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Global estimates and positivity preservation for some elliptic PDEs on Riemannian manifolds

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Abstract

This dissertation deals with certain qualitative properties for the solutions of two elliptic problems on Riemannian manifolds.

In the first part of this work, we focus on solutions to the Poisson equation and investigate the validity of first and second order global regularity estimates, called respectively L^p-gradient and Calderón–Zygmund estimates. On complete non-compact manifolds, their validity might be strongly influenced by the large-scale geometry. The systematic study of the Calderón–Zygmund theory was initiated in this setting by Güneysu and Pigola. Since then, geometric analysts have shown an increasing interest towards the topic, regarding positive results, counterexamples and interactions with other related issues.

In Chapter 1 we prove a number of results that ensure the validity of L^p-gradient and Calderón–Zygmund estimates under comparatively weak geometric assumptions on the Ricci curvature and the injectivity radius. Often, we assume some type of integral lower bound on the Ricci tensor instead of the pointwise bounds that commonly appear in the previous literature. We observe the implications of the above results for the theory of Sobolev spaces, including density properties, and prove the equivalence between Calderón–Zygmund estimates and boundedness properties of the second order Riesz transform. The case of higher order Calderón–Zygmund estimates is also addressed.

In Chapter 2 we prove counterexamples to the L^p-gradient and Calderón–Zygmund estimates. First, we show the failure of these estimates on manifolds where the negative part of the curvature, although unbounded, grows as slowly as desired. This proves the optimality of the curvature bounds assumed in certain results in the literature. The other main contribution of this chapter is the construction of a complete, non-compact manifold with positive sectional curvatures which does not support the L^p-Calderón–Zygmund inequality for large p. This example, which relies on tools from metric geometry, shows the non-equivalence of L^p-gradient and Calderón–Zygmund estimates, thus answering to an open question in the literature.

The results of the first chapter require various lower bounds on the Ricci curvature, whose optimality is testified by the above-mentioned counterexamples. In Chapter 3, nonetheless, by focusing on a special class of manifolds, we are able to prove some of these results even in situations when the Ricci curvature explodes very fast at $-\infty$, albeit in a controlled way. Namely, we consider Cartan–Hadamard manifolds with a polynomial pinching on the Ricci curvature and prove a density result for the Sobolev space $W^{2,p}$ and the validity of an L^2-Calderón–Zygmund inequality. The main tool in these proofs is a carefully constructed sequence of cutoff functions with a second order control.

The second part of this dissertation deals with positivity preservation properties for a Schrödinger operator. More precisely, a manifold has the L^p-positivity preserving property if all the distributional L^p solutions of $(-\Delta + 1)u \geq 0$ are non-negative. This definition was introduced by Güneysu some twenty years ago, although the notion, when $p = 2$, can be traced back to the seminal work of Kato on the essential self-adjointness of certain Schrödinger-type operators with possibly singular non-negative potential. The validity of the L^∞-positivity preserving property, instead, is connected to the stochastic completeness of the manifold at hand, i.e. the fact that the minimal heat kernel preserves probability.

Chapter 4 is devoted to proofs of the L^p-positivity preserving property which rely on the existence of smooth cutoff functions with a control on the gradient and Laplacian. Using the cutoffs developed in Chapter 3, we prove the property for $p \geq 2$ on Cartan–Hadamard manifolds with a polinomially pinched Ricci curvature. On manifolds with a subquadratic growth of the negative part of the Ricci curvature, the L^p-positivity preserving property is verified for any $p \in [1, +\infty]$ thanks to similar cutoffs with a stronger, uniform second order control. When $p = +\infty$, this gives also a new proof of a well-known optimal condition for stochastic completeness due to Hsu.

In Chapter 5, we deal with the limit cases $p = 1$ and $p = +\infty$. Using a monotone approximation result, which is of independent interest, we prove that the stochastic completeness is in fact equivalent to the validity of the L^∞-positivity preserving property. Finally, we exhibit a counterexample to the L^1-positivity preserving property which shows sharpness of our subquadratic bound on the Ricci curvature.

Contents

Basic notation

We begin by fixing some basic notation, which will be used repeatedly in the rest of this thesis.

In the following, unless otherwise stated, (M, g) denotes a smooth, connected Riemannian manifold of dimension $\dim M = n$; the boundary ∂M is always assumed to be empty. Given $x \in M$, $r(x) = d(x, o)$ denotes the *Riemannian distance function* from a fixed reference point $o \in M$. Whenever we state a property or assumption involving $r = r(x)$, it is implicitly meant that said property holds with respect to some fixed pole $o \in M$. We denote with $B_R(x)$ the (open) geodesic ball of radius $R > 0$ and center $x \in M$, if the center is the fixed pole $o \in M$, we simply write B_R.

In the following we use the sub/superscript e to denote objects taken with respect to the Euclidean metric, whenever it is necessary to distinguish between the Riemannian counterparts in a local computation.

Let ∇ be the Levi-Civita connection and $X, Y, Z, W \in \mathfrak{X}(M)$ be smooth vector fields, the *Riemannian curvature tensor* is defined by

$$\mathrm{Riem}(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_y \nabla_x Z - \nabla_{[X,Y]} Z, W).$$

The *sectional curvature* of a 2-plane $\pi_x \subseteq T_x M$ spanned by a a pair of linearly independent vectors $X, Y \in T_x M$ is given by

$$\mathrm{Sect}(\pi_x) = \frac{\mathrm{Riem}(X, Y, X, Y)}{g(X, X) g(Y, Y) - g(X, Y)^2}.$$

The trace of the Riemann tensor give rises to the *Ricci tensor*: given $X, Y \in \mathfrak{X}(M)$ and $\{E_i\}$ some local orthonormal frame, we define

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} \mathrm{Riem}(X, E_i, Y, E_i).$$

Recall that if we fix $x \in M$ and some $X \in T_x M$, the Ricci curvature in the direction of X can also be computed by

$$\mathrm{Ric}(X, X)(x) = \sum_{i=1}^{n-1} \mathrm{Sect}(\pi_i)$$

where $\pi_1, \ldots, \pi_{n-1} \subseteq T_x M$ are the $n-1$ linearly independent 2-planes orthogonal to the direction X. As a result, bounds on the sectional curvatures imply bounds on the Ricci tensor as a quadratic form. Finally, we denote with $\mathrm{Ric}_o = \mathrm{Ric}(\nabla r, \nabla r) : M \to \mathbb{R}$ the radial Ricci curvature.

In this thesis, we say that a manifold has non-negative sectional curvatures, $\mathrm{Sect}(M) \geq 0$, if $\mathrm{Sect}(\pi_x) \geq 0$ for every $x \in M$ and every 2-plane $\pi_x \subseteq T_xM$; more generally, given $f : M \to R$ we say that $\min \mathrm{Sect}(x) \geq f(x)$ if $\mathrm{Sect}(\pi_x) \geq f(x)$ for every $x \in M$ and every 2-plane $\pi_x \subseteq T_xM$. Inequalities involving the Ricci tensor and the Riemannian metric g are intended in the sense of quadratic forms: for instance, if $f : M \to \mathbb{R}$ we say $\mathrm{Ric} \geq f$ if $\mathrm{Ric}(X, X)(x) \geq f(x)g(X, X)$ for every $x \in M$ and $X \in \mathfrak{X}(M)$. Moreover, we denote with $\min \mathrm{Ric}$ the function which associates at every point of M the lowest eigenvalue of Ric. Upper bounds on sectional and Ricci curvatures are defined similarly.

Given $x \in M$ the *injectivity radius at* x, $\mathrm{inj}(x) > 0$, is the largest $r > 0$ such that any geodesic of length less than r and having x as endpoint is minimizing; the *injectivity radius of* M is the infimum of the injectivity radii over M: $r_{\mathrm{inj}}(M) = \inf_{x \in M} \mathrm{inj}(x)$.

If $u \in C^\infty(M)$, $\nabla u \in \mathfrak{X}(M)$ denotes the *gradient* of u while $\nabla^2 u = \mathrm{Hess}\, u$ is the *Hessian*. The *Laplace–Beltrami operator* of u is defined by

$$\Delta u = \mathrm{div}(\nabla u) = \mathrm{tr}_g(\nabla^2 u),$$

where the *divergence* of a vector field $X \in \mathfrak{X}(M)$ is defined (locally) by

$$\mathrm{div}(X) = \sum_{i=1}^n g(\nabla_{E_i} X, E_i)$$

where $\{E_i\}$ is a local orthonormal frame. In particular, $-\Delta$ is a non-negative operator; if the bottom of the L^2 spectrum of $-\Delta$ is strictly positive, we say that (M, g) has a *spectral gap*. Finally, the p-Laplace–Beltrami operator is defined by

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u),$$

for $p \in (1, +\infty)$. In the following, $C_c^\infty(M)$ denotes the space of smooth, real valued functions on M with compact support.

If $(x^1, \ldots, x^n)$ are local coordinates, *the Riemannian volume form* $d\mu_g$ is defined as

$$d\mu_g = \sqrt{\det[g_{ij}]}dx^1 \wedge \ldots \wedge dx^n,$$

where $[g_{ij}]$ is the matrix with coefficients $g_{ij} = g(\partial_i, \partial_j)$. In the rest of this thesis, unless otherwise specified, all integrals are taken with respect to $d\mu_g$. If $\Omega \subset M$ we equivalently use $\mathrm{vol}(\Omega) = \mathrm{vol}_g(\Omega)$ to denote the volume with respect to $d\mu_g$, while $\mathrm{vol}_e(\Omega) = |\Omega|$ represents the Euclidean volume. We denote with $L^p(M; T_r^s M) = \Gamma_{L^p}(M; T_r^s M)$ the space of L^p sections of the fiber bundle $T_r^s M$; if $T \in L^p(M; T_r^s M)$ the L^p norm of T is defined as $\|T\|_{L^p} = \||T|\|_{L^p}$ where $|T|$ is the Hilbert-Schmidt norm of the tensor induced by g. Unless otherwise stated, the L^p norms of tensors and functions $\|\cdot\|_{L^p} = \|\cdot\|_p$ are taken over the whole manifold M.

If $\Omega \subseteq M$ we write $\Omega \Subset M$ to indicate that Ω has compact closure in M.

Throughout this thesis, C will denote a positive constant, whose value may change from place to place. Whenever relevant, we will explicit the dependence of C form the dimension, the curvature, p or other relevant parameters.

Part I

Calderón–Zygmund theory

Introduction to Part I

The aim of Part I of this thesis is the study of first and second order, global L^p estimates for the solution of the Poisson equation on a complete Riemannian manifold. The results henceforth presented are contained in [90, 89, 88] and have been obtained in collaboration with Stefano Meda, Stefano Pigola, and Giona Veronelli.

Let us consider L, an elliptic second order differential operator defined on some relatively compact domain $\Omega \Subset \mathbb{R}^n$. Fix $p \in (1, +\infty)$, if the coefficients of L are regular enough, it is well known that

$$||\nabla^2 u||_{L^p(\Omega')} \leq C \left(||u||_{L^p(\Omega)} + ||Lu||_{L^p(\Omega)} \right)$$

for all $\Omega' \Subset \Omega$ and $u \in L^p$ with $Lu \in L^p$ distributionally. Here the constant $C > 0$ does not depend on u but might depend on the dimension n, on p, on the operator L and on the geometry of the domains Ω and Ω'. These local estimates, central in the regularity theory of elliptic PDEs, are always available as long as L is reasonably well behaved, obtaining global estimates when $\Omega' = \Omega = \mathbb{R}^n$, instead, poses a possibly harder task. In the case of the Laplace operator $L = \Delta$, however, one has in fact the stronger estimate

$$||\nabla^2 u||_{L^p(\mathbb{R}^n)} \leq C(n,p) ||\Delta u||_{L^p(\mathbb{R}^n)} \quad \forall u \in C_c^\infty(\mathbb{R}^n),$$

obtained by Calderón and Zygmund in their seminal work on singular integral operator, [23]. See also [49, Theorem 9.9]. The validity of this functional inequality implies by interpolation a bound on the L^p norm of the gradient

$$||\nabla u||_{L^p(\mathbb{R}^n)} \leq C(||u||_{L^p(\mathbb{R}^n)} + ||\Delta u||_{L^p(\mathbb{R}^n)}) \quad \forall u \in C_c^\infty(\mathbb{R}^n).$$

These last two estimates combined yield

$$||u||_{W^{2,p}(\mathbb{R}^n)} \leq C(||u||_{L^p(\mathbb{R}^n)} + ||\Delta u||_{L^p(\mathbb{R}^n)}) \quad \forall u \in C_c^\infty(\mathbb{R}^n).$$

Note that, since $|\Delta u| \leq \sqrt{n}|\nabla^2 u|$, this implies the equivalence on $C_c^\infty(M)$ of the norms $||u||_{W^{2,p}(\mathbb{R}^n)}$ and $||u||_{L^p(\mathbb{R}^n)} + ||\Delta u||_{L^p(\mathbb{R}^n)}$. This fact has important consequences on the theory of Sobolev spaces and on spectral properties of the Schrödinger operator $-\Delta + 1$.

If we move away from the Euclidean setting, replacing $\mathbb{R}^n$ with a complete Riemannian manifold (M, g) and the Laplacian with the Laplace–Beltrami operator Δ, it is an interesting problem to investigate how these inequalities change. Unsurprisingly, if we stick to local regularity estimates, i.e. on relatively compact domains or if the manifold itself is compact, the Riemannian scenario does not differ much from the Euclidean one: the geometry of the manifold is simply encoded in the value of the constant C. However, the Riemannian picture diverges qualitatively, thus becoming more faceted, if we consider global regularity estimates on complete, non-compact manifolds, on which the geometry at infinity has a much greater influence. We shall mainly focus on this class of manifolds.

We begin with the following

Definition I.1. Let (M, g) be a complete Riemannian manifold and fix $p \in (1, \infty)$. We say that (M, g) supports an *L^p-Calderón–Zygmund inequality* if there exists a constant $C > 0$ such that

$$\|\nabla^2 u\|_{L^p(M)} \leq C(\|u\|_{L^p(M)} + \|\Delta u\|_{L^p(M)}) \quad \forall u \in C_c^\infty(M). \tag{CZ(p)}$$

The terminology in the context of Riemannian manifold was first introduced by Güneysu and Pigola in [63]. Since then, researcher coming from diverse mathematical backgrounds have shown an increasing interest towards the topic, both concerning the validity of CZ(p) on a given manifold, and the interaction of Calderón–Zygmund theory with other related issues. Note that the limit cases CZ(1) and CZ(∞) are disregarded as they fail to be true, also locally, even in $\mathbb{R}^n$, [37, 98]. See also the two examples in [42, Section 2.2]. Similarly to the Euclidean case, Calderón–Zygmund inequalities have companion first order estimates.

Definition I.2. Let (M, g) be a complete Riemannian manifold and fix $p \in (1, \infty)$, we say that (M, g) supports an *L^p-gradient estimate* if there exists a constant $C > 0$ such that

$$\|\nabla u\|_{L^p(M)} \leq C(\|u\|_{L^p(M)} + \|\Delta u\|_{L^p(M)}) \quad \forall u \in C_c^\infty(M). \tag{GE(p)}$$

There is a clear hierarchy between L^p-gradient and Calderón–Zygmund estimates: indeed, the validity of CZ(p) on a complete manifold implies the corresponding L^p-gradient estimate, [63, Corollary 3.11]. In particular, whenever CZ(p) holds we automatically get the following estimate

$$||u||_{L^p} + ||\nabla u||_{L^p} + ||\nabla^2 u||_{L^p} \leq C(||u||_{L^p} + ||\Delta u||_{L^p}) \quad \forall u \in C_c^\infty(M).$$

Beside the interest they have in themselves as global regularity estimates, Calderón–Zygmund and gradient estimates are related to a number of topics of great interest in geometric and harmonic analysis.

The first important feature of these inequalities is their interaction with the theory of Sobolev spaces. Unlike in the Euclidean setting, on a Riemannian manifold there exist several, non necessarily equivalent, definitions of the Sobolev space of order $k \in \mathbb{N}$ and integrability class $p \in [1, +\infty]$. For instance, one can define $W^{k,p}(M)$ as the space of L^p functions whose covariant (distributional) derivatives are in L^p up to the order k:

$$W^{k,p}(M) := \{u \in L^p(M) : \nabla^j u \in L^p(M), \quad j = 0, \dots k\}.$$

This turns out to be a Banach space once endowed with the usual norm

$$\|u\|_{W^{k,p}} := \sum_{j=0}^{k} \|\nabla^j u\|_{L^p}.$$

Thanks to a generalized Meyers–Serrin-type theorem (see e.g. Guidetti, Güneysu and Pallara, [59]), if $p \in [1, +\infty)$, this space can be characterized as the closure of $W^{k,p}(M) \cap C^\infty(M)$ with respect to $\|\cdot\|_{W^{k,p}}$, which is quite useful in applications. Alternatively, one can define the space $W_0^{k,p}(M)$ as the closure of compactly supported smooth functions $C_c^\infty(M)$ with respect to the Sobolev norm $\|\cdot\|_{W^{k,p}}$,

$$W_0^{k,p} := \overline{C_c^\infty(M)}^{\|\cdot\|_{W^{k,p}}}.$$

Finally, for even orders one can consider $H^{2m,p}(M)$ as the space of L^p functions whose iterations of the (distributional) Laplace–Beltrami operator are in L^p up to order m, i.e.,

$$H^{2m,p}(M) := \{u \in L^p(M) : \Delta^j u \in L^p(M), \quad j = 0, \dots m\},$$

endowed with the norm:

$$\|u\|_{H^{2m,p}} := \sum_{j=0}^{m} \|\Delta^j u\|_{L^p}.$$

Note that the space $H^{2,p}$ can also be interpreted as the domain of the m-accretive realization of $-\Delta : C_c^\infty(M) \to L^p(M)$, see [110, p.240].

In the Euclidean setting, $M = \mathbb{R}^n$, and on compact manifolds, the three spaces coincide. On geodesically complete Riemannian manifold one always has $W^{1,p}(M) = W_0^{1,p}(M)$, [7], whereas $k = 2$ is the first non-trivial order where, in general, one can only conclude that

$$W_0^{2,p}(M) \subseteq W^{2,p}(M) \subseteq H^{2,p}(M).$$

Nonetheless, if $|\mathrm{Ric}(x)| \leq br^2(x)$ and the injectivity radius decays in a controlled way, it is possible to prove that $W_0^{2,p}(M) = W^{2,p}(M)$, see [77, 78] as well as [63, 68] for previous results. Actually, when $p \in (1, 2]$ the sole bound on Ric is sufficient, see [78] for $p = 2$ and [74] for the extension to $(1, 2]$. See also [12] for previous results. Examples of manifolds on which $W_0^{2,p}(M) \subsetneq W^{2,p}(M)$ or $W^{2,p}(M) \subsetneq H^{2,p}(M)$ have been found in [121, 74, 41], note that all of these examples are characterized by wildly unbounded geometries. We refer to [121, 74] for an in depth introduction to the topic.

It turns out that if a Riemannian manifold supports CZ(p), hence GE(p), then using a result of Milatovic [65, Appendix] it is possible to prove that the three definitions of Sobolev spaces coincide. We refer to Remark 1.13 for a precise statement of this result.

Calderón–Zygmund inequalities play a central role also in harmonic analysis, where they can be interpreted with the language of Riesz transforms. In the Euclidean space $\mathbb{R}^n$, the second order Riesz transform of component $j, \ell \in \{1, \dots, n\}$ is defined, at least formally, by means of Fourier transform:

$$(\mathscr{R}_{j,\ell} f)\widehat{\ }(\xi) := \frac{\xi_j \xi_\ell}{|\xi|^2} \widehat{f}(\xi);$$

$\mathscr{R}_{j,\ell}$ is a paradigmatic example of Calderón–Zygmund singular integral operator. Such operators are known to be bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and of weak type $(1,1)$ [75]. By virtue of the very special structure of the Euclidean space, this is equivalent to saying that the operator $\nabla^2(-\Delta)^{-1}$ extends to a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n; T_2\mathbb{R}^n)$, here ∇^2 denotes the second covariant derivative associated to the Euclidean metric. This definition extends easily to the setting of Riemannian manifolds by taking the covariant derivative associated to the Riemannian metric and the Laplace–Beltrami operator, in spite of their Euclidean counterparts. More generally, on a Riemannian manifold (M, g) we define the k^{th} *order Riesz transform*, with k a positive integer, as

$$\mathscr{R}^k := \nabla^k(-\Delta)^{-k/2},$$

when k is odd, $(-\Delta)^{-k/2}$ is defined by means of the spectral theorem see for instance [123, Section 7.1]. For every $0 < \tau \in \mathbb{R}$, we also introduce the k^{th} *order local Riesz transform* as

$$\mathscr{R}^k_\tau := \nabla^k(\tau I - \Delta)^{-k/2},$$

here I is the identity operator. These latter operators are sometimes called *shifted* Riesz transform. It is not difficult to see that if $\mathscr{R}^2_\tau$ is a bounded operator from $L^p(M)$ to $L^p(M; T_2M)$, then the corresponding CZ(p) holds. In fact, one can say something more: the validity of CZ(p) is equivalent to the L^p boundedness of the second order local Riesz transform $\mathscr{R}^2_\tau$, see Proposition 1.14 below for a precise statement and proof of this equivalence. Boundedness in L^p of $\mathscr{R}^2$, instead, yields a stronger estimate of the form $\|\nabla^2 u\|_{L^p} \leq C\|\Delta u\|_{L^p}$.

The second order local Riesz transform $\mathscr{R}^2_\tau$ can be decomposed in first order terms

$$\nabla^2(\tau I - \Delta)^{-1} = \nabla(\tau I - \Delta_1)^{-1/2} \circ d(\tau I - \Delta)^{-1/2},$$

here d is the exterior differential and $\Delta_1 = d\delta + \delta d$ is the Hodge Laplacian on 1-forms. The first term $\nabla(\tau I - \Delta_1)^{-1/2}$ is a covariant Riesz transform on 1-forms while $d(\tau I - \Delta)^{-1/2}$ is a Riesz transform on functions. Note that the latter term shares the same boundedness properties of $\mathscr{R}^1_\tau$. The boundedness of these two first order local Riesz transforms, on 1-forms and functions, clearly yields the boundedness of $\mathscr{R}^2_\tau$ on L^p and, thus, CZ(p). This approach to Calderón–Zugmund inequalities was first adopted in [63].

On the other hand, if the first order Riesz transform $\mathscr{R}$ is bounded on L^p, the Moment inequality [67, Proposition 6.6.4] (which we can apply, for $-\Delta$ is a sectorial operator on $L^p(M)$), yields

$$\|\nabla u\|_p \leq C \left\|(-\Delta)^{1/2} u\right\|_p \leq C \left\|u\right\|_p^{1/2} \left\|\Delta u\right\|_p^{1/2},$$

which is a stronger multiplicative version of GE(p). The L^p-gradient estimate also follows if we assume L^p boundedness of the local Riesz transform $\mathscr{R}_\tau$, indeed,

$$\begin{aligned}\|\nabla u\|_{L^p} &\leq C\left(\|u\|_{L^p}+\|(-\Delta)^{1/2}u\|_{L^p}\right) \leq C\left(\|u\|_{L^p}+\|\Delta u\|_{L^p}^{1/2}\|u\|_{L^p}^{1/2}\right)\\ &\leq C\left(\|u\|_{L^p}+\|\Delta u\|_{L^p}\right).\end{aligned}$$

Note that in this case we cannot obtain the stronger multiplicative inequality.

In the last years, the validity of CZ(p) and GE(p) has been proved, for various ranges of p, under several geometric assumptions. In the Hilbertian case $p = 2$, GE(2) and CZ(2) are relatively simple to obtain. Indeed, integration by parts of the Poisson equation $\Delta u = f$ yields the multiplicative version of $GE(2)$:

$$\|\nabla u\|_{L^2}\leq \|u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2},$$

which holds on any complete Riemannian manifold, see [103, Section 3.1]. If we also assume a lower bound on the Ricci curvature, $\mathrm{Ric} \geq -K^2$, integration by part of Bochner's inequality gives an *infinitesimal* CZ(2) inequality:

$$\|\nabla^2 u\|_{L^2}^2\leq \frac{C\varepsilon^2}{2}\|u\|_{L^2}^2+\left(1+\frac{C^2}{2\varepsilon^2}\right)\|\Delta u\|_{L^2}^2$$

which holds for any $\varepsilon > 0$ and any $u \in C_c^\infty(M)$, [63, Proposition 4.15]. We use the term infinitesimal when $\|u\|_{L^2}$ can be made arbitrarily small. If $p \neq 2$, the validity of L^p-gradient and Calderón–Zygmund estimates is considerably more subtle.

If the Ricci curvature of M is bounded from below, by a special case of a celebrated result of Bakry, [8], the first order local Riesz transform is bounded on $L^p(M)$ for every $1 < p < +\infty$, which implies the validity of GE(p). In the case where $p > 2$, this result was also obtained via probabilistic arguments by Cheng, Thalmaier and Thompson [28]. To the best of our knowledge, it is not known whether the first order local Riesz transform is bounded from $L^p(M)$ to $L^p(M, TM)$, $1 < p < 2$, on any complete Riemannian manifold M. However, Coulhon and Duong [32] proved that if $p \in (1, 2]$, then the L^p gradient estimates GE(p) holds on any geodesically complete manifold. A much simpler proof thereof may be found in [74, Lemma 1.6]. It is worthwhile noticing that if one takes $M = \mathbb{R}^2 \sharp \mathbb{R}^2$ the connected sum of two copies of $\mathbb{R}^2$, then the multiplicative estimate

$$\|\nabla u\|_{L^p}\leq C\|u\|_{L^p}^{1/2}\|\Delta u\|_{L^p}^{1/2} \quad \forall u \in C_c^\infty(M)$$

fails for $p > 2$, although M has Ricci curvature bounded from below, whence GE(p) holds for every p in $(1, \infty)$. This example of Coulhon and Duong, [32, second remark after Theorem 4.1], illustrates how sensitive of the geometry of the underlying manifold these inequalities may be.

Calderón–Zygmund inequalities are generally harder to prove and often require more geometrical assumptions. Güneysu and Pigola proved in [63] that when the $C^{1,\alpha}$-harmonic radius of (M, g) is positive, a computation in a harmonic coordinate system together with a covering argument allows to localize the Riemannian problem reducing it to the Euclidean setting. Using this strategy, CZ(p) was proved on the whole range $p \in (1, \infty)$ on manifolds of bounded Ricci curvature (both from above and below) and positive injectivity radius; see [63, Theorem C]. In the same paper, CZ(p) is also approached using the aforementioned decomposition of $\mathscr{R}^2_\tau$. Thanks to boundedness results for the covariant Riesz transform on 1-forms, [118], Güneysu and Pigola were able to prove CZ(p) for $1 < p \leq 2$ when $\|\mathrm{Riem}\|_{L^\infty} + \|\nabla \mathrm{Riem}\|_{L^\infty} \leq C$ and an additional volume doubling condition holds on the manifold, [63, Theorem D]. This latter assumption was subsequently removed by Baumgarth–Devyver–Güneysu, [13]. Finally, a very recent and far-reaching result due to Cao–Cheng–Thalmaier, [24], states that CZ(p) holds in the range $1 < p \leq 2$, under the sole assumption of Ricci curvature bounded from below. We refer to the very recent [29] for results on a Calderón–Zygmund inequality with respect to a weighted Laplace–Beltrami operator, or to [64] for a non-linear version of Calderón–Zygmund theory. In the case of Cartan–Hadamard manifolds (i.e. simply connected, complete manifolds of non-positive sectional curvatures) where $\nabla^i \mathrm{Riem}$ is bounded, $0 \leq i \leq 2$, Calderón–Zygmund theory was also investigated in [86] while the study of boundedness properties for $\mathscr{R}^2$ under the assumption that the Riemann tensor decays quadratically has been announced in [25]. Apart from the case of Ricci-bounded geometry alluded to in the above, the only further set of assumptions ensuring the validity of CZ(p) when $p > 2$ are given in [24, Theorem 1.2]. The manifolds considered therein must satisfy (Kato type) conditions on the curvature and its derivatives but, on the other hand, could have zero injectivity radius. Manifolds which do not support L^p-Calderón–Zygmund inequalities have been constructed in [63, 85, 121]. It is worth mentioning that in these counterexamples the Ricci curvature of the manifold at hand is always unbounded from below, we refer to Chapter 2 for further discussion on counterexamples to both CZ(p) and GE(p). For additional references on Calderón–Zygmund theory we refer to the nice survey of Pigola, [103], complemented with the more recent contributions to the field, [13, 74, 24, 29].

The three chapters of Part I contain our contributions to Calderón–Zygmund theory. We begin with "positive" results on L^p-gradient and Calderón–Zygmund estimates, most of which require the Ricci curvature to satisfy an appropriate lower bound in an integral sense in place of the pointwise bounds that commonly appear in the literature. Chapter 1 collects results obtained in [88].

Given (M, g) a Riemannian manifold, we first prove that if $p_0 > n$, and the Ricci curvature is bounded from below in an appropriate local $L^{p_0/2}$ integral sense (see Definition 1.1), then GE(p) holds for all p in $(1, p_0)$, this is our Theorem 1.6. Our condition is trivially satisfied if we assume standard pointwise lower bounds for the Ricci curvature, so that our result extends [28]. If, instead, p_0 is as in the above, M has positive injectivity radius and nonnegative Ricci curvature in a global $L^{p_0/2}$ integral sense (see again Definition 1.1), then GE(p) holds for all p in $(1, \infty)$, see Theorem 1.12. For large p, the proof of the first result is based upon a related L^∞ estimate of Dai, Wei and Zhang, [35], and a covering argument. The whole range is then obtained by interpolation. The second result, instead, relies on a local computation in $W^{1,p}$-harmonic coordinates.

Next, we move onto L^p-Calderón–Zygmund estimates, i.e. boundedness of the second order local Riesz transform. We prove that if M has a positive injectivity radius and either Ricci curvature pointwise bounded from below or nonnegative in the global $L^{p_0/2}$ sense for some $p_0 > n$, then CZ(p) holds for every $p \in (1, +\infty)$, or equivalently, $\mathscr{R}^2_\tau$ is bounded from $L^p(M)$ to $L^p(M; T_2M)$ for every $\tau > 0$. For a precise statement of this result, we refer to Theorem 1.17 whose proof relies on a computation in harmonic coordinates with a uniform $W^{1,q}$ bound. Note that $W^{1,q}$-harmonic estimates for large enough q imply a $C^{0,\alpha}$ control on the metric coefficients. This is an improvement on previously known results for CZ(p), [63], which relied on the existence of uniform $C^{1,\alpha}$-harmonic coordinates, and thus required stronger geometric assumptions. As we later prove in Chapter 2, for large p, there is no hope to drop the injectivity radius from these assumptions. Next, we show that if a Riemannian manifold has a spectral gap, i.e. the bottom of the L^2 spectrum of $-\Delta$ is strictly positive, then the validity of CZ(p) yields a global $W^{2,p}$ estimate of the form $\|u\|_{W^{2,p}} \leq C\|\Delta u\|_{L^p}$, see Lemma 1.20. In particular, this implies the L^p boundedness of the global Riesz transform $\mathscr{R}^2$. As a consequence of this and [24], we obtain L^p boundedness for $1 < p \leq 2$ of the global Riesz transform $\mathscr{R}^2$ when Ric $\geq -K^2$ and M has a spectral gap, Corollary 1.23; this improves on a result of Mauceri, Meda and Vallarino, [91], which required the additional assumption of a positive injectivity radius.

We conclude Chapter 1 with a boundedness result for higher, even order local Riesz transform. More precisely, given ℓ a positive integer and $\tau > 0$, we show that $\mathscr{R}^{2\ell}_\tau$ is bounded from $L^p(M)$ to $L^p(M; T_{2\ell}M)$ for every p in $(1, 2]$ under the assumption that M has a positive injectivity radius and the Ricci tensor and its derivatives up to order $2\ell - 2$ are uniformly bounded, this is Theorem 1.26. This result too has consequences on the corresponding (higher order) Calderón–Zygmund inequalities. In this thesis, we do not consider Riesz transforms of odd order ≥ 3. We believe that it is an interesting problem to find geometric conditions on M under which either $\mathscr{R}^{2\ell+1}_\tau$ or $\mathscr{R}^{2\ell+1}$ are bounded on L^p, for some positive integer ℓ. We also point out that in the special case that M is a symmetric space of noncompact type, a nice result of Anker, [6], proves that the Riesz transforms of any order are bounded on L^p, $1 < p < \infty$.

Chapter 2 is devoted to the exposition of three counterexamples to the validity of L^p-gradient and Calderón–Zygmund estimates, which have been published in [90] and [88]. As mentioned in the above, a lower bound on the Ricci curvature is sufficient to obtain the validity of CZ(p), at least when $1 < p \leq 2$, [24]. In our first example, we show that this result is optimal. Indeed, for every $p \in (1, +\infty)$ and for every increasing function $\lambda : [0, +\infty) \to \mathbb{R}$ such that $\lambda(t) \to +\infty$ as $t \to +\infty$, we construct a complete, n-

dimensional Riemannian manifold satisfying $\min \mathrm{Sect}(x) \geq -\lambda(r(x))$ outside a compact set, which does not support an L^p-Calderón–Zygmund inequality CZ(p), see Theorem 2.1. In particular, it is not possible to obtain CZ(p) under negative decreasing curvature bounds such as $\mathrm{Ric}(x) \geq -Cr^{\alpha}(x)$ for some $\alpha > 0$, as it is in the case of the closely related problem of the density of smooth compactly supported functions in $W^{2,p}(M)$, see [77, 74]. Under this milder condition, however, a disturbed $CZ(p)$ holds, [78, Section 6.2].

