Computers and Mathematics with Applications 151 (2023) 244–251

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The role of stabilization in the virtual element method: A survey

Lorenzo Mascotto [a,b,c,*]

[a] Dip. di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Italy
[b] Istituto di Matematica Applicata e Tecnologie Informatiche, CNR, Pavia, Italy
[c] Fakultät für Mathematik, Universität Wien, Austria

ARTICLE INFO

Keywords:
Virtual element method
Stability bound
Interpolation estimate

ABSTRACT

The virtual element method was introduced 10 years ago, and has generated a large number of theoretical results and applications ever since. Here, we overview the main mathematical results concerning the stabilization of the method as an introduction for newcomers in the field. In particular, we summarize the proofs of some results for two dimensional "nodal" conforming and nonconforming virtual element spaces to pinpoint the essential tools used in the stability analysis. We discuss their extensions to several other virtual elements. Finally, we show several ways to prove the interpolation estimates, including a recent one that is based on employing the stability bounds.

1. Introduction

The virtual element method was introduced ten years ago [1] as a generalization of the finite element method to polytopic meshes. Typically, local virtual element spaces consist of solutions to local problems with polynomial data. Therefore, virtual element functions are not known in closed form; in the spirit of the mimetic finite differences [2,3], only the evaluation of their degrees of freedom is required in the design of the method.

Consequently, the bilinear forms appearing in the variational formulation of given partial differential equations are not computable and are rather discretized based on two main ingredients: projections from local virtual element spaces onto polynomial spaces; bilinear forms that stabilize the scheme.

This entails that the error analysis for virtual elements has the form of a Strang-type result, where several variational crimes have to be taken into account. In particular, one has to cope with certain stability bounds and interpolation estimates in virtual element spaces.

The first work on the virtual element method contains the following statement concerning the stability estimates [1, Section 4.6]:

"*In general, the choice of the bilinear form $S^K(\cdot,\cdot)$ [the local virtual element stabilization] would depend on the problem and on the degrees of freedom. From* (4.20) *it is clear that $S^K(\cdot,\cdot)$ must scale like $a^K(\cdot,\cdot)$ [the "grad-grad bilinear form] on the kernel of Π^∇_p [an H^1 polynomial projector]. Choosing then the canonical basis $\varphi_1,\dots,\varphi_{N_K}$ as*

$\chi_i(\varphi_j)=\delta_{i,j}, i,j=1,\dots,N_K$, *[$\chi_i$ is the i-th local degree of freedom]*

the local stiffness matrix is given by

$$a_h^K(\varphi_i,\varphi_j)=a^K(\Pi^\nabla_p\varphi_i,\Pi^\nabla_p\varphi_j)+S^K((I-\Pi^\nabla_p)\varphi_i,(I-\Pi^\nabla_p)\varphi_j).$$

In our case it is easy to check that, on a "reasonable" polygon, $a^K(\varphi_i,\varphi_j)\approx 1$. Note that this holds true for all $i=1,2,\dots,N_K$ [N_K is the dimension of the local discrete space] since we defined the local degrees of freedom suitably. [...] However, several types of misbehaviour can occur for awkwardly-shaped polygons, in particular if two or more vertices tend to coalesce, although, in our numerical experiments, the method appears to be quite robust in this respect."

* Corresponding author at: Dip. di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Italy.
E-mail address: lorenzo.mascotto@unimib.it.

https://doi.org/10.1016/j.camwa.2023.09.045

So, some facts were made clear since the very inception of the method:

- the stabilization is not required to have approximation properties: it only has to scale as the corresponding "continuous" bilinear form;
- a reasonable choice for the stabilization is the "dofi-dofi" one given by $S^K(\varphi_i,\varphi_j)=\delta_{i,j}$ for all $i,\ j=1,\dots,N_K$;
- the "dofi-dofi" stabilization needs to be carefully tuned/changed in presence of "*awkwardly-shaped polygons*".

Similar considerations are contained in other works tracing back to the early years of the virtual elements literature, say, the period 2013-2016: there, one can find heuristic motivations but no proofs of the stability bounds, i.e., bounds of the form $\alpha^* a^K(v_h,v_h)\le S^K(v_h,v_h)\le \alpha^* a^K(v_h,v_h)$ for suitable discrete functions v_h and positive constants $\alpha_*<\alpha^*$.

In 2017, the first paper [4] on the theoretical aspects of the stabilization in virtual elements was published: stability properties for two dimensional nodal conforming virtual elements were investigated on rather general geometries. Several related contributions have been proposed ever since; amongst them we mention the other three pioneering works [5–7]. Most of the literature on this topic is concerned with nodal conforming virtual elements; fewer works can be found on other types of virtual elements such as face, edge, Stokes, immersed-like virtual elements.

In light of this, the virtual element literature on the theoretical aspects of the stabilization might be classified into three periods:

- the **early years** (2013-2016), when the properties of the stabilization were introduced and motivated heuristically;
- the **pioneering years** (2017-2018), when the first papers [4–7] on the theoretical aspects of the stabilization were published;
- the **consolidation years** (2019-2023), when several other works on the topic were written, the pioneering analysis was generalized, and other types of virtual elements were considered.

We deemed useful to collect and review all contributions that we are aware of about the theoretical aspects of the stabilization. We hope that this work might represent a gentle introduction to these topics. More details on what we are going to discuss can be found at the end of this section.

Disclaimer. As we are interested here only in reviewing the literature on theoretical aspects of the virtual element stabilization, we do not review contributions on more "practical" issues as those in [8–13].

Notation and assumptions. Given $D\subset\mathbb{R}^d$, $d=1,2,3$, a Lipschitz domain with measure $|D|$, we introduce the Sobolev space $H^s(D)$, $s\ge 0$, and endow it with the usual norm $\|\cdot\|_{s,D}$, seminorm $|\cdot|_{s,D}$, and bilinear form $(\cdot,\cdot)_{s,D}$. If $s=0$, we let the Sobolev space $H^0(D)$ be the Lebesgue space $L^2(D)$. Negative order Sobolev spaces are defined by duality.

We set the spaces $\mathbb{P}_p(K)$ and $\mathbb{P}_p(e)$ of polynomials of maximum degree p over a polygon K and an edge e. We use the notation $\mathbb{P}_{-1}(K)=\{0\}$. For the space $\mathbb{P}_p(K)$, given $\mathbf{x}_K=(x_K,y_K)$ and h_K denote the centroid and diameter of K, it is convenient to introduce the basis $\{m_\alpha\}$ of scaled and centred monomials

$$m_{\boldsymbol\alpha}(\mathbf{x}):=\left(\frac{\mathbf{x}-\mathbf{x}_K}{h_K}\right)^{\boldsymbol\alpha}=\left(\frac{x-x_K}{h_K}\right)^{\alpha_1}\left(\frac{y-y_K}{h_K}\right)^{\alpha_2},$$
$$\boldsymbol\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2,\ |\boldsymbol\alpha|=\alpha_1+\alpha_2=0,\dots,p. \tag{1}$$

Analogously, we can define the basis $\{m_\alpha^e\}$, $\alpha=1,\dots,p$, of shifted and scaled monomials for $\mathbb{P}_p(e)$.

