,6)(8,9)}$ and the following $1/4$-eigenvector of $a_{(4,7)(5,6)(8,9)}$ constructed as in Lemma 1.18:

$$\widetilde{c}_{34} = 4(a_{(4,7)(5,6)(8,9)} \cdot \widetilde{u} - (\widetilde{u}, a_{(4,7)(5,6)(8,9)})a_{(4,7)(5,6)(8,9)})$$

From the orthogonality of $\widetilde{c}_{34}$ and $\overline{v}_{(1,2)(4,6,7,5)}$ we get the result.

□

We can now obtain all the inner products by replacing the values $x_1, ..., x_8$ into the previous lemmas. The following table contains the complete list:

Representative	Orders	Group	Inner Product
$((4,5)(6,7)(8,9),(1,3,6)(2,5,7))$	14, 14	$C_2 \times PSL(3,2)$	11/360
$((4,5)(6,7)(8,9),(1,2,7,4)(3,5))$	14, 14	$C_2 \times PSL(3,2)$	9/256
$((2,7)(3,6)(8,9),(1,3)(4,7,6,5)(8,9))$	7, 7	$C_2 \times PSL(3,2)$	17/256
$((1,4,5)(3,6,7),(1,2,3)(4,5,7))$	3, 7	$C_7 : C_3$	4/27
$((1,4,5)(3,6,7),(1,4,2)(3,5,6))$	4, 4	$PSL(3,2)$	32/225
$((2,4,3,5)(6,7),(1,4,5)(3,6,7))$	3, 4	$PSL(3,2)$	1/4
$((1,3)(4,7,6,5),(2,4,6)(3,5,7))$	7, 7	$PSL(3,2)$	1/20
$((1,3)(4,7,6,5),(1,4,5)(2,3,7))$	3, 7	$PSL(3,2)$	71/360
$((2,4,3,5)(6,7),(1,2,7,4)(3,5))$	3, 4	$PSL(3,2)$	37/96
$((2,4,3,5)(6,7),(1,5,2,6)(4,7)(8,9))$	4, 14	$C_2 \times PSL(3,2)$	55/768
$((2,4,3,5)(6,7),(1,3)(4,7,6,5)(8,9))$	4, 6	$C_2 \times PSL(3,2)$	25/96
$((1,4,5)(3,6,7),(2,4,3,5)(6,7)(8,9))$	4, 6	$C_2 \times PSL(3,2)$	19/108
$((1,2,3)(5,6,7),(1,7)(2,5,4,3)(8,9))$	6, 14	$C_2 \times PSL(3,2)$	61/1080
$((1,3)(4,7,6,5)(8,9),(2,7,3,6)(4,5)(8,9))$	3, 4	$C_2 \times PSL(3,2)$	29/96
$((1,3)(4,7,6,5)(8,9),(1,7)(2,5,4,3)(8,9))$	4, 7	$C_2 \times PSL(3,2)$	79/768
$((1,2,3)(5,6,7),(2,6,4)(3,7,5))$	4, 7	$PSL(3,2)$	76/675
$((2,4,3,5)(6,7),(1,6,3,4)(5,7))$	4, 7	$PSL(3,2)$	53/256
$((1,2,3)(5,6,7),(2,7,3,6)(4,5)(8,9))$	14, 14	$C_2 \times PSL(3,2)$	17/180

Table 5.1: Some inner products in $C_2 \times PSL(3,2)$.

We can now construct the Gram matrix and calculate with GAP that the determinant is 0 so some vectors of $\mathcal{D}$ are linearly dependent on the others.

Proposition 5.19. *The subspace D generated by all the vectors in $\mathcal{D}$ has dimension* 80 *and a basis is given by the following vectors:*

- 22 2-*axes* a_t, $t \in T$,
- 28 3-*axes* u_h, *such that* $\langle h \rangle$ *is a subgroup of order* 3 *of* G,
- 21 4-*axes* v_g, *where* $\langle g \rangle$ *is a subgroup of order* 4 *in* G,
- $\overline{v}_{(2,4,3,5)(6,7)(8,9)}$
- $\overline{v}_{(1,6,3,4)(5,7)(8,9)}$
- $\overline{v}_{(1,2,5,6)(3,7)(8,9)}$
- $\overline{v}_{(1,2,7,4)(3,5)(8,9)}$
- $\overline{v}_{(1,2,4,7)(3,6)(8,9)}$
- $\overline{v}_{(1,5,6,2)(3,4)(8,9)}$
- $\overline{v}_{(1,7,2,4)(5,6)(8,9)}$
- $\overline{v}_{(1,6)(2,3,5,4)(8,9)}$
- $\overline{v}_{(1,5)(2,7,6,3)(8,9)}$

Proof. Using the inner products of Table 5.1 we computed the rank of the Gram matrix with respect to the set $\mathcal{D}$ and find that it is equal to 80. Then, with the same procedure, we checked that the listed vectors are linearly independent.

□

Recall that the alternating group A_{12} has a subgroup K isomorphic to $\overline{G}$ which is generated by $2A$-involutions, namely: $(1,3)(2,7)(4,8)(5,6)$, $(3,8,7)(4,6,5)$, $(9,10)(11,12)$. With the GAP Package "Majorana Algebras" it is possible to compute the Majorana representation of the group

$$2^2 \times L_3(2) = \langle (1,3)(2,7)(4,8)(5,6), (3,8,7)(4,6,5), (9,10)(11,12), (9,11)(10,12) \rangle$$

induced by the saturated Majorana representation of A_{12} and then the dimension of the subalgebra corresponding to the subgroup K, which is equal to 80.

We are therefore induced to pose the following conjecture:

Conjecture 5.20. *The Majorana representation of $C_2 \times PSL(3,2)$ that we are constructing is the same as that found by restricting the representation of A_{12} to the group $C_2 \times PSL(3,2)$.*

5.2. ALGEBRA PRODUCTS

Matters get considerably more difficult when one considers the algebra products. We only have a few ideas on how to proceed in determining the missing products.

One difficulty is the fact that we don't know large 2-closed subalgebras of A generated by 2-axes: actually, the only maximal subgroups of $\overline{G}$ generated by elements of T are those isomorphic to $2 \times S_4$ and, by the results of Chapter 4, the subalgebras corresponding to these subgroups are not 2-closed. In particular, for each of these eleven subalgebras, we need six more w vectors not contained in the 2-closure.

On the other hand, we show that some dependence relations between vectors w can be found.