Taking into account that the result of Cao, Cheng and Thalmaier in [24] is sharp, it is natural to conjecture that geodesic completeness and a lower bound on the Ricci curvature are sufficient to ensure the validity of CZ(p) for large $p > 2$, see [60, p.177]. It should be noted that in all previous counterexample to the validity of CZ(p), [63, 85, 121], the Ricci curvature of the manifold at hand is always (wildly) unbounded from below. In our second counterexample, we show that there is no hope to obtain such result since for every $n \geq 2$ and $p > n$, we construct a complete, non-compact, n-dimensional Riemannian manifold with $\mathrm{Sect}(M) > 0$ such that CZ(p) fails; this is Theorem 2.9. Our construction relies on a deep and recent result of De Philippis and Núñez-Zimbrón, [39, Corollary 1.3]; note also that using a trick introduced in [74], our counterexample extends to $p > 2$. Furthermore, since GE(p) is known to hold when Ric is bounded from below, [28], our construction is an example of a Riemannian manifold which supports GE(p) but not CZ(p).

While several counterexamples to the validity of the L^p-Calderón–Zygmund inequality have been found in recent years, including our contributions, to the best of our knowledge, the literature is lacking regarding counterexamples to the L^p-gradient estimate; see [103, Section 9] for an extensive account of the topic. In the last part of Chapter 2, we fill this gap in the theory. Using a sequence of conformal deformations on separated balls of the Euclidean plane, we are able to construct for every n and $p > 2$, a complete Riemannian manifold on which the L^p gradient estimate fails for every $2 < p < +\infty$, see Theorem 2.13. As we explain in Remark 2.14, it is possible to modify this construction so that the negative part of the curvature grows as slowly as desired, which proves the optimality of the result of Cheng, Thalmaier and Thompson, [28].

While many of the above results concerning CZ(p) and the density of $C_c^{\infty}(M)$ in $W^{2,p}(M)$ require some kind of lower bound on the Ricci curvature, in Chapter 3 we show that several of these properties still hold if one allows the curvature to become increasingly negative at infinity, possibly very fast, but in a controlled way. In particular, we consider a Cartan–Hadamard manifold (M, g) (i.e. a simply connected complete Riemannian manifold of non-positive sectional curvature) and assume that the Ricci curvature of M is controlled both from above and below polynomially at infinity. Namely,

$$-b\, r^{\beta}(x) \leq \mathrm{Ric}(x) \leq -a\, r^{\alpha}(x)$$

holds outside a compact set where $a, b > 0$ are positive constants. We prove that $W_0^{2,p}(M) = W^{2,p}(M)$ for all $p \in (1, +\infty)$ if $\beta = 2\alpha + 2$ and the validity of the L^2-

Calderón–Zygmund inequality if $\alpha = \beta$, see Theorem 3.18 and Theorem 3.23. Both results rely on the construction of an appropriate sequence of second order cutoff functions. The lower bound on the Ricci curvature implies via comparison a control on the Hessian of the cutoffs. This bound, however, might explode at $-\infty$ polinomially in $r(x)$; this possible unboundedness of the Hessian term is dealt with by using second order Hardy-type inequalities which we construct via the Green function of the p-Laplacian of an appropriately constructed auxiliary model manifold $(\widetilde{M}, \widetilde{g})$.

Chapter 3 is based upon results of [90].

Chapter 1

Gradient and Calderón–Zygmund inequalities under Ricci lower bounds

In this first chapter, we prove a number of results ensuring the validity of L^p-gradient and Claderón–Zygmund estimates under comparatively weak geometric assumptions. One recurrent theme is to obtain at least some of our results under the assumption that the Ricci curvature satisfies appropriate L^p lower bounds in place of the pointwise bounds that are more commonly found in the literature. These bounds arise naturally in some isospectral and geometric variational problems as well as in Ricci and Kähler-Ricci flows, [27, 115, 119, 10, 11, 9]. Under integral bounds on Ric, several properties of manifolds whose Ricci curvature is uniformly bounded from below are recovered, such as Laplacian and Bishop-Gromov comparisons. Volume doubling and estimates for the isoperimetric and local Sobolev constants can also be proved; see [46, 125, 126, 100, 101].

In the literature, one can find two notions of integral curvature bounds, one of global nature and one of uniform local nature.

Definition 1.1. Let (M, g) be an n-dimensional Riemannian manifold, suppose that $K \geq 0$, $R > 0$ and $1 < p < +\infty$. Set

$$\rho_K(x) \coloneqq (\min \operatorname{Ric} + (n-1)K^2)_-(x) \tag{1.1}$$

(where f_- denotes the negative part of f),

$$k(x, p, R, K) \coloneqq R^2 \frac{\|\rho_K\|_{L^p(B_R(x))}}{\operatorname{vol}(B_R(x))^{1/p}} \quad \text{and} \quad k(p, R, K) \coloneqq \sup_{x \in M} k(x, p, R, K).$$

We say that:

- M has *Ricci curvature bounded from below by* $-(n-1)K^2$ *in the global* L^p *sense* if $\rho_K \in L^p(M)$.
- M has an $\varepsilon > 0$*-amount of Ricci curvature below* $-(n-1)K^2$ *in the* L^p *sense at the scale* R if $k(p, R, K) < \varepsilon$.

Remark 1.2. Note that $\rho_K(x) = 0$ if and only if the Ricci curvature is bounded from below by $-K^2$. In particular, if the Ricci curvature satisfies the lower bound $\operatorname{Ric} \geq -(n-1)K^2$, then $k(p, R, K) = 0$ for all $R > 0$ and $p \in (1, +\infty)$, hence, both integral Ricci curvature conditions are satisfied. On the other hand, the integral bounds we assume are indeed weaker than the usual pointwise bounds; see Remark 1.7 below.

1.1 Gradient estimates: local uniform L^q Ricci bounds

While L^p-gradient estimates hold on any complete Riemannian manifold if $p \in (1,2]$, [32], a lower bound on the Ricci curvature is necessary to obtain the whole range $p \in (1,+\infty)$, [28]. In this section, we prove that the same conclusion holds if we replace the pointwise bound with a local uniform L^q bound on the Ricci tensor. Before proceeding with the proof of the result, we point out the following facts, which will be repeatedly used in the sequel.

Remark 1.3. If (M,g) is a complete Riemannian manifold supporting an L^p-gradient estimate for some $p \in (1,+\infty)$, then GE(p) extends with the same constant to all functions in $H^{2,p}(M)$. Indeed, if $u \in H^{2,p}(M)$, by a result of Milatovic, [65, Appendix], there exists a sequence $\{u_k\} \subseteq C_c^\infty(M)$ such that $u_k \to u$ with respect to the $H^{2,p}$ norm. Applying GE(p) to u_k, we deduce that ∇u_k is Cauchy and thus converges in the space of L^p vector fields. Testing ∇u_k against a smooth and compactly supported vector field and taking the limit shows in fact that ∇u_k converges in L^p norm to the weak gradient ∇u.

Remark 1.4. As noted in [101, Section 2.3] for the case $K = 0$, smallness of $k(q, R_0, K)$ at a fixed scale R_0 implies a control on $k(q, R, K)$ for all scales $R > 0$. This is a consequence of a volume comparison result contained in [16, Lemma 10]. Indeed, if $q > n/2$ there exists $\varepsilon = \varepsilon(n, q, K) > 0$ such that if $k(q, R_2, K) < \varepsilon$, then for every $0 < R_1 < R_2$ one has

$$k(q, R_1, K) \leq 4 \left(\frac{R_1}{R_2}\right)^2 \left(\frac{v_K(R_2)}{v_K(R_1)}\right)^{\frac{1}{q}} k(q, R_2, K),$$

where $v_K(R)$ is the volume of the geodesic ball of radius R in the n-dimensional space form of constant curvature K. Since $v_K(R_1) \sim R_1^n$, $k(q, R_1, K) \to 0$ as $R_1 \to 0$, i.e., $k(q, R_1, K)$ can be made arbitrarily small. See [16, Corollary 13].

Note also that $k(p, r, K) \leq k(q, r, K)$ whenever $p \leq q$.

Under the assumption that $k(p/2, 1, K)$ is small, we first prove a local L^p-gradient estimate, which is based upon a local gradient estimate of Dai, Wei and Zhang, [35]. In what follows, we use the notation

$$\|u\|^*_{L^p(\Omega)} = \left(\fint_\Omega |u|^p\right)^{1/p} = \left(\frac{1}{\mathrm{vol}(\Omega)} \int_\Omega |u|^p\right)^{1/p}.$$

Lemma 1.5. *Let $p > n$. There exists $\varepsilon = \varepsilon(n, p, K) > 0$, $C(n, p) > 1$ and $0 < R_0 \leq 1$ such that if $k(p/2, 1, K) \leq \varepsilon$, then*

$$\sup_{B_{R/2}(x)} |\nabla u|^2 \leq C R^{-2} \left[\left(\|u\|^*_{L^2(B_R(x))}\right)^2 + \left(\|\Delta u\|^*_{L^p(B_R(x))}\right)^2 \right] \tag{1.2}$$

for all $0 < R \leq R_0$, for all $x \in M$ and for all smooth functions u on $B_1(x)$. Moreover, there exists a constant $D(n, p) > 0$ such that

$$\|\nabla u\|^p_{L^p(B_{R/2}(x))} \leq D R^{-p} \left(\|u\|^p_{L^p(B_R(x))} + \|\Delta u\|^p_{L^p(B_R(x))} \right) \tag{1.3}$$

for all $x \in M$, $0 < R \leq R_0$ and all smooth functions u on $B_1(x)$.

Proof. By [35, Theorem 1.9], there exists a constant $\varepsilon_0(n,p) > 0$ independent of R_0 such that if $k(p/2, R_0, 0) \le \varepsilon_0$, then (1.2) holds for all $0 < R \le R_0$. By Remark 1.4 we know that if $k(p/2, 1, K) \le \varepsilon$, then $k(p/2, R, K) \lesssim R^{2-n/2p}$ as $R \to 0$ and since $\rho_0(x) \le \rho_K(x) + (n-1)|K|$, we have

$$k(p/2, R, 0) \le k(p/2, R, K) + (n-1)|K|R^2.$$

Hence, if we take R_0 small enough, then $k(p/2, R_0, 0) \le \varepsilon_0$, which concludes the first part of the lemma. The constant R_0 depends on K, n, ε and ε_0.

From (1.2) we have

$$\sup_{B_{R/2}(x)} |\nabla u|^p \le C^{p/2} R^{-p} 2^{p/2-1} \left[(\|u\|^*_{L^2(B_R(x))})^p + (\|\Delta u\|^*_{L^p(B_R(x))})^p \right].$$

By Hölder's inequality

$$\left(\fint_{B_R(x)} u^2 \right)^{p/2} \le \fint_{B_R(x)} u^p,$$

whence

$$\int_{B_{R/2}(x)} |\nabla u|^p \le C^{p/2} R^{-p} 2^{p/2-1} \frac{\mathrm{vol}(B_{R/2}(x))}{\mathrm{vol}(B_R(x))} \left(\int_{B_R(x)} |u|^p + \int_{B_R(x)} |\Delta u|^p \right).$$

To conclude the proof of (1.3) recall that, as a consequence of the volume comparison, (M, g) satisfies a uniform local volume doubling property, i.e. there exists $C > 0$ such that

$$\mathrm{vol}(B_{R/2}(x)) \le C \,\mathrm{vol}(B_R(x))$$

for all $x \in M$ and $0 < R \le R_0$. See Lemma 10 and subsequent results in [16]. This completes the proof of Lemma 1.5. □

We are now ready to prove the global L^p-gradient estimate.

Theorem 1.6. *Suppose that $n < p_0 < +\infty$. There exists a constant $\varepsilon = \varepsilon(p_0, n, K) > 0$ such that if $k(p_0/2, 1, K) \le \varepsilon$ for some $K \ge 0$, then the L^p-gradient estimate GE(p) holds on M for every $1 < p \le p_0$.*

Proof. We start by noting that the local L^{p_0} gradient estimate (1.3), $p_0 > n$, extends to the whole manifold using a uniformly locally finite covering of M. The existence of such covering is a formal consequence of the local volume doubling inequality, which, as we have recalled above, holds under local integral Ricci bounds. Thus, let $u \in C^\infty_c(M)$ and $\Omega = \mathrm{supp}(u)$ and let $0 < R \le R_0$ small enough such that $2R \le 1$. Here R_0 is the radius appearing in Lemma 1.5. By local volume doubling, there exist $x_1, \ldots, x_h \in M$ such that

(i) $\Omega \subseteq \bigcup_{i=1}^h B_{R/2}(x_i)$;

(ii) every $x \in \Omega$ intersects at most N balls $B_R(x_i)$.

Then,

$$\begin{aligned}\int_M |\nabla u|^{p_0} &\leq \sum_{i=1}^{h} \int_{B_{R/2}(x_i)} |\nabla u|^{p_0} \leq DR^{-p} \sum_{i=1}^{h} \left(\int_{B_R(x_i)} |u|^{p_0} + \int_{B_R(x_i)} |\Delta u|^{p_0} \right) \\ &\leq DR^{-p_0} \int_M \sum_{i=1}^{h} 1_{B_R(x_i)} \left(|u|^{p_0} + |\Delta u|^{p_0}\right) \leq DR^{-p_0} N \left(\int_M |u|^{p_0} + \int_M |\Delta u|^{p_0} \right),\end{aligned}$$

which proves the gradient estimate GE(p) with $p = p_0 > n$.

Recall that if $p \in (1,2]$, then L^p-gradient estimates always holds on complete Riemannian manifolds [32]. We now interpolate between this and the result for $p > n$ obtained in the first part of the proof.

Suppose that $2 < p \leq n$ and choose $q > n$ and θ in $(0,1)$, so that $1/p = \theta/q + (1-\theta)/2$. It is well known that the heat semigroup is strongly continuous and contractive on $L^r(M)$ for all $r \in [1, +\infty)$ [60, Theorem IV.8]. By the Hille–Yosida Theorem, -1 is in the resolvent set of its infinitesimal generator $-\Delta$. Then $-\Delta + I$ is (surjective and) invertible in $L^q(M)$, hence, $(-\Delta + I)^{-1}$ is bounded on $L^q(M)$ and its range is contained in the domain of Δ where $\mathrm{Dom}_{L^q}(\Delta) = H^{2,q}(M)$. Let $u \in L^q(M)$ and $v = (-\Delta + I)^{-1}u \in H^{2,q}(M)$, by Remark 1.3 and the first part of the theorem we have

$$\|\nabla(-\Delta + I)^{-1}u\|_{L^q} = \|\nabla v\|_{L^q} \leq C\left(\|\Delta v\|_{L^q} + \|v\|_{L^q}\right) \leq C\|(-\Delta + I)v\|_{L^q} = C\|u\|_{L^q},$$

hence, $\nabla(-\Delta + I)^{-1}$ extends to a bounded operator from $L^q(M)$ to $L^q(M; T_1M)$.

On the other hand,

$$\|\nabla(-\Delta + I)^{-1}f\|_{L^2} = \left((-\Delta + I)^{-1}f, \Delta(-\Delta + I)^{-1}f\right)_{L^2}.$$

Since both $(-\Delta + I)^{-1}$ and $\Delta(-\Delta + I)^{-1}$ extend to bounded operators on $L^2(M)$, the operator $\nabla(-\Delta + I)^{-1}$ extends to a bounded operator from $L^2(M)$ to $L^2(M; T_1M)$. By the Riesz–Thorin Interpolation Theorem $\nabla(-\Delta + I)^{-1}$ extends to a bounded linear operator from $L^p(M)$ to $L^p(M; T_1M)$. As a consequence, the L^p-gradient estimate holds on M. □

Note that since local uniform L^q Ricci bounds include the usual pointwise lower bounds, our Theorem 1.6 provides yet another alternative proof of the result by Cheng, Thalmaier and Thompson, [28], using only PDEs methods.

Remark 1.7. As alluded to in the introduction, the integral curvature bounds assumed here are weaker than the classical pointwise bounds. An easy example of a Riemannian manifold (M, g) satisfying $\inf_M \min \mathrm{Ric} = -\infty$ but with $k(p, 1, 0)$ arbitrarily small, can be constructed as follows. We take $M = \mathbb{R}^2$ endowed with the conformally flat metric $g = e^{2\varphi}dx^2$, where φ is a smooth non-positive function. It is easy to see that $\mathrm{vol}_g(K) \leq \mathrm{vol}_e(K)$ for any measurable set $K \subset \mathbb{R}^2$, and $B^g_R(w) \supseteq B^e_R(w)$ for any $R > 0$ and $w \in \mathbb{R}^2$. Suppose now that $\mathrm{supp}\,\varphi \in \cup_{n\in\mathbb{N}} B^e_{1/2}((4n, 0))$. This guarantees that $B^g_1(w) \subseteq B^e_2(w)$ for any $w \in \mathbb{R}^2$. Moreover, given $w \in \mathbb{R}^2$, let n_w be the unique integer (if any) such that $B^e_{1/2}((4n_w, 0))$ intersects $B^e_1(w)$. Then

$$\mathrm{vol}_g\, B^g_1(w) \geq \mathrm{vol}_g\, B^e_1(w) \geq \mathrm{vol}_g(B^e_1(w) \setminus B^e_{1/2}(4n_w, 0)) = \frac{3}{4}\pi. \tag{1.4}$$

Fix $a \in (2 - \frac{2}{p}, 2)$, and $\phi_0 \in C^\infty_c(B^e_{1/2}(0,0))$, we define $\varphi(x,y) = \sum_{n\in\mathbb{N}} \phi_n(x,y)$, where $\phi_n(x,y) = n^{-a}\phi_0(n(x-4n,y))$ if $n \geq 1$. On the one hand, since $\Delta_e\phi_0$ attains positive values and since $\Delta_e\phi_n(x,y) = n^{2-a}\Delta_e\phi_0(n(x-4n,y))$, we have that $\mathrm{Ric}_g = -2\Delta_e\varphi$ is lower unbounded. On the other hand, we have

$$\begin{aligned}\int_{B^g_1(w)} ((\min \mathrm{Ric})_-)^p d\mu_g = 2^p \int_{B^g_1(w)} ((\Delta_e\varphi)_+)^p d\mu_g &\leq 2^p \int_{B^e_2(w)} ((\Delta_e\phi_{n_w})_+)^p dx^2 \\ &= 2^p n_w^{2p-pa-2} \int_{B^e_1(0,0)} ((\Delta_e\phi_0)_+)^p dx^2 \\ &\leq 2^p \int_{B^e_1(0,0)} ((\Delta_e\phi_0)_+)^p dx^2,\end{aligned}$$

which is uniformly bounded independently of w. Moreover, choosing an appropriate ϕ_0, we can assume that the right-hand side of the above estimate is arbitrarily small. Together with the uniform volume lower bound (1.4), this proves that $k(p,1,0) < +\infty$ and can be made arbitrarily small.

1.2 Gradient estimates: global L^q Ricci bounds

In this section, we show that gradient estimates also hold under the assumption of positive injectivity radius and non-negative Ricci curvature in the global integral sense. While the proof of Theorem 1.6 relies on a local L^∞ gradient estimate, this second result relies on a computation in harmonic coordinates with a uniform $W^{1,q}$ bound. Preliminarily, we recall the following

Definition 1.8. Let (M,g) be an n-dimensional Riemannian manifold and let $n < q < +\infty$. The $W^{1,q}$ *harmonic radius at* x, denoted by $r_{W^{1,q}}(x)$, is the supremum of all $R > 0$ such that there exists a coordinate chart $\phi : B_R(x) \to \mathbb{R}^n$ satisfying

(i) $2^{-1}[\delta_{ij}] \leq [g_{ij}] \leq 2[\delta_{ij}]$ in the sense of quadratic forms;

(ii) $R^{1-n/q}\|\partial_k g_{ij}\|_{L^q(B_R(x))} \leq 1$;

(iii) ϕ is a harmonic map.

The following theorem encloses in a single statement classical contributions by Anderson and Cheeger and a more recent result of Hiroshima.

Theorem 1.9 ([4, 71]). *Fix $n \in \mathbb{N}$, $q > n$, $K \geq 0$ and $i > 0$. Let (M,g) be a complete, n-dimensional Riemannian manifold satisfying $r_{\mathrm{inj}} \geq i$ and either of the following assumptions:*

(i) $\mathrm{Ric} \geq -(n-1)K^2$ *or*

(ii) Ric *is non-negative in the global* $L^{q/2}$ *sense, i.e.,* $\lambda = \|(\min \mathrm{Ric})_-\|_{L^{q/2}(M)} < +\infty$.

Then, $r_{W^{1,q}}(z) \geq \bar{r}$ *independently of* $z \in M$*, where* $\bar{r} = \bar{r}(n, q, K, i, \lambda) > 0$.

Note that, by Sobolev embedding, we have for free a $C^{0,\alpha}$ control on the metric coefficients within the ball $B_{\bar{r}/2}(z)$. Moreover, we observe the inclusions $B^e_{\bar{r}/8} \subseteq \phi(B_{\bar{r}/4}(z)) \subseteq B^e_{\bar{r}/2}$, where $B^e \subseteq \mathbb{R}^n$ denotes the Euclidean ball centered at the origin. Since, inside $B^e_{\bar{r}/8}$, the Euclidean and the Riemannian measures are mutually controlled by absolute uniform constants, in performing integrations in local coordinates, the chosen measure is irrelevant.

Remark 1.10. We have already observed that complete manifolds with Ricci lower bounds in the uniform local integral sense, enjoy the uniform local volume doubling property at any fixed scale. In the class of manifolds with positive injectivity radius, the same is true if we consider the case of global L^q conditions. This follows from Croke isoperimetric estimate and volume comparison. In particular, at a sufficiently small scale, we have the existence of a covering with finite intersection multiplicity as in the proof of Theorem 1.6; see e.g. [71, Proposition 1.5]. Conversely, if one assumes a priori that $r_{W^{1,q}}(M) := \inf_{x \in M} r_{W^{1,q}}(x) > 0$, then the double-sided Euclidean control on the volume of the balls at a small scale implies the uniform volume doubling property, and hence the covering property.

Recall that, if $(x^1, \cdots, x^n)$ is a system of harmonic coordinates, then

$$(\nabla u)^j = g^{jk}\partial_k u, \quad \Delta u = g^{ij}\partial^2_{ij} u,$$

where $g = [g_{ij}]$ and $g^{-1} = [g^{ij}]$ are, respectively, the matrix of the metric coefficients and its inverse.

Theorem 1.11. *Suppose that* $r_{W^{1,q}}(M) = \bar{r} > 0$ *for some* $q > n$*. Then for every* $1 < p < +\infty$*, the* L^p*-gradient estimate GE(p) holds on* M*.*

Proof. Fix $0 < r < \bar{r}/16$. Since the metric coefficients in $W^{1,q}$-harmonic coordinates are uniformly $C^{0,\alpha}$-controlled, there exist an absolute constant $C > 1$ such that, for any $u \in C^\infty_c(M)$ and $0 < R \leq r$,

$$C^{-1}\|\nabla^e u\|_{L^p(B^e_R)} \leq \|\nabla u\|_{L^p(B_{2R}(x))} \leq C\|\nabla^e u\|_{L^p(B^e_{4R})}$$

and

$$\|g^{ij}\partial^2_{ij}u\|_{L^p(B^e_R)} \leq C\|\Delta u\|_{L^p(B_{2R}(x))}.$$

On the other hand, by the Euclidean estimates of the gradient, [49, Theorem 9.11], there exists a constant $C = C(n, p, R) > 0$ such that

$$C^{-1}\|\nabla^e u\|_{L^p(B^e_{2r})} \leq \|u\|_{L^p(B^e_{4r})} + \|g^{ij}\partial^2_{ij}u\|_{L^p(B^e_{4r})}.$$

Hence,

$$\begin{aligned}\|\nabla u\|_{L^p(B_r(x))} &\leq C\|\nabla^e u\|_{L^p(B^e_{2r})}\\ &\leq C\left(\|u\|_{L^p(B^e_{4r})}+\|g^{ij}\partial^2_{ij}u\|_{L^p(B^e_{4r})}\right)\\ &\leq C\left(\|u\|_{L^p(B_{8r}(x))}+\|\Delta u\|_{L^p(B_{8r}(x))}\right).\end{aligned}$$

Thanks to the uniform local doubling condition, M has a countable covering by balls $\{B_r(x_j)\}$ such that $\{B_{8r}(x_j)\}$ has finite intersection multiplicity, see Remark 1.10. Then, the global L^p estimate follows by adding the local inequalities similarly as in Theorem 1.6. □

Combining Theorem 1.11 and the result of Hiroshima, Theorem 1.9, we obtain the desired result.

Theorem 1.12. *Suppose $r_{\mathrm{inj}}(M) > 0$ and M has non-negative Ricci curvature in the global $L^{q/2}$ sense for some $n < q < +\infty$. Then, for every $1 < p < +\infty$, GE(p) holds on M.*

1.3 Calderón–Zygmund inequalities

We begin this section with an observation which will be useful in the following chapters.

Remark 1.13. If the manifold at hand supports CZ(p) then GE(p) holds thanks to [63, Corollary 3.11], then $\|u\|_{W^{2,p}}\leq \|u\|_{H^{2,p}}$ at least con smooth and compactly supported functions. However, thanks to a functional analytic result of Milatovic, [65, Appendix], $C^\infty_c(M)$ is dense in $H^{2,p}(M)$ with respect to the norm $\|\cdot\|_{H^{2,p}}$ on all complete manifolds, this allows us to conclude that $H^{2,p}(M) \subseteq W^{2,p}_0(M)$ and, thus, equality of the three definition of Sobolev spaces. See [121, Remark 2.1] or [103, Proposition 4.7]. As we shall see in Chapter 2, the converse is not true: there are examples of Riemannian manifolds which lack CZ(p) but where smooth and compactly supported functions are dense in $W^{2,p}(M)$, see Remark 2.2. Note that the above argument also yields that CZ(p) extends with the same constant to functions in $H^{2,p}(M)$. Compare this with Remark 1.3.

Next, we prove the equivalence between boundedness of the second order local Riesz transform and Calderón–Zygmund inequalities. For later use, we state this equivalence also including the case of higher order Riesz transforms.

Proposition 1.14. *Let $1 < p < \infty$, $\tau > 0$ and let $k \geq 1$ be an integer. The local Riesz transform $\mathscr{R}^{2k}_\tau$ of order $2k$ is bounded from $L^p(M)$ to $L^p(M;T_{2k}M)$ if and only if the L^p-Calderón–Zygmund inequality of order $2k$*

$$\|\nabla^{2k}u\|_p \leq C\,[\,\|u\|_p + \|\Delta^k u\|_p\,] \qquad \forall u \in \mathrm{Dom}_{L^p}(\Delta^k) \tag{1.5}$$

holds on M for some $C > 0$, where

$$\mathrm{Dom}_{L^p}(\Delta^k) = \{u \in L^p(M)\ :\ \Delta^k u \in L^p(M)\}$$

is the domain of the kth power of the Laplacian in L^p. Moreover, when $k = 1$ the latter assertions are also equivalent to

$$\|\nabla^2 u\|_p \leq C\,[\,\|u\|_p + \|\Delta u\|_p\,] \qquad \forall u \in C_c^\infty(M).$$

Proof. Since $-\Delta$ generates a contraction semigroup on $L^p(M)$, by the Hille–Yosida Theorem the operator $-\Delta$ is sectorial in $L^p(M)$ and the resolvent $(-\Delta+\tau I)^{-1}$ is bounded on L^p. Hence, so is $(-\Delta+\tau I)^{-k}$. Set $\psi(\lambda) := (\lambda^k+\tau)(\lambda+\tau)^{-k}$. It is not hard to prove that both ψ and $1/\psi$ are in the extended Dunford class $\mathcal{E}_\theta$ for every θ in $(\pi/2,\pi)$, see [67, p.28] for the precise definition. By the standard functional calculus for sectorial operators, [67, Theorem 2.3.3], $\psi((-\Delta))$ and $(1/\psi)((-\Delta))$ extend to bounded operators on $L^p(M)$.

Suppose first that $\mathscr{R}_\tau^{2k}$ is bounded on $L^p(M)$, i.e. there exists a constant C such that

$$\|\mathscr{R}_\tau^{2k} f\|_{L^p(M)} \leq C\,\|f\|_{L^p(M)} \qquad \forall f \in L^p(M).$$

In particular, if $u \in \mathrm{Dom}_{L^p}((-\Delta)^k)$, then $f := (-\Delta+\tau I)^k u$ is in $L^p(M)$, and

$$\|\nabla^{2k} u\|_{L^p(M)} \leq C\,\|(-\Delta+\tau I)^k u\|_{L^p(M)} \leq C\,[\,\|\Delta^k u\|_{L^p(M)} + \|u\|_{L^p(M)}\,],$$

where C depends on τ. The last inequality is a straightforward consequence of the boundedness in L^p of $(1/\psi)((-\Delta))$.

Conversely, suppose that (1.5) holds. Consider f in $L^p(M)$. Since τ is in the resolvent set of $(-\Delta)$, the operator $(-\Delta+\tau I)^k$ maps $\mathrm{Dom}_{L^p}(\Delta^k)$ onto $L^p(M)$. Therefore, there exists u in $\mathrm{Dom}_{L^p}(\Delta^k)$ such that $u = (-\Delta+\tau I)^{-k} f$. Consequently, (1.5) with u as above, yields

$$\|\nabla^{2k} u\|_{L^p(M)} \leq C\,[\,\|(-\Delta+\tau I)^{-k} f\|_p + \|\Delta^k(-\Delta+\tau I)^{-k} f\|_p\,]. \tag{1.6}$$

Now, both $(-\Delta+\tau I)^{-k}$ and $\Delta^k(-\Delta+\tau I)^{-k}$ are bounded operators on $L^p(M)$, as $\psi((-\Delta))$ is. Furthermore, standard properties of sectorial operators imply that there exists a constant C such that

$$|\!|\!|(-\Delta+\tau I)^{-k}|\!|\!|_{L^p(M)} \leq \left|\!\left|\!\left|(-\Delta+\tau I)^{-1}\right|\!\right|\!\right|^k_{L^p(M)} \leq \frac{C}{\tau^k} \qquad \forall \tau > 0$$

and

$$|\!|\!|\Delta^k(-\Delta+\tau I)^{-k}|\!|\!|_{L^p(M)} \leq \left|\!\left|\!\left|\Delta(-\Delta+\tau I)^{-1}\right|\!\right|\!\right|^k_{L^p(M)} \leq C \qquad \forall \tau > 0.$$

This and (1.6) yield

$$\|\nabla^{2k}(-\Delta+\tau I)^{-k} f\|_{L^p(M)} \leq C\,[\tau^{-k}\,\|f\|_p + \|f\|_p\,] \leq C\,\max(1,\tau^{-k})\,\|f\|_p\,,$$

as required.

The last part of the proof follows from Remark 1.13. □

The rest of this section is devoted to the proof of CZ(p) under the assumption that $r_{W^{1,p}}(M) > 0$; as in Section 1.2, this result relies on local estimates in $W^{1,q}$-harmonic coordinates and on a covering argument made possible by the uniform local volume doubling condition, see Remark 1.10.

The crucial ingredient is the following estimate of the first order term in the local expression of the Hessian of a smooth function. Recall that, if $(x^1, \cdots, x^n)$ is a system of harmonic coordinates, then

$$\nabla^2_{ij} u = \mathrm{Hess}(u)_{ij} = \partial^2_{ij} u - \Gamma^k_{ij}\partial_k u$$

where Γ^k_{ij} denote the Christoffel symbols.

Lemma 1.15. *Let $1 < p < +\infty$. Fix $z \in M$, $q > \max(n,p)$ and let $0 < r = \frac{1}{4} r_{W^{1,q}}(z)$. Finally, denote by Γ^k_{ij} the Christoffel symbols with respect to the $W^{1,q}$ harmonic coordinates system $\phi(x) = (x^1, \cdots, x^n) : B_r(z) \to U \supseteq B^e_{r/2}$. Then, there exists a constant $C = C(n,p,q,r) > 0$ such that, for any $u \in C^\infty(M)$,*

$$C^{-1} \cdot \|\Gamma^k_{ij}\partial_k u\|_{L^p(B^e_{r/2})} \leq \|\mathrm{Hess}^e\, u\|_{L^p(B^e_{r/2})} + \|\nabla u\|_{L^p(B_r(z))}.$$

Proof. We apply Hölder's inequality with conjugate exponents $t = q/(q-p)$ and $t' = q/p$ to get

$$\|\Gamma^k_{ij}\partial_k u\|_{L^p(B^e_{r/2})} \leq \sum_k \|\Gamma^k_{ij}\|_{L^q(B_r(z))} \cdot \|\nabla^e u\|_{L^{pq/(q-p)}(B^e_{r/2})}, \qquad \forall\, i,j = 1,\ldots,n. \tag{1.7}$$

Next, we recall that the Christoffel symbols display a C^1 dependence on the metric coefficients in the form

$$\Gamma = \frac{1}{2} g^{-1} \cdot \partial g.$$

Since $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$ and $\|\partial g\|_{L^q}$ are bounded inside $B_r(z)$ (with a bound depending only on n, q, r), we deduce that there exists a constant $C = C(n,q,r) > 0$ such that

$$\|\Gamma^k_{ij}\|_{L^q(B_r(z))} \leq C. \tag{1.8}$$

It remains to take care of gradient term in (1.7). To this end, for the sake of clarity, we distinguish three cases according to the values of p.