Given two quantities a and b, we write $a\lesssim b$ meaning that there exists c depending on the shape of K, but not on its size, such that $a\le c\ b$.

Outline of the paper. We prove the stability bounds for two dimensional nodal conforming virtual elements in Section 2: here, we focus on two paradigmatic stabilizations; derive the corresponding stability bounds; underline what the main tools in the proof are; review the relevant literature. We discuss the generalization to nonconforming "nodal" elements and arbitrarily regular "elliptic-type" elements in Section 3. "Nonelliptic-type" elements, such as face, edge, Stokes-like, and immersed virtual elements, are overviewed in Section 4. Comments on "p-version" spaces, and the role of the stabilization in relability and efficiency bounds for residual error estimators are given in Section 5. In Section 6, we show that the stability bound implies certain interpolation estimates. We draw some conclusions in Section 7.

2. Basic results: stability in nodal conforming virtual elements

In this section, we review the role of the stabilization in two dimensional nodal conforming virtual elements. After designing local virtual elements with a set of unisolvent degrees of freedom, we describe computable polynomial projectors and discrete bilinear forms in Section 2.1. We introduce two paradigmatic stabilizations and provide a simple proof of the stability bounds in Section 2.2. In Section 2.3, we review the essential literature concerning the theory behind the stabilization in nodal conforming virtual elements.

2.1. Nodal conforming virtual elements

Following [1], given a polygonal element K and a positive integer number p, we define the nodal conforming virtual element

$$V_h(K):=\left\{v_h\in H^1(K)\ \middle|\ \Delta v_h\in\mathbb{P}_{p-2}(K),\ v_{h|e}\in\mathbb{P}_p(e)\quad\forall e\in\mathcal{E}^K\right\}. \tag{2}$$

We endow the space $V_h(K)$ with the following set of unisolvent degrees of freedom: given v_h in $V_h(K)$,

- the point values of v_h at the vertices of K;
- on each edge e of K, the point values of v_h at the $p-1$ internal Gauß-Lobatto nodes;
- given the scaled monomial basis $\{m_\alpha\}$ of $\mathbb{P}_{p-2}(K)$ as in (1), the scaled moments

$$\frac{1}{|K|}\int_K m_\alpha\ v_h.$$

The second set of degrees of freedom can be replaced by suitably scaled edge moments. Given N_K the dimension of $V_h(K)$, we collect the above degrees of freedom in the set $\{\mathrm{dof}_j\}_{j=1}^{N_K}$.

The degrees of freedom of $V_h(K)$ allow for the computation of several polynomial projections [1]. In what follows, we need the operator $\Pi_p^\nabla: H^1(K)\to\mathbb{P}_p(K)$ defined as

$$\begin{cases}(\nabla q_p,\nabla(v-\Pi_p^\nabla v))_{0,K} & \forall q_p\in\mathbb{P}_p(K)\\ \int_{\partial K}(v-\Pi_p^\nabla v)=0,\end{cases} \tag{3}$$

and the operator $\Pi_{p-2}^0: L^2(K)\to\mathbb{P}_{p-2}(K)$ defined as

$$(q_{p-2},v-\Pi_{p-2}^0 v)_{0,K}\qquad \forall q_{p-2}\in\mathbb{P}_{p-2}(K). \tag{4}$$

Given the degrees of freedom of a function v_h in $V_h(K)$, we can compute $\Pi_p^\nabla v_h$ and $\Pi_{p-2}^0 v_h$; see [1].

Other options to fix the constant part of Π_p^∇, see the second condition in (3), can be found in the literature. For instance, it is alternatively possible to fix the average in the bulk or the arithmetic average of the values at the vertices of the polygon.

The standard discretization of the bilinear form $a^K(\cdot,\cdot):=(\nabla\cdot,\nabla\cdot)_{0,K}$ in nodal conforming virtual elements is given by

$$a_h^K(u_h,v_h):=a^K(\Pi_p^\nabla u_h,\Pi_p^\nabla v_h)+S^K((I-\Pi_p^\nabla)u_h,(I-\Pi_p^\nabla)v_h). \tag{5}$$

The bilinear form $S^K(\cdot,\cdot): V_h(K)\times V_h(K)$ is required to be computable via the degrees of freedom and satisfies the following stability bounds: there exist $0<\alpha_*\le\alpha^*$ independent of h_K such that

$$\alpha_*|v_h|^2_{1,K} \le S^K(v_h,v_h) \le \alpha_*|v_h|^2_{1,K} \qquad \forall v_h\in V_h(K)\cap \ker(\Pi^\nabla_p). \tag{6}$$

2.2. Two stabilizations for nodal conforming virtual elements

We introduce two stabilizations that are common in the literature of nodal conforming virtual elements. The first one is known as the "dofi-dofi" stabilization [1] and is given by

$$S^K(u_h,v_h) := \sum_{j=1}^{N_K} \mathrm{dof}_j(u_h)\,\mathrm{dof}_j(v_h) \qquad \forall u_h,v_h\in V_h(K). \tag{7}$$

The second one, which we shall refer to as "projected" stabilization, is [14]

$$S^K(u_h,v_h) := h_K^{-1}(u_h,v_h)_{0,\partial K} + h_K^{-2}(\Pi^0_{p-2}u_h,\Pi^0_{p-2}v_h)_{0,K} \qquad \forall u_h,v_h\in V_h(K). \tag{8}$$

Both stabilizations are computable using the degrees of freedom of $V_h(K)$.

Lemma 1. *The bilinear forms $S^K(\cdot,\cdot)$ in (7) and (8) satisfy (6).*

Proof. We prove the assertion for the "projected" stabilization (8) only; the details for the "dofi-dofi" stabilization (7) are similar but slightly more involved. The assertion for the stabilization in (7) follows combining the bounds for the stabilization in (8) and the techniques, e.g., in [4,5,7].

The lower bound. We recall an inverse estimate for functions with polynomial Laplacian; see, e.g., [15, Lemma 10] or [14, Theorem 2]:

$$\|\Delta v_h\|_{0,K} \lesssim h_K^{-1}|v_h|_{1,K} \qquad \forall v_h\in H^1(K),\ \Delta v_h\in\mathbb{P}_{p-2}(K). \tag{9}$$

Furthermore, we have the polynomial inverse estimate

$$\left|q_p\right|_{\frac12,\partial K} \lesssim h_K^{-\frac12}\left\|q_p\right\|_{0,\partial K} \qquad \forall q_p\in C^0(\partial K),\quad q_{p|e}\in\mathbb{P}_p(e),\quad \forall e\in\mathcal{E}^K. \tag{10}$$

Integrating by parts, recalling that Δv_h belongs to $\mathbb{P}_{p-2}(K)$, applying the Cauchy-Schwarz inequality, invoking the definition of negative Sobolev norms, and using the inverse estimates (9) and (10), the lower bound in (6) follows:

$$\begin{aligned}
|v_h|^2_{1,K} &= -\int_K \Delta v_h\ v_h + \int_{\partial K}(\mathbf{n}_K\cdot\nabla v_h)v_h\\
&= -\int_K \Delta v_h\ \Pi^0_{p-2}v_h + \int_{\partial K}(\mathbf{n}_K\cdot\nabla v_h)v_h\\
&\le \|\Delta v_h\|_{0,K}\left\|\Pi^0_{p-2}v_h\right\|_{0,K} + \|\mathbf{n}_K\cdot\nabla v_h\|_{-\frac12,\partial K}|v_h|_{\frac12,\partial K}\\
&\lesssim \|\Delta v_h\|_{0,K}\left\|\Pi^0_{p-2}v_h\right\|_{0,K} + (h_K\|\Delta v_h\|_{0,K}+|v_h|_{1,K})|v_h|_{\frac12,\partial K}\\
&\lesssim |v_h|_{1,K}\left(h_K^{-1}\left\|\Pi^0_{p-2}v_h\right\|_{0,K} + h_K^{-\frac12}\|v_h\|_{0,\partial K}\right).
\end{aligned}$$