For example, we consider the subalgebra A_H, where $H \cong 2 \times S_4$, with basis:

$a_{(4,5)(6,7)(8,9)}$

$a_{(4,6)(5,7)(8,9)}$

$a_{(4,7)(5,6)(8,9)}$

$a_{(2,3)(6,7)(8,9)}$

$a_{(2,3)(4,5)(8,9)}$

$a_{(1,2)(5,6)(8,9)}$

$a_{(1,2)(4,7)(8,9)}$

$a_{(1,3)(5,7)(8,9)}$

$a_{(1,3)(4,6)(8,9)}$

$u_{(1,2,3)(5,6,7)}$

$u_{(1,2,3)(4,5,7)}$

$u_{(1,2,3)(4,6,5)}$

$u_{(1,2,3)(4,7,6)}$

$v_{(2,3)(4,7,5,6)}$

$v_{(1,3)(4,7,6,5)}$

$v_{(1,2)(4,6,7,5)}$

$\overline{v}_{(2,3)(4,7,5,6)(8,9)}$

$\overline{v}_{(2,6,3,7)(4,5)(8,9)}$

$\overline{v}_{(1,2)(4,6,7,5)(8,9)}$

$w_5 := a_{(2,3)(6,7)(8,9)} \cdot (v_{(2,3)(4,7,5,6)} + v_{(1,3)(4,7,6,5)} + v_{(1,2)(4,6,7,5)})$

$w_6 := a_{(2,3)(4,5)(8,9)} \cdot (v_{(2,3)(4,7,5,6)} + v_{(1,3)(4,7,6,5)} + v_{(1,2)(4,6,7,5)})$

$w_{11} := a_{(1,2)(5,6)(8,9)} \cdot (v_{(2,3)(4,7,5,6)} + v_{(1,3)(4,7,6,5)} + v_{(1,2)(4,6,7,5)})$

$w_{12} := a_{(1,2)(4,7)(8,9)} \cdot (v_{(2,3)(4,7,5,6)} + v_{(1,3)(4,7,6,5)} + v_{(1,2)(4,6,7,5)})$

$w_{13} := a_{(1,3)(5,7)(8,9)} \cdot (v_{(2,3)(4,7,5,6)} + v_{(1,3)(4,7,6,5)} + v_{(1,2)(4,6,7,5)})$

$w_{14} := a_{(1,3)(4,6)(8,9)} \cdot (v_{(2,3)(4,7,5,6)} + v_{(1,3)(4,7,6,5)} + v_{(1,2)(4,6,7,5)})$

Lemma 5.21. *We have* $w_5 + w_6, w_{11} + w_{12}, w_{13} + w_{14} \in D$.

Proof. Let us consider the product between the 2-axis $a_{(4,6)(5,7)(8,9)}$ and the fake 4-axis $\overline{v}_{(2,3)(4,7,5,6)(8,9)}$. On the one hand it lies in the subalgebra A_H, so with the fomulae found in Chapter 4 one can write the product as a function of w vectors:

$$\begin{aligned}
a_{(4,6)(5,7)(8,9)} \cdot \overline{v}_{(2,3)(4,7,5,6)(8,9)} =& \tfrac{5}{36} a_{(4,5)(6,7)(8,9)} + \tfrac{1}{16} a_{(4,6)(5,7)(8,9)} \\
&+ \tfrac{1}{48} a_{(4,7)(5,6)(8,9)} + \tfrac{1}{9}(a_{(2,3)(6,7)(8,9)} + a_{(2,3)(4,5)(8,9)}) \\
&- \tfrac{11}{72}(a_{(1,2)(5,6)(8,9)} + a_{(1,2)(4,7)(8,9)} + a_{(1,3)(5,7)(8,9)} \\
&+ a_{(1,3)(4,6)(8,9)}) + \tfrac{15}{512}(u_{(1,2,3)(5,6,7)} + u_{(1,2,3)(4,5,7)} \\
&+ u_{(1,2,3)(4,6,5)} + u_{(1,2,3)(4,7,6)}) + \tfrac{1}{24} \overline{v}_{(2,3)(4,7,5,6)(8,9)} \\
&- \tfrac{5}{48}(\overline{v}_{(2,6,3,7)(4,5)(8,9)} + \overline{v}_{(1,2)(4,6,7,5)(8,9)}) \\
&+ \tfrac{1}{3}(-w_5 - w_6 + w_{11} + w_{12} + w_{13} + w_{14})
\end{aligned}$$

On the other hand $a_{(4,6)(5,7)(8,9)} \cdot \overline{v}_{(2,3)(4,7,5,6)(8,9)}$ lies inside a subalgebra A_K, where K is a subgroup of $\overline{G}$ isomorphic to S_4 generated by elements of T. So it can be calculated as a linear combination of axes and fake 4-axes by [27]. Hence we can obtain $w_{11}+w_{12}+w_{13}+w_{14}-w_5-w_6$ as a combination of vectors in $\mathcal{D}$.

By applying the appropriate Miyamoto involutions we get that the following vectors are in D:

$$w_5+w_6+w_{13}+w_{14}-w_{11}-w_{12} = \tau_{(1,3)(5,7)(8,9)}(w_{11}+w_{12}+w_{13}+w_{14}-w_5-w_6)$$
$$w_5+w_6+w_{11}+w_{12}-w_{13}-w_{14} = \tau_{(1,2)(5,6)(8,9)}(w_{11}+w_{12}+w_{13}+w_{14}-w_5-w_6)$$

So we obtain that:

$$w_5+w_6 = \tfrac{1}{2}[(w_5+w_6+w_{13}+w_{14}-w_{11}-w_{12})+(w_5+w_6+w_{11}+w_{12}-w_{13}-w_{14})]$$
belongs to D. Similarly we get $w_{11}+w_{12} \in D$ and $w_{13}+w_{14} \in D$.

□

Consequently, for every maximal subgroup isomorphic to $2 \times S_4$, we only need to add 3 vectors w.

A possible next step could be to add to the set $\mathcal{D}$ three vectors w for each maximal subgroup of $\overline{G}$ isomorphic to $2 \times S_4$ and use the standard methods to find relations and identifying the unknown products. Unfortunately, the number of unknown products appearing in each equation we found is large and it is not clear how and if the system could be solved. Another possible approach could be to find all inner products involving the elements in $\mathcal{D}$ and the new vectors w. This is not easy, since we expect that some algebra products are needed for this task. This would allow us to determine the dimension of the subspace generated by them and check if vectors w actually belong to D (as Conjecture 5.20 suggests) or not.

