Università degli studi di Milano-Bicocca
Università degli studi di Pavia
Istituto Nazionale di Alta Matematica

Joint Ph.D. Program in Mathematics
XXXVI Cycle

Unique continuation principles for elliptic and parabolic equations and spectral stability for Aharonov-Bohm operators

Ph.D. candidate:
Giovanni Siclari

Supervisor:
Prof. Veronica Felli

Not for money,
nor love,
nor the heavens.

Non al denaro,
non all'amore,
ne al cielo.

F. De André.

Abstract

The present dissertation consists of three parts.

In the first part we study unique continuation principles and the asymptotic behaviour of weak solutions to some elliptic problems. Our approach is based on the combination of an Almgren-type monotonicity formula with a blow-up analysis. Pohozaev-type identities play a key role in the derivation of Almgren's monotonicity formulas and can be challenging to derive when the solution lacks regularity. Then we also present a regularity result in weighted Sobolev spaces and a Pohozaev identity obtained as an application.

More precisely, in Chapter 2 we derive local asymptotics of solutions to second order elliptic equations at the edge of a $(N-1)$-dimensional crack, with homogeneous Neumann boundary conditions prescribed on both sides of the crack. We provide a complete classification of all possible asymptotic degrees of homogeneities of solutions at the crack's tip, together with a strong unique continuation principle.

In Chapter 3 we recall some useful results about weighted Sobolev spaces which are used throughout the present dissertation. Furthermore, we prove Sobolev-type regularity results for solutions to a class of second order elliptic equations with a singular or degenerate weight, under non-homogeneous Neumann conditions. As an application, we derive a Pohozaev-type identity.

In Chapter 4 we investigate unique continuation properties and asymptotic behaviour at boundary points for solutions to a class of elliptic equations involving the spectral fractional Laplacian. An extension procedure leads us to study a degenerate or singular equation on a cylinder, with a homogeneous Dirichlet boundary condition on the lateral surface and a non homogeneous Neumann condition on the basis. For the extended problem, we classify the local asymptotic profiles at the edge where the transition between boundary conditions occurs. Passing to traces, an analogous blow-up result and its consequent strong unique continuation property are deduced for the non-local fractional equation.

In Chapter 5 strong unique continuation properties and a classification of the asymptotic profiles are established for spectral fractional powers of a Schrödinger operator with a Hardy-type potential singular at 0. Similarly to Chapter 4, we make use of an extension procedure and classify the local behaviour for the extended problem, which turns out to depend on the coefficient of the singular potential.

In the second part of the present dissertation, we investigate unique continuation principles and the asymptotic behaviour of weak solutions to a class of parabolic equations. More precisely, in Chapter 6 we obtain a classification of local asymptotic profiles and strong unique continuation properties for a class of fractional heat equations with a Hardy-type potential. Similarly to the elliptic case, we make use of an extension procedure to localize the problem.

In the third part of the present dissertation, we study the asymptotic behaviour of simple eigenvalues of Aharonov-Bohm operators with half integer circulation on a simple connected

bounded domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary conditions as many poles coalesce to a fixed point. More precisely, in Chapter 7, we make use of a gauge transformation to reformulate the problem as an eigenvalue problem for the Laplacian in a domain with straight cracks, laying along the moving directions of poles. For this problem, we obtain an asymptotic expansion for eigenvalues, in which the dominant term is related to the minimum of an energy functional associated with the configuration of poles and defined on a space of functions suitably jumping through the cracks. Concerning configurations with an odd number of poles, an accurate blow-up analysis identifies the exact asymptotic behaviour of eigenvalues and the sign of the variation in some cases. An application to the special case of two poles is also discussed.

Acknowledgements

I have really loved spending these three years at U5. Bicocca has become a welcoming home to me thanks to the many people that I have meet there. They have taught me so much and for this I am deeply grateful.

The first person that I would like to thank is my Ph.D. supervisor Veronica Felli. It is not easy to sum up all the help and contributions to my academic life she has given me over these three years. I just want to say thanks for her constant patience and for never getting tired of correcting my mistakes. The very same, again and again. I also want to thank Susanna Terracini getting me in touch with Veronica at the very begging of my Ph.D.

Moreover I am very grateful to Ana Primo Ramos for her hospitality in my many visit to the Universidad Autonoma of Madrid and for fruitful collaborations. I also want to thank her for introducing me to many wonderful people in Madrid, in particular the Ph.D students of the Universidad Autonoma. I am also grateful to Boumediene Abdellaoui for many afternoons spent together doing lengthy computations, he has really taught me a lot. Furthermore I am thankful to Benedetta Noris for her good advices and many stimulating discussions.

Another huge thanks goes to my dear friends Federico Bernini, Roberto Ognibene, Giulio Romani and Stefano Vita. I have asked them more things then I can remember during these three years and they have always answered even when the questions made little sense. A special thanks to another one of my best friends, Andrea Bisterzio. We have spent this Ph.D. together and I am very lucky to have meet him. I also want to thank all the postdoc and Ph.D students at the university of Milano-Bicocca. I have made some wonderful friends there and I apologize if I cannot mention you all by name. In particular I want to thank Marta Tameni, Sara Perletti, Alberto Cassella, Simone Blumer, Ettore Marmo, Tommaso Toti, Simone Gallivanone, Fabio Mastrogiacomo, Marco Fusuari and Garbiele Barbieri for their steadfast friendship. Furthermore I am grateful to Anna Donadini, Eleonora Sforza and Claudia Nocita for making my office the best one on the whole floor this year.

Finally I want to express my gratitude to my whole family. They have always been there for me and it really means a lot.

Contents

Chapter 1

Introduction

The present dissertation deals with two subjects. The first one is strong unique continuation and classification of local behaviour of solutions to some elliptic and parabolic equations. We recall that a family of functions $\mathcal{F} = \{f_i\}_{i\in I}$, with $f_i : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^N$, satisfies the *strong unique continuation* property if no function in $\mathcal{F}$, besides possibly the trivial null function, has a zero of infinite order at any point $x_0 \in A$. We study unique continuation properties for several classes of problems, also involving fractional operators, by monotonicity approach. The proof of monotonicity formulas in a fractional setting rises delicate regularity issues; these require the development of ad-hoc Sobolev-type regularity results, which happen to be of independent interest.

The second subject treated in this thesis is spectral stability for Aharonov-Bohm operators, on a simple connected bounded domain $\Omega \subset \mathbb{R}^2$, with Dirichlet boundary condition. We focus on the case of half-integer-circulation, which is of particular interest from a physical and mathematical point of view. More precisely, we consider operators with many poles coalescing to a fixed point and study the sharp asymptotic behaviour of eigenvalues.

In this introduction we give a brief and not exhaustive overview of the vast literature on this two subjects and outline our results and methods.

1.0.1 Part I: Unique continuation for elliptic problems

In the first part of this dissertation we study unique continuation for some elliptic equations. The first result about strong unique continuation for second order problems was obtained by Carleman in [38] for bounded potentials in dimension 2, by means of weighted a priori inequalities. The so-called *Carleman estimates* are still today one of the main techniques used in this research field. They have been adapted by many authors to generalize Carleman's results and prove unique continuation for more general classes of elliptic equations. Among the numerous contributions in this area we mention [16, 88, 122, 133] and in particular [92], where strong unique continuation is established under sharp scale invariant assumptions on the potentials. Garofalo and Lin developed in [79] an alternative approach to the study of unique continuation, based on local doubling inequalities, which are in turn deduced by the monotonicity of an Almgren-type frequency function, see [14]. In the present dissertation we follow this latter approach.

For some fixed $x_0 \in \mathbb{R}^N$, the Almgren frequency function $\mathcal{N}$, associated to the solution u of some problem, can be defined as the ratio between a local energy D and a local mass or height H around x_0. For example, for the model problem

$$-\Delta u = hu$$

and $x_0 = 0$, the energy and height functions are defined as

$$D(r) := \frac{1}{r^{N-2}} \int_{B_r} (|\nabla u|^2 - hu^2)\, dx \quad \text{and} \quad H(r) := \frac{1}{r^{N-1}} \int_{\partial B_r} u^2 \, dS,$$

respectively, while the frequency function $\mathcal{N}$ is given by

$$\mathcal{N}(r) := \frac{D(r)}{H(r)}.$$

For any $r > 0$, we are denoting with B_r the set

$$B_r := \{x \in \mathbb{R}^N : |x| < r\}. \tag{1.1}$$

The notion of frequency introduced above has been adapted to several classes of elliptic problems, see [65, 67, 68] for equations with singular potentials and [63] for domains with corners, to prove not only unique continuation, but also, in combination with a blow-up analysis, a classification of possible asymptotic profiles of solutions.

In Chapter 2 we establish a strong unique continuation principle and analyse the asymptotic behaviour of solutions, from the edge of a flat crack Γ, for the following elliptic problem with homogeneous Neumann boundary conditions on both sides of the crack

$$\begin{cases} -\Delta u = fu, & \text{in } B_R \setminus \Gamma, \\ \dfrac{\partial^+ u}{\partial \nu^+} = \dfrac{\partial^- u}{\partial \nu^-} = 0, & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where $B_R \subset \mathbb{R}^N$ is as in (1.1), $N \geq 2$, Γ is a closed subset of $\mathbb{R}^{N-1} \times \{0\}$ with $C^{1,1}$-boundary, and the potential f is a measurable function satisfying suitable regularity or growth conditions (see (2.6) and (2.7)). The boundary operators $\frac{\partial^+}{\partial \nu^+}$ and $\frac{\partial^-}{\partial \nu^-}$ in (1.2) are defined as

$$\frac{\partial^+ u}{\partial \nu^+} := -\frac{\partial}{\partial x_N} \left(u|_{B_R^+} \right) \quad \text{and} \quad \frac{\partial^- u}{\partial \nu^-} := \frac{\partial}{\partial x_N} \left(u|_{B_R^-} \right),$$

where we are denoting, for all $r > 0$,

$$B_r^+ := \{(x', x_{N-1}, x_N) \in B_r : x_N > 0\}, \quad B_r^- := \{(x', x_{N-1}, x_N) \in B_r : x_N < 0\},$$

being the total variable $x \in \mathbb{R}^N$ written as $x = (x', x_{N-1}, x_N) \in \mathbb{R}^{N-2} \times \mathbb{R} \times \mathbb{R}$.

The interest in elliptic problems in domains with cracks is motivated by elasticity theory, see e.g. [90, 43]. In particular, in crack problems, the coefficients of the asymptotic expansion of solutions near the crack's tip are related to the so called *stress intensity factor*, see [43]. We refer to [40, 41, 52] and references therein for the study of the behaviour of solutions at the edge of a cut.

The derivation of a monotonicity formula around a boundary point presents some additional difficulties with respect to the interior case, due to the role that the regularity and the geometry of the domain may play.

Among papers dealing with unique continuation from the boundary under homogeneous Dirichlet conditions we cite [12, 13, 63, 94, 130]. Instead, for Neumann problems, we refer to [12] and [126] for the homogeneous case and to [49] for unique continuation from the vertex of a cone under non-homogeneous Neumann conditions. We also mention that unique continuation from Dirichlet-Neumann junctions for planar mixed boundary value problems was established in [61].

In order to estimate the derivative of the Almgren frequency function, see Proposition 2.3.10, a Pohozaev type identity is needed. However, the high non-smoothness of the domain $B_R \setminus \Gamma$ at points on the edge of the crack causes two kinds of difficulties in its proof. A first difficulty is a lack of regularity that can prevent us from integrating Rellich-Nečas identities of type (2.67). A second issue is related to the interference with the geometry of the crack, which manifests in the form of extra terms, produced by integration by parts, which could be problematic to estimate.

In [45], where homogeneous Dirichlet conditions on the crack are considered, this latter difficulty is overcome by assuming a local star-shapedness condition for the cracked domain. This geometric assumption forces the extra terms, produced by integration by parts, to have a sign favourable to the desired estimates. The problem produced by lack of regularity is instead solved in [45] by approximating $B_r \setminus \Gamma$ with a sequence of smooth domains $\Omega_{n,r} \subset B_r$. The solutions u_n of approximating problems in $\Omega_{n,r}$ converge in $H^1(B_r)$ to the solution of the original cracked problem for $r \in (0, R)$ small enough. Each function u_n is sufficiently regular to satisfy a Pohozaev type identity, in which it is possible to pass to the limit as $n \to \infty$. In this way it is possible to establish the inequality needed to estimate the derivative of the Almgren frequency function.