$(\mathbf{1 < p < n})$. Since

$$\frac{pq}{q-p} < p^* := \frac{np}{n-p},$$

we can apply directly the Sobolev(–Kondrakov) embedding theorem and deduce that, for some constant $S = S(r,p,q,n) > 0$,

$$S^{-1} \cdot \|\nabla^e u\|_{L^{pq/(q-p)}(B^e_{r/2})} \leq \|\mathrm{Hess}^e\, u\|_{L^p(B^e_{r/2})} + \|\nabla^e u\|_{L^p(B^e_{r/2})}.$$

On the other hand, observe that

$$\|\nabla^e u\|_{L^p(B^e_{r/2})} \leq C \|\nabla u\|_{L^p(B_r(z))}$$

for some absolute constant $C > 0$, whence

$$\|\nabla^e u\|_{L^{\frac{pq}{q-p}}(B^e_{r/2})} \leq C\left(\|\mathrm{Hess}^e\, u\|_{L^p(B^e_{r/2})} + \|\nabla u\|_{L^p(B_r(z))}\right). \tag{1.9}$$

Inserting (1.8) and (1.9) into (1.7), gives the desired inequality when $1 < p < n$.

($\mathbf{p} = \mathbf{n}$). Let $1 < \tilde{p} < n = p$ be defined by

$$\tilde{p} = \frac{nq}{2q-n}.$$

Since

$$\frac{nq}{q-n} = \frac{n\tilde{p}}{n-\tilde{p}} =: \tilde{p}^*,$$

we can apply the Sobolev embedding theorem and the Hölder inequality to deduce that, for some constant $S = S(r, q, n) > 0$,

$$\begin{aligned} S^{-1} \cdot \|\nabla^e u\|_{L^{nq/(q-n)}(B^e_{r/2})} &\leq \|\mathrm{Hess}^e\, u\|_{L^{\tilde{p}}(B^e_{r/2})} + \|\nabla^e u\|_{L^{\tilde{p}}(B^e_{r/2})} \\ &\leq |B^e_{r/2}|^{(n-\tilde{p})/n\tilde{p}}\left(\|\mathrm{Hess}^e\, u\|_{L^n(B^e_{r/2})} + \|\nabla^e u\|_{L^n(B^e_{r/2})}\right). \end{aligned}$$

The conclusion follows exactly as above.

($\mathbf{p} > \mathbf{n}$). In this case, we can use Morrey's and Hölder's inequalities to deduce that, for some constant $S = S(r, p, q, n) > 0$,

$$\begin{aligned} \|\nabla^e u\|_{L^{pq/(q-p)}(B^e_{r/2})} &\leq |B^e_{r/2}|^{(q-p)/pq} \cdot \|\nabla^e u\|_{L^\infty(B^e_{r/2})} \\ &\leq S|B^e_{r/2}|^{(q-p)/qp}\left(\|\mathrm{Hess}^e\, u\|_{L^p(B^e_{r/2})} + \|\nabla^e u\|_{L^p(B^e_{r/2})}\right). \end{aligned}$$

The proof of the lemma is complete. □

We are now in the position to prove the following:

Theorem 1.16. *Let $1 < p < +\infty$. Suppose that $r_{W^{1,q}}(M) > 0$ for some $q > \max(n, p)$. Then the L^p-Calderón–Zygmund estimate CZ(p) holds on M.*

Proof. Set $\bar{r} = r_{W^{1,q}}(M)/4$ and let $u \in C^\infty_c(M)$. We preliminarily observe that there exists a uniform constant $C > 0$ such that, for any $z \in M$,

$$\|\nabla^e u\|_{L^p(B^e_{\bar{r}})} \leq C\|\nabla u\|_{L^p(B_{2\bar{r}}(z))}, \qquad \|g^{ij}\partial^2_{ij} u\|_{L^p(B^e_{\bar{r}})} \leq C\|\Delta u\|_{L^p(B_{2\bar{r}}(z))}.$$

Using the Euclidean Calderón–Zygmund estimate [49, Theorem 9.11] joint with Lemma 1.15, we find a constant $C = C(n, p, \bar{r}) > 0$ such that, for any $z \in M$,

$$\begin{aligned} \|\nabla^2 u\|_{L^p(B_{\bar{r}/4}(z))} &\leq \|\mathrm{Hess}^e\, u\|_{L^p(B^e_{\bar{r}/2})} + \sum_{ij}\|\Gamma^k_{ij}\partial_k u\|_{L^p(B^e_{\bar{r}/2})} \\ &\leq C\left(\|g^{ij}\partial^2_{ij} u\|_{L^p(B^e_{\bar{r}})} + \|u\|_{L^p(B^e_{\bar{r}})} + \|\nabla u\|_{L^p(B_{2\bar{r}}(z))}\right) \\ &\leq C\left(\|\Delta u\|_{L^p(B_{2\bar{r}}(z))} + \|u\|_{L^p(B_{2\bar{r}}(z))} + \|\nabla u\|_{L^p(B_{2\bar{r}}(z))}\right). \end{aligned}$$

Now, according to Remark 1.10, we cover M by a sequence of balls $\{B_{\bar{r}/4}(z_j)\}_{j\in\mathbb{N}}$ with the property that the covering $\{B_{2\bar{r}}(z_j)\}_{j\in\mathbb{N}}$ has finite intersection multiplicity. Summing up the local inequalities and using monotone and dominated convergence, we deduce the existence of a constant $C = C(n, p, K, i) > 0$ such that

$$C^{-1}\|\nabla^2 u\|_{L^p} \leq \|\Delta u\|_{L^p} + \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

To conclude we apply the L^p-gradient estimates of Theorem 1.11. Accordingly, there exists a constant $C = C(n, p, K) > 0$ such that

$$C^{-1}\|\nabla u\|_{L^p} \leq \|u\|_{L^p} + \|\Delta u\|_{L^p}$$

and this completes the proof. □

Combining Proposition 1.14 with Theorem 1.16 yields the following

Theorem 1.17. *Suppose that $r_{\mathrm{inj}}(M) > 0$, then $\mathscr{R}^2_\tau$ is bounded for every $\tau > 0$ from $L^p(M)$ to $L^p(M; T_2M)$*

(i) for every $1 < p < +\infty$, if $\mathrm{Ric} \geq -(n-1)K^2$ for some $K \geq 0$;

(ii) for every $1 < p \leq q$, if the Ricci curvature is non-negative in the global $L^{q/2}$ sense for some $q > n$.

Equivalently, we have the validity of the corresponding CZ(p) inequalities.

As explained in Remark 1.13, the validity of CZ(p) yields a new density result in Sobolev spaces.

Corollary 1.18. *Under the assumptions of Theorem 1.17, $C^\infty_c(M)$ is dense in $W^{2,p}(M)$ in the corresponding ranges of p.*

Remark 1.19. When $1 < p \leq 2$, the conclusions of Theorem 1.17 and Corollary 1.18 were already known in the case $\mathrm{Ric} \geq -K^2$ without assumptions on the injectivity radius, see [74] and [24]. Actually, in the case of the density result, a controlled growth of the negative part of the Ricci curvature can be allowed.

We conclude this section by showing how to pass from a Calderón–Zygmund inequality to a strong $W^{2,p}$ estimate when the underlying manifold has a spectral gap. The main tool is the following result.

Lemma 1.20. *Let $1 < p < \infty$. Suppose that M has spectral gap $b > 0$. Then, there exists a constant $C = C(n, p, b) > 0$ such that, for any $u \in C^\infty_c(M)$ it holds*

$$C^{-1}\|u\|_{L^p} \leq \|\Delta u\|_{L^p}.$$

Proof. As a straightforward application of the spectral theorem, if M has spectral gap $b > 0$, the heat semigroup on M satisfies the following estimate

$$\|e^{-t\mathcal{L}}\|_{L^2\to L^2} \leq e^{-bt},$$

where $\mathcal{L} = -\Delta$ and $\|\cdot\|_{L^2\to L^2}$ denotes the operatorial norm in L^2. On the other hand, the heat semigroup is contractive in L^p for all $1 \le p \le \infty$, i.e.

$$\|e^{-t\mathcal{L}}\|_{L^1\to L^1} \le 1, \qquad \|e^{-t\mathcal{L}}\|_{L^\infty\to L^\infty} \le 1.$$

Hence, an application of the Riesz–Thorin Interpolation Theorem implies that

$$\|e^{-t\mathcal{L}}\|_{L^p\to L^p} \le e^{-bc_pt},$$

for all $1 < p < \infty$, with $c_p = 1 - |(p-2)/p|$. Accordingly, for any $u \in C_c^\infty(M)$ one has the representation formula

$$u = \int_0^\infty e^{-t\mathcal{L}}\mathcal{L}u\,dt,$$

which yields by the Minkowski inequality

$$\|u\|_{L^p} \le \int_0^\infty e^{-bc_pt}\|\mathcal{L}u\|_{L^p}\,dt \le \frac{1}{bc_p}\|\Delta u\|_{L^p}.$$

□

Remark 1.21. In a first draft of [88], Lemma 1.20 was proved under the additional assumption that $\mathrm{Ric} \ge -K^2$. However, thanks to a suggestion of an anonymous referee, this assumption was later removed.

A trivial consequence of Lemma 1.20 is the estimate

$$\|u\|_{L^p} + \|\Delta u\|_{L^p} \le C\|\Delta u\|_{L^p} \qquad \forall u \in C_c^\infty(M).$$

It is precisely this latter that gives improved versions of Calderón–Zygmund inequalities and L^p-gradient estimates whenever M has a spectral gap.

For instance, from a somewhat abstract viewpoint, where curvature restrictions do not appear explicitly, we point out the following direct consequence of Theorems 1.11 and 1.16:

Corollary 1.22. *Let $1 < p < +\infty$. Suppose that $r_{W^{1,q}}(M) > 0$ for some $q > \max(n,p)$ and that M has spectral gap. Then the strong $W^{2,p}$-estimate*

$$\|u\|_{W^{2,p}} \le C\,\|\Delta u\|_{L^p} \qquad \forall u \in C_c^\infty(M). \tag{$W(2,p)$}$$

holds for some constant $C > 0$.

Proof. We start by noting that, by Theorem 1.16, there exists a constant $C > 0$ such that, for every $u \in C_c^\infty(M)$,

$$C^{-1}\|\nabla^2 u\|_{L^p} \le \|u\|_{L^p} + \|\Delta u\|_{L^p}.$$

On the other hand, by Theorem 1.11, the L^p-gradient estimates state that, for a suitable constant $C > 0$,

$$C^{-1}\|\nabla u\|_{L^p} \le \|u\|_{L^p} + \|\Delta u\|_{L^p}.$$

Summarizing

$$C^{-1}\|u\|_{W^{2,p}} \leq \|u\|_{L^p} + \|\Delta u\|_{L^p}.$$

An application of Lemma 1.20 yields the desired strong $W^{2,p}$-estimate. □

On a more concrete side, interesting examples are contained in the next two corollaries. The first one improves a result contained in [91] by removing the injectivity radius assumption. The argument is exactly as in the above proof up to using the L^p-Calderón–Zygmund inequality proved in [24] combined with the gradient estimate originally obtained in [28].

Corollary 1.23. *Suppose that* $\mathrm{Ric} \geq -(n-1)K^2$ *and that* M *has spectral gap. Then, for any fixed* $1 < p \leq 2$*, the strong* $W^{2,p}$*-estimate* $W(2,p)$ *holds.*

Similarly, we have the following straightforward consequences of Theorem 1.6.

Corollary 1.24. *Suppose that* $n < p_0 < +\infty$ *and that* M *has spectral gap. There exists a constant* $\varepsilon = \varepsilon(p_0, n, K) > 0$ *such that if* $k(p_0/2, 1, K) \leq \varepsilon$ *for some* $K \geq 0$*, then the* L^p*-gradient estimate*

$$\|u\|_{W^{1,p}} \leq C \ \|\Delta u\|_{L^p} \qquad \forall u \in C_c^\infty(M).$$

holds on M *for every* $1 < p \leq p_0$ *and for some constant* $C > 0$.

1.4 Higher order Calderón–Zygmund inequalities

We conclude this chapter with the study of higher order Calderón–Zygmund estimates and the related boundedness properties of higher even order, local Riesz transform. We start by recalling the following consequence of [91, Theorem 5.2], proved by Mauceri, Meda and Vallarino.

Theorem 1.25 ([91])**.** *Suppose that* M *has bounded geometry at the order* $2\ell - 2 \in \mathbb{N}$*, namely,*

$$|\nabla^j \mathrm{Ric}| \leq K, \ \forall j = 0, \cdots, 2\ell - 2 \quad \textit{and} \quad r_{\mathrm{inj}}(M) \geq i,$$

for some constants $K \geq 0$ *and* $i > 0$*. Assume also that* M *has spectral gap* $b > 0$*. Then, for any* $1 < p \leq 2$ *there exists a constant* $C = C(n, p, \ell, K, b, i) > 0$ *such that the global Riesz transform* $\mathscr{R}^{2\ell}$ *of order* 2ℓ *is bounded from* $L^p(M)$ *to* $L^p(M; T_{2\ell}M)$

Actually, the result in [91] is stronger, as it establishes that the global covariant Riesz transform $\mathscr{R}^{2\ell}$ is bounded as an operator from a certain Hardy space to L^1. Its L^p boundedness for $1 < p \leq 2$ then follows from an interpolation argument.

It is natural to speculate whether some of the assumptions in Theorem 1.25 can be removed. Our contribution is to allow b to be zero, at the expense of considering local Riesz transforms versus the global version thereof.

Theorem 1.26. *Suppose that ℓ is a positive integer and let $\tau > 0$. Assume that $r_{\mathrm{inj}}(M) > 0$ and that the covariant derivatives of the Ricci tensor are uniformly bounded up to the order $2\ell - 2$. Then $\mathscr{R}^{2\ell}_{\tau}$ is bounded from $L^p(M)$ to $L^p(M; T_{2\ell}M)$ for every p in $(1, 2]$.*

Proof. All over this proof, we denote by $\mathcal{L} := -\Delta$ the positively defined Laplace–Beltrami operator of the underlying manifold. Suppose that (M, g) has bounded geometry at the order $2\ell - 2$. Take the standard hyperbolic plane $\mathbb{H}^2$, and consider the Riemannian product $(M \times \mathbb{H}^2, g + g_{\mathbb{H}^2})$. Then, denoting by $b_M = b$, $b_{\mathbb{H}^2}$ and $b_{M\times\mathbb{H}^2}$ the bottom of the L^2 spectrum of the (positive) Laplace–Beltrami operator on M, $\mathbb{H}^2$ and $M \times \mathbb{H}^2$ respectively, it holds

$$b_{M\times\mathbb{H}^2} = b_M + b_{\mathbb{H}^2} \geq b_{\mathbb{H}^2} = \frac{1}{4}.$$

Moreover

$$|\nabla^j \operatorname{Ric}_N| \leq \max(1, K),\ j = 0, \cdots, 2\ell - 2,$$

and

$$r_{\mathrm{inj}}(M \times \mathbb{H}^2) \geq r_{\mathrm{inj}}(M) \geq i.$$

It follows from Theorem 1.25 and Proposition 1.14 that, if p is in $(1, 2]$, there exists a constant $C > 0$ such that

$$\|\nabla^{2\ell}_{M\times\mathbb{H}^2} w\|_{L^p(M\times\mathbb{H}^2)} \leq C\, \|\mathcal{L}^{\ell}_{M\times\mathbb{H}^2} w\|_{L^p(M\times\mathbb{H}^2)}\ , \forall w \in \mathrm{Dom}_{L^p}(\mathcal{L}_{M\times\mathbb{H}^2}). \tag{1.10}$$

We apply this estimate to functions w of the form $\varphi \otimes \psi$, where $\varphi \in \mathrm{Dom}_{L^p}(\mathcal{L}_M)$ and ψ belongs to $C^\infty_c(\mathbb{H}^2)$. Since

$$\mathcal{L}^{\ell}_{M\times\mathbb{H}^2}(\varphi \otimes \psi) = \sum_{j=0}^{\ell} \binom{\ell}{j} (\mathcal{L}^j_M \varphi) \otimes (\mathcal{L}^{\ell-j}_{\mathbb{H}^2} \psi)$$

and

$$|\nabla^{2\ell}_{M\times\mathbb{H}^2}(\varphi \otimes \psi)|^2_{M\times\mathbb{H}^2} = \sum_{j=0}^{2\ell} 2\binom{\ell}{j} |(\nabla^j_M \varphi) \otimes (\nabla^{2\ell-j}_{\mathbb{H}^2} \psi)|^2_{M\times\mathbb{H}^2},$$

by (1.10) we see that

$$\begin{aligned}
\|\nabla^{2\ell}_M \varphi\|_{L^p(M)}\ \|\psi\|_{L^p(\mathbb{H}^2)} &= \|(\nabla^{2\ell}_M \varphi) \otimes \psi\|_{L^p(M\times\mathbb{H}^2)} \\
&\leq \|\nabla^{2\ell}_{M\times\mathbb{H}^2}(\varphi \otimes \psi)\|_{L^p(M\times\mathbb{H}^2)} \\
&\leq C\, \|\mathcal{L}^{\ell}_{M\times\mathbb{H}^2}(\varphi \otimes \psi)\|_{L^p(M\times\mathbb{H}^2)} \\
&\leq C \sum_{j=0}^{\ell} \binom{\ell}{j}\ \|\mathcal{L}^j_M \varphi\|_{L^p(M)}\ \|\mathcal{L}^{\ell-j}_{\mathbb{H}^2} \psi\|_{L^p(\mathbb{H}^2)}\ .
\end{aligned}$$

Now, suppose that ψ does not vanish identically on $\mathbb{H}^2$. Then divide both sides of the previous inequality by $\|\psi\|_{L^p(\mathbb{H}^2)}$, and obtain that

$$\|\nabla^{2\ell}_M \varphi\|_{L^p(M)} \leq C\, \sigma_{p,\ell} \sum_{j=0}^{\ell} \binom{\ell}{j}\ \|\mathcal{L}^j_M \varphi\|_{L^p(M)} \qquad \forall \varphi \in L^p(M),$$

where

$$\sigma_{p,l} \coloneqq \min_{0\le j\le l} \inf_{\psi\neq 0} \frac{\|\mathcal{L}_{\mathbb{H}^2}^{l-j}\psi\|_{L^p(\mathbb{H}^2)}}{\|\psi\|_{L^p(\mathbb{H}^2)}}$$

is a finite constant. Now, since $\mathcal{L}_M$ is sectorial on $L^p(M)$ (for $\mathcal{L}_M$ generates the contraction semigroup $\{\mathcal{H}_t\}$ on $L^p(M)$), the Moment inequality [67, Theorem 6.6.4] implies that

$$\|\mathcal{L}_M^j\varphi\|_{L^p(M)} \le C\ \|\varphi\|_{L^p(M)}^{1-j/\ell}\ \|\mathcal{L}_M^\ell\varphi\|_{L^p(M)}^{j/\ell},$$

so that

$$\begin{aligned}\sum_{j=0}^{\ell}\binom{\ell}{j}\|\mathcal{L}_M^j\varphi\|_{L^p(M)} &\le C\,(\ \|\varphi\|_{L^p(M)}^{1/l} + \|\mathcal{L}_M^\ell\varphi\|_{L^p(M)}^{1/\ell}\ )^\ell\\ &\le C\,2^\ell(\ \|\varphi\|_{L^p(M)} + \|\mathcal{L}_M^\ell\varphi\|_{L^p(M)}\ ).\end{aligned}$$

By combining the steps above, we find that there exists a constant $C>0$ such that

$$\|\nabla_M^{2\ell}\varphi\|_{L^p(M)} \le C\,(\ \|\varphi\|_{L^p(M)} + \|\mathcal{L}_M^\ell\varphi\|_{L^p(M)}\ ). \tag{1.11}$$

A further application of Proposition 1.14 concludes the proof. □

Remark 1.27. We conclude with some final observation which arise from the proof of Theorem 1.26

(1) It is natural to speculate whether the Riesz transforms of higher odd order $\mathscr{R}_\tau^{2\ell-1}$ are bounded on $L^p(M)$ when $\ell\ge 2$.

(2) It should be possible to give an alternative proof to Theorem 1.26 using $C^{2\ell-1,\alpha}$ harmonic coordinates, which exist in our assumptions, see [4]. Such a proof would likely work also in the case $p>2$, but it would be very technical and involved, due to the large number of terms of the coordinate expression of $\nabla^{2\ell}$ to deal with; compare for instance with the analogous result for the higher order density problem in [78]. For the sake of simplicity, we decided not to investigate such an approach in this thesis.

Chapter 2

Counterexamples

While Chapter 1 is devoted to positive results, in this chapter we focus on three examples where the L^p-gradient or Calderón–Zygmund estimates fail. These counterexamples have been constructed in [90] and [88].

We begin with Calderón–Zygmund inequalities. In the last years, counterexamples to the validity of CZ(p) have been constructed by Güneysu–Pigola, Li, and Veronelli in [63, 85, 121] respectively. The first example is due to Güneysu and Pigola, who proved the existence of a 2 dimensional, complete, parabolic Riemannian manifold on which CZ(2) fails. This construction was later extended by Li to all dimensions and to all integrability orders $p \in (1, +\infty)$. Both examples are warped manifolds with Gaussian/sectional curvature exploding at ∞ and injectivity radius going to zero. The example of [121] is constructed too on a warped manifold. In this work, however, Veronelli proves that smooth and compactly supported functions are not dense in $W^{2,p}(M)$, by Remark 1.13 this is sufficient to prove that CZ(p) fails on M. This example too has curvatures exploding at ∞.

2.1 A counterexample to CZ(p) with arbitrarily small negative curvatures

Our first contribution is to show that it is possible to construct counterexamples to the validity of CZ(p) on manifolds with curvature growing at $-\infty$ as slow as we want. Our construction follows from the ones in [63, 85] which have sectional curvature oscillating increasingly on a sequence of compact annuli going to infinity. Nevertheless, we show that distancing the (disjoint) annuli far enough allows a control on the rate of explosion of Sect at $-\infty$.

Theorem 2.1. *For each $n \geq 2$ and $p \in (1, \infty)$, and for each increasing function $\lambda : [0, +\infty) \to \mathbb{R}$ such that $\lambda(t) \to +\infty$ as $t \to \infty$, there exists a complete n-dimensional Riemannian manifold (M, g) satisfying $\min \mathrm{Sect}(x) \geq -\lambda(r(x))$ for $r(x)$ large enough, and which does not support CZ(p).*

Proof. The counterexamples to CZ(p) in [63, 85] are constructed on a model manifold (M, g), i.e. $M = [0, +\infty) \times \mathbb{S}^{n-1}$ endowed with a warped metric $g = dt^2 + \sigma^2(t) g_{\mathbb{S}^{n-1}}$. By carefully choosing the warping function σ, the authors proved the existence of a sequence of smooth functions $\{u_k\}_{k=1}^{\infty}$ and a sequence of intervals $\{[a_k, b_k]\}_{k=1}^{\infty}$ such that

- $a_{k+1} > b_k$;
- u_k is compactly supported in the annulus $[a_k, b_k] \times \mathbb{S}^{n-1}$;
- the sequence of functions u_k contradicts CZ(p) for any possible constant, i.e.

$$\frac{\|\nabla^2 u_k\|_{L^p}}{\|\Delta u_k\|_{L^p}+\|u_k\|_{L^p}} \to \infty, \qquad \text{as } k \to \infty;$$

- there exists two sequences of intervals $\{[c_k, d_k]\}_{k=1}^{\infty}$ and $\{[e_k, f_k]\}_{k=1}^{\infty}$ with $b_k < c_k < d_k < e_k < f_k < a_{k+1}$ such that σ is linear and increasing on $[c_k, d_k]$ and is linear and decreasing on $[e_k, f_k]$, namely

$$\sigma|_{[c_k,d_k]}(t) = \alpha_k t + \beta_k, \quad \text{and} \quad \sigma|_{[e_k,f_k]}(t) = \gamma_k t + \delta_k$$

for some constants $\alpha_k > 0$, $\gamma_k < 0$ and $\beta_k, \delta_k \in \mathbb{R}$.

Note that, in order to satisfy this latter condition, our $\{u_k\}_{k=1}^{\infty}$ could be a subsequence of the sequence $\{u_k\}_{k=1}^{\infty}$ produced in [85]

Now, for $k \geq 2$, let $0 < \kappa_k < \infty$ be such that

$$\forall\, x \in [e_{k-1}, d_k] \times \mathbb{S}^{n-1}, \quad \min \operatorname{Sect}(x) \geq -\kappa_k.$$

Up to an increase of κ_{k+1}, we can assume that $\kappa_k \leq \kappa_{k+1}$. For $k \geq 2$, let T_k be such that $\lambda(T_k) > \kappa_k$. For later purpose, since λ is increasing, we can assume without loss of generality that $T_{k+1} > T_k + d_{k-1} - e_{k-2}$ and that

$$\alpha_{k-1}(T_{k+1} + e_{k-2} - T_k) + \beta_{k-1} > \sigma(e_{k-1}). \tag{2.1}$$

We define now a new warping function $\tilde{\sigma}(t) : [0, +\infty) \to [0, +\infty)$ and a corresponding model metric $\tilde{g} = dt^2 + \tilde{\sigma}^2(t) g_{\mathbb{S}^{n-1}}$ on M as follows. We define $\tilde{\sigma}(t)$ only for $t \geq T_3$, since the choice of $\tilde{\sigma}$ on $[0, T_3)$ does not affect the conclusion of the theorem. For $t \in [T_k, T_k + d_{k-1} - e_{k-2}]$ define

$$\tilde{\sigma}(t) = \sigma(t + e_{k-2} - T_k),$$

so that

$$\operatorname{Sect}_{\tilde{g}} \geq -\kappa_{k-1}$$

on $[T_k, T_k + d_{k-1} - e_{k-2}] \times \mathbb{S}^{n-1}$. In particular,

$$\operatorname{Sect}_{\tilde{g}}(t, \Theta) \geq -\kappa_k > -\lambda(T_k) \geq -\lambda(t)$$

for any $(t, \Theta) \in ([T_k, T_k + d_{k-1} - e_{k-2}] \cup [T_{k+1}, T_{k+1} + d_k - e_{k-1}]) \times \mathbb{S}^{n-1}$. It remains to prescribe $\tilde{\sigma}$ on the intervals $(T_k + d_{k-1} - e_{k-2}, T_{k+1})$ for $k \geq 3$. Note that on $[T_k + c_{k-1} - e_{k-2}, T_k + d_{k-1} - e_{k-2}]$ we have $\tilde{\sigma}(t) = \alpha_{k-1}(t + e_{k-2} - T_k) + \beta_{k-1}$. Similarly, on $[T_{k+1}, T_{k+1} + f_{k-1} - e_{k-1}]$, we have $\tilde{\sigma}(t) = \gamma_{k-1}(t + e_{k-1} - T_{k+1}) + \delta_{k-1}$. Because of assumption (2.1), we can find a $S_k \in (T_k + d_{k-1} - e_{k-2}, T_{k+1})$ such that

$$\hat{\sigma}(t) = \begin{cases} \alpha_{k-1}(t + e_{k-2} - T_k) + \beta_{k-1} & \text{on } [T_k + c_{k-1} - e_{k-2}, S_k] \\ \gamma_{k-1}(t + e_{k-1} - T_{k+1}) + \delta_{k-1} & \text{on } [S_k, T_{k+1} + f_{k-1} - e_{k-1}] \end{cases}$$

is a well-defined concave continuous piece-wise linear function which coincides with $\tilde{\sigma}$ outside $(T_k + d_{k-1} - e_{k-2}, T_{k+1})$. Let $\epsilon_k > 0$ be a small constant to be fixed later, and define $\tilde{\sigma}$ on $(T_k + d_{k-1} - e_{k-2}, T_{k+1})$ to be a concave smooth approximation of $\hat{\sigma}$ equal to $\hat{\sigma}$ outside $[S_k - \epsilon_k, S_k + \epsilon_k]$ (this can be produced for instance applying [47, Theorem 2.1]). A standard computation shows that the sectional curvatures of $(M, \tilde{g})$ are given by

$$\mathrm{Sect}_{rad}(t, \Theta) = -\frac{\tilde{\sigma}''(t)}{\tilde{\sigma}(t)}, \qquad \mathrm{Sect}_{tg}(t, \Theta) = \frac{1 - (\tilde{\sigma}'(t))^2}{\tilde{\sigma}(t)^2},$$

for tangent planes respectively containing the radial direction, or orthogonal to it. Since $\tilde{\sigma}$ is concave for $t \in (T_k + d_{k-1} - e_{k-2}, T_{k+1})$ then

$$\mathrm{Sect}_{rad}(t, \Theta) \geq 0 \geq -\lambda(t).$$

If $\alpha_{k-1} \leq 1$ and $\gamma_{k-1} \geq -1$ then $\mathrm{Sect}_{tg}(t, \Theta) \geq 0 \geq -\lambda(t)$ in a trivial way. Otherwise,

$$\mathrm{Sect}_{tg}(t, \Theta) > \mathrm{Sect}_{tg}(T_k + d_{k-1} - e_{k-2}, \Theta) \geq -\kappa_k > -\lambda(t)$$

for $t \in (T_k + d_{k-1} - e_{k-2}, S_k - \epsilon_k)$ and

$$\mathrm{Sect}_{tg}(t, \Theta) > \mathrm{Sect}_{tg}(T_{k+1}, \Theta) \geq -\kappa_k > -\lambda(t)$$

for $t \in (S_k + \epsilon_k, T_{k+1})$. Finally, for $t \in [S_k - \epsilon_k, S_k + \epsilon_k]$, by concavity

$$1 - (\tilde{\sigma}'(t))^2 \geq \min\{1 - (\tilde{\sigma}'(S_k - \epsilon_k))^2; 1 - (\tilde{\sigma}'(S_k + \epsilon_k))^2\},$$

while $\tilde{\sigma}(t)$ is arbitrarily close to $\tilde{\sigma}(S_k - \epsilon_k)$ and to $\tilde{\sigma}(S_k + \epsilon_k)$ for ϵ_k small enough. Accordingly, we can choose ϵ_k small enough so that $\mathrm{Sect}_{tg}(t, \Theta) > -\lambda(t)$ also for $t \in [S_k - \epsilon_k, S_k + \epsilon_k]$. Since Sect_{tg} and Sect_{rad} are the extremal values of the sectional curvatures, we have that $\min \mathrm{Sect}$ at (t, Θ) is lower bounded by $-\lambda(t)$ for all $t \geq T_3$. Observe that $([T_k, T_k + d_{k-1} - e_{k,2}] \times \mathbb{S}^{n-1}, \tilde{g})$ is isometric to $([e_{k,2}, d_{k-1}] \times \mathbb{S}^{n-1}, g)$. Then we conclude by defining $w_k(t, \Theta) = u_{k-1}(t + e_{k-2} - T_k, \Theta)$ so that the w_k are smooth, compactly supported in $[T_k + a_{k-1} - e_{k-2}, T_k + b_{k-1} - e_{k-2}] \times \mathbb{S}^{n-1}$ and verify

$$\frac{\|\nabla^2 w_k\|_{L^p}}{\|\Delta w_k\|_{L^p} + \|w_k\|_{L^p}} \to \infty, \qquad \text{as } k \to \infty.$$

□

Remark 2.2. The above construction suggests two considerations.

- Theorem 2.1 implies sharpness of the pointwise lower bound on Ricci curvature in the result of Cao, Cheng and Thalmaier, [24, Theorem 1.1] (Güneysu–Pigola when $p = 2$ [63, Theorem B]). Note also that in our construction the injectivity radius, $r_{\text{inj}}(M)$, vanishes.

- Another consequence is that it is not generally possible to obtain CZ(p) under negative decreasing curvature bounds of the type $\text{Ric}(x) \geq -Cr^{\alpha}(x)$ for some $\alpha > 0$. However, if $\alpha \in (0, 2]$ this bound implies that $W_0^{2,p}(M) = W^{2,p}(M)$ for all $p \in (1, 2]$, see [77, Theorem 1.4] and [74, Theorem 1.3]. Hence, while CZ(p) implies the density result, Remark 1.13, the converse is not true. Under this milder condition, however, a disturbed CZ(p) holds, [78, Section 6.2].