Appendix A

Code for the Gram matrix

In order to demonstrate Proposition 5.19, we have used GAP to construct the Gram matrix and to calculate its rank and determinant.

```
F:=Rationals;;
n:=92;;
A:=FullRowSpace(F,n);;
a:=Basis(A);;

a1:= a[1];; #(8,9)
a2:= a[2];; #(4,5)(6,7)(8,9)
a3:= a[3];; #(4,6)(5,7)(8,9)
a4:= a[4];; #(4,7)(5,6)(8,9)
a5:= a[5];; #(2,3)(6,7)(8,9)
a6:= a[6];; #(2,3)(4,5)(8,9)
a7:= a[7];; #(2,4)(3,5)(8,9)
a8:= a[8];; #(2,5)(3,4)(8,9)
a9:= a[9];; #(2,6)(3,7)(8,9)
a10:= a[10];; #(2,7)(3,6)(8,9)
a11:= a[11];; #(1,2)(5,6)(8,9)
a12:= a[12];; #(1,2)(4,7)(8,9)
a13:= a[13];; #(1,3)(5,7)(8,9)
a14:= a[14];; #(1,3)(4,6)(8,9)
a15:= a[15];; #(1,4)(3,6)(8,9)
a16:= a[16];; #(1,4)(2,7)(8,9)
a17:= a[17];; #(1,5)(3,7)(8,9)
```

```
a18:= a[18];; #(1,5)(2,6)(8,9)
a19:= a[19];; #(1,6)(3,4)(8,9)
a20:= a[20];; #(1,6)(2,5)(8,9)
a21:= a[21];; #(1,7)(3,5)(8,9)
a22:= a[22];; #(1,7)(2,4)(8,9)

a[23];; #(2,4,3,5)(6,7)
a[24];; #(1,6,3,4)(5,7)
a[25];; #(1,2,5,6)(3,7)
a[26];; #(1,2,7,4)(3,5)
a[27];; #(1,2,4,7)(3,6)
a[28];; #(1,5,6,2)(3,4)
a[29];; #(1,7,2,4)(5,6)
a[30];; #(1,6)(2,3,5,4)
a[31];; #(1,5,2,6)(4,7)
a[32];; #(1,4)(2,3,7,6)
a[33];; #(1,3,6,4)(2,5)
a[34];; #(1,7)(2,5,4,3)
a[35];; #(1,3,4,6)(2,7)
a[36];; #(1,5)(2,7,6,3)
a[37];; #(2,3)(4,7,5,6)
a[38];; #(1,5,3,7)(4,6)
a[39];; #(2,6,3,7)(4,5)
a[40];; #(1,3)(4,7,6,5)
a[41];; #(1,3,5,7)(2,6)
a[42];; #(1,3,7,5)(2,4)
a[43];; #(1,2)(4,6,7,5)

a[44];; #(1,4,5)(3,6,7)
a[45];; #(1,2,3)(5,6,7)
a[46];; #(2,4,6)(3,5,7)
a[47];; #(1,6,7)(3,4,5)
a[48];; #(1,3,6)(2,5,7)
```

```
a[49];; #(1,2,3)(4,5,7)
a[50];; #(1,3,4)(2,7,5)
a[51];; #(2,5,6)(3,4,7)
a[52];; #(1,7,3)(2,6,4)
a[53];; #(2,4,7)(3,5,6)
a[54];; #(1,5,3)(2,4,6)
a[55];; #(1,4,7)(3,6,5)
a[56];; #(1,4,5)(2,6,3)
a[57];; #(1,2,7)(3,5,6)
a[58];; #(1,6,7)(2,4,3)
a[59];; #(1,2,3)(4,6,5)
a[60];; #(1,2,5)(3,7,4)
a[61];; #(1,6,4)(2,5,7)
a[62];; #(1,4,5)(2,7,6)
a[63];; #(1,2,3)(4,7,6)
a[64];; #(1,6,5)(3,4,7)
a[65];; #(1,6,7)(2,3,5)
a[66];; #(1,4,5)(2,3,7)
a[67];; #(1,6,7)(2,5,4)
a[68];; #(1,6,2)(3,7,4)
a[69];; #(2,5,7)(3,4,6)
a[70];; #(1,5,7)(2,6,4)
a[71];; #(1,4,2)(3,5,6)

b23:= a[72];; #(2,4,3,5)(6,7)(8,9)
b24:= a[73];; #(1,6,3,4)(5,7)(8,9)
b25:= a[74];; #(1,2,5,6)(3,7)(8,9)
b26:= a[75];; #(1,2,7,4)(3,5)(8,9)
b27:= a[76];; #(1,2,4,7)(3,6)(8,9)
b28:= a[77];; #(1,5,6,2)(3,4)(8,9)
b29:= a[78];; #(1,7,2,4)(5,6)(8,9)
b30:= a[79];; #(1,6)(2,3,5,4)(8,9)
b31:= a[80];; #(1,5,2,6)(4,7)(8,9)
```

```
b32:= a[81];; #(1,4)(2,3,7,6)(8,9)
b33:= a[82];; #(1,3,6,4)(2,5)(8,9)
b34:= a[83];; #(1,7)(2,5,4,3)(8,9)
b35:= a[84];; #(1,3,4,6)(2,7)(8,9)
b36:= a[85];; #(1,5)(2,7,6,3)(8,9)
b37:= a[86];; #(2,3)(4,7,5,6)(8,9)
b38:= a[87];; #(1,5,3,7)(4,6)(8,9)
b39:= a[88];; #(2,6,3,7)(4,5)(8,9)
b40:= a[89];; #(1,3)(4,7,6,5)(8,9)
b41:= a[90];; #(1,3,5,7)(2,6)(8,9)
b42:= a[91];; #(1,3,7,5)(2,4)(8,9)
b43:= a[92];; #(1,2)(4,6,7,5)(8,9)

# L ordered list of all vectors

L:=[ (8,9),(4,5)(6,7)(8,9), (4,6)(5,7)(8,9), (4,7)(5,6)(8,9),
(2,3)(6,7)(8,9), (2,3)(4,5)(8,9), (2,4)(3,5)(8,9), (2,5)(3,4)(8,9),
(2,6)(3,7)(8,9), (2,7)(3,6)(8,9), (1,2)(5,6)(8,9), (1,2)(4,7)(8,9),
(1,3)(5,7)(8,9), (1,3)(4,6)(8,9), (1,4)(3,6)(8,9), (1,4)(2,7)(8,9),
(1,5)(3,7)(8,9), (1,5)(2,6)(8,9), (1,6)(3,4)(8,9), (1,6)(2,5)(8,9),
(1,7)(3,5)(8,9), (1,7)(2,4)(8,9), (2,4,3,5)(6,7), (1,6,3,4)(5,7),
(1,2,5,6)(3,7), (1,2,7,4)(3,5), (1,2,4,7)(3,6), (1,5,6,2)(3,4),
(1,7,2,4)(5,6), (1,6)(2,3,5,4), (1,5,2,6)(4,7), (1,4)(2,3,7,6),
(1,3,6,4)(2,5), (1,7)(2,5,4,3), (1,3,4,6)(2,7), (1,5)(2,7,6,3),
(2,3)(4,7,5,6), (1,5,3,7)(4,6), (2,6,3,7)(4,5), (1,3)(4,7,6,5),
(1,3,5,7)(2,6), (1,3,7,5)(2,4), (1,2)(4,6,7,5), (1,4,5)(3,6,7),
(1,2,3)(5,6,7), (2,4,6)(3,5,7), (1,6,7)(3,4,5), (1,3,6)(2,5,7),
(1,2,3)(4,5,7), (1,3,4)(2,7,5), (2,5,6)(3,4,7), (1,7,3)(2,6,4),
(2,4,7)(3,5,6), (1,5,3)(2,4,6), (1,4,7)(3,6,5), (1,4,5)(2,6,3),
(1,2,7)(3,5,6), (1,6,7)(2,4,3), (1,2,3)(4,6,5), (1,2,5)(3,7,4),
(1,6,4)(2,5,7), (1,4,5)(2,7,6), (1,2,3)(4,7,6), (1,6,5)(3,4,7),
(1,6,7)(2,3,5), (1,4,5)(2,3,7), (1,6,7)(2,5,4), (1,6,2)(3,7,4),
(2,5,7)(3,4,6), (1,5,7)(2,6,4), (1,4,2)(3,5,6), (2,4,3,5)(6,7)(8,9),
```

```
(1,6,3,4)(5,7)(8,9), (1,2,5,6)(3,7)(8,9), (1,2,7,4)(3,5)(8,9),
(1,2,4,7)(3,6)(8,9), (1,5,6,2)(3,4)(8,9), (1,7,2,4)(5,6)(8,9),
(1,6)(2,3,5,4)(8,9), (1,5,2,6)(4,7)(8,9), (1,4)(2,3,7,6)(8,9),
(1,3,6,4)(2,5)(8,9), (1,7)(2,5,4,3)(8,9), (1,3,4,6)(2,7)(8,9),
(1,5)(2,7,6,3)(8,9), (2,3)(4,7,5,6)(8,9), (1,5,3,7)(4,6)(8,9),
(2,6,3,7)(4,5)(8,9), (1,3)(4,7,6,5)(8,9), (1,3,5,7)(2,6)(8,9),
(1,3,7,5)(2,4)(8,9), (1,2)(4,6,7,5)(8,9) ];

# Algebra scalar product

Scal:=List([1..n],i->[]);;

# Scalar Product function

Sprod:=function(u,v)
local ans,rem,i,j,m;
ans:=Zero(F);
rem:=[];
for i in [1..n] do
m:=u[i]*v[i];
if m<>Zero(F) then
if IsBound(Scal[i][i]) then
ans:=ans+m*Scal[i][i];
else
Add(rem,[[i,i],m]);
fi;
fi;
for j in [i+1..n] do
m:=u[i]*v[j]+u[j]*v[i];