In Chapter 2 we use a similar approximation technique, which however entails additional difficulties and requires substantial modifications due to the Neumann boundary conditions. In particular, the existence of an extension operator for Sobolev functions on Ω_n, uniform with respect to n, is obvious under Dirichlet boundary conditions but it turns out to be more delicate in the Neumann case, see Proposition 2.2.11. Furthermore, the different boundary conditions produce remainder terms with different signs, requiring a modified profile for the approximating domains, see Section 2.2.3.

Unlike [45], we do not require any geometric star-shapedness condition on the crack Γ, limiting ourselves to a $C^{1,1}$-regularity assumption, see (2.4) below. The removal of the star-shapedness condition assumed in [45] requires a more sophisticated monotonicity formula, which is developed for the auxiliary problem (2.23), obtained after straightening the crack Γ with a diffeomorphism introduced in [12], see Section 2.2.1. We mention that the same diffeomorphism is used for fractional elliptic equations, with a similar purpose, in [47]. The effect of this transformation straightening the crack is the appearance of a variable coefficient matrix in the divergence-form elliptic operator. As a consequence, an adaption of the definition of the energy D and the height H is needed, see (2.58) and (2.59). Chapter 2 is based on the paper [76].

In Chapter 3 we develop a Sobolev-type regularity theory in some weighted Sobolev spaces which, besides being of independent interest, is a key ingredient to prove a Pohozaev-type identity and a monotonicity formula in a fractional setting, see Chapters 4 and 5. More precisely, we deal with the following class of second order elliptic equations

$$-\operatorname{div}(y^{1-2s}A(x,y)\nabla U(x,y)) + y^{1-2s}c(x,y) = 0, \quad x \in \mathbb{R}^N, \ y \in (0,+\infty), \tag{1.3}$$

with the weight y^{1-2s} (being $s \in (0,1)$) which belongs to the second Muckenhoupt class and is singular if $s > 1/2$ and degenerate if $s < 1/2$; we couple (1.3) with non-homogeneous Neumann conditions

$$\lim_{y \to 0^+} y^{1-2s} A(x,y) \nabla U(x,y) \cdot \nu = hU(x,0) + g(x) \tag{1.4}$$

on the bottom of a half $(N+1)$-dimensional ball.

The interest in such a type of equations and related regularity issues has developed starting from the pioneering paper [58], proving local Hölder continuity results and Harnack's inequalities, and has grown significantly in recent years stimulated by the study of the fractional Laplacian in its realization as a Dirichlet-to-Neumann map [35].

In this context, among recent regularity results for problems of type (1.3)–(1.4), we mention [33] and [89] for Schauder and gradient estimates with A being the identity matrix and $c \equiv 0$. More general degenerate/singular equations of type (1.3), admitting a varying coefficient matrix A, are considered in [120, 121]. In [120], under suitable regularity assumptions on A and c, Hölder continuity and $C^{1,\alpha}$-regularity are established for solutions to (1.3)–(1.4) in the case $h \equiv g \equiv 0$, which, up to a reflection through the hyperspace $y = 0$, corresponds to the study of solutions to the equation $-\operatorname{div}(|y|^{1-2s} A \nabla U) + |y|^{1-2s} c = 0$ which are even with respect to the y-variable; Hölder continuity of solutions which are odd in y is instead investigated in [121]. In addition, in [120] $C^{0,\alpha}$ and $C^{1,\alpha}$ bounds are derived for some inhomogeneous Neumann boundary problems (i.e. for $g \not\equiv 0$) in the case $c \equiv 0$. We also mention [51, 50] for regularity results in weighted Sobolev spaces and mixed-norm weighted Sobolev spaces for a class of singular or degenerate parabolic and elliptic equations in the upper half space.

Our goal in Chapter 3 is to derive Sobolev-type regularity results for solutions to (1.3)–(1.4). Under suitable assumptions on c, h, g, the presence of the singular/degenerate homogeneous weight, involving only the $(N+1)$-th variable y, makes the solutions to have derivates with respect to the first N variables $x_1, x_2, \dots, x_N$ belonging to a weighted H^1-space (with the same weight y^{1-2s}); concerning the regularity of the derivative with respect to y, we obtain instead that the weighted derivative $y^{1-2s} \frac{\partial U}{\partial y}$ belongs to a H^1-space with the dual weight y^{2s-1}, confirming what has already been observed in [120, Lemma 7.1] for even solutions of the reflected problem corresponding to (1.3)–(1.4) with $h \equiv g \equiv 0$.

Our motivation for studying this question lies in the search for the minimal regularity needed to prove Pohozaev-type identities for solutions of the extended problem, resulting from the Caffarelli-Silvestre extension for the fractional Laplacian; Pohozaev-type identities can in turn be used to obtain Almgren-type monotonicity formulas in the spirit of [60]. Indeed, the Sobolev-type regularity results obtained in Theorem 3.2.1 allow us to directly obtain a Pohozaev-type identity (Proposition 3.2.3), without requiring C^1-regularity for the potential h as in [60] and without approximating potentials in Sobolev spaces with smooth ones as done in [47]. Furthermore, the presence of the matrix A makes our results applicable even to the problem modified by a diffeomorphic deformation of the domain, which straightens a $C^{1,1}$-boundary and produces the appearance of a variable coefficient matrix A, satisfying suitable regularity conditions (see (3.14), (3.15), and (3.16)); such a procedure is useful to study the behaviour of solutions at the boundary, see e.g. [47].

For a precise statement of our regularity result and the Pohozaev-type identity see Theorem 3.2.1 and Proposition 3.2.3. The proof of Theorem 3.2.1 is based on the classical Nirenberg difference quotient method, see [107]. Chapter 3 is based on the paper [75].

In Chapter 4 we prove the strong unique continuation property and derive local asymptotics from a point $x_0 \in \partial\Omega$ for the solutions to the following equation

$$(-\Delta)^s u = hu \quad \text{on } \Omega,$$

where $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^N$ is a bounded Lipschitz domain whose boundary is $C^{1,1}$ in a neighbourhood of x_0, $N > 2s$, h is a measurable function on Ω satisfying suitable summability properties, see (4.8), and $(-\Delta)^s$ is the so-called *spectral* fractional Laplacian.

Several results are available in the literature about the spectral fractional Laplacian and its interpretations. See [9], [102], and references therein for a detailed overview. We mention that regularity properties for stationary equations are discussed in [82], while existence and uniqueness results for evolution equations governed by the spectral fractional Laplacian are established in [26]. More closely related to our topic of investigation are the results in [135], where a strong unique continuation principle at nodal points is proved for fractional powers of some divergence-type elliptic operators, including the case of the spectral fractional Laplacian. The techniques used in [135] are inspired by those introduced in [60], which are based on a combination of a monotonicity formula for an Almgren-type frequency function and a blow-up analysis. This local approach is made possible by the extension results by [125, Theorem 1.1] and [35, Theorem 2.5].

As already observed, since the point x_0 from which the unique continuation is sought after lies on $\partial\Omega$, the geometry of $\partial\Omega$ can interfere with the monotonicity argument. In Chapter 4 we face this difficulty by straightening the boundary with a local diffeomorphism in the same spirt of Chapter 2. This transformation transfers the information about the geometry of $\partial\Omega$ into a coefficient matrix in the operator, which turns out to be a perturbation of the identity if the boundary is regular enough, see Section 4.3. Secondly, we make use of the Pohozaev type identity obtained in Proposition 3.2.3 to differentiate the frequency function and to develop the monotonicity argument. Furthermore, a blow-up analysis provides a detailed description of the asymptotic behaviour of solutions to (4.1) at x_0, giving a complete classification of the order of homogeneity of asymptotic profiles, see Theorem 4.1.2 below. For this purpose, an important role is played by an eigenvalue problem on a half-sphere under a symmetry condition, see (4.19).

The extension problem corresponding to (4.1) consists of a degenerate or singular equation on the cylinder $\Omega \times (0, +\infty)$; a homogeneous Dirichlet boundary condition is imposed on the lateral surface $\partial\Omega \times (0, +\infty)$ and a weighted Neumann-type derivative on the basis $\Omega \times \{0\}$ is equal to the right hand side of (4.1), see (4.17). Therefore, the formulation of the problem in terms of the extension leads us to study what happens near a point of the edge, at which a transition between boundary conditions of a different type takes place. We observe that this situation is quite different from the one that occurs in [47], where unique continuation from boundary points is studied for the *restricted fractional Laplacian*; indeed, the extension problem corresponding to the case treated in [47] is a degenerate or singular problem with mixed conditions that vary on a flat basis rather than on an edge. In fact, the analysis carried out in Chapter 4 highlights different asymptotic behaviours at the boundary for the two operators, unlike what happens at internal points, where the locally equivalent form of the extended problems induces the same blow-up profiles. Chapter 4 is based on the paper [46].

In Chapter 5 we deal with fractional powers of the operator

$$L_{\alpha,k} u := -\Delta u - \frac{\alpha}{|x|_k^2} u$$

on a connected bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ and $0 \in \Omega$, where

$$|x|_k^2 = \sum_{i=1}^{k} x_i^2 \quad \text{and} \quad \alpha \in \left(-\infty, \left(\frac{k-2}{2}\right)^2\right)$$

for any $k \in \{3, \dots, N\}$. If $k = N$ we simply write $|x|$ for $|x|_N$.

The operator $L_{\alpha,k}$ is an elliptic operator which is singular on a $N-k$-dimensional set. In view of Hardy-Maz'ja-type inequalities, see Section 5.1, the operator $L_{\alpha,k}$ has a discrete spectrum on $H_0^1(\Omega)$. Hence the fractional powers $L_{\alpha,k}^s$ of $L_{\alpha,k}$ with $s \in (0,1)$ can be defined in a spectral sense, see for example [125]. In the particular case $\alpha = 0$ the operator $L_{\alpha,k}^s$ reduces to the spectral fractional Laplacian $(-\Delta)^s$ consider in Chapter 4.

We give a more precise definition of $L_{\alpha,k}^s$ in Section 5.1. To the best of our knowledge, the operator $L_{\alpha,k}^s$ has not been considered before in the literature with $\alpha \neq 0$ in a bounded domain. In the whole space $\mathbb{R}^N$ the fractional powers of $L_{\alpha,N}$ have already been defined by means of spectral theory, see [78]. In [78], generalised and reversed Hardy types inequalities have been obtained for $L_{\alpha,N}^s$, using semigroup theory and estimates on the corresponding heat kernel.

We establish a unique continuation principle from the singular point 0 and classify the asymptotic profiles for solutions of linear equations involving the operator $L_{\alpha,k}^s$. More precisely, we are interested in the equation

$$L_{\alpha,k}^s u = gu \quad \text{in } \Omega$$

where the potential g is a measurable function satisfying some growth assumption near 0, see (5.3). In particular, we prove that the asymptotic profile of u in 0 is a homogenous function. We also characterize the possible orders of homogeneity of blow-up profiles, which have a non-trivial dependence on the singular potential $\alpha|x|_k^{-2}$.

For the restricted fractional Laplacian with a Hardy-type potential, under similar assumptions on the potential g and with a non-linear term, a complete classification of the possible asymptotic profiles and a unique continuation property from 0 have been obtained in [60]. The asymptotic behaviour of the spectral fractional Laplacian with a Hardy-type potential is identical since the equivalent problem obtained with a Caffarelli-Silvestre extension procedure is locally the same. The restricted fractional Laplacian with a Hardy-type potential has been intensively studied in the literature, see for example [59, 25, 10, 62, 69] and the references within.

If $k = N$, it is interesting to compare our results with [60], in particular as far as the minimal order of homogeneity of the asymptotics profiles are concerned, see (5.23), Theorem 5.1.10 and [60, Proposition 2.3]. In our case, it is possible to compute it explicitly, while for the restricted fractional Laplacian only a more implicit expression is available.

Similar results have been obtained in [68] in the classical case, that is $s = 1$, in the much more general situation of multiple potentials, including cylindrical and multi-body ones, and with the presence of a non-linear term. Furthermore, in [68] the authors also studied regularity properties of the solutions by means of a Brezis-Kato argument and obtained pointwise estimates.