2.2 A counterexample to CZ(p) with positive curvatures

The result of Cao, Cheng and Thalmaier, [24], which is sharp by Theorem 2.1, completes the picture of CZ(p) in the range $p \in (1, 2]$. It is quite reasonable to speculate if these assumptions are sufficient also when $p > 2$. Indeed, Güneysu posed the following

Question 2.3 (Conjectured for $\text{Ric} \geq 0$ in [60], p.177)**.** *Suppose that (M, g) is geodesically complete and has lower bounded Ricci curvature. Does CZ(p) hold on (M, g) for all $p \in (1, \infty)$?*

Strong evidence for a negative answer to the above question comes from a deep and recent result by De Philippis and Núñez-Zimbrón who proved the impossibility to have a Calderón–Zygmund theory on compact manifolds with constants depending only on a lower bound on the sectional curvature, at least when $p > n$. Namely, when $p > n$ one can find a sequence of compact, non-negatively curved Riemannian manifolds $\{(M_j, g_j)\}_{j=1}^{\infty}$ for which the best constant in CZ(p) is at least j; see [39, Corollary 1.3].

To prove this result, the authors considered a sequence of smooth non-negatively curved n dimensional compact manifolds Gromov–Hausdorff approaching, [22, Definition 7.3.10] a compact $RCD(0, n)$ space X with a dense set of singular points. For basic definitions on the theory of RCD spaces, see for instance [113, Chapter 1]. A bound on the constant C in CZ(p) along the sequence, combined with a Morrey inequality, would imply that all functions on X with Laplacian in $L^{p>n}$ are C^1. On the other hand, De Philippis and Núñez-Zimbrón proved in [39, Theorem 1.1] that the gradient of a harmonic function (or more generally of any function whose Laplacian is in $L^{p>n}$) vanishes at singular points of an $RCD(K, n)$ space. By a density argument, this would imply that all harmonic functions on X are constant, a fact we know is false.

In this section, we give a concrete and final answer to Question 2.3, even under the stricter assumption of positive sectional curvature, see Theorem 2.9 below. With respect to the argument in [39], our main contribution consists in proving the existence of a fixed Riemannian manifold on which CZ(p) can not hold, whatever constant C one take.

To achieve our result, we localize the procedure of De Philippis and Núñez-Zimbrón. The key observation is the fact that the above argument is indeed local and can be repeated on infinitely many singular perturbations, suitably distributed over a non-compact manifold. Namely, we begin with a complete non compact manifold (M, g) with $\mathrm{Sect}(M) > 0$. In the interior of infinitely many separated sets $\{\mathfrak{D}_j\}_{j\in\mathbb{N}}$ of M we take sequences of local perturbations $g_{j,k}$ of the original metric g such that all the $g_{j,k}$ have Sect > 0 and $g_{j,k}$ Gromov–Hausdorff converges, as $k \to +\infty$, to an Alexandrov metric $d_{j,\infty}$ on M of non-negative curvature, [99, Section 1.3], (hence $RCD(0, n)$, see [102, 129, 38]). In particular, the metric $d_{j,\infty}$ is singular on a dense subset of $\mathfrak{D}_j$. Next, we observe that a neighborhood of each $\mathfrak{D}_j$ can be seen as a piece of a compact space whose metric is smooth outside $\mathfrak{D}_j$, so that De Philippis and Núñez-Zimbrón's strategy can be applied locally to the sequence $g_{j,k}$. Accordingly, we find a (large enough) k and a function v_j compactly supported in a small neighborhood of $\mathfrak{D}_j$ such that the following estimate holds with respect to the metric $g_{j,k}$

$$\|\nabla^2 v_j\|_{L^p} > j\,(\|\Delta v_j\|_{L^p} + \|v_j\|_{L^p})\,.$$

Gluing together all the local deformations of the metric, we thus obtain a smooth manifold on which no constant C makes CZ(p) true.

2.2.1 The singular space and its smooth approximations

It is well known from previous literature that for every $n \geq 2$ one can always construct a compact, convex set $C \subset \mathbb{R}^{n+1}$ whose boundary $X = \partial C$ is an Alexandrov space with $\mathrm{Curv}(X) \geq 0$ and a dense set of singular points. The first example of such spaces is due to Otsu and Shioya in dimension 2, [99, Example (2)], although the result holds in arbitrary dimension. Observe that the space X can be GH approximated with a sequence of smooth manifolds X_k of non-negative sectional curvature; see the proof of [1, Theorem 1].

In the following, we would like to localize this construction inside a compact set of a complete, non-compact manifold. Indeed, we prove that a smooth and strictly convex function can always be perturbed on a compact set by introducing a dense sequence of singular points. Our construction leaves the function unaltered outside the compact set, preserves smoothness outside the singular set and convexity at a global scale. Furthermore, we prove that such singular perturbation can be locally and uniformly approximated with smooth convex functions in a neighborhood of the singular set. Once again, the difficulty here is to leave the functions unaltered outside the compact set.

Lemma 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth, strictly convex function. For every $x \in \mathbb{R}^n$, $r > 0$ there exists a convex function $f_\infty : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) f_∞ is smooth and equal to f outside $B_r(x)$;

(ii) the graph of f_∞ restricted to $B_r(x)$ has a dense set of singularities.

Furthermore, there exists a sequence of smooth, strictly convex functions $f^k_\infty : \mathbb{R}^n \to \mathbb{R}$ converging uniformly to f_∞ and equal to f outside $B_r(x)$.

Proof. Take $\{y_k\}_{k=1}^{\infty}$ any dense set contained in $S := B_r(x)$. We want to perturb f in S to obtain a new function whose graph has singularities in correspondence with y_k while preserving the convexity properties. To do so, we consider $g : B_1(0) \to \mathbb{R}$ such that

$$\begin{cases} g(x) = |x|+|x|^2-1 & x \in B_{1/2}(0) \\ g \in C^\infty(B_1(0) \setminus \{0\}) \\ \operatorname{supp} g \subseteq B_1(0) \\ g \leq 0. \end{cases}$$

Then, for $\varepsilon > 0$ and $y \in \mathbb{R}^n$ we define $g_{\varepsilon,y} : B_\varepsilon(y) \to \mathbb{R}$ as

$$g_{\varepsilon,y}(x) \coloneqq g\left(\frac{x-y}{\varepsilon}\right),$$

so that $g_{\varepsilon,y}$ is smooth outside $\{y\}$, non-positive and strictly convex on $B_{\varepsilon/2}(y)$.

Let $\varepsilon_1 < 1 - |y_1|$ and define

$$f_1(x) \coloneqq f(x) + \eta_1 g_{\varepsilon_1,y_1}(x).$$

If $\eta_1 > 0$ is small enough then f_1 is strictly convex. Indeed, f_1 is strictly convex in a neighborhood of y_1 because sum of two strictly convex function, on the other hand, g_{ε_1,y_1} is concave in a region outside y_1 where f is smooth and $\nabla^2 f$ is uniformly positive, hence, f_1 is strictly convex provided that η_1 is small enough. Observe that f_1 is smooth outside $\{y_1\}$ and (its graph) has a singular point on y_1.

Recursively, we let $\varepsilon_k < \min\{1 - |y_k|, \operatorname{dist}(y_k, y_1), \dots \operatorname{dist}(y_k, y_{k-1})\}$ and we define

$$f_k(x) \coloneqq f_{k-1}(x) + \eta_k g_{\varepsilon_k,y_k}(x). \tag{2.2}$$

By construction f_k is smooth outside $\{y_1, \dots, y_k\}$, where its graph is singular. Moreover, f_k is strictly convex in a neighbourhood of y_k because sum of two strictly convex functions while g_{ε_k,y_k} is concave in a region where f_{k-1} is smooth and $\nabla^2 f_{k-1}$ is uniformly positive hence f_k is strictly convex provided that η_k is small enough. Furthermore, if η_k are such that $\sum_k \eta_k$ converges, f_k converges uniformly to some f_∞, which is convex, singular on $\{y_k\}_{k=1}^\infty$ and is smooth elsewhere. Moreover, it is equal to f outside S. Observe also that $\{(y_k, f_\infty(y_k))\}_{k=1}^\infty$ is dense in $\operatorname{Graph}(f_\infty|_S)$ since f_∞ is locally Lipschitz.

It remains to show that f_∞ can be smoothly approximated with strictly convex functions. By a diagonalization procedure, it is enough to uniformly approximate each f_k.

For $0 < \delta < \min\{\varepsilon_1, \dots, \varepsilon_k\}$, let $\phi_{\delta,k} = \phi_\delta : \mathbb{R}^n \to \mathbb{R}$ be a smooth convex function such that

$$\phi_\delta(x) = f_k(x) \text{ on } \mathbb{R}^n \setminus \bigcup_{i=1}^k B_\delta(y_k).$$

The existence of ϕ_δ is ensured by [47, Theorem 2.1]. Clearly ϕ_δ converges pointwise to f_k as $\delta \to 0$. Since the functions are all strictly convex, the convergence is actually uniform. This concludes the proof. □

Remark 2.5. Observe that the epigraph of f_∞ is a convex set in $\mathbb{R}^{n+1}$ whose boundary, endowed with the intrinsic distance, is an Alexandrov space of non-negative curvature (see [21]). Its singularities are contained (and dense) in the compact set $\mathrm{Graph}(f_\infty|_S)$. Similarly, the graphs of f_∞^k are smooth hypersurfaces of positive sectional curvature, isometrically immersed in $\mathbb{R}^{n+1}$. Since $f_\infty^k \to f_\infty$ uniformly, their graphs converge with respect to the Hausdorff metric. In the case of convex sets of $\mathbb{R}^n$, it is well known that this implies Gromov–Hausdorff convergence; see [22, Theorem 10.2.6] observing that the proof applies in any dimension. Notice also that the convergence is measured, [122, Definition 27.30], if we endow these spaces with the usual n-dimensional Hausdorff measure $\mathcal{H}^n$. On an isometrically immersed manifold, this is in fact the Riemannian volume.

2.2.2 Convergence of solutions of the Poisson equation

The next step in our proof is a convergence result for the solutions of the Poisson equation on limit spaces. In what follows we mimic, up to minor modifications necessary to our purposes, [103, Proposition B.1], where Pigola collects and develops a series of previous results due to Honda, [72] and [73].

Let us consider the following space

$$\mathcal{M}(n,D) = \{(M,g) \text{ compact } : \dim M = n,\ \mathrm{diam}(M) \le D,\ \mathrm{Sect} \ge 0\},$$

and denote with $\overline{\mathcal{M}(n,D)}$ its closure with respect to the measured Gromov–Hausdorff topology, [122, Definition 27.30]. Note that elements of $\overline{\mathcal{M}(n,D)}$ are in particular Alexandrov spaces with $\mathrm{Curv} \ge 0$ and $\mathrm{diam} \le D$. Note that, by volume comparison and bounds on the diameter, there exists $V > 0$ depending on n, D such that $\mathrm{vol}\, X \le V$ for all $X \in \mathcal{M}(n,D)$.

Remark 2.6. The following proposition actually holds in the more general setting of Ricci limit spaces. To avoid unnecessary complication in notations, we restrict ourselves to the case of Alexandrov spaces which are a special case of the former, [102, 129, 38].

In what follows, all convergences are intended in the sense of Honda, see [72, Section 3], see also [83].

Proposition 2.7. *Let $(M_k, h_k) \in \mathcal{M}(n,D)$ be a sequence of smooth manifolds converging in the mGH topology to an Alexandrov space $(X_\infty, d_\infty, \mu_\infty) \in \overline{\mathcal{M}(n,D)}$ of dimension n and let $x_\infty \in X_\infty$. Up to a subsequence of (M_k, h_k), there exist functions $u_k \in C^2(M_k)$, $g_k \in \mathrm{Lip}(M_k)$ and $u_\infty \in W^{1,2}(X_\infty) \cap L^p(X_\infty)$, $g_\infty \in L^p(X_\infty)$ for all $1 < p < +\infty$, such that u_k, u_∞ are non-constant and $\Delta_{M_k} u_k = g_k$, $\Delta_{X_\infty} u_\infty = g_\infty$. Furthermore,*

(a) *$g_\infty \ge 1/2$ on a neighborhood of x_∞;*
(b) *$g_k \to g_\infty$ in the strong L^p sense;*
(c) *$u_k \to u_\infty$ in the strong $W^{1,2}$ sense;*
(d) *$\|u_k\|_{W^{1,p}} \le L$ for some $L = L(p,n,D,K) > 0$;*
(e) *$u_k \to u_\infty$ in the strong L^p sense;*
(f) *$\nabla^{M_k} u_k \to \nabla^X u_\infty$ in the weak L^p sense.*

These functions satisfy (a) through (f) for all $1 < p < +\infty$.

Proof. Since M_k is bounded, separable and M_k converges to X_∞ with respect to the mGH topology, there exists a sequence of points $x_k \in M_k$ such that the mGH convergence $(M_k, h_k, x_k) \to (X_\infty, \mu_\infty, x_\infty)$ is pointed, [122, Definition 27.13].

Next, using volume comparison and the convergence $\mathrm{vol}(M_k) \to \mathcal{H}^n(X)$ as $k \to \infty$, one can show the existence of a uniform $R > 0$ such that for $k \gg 1$,

$$\mathrm{vol}\, B_R^{M_k}(x_k) \leq \frac{1}{2}\, \mathrm{vol}\, M_k.$$

Let $f_k : M_k \to [0,1]$ be Lipschitz functions compactly supported in $B_R^{M_k}(x_k)$ satisfying

$$i)\; f_k = 1 \quad \text{on } B_{R/2}^{M_k}(x_k), \qquad ii)\; \|\nabla f_k\|_{L^\infty} \leq \frac{2}{R}.$$

Define

$$g_k := f_k - \fint_{M_k} f_k \in \mathrm{Lip}(M_k),$$

and note that

$$0 \leq \fint_{M_k} f_k \leq \frac{\mathrm{vol}\, B_R^{M_k}(x_k)}{\mathrm{vol}\, M_k} \leq \frac{1}{2}.$$

Clearly $\fint_{M_k} g_k = 0$ and $\|g_k\|_{L^\infty} \leq 1$. Moreover, $g_k \geq 1/2$ on $B_{R/2}^{M_k}(x_k)$ so that $g_k \not\equiv 0$. Since $\|g_k\|_{L^\infty} \leq 1$ and the volumes are uniformly bounded, $\|g_k\|_{L^p} \leq V^{1/p}$ for all $p > 1$. Using [72, Proposition 3.19] we conclude that g_k converges weakly, up to subsequences, to some $g_\infty \in L^p(X_\infty)$ ([72, Definition 3.4]). Condition *ii)* ensures that the sequence g_k is asymptotically uniformly continuous in the sense of [72, Definition 3.2]. Hence, g_k converges to g_∞ strongly and in the sense of [72, Definition 3.1], see [72, Remark 3.8]. This ensures that $g_\infty \not\equiv 0$ in a neighborhood of x_∞ and, more importantly, allows us to use [72, Proposition 3.32] which proves strong L^p convergence of g_k to g_∞. It is worthwhile to notice that g_k converges L^p strongly to g_∞ for every $1 < p < +\infty$, in particular, for $p = 2$.

Next, we denote with $u_k \in C^2(M_k)$ the unique (non-constant) solution of the Poisson equation

$$\Delta_{M_k} u_k = g_k \quad \text{on } M_k,$$

satisfying

$$\fint_{M_k} u_k = 0.$$

Since g_k converges to g_∞ in a strong (and thus weak) L^2 sense, [73, Theorem 1.1] ensures $W^{1,2}$ convergence of u_k to the unique (non-constant) solution $u_\infty \in W^{1,2}(X_\infty)$ of the Poisson equation

$$\Delta_{X_\infty} u_\infty = g_\infty \quad \text{on } X_\infty,$$

satisfying

$$\fint_{X_\infty} u_\infty = 0.$$

Finally, we claim that $\{u_k\}$ is bounded in $W^{1,p}$. By [72, Theorem 4.9] this implies L^p strong convergence of u_k to u_∞ and L^p weak convergence of $\nabla^{M_k} u_k$ to $\nabla^X u_\infty$ up to a subsequence, and thus concludes the proof of Proposition 2.7. To prove the claim, we observe that since $u_k \to u_\infty$ in a strong $W^{1,2}$ sense, we have L^2 boundedness of u_k. Applying the estimates in [130, Corollary 4.2] we obtain L^∞ bounds for u_k and ∇u_k, hence, the desired L^p estimates using the uniform bound on volumes. □

2.2.3 Proofs of the results

In Section 2.2.1 we established a method to locally perturb a smooth and strictly convex function by introducing a set of singular points, which is dense inside a given compact. In the following we consider a sequence of infinitely many singular perturbations distributed over a non-compact manifold, each of these perturbations is GH approximated with smooth Riemannian manifolds. For each perturbation, we prove that it is impossible to have the validity of a local (hence of a global) Calderón–Zygmund inequality whose constant is uniformly bounded across the approximating sequence of manifolds. To do so, we show that each singular set together with its corresponding approximation can be seen as a piece of a compact space whose metric is smooth outside the singular part. This observation is a technical device which allows the application of already available results. In particular, we can employ Proposition 2.7 to localize the strategy of De Philippis and Núñez-Zimbrón in a neighborhood of each singular set. Once we have proven that the constants of the local Calderón–Zygmund inequalities cannot be chosen uniformly, we select on the j-th perturbation in the approximating sequence a manifold with CZ(p) constant greater than j.

Lemma 2.8. *Let $n \geq 2$ and $p > n$. There exists a sequence of smooth and strictly convex functions $f_j : \mathbb{R}^n \to \mathbb{R}$, $j \geq 1$ and a monotone increasing sequence of radii $r_j > 0$ such that*

(i) $f_j(x) = f_{j-1}(x)$ for $x \in \mathfrak{B}_{j-1}$;

(ii) $f_j(x) = |x|^2$ for $x \in \mathbb{R}^n \setminus \overline{\mathfrak{B}_j}$;

where $\mathfrak{B}_j = B_{r_j}(0)$ and $\mathfrak{B}_0 = \emptyset$. Furthermore, if we consider $N_j = \mathrm{Graph}(f_j)$ as a Riemannian manifold isometrically immersed in $\mathbb{R}^{n+1}$, there exists some $v_j \in C^2(N_j)$ compactly supported in $\mathrm{Graph}(f_j|_{\mathfrak{B}_j \setminus \overline{\mathfrak{B}_{j-1}}})$ which satisfies

$$\|\nabla^2 v_j\|_{L^p} > j\,(\|\Delta v_j\|_{L^p} + \|v_j\|_{L^p}), \tag{2.3}$$

where $L^p = L^p(N_j)$.

Proof. We begin with a remark on notation: given a subset $A \subset \mathbb{R}^n$ and some function $k : \mathbb{R}^n \to \mathbb{R}$, we denote with $k(A) = \mathrm{Graph}(k|_A) \subset \mathbb{R}^{n+1}$. This abuse of notation is repeatedly used throughout the proof.

To simplify the exposition, the proof proceeds inductively on $j \geq 1$. Set $f_0(x) = |x|^2$. Suppose one has f_{j-1} and wants to build f_j. Let S_j be a Euclidean ball contained in $\mathbb{R}^n \setminus \overline{\mathfrak{B}_{j-1}}$. By Lemma 2.4 there exists a convex function h_j with a dense set of singular points in S_j and equal to f_{j-1} outside S_j. Furthermore, h_j can be approximated with smooth and strictly convex functions $h_{j,k} : \mathbb{R}^n \to \mathbb{R}$ equal to f_{j-1} outside S_j. Note that $h_j(S_j)$ corresponds to the sets $\mathfrak{D}_\mathrm{j}$ mentioned at the beginning of Section 2.2.

Next, let $r_j > 0$ be such that $S_j \subset \mathfrak{B}_j$. For later use, we observe that one can always consider a larger ball T_j such that $S_j \subset T_j$ and $T_j \subset \mathfrak{B}_j \setminus \overline{\mathfrak{B}_{j-1}}$. We want to extend $h_j(\mathfrak{B}_j)$ to a closed (i.e. compact without boundary) Alexandrov space X_j with $\mathrm{Curv}(X_j) \geq 0$. Moreover, we would like the extension to be smooth outside $h_j(S_j)$. To this purpose, let A_j be the upper hemisphere of boundary $h_j(\partial\mathfrak{B}_j)$ in $\mathbb{R}^{n+1}$, so that $\widetilde{X_j} \coloneqq h_j(\mathfrak{B}_j) \cup A_j$ is a convex hypersurface in $\mathbb{R}^{n+1}$. To obtain X_j, one simply needs to smooth $\widetilde{X_j}$ in a neighborhood of $h_j(\partial\mathfrak{B}_j)$. For instance, one can use [47, Theorem 2.1], observing that in this neighborhood, $\widetilde{X_j}$ is obtained by rotation of a piecewise smooth curve. We consider on X_j the metric induced by $\mathbb{R}^{n+1}$. By the same strategy, we extend $h_{j,k}(\mathfrak{B}_j)$ to a compact and smooth Riemannian manifold $M_{j,k}$ with $\mathrm{Sect}(M_{j,k}) > 0$, isometrically immersed in $\mathbb{R}^{n+1}$.

Note that, for all k, $M_{j,k} = X_j$ outside $h_j(S_j)$. Moreover, $M_{j,k}$ converges to X_j in a (measured) Gromov–Hausdorff sense as $k \to \infty$. Then, choosing a point $x_{j,\infty} \in h_j(S_j) \subset X_j$, we apply Proposition 2.7 to deduce the existence, up to subsequences on k, of $u_{j,k} \in C^2(M_{j,k})$, $g_{j,k} \in \mathrm{Lip}(M_{j,k})$ and $u_{j,\infty} \in W^{1,2}(X_j) \cap L^p(X_j)$, $g_{j,\infty} \in L^p(X_j)$ such $\Delta_{M_{j,k}} u_{j,k} = g_{j,k}$ and $\Delta_{X_j} u_{j,\infty} = g_{j,\infty}$. In particular,

(a) $\Delta u_{j,k} \to \Delta u_{j,\infty}$ strongly in L^p, hence, $\|\Delta u_{j,k}\|_{L^p} \leq C_1$;
(b) $\|u_{j,k}\|_{W^{1,p}} \leq C_1$;
(c) $g_{j,\infty} \geq 1/2$ in a neighborhood of $x_{j,\infty}$. In particular, in this neighborhood $u_{j,\infty}$ can not be constant.

Here C_1 depends on n, p and the upper bound $\operatorname{diam} M_{j,k} \leq D_j$ and the norms are intended over $L^p = L^p(M_{j,k})$ and $W^{1,p} = W^{1,p}(M_{j,k})$.

A key element in our proof is the possibility to localize the sequence $u_{j,k}$ without altering its essential properties. This can be done via smooth cutoff functions $\chi_{j,k} \in C^\infty(M_{j,k})$ equal to 1 on $h_{j,k}(S_j)$ and identically 0 outside of $h_{j,k}(T_j)$. Moreover, since the manifolds $M_{j,k}$ are all isometric outside $h_{j,k}(S_j)$, we can choose the functions $\chi_j = \chi_{j,k}$ so that they are equal (independently of k) outside $h_{j,k}(S_j)$. Let $v_{j,k} \coloneqq \chi_j\, u_{j,k} \in C^2(M_{j,k})$ and observe that $v_{j,k}$ preserves the L^p bounds of $u_{j,k}$, indeed:

$$\|v_{j,k}\|_{L^p} \leq \|u_{j,k}\|_{L^p} \leq C_2, \tag{2.4}$$

$$\|\Delta v_{j,k}\|_{L^p} \leq \|\Delta u_{j,k}\|_{L^p} + \|u_{j,k}\Delta\chi_j\|_{L^p} + 2\||\nabla u_{j,k}|\,|\nabla\chi_j|\|_{L^p} \leq C_2, \tag{2.5}$$

where C_2 depends on C_1 as well as on the choice of χ_j.

Next, we need some function theoretic considerations. First, we observe that compactness of $M_{j,k}$ implies the validity of an L^p-Calderón–Zygmund inequality

$$\|\nabla^2\varphi\|_{L^p} \leq E_{j,k}\left(\|\Delta\varphi\|_{L^p} + \|\varphi\|_{L^p}\right), \quad \forall\, \varphi \in C^2(M_{j,k}). \tag{2.6}$$

Second, if $p > n$, we have the validity on the sequence $M_{j,k}$ of a uniform Morrey–Sobolev inequality

$$|\varphi(x) - \varphi(y)| \leq C_3 \|\nabla\varphi\|_{L^p} d_{j,k}(x,y)^{1-\frac{n}{p}}, \quad \forall\, \varphi \in C^1(M_{j,k}), \tag{2.7}$$

where $d_{j,k}$ is the Riemannian distance on $M_{j,k}$, and the constant C_3 depends on n, p and the uniform upper bound on $\operatorname{diam} M_{j,k}$. See [70, Theorem 9.2.14] for reference, observing that the lower bound on the Ricci curvature ensures the validity of a p-Poincaré inequality; see [111, Theorem 5.6.5]. Applying (2.7) to $|\nabla\varphi|$ and combining with the Calderón–Zygmund inequality (2.6) implies the following estimate

$$||\nabla\varphi|(x) - |\nabla\varphi|(y)| \leq C_3 E_{j,k} \left(\|\Delta\varphi\|_{L^p} + \|\varphi\|_{L^p}\right) d_{j,k}(x,y)^{1-\frac{n}{p}}, \tag{2.8}$$

for all $\varphi \in C^2(M_{j,k})$ and all $x, y \in M_{j,k}$.

Applying (2.8) to $v_{j,k}$ and using estimates (2.4) and (2.5) we obtain

$$||\nabla v_{j,k}|(x) - |\nabla v_{j,k}|(y)| \leq C E_{j,k} d_{j,k}(x,y)^{1-\frac{n}{p}} \quad x, y \in M_{j,k}, \tag{2.9}$$

where C depends on C_1, C_2 and C_3, i.e., $C = C(n, p, \chi_j, D_j)$. Suppose by contradiction that $E_{j,k}$ is bounded from above uniformly in k. By (2.9) we deduce that $|\nabla v_{j,k}|$ is uniformly asymptotic continuous in the sense of Honda, hence, from [72, Proposition 3.3] we conclude that $|\nabla v_{j,k}|$ converges pointwise to $|\nabla v_{j,\infty}| \in C^0(X)$. However, since X_j is an n-dimensional Alexandrov space with Curv ≥ 0, it is a $RCD(0,n)$ space. Moreover, $\Delta v_{j,\infty} \in L^{p>n}$. From [39, Theorem 1.1] we then conclude that $|\nabla v_{j,\infty}|(x) = |\nabla u_{j,\infty}|(x) = 0$ whenever x is a singular point. Note here that singular points of Alexandrov spaces are *sharp* in the sense of De Philippis and Núñez-Zimbrón and have finite Bishop–Gromov density. By density, we conclude that $v_{j,\infty}$ must be constant in a neighborhood of $x_{j,\infty}$, thus contradicting (c).

In particular, there exists some $\bar{k}$, which may depend on j, such that

$$\|\nabla^2 v_{j,\bar{k}}\|_{L^p} > j \left(\|\Delta v_{j,\bar{k}}\|_{L^p} + \|v_{j,\bar{k}}\|_{L^p}\right) \tag{2.10}$$

on $M_{j,\bar{k}}$. Finally, we set $f_j = h_{j,\bar{k}}$, since $v_{j,\bar{k}}$ is compactly supported in $h_{j,\bar{k}}(T_j)$, it defines a function $v_j = v_{j,\bar{k}}$ on N_j which satisfies (2.3). □

Note that, while Proposition 2.7 is independent of p, the previous result depends on the initial choice of $p > n$. This has to be attributed to the fact that the constants C_1, C_2, C_3 and C are all dependent on p.

To obtain a contradiction to CZ(p) for $p > n$, we then simply need to glue the manifolds of Lemma 2.8 together.

Theorem 2.9. *For every $n \geq 2$ and $p > n$, there exists a complete, non compact n dimensional Riemannian manifold (M, g) with* $\operatorname{Sect}(M) > 0$ *such that CZ(p) fails.*

Proof. For $p > n$, let f_j be as in Lemma 2.8, and let f be its point-wise limit. Note that the convergence is actually uniform on compact sets. The function f is smooth and strictly convex, thus, $M = \operatorname{Graph}(f)$ is a smooth, non-compact Riemannian manifold isometrically immersed in $\mathbb{R}^{n+1}$ satisfying $\operatorname{Sect}(M) > 0$. Since f is defined on the whole space $\mathbb{R}^n$, M is also a complete manifold. Observe that the sequence v_j as in Lemma 2.8 induces functions in $C^2(M)$ whose supports are compact and disjoint, and which satisfy (2.3) on $L^p(M)$. This sequence clearly contradicts the validity of a global Calderón–Zygmund inequality on M. □

Remark 2.10. Thanks to a trick introduced in [74], it is possible to extend the above counterexample to the case $p > 2$. Indeed, fix $n \geq 2$ and take (Y, h) a closed Riemannian manifold of positive curvature and dimension $n - 2$. If (M, g) is the 2 dimensional Riemannian manifold of Theorem 2.9 on which $\mathrm{CZ}(p)$ fails for every $p > 2$, then, the product $M \times Y$ is an n-dimensional Riemannian manifold of non-negative curvatures. Take $\{v_j\}$ to be the sequence which violates $\mathrm{CZ}(p)$ on (M, g) and extend it to the whole product setting $\widetilde{v}_j(x, y) = v_j(x)$ where $x \in M$ and $y \in Y$. Then, $\{\widetilde{v}_j\}$ violates $\mathrm{CZ}(p)$ on $M \times Y$ for all $p > 2$.

When $p > 2$, the only results ensuring the validity of $\mathrm{CZ}(p)$ are the one of Güneysu–Pigola, [63], and the one of Cao, Cheng and Thalmaier, [24, Theorem 1.2]. This latter, in addition to $\mathrm{Ric} \geq -K^2$, essentially requires a bound of the type $\|\mathrm{Riem}\|_\infty + \|\nabla \mathrm{Ric}\|_\infty \leq H$, here H is a non-negative function belonging to the so-called *Kato class*. Baumgarth, Devyver and Güneysu conjectured that $\|\mathrm{Riem}\|_\infty + \|\nabla \mathrm{Riem}\|_\infty < +\infty$ should be enough to prove $\mathrm{CZ}(p)$ on the whole range $p \in (1, +\infty)$, [13, Conjecture 1.5]. Note that both [24, 13] are subsequent to our [90].

Theorem 2.9 has also consequences on the theory of L^p-gradient estimates. Indeed, since L^p-gradient estimates hold on manifolds with Ricci curvature bounded from below, [32], the manifolds constructed in Theorem 2.9 support $\mathrm{GE}(p)$ for all $p \in (1, +\infty)$ although $\mathrm{CZ}(p)$ fails for $p > n$ (actually $p > 2$ by Remark 2.10).

Corollary 2.11. *For any $n \geq 2$ and $p > 2$, there exists a complete Riemannian manifold (M, g) supporting the L^p-gradient estimate GE(p) on which CZ(p) does not hold.*

This gives a negative answer to a question raised by Devyver on the equivalence of L^p-gradient and Calderón–Zygmund inequalities, see Section 8.1 in [103],

In the proof of Theorem 2.9 we have not exploited to the fullest the fact that the functions v_j have disjoint supports. In fact, not only one has a sequence v_j on which (2.3) holds, but one can actually define a function $F \in C^2(M)$ such that $\|F\|_{L^p} + \|\Delta F\|_{L^p} < +\infty$ but $\|\nabla^2 F\|_{L^p} = +\infty$, which is a stronger condition. This leads to the following consequence on the theory of Sobolev spaces.

Corollary 2.12. *For every $n \geq 2$ and $p > n$ there exists a complete, non-compact n-dimensional Riemannian manifold with $\mathrm{Sect}(M) > 0$ such that $W^{2,p}(M) \subsetneq H^{2,p}(M)$.*

Proof. Fix $p > n$, let (M, g) and $v_j \in C^2(M)$ be as in the proof of Theorem 2.9. Define

$$F := \sum_{j=1}^{\infty} \frac{1}{j^2} \frac{v_j}{\|\Delta v_j\|_{L^p} + \|v_j\|_{L^p}},$$

and observe that the sum converges since it is locally finite. Note that

$$\|\Delta F\|_{L^p} + \|F\|_{L^p} = \sum_{j=1}^{\infty} \frac{1}{j^2},$$

so that $F \in H^{2,p}(M)$. By (2.3), on the other hand, we have

$$\|\nabla^2 F\|_{L^p} \geq \sum_{j=1}^{\infty} \frac{1}{j},$$

hence, $F \notin W^{2,p}(M)$. □

2.3 A counterexample to L^p-gradient estimates

In this last section of Chapter 2, we focus on counterexamples to L^p-gradient estimates.

Theorem 2.13. *Suppose that n is an integer ≥ 2. For any $p > 2$ there exists a complete n-dimensional Riemannian manifold M where the L^p gradient estimate GE(p) fails.*

Proof. First, we prove the result in the case where $n = 2$. Take $(\Sigma, g) = (\mathbb{R}^2, \lambda^2 dx^2)$ where dx^2 is the usual Euclidean metric on $\mathbb{R}^2$ and $\lambda \in C^\infty(\Sigma)$ such that $0 < \lambda \leq 1$. As above, we denote by Δ and ∇ the Laplace–Beltrami operator and gradient with respect to the metric g while we use Δ_e and ∇^e to denote the corresponding Euclidean differential operators. The spaces $L^p(\Sigma)$ are defined in terms of the Riemannian volume form $d\mu_g$, whereas $L^p(\mathbb{R}^2)$ are the spaces with respect to the Lebesgue measure dx^2.

