Stability Estimates

Uniqueness due to a local condition

regular $\ast$ singular

Cauchy problem

Hp:

$u, v \in \mathcal{X}$

$\exists\, \tau \in \bar{T} \ni.$

regular:

$\exists \lim_{x \to x_0^+} u_x(\tau), v_x(\tau)$

$\left|\frac{1}{u_x}\right|(\tau), \left|\frac{1}{v_x}\right|(\tau) \leq_{a.e.} c_v$ in D

$\exists\, a(u,f) \in \mathcal{A}_{ad}$

$B \in \mathcal{B}_{ad}$

singular:

$\bar{I} \subset D$, given

$u_x(\tau), v_x(\tau) \in \mathcal{C}^0(\bar{I})$

$E_u(\tau), E_v(\tau) \neq \emptyset$

$\operatorname{meas} E_u(\tau) = \operatorname{meas} E_v(\tau) = 0$

$E_u(\tau), E_v(\tau) \subseteq \bar{I}$

$\left\|\frac{1}{u_x}\right\|_{0,q}(\tau), \left\|\frac{1}{v_x}\right\|_{0,q}(\tau) \leq c_v$

for some $1 \leq q < \infty$

$\exists\, a \in \mathcal{A}_{ad} \cap \mathcal{C}^0(\bar{I})$

$\exists\, b(v,f) \in L^\infty;\ b \geq_{a.e} a_L (>0)$

Th:

$\|B\|_{0,\infty} \leq$ $\qquad$ $\|B\|_{0,q} \leq$

$$\leq c_v(1+2a_H)\|V\|_{\mathcal{X}(\tau)}$$

$$\|V\|_{\mathcal{X}(\tau)} := \max_{D}|V|(\tau) + \|V_x\|_{0,\infty}(\tau) + \sqrt{|x_1-x_0|}\,\|V_t\|_{0,2}(\tau)$$

L^∞-estimate $\qquad$ L^q-estimate

STABILITY ESTIMATES:
3 – *SINGULAR* CAUCHY $*$ – UNIQUE SOLN.

Cor.

(only different Hp. are shown)

IF $\quad \text{meas}[E_v(\tau)] > 0$

$$\left(\tfrac{1}{v_x}\right)(\tau) \in L^1(D \setminus E_v(\tau)) := \mathbb{Y}(\tau)$$

$$\left\|\tfrac{1}{v_x}\right\|_{\mathbb{Y}_\tau} \leq c_v$$

$$b = a \text{ in } E_v(\tau)$$

THEN

$$\|B\|_{0,1} \leq c_v \, [1 + 2a_H] \, \|V\|_{\mathbb{X}_\tau}$$

STABILITY ESTIMATES:
6 – *SINGULAR* CAUCHY ** – UNIQUE SOLN.

Let $\overline{I} \subset D$

Thm.

IF $\quad u, v \in \mathbb{X}$, u, v comply with (**)

$$\bigcup_t E_u(t)\ ,\ \bigcup_t E_v(t) \subseteq \overline{I}$$

$$\forall t \in T,\quad \left(\tfrac{1}{u_x}\right)(., t) \in L^1(\,D\backslash E_{u(t)}\,)$$

$$\left(\tfrac{1}{v_x}\right)(., t) \in L^1(\,D\backslash E_{v(t)}\,) := \mathbb{Y}(\,t\,)$$

$$\max_{\overline{T}} \left\|\frac{1}{v_x}\right\|_{\mathbb{Y}_t} \leq c_v$$

$$\exists\, a(\,u, f) \in \mathbb{A}_{ad} \cap C^0(\overline{I})$$

$\exists\, b(\,v, f) \in L^{\infty}$, $b \geq_{a.e.} a_L\ (>0)$

THEN

$$\|B\|_{0,1} \leq c_v[1 + 2a_H]\ \|V\|_{\mathbb{X}}$$

Stability Estimates

Synopsis of Proofs

$V := v - u \; ; \quad B := b - a \; ; \quad r := -V_t + (aV_x)_x$

$$(Bv_x)_x =_{d.w.} -r \qquad @\, t = \tau$$

Regular Cauchy Pbm.

$B(x_0^+) = 0$

*** Singular Cauchy Pbm.**

$(Bv_x)(\xi_v(\tau), \tau) = 0$

$$B =_{a.e.} \frac{-r_0^{[-1]} + c(\tau)}{v_x}$$

$$r_0^{[-1]} := -\int_{x_0}^{x} V_t \, d\xi + aV_x - (aV_x)_{0^+}$$

$$r_{0+}^{[-1]} = -\int_{\xi_v(\tau)}^{x} V_t \, d\xi + aV_x - (aV_x)(\xi_v^+(\tau), \tau)$$

$\xi_v(\tau) \le x \le x_1$

$$r_{0-}^{[-1]} = -\int_{-\xi_v(\tau)}^{y} V_t \, d\eta + aV_y - (aV_y)(\xi_v^-(\tau), \tau)$$

$-\xi_v(\tau) \le y \le -x_0$

$\|B\|_{0,\infty} \le$ $\qquad\qquad$ $\|B\|_{0,1} \le$

$$\le c_v \, \| r_0^{[-1]} \|_{0,\infty}(\tau) \le$$

$$\le c_v \left[\|V_t\|_{0,1} + 2a_H \|V_x\|_{0,\infty} \right](\tau)$$

L^∞- estimate

L^1- estimate

(L^q- estimate ; q finite , $q \ge 1$)

Stability Estimates

Uniqueness from a non-local condition

Hp.