Similarly to Chapter 4, in order to obtain an Almgren type monotonicity formula and perform a blow-up analysis, we localize the problem with an extension result, see Theorem 5.1.7 and also [37, 35, 125]. We also need a Pohozaev type identity. The singularity of the

Hardy type potential $\alpha|x|_k^{-2}$, the assumptions (5.3) on g and the singularity or degeneracy of the Muckenhoupt weight y^{1-2s} in the hyperplane $\mathbb{R}^n \times \{0\}$ cause an eventual lack of regularity for solutions to the extended problem. We overcome this difficulty by means of an approximation procedure based on the Implicit Function Theorem and the Pohozaev identity proved in Proposition 3.2.3. Chapter 5 is based on the paper [118].

1.0.2 Part II: Unique continuation for parabolic problems

In the second part of the present dissertation we deal with fractional parabolic equations. There exists a large literature dealing with strong continuation properties in the local parabolic setting. Similarly to the elliptic case, both Carleman estimates and monotonicity methods, have been widely used starting form the pioneristic paper [113]. We mention [100] for unique continuation for parabolic operators with $L^{\frac{N+1}{2}}$ time-independent coefficients and [115, 123] for unique continuation on horizontal components, proved by Carleman weighted inequalities, in the presence of time-dependent coefficients. The paper [39] contains not only a unique continuation result but also some local asymptotic analysis of solutions to parabolic inequalities with bounded coefficients. We quote [53, 54, 55, 56, 77] for unique continuation results for parabolic equations with time-dependent potentials by Carleman inequalities and monotonicity methods. We also refer to [22] for unique continuation properties for the heat operator with a Hardy potential established by Carleman estimates.

In Chapter 6 we deal with the following singular fractional evolution equation

$$(w_t - \Delta w)^s = \frac{1}{\kappa_s}\left(\frac{\mu}{|x|^{2s}}w + gw\right), \quad \text{in } \mathbb{R}^N \times (t_0 - T, t_0), \tag{1.5}$$

where $T > 0$, and, letting Γ be the usual Γ-function,

$$s \in (0,1), \quad N > 2s, \quad \mu < \kappa_s \Lambda_{N,s}, \quad \kappa_s := \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}, \quad \Lambda_{N,s} := 2^{2s}\frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}.$$

The potential g is a measurable function satisfying some regularity and growth assumptions see (6.4) and (6.37). We are interested in studying the asymptotic behaviour of solutions to (1.5) at $(x,t) = (0,t_0)$ along the directions $(\lambda x, t_0 - \lambda^2 t)$ as $\lambda \to 0^+$. Our main result is a classification of possible limiting asymptotic rates and profiles in terms of the eigenfunctions of a weighted Ornstein-Uhlenbeck operator. As a corollary, we obtain a strong space-like unique continuation property from the point $(0, t_0)$.

In the literature one may find many definitions of the operator $H^s(w) := (w_t - \Delta w)^s$ in (1.5), that is of the fractional power of the classical heat operator $H(w) := w_t - \Delta w$. We refer to [19] and [125] for a presentation of the several ways to define H^s corresponding to different functional settings. It is also worth mentioning that a pointwise formula for $H^s u$ is derived in [125]. In Section 6.1 we give a precise definition of H^s and of weak solutions to (1.5) by the Fourier transform.

Our approach is based on an Almgren-Poon type monotonicity formula, see [113], combined with a blow-up argument. We mention that monotonicity methods and blow-up analysis are used in [72] to prove strong unique continuation and classification of blow-up profiles for parabolic equations with a Hardy potential (corresponding to the case $s = 1$ in (1.5)); analogous results are obtained in [73] for a class of parabolic equations with critical electromagnetic potentials.

To deal with the fractional case we introduce an Almgren-Poon frequency function for an equivalent localized problem, constructed by the extension procedure developed in [21, 24, 110, 125], in the spirit of the one introduced by Caffarelli and Silvestre in [35] for the fractional powers of the Laplacian. This leads us to deal with equation (6.10), which is a local degenerate or singular parabolic problem in a one more dimension, see Section 6.1 for the details.

In a fractional parabolic setting, an Almgren-Poon frequency formula is first established in [125] in the absence of potentials, i.e. for $g \equiv 0$ and $\mu = 0$. Subsequently, an Almgren-Poon monotonicity approach is used in [21] to prove unique continuation properties for weak solutions to (1.5) in the case $\mu = 0$, that is without the Hardy singularity, and under C^1 or C^2 regularity assumptions on the potential g, depending on the value of s. In [21] a crucial role is played by a Hölder regularity theory for solutions to the extended problem, which has, in addition to its independent interest, applications to the estimates needed to derive an Almgren-Poon type monotonicity formula. We mention that a space-like strong unique continuation property is established in [17] in the case $\mu = 0$ via a conditional elliptic type doubling property and blow-up analysis. The case $\mu = 0$ is treated also in [19], where, under similar regularity assumptions on the potential g, a fine analysis of the structure of the nodal set and of possible blow-ups of solutions vanishing with a finite order is performed. The approach of [19] is also based on an Almgren-Poon type monotonicity formula and makes use of some uniform Hölder bounds, improving the regularity estimates of [21] and providing an independent proof of the Hölder regularity of weak solutions.

Due to the presence of a Hardy-type potential, there is no hope to obtain similar regularity results, since weak solutions to (1.5) may in general be not bounded, see Theorem 6.1.7. In the spirit of [72], to overcome this difficulty we rely instead on the theory of abstract parabolic equations, once a formulation of the extension problem in a suitable Gaussian space is obtained. Furthermore we also obtain a classification of the asymptotic profiles of weak solutions to (1.5) at $(x,t) = (0,t_0)$ along the directions $(\lambda x, t_0 - \lambda^2 t)$ as $\lambda \to 0^+$, see Theorem 6.1.7 and Theorem 6.1.6 in Section 6.1. Chapter 6 is based on the paper [74].

1.0.3 Part III: Spectral Stability for Aharonov-Bohm operators

In the third part of the present thesis we deal with quantitative spectral stability for Aharonov-Bohm operators with many coalescing poles, half-integer circulation and homogeneous Dirichlet boundary conditions on a simply connected open bounded domain $\Omega \subset \mathbb{R}^2$.

More precisely, in Chapter 7 we consider the case of any number k of poles moving along straight lines towards a collision point $P \in \Omega$, with distances from P vanishing with the same order. Without loss of generality, we assume that $P = 0 \in \Omega$, so that the moving poles can be written as multiples of k fixed points $\{a^j\}_{j=1,\dots,k}$ with the same multiplicative infinitesimal parameter $\varepsilon > 0$.

Since we are interested in the asymptotic behaviour of eigenvalues as $\varepsilon \to 0^+$, it is not restrictive to assume that there exists $R < 1$ such that

$$\{a^j\}_{j=1,\dots,k} \subset B_R(0) \subset \Omega,$$

where, for every $r > 0$ and $x \in \mathbb{R}^2$, we denote $B_r(x) := \{y \in \mathbb{R}^2 : |x - y| < r\}$. Henceforth, we denote $B_r(0)$ simply by B_r.

We assume that, among the k poles, there are k_1 poles that stand alone on their own straight line through the origin, while the remaining ones form k_2 pairs of poles staying on the same straight line but on different sides with respect to the origin. Hence

$$k = k_1 + 2k_2 \quad \text{with } k_1, k_2 \in \mathbb{N}, \ (k_1, k_2) \neq (0, 0),$$

and, for every $j = 1, \ldots, k$, there exist $r_j > 0$ and $\alpha^j \in (-\pi, \pi]$ such that $\alpha^j \neq \alpha^\ell$ if $j \neq \ell$ and

$$a^j = r_j(\cos(\alpha^j), \sin(\alpha^j)), \tag{1.6}$$

where $\alpha^{j_1} \neq \alpha^{j_2} \pm \pi$ if $j_1 \neq j_2$ and $j_1, j_2 \in \{1, \ldots, k_1\}$, while $\alpha^j \in (-\pi, 0]$ and $\alpha^{j+k_2} = \alpha^j + \pi$ for every $j \in \{k_1 + 1, \ldots, k_1 + k_2\}$. For the sake of simplicity, we treat in detail configurations of the type described above, see 1.1; in Section 7.7 we explain how our methods and results can be extended to more general configurations of poles.

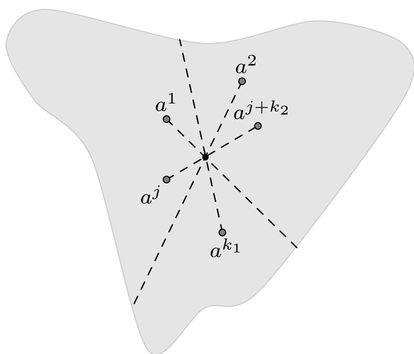


Figure 1.1: Configuration of poles ($k_1 + 1 \leq j \leq k_1 + k_2$).

For every $j = 1, \ldots, k$ and $\varepsilon \in (0, 1]$, we define

$$a^j_\varepsilon := \varepsilon a^j.$$

For every $b = (b_1, b_2) \in \mathbb{R}^2$, the Aharonov-Bohm vector potential with pole b and circulation $\rho \in \mathbb{R}$ is defined as

$$A^\rho_b(x_1, x_2) := \rho \left(\frac{-(x_2 - b_2)}{(x_1 - b_1)^2 + (x_2 - b_2)^2}, \frac{x_1 - b_1}{(x_1 - b_1)^2 + (x_2 - b_2)^2} \right), \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{b\}.$$

In Chapter 7, we consider the case of half-integer circulations $\rho \in \frac{1}{2} + \mathbb{Z}$, which is of particular interest from the mathematical point of view due to applications to the problem of spectral minimal partitions, see [27, 109]. For $\rho = \frac{1}{2}$ we denote

$$A_b := A^{1/2}_b. \tag{1.7}$$

We are interested in the multi-singular vector potential

$$\mathcal{A}^{(n_1, n_2, \ldots, n_k)}_\varepsilon := \sum_{j=1}^k A^{n_j + \frac{1}{2}}_{a^j_\varepsilon} = \sum_{j=1}^k (2n_j + 1) A_{a^j_\varepsilon},$$

having at each pole a_ε^j half-integer circulation $n_j + \frac{1}{2}$ with $n_j \in \mathbb{Z}$, and in the corresponding eigenvalue problem

$$\begin{cases} \left(i\nabla + \mathcal{A}_\varepsilon^{(n_1,n_2,\dots,n_k)}\right)^2 u = \lambda\, u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where the magnetic Schrödinger operator $(i\nabla + \mathcal{A}_\varepsilon^{(n_1,n_2,\dots,n_k)})^2$ acts as

$$\left(i\nabla + \mathcal{A}_\varepsilon^{(n_1,n_2,\dots,n_k)}\right)^2 u := -\Delta u + 2i\,\mathcal{A}_\varepsilon^{(n_1,n_2,\dots,n_k)} \cdot \nabla u + \left|\mathcal{A}_\varepsilon^{(n_1,n_2,\dots,n_k)}\right|^2 u.$$

Since $n_j \in \mathbb{Z}$, $\mathcal{A}_\varepsilon^{(n_1,n_2,\dots,n_k)}$ is gauge equivalent to the vector potential

$$\mathcal{A}_\varepsilon := \sum_{j=1}^{k} (-1)^{j+1} A_{a_\varepsilon^j}.$$

Therefore the operators $(i\nabla + \mathcal{A}_\varepsilon^{(n_1,n_2,\dots,n_k)})^2$ and $(i\nabla + \mathcal{A}_\varepsilon)^2$ are unitarily equivalent (see [96, Theorem 1.2] and [97, Proposition 2.2]), and consequently the spectrum of (1.8) coincides with that of

$$\begin{cases} (i\nabla + \mathcal{A}_\varepsilon)^2 u = \lambda\, u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Hence, to study the behaviour as $\varepsilon \to 0^+$ of the spectrum of (1.8), it is not restrictive to consider problem (1.9). We refer to (7.2) for the variational formulation of (1.9). From classical spectral theory, problem (1.9) has a diverging sequence of real positive eigenvalues $\{\lambda_{\varepsilon,n}\}_{n\in\mathbb{N}\setminus\{0\}}$; in the sequence $\{\lambda_{\varepsilon,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ we repeat each eigenvalue according to its multiplicity. Moreover, the eigenspace associated to each eigenvalue has finite dimension.