The upper bound. Using the stability of Π^0_{p-2} in the L^2 norm, the trace inequality, and the Poincaré inequality (recall we are assuming that v_h belongs to $\ker(\Pi^\nabla_p)$), the upper bound in (6) follows:

$$\begin{aligned}
S^K(v_h,v_h) &:= h_K^{-1}\|v_h\|^2_{0,\partial K} + h_K^{-2}\left\|\Pi^0_{p-2}v_h\right\|^2_{0,K}\\
&\le h_K^{-1}\|v_h\|^2_{0,\partial K} + h_K^{-2}\|v_h\|^2_{0,K} \lesssim |v_h|^2_{1,K}. \qquad \square
\end{aligned}$$

From the proof of Lemma 1, we can see that

- the lower bound in (6) follows from polynomial and virtual inverse estimates, but no zero average condition (based on the fact that v_h belongs also to $\ker(\Pi^\nabla_p)$) is used;
- the upper bound in (6) is proven based only on "direct estimates", such as the Poincaré and the trace inequalities, and is therefore valid for H^1 functions with zero average.

For these reasons, Lemma 1 immediately generalizes as follows.

Corollary 2. *The bilinear forms $S^K(\cdot,\cdot)$ in (7) and (8) satisfy the following stability bounds: there exist positive constants $\alpha_*\le\alpha^*$ such that*

$$\alpha_*|v_h|^2_{1,K} \le S^K(v_h,v_h) \quad \forall v_h\in V_h(K),$$
$$S^K(v,v)\le\alpha^*|v|^2_{1,K} \quad \forall v\in H^1(K),\ (v,1)_{0,D}=0,$$

where D is any subset with nonzero measure of either K or ∂K.

The stability bounds in Lemma 1 and Corollary 2 can be extended to the case of the so-called enhanced virtual element spaces introduced in [16]. More precisely, define

$$\begin{aligned}
V_h(K) := \Big\{ v_h\in H^1(K)\ \Big|\ &\Delta v_h\in\mathbb{P}_p(K);\quad v_{h|e}\in\mathbb{P}_p(e);\quad v_{h|\partial K}\in C^0(\partial K);\\
&\int_K (v_h-\Pi^\nabla_p v_h)m_\alpha = 0\ \forall|\boldsymbol{\alpha}|=p-1,p\Big\}.
\end{aligned} \tag{11}$$

This space can be endowed with the same degrees of freedom of the standard space virtual element space in (2). Such degrees of freedom allow us to compute the projectors Π^∇_{p+2} and Π^0_p in (3) and (4). It can be checked that the stabilization

$$S^K(u_h,v_h) := h_K^{-1}(u_h,v_h)_{0,\partial K} + h_K^{-2}(\Pi^0_p u_h,\Pi^0_p v_h)_{0,K} \qquad \forall u_h,v_h\in V_h(K)$$

satisfies Lemma 1 and Corollary 2.

2.3. Pioneering results and later contributions

The first work containing mathematical proofs on the stabilization is [4]. This and other three works constitute the early backbone of the stability analysis of the virtual element method. In particular, we pinpoint and describe these four works, which were published in 2017-2018:

- [4] contains the first analysis of the stabilization in two dimensional nodal conforming virtual elements; an arbitrary number of edges is allowed; the stability bounds are derived for different stabilizations (including the "dofi-dofi" one);
- [6,7] contain the stability analysis for two and three dimensional nodal conforming virtual elements; arbitrary numbers of edges and faces are allowed; the stability bounds are derived for different stabilizations (including the "dofi-dofi" one);
- [5] contains the stability analysis for two dimensional nodal conforming virtual elements; more standard geometries are employed.

It is our opinion that the presentation in [5] is the simplest one and is therefore recommended to virtual elements novices; the presentation in [4] and [6,7] is more technical and applies to more general geometries, whence these contributions are recommended to more expert readers.

After 2018, other works were devoted to additional mathematical aspects of the stability in nodal conforming virtual elements. Amongst others, we mention nodal conforming virtual elements on curved domains [17]; robustness with respect to anisotropic elements [18,19]; virtual elements stabilized by means of higher-order polynomial energy projection terms [20–23] or projections onto Raviart-Thomas polynomials over subtriangulations [24]; lack of robustness with respect to the degree of accuracy of the method [14]; nodal serendipity virtual elements [25]; presence of stabilization terms in residual error estimators [15,26]. We shall elaborate more on the two last topics in Section 5 below.

The literature stability analysis for other types of elements is more limited. The reason for this is that nodal conforming virtual element functions have a "second order elliptic structure". In other virtual elements, local spaces consist of functions solving local problems (with polynomial data) that are in general more technical to handle.

3. Stability in other "elliptic-type" virtual elements

In this section, we investigate the role of the stabilization for other "elliptic-type" virtual elements. In particular, we focus on "second order" nonconforming virtual elements in Section 3.1; investigate their stability properties in Section 3.2; review conforming and nonconforming arbitrarily regular virtual element spaces in Section 3.3.

3.1. "Nodal" nonconforming virtual elements

Following [27], given a polygonal element K, $\mathcal{E}^K$ its set of edges, and a positive integer number p, we define the "nodal" nonconforming virtual element

$$V_h(K) := \left\{ v_h \in H^1(K) \;\middle|\; \Delta v_h \in \mathbb{P}_{p-2}(K),\, \mathbf{n}_K \cdot \nabla v_{h|e} \in \mathbb{P}_{p-1}(e) \quad \forall e \in \mathcal{E}^K \right\}. \tag{12}$$

We endow the space $V_h(K)$ with the following set of unisolvent degrees of freedom: given v_h in $V_h(K)$,

- on each edge e of K, given the scaled monomial basis $\{m_\alpha^e\}$ of $\mathbb{P}_{p-1}(e)$ discussed in Section 1, the scaled moments
$$\frac{1}{|e|}\int_e m_\alpha^e\, v_h;$$
- given the scaled monomial basis $\{m_{\boldsymbol\alpha}\}$ of $\mathbb{P}_{p-2}(K)$ as in (1), the scaled moments
$$\frac{1}{|K|}\int_K m_{\boldsymbol\alpha}\, v_h.$$

Given N_K the dimension of $V_h(K)$, we collect the above degrees of freedom in the set $\{\mathrm{dof}_j\}_{j=1}^{N_K}$.