if m<>Zero(F) then
if IsBound(Scal[i][j]) then
ans:=ans+m*Scal[i][j];
else
```

```
Add(rem,[[i,j],m]);
fi;
fi;
od;
od;
if rem=[] then
return ans;
else
return [ans,rem];
fi;
end;;

for i in [1..22] do
Scal[i][i]:=1*One(F);
od;;

for i in [23..43] do
Scal[i][i]:=2*One(F);
od;;

for i in [44..71] do
Scal[i][i]:=8/5*One(F);
od;;

for i in [2..92] do
Scal[1][i]:=0*One(F);
Scal[i][1]:=0*One(F);
od;;

for i in [72..92] do
Scal[i][i]:=2*One(F);
od;;
```

```
# Products between two 2-axes

# 2B
for i in [2..22] do
for j in [2..22] do
if i <> j then
if L[i]*L[j]=L[j]*L[i] then
Scal[i][j]:=0*One(F);
Scal[j][i]:=0*One(F);
fi;
fi;
od;
od;

# 3A
for i in [2..22] do
for j in [2..22] do
if Order(L[i]*L[j])=3 then
Scal[i][j]:=13/2^8*One(F);
Scal[j][i]:=13/2^8*One(F);
fi;
od;
od;

# 4A
for i in [2..22] do
for j in [2..22] do
if Order(L[i]*L[j])=4 then
Scal[i][j]:=1/2^5*One(F);
Scal[j][i]:=1/2^5*One(F);
fi;
od;
od;
```

```
# Products between a 2-axis and a 3-axis

g:=Group([(8,9), (4,6)(5,7), (1,2,4)(3,6,5) ]);

3axes:=AsSet(Group([(1,4,5)(3,6,7)])^g);
c:=Centralizer(g, (4,5)(6,7)(8,9));
orbc3:=Orbits(c, 3axes);
List([1..Length(orbc3)], x->orbc3[x][1]);
#[ Group([ (2,6,4)(3,7,5) ]), Group([ (1,3,2)(5,7,6) ]),
#  Group([ (1,2,4)(3,6,5) ]), Group([ (1,4,5)(3,6,7) ]),
#  Group([ (1,7,4)(3,5,6) ]), Group([ (1,4,5)(2,3,7) ]) ]

rep23:=[ (2,6,4)(3,7,5), (1,3,2)(5,7,6), (1,2,4)(3,6,5), (1,4,5)(3,6,7),
(1,7,4)(3,5,6), (1,4,5)(2,3,7)];

List(rep23, x->StructureDescription(Group([(4,5)(6,7)(8,9), x])));
#[ "C2 x A4", "C2 x A4", "C2 x PSL(3,2)", "S3", "S4", "S4" ]

orb23:=List([1..Length(rep23)], y->AsSet(List(g, x->[(4,5)(6,7)(8,9)^x,
Subgroup(g, [rep23[y]^x])])));;

# 3A
for i in [2..22] do
for j in [44..71] do
if L[j]^L[i]=L[j]^-1 then
Scal[i][j]:=1/4*One(F);
Scal[j][i]:=1/4*One(F);
fi;
od;
od;

# S4
```

```
for i in [2..22] do
for j in [43..71] do
if [L[i], Subgroup(g, [L[j]])] in Union(orb23[5], orb23[6]) then
Scal[i][j]:=13/180*One(F);
Scal[j][i]:=13/180*One(F);
fi;
od;
od;

# C2 x A4
for i in [2..22] do
for j in [43..71] do
if [L[i], Subgroup(g, [L[j]])] in Union(orb23[1], orb23[2]) then
Scal[i][j]:=2/45*One(F);
Scal[j][i]:=2/45*One(F);
fi;
od;
od;

# C2 x PSL(3,2)
for i in [2..22] do
for j in [43..71] do
if [L[i], Subgroup(g, [L[j]])] in orb23[3] then
Scal[i][j]:=11/360*One(F);
Scal[j][i]:=11/360*One(F);
fi;
od;
od;

# Products between a 2-axis and a 4-axis

4axes:=AsSet(Group([(2,4,3,5)(6,7)])^g);
orbc4:=Orbits(c, 4axes);
```

```
List([1..Length(orbc4)], x->orbc4[x][1]);
#[ Group([ (2,3)(4,6,5,7) ]), Group([ (1,3)(4,5,6,7) ]),
#  Group([ (2,6,3,7)(4,5) ]), Group([ (1,7)(2,3,4,5) ]),
#  Group([ (1,5,2,6)(4,7) ]), Group([ (1,2,4,7)(3,6) ]) ]

rep24:=[ (2,3)(4,6,5,7), (1,3)(4,5,6,7), (2,6,3,7)(4,5),
(1,7)(2,3,4,5), (1,5,2,6)(4,7), (1,2,4,7)(3,6)];

List(rep24, x->StructureDescription(Group([(4,5)(6,7)(8,9), x])));
#[ "C4 x C2", "D8", "D8", "C2 x S4", "C2 x S4", "C2 x PSL(3,2)" ]

orb24:=List([1..Length(rep24)], y->AsSet(List(g, x->[(4,5)(6,7)(8,9)^x,
Subgroup(g, [rep24[y]^x])])));;

# 4A
for i in [2..22] do
for j in [23..43] do
if L[j]^L[i]=L[j]^-1 then
Scal[i][j]:=3/8*One(F);
Scal[j][i]:=3/8*One(F);
fi;
od;
od;

# C4 x C2
for i in [2..22] do
for j in [23..43] do
if [L[i], Subgroup(g, [L[j]])] in orb24[1] then
Scal[i][j]:=0*One(F);
Scal[j][i]:=0*One(F);
fi;
od;
od;
```

```
# C2 x S4
for i in [2..22] do
for j in [23..43] do
if [L[i], Subgroup(g, [L[j]])] in Union(orb24[4], orb24[5]) then
Scal[i][j]:=5/64*One(F);
Scal[j][i]:=5/64*One(F);
fi;
od;
od;

# C2 x PSL(3,2)
for i in [2..22] do
for j in [23..43] do
if [L[i], Subgroup(g, [L[j]])] in orb24[6] then
Scal[i][j]:=9/256*One(F);
Scal[j][i]:=9/256*One(F);
fi;
od;
od;

# Products between a 2-axis and a fake 4-axis

4axesf:=AsSet(Group([(2,3)(4,7,5,6)(8,9)])^g);
orbc4f:=Orbits(c, 4axesf);
List([1..Length(orbc4f)], x->orbc4f[x][1]);
#[ Group([ (2,3)(4,7,5,6)(8,9) ]), Group([ (1,3)(4,7,6,5)(8,9) ]),
#  Group([ (2,7,3,6)(4,5)(8,9) ]), Group([ (1,7)(2,5,4,3)(8,9) ]),
#  Group([ (1,6,2,5)(4,7)(8,9) ]), Group([ (1,7,4,2)(3,6)(8,9) ]) ]

rep24f:=[ (2,3)(4,7,5,6)(8,9), (1,3)(4,7,6,5)(8,9), (2,7,3,6)(4,5)(8,9),
(1,7)(2,5,4,3)(8,9), (1,6,2,5)(4,7)(8,9), (1,7,4,2)(3,6)(8,9) ];
```

```
List(rep24f, x->StructureDescription(Group([(4,5)(6,7)(8,9), x])));
#[ "C4 x C2", "D8", "D8", "S4", "S4", "C2 x PSL(3,2)" ]

orb24f:=List([1..Length(rep24f)], y->AsSet(List(g, x->[(4,5)(6,7)(8,9)^x,
Subgroup(g, [rep24f[y]^x])])));;

# D8
for i in [2..22] do
for j in [72..92] do
if [L[i], Subgroup(g, [L[j]])] in Union(orb24f[2], orb24f[3]) then
Scal[i][j]:=1/24*One(F);
Scal[j][i]:=1/24*One(F);
fi;
od;
od;

# C4 x C2
for i in [2..22] do
for j in [72..92] do
if [L[i], Subgroup(g, [L[j]])] in orb24f[1] then
Scal[i][j]:=0*One(F);
Scal[j][i]:=0*One(F);
fi;
od;
od;

# S4
for i in [2..22] do
for j in [72..92] do