As $\varepsilon \to 0^+$, the following limit eigenvalue problem comes into play:

$$\begin{cases} \left(i\nabla + \frac{1+(-1)^{k+1}}{2} A_0\right)^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

with A_0 defined as in (1.7) with $b = 0$. If k is odd, the operator in (1.10) is the Aharonov-Bohm operator with one pole in 0 and circulation $\frac{1}{2}$; as above, the classical Spectral Theorem applies and provides a diverging sequence of real positive eigenvalues $\{\lambda_{0,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ with finite multiplicity. Furthermore, it is well-known that, in this case, eigenfunctions vanish in 0 with order $\frac{m}{2}$, for some odd $m \in \mathbb{N}$, and have exactly m nodal lines meeting at 0 and dividing the whole 2π-angle into m equal parts; see [66, Theorem 1.3, Section 7] and (7.55)–(7.56) for a description of the asymptotic behaviour at 0 of eigenfunctions of (1.10).

If k is even the nature of the limit eigenvalue problem undergoes a significant mutation. Indeed, for k even, the operator in (1.10) is the classical Dirichlet Laplacian and the eigenvalue problem (1.10) can be rewritten as

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

We conclude that, for every $k \in \mathbb{N} \setminus \{0\}$, the spectrum of (1.10) is a diverging sequence $\{\lambda_{0,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ of positive real eigenvalues.

We recall from [97, Theorem 1.2] that, whatever the number k of poles is,

$$\text{the function} \quad \varepsilon \mapsto \lambda_{\varepsilon,n} \quad \text{is continuous on } [0,1],$$

so that, in particular,

$$\lim_{\varepsilon \to 0^+} \lambda_{\varepsilon,n} = \lambda_{0,n} \tag{1.12}$$

for every $n \in \mathbb{N} \setminus \{0\}$. In Chapter 7 we aim to give a sharp asymptotic expansion for the variation $\lambda_{\varepsilon,n} - \lambda_{0,n}$ of simple eigenvalues with respect to the moving configuration of poles.

In the case of one moving pole, [28] establishes a first relation between the rate of convergence (1.12) and the number of the nodal lines of the corresponding eigenfunction. Sharper asymptotic expansions for simple eigenvalues are obtained in [1], in the case of one pole moving along the tangent to a nodal line of the limit eigenfunction, and in [2], in the case of one pole moving along any direction. The case of one pole approaching the boundary is treated in [6] and [108]. The methods developed in [1], [6], and [108] are based on an Almgren type frequency formula, which provides local energy bounds for eigenfunctions. These are used to estimate the Rayleigh quotient, whose minimax levels characterize the eigenvalues, and to prove the convergence of a family of blown-up eigenfunctions to some non trivial limit profile. In particular, using the notation introduced above, in [1] it is proved that, for $k = k_1 = 1$ and $a^1_\varepsilon = \varepsilon a^1 = \varepsilon\, r_1(\cos(\alpha^1), \sin(\alpha^1))$ moving along the tangent to one of the m nodal lines of the limit eigenfunction u_0, if $\lambda_{0,n}$ is a simple, then

$$\lambda_{\varepsilon,n} - \lambda_{0,n} = 4\, r_1^m (|\beta_1|^2 + |\beta_2|^2)\, \mathfrak{M}\, \varepsilon^m + o(\varepsilon^m) \quad \text{as } \varepsilon \to 0^+. \tag{1.13}$$

In (1.13) $(\beta_1, \beta_2) \neq (0,0)$ is such that

$$\lim_{r \to 0^+} r^{-\frac{m}{2}} u_0(r\cos t, r\sin t) = \beta_1 e^{i\frac{t}{2}} \cos\left(\tfrac{m}{2}t\right) + \beta_2 e^{i\frac{t}{2}} \sin\left(\tfrac{m}{2}t\right),$$

see (7.55), and $\mathfrak{M} < 0$ is a negative constant depending only on m, which has the following variational characterization:

$$\mathfrak{M} = \min\left\{ \frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u(x)|^2\, dx - \frac{m}{2}\int_0^1 x_1^{\frac{m}{2}-1} u(x_1, 0)\, dx_1 : u \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+) \right\}, \tag{1.14}$$

where $s := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geq 1\}$, $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0)\}$, and $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ is the completion of $C^\infty_c(\overline{\mathbb{R}^2_+} \setminus s)$ with respect to the norm $(\int_{\mathbb{R}^2_+} |\nabla u|^2\, dx)^{1/2}$. For an explicit formula for $\mathfrak{M}$ we refer to [5, Theorem 2.3]. The quantity appearing in (1.14) can be interpreted as a weighted *torsional rigidity* of the segment along which the pole is moving. Concerning the classical notion of torsional rigidity of a set, the literature is vast; among many others, we cite the classical books [112, 85] for the basic definitions and some possible application in shape optimization and [131, 32, 30] for more recent investigations in the field. We also point out [8], where a notion of *thin torsional rigidity* is exploited in the study of spectral stability for some singularly perturbed problems.

In the case of one single pole, the study of Aharonov-Bohm eigenvalues benefits from the known regularity of the eigenvalue as a function of the pole position. Indeed, in [97] it is proved that, in the case of one moving pole, eigenvalues are analytic as functions of the pole, so that the eigenvalue variation admits a Taylor expansion. The sharp asymptotics on nodal lines (1.13) obtained in [1] is used in [2] to compute the leading term of such Taylor expansion, exploiting symmetry and periodicity properties of the Fourier coefficients of the blow-up profile with respect to the moving direction. In the case of many poles, the analyticity property is maintained as long as the poles are away from each other (see again [97]), but is lost in the case of a collision; indeed in [4] (and in [3] for symmetric domains) it is proved that, in the case of two poles colliding at a point outside the nodal set of the limit eigenfunction, the eigenvalue variation is asymptotic to the logarithm of the distance.

From the above discussion it therefore emerges that the case of multiple colliding poles presents additional significant difficulties. So far, up to our knowledge, in the literature only the case of two coalescing poles has been addressed with the aim of deriving precise asymptotic estimates in terms of the distance between the two poles. The paper [3] derives the asymptotic behaviour of eigenvalues of Aharonov–Bohm operators with two colliding poles moving on an axis of symmetry of the domain, which is assumed not to be tangent to any nodal line of the limit eigenfunction. The argument used in [3] is based on isospectrality with the Dirichlet Laplacian on the domain with a small segment removed, for which an asymptotic expansion of the eigenvalue variation is obtained by a capacity argument, in the spirit of [42]. The complementary case of two colliding poles, which move on an axis of symmetry coinciding with a nodal line of the limit eigenfunction, is treated in [5], exploiting an isospectrality result and a monotonicity formula in the spirit of [1]. The assumption of symmetry of the domain is removed in [4], in the case of two poles collapsing at an interior point out of nodal lines of the limit eigenfunction; this is possible thanks to an estimate of the diameter of the nodal set of magnetic eigenfunctions close to the collision point.

In Chapter 7 we develop a new approach that provides asymptotic expansions of the eigenvalue variation in the most general case of any number of poles moving towards a collision point. We propose a method which combines the idea of torsional rigidity, naturally appearing in [1] (see also [6, Theorem 2.2]) to variationally characterize the coefficient of the leading term as in (1.14), with that of capacity, which [42] and [3] show to be the good small parameter in a spectral perturbation theory in domains with small holes.

Let us assume that there exists $n_0 \in \mathbb{N} \setminus \{0\}$ such that

$$\lambda_{0,n_0} \text{ is a simple eigenvalue of (1.10).} \tag{1.15}$$

In view of (1.12), assumption (1.15) implies that also $\lambda_{\varepsilon,n_0}$ is simple as an eigenvalue of (1.9), provided ε is sufficiently small. Simplicity of the spectrum is a *generic* property for many differential operators. We refer e.g. to [129], where the author exhibits sufficient conditions for genericity of simplicity of the spectrum for various families of differential operators (including Aharonov-Bohm operators with a single pole). See also [7] for a focus on the particular case of Aharonov-Bohm operators.

The first step in our approach is to perform some gauge transformation, making the magnetic eigenvalue problem (1.9), and its corresponding limit one (1.10), equivalent to eigenvalue problems for the Laplacian in domains with straight cracks, laying along the moving directions of poles, see (7.10) and (7.14). Fixing a L^2-normalized eigenfunction v_0 of the equivalent limit eigenvalue problem (7.14) associated to the eigenvalue λ_{0,n_0}, we prove in Theorem 7.1.1 the following asymptotic expansion:

$$\lambda_{\varepsilon,n_0} - \lambda_{0,n_0} = 2\big(\mathcal{E}_\varepsilon - L_\varepsilon(v_0)\big) + o\big(\|\nabla V_\varepsilon\|^2_{L^2(\Omega)}\big) \quad \text{as } \varepsilon \to 0^+, \tag{1.16}$$

where L_ε is the linear functional defined in (7.16), $\mathcal{E}_\varepsilon$ is the minimum of an energy functional associated with the configuration of poles and defined on a space of functions suitably jumping through the cracks, see (7.19), and V_ε is the potential attaining such a minimum. We observe that $\mathcal{E}_\varepsilon$ is a kind of intermediate quantity between torsional rigidity and capacity of the set obtained as the union of the segments connecting the poles to the origin. Indeed, the capacity of a set is defined by minimizing the L^2-norm of the gradient among functions which are prescribed on the set; the torsional rigidity, instead, is constructed by minimizing an energy functional, which contains a linear term involving an integral on the set, without prescribing any condition. In the definition of $\mathcal{E}_\varepsilon$ given in (7.19), we minimize an energy functional over a family of functions which are only partially prescribed on the cracks, in the sense that we impose a jump condition on the functions across the segments, obtaining a jump of the normal derivatives as a consequent natural condition. The development of such an intermediate notion provides a unified approach, which does not require an a priori relation between the configuration of poles and the orientation of the nodal set of the limit eigenfunction. We mention that elliptic problems in cracked domains, with jumps of the unknown function and its normal derivative prescribed on the cracks, are studied in [105].

For k odd, a blow-up analysis allows us to identify the exact asymptotic behaviour of the quantities appearing in the right hand side of (1.16). In Theorem 7.1.2 we prove that $\lim_{\varepsilon\to 0^+} \varepsilon^{-m}\mathcal{E}_\varepsilon = \mathcal{E}$, where m is the vanishing order of v_0 at 0 and $\mathcal{E}$ is the minimum of the energy functional defined in (7.28) over a space of suitably jumping functions, see (7.31). Thus we generalize (1.13) in the multipolar case, obtaining the following explicit expansion

$$\lambda_{\varepsilon,n_0} - \lambda_{0,n_0} = 2\,\varepsilon^m(\mathcal{E} - L(\Psi_0)) + o(\varepsilon^m) \tag{1.17}$$

as $\varepsilon \to 0^+$, where L is the linear functional defined in (7.27) and Ψ_0 is the $\frac{m}{2}$-homogeneous harmonic function introduced in (7.26). We note that the assumption that k is odd is crucial in the blow-up analysis, since it guarantees the validity of the Hardy-type inequality proved in Proposition 7.5.2, needed to characterize the functional space containing the limiting blow-up profile. In the particular case of all poles moving either along the tangents to nodal lines or along the bisectors between nodal lines of the limit eigenfunction, we can prove that the quantity $\mathcal{E} - L(\Psi_0)$, appearing as the coefficient of the leading term of the asymptotic expansion (1.17), does not vanish, see Proposition 7.1.3; this shows that m is exactly the vanishing order of the eigenvalue variation. On the other hand, the study of the continuity properties of the coefficients appearing in (1.17), see Theorem 7.5.8, allows us to prove the existence of configurations of poles for which $\mathcal{E} - L(\Psi_0) = 0$ and hence $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0}$ is an infinitesimal of higher order than m.

If k is even, a Hardy type inequality is no more available, and therefore the blow-up analysis meets the technical difficulty of identifying the limiting profile in an appropriate functional space. In spite of that, in the case of two poles colliding in a point of the nodal set of the limit eigenfunction and moving either along the tangents to its nodal lines or along its bisectors, in Theorems 7.1.6 and 7.1.7 we are able to derive the exact asymptotic behaviour of $\mathcal{E}_\varepsilon - L_\varepsilon(v_0)$, and consequently of $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0}$ thanks to the use of elliptic coordinates; in this way we generalize the results of [3] and [5], which require an axial symmetry of the domain as a further hypothesis. Chapter 7 is based on the paper [70].

Finally, we mention that the case of arbitrary real circulations is the object of current investigation, see [71].