The degrees of freedom of $V_h(K)$ allow for the computation of several polynomial projections [27]. In what follows, we need the operators $\Pi_p^\nabla : H^1(K) \to \mathbb{P}_p(K)$ as in (3), $\Pi_{p-2}^0 : L^2(K) \to \mathbb{P}_{p-2}(K)$ as in (4), and $\Pi_{p-1}^{0,e} : L^2(e) \to \mathbb{P}_{p-1}(e)$ defined as

$$(q_{p-1}^e, v - \Pi_{p-1}^{0,e} v)_{0,e} \qquad \forall q_{p-1}^e \in \mathbb{P}_{p-1}(e).$$

Given the degrees of freedom of a function v_h in $V_h(K)$, we can compute $\Pi_p^\nabla v_h$ and $\Pi_{p-2}^0 v_h$, and $\Pi_{p-1}^{0,e} v_h$ for all edges e; see [27]. The standard discretization of the bilinear form $a^K(\cdot,\cdot) := (\nabla\cdot,\nabla\cdot)_{0,K}$ in "nodal" nonconforming virtual elements is as in (5).

Also in the "nodal" nonconforming virtual element setting, the stabilization $S^K(\cdot,\cdot) : V_h(K)\times V_h(K)$ is a bilinear form that is computable via the degrees of freedom, and is such that there exist positive constants $\alpha_* \le \alpha^*$ independent of h_K for which the stability bounds in (6) are valid.

3.2. Stability bounds in "nodal" nonconforming virtual elements

We introduce two stabilizations for "nodal" nonconforming virtual elements. The first one is the "dofi-dofi" stabilization as in (7). The second one, which we shall refer to as "projected" stabilization, is

$$S^K(u_h,v_h) := h_K^{-1}\sum_{e\in\mathcal{E}^K}(\Pi_{p-1}^{0,e}u_h, \Pi_{p-1}^{0,e}v_h)_{0,e} + h_K^{-2}(\Pi_{p-2}^0 u_h, \Pi_{p-2}^0 v_h)_{0,K} \qquad \forall u_h, v_h \in V_h(K). \tag{13}$$

Both stabilization are computable using the degrees of freedom of $V_h(K)$.

Lemma 3. *The bilinear forms $S^K(\cdot,\cdot)$ in (7) and (13) satisfy (6).*

Proof. We prove only the assertion for the "projected" stabilization (13); to this aim, we follow the guidelines in [28, Theorem 3.2]. The details for the "dofi-dofi" stabilization (7) are similar but slightly more involved.

The following polynomial inverse inequality holds true [28, Theorem 3.2]:

$$\left\| q_p \right\|_{0,\partial K} \lesssim h_K^{-\frac12} \left\| q_p \right\|_{-\frac12,\partial K} \quad \forall q_p \in C^0(\partial K), \quad q_p|_e \in \mathbb{P}_p(e)\, \forall e \in \mathcal{E}^K. \tag{14}$$

The lower bound. Integrating by parts, recalling the properties of functions in "nodal" nonconforming virtual element spaces in (12), applying the Cauchy-Schwarz inequality twice, using the polynomial inverse inequality (14), invoking the Neumann trace inequality [29, Theorem A.33], and recalling the virtual inverse estimate (9), we can write

$$\begin{aligned}
|v_h|_{1,K}^2 &= -\int_K \Delta v_h\, v_h + \sum_{e\in\mathcal{E}^K}\int_e (\mathbf{n}_K\cdot\nabla v_h)v_h \\
&= -\int_K \Delta v_h\, \Pi_{p-2}^0 v_h + \sum_{e\in\mathcal{E}^K}\int_e (\mathbf{n}_K\cdot\nabla v_h)\Pi_{p-1}^{0,e} v_h \\
&\le \|\Delta v_h\|_{0,K}\left\|\Pi_{p-2}^0 v_h\right\|_{0,K} + \sum_{e\in\mathcal{E}^K}\|\mathbf{n}_K\cdot\nabla v_h\|_{0,e}\left\|\Pi_{p-1}^{0,e} v_h\right\|_{0,e} \\
&\le \|\Delta v_h\|_{0,K}\left\|\Pi_{p-2}^0 v_h\right\|_{0,K} + \|\mathbf{n}_K\cdot\nabla v_h\|_{0,\partial K}\Big(\sum_{e\in\mathcal{E}^K}\left\|\Pi_{p-1}^{0,e} v_h\right\|_{0,e}^2\Big)^{\frac12} \\
&\lesssim \|\Delta v_h\|_{0,K}\left\|\Pi_{p-2}^0 v_h\right\|_{0,K} \\
&\quad + \|\mathbf{n}_K\cdot\nabla v_h\|_{-\frac12,\partial K} h_K^{-\frac12}\Big(\sum_{e\in\mathcal{E}^K}\left\|\Pi_{p-1}^{0,e} v_h\right\|_{0,e}^2\Big)^{\frac12} \\
&\lesssim \|\Delta v_h\|_{0,K}\left\|\Pi_{p-2}^0 v_h\right\|_{0,K} \\
&\quad + (h_K\|\Delta v_h\|_{0,K} + |v_h|_{1,K}) h_K^{-\frac12}\Big(\sum_{e\in\mathcal{E}^K}\left\|\Pi_{p-1}^{0,e} v_h\right\|_{0,e}^2\Big)^{\frac12} \\
&\lesssim |v_h|_{1,K}\left(h_K^{-1}\left\|\Pi_{p-2}^0 v_h\right\|_{0,K} + h_K^{-\frac12}\Big(\sum_{e\in\mathcal{E}^K}\left\|\Pi_{p-1}^{0,e} v_h\right\|_{0,e}^2\Big)^{\frac12}\right).
\end{aligned}$$

The upper bound. Using the stability of Π_{p-2}^0 and $\Pi_{p-1}^{0,e}$ in the $L^2(K)$ and $L^2(e)$ (for all edges e of K) norms, respectively, the trace inequality, and the Poincaré inequality, the upper bound in (6) follows:

$$\begin{aligned}
S^K(v_h,v_h) &:= h_K^{-1}\sum_{e\in\mathcal{E}^K}\left\|\Pi_{p-1}^{0,e}v_h\right\|_{0,e}^2 + h_K^{-2}\left\|\Pi_{p-2}^0 v_h\right\|_{0,K}^2 \\
&\le h_K^{-1}\sum_{e\in\mathcal{E}^K}\|v_h\|_{0,e}^2 + h_K^{-2}\|v_h\|_{0,K}^2 \\
&= h_K^{-1}\|v_h\|_{0,\partial K}^2 + h_K^{-2}\|v_h\|_{0,K}^2 \lesssim |v_h|_{1,K}^2. \qquad \square
\end{aligned}$$

As in the case of nodal conforming virtual elements, Lemma 3 can be generalized in the sense of Corollary 2.

Moreover, as in Section 2.2, it is possible to define an enhanced version of the space in (12):

$$V_h(K) := \Big\{ v_h \in H^1(K) \;\Big|\; \Delta v_h \in \mathbb{P}_p(K); \quad \mathbf{n}_K\cdot\nabla v_{h|e} \in \mathbb{P}_{p-1}(e) \quad \forall e\in\mathcal{E}^K; \quad \int_K (v_h - \Pi_p^\nabla v_h) m_{\boldsymbol\alpha} = 0 \quad \forall |\boldsymbol\alpha| = p-1, p \Big\}.$$

We endow this space with the same degrees of freedom as for the space in (12), which allow for the computation of higher order polynomial projectors Π^{∇}_{p+2} and Π^0_p; see (3) and (4).