if [L[i], Subgroup(g, [L[j]])] in Union(orb24f[4], orb24f[5]) then
Scal[i][j]:=31/192*One(F);
Scal[j][i]:=31/192*One(F);
fi;
```

```
od;
od;

# C2 x PSL(3,2)
for i in [2..22] do
for j in [72..92] do
if [L[i], Subgroup(g, [L[j]])] in orb24f[6] then
Scal[i][j]:=17/256*One(F);
Scal[j][i]:=17/256*One(F);
fi;
od;
od;

# Products between two 3-axes

d:=Normalizer(g,(1,2,3)(5,6,7));
orbd3:=Orbits(d, 3axes);
List([1..Length(orbd3)], x->orbd3[x][1]);
#[ Group([ (2,6,4)(3,7,5) ]), Group([ (2,4,7)(3,5,6) ]),
#  Group([ (1,3,2)(5,7,6) ]), Group([ (1,3,2)(4,7,5) ]),
#  Group([ (1,2,4)(3,6,5) ]), Group([ (1,2,5)(3,7,4) ]),
#  Group([ (1,2,7)(3,5,6) ]) ]

rep33:=[ (2,6,4)(3,7,5), (2,4,7)(3,5,6), (1,3,2)(5,7,6), (1,3,2)(4,7,5),
(1,2,4)(3,6,5), (1,2,5)(3,7,4), (1,2,7)(3,5,6)];

List(rep33, x->[StructureDescription(Group([(1,2,3)(5,6,7), x])),
Order((1,2,3)(5,6,7)*x),Order((1,2,3)(5,6,7)*x^-1)]);
#[ [ "PSL(3,2)", 7, 4 ], [ "C7 : C3", 7, 3 ], [ "C3", 1, 3 ], [ "A4", 2, 3 ],
#  [ "PSL(3,2)", 4, 4 ], [ "C7 : C3", 7, 3 ], [ "A4", 3, 2 ] ]

orb33:=List([1..Length(rep33)], y->AsSet(List(g, x->
[Subgroup(g, [(1,2,3)(5,6,7)^x]),Subgroup(g, [rep33[y]^x])])));;
```

```
# A4
for i in [44..71] do
for j in [44..71] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb33[4], orb33[7]) then
Scal[i][j]:=56/675*One(F);
Scal[j][i]:=56/675*One(F);
fi;
od;
od;

# C7 : C3
for i in [44..71] do
for j in [44..71] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb33[2], orb33[6]) then
Scal[i][j]:=4/27*One(F);
Scal[j][i]:=4/27*One(F);
fi;
od;
od;

# PSL(3,2) order 4,7
for i in [44..71] do
for j in [44..71] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb33[1] then
Scal[i][j]:=76/675*One(F);
Scal[j][i]:=76/675*One(F);
fi;
od;
od;
```

```
# PSL(3,2) order 4,4
for i in [44..71] do
for j in [44..71] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb33[5] then
Scal[i][j]:=32/225*One(F);
Scal[j][i]:=32/225*One(F);
fi;
od;
od;

# Products between a 3-axis and a 4-axis

orbd4:=Orbits(d, 4axes);
List([1..Length(orbd4)], x->orbd4[x][1]);
#[ Group([ (2,3)(4,6,5,7) ]), Group([ (2,6,3,7)(4,5) ]),
#  Group([ (2,4,3,5)(6,7) ]), Group([ (1,7)(2,3,4,5) ]),
#  Group([ (1,5)(2,3,6,7) ]), Group([ (1,4)(2,6,7,3) ]) ]

rep34:=[ (2,3)(4,6,5,7), (2,6,3,7)(4,5), (2,4,3,5)(6,7),
(1,7)(2,3,4,5), (1,5)(2,3,6,7), (1,4)(2,6,7,3)];

List(rep34, x->[StructureDescription(Group([(1,2,3)(5,6,7), x])),
Order((1,2,3)(5,6,7)*x),Order((1,2,3)(5,6,7)*x^-1)]);
#[ [ "S4", 2, 4 ], [ "PSL(3,2)", 7, 7 ], [ "PSL(3,2)", 3, 4 ],
#  [ "PSL(3,2)", 7, 3 ], [ "S4", 4, 2 ], [ "PSL(3,2)", 4, 3 ] ]

orb34:=List([1..Length(rep34)], y->AsSet(List(g, x->
[Subgroup(g, [(1,2,3)(5,6,7)^x]), Subgroup(g, [rep34[y]^x])])));;

# S4
for i in [44..71] do
for j in [23..43] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
```

```
Union(orb34[1], orb34[5]) then
Scal[i][j]:=1/9*One(F);
Scal[j][i]:=1/9*One(F);
fi;
od;
od;

# PSL(3,2) order 3,4
for i in [44..71] do
for j in [23..43] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb34[3], orb34[6]) then
Scal[i][j]:=1/4*One(F);
Scal[j][i]:=1/4*One(F);
fi;
od;
od;

# PSL(3,2) order 7,7
for i in [44..71] do
for j in [23..43] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb34[2] then
Scal[i][j]:=1/20*One(F);
Scal[j][i]:=1/20*One(F);
fi;
od;
od;

# PSL(3,2) order 3,7
for i in [44..71] do
for j in [23..43] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb34[4] then
Scal[i][j]:=71/360*One(F);
```

```
Scal[j][i]:=71/360*One(F);
fi;
od;
od;

# Products between a 3-axis and a fake 4-axis

orbd4f:=Orbits(d, 4axesf);
List([1..Length(orbd4f)], x->orbd4f[x][1]);
#[ Group([ (2,3)(4,7,5,6)(8,9) ]), Group([ (2,7,3,6)(4,5)(8,9) ]),
#  Group([ (2,5,3,4)(6,7)(8,9) ]), Group([ (1,7)(2,5,4,3)(8,9) ]),
#  Group([ (1,5)(2,7,6,3)(8,9) ]), Group([ (1,4)(2,3,7,6)(8,9) ]) ]

rep34f:=[ (2,3)(4,7,5,6)(8,9), (2,7,3,6)(4,5)(8,9), (2,5,3,4)(6,7)(8,9),
(1,7)(2,5,4,3)(8,9), (1,5)(2,7,6,3)(8,9), (1,4)(2,3,7,6)(8,9)];

List(rep34f, x->[StructureDescription(Group([(1,2,3)(5,6,7), x])),
Order((1,2,3)(5,6,7)*x),Order((1,2,3)(5,6,7)*x^-1)]);
#[ [ "S4", 4, 2 ], [ "C2 x PSL(3,2)", 14, 14 ], [ "C2 x PSL(3,2)", 4, 6 ],
#  [ "C2 x PSL(3,2)", 6, 14 ], [ "S4", 2, 4 ], [ "C2 x PSL(3,2)", 6, 4 ] ]

orb34f:=List([1..Length(rep34f)], y->AsSet(List(g, x->
[Subgroup(g, [(1,2,3)(5,6,7)^x]), Subgroup(g, [rep34f[y]^x])])));;

# S4
for i in [44..71] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb34f[1], orb34f[5]) then
Scal[i][j]:=11/27*One(F);
Scal[j][i]:=11/27*One(F);
fi;
od;
```

```
od;

# C2 x PSL(3,2) order 4,6
for i in [44..71] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb34f[3], orb34f[6]) then
Scal[i][j]:=19/108*One(F);
Scal[j][i]:=19/108*One(F);
fi;
od;
od;

# C2 x PSL(3,2) order 14,14
for i in [44..71] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb34f[2] then
Scal[i][j]:=17/180*One(F);