Part I

Unique continuation for elliptic problems

Chapter 2

Unique continuation from a crack's tip under Neumann boundary conditions

2.1 Statements of the main results

In this chapter we establish strong unique continuation properties and classify the asymptotic behaviour of solutions, from the edge of a flat crack Γ, for the problem

$$\begin{cases} -\Delta u = fu, & \text{in } B_R \setminus \Gamma, \\ \dfrac{\partial^+ u}{\partial \nu^+} = \dfrac{\partial^- u}{\partial \nu^-} = 0, & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where B_R is as in (1.1), Γ is a closed subset of $\mathbb{R}^{N-1} \times \{0\}$ with $C^{1,1}$-boundary, and the potential f satisfies either assumption (2.6) or assumption (2.7) below. We recall that the boundary operators $\frac{\partial^+}{\partial \nu^+}$ and $\frac{\partial^-}{\partial \nu^-}$ in (2.1) are defined as

$$\frac{\partial^+ u}{\partial \nu^+} := -\frac{\partial}{\partial x_N}\left(u|_{B_R^+}\right) \quad \text{and} \quad \frac{\partial^- u}{\partial \nu^-} := \frac{\partial}{\partial x_N}\left(u|_{B_R^-}\right),$$

where for all $r > 0$,

$$B_r^+ := \{(x', x_{N-1}, x_N) \in B_r : x_N > 0\}, \quad B_r^- := \{(x', x_{N-1}, x_N) \in B_r : x_N < 0\},$$

being the total variable $x \in \mathbb{R}^N$ written as $x = (x', x_{N-1}, x_N) \in \mathbb{R}^{N-2} \times \mathbb{R} \times \mathbb{R}$.

To state the main results of this chapter, we introduce now our assumptions on the crack Γ and the potential f. We suppose that Γ is a closed set of the form

$$\Gamma := \{(x_1, 0) : x_1 \in [0, +\infty)\} \quad \text{if } N = 2 \tag{2.2}$$

and

$$\Gamma := \{(x', x_{N-1}, 0) \in \mathbb{R}^N : g(x') \le x_{N-1}\} \quad \text{if } N \ge 3, \tag{2.3}$$

where

$$g : \mathbb{R}^{N-2} \to \mathbb{R}, \quad g \in C^{1,1}(\mathbb{R}^{N-2}), \tag{2.4}$$

and

$$g(0) = 0, \quad \nabla g(0) = 0. \tag{2.5}$$

Assumption (2.5) is not restrictive, being a free consequence of an appropriate choice of the Cartesian coordinate system. We are going to study the behaviour of solutions to (2.1) near 0, which belongs to the edge of the crack Γ defined in (2.2)–(2.3).

Furthermore we assume that $f : B_R \to \mathbb{R}$ is a measurable function for which there exists $\epsilon \in (0,1)$ such that either

$$f \in W^{1,\frac{N}{2}+\epsilon}(B_R \setminus \Gamma), \tag{2.6}$$

or

$$N \geq 3 \quad \text{and} \quad |f(x)| \leq c|x|^{-2+2\epsilon} \quad \text{for some } c > 0 \text{ and for all } x \in B_R. \tag{2.7}$$

For every closed set $K \subseteq \mathbb{R}^{N-1} \times \{0\}$ and $r > 0$, we define the functional space $H^1_{0,\partial B_r}(B_r \setminus K)$ as the closure in $H^1(B_r \setminus K)$ of the set

$$\{v \in H^1(B_r \setminus K) : v = 0 \text{ in a neighbourhood of } \partial B_r\}.$$

A weak solution to (2.1) is a function $u \in H^1(B_R \setminus \Gamma)$ such that

$$\int_{B_R \setminus \Gamma} (\nabla u \cdot \nabla \phi - f u \phi)\, dy = 0,$$

for all $\phi \in H^1_{0,\partial B_R}(B_R \setminus \Gamma)$.

The following unique continuation principle for solutions to (2.1) is our main result.

Theorem 2.1.1. *Let u be a weak solution to (2.1) with Γ as in (2.2)–(2.3) and f satisfying either (2.6) or (2.7). If $u(x) = O(|x|^k)$ as $|x| \to 0^+$ for all $k \in \mathbb{N}$, then $u \equiv 0$ in B_R.*

In Theorem 2.4.8 we provide a classification of blow-up limits in terms of the eigenvalues of the following problem

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}} \psi = \mu \psi, & \text{on } \mathbb{S}^{N-1} \setminus \Sigma, \\ \dfrac{\partial^+ \psi}{\partial \nu^+} = \dfrac{\partial^- \psi}{\partial \nu^-} = 0, & \text{on } \Sigma, \end{cases} \tag{2.8}$$

on the unit $(N-1)$-dimensional sphere $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ with a cut on the half-equator

$$\Sigma := \{(x', x_{N-1}, 0) \in \mathbb{S}^{N-1} : x_{N-1} \geq 0\},$$

where, letting $e_N := (0, \dots, 1)$,

$$\mathbb{S}^{N-1}_+ := \left\{(x', x_{N-1}, x_N) \in \mathbb{S}^{N-1} : x_N > 0\right\}, \quad \mathbb{S}^{N-1}_- := \left\{(x', x_{N-1}, x_N) \in \mathbb{S}^{N-1} : x_N < 0\right\},$$

the boundary operators $\frac{\partial^\pm}{\partial \nu^\pm}$ are defined as

$$\frac{\partial^+ \psi}{\partial \nu^+} := -\nabla_{\mathbb{S}^{N-1}_+} \left(\psi|_{\mathbb{S}^{N-1}_+}\right) \cdot e_N \quad \text{and} \quad \frac{\partial^- \psi}{\partial \nu^-} := \nabla_{\mathbb{S}^{N-1}_-} \left(\psi|_{\mathbb{S}^{N-1}_-}\right) \cdot e_N,$$

see Section 2.4.1 for the weak formulation of (2.8). In Section 2.4.1 we prove that the set of the eigenvalues of (2.8) is $\{\mu_k : k \in \mathbb{N}\}$ where

$$\mu_k = \frac{k(k + 2N - 4)}{4}, \quad k \in \mathbb{N}.$$

As a consequence of the classification of blow-up limits, we obtain the following unique continuation result from the edge with respect to crack points.

Theorem 2.1.2. *Let u be a weak solution to (2.1) with Γ as in (2.2)–(2.3) and f satisfying either (2.6) or (2.7). Let us also assume that u vanishes at 0 at any order with respect to crack points, namely that either $\mathrm{Tr}_\Gamma^+ u(z) = O(|z|^k)$ as $|z| \to 0^+$, $z \in \Gamma$, for all $k \in \mathbb{N}$ or $\mathrm{Tr}_\Gamma^- u(z) = O(|z|^k)$ as $|z| \to 0^+$, $z \in \Gamma$, for all $k \in \mathbb{N}$, where $\mathrm{Tr}_\Gamma^+ u$, respectively $\mathrm{Tr}_\Gamma^- u$, denotes the trace of $u\big|_{B_R^+}$, respectively $u\big|_{B_R^-}$, on Γ. Then $u \equiv 0$ in B_R.*

If $N \geq 3$, we can combine the blow-up analysis with an expansion in Fourier series with respect to a orthonormal basis made of eigenfunctions of (2.8). This allows us to classify the possible asymptotic homogeneity degrees of solutions at 0.

Theorem 2.1.3. *Let $N \geq 3$ and let $u \in H^1(B_R \setminus \Gamma)$, $u \not\equiv 0$, be a non-trivial weak solution to (2.1), with Γ defined in (2.2)–(2.3) and f satisfying either assumption (2.6) or assumption (2.7). Then there exist $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem (2.8), associated to the eigenvalue μ_{k_0}, such that, letting*

$$\Phi(x) := |x|^{\frac{k_0}{2}} Y\left(\frac{x}{|x|}\right),$$

we have that

$$\lambda^{-\frac{k_0}{2}} u(\lambda \cdot) \to \Phi \quad \text{and} \quad \lambda^{1-\frac{k_0}{2}} \left(\nabla_{B_R \setminus \Gamma} u\right)(\lambda \cdot) \to \nabla_{\mathbb{R}^N \setminus \tilde{\Gamma}} \Phi \quad \text{in } L^2(B_1)$$

as $\lambda \to 0^+$, where

$$\tilde{\Gamma} := \left\{x = (x', x_{N-1}, 0) \in \mathbb{R}^N : x_{N-1} \geq 0\right\} \tag{2.9}$$

and $\nabla_{B_R \setminus \Gamma}$ and $\nabla_{\mathbb{R}^N \setminus \tilde{\Gamma}}$ denote the distributional gradients in $B_R \setminus \Gamma$ and $\mathbb{R}^N \setminus \tilde{\Gamma}$ respectively.

A more precise version of Theorem 2.1.3, relating k_0 to the limit of a frequency function and characterizing the eigenfunction Y, will be proved in Section 2.5, see Theorem 2.5.3.

Chapter 2 is organized as follows. In Section 2.2.1 an equivalent problem in a domain with a straightened crack is constructed. Sections 2.2.2 contains some trace and embedding inequalities for the space $H^1(B_r \setminus \tilde{\Gamma})$. Section 2.2.3 is devoted to the construction of the approximating problems. In Section 2.3 we develop the monotonicity argument, which is first used to prove Theorem 2.1.1 and later, in Section 2.4.2, to perform a blow-up analysis and prove Theorem 2.1.2, taking into account the structure of the spherical eigenvalue problem (2.8) studied in Section 2.4.1. Finally Theorem 2.1.3 is proved in Section 2.5.

2.2 An equivalent problem with straightened crack and approximation procedure

In this section we first introduce an equivalent problem with a straightened crack; then we develop an approximation procedure regularizing the domain, for which suitable trace and embedding inequalities are needed.

2.2.1 An equivalent problem with straightened crack

In this section we straighten the boundary of the crack in a neighbourhood of 0. If $N \geq 3$ we use the local diffeomorphism F defined in [47, Section 2], see also [12]; for the sake of clarity and completeness we summarize its properties in Propositions 2.2.1 and 2.2.2 below, referring to [47, Section 2] for their proofs. If $N = 2$, the crack is a segment and we simply take $F = \mathrm{Id}$, where Id is the identity function on $\mathbb{R}^2$.

Proposition 2.2.1. *[47, Section 2] Let $N \geq 3$ and Γ be defined in (2.3) with g satisfying (2.4) and (2.5). There exist $F = (F_1, \dots, F_N) \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ and $r_1 > 0$ such that $F\big|_{B_{r_1}} : B_{r_1} \to F(B_{r_1})$ is a diffeomorphism of class $C^{1,1}$,*

$$F(y', 0, 0) = (y', g(y'), 0) \text{ for any } y' \in \mathbb{R}^{N-1}, \quad \text{and} \quad F(\tilde{\Gamma} \cap B_{r_1}) = \Gamma \cap F(B_{r_1}),$$

with $\tilde{\Gamma}$ as in (2.9). Furthermore, letting $y = (y', y_{N-1}, y_N) \in B_{r_1}$ and $J_F(y)$ be the Jacobian matrix of F at y

$$A(y) := |\det J_F(y)|(J_F(y))^{-1}((J_F(y))^{-1})^T, \tag{2.10}$$

the following properties hold:

i) *J_F depends only on the variable $y'' = (y', y_{N-1})$ and*

$$J_F(y) = J_F(y'') = \mathrm{Id}_N + O(|y''|) \quad \text{as } |y''| \to 0^+,$$

where Id_N denotes the identity $N \times N$ matrix and $O(|y''|)$ denotes a matrix with all entries being $O(|y''|)$ as $|y''| \to 0^+$;

ii) $\det J_F(y) = \det J_F(y', y_{N-1}) = 1 + O(|y'|^2) + O(y_{N-1})$ *as $|y'| \to 0^+$ and $y_{N-1} \to 0$;*

iii) $\frac{\partial F_i}{\partial y_N} = \frac{\partial F_N}{\partial y_i} = 0$ *for any $i = 1, \dots, N-1$ and $\frac{\partial F_N}{\partial y_N} = 1$;*

iv) *the matrix-valued function A can be written as*

$$A(y) = A(y', y_{N-1}) = \left(\begin{array}{c|c} D(y', y_{N-1}) & 0 \\ \hline 0 & \det J_F(y', y_{N-1}) \end{array}\right), \tag{2.11}$$

with

$$D(y', y_{N-1}) = \left(\begin{array}{c|c} \mathrm{Id}_{N-2} + O(|y'|^2) + O(y_{N-1}) & O(y_{N-1}) \\ \hline O(y_{N-1}) & 1 + O(|y'|^2) + O(y_{N-1}) \end{array}\right), \tag{2.12}$$

where Id_{N-2} denotes the identity $(N-2) \times (N-2)$ matrix and $O(y_{N-1})$, respectively $O(|y'|^2)$, denotes blocks of matrices with all entries being $O(y_{N-1})$ as $y_{N-1} \to 0$, respectively $O(|y'|^2)$ as $|y'| \to 0$.