For each non-negative integer m, consider the point x_m in $\mathbb{R}^2$, with coordinates $(m, 0)$. Take $\lambda(x) = 1$ for all $x \in \Sigma \setminus \bigcup_{m\in\mathbb{N}} B_{1/8}(x_m)$. Since (Σ, g) is isometric to $(\mathbb{R}^2, dx^2)$ outside a countable union of bounded sets whose pairwise distance is uniformly lower bounded, it is a complete Riemannian manifold. Next, take $\varphi_0 \in C_c^\infty(\Sigma)$ such that

$$\begin{cases} \varphi_0(u, v) = u + 1 \text{ on } B_{1/4}(x_0) \\ \operatorname{supp}(\varphi_0) \Subset B_{1/2}(x_0) \end{cases}$$

and let $\varphi_m(u, v) = \varphi_0(u - m, v)$, for all positive integers m. Then, for every positive integer k define

$$u_k := \sum_{m=0}^{k} 2^{-m} \varphi_m.$$

Clearly $u_k \in C_c^\infty(\Sigma)$. Notice that

$$\begin{aligned} \|u_k\|^p_{L^p(\Sigma)} &= \sum_{m=0}^{k} 2^{-mp} \int_\Sigma |\varphi_m|^p \lambda^2 \, dx \\ &\leq \sum_{m=0}^{k} 2^{-mp} \|\varphi_m\|^p_{L^p(\mathbb{R}^2)} = \|\varphi_0\|^p_{L^p(\mathbb{R}^2)} \sum_{m=0}^{+\infty} 2^{-mp} < +\infty. \end{aligned}$$

Now observe that $\Delta\varphi_m = \lambda^{-2}\Delta_e\varphi_m$. Hence,

$$\|\Delta u_k\|^p_{L^p(\Sigma)} = \sum_{m=0}^{k} 2^{-mp} \int_\Sigma |\Delta\varphi_m|^p \lambda^2 \, dx = \sum_{m=0}^{k} 2^{-mp} \int_\Sigma |\Delta_e\varphi_m|^p \, \lambda^{2(1-p)} \, dx.$$

Moreover, we have that $\Delta_e\varphi_m(u,v) = (\Delta_e\varphi_0)(u-m,v)$. Since $\Delta_e\varphi_0$ vanishes on $B_{1/4}(x_0)$, the function $\Delta_e\varphi_m$ vanishes on $B_{1/4}(x_m)$. This and the fact that the support of φ_0 is contained in $B_{1/2}(x_0)$ yield

$$\|\Delta u_k\|^p_{L^p(\Sigma)} = \int_{B_{1/2}(x_0)\setminus B_{1/4}(x_0)} |\Delta_e\varphi_0|^p\, \lambda^{2(1-p)}\, dx = \int_{B_{1/2}(x_0)\setminus B_{1/4}(x_0)} |\Delta_e\varphi_0|^p\, dx,$$

where the last equality holds, because $\lambda = 1$ on $B_{1/2}(x_0)\setminus B_{1/4}(x_0)$. Altogether, we obtain that

$$\|\Delta u_k\|^p_{L^p(\Sigma)} \le \|\Delta_e\varphi_0\|^p_{L^p(\mathbb{R}^2)} \sum_{m=0}^{+\infty} 2^{-mp} < +\infty.$$

Now, recall that $p > 2$ is given. Choose $\beta > 1/(p-2)$, and consider $\lambda_\infty(x) := |x|^{2\beta}$ in $B_\delta(x_0)$ for some $0 \le \delta \ll 1/8$. Note that $|\nabla^e\varphi_0| = 1$ on $B_{1/8}(x_0)$, whence

$$\int_{B_\delta(x_0)} |\nabla^e\varphi_0|^p_e\, \lambda_\infty^{2-p} dx = 2\pi \int_0^\delta r^{1-2\beta(p-2)} dr = +\infty.$$

Here $|x| = r$ denotes the Euclidean distance from the origin. Then for any $m \in \mathbb{N}$ we can find $\varepsilon_m > 0$, such that $\varepsilon_m \to 0$ as $m \to +\infty$, and

$$\int_{B_\delta(x_0)} |\nabla^e\varphi_0|^p\, (|x|^2 + \varepsilon_m)^{(2-p)\beta} dx \ge 2^{mp}.$$

For $x \in B_{1/8}(x_0)$ and $\varepsilon \in [0,1]$ we define a function $\lambda_\varepsilon \in C^\infty(B_{1/8}(x_0))$ by

$$\begin{cases} 0 < \lambda_\varepsilon \le 1 \\ \lambda_\varepsilon(x) = (|x|^2 + \varepsilon)^\beta \text{ if } x \in B_\delta(x_0) \\ \operatorname{supp}(1-\lambda_\varepsilon) \subseteq B_{1/8}(x_0). \end{cases}$$

Now define $\lambda \in C^\infty(\Sigma)$ by

$$\begin{cases} 0 < \lambda \le 1 \\ \lambda(x) = 1 \text{ if } x \in \Sigma \setminus \bigcup_{m\in\mathbb{N}} B_{1/8}(x_m) \\ \lambda(x) = \lambda_{\varepsilon_m}(x - x_m) \text{ if } x \in B_\delta(x_m). \end{cases}$$

Then, arguing much as above,

$$\begin{aligned} \|\nabla u_k\|^p_{L^p(\Sigma)} &= \int_\Sigma \sum_{m=0}^k \frac{|\nabla\varphi_m|^p}{2^{mp}} \lambda^2 dx \ge \sum_{m=0}^k 2^{-mp} \int_{B_\delta(x_0)} |\nabla\varphi_o|^p \lambda_m^2 dx \\ &= \sum_{m=0}^k 2^{-mp} \int_{B_\delta(x_0)} |\nabla^e\varphi_0|^p (|x|^2 + \varepsilon_m)^{(2-p)\beta} dx \ge k. \end{aligned}$$

Since $\{\|u_k\|_{L^p(\Sigma)}\}$ and $\{\|\Delta u_k\|_{L^p(\Sigma)}\}$ are bounded, the gradient estimate fails on Σ.

This concludes the proof of Theorem 2.13 in the case where $n = 2$.

Suppose now that $n \geq 3$. We proceed as in [74], see also Remark 2.10. Let (Σ, g) be the Riemannian manifold considered above and (N, h) any $n-2$ dimensional closed Riemannian manifold. Consider the product manifold $M = \Sigma \times N$ and define

$$v_k(x, y) = u_k(x) \qquad \forall (x, y) \in \Sigma \times N.$$

Clearly $\{v_k\} \subseteq C_c^\infty(M)$. It is straightforward to check that the sequences $\{\|v_k\|_{L^p(M)}\}$ and $\{\|\Delta v_k\|_{L^p(M)}\}$ are bounded, whereas $\{\||\nabla v_k|\|_{L^p(M)}\}$ is unbounded. Hence, the gradient estimate fails on M.

This concludes the proof of Theorem 2.13. $\square$

Remark 2.14. We observe that the choice of the sequence $\{x_m\}$ is quite arbitrary. In particular, let $\alpha : [0, +\infty) \to [0, +\infty)$ be an arbitrary increasing function such that $\alpha(t) \to \infty$ as $t \to +\infty$. If we choose x_m which diverges quick enough to infinity, we can make the lower bound on Ricci arbitrarily small so that

$$\mathrm{Ric}(x) \geq -\alpha(r(x)).$$

This can be done with a similar strategy as in the proof of Theorem 2.1. As a consequence, the result by Cheng, Thalmaier and Thompson, [28], is sharp with respect to pointwise lower bounds.

We also point out the following corollary of the proof of Theorem 2.13.

Corollary 2.15. *For any $n \geq 2$ and $p > 2$, there exists a Riemannian manifold M and a function $v_\infty \in H^{2,p}(M)$ such that $v_\infty \notin W^{1,p}(M)$.*

Proof. For $n > p$, it is enough to define

$$u_\infty = \sum_{m=0}^{+\infty} 2^{-m} \varphi_m;$$

then $u_\infty, \Delta u_\infty \in L^p(\Sigma)$ while $|\nabla u_\infty| \notin L^p(\Sigma)$. In particular, $u_\infty \in H^{2,p}(\Sigma)$ while $u_\infty \notin W^{1,p}(\Sigma)$. The case $2 < p \leq n$ can be dealt with the same trick as in the proof of Theorem 2.13. $\square$

Chapter 3

The case of Cartan–Hadamard manifolds

In the previous chapters, we have shown that lower bounds on the Ricci curvature of various types are required to ensure the validity of L^p-gradient and Calderón–Zygmund estimates, as well as in results concerning the closely related problem of the density of $C_c^\infty(M)$ in $W^{2,p}(M)$. Many of these bounds, moreover, are optimal by the counterexamples of Chapter 2. In this last chapter of Part I, we focus on Cartan–Hadamard manifolds (i.e. simply-connected complete manifolds with non-positive sectional curvature) with pinched Ricci curvature exploding at $-\infty$, possibly very fast, although in a controlled way. More precisely, we take (M, g) a Cartan–Hadamard manifold satisfying

$$-b\, r^\beta(x) \leq \mathrm{Ric}(x) \leq -a\, r^\alpha(x), \tag{3.1}$$

holds outside a compact set. Here α and β are positive constants. For suitable values of α and β we can still obtain the density result for Sobolev spaces as well as the validity of CZ(2).

Thanks to their simple topology and rich structure, Cartan–Hadamard manifolds with suitable curvature bounds have several interesting functional analytic properties. A non-exhaustive list of well-known results includes [33, 5, 3, 87, 81, 31, 95, 30, 54, 48] see also [56, 94, 58, 96, 15, 80, 57] for more recent works quite in the same spirit as ours. Beyond their obvious topological triviality, the Cartan–Hadamard manifolds we take into account have also quite strong metrical features. On the one hand, the lower bound $-br^\beta(x)$ for the Ricci curvature implies a Laplacian comparison, i.e, an upper control on Δr. This, in turn, permits to construct a suitable sequence of Hessian cutoff functions. Namely, one gets the existence of a family of smooth cutoffs $\{\chi_R\} \in C_c^\infty(M)$ such that

(1) $\chi_R \equiv 1$ on B_R and $\chi_R \equiv 0$ on $M \setminus \overline{B_{2R}}$;
(2) $|\nabla \chi_R| \leq \frac{C}{R}$;
(3) $|\nabla^2 \chi_R| \leq C R^{\frac{\beta}{2}-1}$,

with $C > 0$ (see Lemma 3.16). Most of the strategies proposed in previous literature to approach the density problem or CZ(2), are precisely based on the existence of suitable cutoff functions which have bounded covariant derivatives up to the second order, see for instance [77, 78, 17, 60].

Conversely, the control that we get on $|\nabla^2 \chi_R|$ under our assumptions is not strong enough to allow us to obtain the desired results. The reason is essentially that, when $\beta > 2$, the sole lower bound $\mathrm{Ric} \geq -br^\beta$ cannot guarantee that for any function $f \in W^{2,p}$, then $|\nabla^2 \chi_R| f$ is uniformly bounded in L^p. Instead, assuming also that $\mathrm{Ric} \leq -ar^\alpha$, one

gets
$$f \in W^{2,p} \implies (r^\alpha f) \in L^p, \tag{3.2}$$
see Theorem 3.13. This latter relation, combined with the properties of the Hessian cutoff functions, yields a uniform L^p bound on $|\nabla^2 \chi_R| f$.

To obtain (3.2), we exploit the validity on $\Omega \subset M$ of certain Hardy-type inequalities (obtained elaborating on ideas by L. D'Ambrosio and S. Dipierro, [36]) of the form
$$\int_\Omega \frac{|\nabla G|^p}{|G|^p}(-\log G)^{\beta p}|f|^p d\mu_g \leq \left(\frac{p}{p-1}\right)^p \int_\Omega (-\log G)^{\beta p}|\nabla f|^p d\mu_g \qquad \forall f \in C_c^\infty(\Omega),$$
where $G \in C^\infty(\Omega)$ satisfies
(i) $-\Delta_p G \geq 0$ on Ω;
(ii) $0 \leq G \leq c < 1$;
and $p \in (1, +\infty)$, see Theorem 3.7 and Theorem 3.13. Using a Laplacian comparison for Cartan–Hadamard manifolds, it turns out that an appropriate choice for G is the Green function for the p-Laplacian of the model manifold $\widetilde{M}$ whose (radial) Ricci curvature is precisely $-ar^\alpha$.

While this is sufficient to prove the density result, a further ingredient is needed in the case of CZ(2). Using Bochner inequality, we first prove that the lower bound Ric $\geq -b\, r^\beta$ implies the validity of the disturbed infinitesimal Calderón–Zygmund inequality
$$\|\nabla^2 \varphi\|_{L^2} \leq A_1(\varepsilon)\,[\|\Delta\varphi\|_{L^2} + \|\varphi\|_{L^2}] + A_2 \varepsilon^2 \|r^\beta \varphi\|_{L^2} \qquad \forall \varphi \in C_c^\infty(M);$$
see Theorem 3.20. Then, one can conclude using again the Hardy-type inequalities ensured by the upper bound on Ricci.

3.1 Estimates on Cartan–Hadamard manifolds

The goal of this first section is to obtain asymptotic estimates for several geometric objects on Cartan–Hadamard manifolds whose (radial) Ricci curvature behaves polinomially at $-\infty$. These estimates will be crucial in the rest of the chapter.

We begin by obtaining some asymptotic estimates for the solutions of
$$j''(t) - A^2 t^\alpha j(t) = 0 \tag{3.3}$$
on $[0, +\infty)$, where $A > 0$ and $\alpha \geq 0$. The solutions of (3.3) are of the form
$$j(t) = \sqrt{t}\left(D_1 I_\nu\left(2A\nu t^{1/2\nu}\right) + D_2 K_\nu\left(2A\nu t^{1/2\nu}\right)\right), \tag{3.4}$$
where $D_1, D_2 \in \mathbb{R}$, $\nu = \frac{1}{2+\alpha}$ and I_ν, K_ν are modified Bessel functions of the first and second kind and order ν, respectively. That is, I_ν and K_ν are independent solutions on $[0, +\infty)$ of the following equation:
$$t^2 x''(t) + t x'(t) - (t^2 + \nu^2) x(t) = 0.$$

See Section 5.7 of [84] for further references on Bessel functions. Note also that I_ν and K_ν satisfy the following relations

$$I_\nu(t) \sim \frac{e^t}{\sqrt{2\pi t}} \quad \text{and} \quad K_\nu(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t} \quad \text{when} \quad t \to +\infty, \tag{3.5}$$

$$I_\nu(t) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{t}{2}\right)^\nu \quad \text{and} \quad K_\nu(t) \sim \frac{\Gamma(\nu)}{2} \left(\frac{2}{t}\right)^\nu \quad \text{when} \quad t \to 0, \tag{3.6}$$

$$\frac{d}{dt}\left(t^\nu I_\nu(t)\right) = t^\nu I_{\nu-1}(t) \quad \text{and} \quad \frac{d}{dt}\left(t^\nu K_\nu(t)\right) = -t^\nu K_{\nu-1}(t). \tag{3.7}$$

Using (3.5) and (3.7), at least when $D_1 \neq 0$, we have

$$j(t) \sim \frac{D_1}{\sqrt{4A\nu\pi}} t^{-\frac{\alpha}{4}} \exp\left(\frac{2A}{\alpha+2} t^{1+\frac{\alpha}{2}}\right), \tag{3.8}$$

and

$$j'(t) \sim A \frac{D_1}{\sqrt{4A\nu\pi}} t^{\frac{\alpha}{4}} \exp\left(\frac{2A}{\alpha+2} t^{1+\frac{\alpha}{2}}\right), \tag{3.9}$$

hence,

$$\frac{j'(t)}{j(t)} \sim A t^{\frac{\alpha}{2}}. \tag{3.10}$$

3.1.1 Asymptotic estimates on model manifolds

In this section, we study model manifolds with a prescribed asymptotic growth on the Ricci curvature and estimate several geometric objects therein. The manifolds we consider will be useful to prove asymptotic comparison results for Cartan–Hadamard manifolds.

According to Greene and Wu, [50], a model manifold (M, g) is (the smooth extension at the origin) of $(0, +\infty) \times \mathbb{S}^{n-1}$ endowed with the metric

$$g = dt^2 + j^2(t) d\theta^2.$$

Here $d\theta^2$ is the standard round metric on $\mathbb{S}^{n-1}$ and the warping function $j \in C^\infty([0, +\infty))$ satisfies $j > 0$ on $(0, +\infty)$, $j(0) = 0$, $j'(0) = 1$ and $j^{(2k)}(0) = 0$ for $k \in \mathbb{N}$. In the following, $r(x)$ denotes the Riemannian distance form the origin, so that $r(x) = t$ when $x = (t, \theta)$.

Let $H : [0, +\infty) \to [0, +\infty)$ be a smooth, non-negative function such that

$$H(t) = \begin{cases} K^2 & t \leq R_0 \\ A^2 (t - R_0)^\alpha & t \geq R_0 + \delta \end{cases}$$

for $A, K, R_0, \delta > 0$ and $\alpha \geq 0$. Then, we take as warping function j the solution of

$$\begin{cases} j''(t) - H(t) j(t) = 0 \\ j(0) = 0, \quad j'(0) = 1 \end{cases} \tag{3.11}$$

on $[0, +\infty)$, so that

$$\mathrm{Ric}_o(x) = -(n-1)\frac{j''(t)}{j(t)} = -(n-1)H(t)$$

for every $x = (t, \theta) \in M$. By uniqueness, when $0 \leq t \leq R_0$ we have $j(t) = t$ if $K = 0$, and $j(t) = \frac{1}{K}\sinh(Kt)$ otherwise. On the other hand, if $t \geq R_0 + \delta$, the function j is of the form $j(t) = h(t - R_0)$ where h is a solution of

$$\begin{cases} h''(t) - A^2 t^\alpha h(t) = 0 \\ h(\delta) = a \quad h'(\delta) = b, \end{cases}$$

where a, b are the values of j, j' at $R_0 + \delta$. Note that $b \geq 1$ since j is a convex function. In particular, h is given by (3.4) where the constants $D_1 \neq 0$ and D_2 depend on a, b. Since $j(t) \sim h(t)$ for $t \to +\infty$, we can use (3.8), (3.9) and (3.10) to control several geometric objects on the manifold.

First, we use (3.10) to estimate the Laplacian of the Riemannian distance r, which on a model manifold is given by

$$\Delta r = (n-1)\frac{j'(r)}{j(r)} \sim (n-1)A r^{\alpha/2}. \tag{3.12}$$

Next we consider the volume of the geodesic spheres centered at the origin. Recall that on models $\mathrm{vol}(\partial B_t) = \omega_n j^{n-1}(t)$, where ω_n is the volume of the n-dimensional unit ball. Hence, (3.8) implies

$$\left(\frac{1}{\mathrm{vol}(\partial B_t)}\right)^{\frac{1}{p-1}} \in L^1(+\infty)$$

for all $p > 1$. By [120, Corollary 5.2], we deduce that (M, g) is p-hyperbolic, i.e., there exists a symmetric positive Green function for the p-Laplacian. Specifically, if we fix the origin as a pole, this function is radial and its expression is given by

$$G_p(x) = G_p(t) = \int_t^{+\infty} \left(\frac{1}{j(s)}\right)^{\frac{n-1}{p-1}} ds \tag{3.13}$$

where $x = (t, \theta) \in M$. Using (3.8), we obtain

$$\partial_t G_p(t) \sim -D_3 t^{\frac{\alpha}{4}\frac{n-1}{p-1}} \exp\left(-A\frac{2}{\alpha+2}\frac{n-1}{p-1}t^{1+\frac{\alpha}{2}}\right) \quad t \to +\infty, \tag{3.14}$$

and applying de l'Hôpital rule

$$G_p(t) \sim D_4 t^{\frac{\alpha}{4}\left(\frac{n-1}{p-1}-2\right)} \exp\left(-A\frac{2}{\alpha+2}\frac{n-1}{p-1}t^{1+\frac{\alpha}{2}}\right) \quad t \to +\infty, \tag{3.15}$$

where D_3, D_4 are positive constants depending on D_1, α, n and p. Note that $\partial_t G_p(t) < 0$ for all $t > 0$. Finally, using (3.8) once again we deduce that

$$\int_0^t j^{n-1}(s)ds \sim D_5 t^{-\frac{\alpha}{4}(n+1)} \exp\left(\frac{2A}{\alpha+2}(n-1)t^{1+\frac{\alpha}{2}}\right) \tag{3.16}$$

for some positive constant D_5, hence,

$$\frac{\int_0^t j^{n-1}(s)ds}{j^{(n-1)}(t)} \sim D_6 t^{-\frac{\alpha}{2}}. \tag{3.17}$$

This estimate will be useful later on.

3.1.2 Asymptotic comparison results for Cartan–Hadamard manifolds

Next, we relate via comparisons the above estimates to a Cartan–Hadamard manifold with suitable asymptotic bounds on the radial Ricci curvature.

Upper bounds

Let (M, g) be a Cartan–Hadamard manifold of dimension $n \geq 2$ with a fixed pole and suppose

$$\mathrm{Ric}_o(x) \leq -ar^{\alpha}(x)$$

holds outside of a compact set containing the pole for some $a > 0$ and $\alpha \geq 0$. Here $r(x)$ denotes the Riemannian distance from the pole. Let $(\widehat{M}, \widehat{g})$ be the model manifold of radial Ricci curvature $\widehat{\mathrm{Ric}}_o(\hat{x}) = -(n-1)\widehat{H}(\widehat{r}(\hat{x}))$ where $\widehat{H}(t)$ is a non-negative smooth function satisfying

$$\widehat{H}(t) = \begin{cases} 0 & t \leq R_0 \\ a(t-R_0)^{\alpha} & t \geq R_0 + \delta. \end{cases}$$

for some $R_0 > 0$ and $\delta > 0$, so that

$$\mathrm{Ric}_o(x) \leq \frac{1}{n-1}\widehat{\mathrm{Ric}}_o(\hat{x}),$$

for all $x \in M$ and $\hat{x} \in \widehat{M}$ with $r(x) = \hat{r}(\hat{x})$. Let $\widehat{j}$ be the corresponding warping function, by (3.12) we have

$$\widehat{\Delta}\widehat{r}(\widehat{x}) = (n-1)\frac{\widehat{j}'(t)}{\widehat{j}(t)} \sim (n-1)\sqrt{a}t^{\frac{\alpha}{2}}.$$

Then, since the bound

$$\mathrm{Ric}_o \leq \frac{1}{n-1}\widehat{\mathrm{Ric}}_o,$$

holds globally, by [124, Theorem 2.15] we have

$$\Delta r \geq \frac{\widehat{j}'(r)}{\widehat{j}(r)} \sim \sqrt{a}r^{\frac{\alpha}{2}} \qquad r \to +\infty.$$

Next, we consider the model manifold (N, h) with Ricci curvature

$$\mathrm{Ric}_o^N(y) = -(n-1)H(r_N(y)). \tag{3.18}$$

Here $r_N(y)$ is the Riemannian distance from the origin and $H : [0, +\infty) \to [0, +\infty)$ is a non-negative smooth function satisfying

$$H(t) = \begin{cases} 0 & t \leq R_0 \\ A^2(t - R_0)^\alpha & t \geq R_0 + \delta \end{cases} \tag{3.19}$$

with $(n-1)A \leq \sqrt{a}$. We denote with j its warping function, by (3.12) we have

$$\frac{j'(t)}{j(t)} \sim At^{\frac{\alpha}{2}},$$

hence,

$$\Delta r \geq \frac{\widehat{j}'(r)}{\widehat{j}(r)} \geq (n-1)\frac{j'(r)}{j(r)}$$

for large $r >> 1$. In summary, we have the following comparison result, which holds outside a compact set.

Proposition 3.1. *Let (M, g) be a Cartan–Hadamard manifold of dimension $n \geq 2$ such that*

$$\mathrm{Ric}_o(x) \leq -ar^\alpha(x) \tag{3.20}$$

holds outside of a compact set for some $a > 0$, $\alpha \geq 0$. Let j be the warping function of the model (N, h) with radial Ricci curvature prescribed by (3.18),(3.19). *Then,*

$$\Delta r \geq (n-1)\frac{j'(r)}{j(r)} \tag{3.21}$$

for $r(x) >> 1$.

As a corollary, we deduce a comparison result for the p-Laplacian of (radially) monotonic functions. We begin with the following:

Lemma 3.2. *Let (M, g) be a Cartan–Hadamard manifold and suppose that*

$$\Delta r \geq \phi(r) \text{ on } \Omega \subseteq M, \tag{3.22}$$

for some $\phi \in C^0((0, +\infty))$ and Ω open. Let $v \in C^2(\mathbb{R})$ non-negative and define $u(x) = v(r(x))$ for $x \in \Omega$. If $v' < 0$, then for all $p > 1$ we have

$$\Delta_p u \leq |v'|^{p-2}(v'\phi(r) + (p-1)v''), \tag{3.23}$$

on $\Omega \setminus \{o\}$.

Proof. Since (M, g) is Cartan–Hadamard, then $r \in C^\infty(M \setminus \{o\})$ so that $u \in C^2(M \setminus \{o\})$. Suppose $v' < 0$, then

$$\begin{aligned} \Delta_p u &= \mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div}(|v'|^{p-2}v'\nabla r) = |v'|^{p-2}(v'\Delta r + (p-1)v'') \\ &\leq |v'|^{p-2}(v'\phi(r) + (p-1)v''), \end{aligned}$$

on $\Omega \setminus \{o\}$. □

Remark 3.3. Although it is not relevant to our work, we observe that if $v' > 0$, then (3.23) holds with the opposite sign.

Combining Lemma 3.2 with Proposition 3.1, we obtain the following

Proposition 3.4. *Let* (M, g) *be a Cartan–Hadamard manifold satisfying* (3.20) *and let* (N, h) *be the model manifold with radial Ricci curvature prescribed by* (3.18), (3.19). *Let* $v \in C^2(\mathbb{R})$ *non-negative with* $v' < 0$ *and define* $u(x) = v(r(x))$ *and* $w(y) = v(r_N(y))$. *Then* $\Delta_p u(x) \le \Delta_p^N w(y)$ *for all* $x \in M$ *and* $y \in N$ *such that* $r(x) = r_N(y) >> 1$.

Proof. By Proposition 3.1, if $r(x) >> 1$, then $\Delta r \ge (n-1)j'(r)/j(r)$, hence

$$\Delta_p u(x) \le |v'(r(x))|^{p-2}\left[v'(r(x))(m-1)\frac{j'(r(x))}{j(r(x)} + (p-1)v''(r(x))\right] = \Delta_p^N w(y).$$

□

In particular, if we take $v(t) = G_p(t)$ as in (3.13) where G_p defines the p-Green function on (N, h) we have the following corollary.

Corollary 3.5. *Let* (M, g) *be a Cartan–Hadamard manifold satisfying* (3.20), *then* $G_p(x)$ *is* p*-superharmonic on* (M, g) *outside of a large enough compact set.*

Lower bounds

Let (M, g) be a Cartan–Hadamard manifold of dimension $n \ge 2$ with a fixed pole and suppose

$$\mathrm{Ric}_o(x) \ge -br^\beta(x)$$

holds outside of a compact set containing the pole for some $b > 0$ and $\beta \ge 0$. Let $(\widetilde{M}, \widetilde{g})$ be the model manifold as in Section 3.1.1 with H given by

$$H(t) = \begin{cases} K^2 & t \le R_0 \\ B^2(t-R_0)^\beta & t \ge R_0 + \delta \end{cases}$$

with K, R_0 and $B^2 = b$ so that $\mathrm{Ric}_o(x) \ge -(n-1)H(r(x))$ holds globally on M. Denote with j the corresponding warping function. We have the following (asymptotic) comparison result.

Proposition 3.6. *Let* (M, g) *be a Cartan–Hadamard manifold satisfying*

$$\mathrm{Ric}_o(x) \ge -br^\beta(x) \tag{3.24}$$

outside of a compact set for some $b > 0$ *and* $\beta \ge 0$. *Then, there exist* $R_1 >> 1$ *and* $C > 0$ *such that*

$$|\nabla^2 r|(x) \le Cr^{\frac{\beta}{2}}(x) \tag{3.25}$$

for $r(x) \ge R_1$.

Proof. Since M has non-positive sectional curvatures, by the Hessian comparison ([106, Theorem 2.3]), $\nabla^2 r$ has non-negative eigenvalues at every point in M and, in particular,

$$|\nabla^2 r| \leq \Delta r.$$

Then, since $\mathrm{Ric}_o(x) \geq -(n-1)H(r(x))$ holds globally on M, by Laplacian comparison we conclude that

$$|\nabla^2 r| \leq (n-1)\frac{j'(r)}{j(r)}.$$

Since

$$\frac{j'(r)}{j(r)} \sim Br^{\frac{\beta}{2}},$$

there exist some $R_1 >> 1$ and some positive constant C, depending on B, β, R_1, such that

$$\frac{j'(t)}{j(t)} \leq Ct^{\frac{\beta}{2}}$$

for $t \geq R_1$. This concludes the proof. $\square$

3.2 Hardy inequalities via Green function estimates

We now turn to the study of a class of functional inequalities on Riemannian manifolds, which go under the name of Hardy-type inequalities. These inequalities have an interest of their own and are extensively studied in literature, especially in the case of Cartan–Hadamard manifolds. See [14, 36, 40, 44, 82, 96, 127] among others. With the help of a result by D'Ambrosio and Dipierro, [36], we establish a new Hardy-type inequality on complete Riemannian manifolds possessing a non-negative p-superharmonic function G.

Theorem 3.7. *Let (M, g) be a complete Riemannian manifold and $\Omega \subseteq M$ open. Fix $p > 1$ and let $G \in C^\infty(\Omega)$ such that*
(i) $-\Delta_p G \geq 0$ on Ω;
(ii) $0 \leq G \leq c < 1$.
Then, for any $\beta \geq 0$,

$$\int_\Omega \frac{|\nabla G|^p}{|G|^p}(-\log G)^{\beta p}|f|^p d\mu_g \leq \left(\frac{p}{p-1}\right)^p \int_\Omega (-\log G)^{\beta p}|\nabla f|^p d\mu_g \qquad \forall f \in C_c^\infty(\Omega). \tag{3.26}$$

Proof. Let $\delta > 0$ such that $G_\delta := G + \delta < 1$ and define

$$h := -\frac{|\nabla G_\delta|^{p-2}\nabla G_\delta}{G_\delta^{p-1}}(-\log G_\delta)^{\beta p}, \qquad A_h := (p-1)\frac{|\nabla G_\delta|^p}{G_\delta^p}(-\log G_\delta)^{\beta p}.$$

Since $G \in C^\infty(\Omega)$ and $G_\delta \geq \delta$ we have $|h|, A_h \in L^1_{\mathrm{loc}}(\Omega)$, furthermore,

$$\frac{|h|^p}{A_h^{p-1}} = (p-1)^{1-p}(-\log G_\delta)^{\beta p} \in L^1_{\mathrm{loc}}(\Omega).$$

Next, we estimate

$$\begin{aligned}\operatorname{div}(h) &= -\frac{(-\log G_\delta)^{\beta p}}{G_\delta^{p-1}}\Delta_p G_\delta + (p-1)\frac{|\nabla G_\delta|^p}{G_\delta^p}(-\log G_\delta)^{\beta p} + \beta p\frac{|\nabla G_\delta|^p}{G_\delta^p}(-\log G_\delta)^{\beta p-1}\\ &\geq (p-1)\frac{|\nabla G_\delta|^p}{G_\delta^p}(-\log G_\delta)^{\beta p} = A_h.\end{aligned}$$

Thanks to [36, Lemma 2.10] we have

$$\int_\Omega \frac{|\nabla G_\delta|^p}{|G_\delta|^p}(-\log G_\delta)^{\beta p}|f|^p d\mu_g \leq \left(\frac{p}{p-1}\right)^p \int_\Omega (-\log G_\delta)^{\beta p}|\nabla f|^p d\mu_g \qquad \forall f \in C_c^\infty(\Omega).$$

Since $-\log G_\delta \leq -\log G$ and $\nabla G_\delta = \nabla G$, letting $\delta \to 0$ and using Fatou's lemma yields the desired estimate. □

Remark 3.8. It is worth noticing that the results of Theorem 3.7 still hold under more relaxed regularity assumptions. Notably, it suffices to have $G \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $-\Delta_p G \geq 0$ weakly on Ω to have the validity of (3.26). Assumption *(ii)* still needs to hold, although it is always satisfied in applications.

Once we have the quite general (3.26), we return to our setting, that is, (M, g) is a Cartan–Hadamard manifold satisfying the Ricci upper bound (3.20). Under such curvature assumptions one easily gets that (M, g) is a p-hyperbolic manifold. Indeed, if $\mathcal{G}_p(x)$ is the p-Green function with pole $o \in M$, it satisfies $\Delta_p \mathcal{G}_p(x) = 0$ for all $x \neq o$ and, thus, can be used as weight in Theorem 3.7. Our interest is then to look for asymptotic estimates for the p-Green function of (M, g) and its gradient, so to better control the growth at infinity of the weights in (3.26). One possibility is to use Li–Yau type estimates, which are ensured under several lower bounds on Ricci. These, however, are not sufficient because they provide only an upper bound on $\nabla \log \mathcal{G}_p$.