Scal[j][i]:=17/180*One(F);
fi;
od;
od;

# C2 x PSL(3,2) order 6,14
for i in [44..71] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb34f[4] then
Scal[i][j]:=61/1080*One(F);
Scal[j][i]:=61/1080*One(F);
fi;
od;
od;
```

```
# Products between two 4-axes

e:=Normalizer(g,(1,6,3,4)(5,7));
orbe4:=Orbits(e, 4axes);
List([1..Length(orbe4)], x->orbe4[x][1]);
#[ Group([ (2,3)(4,6,5,7) ]), Group([ (1,3)(4,5,6,7) ]),
#  Group([ (2,6,3,7)(4,5) ]), Group([ (1,6)(2,4,5,3) ]),
#  Group([ (1,6,3,4)(5,7) ]), Group([ (1,3,6,4)(2,5) ]) ]

rep44:=[ (2,3)(4,6,5,7), (1,3)(4,5,6,7), (2,6,3,7)(4,5),
(1,6)(2,4,5,3), (1,6,3,4)(5,7), (1,3,6,4)(2,5)];

List(rep44, x->[StructureDescription(Group([(1,6,3,4)(5,7), x])),
Order((1,6,3,4)(5,7)*x),Order((1,6,3,4)(5,7)*x^-1)]);
#[ [ "PSL(3,2)", 3, 4 ], [ "S4", 3, 3 ], [ "PSL(3,2)", 7, 4 ],
#  [ "PSL(3,2)", 3, 4 ], [ "C4", 2, 1 ], [ "S4", 3, 3 ] ]

orb44:=List([1..Length(rep44)], y->AsSet(List(g, x->
[Subgroup(g, [(1,6,3,4)(5,7)^x]), Subgroup(g, [rep44[y]^x])])));;

# S4
for i in [23..43] do
for j in [23..43] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb44[2], orb44[6]) then
Scal[i][j]:=11/48*One(F);
Scal[j][i]:=11/48*One(F);
fi;
od;
od;

# PSL(3,2) order 3,4
for i in [23..43] do
```

```
for j in [23..43] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb44[1], orb44[4]) then
Scal[i][j]:=37/96*One(F);
Scal[j][i]:=37/96*One(F);
fi;
od;
od;

# PSL(3,2) order 4,7
for i in [23..43] do
for j in [23..43] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb44[3] then
Scal[i][j]:=53/256*One(F);
Scal[j][i]:=53/256*One(F);
fi;
od;
od;

# Products between a 4-axis and a fake 4-axis

orbe4f:=Orbits(e, 4axesf);
List([1..Length(orbe4f)], x->orbe4f[x][1]);
#[ Group([ (2,3)(4,7,5,6)(8,9) ]), Group([ (1,3)(4,7,6,5)(8,9) ]),
#  Group([ (2,7,3,6)(4,5)(8,9) ]), Group([ (1,6)(2,3,5,4)(8,9) ]),
#  Group([ (1,4,3,6)(5,7)(8,9) ]), Group([ (1,4,6,3)(2,5)(8,9) ]) ]

rep44f:=[ (2,3)(4,7,5,6)(8,9), (1,3)(4,7,6,5)(8,9), (2,7,3,6)(4,5)(8,9),
(1,6)(2,3,5,4)(8,9), (1,4,3,6)(5,7)(8,9), (1,4,6,3)(2,5)(8,9)];

List(rep44f, x->[StructureDescription(Group([(1,6,3,4)(5,7), x])),
Order((1,6,3,4)(5,7)*x),Order((1,6,3,4)(5,7)*x^-1)]);
#[ [ "C2 x PSL(3,2)", 4, 6 ], [ "C2 x S4", 6, 6 ], [ "C2 x PSL(3,2)", 4, 14 ],
```

```
#  [ "C2 x PSL(3,2)", 4, 6 ], [ "C4 x C2", 2, 2 ], [ "C2 x S4", 6, 6 ] ]

orb44f:=List([1..Length(rep44f)], y->AsSet(List(g, x->
[Subgroup(g, [(1,6,3,4)(5,7)^x]), Subgroup(g, [rep44f[y]^x])])));;

# C2 x S4
for i in [23..43] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb44f[2], orb44f[6]) then
Scal[i][j]:=11/48*One(F);
Scal[j][i]:=11/48*One(F);
fi;
od;
od;

# C4 x C2
for i in [23..43] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb44f[5] then
Scal[i][j]:=0*One(F);
Scal[j][i]:=0*One(F);
fi;
od;
od;

# C2 x PSL(3,2) order 4,14
for i in [23..43] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb44f[3] then
Scal[i][j]:=55/768*One(F);
Scal[j][i]:=55/768*One(F);
fi;
```

```
od;
od;

# C2 x PSL(3,2) order 4,6
for i in [23..43] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb44f[1], orb44f[4]) then
Scal[i][j]:=25/96*One(F);
Scal[j][i]:=25/96*One(F);
fi;
od;
od;

# Products between two fake 4-axes

f:=Normalizer(g,(1,3)(4,7,6,5)(8,9));
orbf4f:=Orbits(f, 4axesf);
List([1..Length(orbf4f)], x->orbf4f[x][1]);
#[ Group([ (2,3)(4,7,5,6)(8,9) ]), Group([ (1,3)(4,7,6,5)(8,9) ]),
#  Group([ (2,7,3,6)(4,5)(8,9) ]), Group([ (1,7)(2,5,4,3)(8,9) ]),
#  Group([ (1,7,3,5)(4,6)(8,9) ]), Group([ (1,5,7,3)(2,4)(8,9) ]) ]

rep4f4f:=[ (2,3)(4,7,5,6)(8,9), (1,3)(4,7,6,5)(8,9), (2,7,3,6)(4,5)(8,9),
(1,7)(2,5,4,3)(8,9), (1,7,3,5)(4,6)(8,9), (1,5,7,3)(2,4)(8,9)];

List(rep4f4f, x->[StructureDescription(Group([(1,3)(4,7,6,5)(8,9), x])),
Order((1,3)(4,7,6,5)(8,9)*x),Order((1,3)(4,7,6,5)(8,9)*x^-1)]);
#[ [ "S4", 3, 3 ], [ "C4", 2, 1 ], [ "C2 x PSL(3,2)", 4, 3 ],
#  [ "C2 x PSL(3,2)", 7, 4 ], [ "S4", 3, 3 ], [ "C2 x PSL(3,2)", 4, 3 ] ]

orb4f4f:=List([1..Length(rep4f4f)], y->AsSet(List(g, x->
[Subgroup(g,[(1,3)(4,7,6,5)(8,9)^x]), Subgroup(g, [rep4f4f[y]^x])])));;
```

```
# S4
for i in [72..92] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb4f4f[1], orb4f4f[5]) then
Scal[i][j]:=9/16*One(F);
Scal[j][i]:=9/16*One(F);
fi;
od;
od;

# C2 x PSL(3,2) order 3,4
for i in [72..92] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in
Union(orb4f4f[3], orb4f4f[6]) then
Scal[i][j]:=29/96*One(F);
Scal[j][i]:=29/96*One(F);
fi;
od;
od;

# C2 x PSL(3,2) order 4,7
for i in [72..92] do
for j in [72..92] do
if [Subgroup(g, [L[i]]), Subgroup(g, [L[j]])] in orb4f4f[4] then
Scal[i][j]:=79/768*One(F);
Scal[j][i]:=79/768*One(F);
fi;
od;
od;
```

```
Gram:=List([1..92],x->List([1..92],y->Sprod(a[x],a[y])));

Determinant(Gram);
# 0

Rank(Gram);
# 80

qq:=Union([1..79],[85]);;

gram:=List(qq,x->List(qq,y->Sprod(a[x],a[y])));

Rank(gram);
# 80
```