v) *A is symmetric with coefficients of class $C^{0,1}$ and*

$$\frac{1}{2}|z|^2 \leq A(y)z \cdot z \leq 2|z|^2 \quad \text{for all } z \in \mathbb{R}^N \text{ and } y \in B_{r_1}. \tag{2.13}$$

We note that (2.13) implies that $\|A(y)\|_{\mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)} \leq 2$ for all $y \in B_{r_1}$. We also observe

$$A = \mathrm{Id}_2 \quad \text{if } N = 2. \tag{2.14}$$

Moreover (2.11)– (2.12) easily imply that

$$A(y) = A(y'') = \mathrm{Id}_N + O(|y''|) \quad \text{as } |y''| \to 0^+. \tag{2.15}$$

Under the same assumptions and with the same notation of Proposition 2.2.1, we define

$$\mu(y) := \frac{A(y)y \cdot y}{|y|^2} \quad \text{and} \quad \beta(y) := \frac{A(y)y}{\mu(y)} \quad \text{for any } y \in B_{r_1} \setminus \{0\}. \tag{2.16}$$

Proposition 2.2.2. *[47, Section 2] Under the same assumptions as Proposition 2.2.1, let μ and β be as in (2.16). Then, possibly choosing r_1 smaller from the beginning,*

$$\frac{1}{2} \leq \mu(y) \leq 2 \quad \textit{for any } y \in B_{r_1} \setminus \{0\}, \tag{2.17}$$

$$\mu(y) = 1 + O(|y|) \quad \textit{as } |y| \to 0^+, \tag{2.18}$$

$$\nabla\mu(y) = O(1) \quad \textit{as } |y| \to 0^+. \tag{2.19}$$

Moreover β is well-defined and

$$\beta(y) = y + O(|y|^2) = O(|y|) \quad \textit{as } |y| \to 0^+, \tag{2.20}$$

$$J_\beta(y) = A(y) + O(|y|) = \mathrm{Id}_N + O(|y|) \quad \textit{as } |y| \to 0^+,$$

$$\mathrm{div}(\beta)(y) = N + O(|y|) \quad \textit{as } |y| \to 0^+. \tag{2.21}$$

We also define $dA(y)zz$, for every $z = (z_1, \dots, z_N) \in \mathbb{R}^N$ and $y \in B_{r_1}$, as the vector of $\mathbb{R}^N$ with i-th component, for $i = 1, \dots, N$, given by

$$(dA(y)zz)_i = \sum_{h,k=1}^{N} \frac{\partial a_{kh}}{\partial y_i} z_h z_k, \tag{2.22}$$

where we have defined the matrix $A = (a_{k,h})_{k,h=1,\dots,N}$ in (2.10).

Remark 2.2.3. For any measurable function $f : F(B_{r_1}) \to \mathbb{R}$ we set

$$\tilde{f} : B_{r_1} \to \mathbb{R}, \quad \tilde{f} := |\det J_F| \, (f \circ F).$$

Then, in view of i) and ii) in Proposition 2.2.1, the function $\tilde{f}$ satisfies assumptions (2.6) or (2.7) on B_{r_1} if and only if f satisfies assumptions (2.6) or (2.7) on $F(B_{r_1})$.

It is easy to see that, if u is a solution to (2.1), then the function $U := u \circ F$ belongs to $H^1(B_{r_1} \setminus \tilde{\Gamma})$ and is a weak solution of the problem

$$\begin{cases} -\mathrm{div}(A\nabla U) = \tilde{f}u, & \text{in } B_{r_1} \setminus \tilde{\Gamma}, \\ A\nabla^+ U \cdot \nu^+ = A\nabla^- U \cdot \nu^- = 0, & \text{on } \tilde{\Gamma}, \end{cases} \tag{2.23}$$

where

$$\nabla^+ U = \nabla \left(U|_{B^+_{r_1}} \right), \quad \nabla^- U = \nabla \left(U|_{B^-_{r_1}} \right), \quad \text{and } \nu^- = -\nu^+ = (0, \dots, 1).$$

By saying that U is a weak solution to (2.23) we mean that $U \in H^1(B_{r_1} \setminus \tilde{\Gamma})$ and

$$\int_{B_{r_1} \setminus \tilde{\Gamma}} (A\nabla U \cdot \nabla\phi - \tilde{f} U \phi) \, dy = 0$$

for all $\phi \in H^1_{0,\partial B_{r_1}}(B_{r_1} \setminus \tilde{\Gamma})$.

2.2.2 Traces and embeddings results

In this section, we present some trace and embedding inequalities for the space $H^1(B_{r_1} \setminus \tilde{\Gamma})$ which will be used throughout this Chapter.

We define the even reflection operators

$$\mathcal{R}^+(v)(y', y_{N-1}, y_N) = v(y', y_{N-1}, |y_N|),$$
$$\mathcal{R}^-(v)(y', y_{N-1}, y_N) = v(y', x_{N-1}, -|y_N|),$$

and observe that, for all $r > 0$, $\mathcal{R}^+ : H^1(B_r \setminus \tilde{\Gamma}) \to H^1(B_r)$ and $\mathcal{R}^- : H^1(B_r \setminus \tilde{\Gamma}) \to H^1(B_r)$. We have that $\mathcal{R}^+(v), \mathcal{R}^-(v) \in L^p(B_r)$ for some $p \in [1, \infty)$ if and only if $v \in L^p(B_r)$; in such a case we have that

$$\left\|\mathcal{R}^+(v)\right\|^p_{L^p(B_r)} = 2\left\|v\right\|^p_{L^p(B_r^+)}, \quad \left\|\mathcal{R}^-(v)\right\|^p_{L^p(B_r)} = 2\left\|v\right\|^p_{L^p(B_r^-)}, \tag{2.24}$$

and

$$\|v\|^p_{L^p(B_r)} = \frac{1}{2}\left(\left\|\mathcal{R}^+(v)\right\|^p_{L^p(B_r)} + \left\|\mathcal{R}^-(v)\right\|^p_{L^p(B_r)}\right). \tag{2.25}$$

Furthermore, for every $v \in H^1(B_r \setminus \tilde{\Gamma})$,

$$\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 \, dy = \frac{1}{2}\left(\int_{B_r} |\nabla \mathcal{R}^+(v)|^2 \, dy + \int_{B_r} |\nabla \mathcal{R}^-(v)|^2 \, dy\right). \tag{2.26}$$

Proposition 2.2.4. *For any $r > 0$ there exists a linear continuous trace operator*

$$\gamma_r : H^1(B_r \setminus \tilde{\Gamma}) \to L^2(\partial B_r).$$

Furthermore γ_r is compact.

Proof. Since B_r^+ and B_r^- are Lipschitz domains, there exist two linear, continuous and compact trace operators $\gamma_r^+ : H^1(B_r^+) \to L^2(\partial B_r^+ \cap \partial B_r)$ and $\gamma_r^- : H^1(B_r^-) \to L^2(\partial B_r^- \cap \partial B_r)$. By setting

$$\gamma_r(v)(y) := \begin{cases} \gamma_r^+(v)(y), & \text{if } y_N > 0, \\ \gamma_r^-(v)(y), & \text{if } y_N < 0, \end{cases}$$

we complete the proof. □

Letting γ_r be the trace operator introduced in Proposition 2.2.4, we observe that

$$\int_{\partial B_r} |\gamma_r(v)|^2 \, dS = \frac{1}{2}\left(\int_{\partial B_r} |\gamma_r(\mathcal{R}^+(v))|^2 \, dS + \int_{\partial B_r} |\gamma_r(\mathcal{R}^-(v))|^2 \, dS\right) \tag{2.27}$$

for every $v \in H^1(B_r \setminus \tilde{\Gamma})$. With a slight abuse of notation we will often write v instead of $\gamma_r(v)$ on ∂B_r.

Proposition 2.2.5. *If $N \geq 3$ and $r > 0$, then, for any $v \in H^1(B_r \setminus \tilde{\Gamma})$,*

$$\left(\frac{N-2}{2}\right)^2 \int_{B_r} \frac{v^2}{|x|^2} \, dx \leq \int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 \, dx + \frac{N-2}{2r} \int_{\partial B_r} v^2 \, dS. \tag{2.28}$$

Proof. By scaling, [132, Theorem 1.1] proves the claim for $\mathcal{R}^+(v)$ and $\mathcal{R}^-(v)$. Then we conclude by (2.25), (2.26), and (2.27). □

Proposition 2.2.6. *Let $N \geq 2$ and $q \geq 1$ be such that $q \leq 2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $q < \infty$ if $N = 2$. Then*

$$H^1(B_r \setminus \tilde{\Gamma}) \subset L^q(B_r) \quad \textit{for every } r > 0$$

and there exists $\mathcal{S}_{N,q} > 0$ (depending only on N and q) such that

$$\|v\|^2_{L^q(B_r)} \leq \mathcal{S}_{N,q}\, r^{\frac{N(2-q)+2q}{q}} \left(\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dx + \frac{1}{r} \int_{\partial B_r} v^2\, dS \right), \tag{2.29}$$

for all $r > 0$ and $v \in H^1(B_r \setminus \tilde{\Gamma})$.

Proof. Since

$$\left(\int_{B_1} |\nabla v|^2\, dx + \int_{\partial B_1} v^2\, dS \right)^{\frac{1}{2}}$$

is an equivalent norm on $H^1(B_1)$, from a scaling argument and Sobolev embedding Theorems it follows that, for all $q \in [1, 2^*]$ if $N \geq 3$ and $q \in [1, \infty)$ if $N = 2$, there exists $\mathcal{S}_{N,q} > 0$ such that, for all $r > 0$ and $v \in H^1(B_r)$,

$$\|v\|^2_{L^q(B_r)} \leq \mathcal{S}_{N,q} r^{\frac{N(2-q)+2q}{q}} \left(\int_{B_r} |\nabla v|^2\, dx + \frac{1}{r} \int_{\partial B_r} v^2\, dS \right).$$

Using (2.24), (2.25), (2.26) and (2.27) we complete the proof. □

Proposition 2.2.7. *For any $r > 0$, $h \in L^{\frac{N}{2}+\epsilon}(B_r)$ with $\epsilon > 0$, and $v \in H^1(B_r \setminus \tilde{\Gamma})$, there holds*

$$\int_{B_r} |h| v^2 \leq \eta_h(r) \left(\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dx + \frac{1}{r} \int_{\partial B_r} v^2\, dS \right), \tag{2.30}$$

where

$$\eta_h(r) = \mathcal{S}_{N,q_\epsilon} \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} r^{\frac{4\epsilon}{N+2\epsilon}} \quad \textit{and} \quad q_\epsilon := \frac{2N+4\epsilon}{N-2+2\epsilon}. \tag{2.31}$$

Proof. For any $v \in H^1(B_r \setminus \tilde{\Gamma})$

$$\begin{aligned}\int_{B_r} |h| v^2\, dx &\leq \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \left(\int_{B_r} |v|^{q_\epsilon}\, dx \right)^{2/q_\epsilon} \\ &\leq \mathcal{S}_{N,q_\epsilon} \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} r^{\frac{4\epsilon}{N+2\epsilon}} \left(\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dx + \frac{1}{r} \int_{\partial B_r} v^2 dS \right)\end{aligned}$$

thanks to Hölder's inequality and (2.29). □

Remark 2.2.8. If f satisfies (2.7), then $f \in L^{\frac{N}{2}+\epsilon}(B_R)$, so that Proposition 2.2.7 applies to potentials satisfying either (2.6) or (2.7).

