ACC 92

Mr Chairman, Ladies & gentlemen!

First of all let me thank the Conference organizers for accepting this contribution

I also have to acknowledge the support of these institutions ACK this research work is part of these two projects, funded resp by the Italian Ministry of Higher Education and by the Italian Nat'l Research Council

The distributed parameter system I am considering is DPS formed by a parabolic PDE, defined in a space-time domain. x represents the scalar space variable

Already at this point many questions arise

1) Why distributed parameter i.e. infinite dimensional systems at all?

If time is left, I leave a tentative answer to this question at the end of the talk.

Let me proceed. I want to identify the leading coefficient (conductivity) from knowledge of the {potential, source term} data pair

The next question is: why inverse problems for PDES? Well, for the same reason as parameter identification arises with finite dimensional systems.

Almost All inverse problems are known to be ill-posed (otherwise they would not be so interesting). Existence, uniqueness and stability have to be obtained at a price.

I have mentioned stability! In this context, stability will denote a property of the map which relates the data to the parameters. Stability here is a synonym for the type of continuity exhibited by the map.

DEF

In detail, consider the space-time domain Q and the parab. eqn., which shall be understood in the distribution sense.

The inverse problem consists of finding the conductivity a in this admissible set A, given f in this space and u in X. Note that X is made of continuous trajectories s.t. spatial derivatives are in L^2 up to the 2^{nd} order.
The admissible set A_{ad} is made of bounded measurable functions, which are bounded away from 0 and from above by given constants.
The key technical tool in the following is the defect equation, also to be understood in the distribution sense, which relates the difference V between the 2^{nd} & the reference potential to the difference B betw the 2^{nd} and the ref conductivity.
It is assumed throughout that admissible conductivities a, b exist. This is important.
The purpose of this talk is to provide a unifying UNI view on both uniqueness and stability of the solution to the considered inverse problem.
Uniqueness is obtained by providing a Cauchy datum for the defect equation, which is an ODE wr to B.
In general Cauchy data may be available either at regular or singular points.
A singular point a pt where the partial derivative of potential wr to x vanishes.

UREG

I can now state the uniqueness result for the regular Cauchy problem. Introduce the defect r and this quotient, which makes sense, provided the denominator does not vanish. I define B_{ad} by requiring that B be continuous at the left endpt of the interval and vanish there.
The uniqueness result perhaps is more easily read than written.

It requires some conditions to be met. By the way, $R^{[-1]}$ is the spatial antiderivative of a distribution, since all antider. of a distribution differ by a constant, then antidifferentiation is a set valued map, as it is for ordinary functions - Now, these antiderivatives shall be continuous @ x_0^+.

If at the instant τ these hypotheses hold, then B is zero a.e. in D.

USIN.

The uniqueness statement corresponding to the singular problem relies on substantially different hypotheses - more precisely it is based on the properties of the set of pts where potential is spatially stationary.

I assume that in addition to $\hat{a}$ (the reference cond) there ∃ another admissible cond $\tilde{a}$ s.t. the corresponding potentials coincide a.e. in Q -

Here f can be less regular than specified before. In fact, also u may be less regular. U_{ad} is a space larger than X.

I do not give the details here and refer you to the proceedings

Uniqueness is obtained

either if at a given τ, $E_u(\tau)$ is non empty but has zero Lebesgue measure

or if E_u is never empty and the measure of this intersection is zero.

There is another condition, which is non- USEL local and which I call self-identifiability - It may be interesting to examine it, however, I skip it for the sake of conciseness

When it comes to stability, additional cons- SREG traints have to be introduced i.e., the problem must be regularized.

Assume that uniqueness comes from a regular Cauchy pblm.
the previous hyp are not sufficient - some more must be added - these reciprocals shall eg be bounded above by given constants - this is an example of regularisation.
similarly for this L^2 norm -
But the most relevant restriction affects the reference conductivity. It shall be piecewise regular, although it may have a countable infinity of discontinuity pts.
Here $\delta(.)$ is Dirac's measure - Moreover the series of these absol values shall converge
under these hyp., there is a uniform (ie L^∞) stability estimate for B. the L^∞ norm of B is bounded above by the product of some constants times the X_T norm of V -
in fact, the quantity between square brackets can be replaced by a suitable constant - what matters is that it only depends on the reference conduc, $\hat{a}$, not on b.

the hyp. which affect conductiv may seem restric APPL. tive - however, there ∃ many applications where they are met - Consider e.g. a layered medium - Conductivity satisfying this condition is such that the anti derivatives of the abs value of a' are all in L^∞ - Moreover the reference $\hat{a}$ is continuous at x_0 and x_1 -

SSIN

If uniqueness is due to a singular Cauchy problem, we shall expect two things
i) that an estimate be obtained, provided the reference conductivity is continuous at the singular point
ii) that the type of stab estimate differs from the previous one, in spite of increased local regularity

of $\hat{a}$,

I have to introduce the closed interval I. I require that $1/u_x$ be in L^1, at least at that instant of time τ, when the set E_u complies with this condition, which is sufficient to yield uniqueness.

Now the interesting fact is that $E_v(\tau)$ may have positive measure. Both sets shall be contained in I, however $1/v_x$ shall be in L^1 of this interval. I assume $\hat{a}$, besides being admissible, is continuous on I and that b, besides being admissible, is made to coincide with $\hat{a}$ inside $E_v(\tau)$. Namely b would not be uniquely determined there.

Under these circumstances, there is an L^1 estimate for B, where the estimation constant is the product of C_v, introduced here, and the quantity 1 + twice the L^∞ norm of $\hat{a}$. This norm can be replaced by a_H.

SYN

Why do stability inequalities differ?

In both cases the starting pt is the defect equation at time τ.

The regular Cauchy pbm corresponds to this condition: $B(x_0)$ vanishes. I can then divide all terms in the eqn by v_x, which vanishes nowhere (as requested by the Hp), then estimate the growth of B. Standard stability theory for ODEs, based on a generalization of Gronwall-Bellman's inequality, yields this estimate, where the antiderivative of R must be rewritten as a fcn of V. This eventually yields the L^∞ estimate you have seen a few slides ago

On the other hand, if uniqueness comes from a singular Cauchy problem, the quantity which vanishes at time τ and at the point $\xi(\tau)$ is this product.
I cannot divide by v_x ( ) and carry out the same estimate as in the previous case, because v_x vanishes at some point.
uniform
I do divide by v_x, but then I have to integrate the quotient over the interval D, which gives me an L^1-estimate for B.
The remaining step is to replace this antiderivative by a function of V

We have seen the role played by the defect CONC. equation and the related Cauchy problems
They lead to a comprehensive view of these uniqueness/& stability estimates,
conditions
Also, I have shown the role of regularization conditions required to attain stability estimates.
I hope the synopsis of the proof has given an idea of the technical details.
The last remark in the list is about the time independent case: everything applies in a straightforward way.

I can now answer the question: why infinite dimensional systems? In practical simulations all PDES are replaced by algebraic systems. It may seem that finite dim systems suffice. On the other hand we often want to refine the grid over which our algebraic system is defined. We want therefore to predict what happens as the grid is made finer & finer. This implies passing to the limit, hence investigating an infinite-dimensional system, governed by a PDE. Thank You!