Bibliography

[1] R.E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math., 109, 405–44 (1992).

[2] A.E. Brouwer, A.A. Ivanov, *Majorana algebra for the Hoffman–Singleton graph*, Geom Dedicata 217, 4 (2023).

[3] A. Castillo-Ramirez, *Idempotents of the Norton-Sakuma Algebras*, J. Group Theory, 16, 419-444 (2013).

[4] A. Castillo-Ramirez and A. A. Ivanov, *The axes of a Majorana representation of A_{12}*, Groups of exceptional type, Coxeter groups and related geometries, Springer Proc. Math. Stat., vol. 82, Springer, New Delhi, 159–188 (2014).

[5] A. Castillo-Ramirez, *On Majorana Algebras and Representations*, Ph.D. Thesis, Imperial College London (2014).

[6] A. Castillo-Ramirez, *Associative subalgebras of low-dimensional Majorana algebras*, J. Algebra, 421, 119-135 (2015).

[7] H.Y. Chen and C. Lam, *An explicit Majorana representation of the group $3^2 : 2$ of $3C$-pure type*, Pacific J. Math. 271, 25-51 (2014).

[8] J.H. Conway, *A simple construction for the Fischer-Griess monster group*, Invent. Math., 79, 513–40 (1984).

[9] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford (1985).

[10] S. Decelle, *Majorana representations and the Coxeter groups $G^{(m,n,p)}$*, PhD Thesis, Imperial College (2013).

[11] S. Decelle, *The $L_2(11)$-subalgebra of the monster algebra*, Ars Math. Contemp. 7 (1), 83–103 (2014).

[12] C. Franchi, A. A. Ivanov and M. Mainardis, *The $2A$ Majorana representations of the Harada-Norton group*, Quad. del Sem. Matem., 1-7 (2013).

[13] C. Franchi, A. A. Ivanov and M. Mainardis, *Computing the dimension of a Majorana representation of the Harada-Norton group*, Ars Mathematica Contemporanea, 11, 175-187 (2016).