It can be checked that the following stabilization satisfies Lemma 3:

$$S^K(u_h, v_h) := h_K^{-1} \sum_{e\in\mathcal{E}^K} (\Pi^{0,e}_{p-1} u_h, \Pi^{0,e}_{p-1} v_h)_{0,e} + h_K^{-2} (\Pi^0_p u_h, \Pi^0_p v_h)_{0,K} \qquad \forall u_h, v_h \in V_h(K).$$

Other theoretical results on the stabilization for "nodal" nonconforming virtual elements can be found in [30].

3.3. Arbitrarily regular virtual element spaces

Arbitrarily regular virtual element spaces were one of the first generalization of the nodal conforming virtual element method in Section 2.1 and trace back to almost the inception of the method; see [31,32]. Over the years, several conforming and nonconforming variants have been proposed; see, e.g., [33–35].

Differently from standard C^0 elements, arbitrarily regular virtual elements are designed so as global C^k spaces, $k > 0$, can be constructed. This is accomplished by an enrichment of interface and bulk degrees of freedom. In particular, local virtual element spaces are the set of solutions to polyharmonic problems with polynomial data.

Thence, the "elliptic structure" of the spaces can be still employed while deriving the stability bounds. For this reason, the analysis is quite similar to that for the standard "nodal" conforming and nonconforming virtual elements. We refer to [36] and [37] for the stability bounds in nonconforming and conforming arbitrarily regular virtual elements. It is quite interesting that both references are rather recent (2020 and 2022, respectively). This is probably motivated by the fact that (*i*) the stability bounds are derived in any dimension; (*ii*) even though the structure of the arbitrarily regular virtual element spaces is still "elliptic", the interaction of high-order derivatives and the presence of multiple boundary conditions render the analysis of the stability bounds quite involved.

We report here, for the 2D case, a stability result for C^1 virtual elements. Let $p \geq 3$[1] and consider the space

$$V_h(K) := \left\{ v_h \in H^2(K) \;\middle|\; \Delta^2 v_h \in \mathbb{P}_{p-4}(K),\ v_{h|e} \in \mathbb{P}_p(e),\ \mathbf{n}_K \cdot \nabla v_h \in \mathbb{P}_{p-1}(e) \quad \forall e \in \mathcal{E}^K \right\}.$$

We endow this space with the following set of degrees of freedom [31, 38]: given v_h in $V_h(K)$,

- the point values of v_h at the vertices of K;
- the point values of $h_K \partial_1 v_h$ and $h_K \partial_2 v_h$ at the vertices of K, where ∂_1 and ∂_2 are the derivatives along the edges matching at such vertices;
- for all edges e in $\mathcal{E}^K$, given $\{m^e_\alpha\}$ the scaled and shifted monomial basis of $\mathbb{P}_{p-4}(e)$, the moments
  $$\frac{1}{h_e} \int_e m^e_\alpha\, v_h;$$
- for all edges e in $\mathcal{E}^K$, given $\{m^e_\alpha\}$ the scaled and shifted monomial basis of $\mathbb{P}_{p-3}(e)$, the moments
  $$\int_e m^e_\alpha\, \mathbf{n}_K \cdot \nabla v_h;$$
- given $\{m_\alpha\}$ the scaled and shifted monomial basis of $\mathbb{P}_{p-4}(e)$, the moments
  $$\frac{1}{|K|} \int_K m_\alpha\, v_h.$$

Based on such degrees of freedom, the following stabilization is computable:

$$S^K(u_h, v_h) := h_K^{-4} (\Pi^0_{p-4} u_h, \Pi^0_{p-4} v_h)_{0,K} + h_K^{-3} (u_h, v_h)_{0,\partial K} + h_K^{-1} (\mathbf{n}_K \cdot \nabla u_h, \mathbf{n}_K \cdot \nabla v_h)_{0,\partial K}. \tag{15}$$

Proceeding as in Section 2.2 and [37] yields the following result.

Lemma 4. *The stabilization in* (15) *satisfies the two following bounds: there exist positive constants* $\alpha_* \leq \alpha^*$ *such that*

$$\alpha_* |v_h|^2_{2,K} \leq S^K(v_h, v_h) \quad \forall v_h \in V_h(K),$$
$$S^K(v, v) \leq \alpha^* |v|^2_{2,K} \qquad \forall v \in H^1(K) \quad \text{such that}$$
$$(v, 1)_{0,D} = 0,\ (\nabla v, (1,1))_{0,D} = 0,$$

where D *is any subset with nonzero measure of either* K *or* ∂K.

4. Stability in "nonelliptic" virtual elements

In this section, we review the literature on the stabilization for some "nonelliptic" virtual elements: face (Section 4.1); edge (Section 4.2); Stokes-like (Section 4.3); immersed (Section 4.4) virtual elements.

4.1. Face virtual elements

Face virtual elements were designed in [39–41] and generalize the Raviart-Thomas elements to polytopic meshes. Both in two and three dimensions, local face spaces consist of functions solving local "div-rot" problems and with polynomial normal components on faces. The degrees of freedom consist of bubble-like moments in the element and moments of normal components over faces.

The first stability analysis (on 2D curved elements) can be found in [42]. Lowest order two and three dimensional face spaces were analyzed in [43], general order two and three dimensional (standard and serendipity) face spaces in [44], and three dimensional face spaces on curved polyhedra in [45]. Related results are discussed in [46].

Compared to the elliptic case, the stability analysis is here complicated by the "div-rot" structure of the spaces. Notably, the analysis needs the design of certain Helmholtz-like decompositions, as well as employing more sophisticated results from the theory of vector potential and div-rot systems [47,48].

4.2. Edge virtual elements

Edge virtual elements were first designed in [41] and generalize the Nédélec element to polytopic elements. In three dimensions, local edge spaces consist of functions solving local "div-curlcurl" problems inside the element; local "div-rot" problems on faces; polynomial tangential traces over edges. The degrees of freedom consist of bubble-like moments in the element and on faces, and moments of tangential components over edges.

The first stability analysis (lowest order, in two and three dimensions) can be found in [43]; the general order (standard and serendipity) case was later tackled in [44].

Compared to the case of face elements, the analysis of edge elements in three dimensions is further complicated by the fact that virtual element functions are solutions to different types of problems inside the element ("div-curlcurl" problems) and on faces ("div-rot" problems), which require a different treatment to establish suitable stability estimates.

[1] The case $p = 2$ requires a slightly different but similar treatment, and is therefore omitted.

4.3. Stokes-like virtual elements

Lowest order Stokes-like virtual elements appeared in [49] and were later generalized to the general order case in [50]. Local spaces consist of functions solving local Stokes-like with polynomial data; in particular, the divergence of Stokes-like virtual element functions is polynomial. This allows for an immediate design of divergence free velocity spaces. In two dimensions, the degrees of freedom consist of bubble-like moments in the element and point values on the boundary; in three dimensions, of element and face moments, and point values on the edges.

Stokes-like virtual elements are very successful and were the topic of a plethora of articles. However, to the best of our knowledge, the first and only paper dealing with the stability analysis of Stokes-like virtual element spaces is [51]. The stability bounds are essentially derived based on the stable inf-sup structure of the local Stokes problems, and polynomial and (novel) virtual element inverse estimates.