Thus, instead of using the p-Green function of (M, g) directly, we use the p-Green function of the model manifold (N, h) constructed in Section 3.1.2 which is p-superharmonic outside a large enough compact set and whose estimates are already available, see Corollary 3.5.

Notice also that $G_p(x) \to 0$ as $r(x) \to +\infty$, hence, $G_p(x)$ is distant from 1 provided that $r(x) >> 1$. In other words, $G_p(x)$ is a suitable weight in Theorem 3.7 as long as we are outside a large enough compact set containing the pole.

Proposition 3.9. *Let (M, g) be a Cartan–Hadamard manifold satisfying* (3.20).
For $p > 1$ and $\beta \geq 0$ there exists a compact K containing the pole such that

$$\int_\Omega \frac{|\nabla G_p|^p}{|G_p|^p}(-\log G_p)^{\beta p}|f|^p d\mu_g \leq \left(\frac{p}{p-1}\right)^p \int_\Omega (-\log G_p)^{\beta p}|\nabla f|^p d\mu_g, \tag{3.27}$$

for all $f \in C_c^\infty(\Omega)$ where $\Omega = M \setminus K$.

Using estimates (3.14) and (3.15) we deduce

$$\frac{|\nabla G_p|}{|G_p|}(r(x)) \sim D_5 r(x)^{\frac{\alpha}{2}}, \tag{3.28}$$

$$(-\log G_p(r(x))) \sim D_6 r(x)^{1+\frac{\alpha}{2}} \tag{3.29}$$

so that

$$|\log G_p|^{\beta} = \mathcal{O}(|\nabla \log G_p|) \tag{3.30}$$

provided that $\beta \leq \frac{\alpha}{2+\alpha}$.

Note that Proposition 3.9 requires f to be smooth and compactly supported in Ω. Both assumptions, however, can be weakened as long as the support of f is far away from the pole o.

Theorem 3.10. *Let (M, g) be a Cartan–Hadamard manifold satisfying* (3.20)*. For $p > 1$ and $0 \leq \beta \leq \frac{\alpha}{\alpha+2}$ there exists a compact K containing the pole such that*

$$\int_M \frac{|\nabla G_p|^p}{|G_p|^p}(-\log G_p)^{\beta p}|f|^p d\mu_g \leq \left(\frac{p}{p-1}\right)^p \int_M (-\log G_p)^{\beta p}|\nabla f|^p d\mu_g, \tag{3.31}$$

for all $f \in W^{1,p}(M)$ with $\mathrm{supp}(f) \cap K = \emptyset$.

Proof. We proceed by steps, gradually weakening the assumptions on f.

Step 1 We begin by considering $f \in W^{1,p}(M)$ compactly supported in $\Omega = M \setminus K$ so that $f \in W_0^{1,p}(\Omega)$, i.e., there exists $u_n \in C_c^\infty(\Omega)$ such that $u_n \to f$ in $W^{1,p}$ norm. Note that the u_n can be chosen so that $\mathrm{supp}(u_n)$ and $\mathrm{supp}(f)$ are all contained in a compact $\Omega' \subset \Omega$. Then, by (3.27) we have

$$\int_M \frac{|\nabla G_p|^p}{|G_p|^p}(-\log G_p)^{\beta p}|u_n|^p d\mu_g \leq \left(\frac{p}{p-1}\right)^p \int_M (-\log G_p)^{\beta p}|\nabla u_n|^p d\mu_g. \tag{3.32}$$

Note that

$$\left|\int_M (-\log G_p)^{\beta p}(|\nabla u_n|^p - |\nabla f|^p) d\mu_g\right| \leq \sup_{\Omega'}(-\log G_p)^{\beta p} \int_M ||\nabla u_n|^p - |\nabla f|^p| d\mu_g,$$

so that

$$\int_M (-\log G_p)^{\beta p}|\nabla u_n|^p d\mu_g \to \int_M (-\log G_p)^{\beta p}|\nabla f|^p d\mu_g.$$

Similarly,

$$\int_M \frac{|\nabla G_p|^p}{|G_p|^p}(-\log G_p)^{\beta p}|u_n|^p d\mu_g \to \int_M \frac{|\nabla G_p|^p}{|G_p|^p}(-\log G_p)^{\beta p}|f|^p d\mu_g.$$

Hence, passing to the limit in (3.32) we obtain the validity of (3.31) for all $f \in W^{1,p}(M)$ compactly supported in Ω.

Step 2 Next, let $f \in W^{1,p}(M)$ such that $\mathrm{supp}(f) \cap K = \emptyset$ and consider a family of cutoffs $\chi_R \in C^\infty(M)$ such that $\chi \equiv 1$ on B_R, $\chi_R \equiv 0$ outside B_{2R} and $|\nabla \chi_R| \leq C$ uniformly on R. Such a family exists on any complete Riemannian manifold, see [45]. Consider $f\chi_R \in W^{1,p}(M)$, clearly $\mathrm{supp}(f\chi_R) \subseteq M \setminus K$ is compact. Then, by Step 1 (with $\beta = 0$) we have

$$\begin{aligned}\int_M \frac{|\nabla G_p|^p}{|G_p|^p}|f|^p|\chi_R|^p d\mu_g &\leq \left(\frac{p}{p-1}\right)^p 2^{p-1}\left(\int_M |f|^p|\nabla\chi_R|^p d\mu_g + \int_M |\nabla f|^p|\chi_R|^p d\mu_g\right)\\ &\leq \left(\frac{p}{p-1}\right)^p 2^{p-1}\left(\int_M |\nabla f|^p d\mu_g + \int_{B_{2R}\setminus B_R}|f|^p d\mu_g\right).\end{aligned}$$

Notice that the LHS converges to $\int_M |\nabla \log G_p|^p|f|^p d\mu_g$ by monotone convergence, on the other hand $\int_{B_{2R}\setminus B_R}|f|^p d\mu_g \to 0$ since $f \in L^p(M)$. We conclude that

$$\int_M |\nabla \log G_p|^p|f|^p d\mu_g \leq \left(\frac{p}{p-1}\right)^p 2^{p-1}\int_M |\nabla f|^p d\mu_g \tag{3.33}$$

for all $f \in W^{1,p}(M)$ with $\mathrm{supp} \cap K = \emptyset$.

Step 3 Using Step 2, we now prove the more general (3.31) under the assumptions that $f \in W^{1,p}(M)$ and $\mathrm{supp}(f) \cap K = \emptyset$. Indeed, let $\chi_R \in C^\infty(M)$ be as in Step 2 so that $f\chi_R$ is compactly supported in $M \setminus K$, by Step 1 we have

$$\begin{aligned}&\int_M |\nabla \log G_p|^p(-\log G_p)^{\beta p}|f|^p|\chi_R|^p d\mu_g \leq\\ &\left(\frac{p}{p-1}\right)^p 2^{p-1}\left(\int_M (-\log G_p)^{\beta p}|f|^p|\nabla\chi_R|^p d\mu_g + \int_M (-\log G_p)^{\beta p}|\nabla f|^p|\chi_R|^p d\mu_g\right).\end{aligned}$$

Here, we reason as in Step 2. The only difference is the following estimate which is a consequence of (3.30) and (3.33):

$$\int_M (-\log G_p)^{\beta p}|f|^p d\mu_g \leq C\int_M |\nabla \log G_p|^p|f|^p d\mu_g \leq C\left(\frac{p}{p-1}\right)^p 2^{p-1}\int_M |\nabla f|^p d\mu_g$$

where $C > 0$. Since $|\nabla f| \in L^p(M)$ we are still able to conclude that

$$\int_{B_{2R}\setminus B_R}(-\log G_p)^{\beta p}|f|^p d\mu_g \to 0 \quad R \to +\infty.$$

□

Remark 3.11. Note that both in Proposition 3.9 and Proposition 3.9, we have not used the estimates (3.15), (3.14) or (3.30) so a bound of the form (3.20) is not really necessary. It would be sufficient to have sectional curvatures bounded by -1, at least asymptotically. This is enough to ensure that the function G_p is p-superharmonic and distant from 1. We also point out that if $W^{1,p}(M)$ the RHS of (3.31) can be infinite although in this case the inequality holds trivially.

If we require $f \in W^{2,p}(M)$ and apply (3.31) twice, we obtain the following second order Hardy-type inequality.

Theorem 3.12. *Let (M,g) be a Cartan–Hadamard manifold satisfying (3.20). For $p>1$ and $0 \le \beta \le \frac{\alpha}{2+\alpha}$ there exists a compact K containing the pole such that*

$$\int_M \frac{|\nabla G_p|^p}{|G_p|^p}(-\log G_p)^{\beta p}|f|^p d\mu_g \le C \int_M |\nabla^2 f|^p d\mu_g, \tag{3.34}$$

for all $f \in W^{2,p}(M)$ such that $\operatorname{supp}(f) \cap K = \emptyset$, where and $C = C(p,K) > 0$.

Proof. Using Theorem 3.10 and (3.30) we have

$$\int_M \frac{|\nabla G_p|^p}{|G_p|^p}(-\log G_p)^{\beta p}|f|^p d\mu_g \le C \int_M |\nabla \log G_p|^p |\nabla f|^p d\mu_g.$$

Since $|\nabla f| \in W^{1,p}(M)$ with $\operatorname{supp}(|\nabla f|) \cap K = \emptyset$ we apply (3.31) with $\beta = 0$ to $|\nabla f|$ and conclude using Kato's inequality $|\nabla|\nabla f|| \le |\nabla^2 f|$. $\square$

Note that (3.34) is more of a second-order Hardy-type inequality rather then a proper Rellich inequality. The reason being that the RHS is estimated with the L^p-norm of the Hessian rather than the Laplacian of f. The optimal value for β in (3.34) is $\beta = \frac{\alpha}{2+\alpha}$, in this case we have:

$$\frac{|\nabla G_p|}{|G_p|}(-\log G_p)^{\frac{\alpha}{2+\alpha}} \sim D_7 r(x)^{\alpha}$$

which is the fastest growth we are able to control via (3.34). Finally, we observe that no assumption on the support of f is needed as long as the weight has support distant from the pole. This is the kind of control needed for applications.

Theorem 3.13. *Let (M,g) be a Cartan–Hadamard manifold satisfying (3.20). For $p>1$ and K as in Theorem 3.12, let $\omega \ge 0$ be a measurable function such that $\operatorname{supp}(\omega) \cap K = \emptyset$ and $\omega(x) = \mathcal{O}(r^{\alpha}(x))$ on M, then $W^{2,p}(M) \hookrightarrow L^p(M, \omega^p d\mu_g)$.*

Proof. In order to extend the support of f, we need to remove the possible problems around the pole. To do so, let K' a compact set such that $K \subseteq K' \subseteq M \setminus \operatorname{supp}(\omega)$ and let $\varphi \in C^{\infty}(M)$ be a cutoff function such that $\varphi \equiv 0$ on K and $\varphi \equiv 1$ outside K'. Note that $|\nabla \varphi|$ and $|\nabla^2 \varphi|$ are bounded and that $f\varphi \in W^{2,p}(M)$ with $\operatorname{supp}(f\varphi) \cap K = \emptyset$, then by Theorem 3.12 we have

$$\begin{aligned}
\int_M \omega^p |f|^p d\mu_g = \int_\Omega \omega^p |f\varphi|^p d\mu_g &\le C' \int_M \frac{|\nabla G_p|^p}{|G_p|^p}(-\log G_p)^{\frac{\alpha}{2+\alpha}p} |\varphi f|^p d\mu_g \\
&\le C \int_\Omega |\nabla^2(\varphi f)|^p d\mu_g \\
&\le C \int_\Omega |\nabla^2 f|^p d\mu_g + C \int_\Omega |\nabla \varphi|^p |\nabla f|^p d\mu_g + C \int_\Omega |\nabla^2 \varphi|^p |f|^p d\mu_g \\
&\le C \|f\|^p_{W^{2,p}(M)}.
\end{aligned}$$

$\square$

As a direct consequence, if we have a family of weights $\{\omega_R\}$ whose growth is suitably controlled and whose supports vanish at $+\infty$ then $\|\omega_R f\|_{L^p} \to 0$.

Corollary 3.14. *Let $p > 1$ and (M, g) as in Theorem 3.13. Let $f \in W^{2,p}(M)$ and $\{\omega_R\} \subseteq C^\infty(M)$ non-negative such that* $\mathrm{supp}(\omega_R) \subseteq M \setminus \overline{B}_R$ *with $R >> 1$ and $\omega_R(x) \leq Cr^\alpha(x)$, then*

$$\int_M \omega_R^p |f|^p d\mu_g \to 0$$

as $R \to +\infty$.

Remark 3.15. Note that if we assume lower regularity in f, namely, $f \in W^{1,p}(M)$ we are still able to control $\|\omega f\|_{L^p(M)}$ as long as $\omega(x) \leq Cr^{\frac{\alpha}{2}}(x)$. The strategy here is the same of Theorem 3.13 but instead of the second order Hardy-type inequality (3.34), we use the first order inequality (3.31) with $\beta = 0$. Similarly, if we take a family of weights $\{\omega_R\}$ such that $\omega_R(x) \leq Cr^{\frac{\alpha}{2}}(x)$ and $\mathrm{supp}(\omega_R) \subseteq M \setminus \overline{B}_R$, we are still able to conclude that $\|\omega_R f\|_{L^p} \to 0$.

3.3 Density in $W^{2,p}$

In the following section, we apply the estimates developed in Section 3.2 to the density problem of smooth and compactly supported functions in the Sobolev space $W^{2,p}(M)$. To this aim, we construct via the Riemannian distance a family of smooth cutoff functions $\{\chi_R\}$, which we control up to the second covariant derivative. On arbitrary Riemannian manifolds there are two obstacles to this construction: the Riemannian distance might fail to be smooth on $M \setminus \{o\}$ and, while $|\nabla r|$ is always bounded, $|\nabla^2 r|$ might grow uncontrollably. In the case of Cartan–Hadamard manifolds, however, both difficulties can be overcome. Indeed, the cut locus of M is empty, which implies smoothness of the Riemannian distance. Furthermore, a lower bound on the radial Ricci curvature allows to control the Hilbert-Schmidt norm of $\nabla^2 r$ as proved in Proposition 3.6. Using these second order estimates on the Riemannian distance, we construct $\{\chi_R\}$ by composing with a sequence of real cutoffs.

Lemma 3.16. *Let (M, g) be a Cartan–Hadamard manifold satisfying*

$$\mathrm{Ric}_o(x) \geq -br^\beta(x)$$

outside a compact set for some $b > 0$ and $\beta \geq 0$. Then, there exists a family of smooth cutoffs $\{\chi_R\} \in C_c^\infty(M)$ with $R >> 1$ such that
(1) $\chi_R \equiv 1$ on B_R and $\chi_R \equiv 0$ on $M \setminus \overline{B_{2R}}$;
(2) $|\nabla \chi_R| \leq \frac{C}{R}$;
(3) $|\nabla^2 \chi_R| \leq CR^{\frac{\beta}{2}-1}$,
with $C > 0$.

Proof. Fix $\phi : \mathbb{R} \to [0,1]$ a smooth function such that $\phi \equiv 1$ on $(-\infty, 1]$ and $\phi \equiv 0$ on $[2, +\infty)$, and let $a > 0$ such that $|\phi'|+|\phi''| \leq a$ uniformly on $\mathbb{R}$. For $R >> 1$ (it suffices $R \geq R_1$, with R_1 as in Proposition 3.6), let

$$\phi_R(t) \coloneqq \phi\left(\frac{t}{R}\right)$$

so that

$$|\phi'_R| \leq \frac{a}{R}, \qquad |\phi''_R| \leq \frac{a}{R^2}.$$

Then, define $\chi_R(x) \coloneqq \phi_R \circ r(x)$, we have $\chi_R \equiv 1$ on B_R and $\chi_R \equiv 0$ on $M \setminus B_{2R}$. Furthermore,

$$\begin{aligned}
|\nabla \chi_R| &\leq |\phi'_R(r(x))||\nabla r(x)| \leq \frac{C}{R} \\
|\nabla^2 \chi_R| &\leq |\phi'_R(r(x))||\nabla^2 r(x)|+|\phi''_R(r(x))||\nabla r(x)|^2 \leq C R^{\frac{\beta}{2}-1},
\end{aligned}$$

where the constant C depends on a and on the constant of Proposition 3.6. $\square$

Remark 3.17. The above construction of the Hessian cutoffs is not the only possible one. It is worth noticing that the family $\{\chi_R\}$ can be constructed on Riemannian manifolds without any topological restrictions as long as one of the following assumptions holds:

(a) $|\mathrm{Ric}|(x) \leq B^2 r^{\beta}(x)$ and $\mathrm{inj}(x) \geq i_0 r^{-\frac{\beta}{2}}(x) > 0$
(b) $|\mathrm{Sect}|(x) \leq B^2 r^{\beta}(x)$,

for some $B, i_0 > 0$ and $\beta \geq 0$. In this setting, although the Riemannian distance might lose smoothness, it is possible to construct a distance-like function $H \in C^{\infty}(M)$ such that

(i) $C^{-2} r(x) \leq H(x) \leq \max\{r(x), 1\}$;
(ii) $|\nabla H(x)| \leq 1$;
(iii) $|\nabla^2 H(x)| \leq C \max\{r^{\frac{\beta}{2}}(x), 1\}$,

for some $C > 1$, see [78, Theorem 1.2]. Then, one defines $\chi_R = \phi_R \circ H(x)$ where ϕ_R is a family of real cutoffs in a similar fashion to Lemma 3.16.

We can now prove the density of smooth compactly supported functions in the Sobolev space $W^{2,p}$. To obtain this, we assume a double-sided bound on the radial Ricci curvature. The bound from below allows the construction of the smooth cutoff functions, while the bound from above ensures the validity of the functional estimates in Section 3.2.

Theorem 3.18. *Let (M, g) be a Cartan–Hadamard manifold satisfying*

$$-b r^{2\alpha+2}(x) \leq \mathrm{Ric}_o(x) \leq -a r^{\alpha}(x)$$

outside a compact set, for some $a, b > 0$ and $\alpha \geq 0$. Then $W_0^{2,p}(M) = W^{2,p}(M)$ for all $1 < p < +\infty$.

Proof. Since $C^\infty(M) \cap W^{2,p}(M)$ is dense in $W^{2,p}(M)$ (see [59]), it suffices to show that $C_c^\infty(M)$ is dense in $C^\infty(M) \cap W^{2,p}(M)$ with respect to the $W^{2,p}$ norm. To this goal, take $f \in C^\infty(M) \cap W^{2,p}(M)$ and consider a family of cutoffs $\{\chi_R\} \subseteq C^\infty(M)$ as in Lemma 3.16. Define $f_R \coloneqq \chi_R f \in C_c^\infty(M)$ and observe that

$$\|(f_R - f)\|_{L^p} = \|(\chi_R - 1)f\|_{L^p} \tag{3.35}$$

$$\|\nabla(f_R - f)\|_{L^p} \leq \|f\nabla\chi_R\|_{L^p} + \|(\chi_R - 1)\nabla f\|_{L^p} \tag{3.36}$$

$$\|\nabla^2(f_R - f)\|_{L^p} \leq 2\||\nabla f||\nabla\chi_R|\|_{L^p} + \|(\chi_R - 1)\nabla^2 f\|_{L^p} + \|f\nabla^2\chi_R\|_{L^p}. \tag{3.37}$$

Since $\nabla\chi_R$ and $(\chi_R - 1)$ are uniformly bounded and supported in $M \setminus \overline{B}_R$, $f \in W^{2,p}(M)$ implies that the RHS of (3.35), (3.36), and (3.37) except the last term, vanish as $R \to +\infty$. We only need to show that $||f\nabla^2\chi_R||_{L^p} \to 0$ as $R \to 0$. To see this, it is sufficient to observe that $|\nabla^2\chi_R| \leq Cr^\alpha$ and $\operatorname{supp}(\chi_R) \subseteq M \setminus \overline{B}_R$, then by Corollary 3.14 we conclude the proof (the upper bound on Ricci assumed in this theorem is of the type (3.20) for appropriate values of A and R_0). □

Remark 3.19. When $p = 1$ our strategy to construct Hardy-type inequalities fails. Note for instance that the constant in (3.26) and subsequent derived inequalities explodes as $p \to 1$. Nevertheless, we expect the density result to hold even when $p = 1$.

3.4 An L^2-Calderón–Zygmund inequality

As a further application of the tools developed in Section 3.2, we prove the validity of a L^2-Calderón–Zygmund inequality on Cartan–Hadamard manifolds with bounds on Ricci curvature. Using Bochner inequality and integration by part, we first prove a weighted CZ(2) inequality, which holds under lower bounds on the Ricci curvature.

Theorem 3.20. *Let (M, g) be a Cartan–Hadamard manifold with a fixed pole $o \in M$. Suppose*

$$\operatorname{Ric}(x) \geq -br^\beta(x)$$

holds outside a compact set in the sense of quadratic forms for some $b > 0$ and $\beta \geq 0$. Then, for every $\varepsilon > 0$ there exists a constant $A_1 = A_1(\varepsilon) > 0$ such that

$$\|\nabla^2\varphi\|_{L^2} \leq A_1 \left[\|\Delta\varphi\|_{L^2} + \|\varphi\|_{L^2}\right] + A_2\varepsilon^2\|r^\beta\varphi\|_{L^2} \qquad \forall\varphi \in C_c^\infty(M). \tag{3.38}$$

Here A_2 is a fixed positive constant independent of ε.

Proof. Take $\varphi \in C_c^\infty(M)$ and let $K \subseteq M$ be a compact set such that $\operatorname{Ric}(x) \geq -br^\beta(x)$

on $M \setminus K$. Using Bochner's inequality and integration by parts we have

$$\begin{aligned}\int_M |\nabla^2\varphi|^2 &= -\int_M \langle \nabla\varphi, \nabla\Delta\varphi\rangle - \int_M \mathrm{Ric}(\nabla\varphi,\nabla\varphi)\\ &= \int_M (\Delta\varphi)^2 - \int_K \mathrm{Ric}(\nabla\varphi,\nabla\varphi) - \int_{M\setminus K}\mathrm{Ric}(\nabla\varphi,\nabla\varphi)\\ &\le \int_M (\Delta\varphi)^2 + C\int_K |\nabla\varphi|^2 + b\int_{M\setminus K} r^\beta |\nabla\varphi|^2\\ &\le \int_M (\Delta\varphi)^2 + C\int_K |\nabla\varphi|^2 + b\int_{M\setminus K} r^\beta \left(\frac{1}{2}\Delta\varphi^2 - \varphi\Delta\varphi\right)\end{aligned}$$

where $C = -\min_K \mathrm{Ric} > 0$. Fix $\varepsilon > 0$, by Hölder we have

$$\left|\int_{M\setminus K} r^\beta\varphi\Delta\varphi\right| \le \left|\int_M r^\beta\varphi\Delta\varphi\right| \le ||r^\beta\varphi||_{L^2}||\Delta\varphi||_{L^2} \le \frac{1}{4\varepsilon^2}||\Delta\varphi||^2_{L^2} + \varepsilon^2||r^\beta\varphi||^2_{L^2}.$$

We only need to estimate the integral of $r^\beta\Delta\varphi^2$. To this end, let $\rho \in C^\infty(M)$ be a positive function such that $\rho = r^\beta$ on $M \setminus K$. Then, there exists some constant $C > 0$ such that $|\nabla\rho| \le C\rho$. Hence,

$$\begin{aligned}\left|\int_{M\setminus K}\frac{1}{2}r^\beta\Delta\varphi^2\right| &\le \left|\int_M \frac{1}{2}\rho\Delta\varphi^2\right| = \left|\int_M \varphi\langle\nabla\rho,\nabla\varphi\rangle\right| \le C\int_M \rho|\varphi||\nabla\varphi|\\ &\le C||\rho\varphi||_{L^2}||\nabla\varphi||_{L^2} \le \frac{C^2}{4\varepsilon^2}||\nabla\varphi||^2_{L^2} + \varepsilon^2||\rho\varphi||^2_{L^2}\\ &\le \frac{C^2}{4\varepsilon^2}||\nabla\varphi||^2_{L^2} + \varepsilon^2||r^\beta\varphi||^2_{L^2} + \varepsilon^2 C||\varphi||^2_{L^2}.\end{aligned}$$

Combining the last two estimates and using the L^2-gradient estimate, which holds on any complete manifold, yields

$$\int_M |\nabla^2\varphi|^2 \le A_1(\varepsilon)\left(\int_M |\Delta\varphi|^2 + \int_M |\varphi|^2\right) + b\varepsilon^2\int_M r^{2\beta}|\varphi|^2.$$

This conclude the proof. □

Remark 3.21. In a first draft, Theorem 3.20 was proved using a different strategy. The proof presented in the above was suggested to us by G. Carron during his review of this thesis. We refer to [89] for the original proof, which relies on a carefully constructed conformal deformation. It should be noted that the strategy of [89] does not use the Bochner inequality and is thus better suited to be extended to $p \neq 2$. To do this, however, one would need the validity of an *infinitesimal* $CZ(p)$ inequality of the type

$$||\nabla^2\varphi||^p_{L^p} \le \varepsilon^2 C||\varphi||^p_{L^p} + C\left(1 + \frac{1}{\varepsilon^2}\right)||\Delta\varphi||^p_{L^p}$$

which, to the best of our knowledge, is not known when $p \neq 2$.

Remark 3.22. Note that in Theorem 3.20 we require a bound on Ricci in the sense of quadratic forms, that is

$$\mathrm{Ric}(X,X)(x) \geq -br^{\beta}(x)g(X,X)$$

for any $X \in T_xM$. This is a stronger assumption than the previous bounds on radial Ricci curvature, needed to use the Bochner formula.

If we also assume an upper bound on the Ricci curvature, using the second order Hardy-type inequality (3.34) we can estimate the last term on (3.38) thus proving CZ(2).

Theorem 3.23. *Let (M,g) be a Cartan–Hadamard manifold. Suppose*

$$-br^{\alpha}(x) \leq \mathrm{Ric}(x) \leq -ar^{\alpha}(x) \tag{3.39}$$

holds outside a compact set, for some $a,b > 0$ and $\alpha \geq 0$. Then, the following L^2-Calderón–Zygmund inequality holds on M:

$$\|\nabla^2\varphi\|_{L^2} \leq C\left(\|\Delta\varphi\|_{L^2} + \|\varphi\|_{L^2}\right) \tag{3.40}$$

for all $\varphi \in C_c^{\infty}(M)$.

Proof. By Theorem 3.20 we have the validity of (3.38), thus, we only need to estimate the weighted term $\|r^{\beta}\varphi\|^2_{L^2}$. Let K be a compact large enough (see Theorem 3.13), then

$$\|r^{\beta}\varphi\|^2_{L^2} = \int_M r^{2\beta}\varphi^2 d\mu_g \leq \max_K r^{2\beta} \int_K \varphi^2 d\mu_g + \int_{M\setminus K} r^{2\beta}\varphi^2 d\mu_g.$$

Thanks to Theorem 3.13 we have

$$\int_{M\setminus K} r^{2\beta}\varphi^2 d\mu_g \leq C' \int_M |\nabla^2\varphi|^2 d\mu_g,$$

so that

$$\|\nabla^2\varphi\|_{L^2} \leq A'\left(\|\Delta\varphi\|_{L^2} + \|\varphi\|_{L^2}\right) + A''\varepsilon^2\left(\|\varphi\|^2_{L^2} + \|\nabla^2\varphi\|^2_{L^2}\right).$$

Since ε can be made arbitrarily small and A'' is a fixed constant, this last estimate yields (3.40). $\square$

Since CZ(2) implies equality of the various definitions of Sobolev spaces, Remark 1.13, Theorem 3.23 implies $W_0^{2,2}(M) = W^{2,2}(M)$, although under a stricter pinching than the one of Theorem 3.18.

Corollary 3.24. *Let (M,g) be a Cartan–Hadamard manifold as in Theorem 3.23, then*

$$W_0^{2,2}(M) = W^{2,2}(M) = H^{2,2}(M).$$

Part II

Positivity preservation for Schrödinger operators

Introduction to Part II

Part II of this thesis deals with positivity preservation properties of Schrödinger operators on Riemannian manifolds. The results of the next chapters have been obtained in collaboration with Andrea Bisterzo and Giona Veronelli in [89, 18].

Self-adjoint operators on Hilbert spaces play a key role in mathematical physics, as they represent the observables in the Dirac–Von Neumann interpretation of quantum mechanics. The energy, for instance, is represented by a class of second-order differential operators on L^2, known in literature as Schrödinger-type operators. These operators usually come with a natural domain of definition on which, although symmetric, might lack self-adjointness. To recover this important property, one usually starts by defining the operator on smooth and compactly supported functions and tries to construct a self-adjoint extension. When the extension is unique, the operator is said to be essentially self-adjoint. This property, especially in the case of Schrödinger-type operators, is extensively studied both in functional analysis and mathematical physics. In the case of Riemannian manifolds, the first results on the topic are due to Gaffney, [45], who dealt with the case of the Laplace–Beltrami operator on geodesically complete manifolds, see also Strichartz, [117]. Since then, the problem has seen numerous contributions with a variety of approaches, we refer to [20, Appendix D] for a nice historical account.

A possible strategy to prove the essential self-adjointness of Schrödinger-type operators, first proposed in [79] for the Euclidean case, is to use the so called Kato's inequality, which states that if $u \in L^1_{\text{loc}}(M)$ and $\Delta u \in L^1_{\text{loc}}(M)$ then

$$\Delta |u| \geq \operatorname{sign}(u) \Delta u$$

where the inequality is intended in the distributional sense. Indeed, consider the operator $H := -\Delta + V : C_c^\infty(M) \to L^2(M)$ with a potential $0 \leq V \in L^2_{\text{loc}}(M)$. By functional analysis, its essential self-adjointness is equivalent to the fact that the only $L^2(M)$ distributional solution of

$$(-\Delta + V + 1) f = 0$$

is $f \equiv 0$, see [110, Theorem X.26]. Applying Kato's inequality and using the fact that $V \geq 0$ yields

$$(-\Delta + 1)(-|f|) \geq 0,$$

which motivates the case $p = 2$ of the following definition, proposed by Güneysu in [60].

Definition II.1. Let $p \in [1, +\infty]$, we say that (M, g) has the *L^p-positivity preserving property* if every $u \in L^p(M)$ satisfying

$$(-\Delta + 1)u \geq 0$$

in the sense of distributions is non-negative a.e.

According to the above argument, if a manifold has the L^2-positivity preserving property then H is clearly essentially self-adjoint. This fact was first observed by Braverman, Milatovic and Shubin in [20], while dealing with the more general case of covariant Schrödinger operators on Hermitian vector bundles. On the other hand, if the underlying manifold is geodesically complete, the essential self-adjointness of H and their covariant counterparts can be proved by other means, see [114, Theorem 1.1] or [20] and [66]; if $V = 0$ this is in fact Gaffney and Strichartz's result, [45, 117]. This lead Braverman, Milatovic and Shubin to formulate the following

Conjecture (BMS). *If (M, g) is a geodesically complete Riemannian manifold then the L^2-positivity preserving property holds.*

The BMS conjecture has remained open for 20 years and has only recently been solved in the positive by Pigola and Veronelli, [107]. For further reference on the topic, we refer to the nice survey of Güneysu [61], see also [62, Section XIV.5] and [20, Appendix B].

While the problem of essential self-adjointness of Schrödinger-type operators motivates the case $p = 2$, it is reasonable to consider the L^p-positivity preserving property on the whole L^p scale. Indeed, the case $p \in (1, +\infty)$ has consequences on the m-accretivity of Schrödinger-type operators on $L^p(M)$, see [110, p.240] for the precise definitions. Consider $E \coloneqq -\Delta + V : C_c^\infty(M) \to L^p(M)$ with $0 \leq V \in L^\infty_{\mathrm{loc}}(M)$ and denote with $E_{p,\min}$ the closure of E in $L^p(M)$ and with $E_{p,\max}$ the extension of E to $\mathrm{Dom}(E_{p,\max}) = \{u \in L^p(M) : Eu \in L^p(M) \text{ distributionally}\}$. The validity of the L^p-positivity preserving property for $p \in (1, +\infty)$ ensures that $E_{p,\max}$ is m-accretive and $E_{p,\max} = E_{p,\min}$ which implies that $C_c^\infty(M)$ is an operator core for $E_{p,\max}$, see [92, 93] or [65, Appendix A].

The case $p = +\infty$, instead, is related to another functional property of manifolds: stochastic completeness. From a probabilistic perspective, stochastic completeness is the property of Brownian paths to have almost surely infinite lifetime, or equivalently, the fact that the (minimal) positive heat kernel of the Laplace–Beltrami operator preserves probability. For our scopes, however, we shall adopt the following (equivalent) definition, which is more relevant from the point of view of PDEs. We refer to Section 5.1 for more details on the characterizations of stochastic completeness.

Definition II.2. A Riemannian manifold (M, g) is said to be *stochastically complete* if the only bounded, non-negative C^2 solution of $\Delta u \geq u$ on M is $u \equiv 0$.