[14] C. Franchi, A.A. Ivanov and M. Mainardis, *Standard Majorana representations of the symmetric groups*, J. Algebraic Combin. 44, no. 2, 265–292 (2016).

[15] C. Franchi, A.A. Ivanov, and M. Mainardis, *Permutation modules for the symmetric group*, Proc. Amer. Math. Soc. 145, no. 8, 3249–3262 (2017).

[16] C. Franchi, A.A. Ivanov, and M. Mainardis, *Radicals of S_n-invariant positive semidefinite hermitian forms*, Algebr. Comb. 1, no. 4, 425–440 (2018).

[17] C. Franchi, A.A. Ivanov, and M. Mainardis, *Majorana representations of finite groups*, Algebra Colloq. 27, no. 1, 31–50 (2020).

[18] C. Franchi, M. Mainardis, S. Shpectorov, *2-generated axial algebras of Monster type*, ArXiv:2101.10315 [math.RA] (2021).

[19] C. Franchi, A.A. Ivanov, and M. Mainardis, *Saturated Majorana representations of A_{12}*, Trans. Amer. Math. Soc. 375, 5753-5801 (2022).

[20] C. Franchi, Private communication (2022).

[21] GAP. Groups, Algorithms, Programming - GAP Group. http://www.gap-system.org, Ver. 4.11.1, 2021.

[22] A. Gevorgyan, *Towards the standard Majorana representations of 3-transposition groups*, Algebra Colloquium 217, 4 (2023).

[23] R.L. Griess, *The Friendly Giant*, Invent. Math, 69, 1-102 (1982).

[24] J.I. Hall, F. Rehren, and S. Shpectorov, *Universal axial algebras and a theorem of Sakuma*, J. Algebra, 421, 394 – 424 (2015).

[25] J.I. Hall, F. Rehren, and S. Shpectorov, *Primitive axial algebras of Jordan type*, J. Algebra, 437, 79 – 115 (2015).

[26] A.A. Ivanov, *The Monster Group and Majorana Involutions*, Vol. 176 of Cambridge Tracts In Mathematics, Cambridge Univ. Press, Cambridge (2009).

[27] A.A. Ivanov, D.V. Pasechnik, Á. Seress, S. Shpectorov, *Majorana representations of the symmetric group of degree* 4, J. Algebra 324, 2432–2463 (2010).

[28] A.A. Ivanov, *On Majorana representations of A_6 and A_7*, Comm. Math. Phys. 307, 1–16 (2011).

[29] A.A. Ivanov, *Majorana representations of A_6 involving $3C$-algebras*, Bull. Math. Sci. 1, 356– 378 (2011).

[30] A.A. Ivanov, Á. Seress, *Majorana representations of A_5*, Math. Z. 272, 269–295 (2012).

[31] A.A. Ivanov, S. Shpectorov, *Majorana representations of $L_3(2)$*, Adv. Geom. 12, no. 4, 717–738 (2012).

[32] A. A. Ivanov, *Majorana representation of the Monster group*, Finite simple groups: thirty years of the atlas and beyond, Contemp. Math., vol. 694, Amer. Math. Soc., Providence, RI, pp. 11–17 (2017).

[33] A.A. Ivanov, *The future of Majorana theory*, Group theory and computation, Indian Stat. Inst. Ser., Springer, Singapore, pp. 107–118 (2018).

[34] A.A Ivanov, *A pratical course in Majorana Theory*, Lecture Notes, TGMC, Yichang, China (2021).

[35] A.A. Ivanov, *Closed Majorana representations of $3, 4^+$-transposition groups*, Adv. Geom. 22, 487-494 (2022).

[36] A.A. Ivanov, *Algebraic Combinatorics and the Monster Group*, London Mathematical Society Lecture Note Series, Series Number 487 (2023).

[37] N. Jacobson, *Structure and Representations of Jordan Algebras*, American Mathematical Society Colloquium Publications, Volume 39 (1968).

[38] G.D. James and M. W. Liebeck, *Representations and Characters of Groups*, Cambridge Univ. Press, Cambridge, 2nd edition (2001).

[39] P. Jordan, *Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik*, Nachr. Ges. Wiss. Gottinga (1933).

[40] S.M.S. Khasraw, *M-axial algebras related to* 4*-transposition groups*, PhD thesis, University of Birmingham (2015).

[41] S.M.S. Khasraw, J. McInroy, S. Shpectorov, *Enumerating* 3*-generated axial algebras of Monster type*, Journal of Pure and Applied Algebra, Volume 226, Issue 2 (2022).

[42] C.S. Lim, *From the Monster to Majorana: A study of the* 3*A-axes*, PhD thesis, Imperial College London (2017).

[43] A. Mamontov, A. Staroletov and M. Whybrow, *Minimal* 3*-generated Majorana algebras*, J. Algebra 524, 367-396 (2019).

[44] J. McInroy and S. Shpectorov, *An expansion algorithm for constructing axial algebras*, J. Algebra, 550, 379-409 (2020).

[45] J. McInroy and S. Shpectorov *From forbidden configurations to a classification of some axial algebras of Monster type*, J. Algebra (2021).

[46] J. McInroy and S. Shpectorov, *Axial algebras of Jordan and Monster type*, ArXiv (2022).

[47] M. Miyamoto, *Griess algebras and conformal vectors in vertex operator algebras*, J. Algebra, 179, 523–48 (1996).

[48] S.P. Norton, *The Monster algebra: some new formulae*, Moonshine, the Monster and Related Topics, Comtemp. Math, 193, AMS, Providence, RI, 297-306 (1996).

[49] M. Pfeiffer and M. Whybrow, MajoranaAlgebras, a package for constructing Majorana algebras and representations, Version 1.4, GAP package (2018).

[50] M. Pfeiffer and M. Whybrow, *Constructing Majorana representations*, ArXiv (2018).

[51] F. Rehren, *Axial algebras*, Ph.D. Thesis, University of Birmingham (2015).

[52] F. Rehren, *Generalised dihedral subalgebras from the Monster*, Trans. Amer. Math. Soc. 369, 6953-6986 (2017).

[53] S. Sakuma, *6-Transposition property of τ-involutions of vertex operator algebras*, Int. Math. Res. Not., Article rnm030, 19 pp (2007).

[54] Á. Seress, *Construction of 2-closed M-representations*, Proc. of the 37th Int. Symp. Symb. and Alg. Comp., 311–318 (2012).

[55] G. E. Shilov, *An Introduction to the Theory of Linear Spaces*, Prentice-Hall, New Jersey, (1961).

[56] M. Whybrow, *Majorana algebras generated by a $2A$ algebra and one further axis*, J. Group Theory 21, no. 3, 417–437 (2018).

[57] M. Whybrow, *Majorana algebras and subgroups of the Monster*, PhD Thesis, Imperial College (2018).

[58] M. Whybrow, *An infinite family of axial algebras*, J. Algebra 577, 1–31 (2021).