4.4. Immersed-like virtual elements

More recently, immersed-like virtual elements have been introduced in two and three dimensions for elliptic and Maxwell problems; see [52,53]. Such spaces are particularly effective when handling problems with nonsmooth and in particular discontinuous coefficients. The irregular behaviour of the exact solution is captured by local immersed-like virtual element functions in a Trefftz fashion, as they are solutions to local problems involving the same nonsmooth coefficients as in the continuous problem.

5. Dependence on the degree of accuracy and error estimators

Two situations where the presence of the stabilization might be troublesome are the p- and hp-versions of the method, and when proving the reliability and efficiency of error estimators. We elaborate on these two topics in Sections 5.1 and 5.2, respectively.

5.1. Stability bounds depending on the degree of accuracy

The p- and hp-versions of a Galerkin method aim at achieving convergence by increasing the dimension of the local approximation spaces (while keeping the mesh fixed), and by mesh refinement and p-version at once, respectively.

Stability bounds with explicit dependence on the degree of accuracy p were investigated in several works. Amongst them, we recall the following contributions: p-explicit stability bounds were first derived in [14] for the “projected” stabilization in (8); this result was extended [54] to the “dofi-dofi” stabilization in (7); the p-version nonconforming virtual elements was addressed in [28].

In all cases, at least one of the constants α_* and α^* in (6) depend on p. This is not surprising as several inverse estimates are employed to derive the lower bound in (6); see the lower bounds in Lemmas 1 and 3. However, the suboptimality with respect to the degree of accuracy is rather mild in practice; see [14, Section 4.1].

5.2. The role of the stability in residual error estimators

In finite elements, an error estimator η given by the combination of local error estimators η_K is a quantity that can be computed by means of the solution u_h to the method and has to be designed so as to be comparable to the error of the method.

For the Poisson problem, standard reliability and efficiency estimates read

$$|u-u_h|_{1,\Omega} \lesssim \eta(u_h) + \mathrm{HOT}_\Omega(f), \qquad \eta_K(u_h) \lesssim |u-u_h|_{1,\omega_K} + \mathrm{HOT}_{\omega_K}(f), \tag{16}$$

where ω_K is the patch of elements around K and $\mathrm{HOT}_D(f)$ is a high-order oscillation term on a domain D involving the right-hand side f.

Residual error estimators in virtual elements [15,26] satisfy analogous bounds that differ from (16) in two aspects: (*i*) the solution u_h to the VEM is not available in closed form, so it must be replaced by a polynomial projection; (*ii*) the stabilization appears in the error estimator and error estimates.

For a computable virtual element error estimator $\widetilde{\eta}$ given by the combination of local virtual element error estimators $\widetilde{\eta}_K$, the virtual element counterpart of (16) has the following form:

$$|u-u_h|_{1,\Omega} \lesssim \widetilde{\eta}(\Pi_p^\nabla u_h) + \sum_{K\in\mathcal{T}_h} S^K((I-\Pi_p^\nabla)u_h,(I-\Pi_p^\nabla)u_h)^{\frac12} + \mathrm{HOT}_\Omega(f),$$

$$\widetilde{\eta}_K(\Pi_p^\nabla u_h) \lesssim |u-u_h|_{1,\omega_K} + \sum_{K\in\mathcal{T}_h,\ K\subset\omega_K} S^K((I-\Pi_p^\nabla)u_h,(I-\Pi_p^\nabla)u_h)^{\frac12} + \mathrm{HOT}_{\omega_K}(f).$$

The presence of the extra stabilization term in the bounds above might be inauspicious. On the one hand, such term is not robust with respect to the degree of accuracy as discussed in Section 5.1. On the other hand, adaptive mesh refinements may lead to polygonal meshes with several hanging nodes and possibly very small facets; in that case, one should be careful in designing stabilization that are robust with respect to badly-shaped elements.

A partial breakthrough facing these issues is contained in [55], where the stabilization term is removed in the reliability and efficiency estimates. However, this has been proved under restrictive assumptions: meshes must consist of elements with triangular shape; the 3D version is not covered; lowest order elements are employed. So, general reliability and efficiency estimates that are robust with respect to the stabilization are currently not available; however, they are the subject of current study.

6. The stability bounds may imply certain interpolation estimates

Interpolation estimates by means of functions in the virtual element space are an important ingredient in the error analysis of the method. In this section, we first report standard ways of proving the interpolation estimates in conforming and nonconforming virtual elements; see Sections 6.1 and 6.2, respectively. Next, in Section 6.3, we show a more recent strategy to derive the interpolation estimates based on the stability bounds covering the conforming and nonconforming cases at once.

6.1. Standard proof for the interpolation estimates: conforming elements

The first proof of the interpolation estimates for nodal conforming virtual elements traces back to 2015; see [56]. Several variants have been proposed ever since, most of them extending the ideas therein.

Here, we only provide details on the two dimensional nodal conforming virtual elements; see [56, Proposition 4.2]. Given an element K split into a shape-regular triangulation $\mathcal{T}(K)$, v in $H^1(K)$, q_p in $\mathbb{P}_p(K)$, and v_C the Clément quasi-interpolant of v over $\mathcal{T}(K)$, we define v_I over K as the weak solution to

$$\begin{cases} -\Delta v_I = -\Delta q_p & \text{in } K \\ v_I = v_C & \text{on } \partial K. \end{cases}$$

A minimum energy argument entails

$$|v-v_I|_{1,K} \le 2|v-q_p|_{1,K} + |v-v_C|_{1,K}.$$

Therefore, the interpolation estimates follow from polynomial quasi-interpolation and approximation estimates.

Thus, deriving the interpolation estimates for conforming virtual elements strongly hinges upon using the structure of the local virtual element spaces, i.e., the stability of the continuous formulation. When considering other types of virtual elements, this may result in interpolation estimates that are harder to derive. Prototypical examples of this fact are the interpolation estimates in Stokes-like [50]; Hellinger-Reissner-like [57,58]; face and edge [43,44] virtual elements.

6.2. Standard proof for the interpolation estimates: nonconforming elements

The first proof of the interpolation estimates for "nodal" nonconforming virtual elements traces back to 2018; see [28]. Several variants have been proposed ever since, most of them extending the ideas therein.

Here, we only provide details on the two dimensional "nodal" nonconforming virtual elements; see [28, Proposition 3.1] and the later work [38, Corollary 4.1].