Remark 2.2.9. By (2.30), (2.17) and (2.13), for any $r \in (0, r_1)$, $h \in L^{\frac{N}{2}+\epsilon}(B_r)$, and $v \in H^1(B_r \setminus \tilde{\Gamma})$, we have that

$$\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dy \leq 2 \int_{B_r \setminus \tilde{\Gamma}} (A \nabla v \cdot \nabla v - h v^2)\, dy + 2\eta_h(r) \left(\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dy + \frac{2}{r} \int_{\partial B_r} \mu v^2\, dS \right)$$

and therefore, if $\eta_h(r) < \frac{1}{2}$,

$$\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dy \leq \frac{2}{1 - 2\eta_h(r)} \int_{B_r \setminus \tilde{\Gamma}} (A \nabla v \cdot \nabla v - h v^2)\, dy + \frac{4\eta_h(r)}{(1 - 2\eta_h(r)) r} \int_{\partial B_r} \mu v^2\, dS. \tag{2.32}$$

2.2.3 Approximating problems

In this section we construct a sequence of problems in smooth sets approximating the straightened cracked domain. We define, for any $n \in \mathbb{N} \setminus \{0\}$,

$$g_n : \mathbb{R} \to \mathbb{R}, \quad g_n(t) := nt^4$$

and, for any $r \in (0, r_1]$,

$$\Omega_{n,r} := \{(y', y_{N-1}, y_N) \in B_r : y_{N-1} < g_n(y_N)\}$$

and

$$\Gamma_{n,r} := \{(y', y_{N-1}, y_N) \in B_r : y_{N-1} = g_n(y_N)\} = \partial\Omega_{n,r} \cap B_r.$$

The domains $\Omega_{n,r}$ approximate $B_r \setminus \tilde{\Gamma}$ in the following sense: for every $y \in B_r \setminus \tilde{\Gamma}$, there exists $\bar{n} \in \mathbb{N} \setminus \{0\}$ such that $y \in \Omega_{n,r}$ for all $n \geq \bar{n}$. Moreover $\Omega_{n,r} \cap \tilde{\Gamma} = \emptyset$ for any $r \in (0, r_1]$ and $n \in \mathbb{N} \setminus \{0\}$. We also note that $\Omega_{n,r}$ is a Lipschitz domain and $\Gamma_{n,r}$ is a C^2-smooth portion of its boundary.

Proposition 2.2.10. *Let $\nu(y)$ be the outward normal vector to $\partial\Omega_{n,r_1}$ in y. Then*

$$y \cdot \nu(y) \leq 0 \quad \textit{for all } y \in \Gamma_{n,r_1}, \tag{2.33}$$

$$A(y)y \cdot \nu(y) \leq 0 \quad \textit{for all } y \in \Gamma_{n,r_1}. \tag{2.34}$$

Proof. As a first step we notice that

$$g_n(t) - \frac{1}{3}tg_n'(t) = nt^4 - \frac{4}{3}nt^4 = -\frac{1}{3}nt^4 \leq 0, \quad g_n(t) - tg_n'(t) \leq 0 \tag{2.35}$$

and that

$$\nu(y) = \frac{(0, 1, -g_n'(y_N))}{\sqrt{1 + (g_n'(y_N))^2}} \quad \text{for all } y \in \Gamma_{n,r_1}.$$

Then, for all $y \in \Gamma_{n,r_1}$,

$$\nu(y) \cdot y = \frac{(0, 1, -g_n'(y_N))}{\sqrt{1 + (g_n'(y_N))^2}} \cdot (y', g_n(y_N), y_N) = \frac{g_n(y_N) - y_N g_n'(y_N)}{\sqrt{1 + (g_n'(y_N))^2}} \leq 0$$

due to (2.35). We have then proved (2.33) (and (2.34) in the case $N = 2$ in view of (2.14)). If $N \geq 3$, possibly choosing r_1 smaller in Proposition 2.2.1, for all $y \in \Gamma_{n,r_1}$ we have that

$$\begin{aligned}\sqrt{1 + (g_n'(y_N))^2} A(y)y \cdot \nu(y) &= g_n(y_N)(1 + O(|y'|) + O(y_{N-1})) - \det J_F(y)\, y_N g_n'(y_N) \\ &\leq \frac{3}{2}g_n(y_N) - \frac{1}{2}y_N g_n'(y_N) = \frac{3}{2}(g_n(y_N) - \frac{1}{3}y_N g_n'(y_N)),\end{aligned}$$

thanks to ii) in Proposition 2.2.1, (2.11) and (2.12). Then, by (2.35) we finally obtain (2.34) also for $N \geq 3$. □

Let

$$\mathbb{R}^N_+ := \{y = (y', y_{N-1}, y_N) \in \mathbb{R}^N : y_N > 0\} \text{ and } \mathbb{R}^N_- := \{y = (y', y_{N-1}, y_N) \in \mathbb{R}^N : y_N < 0\}.$$

For any $r \in (0, r_1]$ and $n \in \mathbb{N} \setminus \{0\}$ let

$$\Omega_{n,r}^+ := \Omega_{n,r} \cap B_r^+, \quad \Omega_{n,r}^- := \Omega_{n,r} \cap B_r^-, \quad S_{n,r} := \partial\Omega_{n,r} \cap \partial B_r. \tag{2.36}$$

For all $n \in \mathbb{N} \setminus \{0\}$ we also define

$$K_{n,r_1}^+ := \{y = (y', y_{N-1}, y_N) \in \mathbb{R}_+^N : \text{either } y_{N-1} < g_n(y_N) \text{ or } |y| > r_1\},$$
$$K_{n,r_1}^- := \{y = (y', y_{N-1}, y_N) \in \mathbb{R}_-^N : \text{either } y_{N-1} < g_n(y_N) \text{ or } |y| > r_1\}.$$

Since $\Omega_{n,r}$ is a Lipschitz domain, for any $r \in (0, r_1]$ and $n \in \mathbb{N} \setminus \{0\}$ there exists a trace operator

$$\gamma_{n,r} : H^1(\Omega_{n,r}) \to L^2(\partial\Omega_{n,r}).$$

We define

$$H^1_{0,S_{n,r}}(\Omega_{n,r}) := \{u \in H^1(\Omega_{n,r}) : \gamma_{n,r}(u) = 0 \text{ on } S_{n,r}\}.$$

The following proposition provides an extension operator from $H^1_{0,S_{n,r}}(\Omega_{n,r})$ to $H^1(B_{r_1} \setminus \tilde{\Gamma})$ with an operator norm bounded uniformly with respect to n.

Proposition 2.2.11. *For any $r \in (0, r_1)$ and $n \in \mathbb{N} \setminus \{0\}$ there exists an extension operator*

$$\xi_{n,r}^0 : H^1_{0,S_{n,r}}(\Omega_{n,r}) \to H^1(B_{r_1} \setminus \tilde{\Gamma})$$

such that, for any $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$,

$$\xi_{n,r}^0(\phi)|_{\Omega_{n,r}} = \phi, \quad \xi_{n,r}^0(\phi) = 0 \text{ on } \Omega_{n,r_1} \setminus \Omega_{n,r}, \quad \xi_{n,r}^0(\phi) \in H^1_{0,\partial B_{r_1}}(B_{r_1} \setminus \tilde{\Gamma}), \tag{2.37}$$

and

$$\left\|\xi_{n,r}^0(\phi)\right\|_{H^1(B_{r_1}\setminus\tilde{\Gamma})} \le c_0 \|\phi\|_{H^1(\Omega_{n,r})} = c_0\left(\int_{\Omega_{n,r}} (\phi^2 + |\nabla\phi|^2)\, dy\right)^{1/2}, \tag{2.38}$$

where $c_0 > 0$ is independent of n, r, and ϕ.

Proof. It is well known that, since K_{n,r_1}^+ and K_{n,r_1}^- are uniformly Lipschitz domains, there exist continuous extension operators $\xi_n^+ : H^1(K_{n,r_1}^+) \to H^1(\mathbb{R}_+^N)$ and $\xi_n^- : H^1(K_{n,r_1}^-) \to H^1(\mathbb{R}_-^N)$, see [124], [36] and [99]. Furthermore, since the Lipschitz constants of the parameterization of $\partial K_{n,r_1}^+$ and $\partial K_{n,r_1}^-$ are bounded uniformly with respect to n, there exists a constant $C > 0$, which does not depend on n, such that

$$\left\|\xi_n^+(v)\right\|_{H^1(\mathbb{R}_+^N)} \le C\|v\|_{H^1(K_{n,r_1}^+)} \quad \text{and} \quad \|\xi_n^-(w)\|_{H^1(\mathbb{R}_-^N)} \le C\|w\|_{H^1(K_{n,r_1}^-)} \tag{2.39}$$

for all $v \in H^1(K_{n,r_1}^+)$ and $w \in H^1(K_{n,r_1}^-)$.

If $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$ then the trivial extension $\bar{\phi}_+$ of $\phi|_{\Omega_{n,r}^+}$ to K_{n,r_1}^+ belongs to $H^1(K_{n,r_1}^+)$ and the trivial extension $\bar{\phi}_-$ of $\phi|_{\Omega_{n,r}^-}$ to K_{n,r_1}^- belongs to $H^1(K_{n,r_1}^-)$. Then we define

$$\xi_{n,r}^0(\phi)(y) := \begin{cases} \xi_n^+(\bar{\phi}_+)(y), & \text{if } y \in B_{r_1}^+, \\ \xi_n^-(\bar{\phi}_-)(y), & \text{if } y \in B_{r_1}^-, \end{cases}$$

which belongs to $H^1(B_{r_1} \setminus \tilde{\Gamma})$ and satisfies (2.38) in view of (2.39). Furthermore (2.37) follows directly from the definition of $\xi_{n,r}^0$. □

The next proposition establishes a Poincaré type inequality for $H^1_{0,S_{n,r}}(\Omega_{n,r})$-functions, with a constant independent of n.

Proposition 2.2.12. *For any $r \in (0, r_1]$, $n \in \mathbb{N} \setminus \{0\}$, and $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$*

$$\int_{\Omega_{n,r}} \phi^2 dy \leq \frac{r^2}{N-1} \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy \tag{2.40}$$

and

$$\|\phi\|_{H^1_{0,S_{n,r}}(\Omega_{n,r})} := \left(\int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy \right)^{\frac{1}{2}} \tag{2.41}$$

is an equivalent norm on $H^1_{0,S_{n,r}}(\Omega_{n,r})$.

Proof. For any $\phi \in C^\infty(\overline{\Omega}_{n,r})$ such that $\phi = 0$ in a neighbourhood of $S_{n,r}$ we have that

$$\operatorname{div}(\phi^2 y) = 2\phi \nabla \phi \cdot y + N\phi^2$$

so that

$$N \int_{\Omega_{n,r}} \phi^2 \, dy = -2 \int_{\Omega_{n,r}} \phi \nabla \phi \cdot y \, dy + \int_{\Gamma_{n,r}} \phi^2 y \cdot \nu \, dS \leq \int_{\Omega_{n,r}} \phi^2 \, dy + r^2 \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy,$$

since $y \cdot \nu \leq 0$ on $\Gamma_{n,r}$ by(2.33). Then we may conclude that

$$\int_{\Omega_{n,r}} \phi^2 \, dy \leq \frac{r^2}{N-1} \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy,$$

for all $\phi \in C^\infty(\overline{\Omega}_{n,r})$ such that $\phi = 0$ in a neighbourhood of $S_{n,r}$. Since $\Omega_{n,r}$ is a Lipschitz domain, (2.40) holds for any $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$ by [23, Theorem 3.1]. The second claim is now obvious. □

From now on we consider on $H^1_{0,S_{n,r}}(\Omega_{n,r})$ the norm $\|\cdot\|_{H^1_{0,S_{n,r}}}$ defined in (2.41).