It was first observed by Güneysu, [60], that the L^∞-positivity preserving property implies stochastic completeness of the manifold at hand. In particular, stochastically incomplete manifolds, which can be easily constructed, provide counterexamples to the validity of the L^∞-positivity preserving property.

In the last years there have been significant efforts to better understand the L^p-positivity preserving property and to find geometric and analytic conditions ensuring its validity. In the Euclidean case, $M = \mathbb{R}^n$, the L^2-positivity preserving property was first proved by Kato, [79], long before the introduction of Güneysu's terminology. The proof relies on the fact that $-\Delta + 1$ induces an isomorphism on the space of tempered distributions whose inverse has a positive kernel. Note that this completes Kato's proof of the essential self-adjointness of H. In a Riemannian setting, however, tempered distributions are ill-defined, and it is necessary to adopt different strategies; in recent literature, we can identify two.

The first approach follows an idea of Davies contained in [20], and relies on the existence of a family of smooth cutoff functions with a uniform control on the gradient and Laplacian. Note that on geodesically complete manifolds one can always construct sequences of cutoffs with a uniformly bounded gradient, but controlling the Laplacian requires some geometrical assumption on curvature and injectivity radius. Using these Laplacian cutoffs, the L^p-positivity preserving property has been proved in the following settings for various ranges of p.

- Braverman, Milatovic and Shubin showed in [20] that complete manifolds with bounded geometry, i.e. $||\nabla^j \operatorname{Riem}||_{L^\infty} < +\infty$ for all $j \in \mathbb{N}$ and $\operatorname{inj}(M) > 0$, have the L^2-positivity preserving property.
- If $\operatorname{Ric} \geq -K^2$, the L^p-positivity preserving property was proved on the whole scale $p \in [1, +\infty]$ by Güneysu, [62]. See also [60] for a previous proof in the case $K = 0$.
- Bianchi and Setti further refined this result in [17], verifying that the BMS conjecture holds under the additional assumption that $\operatorname{Ric}(x) \geq -C(1 + r^2(x))$. Under these assumptions, in fact, the same proof yields the L^p-positivity preserving property of the manifold at hand for all $p \in [2, +\infty)$

Note that all of these results require some type of lower bound on the Ricci curvature which cannot grow too fast at $-\infty$.

Inspired by the approach based on Laplacian cutoff functions, in Chapter 4 we present two contributions to the problem of L^p-positivity preserving properties on Riemannian manifold; these results are collected in [89]. In Chapter 3 we proved that Cartan–Hadamard manifolds satisfying a lower bound on the Ricci curvature of the type $\operatorname{Ric}(x) \geq -br^\beta(x)$, admit sequences of cutoff functions with a uniformly bounded gradient and whose Laplacian, although non-bounded uniformly, grows at most like $r^{\beta/2-1}$, see Lemma 3.16. On the other hand, an upper bound of the form $\operatorname{Ric}(x) \leq -ar^\alpha(x)$ yields the inclusion of $W^{1,p}(M)$ in the weighted space $L^p(M, \mu)$ where $\mu = r^{\alpha p/2}$, see Remark 3.15. The combination of these two facts allows us to prove the following

Theorem II.3. *Let (M, g) be a Cartan–Hadamard manifold satisfying*

$$-b\,r^{\alpha+2}(x) \leq \mathrm{Ric}(x) \leq -a\,r^{\alpha}(x),$$

outside a compact set for some $a, b > 0$ and $\alpha \geq 0$. Then M has the L^p-positivity preserving property for all $p \in [2, +\infty)$.

Note that it is impossible to extend this result to $p = +\infty$. Indeed, Cartan–Hadamard manifolds satisfying $\mathrm{Ric}(x) \leq -ar^{\alpha}(x)$ for $\alpha > 2$ are stochastically incomplete, see Theorem 4.5 below, hence, the L^∞-positivity preserving property must fail. Conversely, we prove

Theorem II.4. *Let (M, g) be a complete Riemannian manifold satisfying*

$$-\lambda^2(r(x)) \leq \mathrm{Ric}(x)$$

outside a compact set, with λ given by

$$\lambda(t) = At\prod_{j=0}^{k}\log^{[j]}(t)$$

where $A > 0$, $k \in \mathbb{N}$ and $\log^{[j]}(t)$ stands for the j-th iterated logarithm. Then M has the L^p-positivity preserving property for any $p \in [1, \infty]$.

This theorem improves on previous results, both relaxing the lower bound and improving the range of p. Furthermore, it is sharp when $p = 1$ or $p = +\infty$, see Theorem II.6 and Remark 4.9 below. In addition to the use of a suitable family of Laplacian cutoffs constructed by Impera, Rimoldi and Veronelli in [78], a key step in the proof of Theorem II.4 is to show that for a given $0 \leq \varphi \in C_c^\infty(M)$, there exists a positive solution $v \in C^\infty(M) \cap W^{1,q}(M)$ of $(-\Delta + 1)v = \varphi$, $1/q = 1 - 1/p$. While standard elliptic regularity theory ensures that $v \in L^q(M)$ and thus $\Delta v \in L^q(M)$, the fact that $\nabla v \in L^q(M)$ is non-trivial. The results of Braverman, Milatovic and Shubin, Güneysu, Bianchi and Setti, as well as our Theorem II.3, all rely on the validity of L^q-gradient estimate GE(p) which are known to hold on any complete manifold if $q \in (1, 2]$ ($p \in [2, +\infty)$), and for all $q \in (1, +\infty)$ if the Ricci curvature is bounded from below either pointwise or in some integral sense. See [28, 32] as well as our Theorem 1.6 and Theorem 1.12. In the proof of Theorem II.4, instead, we rely on a refined Li–Yau gradient estimate proved in [17] to conclude that $|\nabla v|(x) \leq \lambda(r(x))v(x)$ outside of a compact set. Hence, ∇v is almost in L^q, which is enough for our purpose.

Using a completely different strategy, Pigola and Veronelli, [107], were finally able to prove the L^p-positivity preserving property for $p \in (1, +\infty)$ on any geodesically complete manifold, thus verifying that the BMS conjecture is true. Their proof uses some new regularity results for non-negative subharmonic distributions to prove that the L^p-positivity preserving property is implied by a Liouville-type property for L^p-subharmonic distributions. When $p \in (1, +\infty)$, this property is known to hold on geodesically complete manifolds thanks to a result of Yau, [128]. It should be noted that the result of Pigola and Veronelli includes our Theorem II.3, since Cartan–Hadamard manifolds are complete, as well as the range $p \in (1, +\infty)$ of Theorem II.4. However, the proof in [107] relies on potential theory results for subharmonic functions, we believe the approach via cutoffs might be suitable for more general operators. Furthermore, the result of Pigola and Veronelli fails when $p = 1$ and $p = +\infty$ because the improved regularity cannot be extended to these cases but also because there are known counterexample to the Liouville property of Yau.

Chapter 5 is devoted to the study of these limit cases and is based upon [18]. To the best of our knowledge, the most general condition ensuring L^1 and L^∞-positivity preserving properties is expressed by Theorem II.4. In particular, when $p = +\infty$ this is essentially the celebrated condition for stochastic completeness due to Hsu, [76], which is known to be optimal with respect to bounds on the Ricci curvature. This suggests a much closer relation between stochastic completeness and the L^∞-positivity preserving property. Indeed, we have the following

Theorem II.5. *Let (M, g) be a (possibly non-complete) Riemannian manifold, then M has the L^∞-positivity preserving property if and only if it is stochastically complete.*

This characterization together with the result of Pigola and Veronelli paint a full picture of the L^p-positivity preserving property when $p \in (1, +\infty]$, which is completed when $p = 1$ by showing that the condition expressed by Theorem II.4 is optimal in the following sense.

Theorem II.6. *For every $\varepsilon > 0$, there exists a 2-dimensional Riemannian manifold (M, g) whose Gaussian curvature satisfies*

$$K(x) \sim -Cr(x)^{2+\varepsilon},$$

such that the L^1-positivity preserving property fails on M.

The proof that stochastic completeness implies the L^∞-positivity preserving property is essentially a problem of regularity for the distributional, $L^\infty(M)$ solutions of $\mathcal{L}u \geq 0$, where

$$\mathcal{L} \coloneqq \Delta - 1.$$

Indeed, using a Brezis–Kato inequality we show that the desired result follows if we prove that for every bounded distributional solution of $\mathcal{L}u \geq 0$, there exists some $w \in C^\infty(M)$ with $u \leq w \leq C < +\infty$ which solves $\mathcal{L}w \geq 0$ in a strong sense. This latter result, then, follows from a monotone approximation theorem for the distributional solutions of $\mathcal{L}u \geq 0$ which might be of independent interest.

Theorem II.7. *Let (M, g) be a Riemannian manifold and let $u \in L^1_{\mathrm{loc}}(M)$ be a solution of $\mathcal{L}u \geq 0$ in the sense of distributions. Then for every $\Omega \Subset M$ there exists a sequence $\{u_k\} \subset C^\infty(\Omega)$ such that:*

(i) $u_k \searrow u$ *pointwise a.e.;*

(ii) $\mathcal{L}u_k \geq 0$ *for all* k*;*

(iii) $u_k \to u$ *in* $L^1(\Omega)$*;*

(iv) $\|u_k\|_{L^\infty(\Omega)} \leq 2\|u\|_{L^\infty(\Omega)}$ *and, if* $u \geq 0$, $\sup_\Omega u_k \leq 2\,\mathrm{ess\ sup}_\Omega\, u$.

Instead of proving this theorem directly for the operator $\mathcal{L}$, we rely on a trick due to Protter and Weinberger, [108], and prove an analogous monotone approximation result for an appropriate weighted Laplacian. The monotone approximation is obtained adapting a strategy proposed by Bonfiglioli and Lanconelli in [19], together with some mean value representation formulas for weighted harmonic functions.

Chapter 4

Positivity preserving properties via cutoff functions

In this chapter, we present two results on the L^p-positivity preserving property of certain Riemannian manifolds, which rely on the existence of a suitable family of smooth Laplacian cutoffs.

Let us fix (M, g) an arbitrary Riemannian manifold. Recall that $u \in L^p(M)$ (or more generally $u \in L^1_{\mathrm{loc}}(M)$) satisfies $(-\Delta + 1)u \geq 0$ *in the sense of distributions*, if

$$\int_M u(-\Delta + 1)\varphi \geq 0$$

for all test functions $\varphi \in C_c^\infty(M)$ with $\varphi \geq 0$. By Riesz Representation Theorem, this is equivalent to say that $\nu = (-\Delta + 1)u$ is a positive Radon measure. If one assumes stronger regularity, say $\phi = (-\Delta + 1)u$ is a (positive) smooth function, proving positivity preservation becomes a much simpler task. Indeed, following [20, Theorem B.1], we prove

Lemma 4.1. *Let $\phi \in C_c^\infty(M)$, $\phi \geq 0$, then there exists a unique $v \in C^\infty(M) \cap L^p(M)$ $\forall p \in [1, +\infty]$, $v > 0$, such that*

$$(-\Delta + 1)v = \phi. \tag{4.1}$$

Proof. Let $\{\Omega_k\}$ be an exhaustion of M by relatively compact, open sets of smooth boundary satisfying

$$\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega_k \Subset \Omega_{k+1} \Subset \cdots,$$

that is, Ω_k is relatively compact in Ω_{k+1} for all $k \in \mathbb{N}$. Furthermore, assume Ω_1 is large enough so that $\mathrm{supp}(\phi) \subseteq \Omega_1$. Let v_k be a smooth solution of the following Dirichlet problem:

$$\begin{cases} (-\Delta + 1)v_k = \phi & \text{on } \Omega_k \\ v_k = 0 & \text{on } \partial\Omega_k. \end{cases} \tag{4.2}$$

By strong maximum principle we immediately get that $v_k > 0$ in the interior of Ω_k and $v_{k+1} \geq v_k$ for all k, hence, $\{v_k\}$ admits a (possibly infinite) pointwise limit

$$0 < v(x) = \lim_{k \to +\infty} v_k(x).$$

Then, we multiply (4.2) by v_k^{p-1} and integrate over Ω_k

$$
\begin{aligned}
\int_{\Omega_k} v_k^{p-1}(-\Delta+1)v_k &= \int_{\Omega_k} v_k^p - \int_{\Omega_k} v_k^{p-1}\Delta v_k \\
&= \int_{\Omega_k} v_k^p + \int_{\Omega_k} \langle \nabla v_k^{p-1}, \nabla v_k\rangle \\
&= \int_{\Omega_k} v_k^p + (p-1)\int_{\Omega_k} v_k^{p-2}|\nabla v_k|^2 \geq \int_{\Omega_k} v_k^p.
\end{aligned}
$$

By Hölder's inequality, we conclude that $\|v_k\|_{L^p(\Omega_k)} \leq \|\phi\|_{L^p(M)}$. Since $\{v_k\}$ is uniformly bounded in L^p on any compact set, by standard interior regularity we deduce that $\{v_k\}$ is uniformly bounded in $W^{h,p}_{\mathrm{loc}}(M)$ for any order h and $p \in [1, +\infty)$. As a consequence of the compact embedding of Sobolev spaces, all the covariant derivatives of $\{v_k\}$ converge up to a subsequence uniformly on compact sets, i.e., v_k converges in $C^\infty(M)$ topology. In particular v is positive, smooth and satisfies (4.1); by Fatou's lemma we also have that $v \in L^p(M)$ for any $p \in [1, +\infty)$. For $p = +\infty$, let $x^* \in \overline{\Omega}_k$ be such $v_k(x^*) = \max_{\overline{\Omega}_k} v_k$, by weak maximum principle we get $v_k(x^*) \leq \phi(x^*) \leq \|\phi\|_{L^\infty(M)}$, hence, $||v_k||_{L^\infty(\Omega_k)} \leq ||\phi||_{L^\infty(M)}$. Letting $k \to +\infty$ we conclude that $v \in L^\infty(M)$. □

Remark 4.2. An alternative and shorter proof of Lemma 4.1 is possible using properties of the Firedrichs realization of $-\Delta$ and functional analytic arguments. See for example the first part of [62, Theorem XIV.31].

Next, with the aid of suitable Laplacian cutoffs, we extend the above result to the case where $(-\Delta+1)u$ is only a positive Radon measure. This requires further restrictions on the geometry of the manifolds we consider.

4.1 Cartan–Hadamard manifolds

Suppose (M, g) is a Cartan–Hadamard manifolds and denote $r(x) = d(x, o)$ the Riemannian distance from a fixed pole $o \in M$. Then we have

Theorem 4.3. *Let (M, g) be a Cartan–Hadamard manifold satisfying*

$$
-br^{\alpha+2}(x) \leq \mathrm{Ric}(x) \leq -ar^\alpha(x) \tag{4.3}
$$

outside a compact set for some constants $a, b > 0$ and some $\alpha \geq 0$. Then, M has the L^p-positivity preserving property for all $2 \leq p < +\infty$

Proof. Let $u \in L^p(M)$ such that $(-\Delta+1)u \geq 0$ in the sense of distributions. We need to show that

$$
\int_M \phi u \geq 0 \qquad \forall \phi \in C_c^\infty(M), \phi \geq 0.
$$

By Lemma 4.1, let $v \in C^\infty(M)$, $v > 0$ such that $(-\Delta+1)v = \phi$ and let $\{\chi_R\} \in C^\infty(M)$ be a family of cutoffs as in Lemma 3.16. Since $v\chi_R \in C_c^\infty(M)$, $v\chi_R \geq 0$ we have

$$\begin{aligned} 0 \leq \int_M u(-\Delta+1)(v\chi_R) &= \int_M [-u\Delta(v\chi_R) + v\chi_R u] \\ &= -\int_M u\chi_R \Delta v - \int_M uv\Delta\chi_R \\ &\quad - 2\int_M u\langle \nabla\chi_R, \nabla v\rangle + \int_M u\chi_R v \\ &= -\int_M uv\Delta\chi_R - 2\int_M u\langle\nabla\chi_R,\nabla v\rangle + \int_M u\chi_R\phi. \end{aligned}$$

Since ϕ has compact support, for R large enough we have

$$\int_M u\chi_R\phi = \int_M u\phi.$$

Moreover, $v \in L^q(M)$, hence $\Delta v \in L^q(M)$, for every $q \in [1,+\infty]$. In particular, this holds for $q = p/(p-1) \in (1,2]$ so that $|\nabla v| \in L^q(M)$ thanks to the validity of L^q-gradient estimates, see [32] or Remark 1.3. Since $u\nabla v \in L^1(M)$, we have

$$\int_M u\langle\nabla\chi_R,\nabla v\rangle \leq \int_M |u||\nabla v||\nabla\chi_R| \to 0$$

for $R \to +\infty$. Finally, by Holder's inequality we have

$$\left|\int_M uv\Delta\chi_R\right| \leq \left\{\int_M |u|^p\right\}^{\frac{1}{p}} \left\{\int_M |v\Delta\chi_R|^q\right\}^{\frac{1}{q}}.$$

Since $|\Delta\chi_R| \leq Cr^{\frac{\alpha}{2}}(x)$ and $v \in W^{1,q}(M)$, the upper bound on Ricci curvature, Remark 3.15, yields

$$\int_M |v\Delta\chi_R|^q \to 0$$

as $R \to +\infty$. Hence,

$$\int_M \phi u = \lim_{R\to+\infty} \int_M u(-\Delta+1)(v\chi_R) \geq 0,$$

which concludes the proof. □

Remark 4.4. Both the existence of the cutoffs and the inclusion of $W^{1,q}$ in the weighted space $L^q(M,\mu)$ where $\mu = r^{\alpha q/2}$, only require a bound on the Ricci tensor in the radial direction. As a consequence, Theorem 4.3 holds also if we require (4.3) only in the radial direction.

Although Lemma 4.1 holds on the whole L^p scale, the case $p = +\infty$ and $1 \leq p < 2$ have been left out in the previous theorem. For these values of p, we generally lack the L^q-gradient estimates for the conjugate exponent of p. Indeed, when $2 < q < +\infty$ the bound (4.3) does not fall in the assumptions of the result of Cheng, Thalmaier and Thompson [28] or of our Theorem 1.6 and Theorem 1.12. On the other hand, the L^1 and L^∞ gradient estimates are false even in the Euclidean setting.

In the case $p = +\infty$, there is an additional reason which prevents a proof of the L^∞-positivity preserving property under the assumptions of Theorem 4.3. As noted in the Introduction to Part II, the validity of the L^∞-positivity preserving property implies stochastic completeness of the manifold at hand. In the case of Cartan–Hadamard manifolds, however, if the Ricci curvature lies below a certain threshold we loose stochastic completeness.

Theorem 4.5. *Let (M, g) be a Cartan–Hadamard manifold whose radial Ricci curvature satisfies*

$$\mathrm{Ric}_o(x) \leq -ar^\alpha(x)$$

outside a compact set, with $a > 0$. If $\alpha > 2$, then (M, g) is not stochastically complete.

Proof. Let j be the warping function of the model (N, h) defined in Section 3.1.2. Take

$$v(t) = \int_0^t j^{1-n}(s) \left(\int_0^s j^{n-1}(\tau) d\tau \right) ds$$

then $u(x) = v(r(x))$ is a C^2 function on M. By (3.17) we have

$$\frac{\int_0^s j^{n-1}(\tau)d\tau}{j^{n-1}(s)} \in L^1(+\infty),$$

hence, u is bounded. Since $v' \geq 0$, by Proposition 3.1 we have

$$\Delta u(x) = v''(r(x)) + \Delta r(x) v'(r(x)) \geq v''(r(x)) + (n-1)\frac{j'(r(x))}{j(r(x))} v'(r(x))$$

for $r >> 1$. By direct computation, this implies that $\Delta u \geq 1$ outside of a compact set. Let $\{x_k\} \subset M$ be a sequence of points such that $u(x_k)$ converges to $\sup_M u$. The monotonicity of v implies that $r(x_k) \to +\infty$, hence, $\Delta u(x_k) \geq 1$ for k large enough which implies that M is stochastically incomplete, see [104, Theorem 1.1]. □

4.2 Manifolds with subquadratic Ricci curvature

The case of Cartan–Hadamard manifolds suggests that a quadratic growth of Ric at $-\infty$ is the threshold distinguishing between different behavior with respect to positivity preservation properties. Note that in the subquadratic case, the cutoff functions constructed in Lemma 3.16 have a uniformly bounded Laplacian. As a consequence, one does not need to impose upper bounds on the Ricci curvature in order to control the terms involving $\Delta\chi_R$. It turns out that such Laplacian cutoffs exist on arbitrary complete Riemannian manifolds without topological assumptions as long as the Ricci curvature satisfies

$$\mathrm{Ric}(x) \geq -\lambda^2(r(x)) \tag{4.4}$$

outside a compact set. Here λ is a C^∞ function given by

$$\lambda(t) = At\prod_{j=0}^{k}\log^{[j]}(t) \tag{4.5}$$

for t large enough, where $A > 0$, $k \in \mathbb{N}$ and $\log^{[j]}(t)$ stands for the j-th iterated logarithm. Indeed, we have the following result of Impera, Rimoldi and Veronelli, which slightly generalizes a previous result of Bianchi and Setti.

Theorem 4.6 ([78, 17])**.** *Let (M, g) be a complete Riemannian manifold satisfying* (4.4)*. Then, there exists a family of smooth cut-off functions $\{\chi_R\} \subseteq C_c^\infty(M)$ with $R >> 1$ large enough, such that*
(1) $\chi_R \equiv 1$ on B_R and $\chi_R \equiv 0$ on $M \setminus \overline{B_{\gamma R}}$;
(2) $||\nabla\chi_R||_\infty \leq \frac{C}{\lambda(R)}$;
(3) $||\Delta\chi_R||_\infty \leq C$;
where $C > 0$, $\gamma > 1$ and λ is the function defined in (4.5)*.*

Using these cutoff functions, one can easily prove the L^p-positivity preserving property for $p \in [2, +\infty)$, where the L^q-gradient estimates for the conjugate exponent are available. This fact was first observed by Güneysu. However, there is no need to use L^p-gradient estimates: we can use a uniform Li–Yau gradient estimate, which is a special case of a result by Bianchi and Setti.

Theorem 4.7 ([17])**.** *Let (M, g) be a complete Riemannian manifold satisfying* (4.4)*. Let $R > r > 0$ and let $\gamma > 1$ and let $v : M \setminus \overline{B_r} \to \mathbb{R}$ be a C^2 function satisfying*

$$\begin{cases} v > 0 & \text{on } M \setminus \overline{B_r} \\ \Delta v = v. \end{cases} \tag{4.6}$$

Then, there exists a positive constant $C = C(n, \gamma, B) > 0$ such that

$$\frac{|\nabla v(x)|}{\lambda(R)} \leq Cv(x) \quad \forall x \in B_{\gamma R} \setminus \overline{B_R}. \tag{4.7}$$

Using these two results, we can prove the following:

Theorem 4.8. *Let (M, g) be a complete Riemannian manifold satisfying* (4.4)*. Then M has the L^p-positivity preserving property for all $p \in [1, +\infty]$.*

Proof. Let $u \in L^p(M)$, $p \in [1, +\infty]$ such that $(-\Delta+1)u \geq 0$ in the sense of distributions. Take $\phi \in C_c^\infty(M)$, $\phi \geq 0$ we need to show that

$$\int_M u\phi \geq 0.$$

Let $v \in C^\infty(M)$, $v > 0$ such that $-\Delta v + v = \phi$ and $v, \Delta v \in L^q(M)$ $\forall q \in [1, +\infty]$. We proceed as in the proof of Theorem 4.3, taking $v\chi_R$ as a test function in $(-\Delta + 1)u \geq 0$, where $\{\chi_R\}$ are the cutoff functions of Theorem 4.6 instead of the ones of Lemma 3.16. The proof differs from the one of Theorem 4.3 only in the estimates of the terms containing $\Delta\chi_R$ and $\nabla\chi_R$. The former is immediate: since $|\Delta\chi_R| \leq C$ we have $|uv\Delta\chi_R| \leq C|uv| \in L^1(M)$ hence

$$\int_M uv\Delta\chi_R \to 0$$

by dominated convergence as $R \to +\infty$. For the latter term, we observe that if $r > 0$ is large enough then $\Delta v = v$ on $M \setminus \overline{B_r}$ and $v > 0$, thus, we have the validity of the Li–Yau estimate of Theorem 4.7. Since $\nabla\chi_R$ is compactly supported in $B_{\gamma R} \setminus \overline{B_R}$, then

$$|u\langle\nabla\chi_R, \nabla v\rangle| \leq C_2|u|\frac{|\nabla v|}{\lambda(R)} \leq CC_2|u||v| \in L^1(M) \quad \forall x \in M.$$

It follows that

$$\int_M u\langle\nabla\chi_R, \nabla v\rangle \to 0$$

as $R \to +\infty$ which concludes the proof of the theorem. □

Remark 4.9. As a consequence of the case $p = +\infty$, we immediately get that the manifold at hand is stochastically complete. Note that Hsu proved in [76] that a manifold is stochastically complete if it is geodesically complete and $\mathrm{Ric}(x) \geq -\kappa(r(x))$, where κ is non-decreasing and $\int^\infty \kappa^{-1} = \infty$. Keeping in account that the choice of λ in our result can be slightly generalized, [78, Proposition 1.1], our function λ is essentially the maximal one admissible in order to fulfill $\int^\infty \lambda^{-1} = \infty$. Observe also, in light of Theorem 4.5, that the condition of Hsu is optimal with respect to bounds on Ricci curvature.

Chapter 5

The extremal cases $p = 1, +\infty$

As mentioned in the Introduction to Part II, after the completion of [89], the L^p-positivity preserving property was proved by Pigola and Veronelli for all $p \in (1, +\infty)$ on complete Riemannian manifolds, [107]. It should be noted that without geodesic completeness, the L^p-positivity preserving property fails. For instance, take $\mathbb{B}_1 \subseteq \mathbb{R}^2$ the Euclidean open ball of radius 1. The radial function $u(r) = -r$, which belongs to all L^p spaces for $p \in [1, +\infty]$, is a non-positive function satisfying $(-\Delta + 1)u \leq 0$ and thus contradicts the L^p-positivity preserving property. In light of this observation, the only cases left out by the result of Pigola and Veronelli are $p = 1, +\infty$. For these values of p, the best conditions ensuring the corresponding positivity preserving property is the one expressed in Theorem 4.8. In the case $p = +\infty$, Remark 4.9 suggests a much closer relation between the L^∞-positivity preserving property and stochastic completeness, which we investigate in this chapter.

5.1 L^∞-positivity preserving property and stochastic completeness

There are countless characterizations of stochastic completeness, a comprehensive account is beyond our scope, and we refer the reader to [51, 53, 104, 105] or the very recent [55]. Here, we recall only the characterizations relevant to our exposition. Given (M, g) a Riemannian manifold, the following are equivalent

(i) M is stochastically complete;
(ii) for every $\lambda > 0$, the only bounded, non-negative C^2 solution of $\Delta u \geq \lambda u$ is $u \equiv 0$;
(iii) for every $\lambda > 0$, the only bounded, non-negative C^2 solution of $\Delta u = \lambda u$ is $u \equiv 0$;
(iv) the only bounded, non-negative C^2 solution of $\Delta u = u$ is $u \equiv 0$.

For a proof of the equivalence, we refer to Theorem 6.2 in [51].

Remark 5.1. Note that the regularity required in the above and in Definition II.2 can be relaxed to $C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$, see for instance Section 2 of [2]. This fact is a consequence of a stronger version of Theorem 5.6 below.

As noted in the Introduction to Part II, a sufficient condition to stochastic completeness is the validity of the L^∞-positivity preserving property.

Proposition 5.2 ([60])**.** *If (M, g) has the L^∞-positivity preserving property, then it is stochastically complete.*

Proof. To see this, take $u \in C^2(M)$ a bounded and non-negative function satisfying $\Delta u \geq u$. Set $v = -u$ and apply the L^∞-positivity preserving property to v to conclude that $u \equiv 0$. $\square$

Remark 5.3. In contrast to the BMS conjecture and the result of Pigola and Veronelli, it is worthwhile noticing that stochastic completeness is generally unrelated to geodesic completeness. For example, warped products with Ricci curvature diverging at $-\infty$ faster than quadratically are geodesically but not stochastically complete while $\mathbb{R}^n \setminus \{0\}$ endowed with the Euclidean metric is stochastically complete but geodesically incomplete.

5.1.1 From stochastic completeness to the L^∞-positivity preserving property

In the following, we prove the converse of Proposition 5.2. Let (M, g) be a stochastically complete Riemannian manifold and take $u \in L^\infty(M)$ satisfying $(-\Delta + 1)u \geq 0$ in the sense of distributions. We want to show that u is non-negative almost everywhere or, equivalently, that the negative part $u_- = \max\{0, -u\} = (-u)_+$ vanishes a.e.. To this end, we use the following Brezis–Kato inequality for the operator

$$\mathcal{L} \coloneqq \Delta - 1. \tag{5.1}$$

due to Pigola and Veronelli.

Theorem 5.4 ([107]). *Given a Riemannian manifold (M, g), if $f \in L^1_{\mathrm{loc}}(M)$ satisfies $\mathcal{L}f \geq 0$ in the sense of distributions, then $f_+ \in L^1_{\mathrm{loc}}(M)$ and $\mathcal{L}f_+ \geq 0$ in the sense of distributions.*

Since $\mathcal{L}(-u) \geq 0$ we conclude that $\mathcal{L}u_- \geq 0$ in the sense of distributions. If u_- was a $C^2(M)$ function, stochastic completeness (see (i) at the beginning of Section 5.1) would allow us to conclude that $u_- \equiv 0$, hence $u \geq 0$. Note that, according to Remark 5.1, $u_- \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$ would be sufficient. In general, however, this is not the case and, as a matter of fact, it is a stronger requirement than what we actually need. Indeed, if we find $w \in C^2(M)$ such that $\sup_M w < +\infty$, $0 \leq u_- \leq w$ and $\mathcal{L}w \geq 0$, then stochastic completeness applied to w implies that w hence u_- are identically zero.

The existence of such function w is implied by the following corollary of Theorem II.7, whose proof is postponed to the next section.

Corollary 5.5. *Let (M, g) be a Riemannian manifold and let $u \in L^\infty(M)$ be a distributional solution of $\mathcal{L}u \geq 0$. Then, for every relatively compact $\Omega \Subset M$ there exists some $u_\Omega \in C^\infty(\Omega)$ which solves $\mathcal{L}u_\Omega \geq 0$ in a strong sense, such that $u \leq u_\Omega$ and $||u_k||_{L^\infty(\Omega)} \leq 2||u||_{L^\infty(\Omega)}$.*

Via a compactness argument, we use the functions u_Ω to construct the function w. The following theorem, proved by Sattinger in [112], also comes into aid as it allows to obtain $\mathcal{L}$-harmonic function from super and sub solutions of $\mathcal{L}u = 0$.

Theorem 5.6 ([112])**.** *Let $u_1, u_2 \in C^\infty(M)$ satisfy*

$$\mathcal{L}u_1 \geq 0, \quad \mathcal{L}u_2 \leq 0, \quad u_1 \leq u_2$$

on M. Then, there exists some $w \in C^\infty(M)$ such that

$$u_1 \leq w \leq u_2 \quad \text{and} \quad \mathcal{L}w = 0.$$

Remark 5.7. Theorem 5.6 is a weaker formulation of a much more general theorem, proved by Ratto, Rigoli and Véron, [109], for a wider class of functions, namely $u_1, u_2 \in C^0(M) \cap W^{1,2}_{\text{loc}}(M)$. This result goes under the name of sub and supersolution method or monotone iteration scheme. Note that the results of [109] hold for a larger class of second order elliptic operators. For a survey on the subject, we refer to Heikkilä and Lakshmikantham, [69].

Using the functions constructed locally in Corollary 5.5 together with an exhaustion procedure, we obtain the following:

Theorem 5.8. *Let (M, g) be a Riemannian manifold and let $u \in L^\infty(M)$ satisfy $\mathcal{L}u \geq 0$ in the sense of distributions. Then, there exists $w \in C^\infty(M)$ such that $u \leq w$, $\mathcal{L}w \geq 0$ in a strong sense and* $\sup_M w < +\infty$.

Proof. We begin by observing that if $u \in L^\infty(M)$ then, setting $c = \|u\|_{L^\infty(M)}$, we have

$$\mathcal{L}c = -c \leq 0 \text{ on } M.$$

Next, take $\{\Omega_h\}$ an exhaustion of M by relatively compact sets such that

$$\Omega_1 \Subset \Omega_2 \Subset \ldots \Subset \Omega_h \Subset \Omega_{h+1} \Subset \ldots \Subset M,$$

$\partial\Omega_h$ is smooth and $M = \cup_h \Omega_h$. On each set Ω_h we apply Corollary 5.5 and we obtain a sequence of functions $u_h \in C^\infty(\Omega_h)$ such that
(1) $u \leq u_h$ in Ω_h;
(2) $\mathcal{L}u_h \geq 0$ strongly on Ω_h;
(3) $||u_h||_{L^\infty(\Omega_h)} \leq 2c$
Since $\mathcal{L}(2c) \leq 0$, we use Theorem 5.6 on each Ω_h to obtain $w_h \in C^\infty(\Omega_h)$ satisfying
(1) $\mathcal{L}w_h = 0$;
(2) $u_h \leq w_h$;
(3) $\|w_h\|_{L^\infty(\Omega_h)} \leq 2c$.
We conclude by showing that $\{w_h\}_h$ is bounded in respect to the $C^\infty(M)$-topology and thus converges, up to a subsequence, to some $w \in C^\infty(M)$.