Given K in $\mathcal{T}_h$ and v in $H^1(K)$, we define v_I in the "nodal" nonconforming space $V_h(K)$ defined in (12) as the degrees of freedom interpolant of v. Notably, v_I is uniquely identified by

$$\int_K (v-v_I)q_{p-2}^K = 0 \quad \forall q_{p-2}^K \in \mathbb{P}_{p-2}(K),$$

$$\int_e (v-v_I)q_{p-1}^e = 0 \quad \forall q_{p-1}^e \in \mathbb{P}_{p-1}(e) \qquad\qquad \forall e \in \mathcal{E}^K.$$

For any q_p^K in $\mathbb{P}_p(K)$, recalling that Δv_I belongs to $\mathbb{P}_{p-2}(K)$ and $\mathbf{n}_K \cdot \nabla v_{I|e}$ belongs to $\mathbb{P}_{p-1}(e)$ for all the edges e of K, we readily deduce

$$\begin{aligned}
|v-v_I|_{1,K} &= -\int_K \Delta(v-v_I)\,(v-v_I) + \sum_{e\in\mathcal{E}^K}\int_e \mathbf{n}_K\cdot\nabla(v-v_I)\,(v-v_I)\\
&= -\int_K \Delta(v-q_p^K)\,(v-v_I) + \sum_{e\in\mathcal{E}^K}\int_e \mathbf{n}_K\cdot\nabla(v-q_p^K)\,(v-v_I)\\
&= \int_K \nabla(v-q_p^K)\cdot\nabla(v-v_h) \le \left|v-q_p^K\right|_{1,K}|v-v_I|_{1,K}.
\end{aligned}$$

We deduce that

$$\inf_{v_h\in V_h(K)} |v-v_h|_{1,K} \le |v-v_I|_{1,K} \le \inf_{q_p\in\mathbb{P}_p(K)} \left|v-q_p^K\right|_{1,K}.$$

Extending this approach to other types of virtual elements may be not immediate.

6.3. A more recent proof for the interpolation estimates

We discuss a recent alternative approach originally employed in [51] to derive the interpolation estimates. It applies to conforming and nonconforming virtual elements at once, and is based on having the stability bounds at hand.

To simplify the presentation, we stick to the two dimensional nodal conforming virtual element case and consider the stabilization in (7). The following procedure might be generalized to several other elements and dimensions, as well as to other stabilizations. Let K be a given element, v be in $H^s(K)$, $s>1$, and q_p be any function in $\mathbb{P}_p(K)$ with the same average of v over K. We denote the degrees of freedom interpolant of any sufficiently smooth function by adding a subscript $\cdot_I$ to such a function. Observe that $q_p = (q_p)_I$. Recalling Corollary 2 and using that $S^K(v_I,v_I) = S^K(v,v)$ by definition (7), we readily get

$$\begin{aligned}
|v-v_I|_{1,K} &\le \left|v-q_p\right|_{1,K} + \left|v_I-q_p\right|_{1,K}\\
&\le \left|v-q_p\right|_{1,K} + \alpha_*^{-1} S^K(v_I-q_p, v_I-q_p)^{\frac12}\\
&= \left|v-q_p\right|_{1,K} + \alpha_*^{-1} S^K((v-q_p)_I, (v-q_p)_I)^{\frac12}\\
&= \left|v-q_p\right|_{1,K} + \alpha_*^{-1} S^K(v-q_p, v-q_p)^{\frac12}\\
&\le \left(1+\frac{\alpha^*}{\alpha_*}\right)\left|v-q_p\right|_{1,K}.
\end{aligned}$$

Thus, upon having at disposal stability bounds of the form (6), we proved that the best interpolation error is bounded by the best polynomial error up to the stability constants. The above estimates readily extend to nonconforming nodal virtual elements.

7. Conclusions and outlook

The first paper on the virtual element method was published ten years ago. The analysis of the stabilization of the method is more recent and is still a current area of research. Several works on the stability in nodal conforming virtual elements have been published; fewer on other types of virtual elements. In this contribution, we reviewed the literature about the role and theoretical analysis of the stabilization in the virtual element method. We presented paradigmatic proofs for "nodal" conforming and nonconforming virtual elements; we only mentioned the main results for other types of elements. We further underlined that the stability bounds imply certain interpolation estimates.

Data availability

No data was used for the research described in the article.

References

[1] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. Marini, A. Russo, Basic principles of virtual element methods, Models Methods Appl. Sci. 23 (01) (2013) 199–214.
[2] K. Lipnikov, G. Manzini, M. Shashkov, Mimetic finite difference method, J. Comput. Phys. 257 (2014) 1163–1227.
[3] L. Beirão da Veiga, K. Lipnikov, G. Manzini, The Mimetic Finite Difference Method for Elliptic Problems, vol. 11, Springer, 2014.
[4] L. Beirão da Veiga, C. Lovadina, A. Russo, Stability analysis for the virtual element method, Math. Models Methods Appl. Sci. 27 (13) (2017) 2557–2594.
[5] L. Chen, J. Huang, Some error analysis on virtual element methods, Calcolo 55 (1) (2018) 1–23.
[6] S.C. Brenner, Q. Guan, L.Y. Sung, Some estimates for virtual element methods, Comput. Methods Appl. Math. 17 (4) (2017) 553–574.
[7] S.C. Brenner, L.-Y. Sung, Virtual element methods on meshes with small edges or faces, Math. Models Methods Appl. Sci. 268 (07) (2018) 1291–1336.
[8] P. Wriggers, B. Hudobivnik, A low order virtual element formulation for finite elasto-plastic deformations, Comput. Methods Appl. Mech. Eng. 327 (2017) 459–477.
[9] P. Wriggers, B.D. Reddy, W. Rust, B. Hudobivnik, Efficient virtual element formulations for compressible and incompressible finite deformations, Comput. Mech. 60 (2) (2017) 253–268.
[10] L. Beirão da Veiga, F. Dassi, A. Russo, High-order virtual element method on polyhedral meshes, Comput. Appl. Math. 74 (5) (2017) 1110–1122.
[11] L. Mascotto, Ill-conditioning in the virtual element method: stabilizations and bases, Numer. Methods Partial Differ. Equ. 34 (4) (2018) 1258–1281.
[12] F. Dassi, L. Mascotto, Exploring high-order three dimensional virtual elements: bases and stabilizations, Comput. Math. Appl. 75 (9) (2018) 3379–3401.
[13] A. Russo, N. Sukumar, Quantitative study of the stabilization parameter in the virtual element method, https://arxiv.org/abs/2304.00063, 2021.
[14] L. Beirão da Veiga, A. Chernov, L. Mascotto, A. Russo, Exponential convergence of the hp virtual element method with corner singularity, Numer. Math. 138 (3) (2018) 581–613.
[15] A. Cangiani, E.H. Georgoulis, T. Pryer, O.J. Sutton, A posteriori error estimates for the virtual element method, Numer. Math. 137 (4) (2017) 857–893.
[16] B. Ahmad, A. Alsaedi, F. Brezzi, L.D. Marini, A. Russo, Equivalent projectors for virtual element methods, Comput. Math. Appl. 66 (3) (2013) 376–391.
[17] L. Beirão da Veiga, A. Russo, G. Vacca, The virtual element method with curved edges, ESAIM: Math. Model. Numer. Anal. 53 (2) (2019) 375–404.
[18] S. Cao, L. Chen, Anisotropic error estimates of the linear virtual element method on polygonal meshes, SIAM J. Numer. Anal. 56 (5) (2018) 2913–2939.
[19] S. Cao, L. Chen, Anisotropic error estimates of the linear nonconforming virtual element methods, SIAM J. Numer. Anal. 57 (3) (2019) 1058–1081.
[20] S. Berrone, A. Borio, F. Marcon, Lowest order stabilization free virtual element method for the Poisson equation, https://arxiv.org/abs/2103.16896, 2021.
[21] S. Berrone, A. Borio, F. Marcon, Comparison of standard and stabilization free virtual elements on anisotropic elliptic problems, Appl. Math. Lett. 129 (2022), Paper No. 107971, 5.
[22] A. Chen, N. Sukumar, Stabilization-free serendipity virtual element method for plane elasticity, Comput. Methods Appl. Mech. Eng. 404 (2023), Paper No. 115784, 19.
[23] A. Lamperti, M. Cremonesi, U. Perego, A. Russo, C. Lovadina, A Hu–Washizu variational approach to self-stabilized virtual elements: 2D linear elastostatics, Comput. Mech. (2023) 1–21.