$u, v \in \mathcal{X}$

$a(u,f) \in \mathcal{A}_{ad}$

$b(v,f) \in L^{\infty}$, $b \geq_{a.e.} a_L \ (>0)$

$\exists \tau \in \overline{T} \ . \ni .$

$\langle a u_x \rangle(\tau) = \langle b v_x \rangle(\tau) = k(\tau)$

$\exists\, V_x(x_0^+, \tau)$

$\left|\frac{1}{v_x}\right|(\tau) \leq_{a.e.} c_v$ in D

$\left\|\frac{1}{v_x}\right\|_{0,q}(\tau) \leq c_v$

$1 \leq q < \infty$

Th.

$\|B\|_{0,\infty} \leq$

$\|B\|_{0,q} \leq$

$\leq c_v (2 + a_H) \|V\|_{\mathcal{X}(\tau)}$

Biblioteca

Quadrelli

Crosta

Stability Estimates

Uniqueness from a non-local condition
the rôle of procedure

Hp.

$\exists\, \tau \in \overline{T} \ni$:

$$\langle a u_x \rangle(\tau) = \langle b v_x \rangle(\tau) = 0$$

$$B = \frac{r_0^{[-1]} + c}{v_x}(\tau)$$

$g := u_t + f$; $h := v_t + f$

$$\begin{cases} a =_{a.e.} \dfrac{g_0^{[-1]} - \langle g_0^{[-1]} \rangle}{u_x}(\tau) \\ b =_{a.e} \dfrac{h_0^{[-1]} - \langle h_0^{[-1]} \rangle}{v_x}(\tau) \end{cases}$$

Th.

$$\|B\|_{0,1} \le$$

$$\le C_v(2 + a_H)\|V\|_{\mathcal{X}(\tau)}$$

provided :

$u, v \in \mathcal{X}$

$\left\|\frac{1}{u_x}\right\|_{0,1}(\tau), \left\|\frac{1}{v_x}\right\|_{0,1}(\tau) \le C_v$

$\exists \lim_{x \to x_0^+} V_x(\cdot, \tau)$

$$\le 2C_v^2(C_g + C_4)\left[\|V\|_{1,4} + \|(V_t)_0^{[-1]}\|_{0,4}\right]$$

@ $t = \tau$

provided :

$\exists \lim_{x \to x_0^+} (v_t)^{(-1)}, (u_t)^{[-1]}, (f)^{[-1]}$;

$\left\|\frac{1}{u_x}\right\|_{0,4}(\tau), \left\|\frac{1}{v_x}\right\|_{0,4}(\tau) \le C_v$

$\|u_x\|_{0,4}(\tau) \le C_4$

$\|g_0^{[-1]}\|_{0,4}(\tau) \le C_g$

3.4 - Stability estimate for the self-identifiable solution

<u>Hp</u> $u, v \in \mathscr{C}^0(\bar{T}, W^{1,4}(D))$

$(u_t)^{[-1]}, (v_t)^{[-1]}, f^{[-1]} \subset \mathscr{C}^0(\bar{T}, L^4(D))$

$\exists \lim_{x \to x_0^+} u_t^{[-1]}, v_t^{[-1]}, f^{[-1]}$

$\frac{1}{u_x}, \frac{1}{v_x} \in \mathscr{C}^0(\bar{T}, L^4(D))$;

$\exists \tau \in \bar{T} \ni \| \frac{1}{u_x} \|_{0,4}, \| \frac{1}{v_x} \|_{0,4} (\tau) \leq C_v$

$\exists \hat{a} \in \mathcal{A}_{ad}, \exists b(v, f) \in \mathcal{A}_{ad} \ni$

$\langle \hat{a} u_x \rangle (\tau) = \langle b v_x \rangle (\tau) = 0$

<u>Th</u> : $\|B\|_{0,1} \leq 2 C_v^2 [\| g_0^{[-1]} \|_{0,4} + \| u_x \|_{0,4}] \cdot$

$\cdot [\| v - u \|_{1,4} + \| (v_t - u_t)_0^{[-1]} \|_{0,4}]$

all quantities on the r.h.s. to be evaluated at $t = \tau$.

<u>Pf</u> [treat $a u_x$ as a whole, to avoid splitting into (meaningless) sum of products $(a_x u_x + a u_{xx})$]

4- Recent developments

T.I. Seidman's approach (Arcata, 1989)

Composite "identification - control" map

data	parameters	results
$\{u,f\} \mapsto$	$a(u,f) \mapsto$	$v(a,g)$
	controls	
	g	

General result

the stability of the data-to-results map is determined by the data-to-parameters map

parameter uniqueness from	stability estimate
regular Cauchy datum "independent" data pairs	L^∞
KN cond.	L^1

CONCLUSION

i) Local *vs*. non local information

ii) Regular *vs*. singular Cauchy problems

iii) Unified view over

uniqueness conditions and stability estimates.

iv) Supplementary (regularization) conditions needed to attain stability estimates

v) Admissible a in stability estimates affected by type of Cauchy problem:

$a \in L^{\infty}$ when Cauchy pbm. regular;

continuity of a @ critical points cannot be relaxed.

vi) Application to time – independent inverse problems:

technically straightforward;

distinction between regular and singular Cauchy problems left unaltered;

same stability estimates

vii) Extension: stability of (Seidman' s) composite map

(*coefficients* to *results*)•(*data* to *coefficients*)