To this end, let $K \subseteq V \subseteq M$ be a compact subset of a relatively compact open set V and $k \in \mathbb{N}$, $k \geq 2$. By Schauder estimates for the operator $\mathcal{L}$ we have

$$\|w_h\|_{C^{k,\alpha}(K)} \leq A \left(\|w_h\|_{L^\infty(V)} + \|\mathcal{L}w_h\|_{C^{k-2,\alpha}(V)} \right)$$

for some $\alpha \in (0,1)$ and for h large enough so that $V \subseteq \Omega_h$. See for instance Section 6.1 of [49]. In particular, there exists a constant $C = C(K, n, k) > 0$ such that $\|w_h\|_{C^k(K)} < C$ for every $h \in \mathbb{N}$. Here

$$||w_h||_{C^k(K)} = ||w_h||_{L^\infty(K)} + ||\nabla w_h||_{L^\infty(K)} + \cdots + ||\nabla^k w_h||_{L^\infty(K)}.$$

Since $\{w_h\}_h$ is pre-compact, it converges in the $C^\infty(M)$ topology up to a subsequence, denoted again with $\{w_h\}_h$. Let $w \in C^\infty(M)$ be the C^∞ limit, we have that

$$u \le w, \quad \sup_M w < +\infty \quad \text{and} \quad \mathcal{L}w = 0.$$

□

This concludes the proof of Theorem II.5, apart from the proof of Corollary 5.5.

5.2 Monotone approximation results

This section is devoted to the proof of Theorem II.7. Instead of proving Theorem II.7 directly, we prove an equivalent monotone approximation result for another elliptic differential operator closely related to $\mathcal{L}$. We begin by taking a function $\alpha \in C^\infty(M)$ satisfying

$$\begin{cases} \mathcal{L}\alpha = 0 \\ \alpha > 0. \end{cases} \tag{5.2}$$

The existence of such function is ensured by [43], and is equivalent to the fact that $\lambda_1^{-\mathcal{L}}(D) \ge 0$ for any bounded domain $D \subseteq M$, where $\lambda_1^{-\mathcal{L}}(D)$ denotes the first Dirichlet eigenvalue of $-\mathcal{L}$ on D. In our case, it is easy to see that $\lambda_1^{-\mathcal{L}}(D) \ge 1$ over any bounded domain $D \subseteq M$.

Using α we define the following drifted Laplacian

$$\Delta_\alpha : u \mapsto \alpha^{-2} \operatorname{div}(\alpha^2 \nabla u). \tag{5.3}$$

With a trivial density argument, one has that Δ_α is symmetric in L^2 with respect to the measure $\alpha^2 d\mu_g$. Then, using the following idea due to Protter and Weinberger, [108], we establish the relation between Δ_α and $\mathcal{L}$. See also Lemma 2.3 of [107].

Lemma 5.9. *If $u \in L^1(\Omega)$ with $\Omega \Subset M$, then*

$$(\Delta - 1)u \ge 0 \qquad \Leftrightarrow \qquad \Delta_\alpha\left(\frac{u}{\alpha}\right) \ge 0,$$

where both inequalities are intended in the sense of distributions.

Proof. Fix $0 \le \varphi \in C_c^\infty(\Omega)$, by direct computation we have

$$\begin{aligned} \alpha\Delta_\alpha\left(\frac{\varphi}{\alpha}\right) &= \alpha^{-1}\operatorname{div}\left[\alpha^2\nabla\left(\frac{\varphi}{\alpha}\right)\right] \\ &= \alpha^{-1}\operatorname{div}\left(\alpha\nabla\varphi - \varphi\nabla\alpha\right) \\ &= \Delta\varphi - \varphi\frac{\Delta\alpha}{\alpha} \\ &= \mathcal{L}\varphi, \end{aligned} \tag{5.4}$$

where in the last equation we have used (5.2). Thus, using (5.4) and the symmetry of Δ_α we conclude

$$\begin{aligned} \left(\Delta_\alpha\left(\frac{u}{\alpha}\right), \alpha\varphi\right)_{L^2} &= \int_\Omega \frac{u}{\alpha}\,\Delta_\alpha\left(\frac{\varphi}{\alpha}\right)\,\alpha^2 d\mu_g \\ &= \int_\Omega u\,(\Delta-1)\varphi\,d\mu_g = ((\Delta-1)u,\varphi)_{L^2}\,. \end{aligned}$$

□

Using Equation (5.4) and setting $v = \alpha^{-1}u$, it is possible to obtain Theorem II.7 from an equivalent statement for the operator Δ_α. In this perspective, our goal is to prove the following:

Theorem 5.10. *Let (M, g) be a Riemannian manifold and let $v \in L^1_{\mathrm{loc}}(M)$ be a solution of $\Delta_\alpha v \ge 0$ in the sense of distributions. Then, for every $\Omega \Subset M$ there exists a sequence $\{v_k\} \subset C^\infty(\Omega)$ such that:*
- *(i)* $v_k \searrow v$ *pointwise a.e.;*
- *(ii)* $\Delta_\alpha v_k \ge 0$ *for all* k*;*
- *(iii)* $v_k \to v$ *in* $L^1(\Omega)$*;*
- *(iv)* $\sup_\Omega v_k \le \operatorname{ess\,sup}_\Omega v$.

5.2.1 Representation formula for α-harmonic functions

Let $\Omega \Subset M$ be a relatively compact subset of M. We begin by establishing some mean value representation formulae involving the Green function of the operator Δ_α on Ω with Dirichlet boundary conditions. Recall that $G : \overline{\Omega} \times \overline{\Omega} \setminus \{x = y\} \to \mathbb{R}$ is a symmetric, $L^1(\Omega \times \Omega)$ function satisfying the following properties:

- (a) $G \in C^\infty(\Omega \times \Omega \setminus \{x = y\})$ and $G(x, y) > 0$ for all $x, y \in \Omega$ with $x \ne y$;
- (b) $\lim_{x\to y} G(x, y) = +\infty$ and $G(x, y) = 0$ if $x \in \partial\Omega$ (or $y \in \partial\Omega$);
- (c) $\Delta_\alpha G(x, y) = -\delta_x(y)$ with respect to $\alpha^2 d\mu_g$, that is,

$$\varphi(x) = -\int_\Omega G(x, y)\,\Delta_\alpha\,\varphi(y)\alpha^2(y)d\mu_y \qquad \forall\varphi \in C_C^\infty(\Omega)$$

.

For $r > 0$ and $x \in \Omega$, we define the following set

$$\mathcal{B}_r(x) \coloneqq \{y \in \Omega \mid G(x,y) > r^{-1}\} \cup \{x\}. \tag{5.5}$$

We adopt the convention $G(x,x) = +\infty$ so that $\mathcal{B}_r(x) = \{y \in \Omega \mid G(x,y) > r^{-1}\}$. Observe that $\mathcal{B}_r(x) \subset \Omega$ are open and relatively compact sets, moreover, for almost all $r > 0$, $\partial\mathcal{B}_r(x)$ is a smooth hypersurface. This is a consequence of Sard's theorem. In the following, $d\sigma$ and $d\mu$ represent the Riemannian surface and volume measure of $\partial\mathcal{B}_r(x)$ and $\mathcal{B}_r(x)$ respectively.

Proposition 5.11. *For every $v \in C^\infty(\Omega)$ and almost every $r > 0$, the following representation formula holds*

$$v(x) = \int_{\partial\mathcal{B}_r(x)} v(y)|\nabla G(x,y)|\alpha^2(y)d\sigma_y - \int_{\mathcal{B}_r(x)} \left[G(x,y) - \frac{1}{r}\right] \Delta_\alpha v(y)\alpha^2(y)d\mu_y \tag{5.6}$$

Proof. By the Green identity we have

$$\begin{aligned} v(x) = &- \int_{\mathcal{B}_r(x)} G(x,y)\, \Delta_\alpha\, v(y)\alpha^2(y)d\mu_y \\ &+ \int_{\partial\mathcal{B}_r(x)} \Big(G(x,y)\frac{\partial v}{\partial \nu}(y) - v(y)\frac{\partial G}{\partial \nu}(x,y)\Big)\alpha^2(y)d\sigma_y. \end{aligned}$$

Since $\mathcal{B}_r(x)$ are level sets of G, we have $\frac{\partial G}{\partial \nu} = -|\nabla G|$ thus

$$\begin{aligned} v(x) &= \int_{\partial\mathcal{B}_r(x)} v(y)\Big|\nabla G(x,y)\Big|\alpha^2(y)d\sigma_y + \frac{1}{r}\int_{\partial\mathcal{B}_r(x)} \frac{\partial v}{\partial \nu}(y)\alpha^2(y)d\sigma_y \\ &\quad - \int_{\mathcal{B}_r(x)} G(x,y)\Delta_\alpha v(y)\alpha^2(y)d\mu_y \\ &= \int_{\partial\mathcal{B}_r(x)} v(y)\Big|\nabla G(x,y)\Big|\alpha^2(y)d\sigma_y - \int_{\mathcal{B}_r(x)} \Big[G(x,y) - \frac{1}{r}\Big]\Delta_\alpha v(y)\alpha^2(y)d\mu_y. \end{aligned}$$

$\square$

In particular, if $v \in C^2(\Omega)$ is α-harmonic, i.e. $\Delta_\alpha u = 0$ on Ω, then

$$v(x) = \int_{\partial\mathcal{B}_r(x)} |\nabla G(x,y)|\; v(y)\; \alpha^2(y)\; d\sigma_y. \tag{5.7}$$

The formulae (5.7) and (5.6) are a generalization of some standard representation formula for the Laplace–Beltrami operator. See for instance the Appendix of [19], [97] or the very recent [34].

5.2.2 Distributional vs. potential α-subharmonic solutions

Before proving the monotone approximation result, we observe that the notion of α-subharmonicity in the distributional sense is closely related to the notion of α-subharmonic solutions in the sense of potential theory.

Definition 5.12. We say that an upper semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is *α-subharmonic in the sense of potential theory* on Ω if the following conditions hold

(i) $\{x \in \Omega \mid u(x) > -\infty\} \neq \emptyset$;

(ii) for all $V \Subset \Omega$ and for every $h \in C^2(V) \cap C^0(\overline{V})$ such that $\Delta_\alpha h = 0$ in V with $u \leq h$ on ∂V, then
$$u \leq h \qquad \text{in } V.$$

The key observation, first noted by Sjörgen in [116, Theorem 1] in the Euclidean setting, is that every distributional α-subharmonic function is almost everywhere equal to a function which is α-subharmonic in the sense of potential theory. Note that in [116, Theorem 1], Sjörgen considers a wider class of elliptic differential operators. The drifted Laplace–Beltrami operator falls into that class.

More precisely, if $v \in L^1(\Omega)$ satisfies $\Delta_\alpha v \geq 0$ in the sense of distributions, then v is equal almost everywhere to an α-subharmonic function in the sense of potential theory. Naturally, if v has some better regularity property, for example it is continuous, the equality holds everywhere. This fact holds true also in the Riemannian case, we sketch here the proof for clarity of exposition.

Recall that for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\varphi(x) = -\int_\Omega G(x,y)\Delta_\alpha\varphi(y)\ \alpha^2(y)d\mu_y.$$

Furthermore, since $\Delta_\alpha v = d\nu^v$ is a positive Radon measure, we have

$$\int_\Omega v(x)\Delta_\alpha\varphi(x)\ \alpha^2(x)d\mu_x = \int_\Omega \varphi(x)\ d\nu^v_x$$

for every $\varphi \in C_c^\infty(\Omega)$. The measure $d\nu^v$ is often referred to as the Δ_α-Riesz measure of v. By a direct computation we have

$$\begin{aligned}
\int_\Omega v(x)\Delta_\alpha\varphi(x)\ \alpha^2(x)d\mu_x &= \int_\Omega \varphi(x)\ d\nu^v_x \\
&= -\int_\Omega\int_\Omega G(x,y)\Delta_\alpha\varphi(y)\ \alpha^2(y)d\mu_y\ d\nu^v_x \\
&= \int_\Omega -\left(\int_\Omega G(x,y)d\nu^v_x\right)\Delta_\alpha\varphi(y)\alpha^2(y)d\mu_y,
\end{aligned}$$

hence,

$$\int_\Omega \left(v(y) + \int_\Omega G(x,y)\ d\nu^v_x\right)\Delta_\alpha\varphi(y)\alpha^2(y)d\mu_y = 0,$$

for every $0 \le \varphi \in C_c^\infty(\Omega)$. In other words, the function

$$v + \int_\Omega G(x, \cdot) d\nu_x^v$$

is α-harmonic in the sense of distributions. By [116, Theorem 1] of Sjörgen we know that α-harmonic functions are almost everywhere equal to a function which is α-harmonic in the sense of potential theory. When the operator at hand is the Euclidean Laplacian, this result is usually referred as Weyl's lemma. We conclude that

$$v \overset{a.e.}{=} h - \int_\Omega G(x, \cdot) d\nu_x^v, \tag{5.8}$$

where h is α-harmonic in a strong sense. On the other hand, one can prove that the function

$$-G * d\nu^v = -\int_\Omega G(x, \cdot) d\nu_x^v \tag{5.9}$$

is α-subharmonic in the sense of potential theory, which concludes the sketch of the proof. For this latter statement, we refer to Section 6 of [19].

5.2.3 Proof of Theorem 5.10

In order to prove Theorem 5.10, we adopt a strategy laid out by Bonfiglioli and Lanconelli in [19], where they obtained some monotone approximation results for a wide class of second order elliptic operators on $\mathbb{R}^n$. To do so, we begin by defining the following mean integral operators. If v is an upper semicontinuous function on Ω, $x \in \Omega$ and $r > 0$, we set

$$m_r(v)(x) := \int_{\partial\mathcal{B}_r(x)} v(y) |\nabla_y G(x, y)| \alpha^2(y) \, d\sigma_y. \tag{5.10}$$

In particular, if v is an α-subharmonic function in the sense of distributions, we prove the following results.

Proposition 5.13. *Given a Riemannian manifold (M, g) and $\Omega \Subset M$, if $v \in L^1(\Omega)$ is α-subharmonic in the sense of distributions, then*

(a) $v(x) \le m_r(v)(x)$ for almost every $x \in \Omega$ and almost every $r > 0$;
(b) let $0 < s < r$ then $m_s(v)(x) \le m_r(v)(x)$ almost everywhere in Ω;
(c) for almost every $x \in \Omega$ we have $\lim_{r \to 0} m_r(v)(x) = v(x)$;
(d) for every $r > 0$ $m_r(v)$ is α-subharmonic in the sense of potential on Ω.

Proof. By the observation in the previous section, up to a choice of a good representative, we can assume that v is α-subharmonic in the sense of potential, cf. Definition 5.12.
(a) Fix $x_0 \in \Omega$ and $r > 0$, consider $\varphi \in C^0(\partial\mathcal{B}_r(x_0))$ such that $v \le \varphi$ on $\partial\mathcal{B}_r(x_0)$. Let $h : \mathcal{B}_r(x_0) \to \mathbb{R}$ be the (classical) solution of

$$\begin{cases} \Delta_\alpha h = 0 & \text{in } \mathcal{B}_r(x_0) \\ h = \varphi & \text{on } \partial\mathcal{B}_r(x_0) \end{cases}. \tag{5.11}$$

Since v is α-subharmonic in the sense of potential, then $v \le h$ in $\mathcal{B}_r(x_0)$. By Proposition 5.11 we have

$$v(x_0) \le h(x_0) = \int_{\partial\mathcal{B}_r(x_0)} \varphi(y)|\nabla_y G(x_0,y)| d\sigma_y^\alpha \tag{5.12}$$

where $d\sigma_y^\alpha = \alpha^2(y)\ d\sigma_y$. Since v is upper semicontinuous on $\partial\mathcal{B}_r(x_0)$, there exists a sequence $\{\varphi_i\}_i \subset C^0(\partial\mathcal{B}_r(x_0))$ such that $\varphi_i(y) \searrow v(y)$ almost everywhere on $\partial\mathcal{B}_r(x_0)$. Applying (5.12) to each φ_i we obtain by Dominated Convergence that

$$v(x_0) \le \int_{\partial\mathcal{B}_r(x_0)} v(y)|\nabla_y G(x_0,y)| d\sigma_y^\alpha = m_r(v)(x_0).$$

(b) Fix $0 < s < r$, let φ and h be as in (a) so that $v \le h$ on $\mathcal{B}_r(x_0)$. By Proposition 5.11 we have

$$m_s(v)(x_0) \le \int_{\partial\mathcal{B}_s(x_0)} h(y)|\nabla_y G(x_0,y)| d\sigma_y^\alpha = h(x_0) = \int_{\partial\mathcal{B}_r(x_0)} \varphi(y)|\nabla_y G(x_0,y)| d\sigma_y^\alpha.$$

Taking a monotone sequence of continuous functions on the boundary $\varphi_i \searrow u$ and proceeding as above, we conclude

$$m_s(v)(x_0) \le \int_{\partial\mathcal{B}_r(x_0)} \varphi_i(y)|\nabla_y G(x_0,y)| d\sigma_y^\alpha \longrightarrow m_r(v)(x_0).$$

(c) This property is a consequence of the fact that v is (almost everywhere) equal to an upper semicontinuous function. Fix $x_0 \in \Omega$ and $\varepsilon > 0$ there exists a small enough neighborhood of x_0, $V(x_0)$, such that

$$v(y) < v(x_0) + \varepsilon$$

on $V(x_0)$. Taking for $r > 0$ small enough so that $\partial\mathcal{B}_r(x_0) \subseteq V(x_0)$, we have

$$m_r(v)(x_0) \le v(x_0) + \varepsilon.$$

Recall that the function constant to 1 is α-harmonic on Ω. By (i), $v(x_0) \le m_r(v)(x_0)$ hence

$$m_r(v)(x_0) - \varepsilon \le v(x_0) \le m_r(v)(x_0).$$

Letting ε, and thus r go to 0, we obtain desired property.
(d) This last property is a consequence of the decomposition of α-subharmonic functions observed in (5.8). Integrating against $|\nabla G|\alpha^2$ both sides of (5.8) we obtain

$$m_r(v)(x) = h(x) - m_r(G * d\nu^v)(x).$$

The desired property follows from the fact that the mean integral $-m_r(G * d\nu^v)$ is α-subharmonic in the sense of potential. For details, we refer to Section 6 of [19].

$\square$

The next step is to take a convolution of the mean integral functions $m_r(v)$ so to obtain smooth functions which produce the desired approximating sequence $\{v_k\}_k$.

Proof of Theorem 5.10. Let $\varphi \in C_c^1([0,1])$ be a non-negative function with unitary L^1-norm, we define

$$v_k(x) := k \int_0^{+\infty} \varphi(ks)\ m_s(v)(x) ds. \tag{5.13}$$

As shown in [19] the functions defined by (5.13) are smooth.

The monotonicity of $\{v_k\}$ follows immediately from the monotonicity of $m_r(v)$ with respect to r. Combining this with property (c) and (a) of Proposition 5.13 we obtain (i) by monotone convergence. The proof of (ii) is a consequence of (d) in Proposition 5.13. To see this, let $\psi \in C_c^\infty(M)$, then by Fubini-Tonelli we have

$$\begin{aligned}\int_M v_k(x)\, \Delta_\alpha\, \psi(x) &= \int_M \left(k \int_0^{+\infty} \varphi(ks) m_s(v)(x) ds \right) \Delta_\alpha\, \psi(x) \\ &= k \int_0^{+\infty} \varphi(ks) \left(\int_M m_s(v)(x) \Delta_\alpha \psi(x) \right) ds \geq 0.\end{aligned}$$

Note that φ is compactly supported on $[0,1]$, $\psi \in C_c^\infty(M)$ and $m_s(v)(x)$ are upper semi-continuous functions bounded from below by $v \in L^1(M)$. For details on the proof of (i) and (ii) we refer to [19, Theorem 7.1]. The convergence in $L^1(\Omega)$ follows from (i), using the fact that $|v_k| \leq \max\{|v|, |v_1|\} \in L^1(\Omega)$ and the dominated convergence theorem. For the uniform estimate of (iv), it is enough to observe that 1 is an α-harmonic function on Ω and φ has unitary L^1 norm, hence,

$$v_k(x) \leq (\text{ess sup}_\Omega\, v) k \int_0^{+\infty} \varphi(ks)\ m_s(1)(x)\ ds = \text{ess sup}_\Omega\, v.$$

This concludes the proof of Theorem 5.10. $\square$

Remark 5.14. Note that in the last estimate, one actually has

$$\sup_\Omega v_k \leq \text{ess sup}_{\mathcal{B}_{1/k}}\, v \leq \text{ess sup}_\Omega\, v.$$

This observation will be crucial later on.

5.2.4 Proof of Theorem II.7

Finally, we deduce the proof of Theorem II.7 from Theorem 5.10. If $\{v_k\}_k$ is the approximating sequence for the function $v = \frac{u}{\alpha}$, we define $u_k := \alpha v_k$. By Equation (5.4), $\{u_k\}_k$ is an approximating sequence for u as it satisfies $(i) - (iii)$ of Theorem II.7. The proof is trivial and is therefore omitted. A little more effort is required to show that if $\sup_\Omega v_k \leq \text{ess sup}_\Omega\, v$, then $\sup_\Omega u_k \leq 2\, \text{ess sup}_\Omega\, u$ for k large enough, at least when $u \geq 0$.

To this end, fix $x \in \Omega$. As noted in Remark 5.14, we have

$$u_k(x) = \alpha(x) v_k(x) \leq \alpha(x) \operatorname{ess\,sup}_{\mathcal{B}_{1/k})} v \leq \frac{\alpha(x)}{\inf_{\mathcal{B}_{1/k}} \alpha} \operatorname{ess\,sup}_\Omega u.$$

Furthermore, for every $y \in \mathcal{B}_{1/k}(x)$, we estimate

$$\frac{\alpha(x)}{\alpha(y)} \leq \frac{|\alpha(x) - \alpha(y)|}{\alpha(y)} + 1 \leq \frac{r_k(x) \sup_\Omega |\nabla \alpha|}{\inf_\Omega \alpha} + 1 \tag{5.14}$$

where $r_k(x) = \sup\{d(x,z) : z \in \mathcal{B}_{1/k}(x)\}$. Next, we show that the function $r_k(x)$ can be uniformly bounded so that (5.14) is bounded above by 2.

Lemma 5.15. *There exists some $k_0 \in \mathbb{N}$ such that*

$$r_k(x) \leq \frac{\inf_\Omega \alpha}{\sup_\Omega |\nabla \alpha|} =: c \qquad \forall x \in \Omega, \quad \forall k \geq k_0.$$

Proof. Suppose by contradiction that there exists a sequence of points $\{x_k\}_k \subset \Omega$ such that $r_k(x_k) > c$ for every $k \in \mathbb{N}$. By definition of $r_k(x_k)$, there exists a sequence of points $\{y_k\}_k \subset \mathcal{B}_{1/k}(x_k)$ such that $d(y_k, x_k) > c$. Since Ω is relatively compact, up to a subsequence, we can assume that $x_k \to x_\infty \in \overline{\Omega}$ and $y_k \to y_\infty \in \overline{\Omega}$. Since $y_k \in \mathcal{B}_{1/k}(x_k)$ we have

$$G(x_k, y_k) > k \to +\infty. \tag{5.15}$$

Note also that the Green function G is smooth and hence continuous on $\Omega \times \Omega \setminus \{x = y\}$. Note that since $d(x_k, y_k) > c$, then $d(x_\infty, y_\infty) \geq c$, in particular, we deduce that $x_\infty \notin \partial\Omega$ because the Green function G vanishes on the boundary of Ω. If $x_\infty \in \Omega$ is not on the boundary, fix $\overline{k} \in \mathbb{N}$. By (5.15) and continuity of the Green function we have $G(y_\infty, x_\infty) > \overline{k}$ which implies that $y_\infty \in \mathcal{B}_{1/\overline{k}}(x_\infty)$. In particular, we have $d(x_\infty, y_\infty) \leq r_{\overline{k}}(x_\infty) \to 0$ as $\overline{k} \to +\infty$, which is a contradiction since $d(x_\infty, y_\infty) \geq c$. Indeed, for every $x \in \Omega$,

$$\lim_{k \to +\infty} r_k(x) = 0.$$

Clearly, $r_k(x)$ is a monotone decreasing sequence in k. Suppose its limit is some $r_0 > 0$, this implies that $r_k(x) \geq r_0$ for all k. In particular, for every k there exists some $z_k \in \mathcal{B}_{1/k}(x)$ such that $d(z_k, x) = \frac{r_0}{2}$. Up to subsequences, $z_k \to \overline{z}$ and $\overline{z} \in \mathcal{B}_{1/k}(x)$ for every k. However,

$$\bigcap_{k=1}^{\infty} \mathcal{B}_{1/k}(x) = \{x\},$$

so $\overline{z} = x$ which is a contradiction since $d(\overline{z}, x) = \frac{r_0}{2}$. $\square$

Thanks to Lemma 5.15, up to taking k large enough, we have

$$\alpha(x) \leq 2\alpha(y) \qquad \forall x \in \Omega \text{ and } \forall y \in \mathcal{B}_{1/k}(x),$$

hence,

$$u_k(x) \leq \frac{\alpha(x)}{\inf_{\mathcal{B}_{1/k}} \alpha} \operatorname{ess\,sup}_\Omega u \leq 2 \operatorname{ess\,sup}_\Omega u \qquad \forall x \in \Omega.$$

Clearly, if we don't assume $u \geq 0$, the estimate in term of L^∞ norms easily follows. This concludes the proof of Theorem II.7.

5.2.5 Remarks on global monotone approximation

A careful analysis of above proofs shows that the monotone approximation results can be obtained globally on the whole manifold M as long as there exists a minimal positive Green function for the operator Δ_α and the super level sets $\mathcal{B}_r(x)$ are compact. Not all Riemannian manifolds, however, satisfy these conditions. We recall the following

Definition 5.16. A Riemannian manifold (M, g) is said to be *α-non-parabolic* if there exists a minimal positive Green function G for the operator Δ_α. Moreover, if this Green function satisfies

$$\lim_{y \to \infty} G(x, y) = 0, \tag{5.16}$$

the manifold M is said to be *strongly α-non-parabolic.*

Note that compact Riemannian manifold are always α-parabolic, thus, we focus on the complete, non-compact case. It is also known that if (M, g) is a geodesically complete, α-non-parabolic manifold, then

$$\int_1^\infty \frac{t}{\mathrm{vol}_\alpha(B_t(x))} dt < \infty \tag{5.17}$$

where $\mathrm{vol}_\alpha(B_t(p))$ is the volume of the geodesic ball of radius t and center x with respect to the measure $\alpha^2 d\mu_g$. See for instance Theorem 9.7 of [52]. Furthermore, if we assume a non-negative m-Bakry-Émery Ricci tensor $\mathrm{Ric}_f^m := \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{m} df \otimes df \geq 0$ with $f = -2 \log \alpha$, it is possible to prove some Li-Yau type estimates for the heat kernel, see Theorems 5.6 and 5.8 in [26]. Integrating in time these estimates we obtain the following bounds for the Green function

$$C^{-1} \int_{d(x,y)}^\infty \frac{t}{\mathrm{vol}_\alpha(B_t(x))} dt \leq G(x, y) \leq C \int_{d(x,y)}^\infty \frac{t}{\mathrm{vol}_\alpha(B_t(x))} dt.$$

In particular, if (5.17) holds true and $\mathrm{Ric}_f^m \geq 0$, the previous estimate implies that the manifold at hand is strongly α-non-parabolic. It would be interesting to investigate which geometric conditions on the manifold (M, g) imply the existence of a function α such that (5.17) and $\mathrm{Ric}_f^m \geq 0$ hold true.

5.3 A counterexample to the L^1-positivity preserving property

This section is devoted to the proof of Theorem II.6. Fix $\varepsilon > 0$ and consider the 2-dimensional model manifold $M = \mathbb{R}_+ \times_\sigma \mathbb{S}^1$, that is $\mathbb{R}_+ \times \mathbb{S}^1$ with the metric $g = dt^2 +$

$\sigma^2(t)d\theta^2$. Here $d\theta^2$ is the standard round metric on $\mathbb{S}^1$ and $\sigma = \sigma_\varepsilon$ is a $C^\infty((0,+\infty))$ function satisfying

$$\sigma(t) = \begin{cases} j(t) & t > t_\varepsilon \\ t & t < \frac{1}{4} \end{cases}.$$

Here $t_\varepsilon = (2(1+\varepsilon)\varepsilon)^{-1/2\varepsilon}$ and the function j is defined as

$$j(t) = \frac{e^{-t^{2+2\varepsilon}}}{t^{1+\varepsilon}}.$$

By a direct computation we have

$$\begin{aligned} j'(t) &= -(1+\varepsilon)e^{-t^{2+2\varepsilon}}\left(2t^\varepsilon + \frac{1}{t^{2+\varepsilon}}\right) \\ j''(t) &= (1+\varepsilon)e^{-t^{2+2\varepsilon}}\left[2t^{\varepsilon-1} + 4(1+\varepsilon)t^{1+3\varepsilon} + (2+\varepsilon)\frac{1}{t^{3+\varepsilon}}\right]. \end{aligned}$$

As a result, outside a compact set, we have the following asymptotic estimate for the Gaussian curvature:

$$\begin{aligned} K(t,\theta) &= -\frac{j''(t)}{j(t)}g \\ &= -(1+\varepsilon)\left[2t^{2\varepsilon} + 4(1+\varepsilon)t^{2+4\varepsilon} + (2+\varepsilon)\frac{1}{t^2}\right]g \\ &\sim -4(1+\varepsilon)^2t^{2+4\varepsilon}g \end{aligned}$$

as $t \to +\infty$. Next, we define the function $U(t,\theta) = u(t) = (e^{t^{2+2\varepsilon}} - e^{t_\varepsilon^{2+2\varepsilon}})_+$ and prove that it satisfies

$$\Delta U \geq U$$

in the sense of distributions. If $t > t_\varepsilon$, by direct computation, we have

$$\begin{aligned} u'(t) &= 2(1+\varepsilon)t^{1+2\varepsilon}e^{t^{2+2\varepsilon}} \\ u''(t) &= 2(1+\varepsilon)e^{t^{2+2\varepsilon}}\left[2(1+\varepsilon)t^{2+4\varepsilon} + (1+2\varepsilon)t^{2\varepsilon}\right] \end{aligned}$$

thus

$$\Delta U - U = u''(t) + \frac{j'(t)}{j(t)}u'(t) - u(t) = e^{t^{2+2\varepsilon}}\left[2(1+\varepsilon)\varepsilon t^{2\varepsilon} - 1\right] + e^{t_\varepsilon^{2+2\varepsilon}} \geq 0.$$

On the other hand, if $t < t_\varepsilon$ the function U is identically zero, so that $\Delta U - U \geq 0$ also for $t \in (0, t_\varepsilon)$. To see that $\Delta U \geq U$ in the sense of distributions on the whole manifold, we take $0 \leq \varphi \in C_c^\infty(M)$ and set $\overline{M} \coloneqq M \setminus B_{t_\varepsilon}(0)$. Then we compute

$$\begin{aligned}
\int_M U(\Delta\varphi - \varphi) &= \int_{\overline{M}} U(\Delta\varphi - \varphi) \\
&= -\int_{\overline{M}} g(\nabla\varphi, \nabla U) + \int_{\partial\overline{M}} U\frac{\partial\varphi}{\partial\nu} - \int_{\overline{M}} U\varphi \\
&= -\int_{\overline{M}} g(\nabla\varphi, \nabla U) - \int_{\overline{M}} U\varphi \\
&= \int_{\overline{M}} \Delta U\varphi - \int_{\partial\overline{M}} \frac{\partial U}{\partial\nu}\varphi - \int_{\overline{M}} U\varphi \\
&= \int_{\overline{M}} \Delta U\varphi + \int_{\partial B_{t_\varepsilon}(0)} \frac{\partial U}{\partial t}\varphi - \int_{\overline{M}} U\varphi \\
&= \int_{\overline{M}} (\Delta U - U)\varphi + \int_{\partial B_{t_\varepsilon}(0)} u'\varphi \geq 0.
\end{aligned}$$

On the other hand we have:

$$\int_M |U| = \omega_m \int_0^{+\infty} u(t)j(t)dt = \int_{t_\varepsilon}^{+\infty} \frac{1}{t^{1+\varepsilon}}dt < +\infty.$$

In conclusion, if we set $V = -U$, we have $V \in L^1(M)$ and $(-\Delta + 1)V \geq 0$ but $V \leq 0$, which contradicts the validity of the L^1-positivity preserving property on M.

Remark 5.17. Using a simple trick introduced in [74], the counterexample in dimension 2 of Theorem II.6 can be used to construct counterexamples to the L^1-positivity preserving property in arbitrary dimensions $n \geq 2$. It suffices to take the product of the 2-dimensional model manifold M with an arbitrary $n-2$ dimensional closed Riemannian manifold. Extending the function which provides the counterexample on M to the whole product produces a counterexample in a manifold of dimension n.

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