[24] C. Chen, X. Huang, H. Wei, Virtual element methods without extrinsic stabilization, https://arxiv.org/abs/2212.01720, 2023.
[25] L. Beirão da Veiga, L. Mascotto, Stability and interpolation properties of serendipity nodal virtual elements, Appl. Math. Lett. 142 (2023) 108639.
[26] L. Beirão da Veiga, G. Manzini, Residual a posteriori error estimation for the virtual element method for elliptic problems, ESAIM: Math. Model. Numer. Anal. 49 (2) (2015) 577–599.
[27] B.P. Ayuso de Dios, K. Lipnikov, G. Manzini, The nonconforming virtual element method, ESAIM: Math. Model. Numer. Anal. 50 (3) (2016) 879–904.
[28] L. Mascotto, I. Perugia, A. Pichler, Non-conforming harmonic virtual element method: h- and p-versions, J. Sci. Comput. 77 (3) (2018) 1874–1908.
[29] C. Schwab, p-and hp-Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics, Oxford University Press, 1998.
[30] S. Bertoluzza, G. Manzini, M. Pennacchio, D. Prada, Stabilization of the nonconforming virtual element method, Comput. Math. Appl. 116 (2022) 25–47.
[31] L. Beirão da Veiga, G. Manzini, A virtual element method with arbitrary regularity, IMA J. Numer. Anal. 34 (2) (2014) 759–781.
[32] F. Brezzi, L.D. Marini, Virtual element methods for plate bending problems, Comput. Methods Appl. Mech. Eng. 253 (2013) 455–462.
[33] P.F. Antonietti, G. Manzini, M. Verani, The fully nonconforming virtual element method for biharmonic problems, Math. Models Methods Appl. Sci. 28 (02) (2018) 387–407.
[34] P.F. Antonietti, G. Manzini, M. Verani, The conforming virtual element method for polyharmonic problems, Comput. Math. Appl. 79 (7) (2020) 2021–2034.
[35] J. Zhao, S. Chen, B. Zhang, The nonconforming virtual element method for plate bending problems, Math. Models Methods Appl. Sci. 26 (09) (2016) 1671–1687.
[36] L. Chen, X. Huang, Nonconforming virtual element method for $2m$th order partial differential equations in $\mathbb{R}^n$, Math. Comp. 89 (324) (2020) 1711–1744.
[37] C. Chen, X. Huang, H. Wei, H^m-conforming virtual elements in arbitrary dimension, SIAM J. Numer. Anal. 60 (6) (2022) 3099–3123.
[38] J. Huang, Y. Yu, A medius error analysis for nonconforming virtual element methods for Poisson and biharmonic equations, J. Comput. Appl. Math. 386 (2021) 113229.
[39] F. Brezzi, R.S. Falk, L.D. Marini, Basic principles of mixed virtual element methods, ESAIM: Math. Model. Numer. Anal. 48 (4) (2014) 1227–1240.
[40] L. Beirão da Veiga, F. Brezzi, L.D. Marini, A. Russo, Mixed virtual element methods for general second order elliptic problems on polygonal meshes, ESAIM: Math. Model. Numer. Anal. 50 (3) (2016) 727–747.
[41] L. Beirão da Veiga, F. Brezzi, L.D. Marini, A. Russo, $H(\mathrm{div})$ and $H(\mathrm{curl})$-conforming virtual element methods, Numer. Math. 133 (2) (2016) 303–332.
[42] F. Dassi, A. Fumagalli, D. Losapio, S. Scialò, A. Scotti, G. Vacca, The mixed virtual element method on curved edges in two dimensions, Comput. Methods Appl. Mech. Eng. 386 (2021) 114098.
[43] L. Beirão da Veiga, L. Mascotto, Interpolation and stability properties of low order face and edge virtual element spaces, IMA J. Numer. Anal. 43 (2023) 828–851.
[44] L. Beirão da Veiga, L. Mascotto, J. Meng, Interpolation and stability estimates for edge and face virtual elements of general order, Math. Models Methods Appl. Sci. 32 (8) (2022) 1589–1632.
[45] F. Dassi, A. Fumagalli, A. Scotti, G. Vacca, Bend 3D mixed virtual element method for Darcy problems, Comput. Math. Appl. 119 (2022) 1–12.
[46] J. Zhao, B. Zhang, The curl–curl conforming virtual element method for the quad-curl problem, Math. Models Methods Appl. Sci. 31 (08) (2021) 1659–1690.
[47] M. Costabel, A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, Math. Methods Appl. Sci. 12 (4) (1990) 365–368.
[48] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci. 21 (9) (1998) 823–864.
[49] P.F. Antonietti, L. Beirão da Veiga, D. Mora, M. Verani, A stream virtual element formulation of the Stokes problem on polygonal meshes, SIAM J. Numer. Anal. 52 (1) (2014) 386–404.
[50] L. Beirão da Veiga, C. Lovadina, G. Vacca, Divergence free virtual elements for the Stokes problem on polygonal meshes, ESAIM: Math. Model. Numer. Anal. 51 (2) (2017) 509–535.
[51] L. Beirão da Veiga, L. Mascotto, J. Meng, Stability and interpolation properties for Stokes-like virtual element spaces, J. Sci. Comput. 94 (3) (2023).
[52] S. Cao, L. Chen, R. Guo, Immersed virtual element methods for electromagnetic interface problems in three dimensions, Math. Models Methods Appl. Sci. 33 (3) (2023) 455–503.
[53] S. Cao, L. Chen, R. Guo, F. Lin, Immersed virtual element methods for elliptic interface problems in two dimensions, J. Sci. Comput. 93 (1) (2022) 1–41.
[54] P.F. Antonietti, L. Mascotto, M. Verani, A multigrid algorithm for the p-version of the virtual element method, ESAIM: Math. Model. Numer. Anal. 52 (1) (2018) 337–364.
[55] L. Beirão da Veiga, C. Canuto, R.H. Nochetto, G. Vacca, M. Verani, Adaptive VEM: stabilization-free a posteriori error analysis and contraction property, SIAM J. Numer. Anal. 61 (2) (2023) 457–494.
[56] D. Mora, G. Rivera, R. Rodríguez, A virtual element method for the Steklov eigenvalue problem, Math. Models Methods Appl. Sci. 25 (08) (2015) 1421–1445.
[57] E. Artioli, S. De Miranda, C. Lovadina, L. Patruno, A stress/displacement virtual element method for plane elasticity problems, Comput. Methods Appl. Mech. Eng. 325 (2017) 155–174.
[58] F. Dassi, C. Lovadina, M. Visinoni, A three-dimensional Hellinger–Reissner virtual element method for linear elasticity problems, Comput. Methods Appl. Mech. Eng. 364 (2020) 112910.