Proposition 2.2.13. *Let $r \in (0, r_1)$, $n \in \mathbb{N} \setminus \{0\}$, $h \in L^{\frac{N}{2}+\epsilon}(B_r)$ with $\epsilon > 0$, and q_ϵ be as in* (2.31)*. Then, for any $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$,*

$$\int_{\Omega_{n,r}} |h|\phi^2 \, dy \leq c_0^2 \frac{N-1+r_1^2}{N-1} \mathcal{S}_{N,q_\epsilon} r_1^{\frac{4\epsilon}{N+2\epsilon}} \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy. \tag{2.42}$$

Proof. We have, for every $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$,

$$\begin{aligned}
\int_{\Omega_{n,r}} |h|\phi^2 \, dy &\leq \int_{B_r} |h||\xi^0_{n,r}(\phi)|^2 \, dy \leq \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \left(\int_{B_{r_1}} |\xi^0_{n,r}(\phi)|^{q_\epsilon} \, dy \right)^{\frac{2}{q_\epsilon}} \\
&\leq \mathcal{S}_{N,q_\epsilon} r_1^{\frac{4\epsilon}{N+2\epsilon}} \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \int_{B_{r_1} \setminus \tilde{\Gamma}} |\nabla \xi^0_{n,r}(\phi)|^2 \, dy \\
&\leq c_0^2 \frac{N-1+r^2}{N-1} \mathcal{S}_{N,q_\epsilon} r_1^{\frac{4\epsilon}{N+2\epsilon}} \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy,
\end{aligned}$$

thanks to Hölder's inequality, (2.29), Proposition 2.2.11, and Proposition 2.2.12. □

Hereafter we fix a potential f satisfying either (2.6) or (2.7) and define $\tilde{f} := |\det J_F|\,(f\circ F)$ as in Remark 2.2.3. Thanks to Remark 2.2.3 we have that $\tilde{f}$ satisfies either (2.6) or (2.7) as well. If f (and consequently $\tilde{f}$) satisfies (2.7), we define

$$f_n(y) = \begin{cases} n, & \text{if } \tilde{f}(y) > n, \\ \tilde{f}(y), & \text{if } |\tilde{f}(y)| \le n, \\ -n, & \text{if } \tilde{f}(y) < -n, \end{cases} \tag{2.43}$$

so that

$$f_n \in L^\infty(B_{r_1}) \quad \text{and} \quad |f_n| \le |\tilde{f}| \text{ a.e. in } B_{r_1} \quad \text{for all } n \in \mathbb{N} \setminus \{0\} \tag{2.44}$$

and

$$f_n \to \tilde{f} \text{ a.e. in } B_{r_1}. \tag{2.45}$$

If f satisfies (2.6), we just let

$$f_n := \tilde{f} \quad \text{for any } n \in \mathbb{N}. \tag{2.46}$$

We observe that

$$f_n \to \tilde{f} \quad \text{in } L^{\frac{N}{2}+\epsilon}(B_{r_1}) \quad \text{as } n \to \infty \tag{2.47}$$

as a consequence of (2.44), (2.45) and the Dominated Convergence Theorem if assumption (2.7) holds and f_n is defined in (2.43), in view of Remark 2.2.8; on the other hand (2.47) is obvious if assumption (2.6) holds and f_n is defined in (2.46).

Since under both assumptions (2.6) and (2.7) we have that $\tilde{f} \in L^{\frac{N}{2}+\epsilon}(B_{r_1})$ (see Remark 2.2.8), by the absolute continuity of the Lebesgue integral we can choose $r_0 \in (0, \min\{1, r_1\})$ such that

$$\eta_{\tilde{f}}(r_0) < \frac{1}{2} \quad \text{and} \quad c_0^2 \frac{N-1+r_1^2}{N-1} \mathcal{S}_{N,q_\epsilon} r_1^{\frac{4\epsilon}{N+2\epsilon}} \|\tilde{f}\|_{L^{\frac{N}{2}+\epsilon}(B_{r_0})} < \frac{1}{4}, \tag{2.48}$$

where q_ϵ and $\eta_{\tilde{f}}$ are defined in (2.31).

Let $U = u \circ F$, where u is a fixed weak solution to (2.1) and F is the diffeomorphism introduced in Section 2.2.1, so that U weakly solves (2.23). For any $n \in \mathbb{N} \setminus \{0\}$, we consider the following sequence of approximating problems, with potentials f_n defined in (2.43)–(2.46):

$$\begin{cases} -\operatorname{div}(A\nabla U_n) = f_n U_n, & \text{in } \Omega_{n,r_0}, \\ A\nabla U_n \cdot \nu = 0, & \text{on } \Gamma_{n,r_0}, \\ \gamma_{n,r_0}(U_n) = \gamma_{n,r_0}(U), & \text{on } S_{n,r_0}, \end{cases} \tag{2.49}$$

with r_0 as in (2.48). A weak solution to problem (2.49) is a function $U_n \in H^1(\Omega_{n,r_0})$ such that $U_n - U \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ and

$$\int_{\Omega_{n,r_0}} (A\nabla U_n \cdot \nabla\phi - f_n U_n \phi)\, dy = 0$$

for all $\phi \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$. If U_n weakly solves (2.49), then $W_n := U - U_n \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ and

$$\int_{\Omega_{n,r_0}} (A\nabla W_n \cdot \nabla\phi - f_n W_n \phi)\, dy = \int_{\Omega_{n,r_0}} (A\nabla U \cdot \nabla\phi - f_n U \phi)\, dy \tag{2.50}$$

for any $\phi \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$.

For every $n \in \mathbb{N} \setminus \{0\}$, let us consider the bilinear form

$$B_n : H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}) \times H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}) \to \mathbb{R}, \quad B_n(v,\phi) := \int_{\Omega_{n,r_0}} (A\nabla v \cdot \nabla \phi - f_n v \phi)\, dy, \tag{2.51}$$

and the functional

$$L_n : H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}) \to \mathbb{R}, \quad L_n(\phi) := \int_{\Omega_{n,r_0}} (A\nabla U \cdot \nabla \phi - f_n U \phi)\, dy. \tag{2.52}$$

Proposition 2.2.14. *The bilinear form B_n defined in (2.51) is continuous and coercive; more precisely*

$$B_n(\phi,\phi) \geq \frac{1}{4} \|\phi\|^2_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})} \quad \text{for all } \phi \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}). \tag{2.53}$$

Furthermore the functional L_n defined in (2.52) belongs to $(H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}))^*$ and there exists a constant $\ell > 0$ independent of n such that

$$|L_n(\phi)| \leq \ell \|\phi\|_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})} \quad \text{for all } \phi \in H^1_{0,S_{n,r}}(\Omega_{n,r_0}). \tag{2.54}$$

Proof. The continuity of B_n and (2.53) easily follow from (2.13),(2.44), (2.42) and (2.48). Thanks to Hölder's inequality, (2.44), (2.13), (2.30), (2.42) and (2.48)

$$|L_n(\phi)| \leq 2\|\nabla U\|_{L^2(\Omega_{n,r_0})} \|\phi\|_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})} + \left(\int_{B_{r_0}} |\tilde f| U^2\, dx\right)^{\frac{1}{2}} \left(\int_{\Omega_{n,r_0}} |\tilde f| \phi^2\, dx\right)^{\frac{1}{2}}$$
$$\leq \left(2\|\nabla U\|_{L^2(B_{r_0}\setminus\tilde\Gamma)} + \frac{1}{2}\sqrt{\eta_{\tilde f}(r_0)} \left(\int_{B_{r_0}\setminus\tilde\Gamma} |\nabla U|^2\, dx + \frac{1}{r_0}\int_{\partial B_{r_0}} U^2\, dS\right)^{\frac{1}{2}}\right) \|\phi\|_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})},$$

thus implying (2.54). □

Corollary 2.2.15. *Let u be a weak solution to (2.1) and $U = u \circ F$. Let either (2.6) hold and $\{f_n\}$ be as in (2.46), or (2.7) hold and $\{f_n\}$ be as in (2.43). Let r_0 be as in (2.48) and ℓ be as in Proposition 2.2.14. Then, for any $n \in \mathbb{N} \setminus \{0\}$, there exists a solution $W_n \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ of (2.50) such that*

$$\|W_n\|_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})} \leq 4\ell. \tag{2.55}$$

Proof. The existence of a solution W_n of (2.50) follows from the Lax-Milgram Theorem, taking into account Proposition 2.2.14. Estimate (2.55) follows from (2.53) and (2.54) with $\phi = W_n$. □

We are now in position to prove the main result of this section.

Theorem 2.2.16. *Suppose that f satisfies either (2.6) or (2.7), u is a weak solution of (2.1), and $U = u \circ F$ with F as in Section 2.2.1. Let $\{f_n\}_{n\in\mathbb{N}}$ satisfies (2.46) under hypothesis (2.6) or (2.43) under hypothesis (2.7). Let $r_0 \in (0, r_1)$ be as (2.48). Then there exists $\{U_n\}_{n\in\mathbb{N}\setminus\{0\}} \subset H^1(B_{r_0}\setminus\tilde\Gamma)$ such that U_n weakly solves (2.49) for any $n \in \mathbb{N}\setminus\{0\}$ and $U_n \to U$ in $H^1(B_{r_0}\setminus\tilde\Gamma)$ as $n \to \infty$. Furthermore $U_n \in H^2(\Omega_{n,r})$ for any $r \in (0, r_0)$ and $n \in \mathbb{N}\setminus\{0\}$.*

Proof. Let $r_0 \in (0, r_1)$ be as in (2.48). For any $n \in \mathbb{N} \setminus \{0\}$, let $W_n \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ be the solution to (2.50) given by Corollary 2.2.15. Then $U - W_n$ weakly solves problem (2.49) and we define $U_n := U - \xi^0_{n,r_0}(W_n)$, with ξ^0_{n,r_0} being the extension operator introduced in Proposition 2.2.11. We observe that $U_n \in H^1(B_{r_0} \setminus \tilde{\Gamma})$. To prove that U_n converges to U in $H^1(B_{r_0} \setminus \tilde{\Gamma})$ as $n \to \infty$, we notice that

$$\|U - U_n\|^2_{H^1(B_{r_0}\setminus\tilde{\Gamma})} \le c_0^2 \, \|W_n\|^2_{H^1(\Omega_{n,r_0})} \le 4\, c_0^2 \, \frac{N - 1 + r_0^2}{N-1} \int_{\Omega_{n,r_0}} (A\nabla W_n \cdot \nabla W_n - f_n W_n^2)\, dy,$$

by Proposition 2.2.11, (2.40), and (2.53). Therefore it is enough to prove that

$$\lim_{n\to\infty} \int_{\Omega_{n,r_0}} (A\nabla W_n \cdot \nabla W_n - f_n W_n^2)\, dy = 0. \tag{2.56}$$

Let

$$O_n := (B_{r_1} \setminus \tilde{\Gamma}) \setminus \Omega_{n,r_1} \tag{2.57}$$

for any $n \in \mathbb{N} \setminus \{0\}$. Since $W_n \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ solves (2.50) and U is a solution to (2.23), by Hölder's inequality, (2.13) and Proposition 2.2.11 we have that

$$\begin{aligned}
&\left| \int_{\Omega_{n,r_0}} (A\nabla W_n \cdot \nabla W_n - f_n W_n^2)\, dy \right| = \left| \int_{\Omega_{n,r_1}} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\, \xi^0_{n,r_0}(W_n))\, dy \right| \\
&= \Bigg| \int_{B_{r_1}\setminus\tilde{\Gamma}} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\, \xi^0_{n,r_0}(W_n))\, dy \\
&\qquad - \int_{O_n} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\, \xi^0_{n,r_0}(W_n))\, dy \Bigg| \\
&= \Bigg| \int_{B_{r_1}\setminus\tilde{\Gamma}} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - \tilde{f} U\, \xi^0_{n,r_0}(W_n))\, dy + \int_{B_{r_1}\setminus\tilde{\Gamma}} (\tilde{f} - f_n) U\, \xi^0_{n,r_0}(W_n))\, dy \\
&\qquad - \int_{O_n} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\, \xi^0_{n,r_0}(W_n))\, dy \Bigg| \\
&\le \left| \int_{O_n} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\, \xi^0_{n,r_0}(W_n))\, dy \right| + \left| \int_{B_{r_1}\setminus\tilde{\Gamma}} (\tilde{f} - f_n) U\, \xi^0_{n,r_0}(W_n)\, dy \right| \\
&\le 2\, \|\nabla U\|_{L^2(O_n)} \left\| \nabla \xi^0_{n,r_0}(W_n) \right\|_{L^2(B_{r_1}\setminus\tilde{\Gamma})} + \|f_n\|_{L^{\frac{N}{2}+\epsilon}(O_n)} \|U\|_{L^{q_\epsilon}(O_n)} \left\| \xi^0_{n,r_0}(W_n) \right\|_{L^{q_\epsilon}(B_{r_1})} \\
&\quad + \|\tilde{f} - f_n\|_{L^{\frac{N}{2}+\epsilon}(B_{r_1})} \|U\|_{L^{q_\epsilon}(B_{r_1})} \left\| \xi^0_{n,r_0}(W_n) \right\|_{L^{q_\epsilon}(B_{r_1})} \\
&\le 4 c_0 \ell \frac{\sqrt{N - 1 + r_0^2}}{\sqrt{N-1}} \Big(2\, \|\nabla U\|_{L^2(O_n)} + \sqrt{\mathcal{S}_{N,q_\epsilon}}\, r_1^{\frac{2\epsilon}{N+2\epsilon}} \|\tilde{f}\|_{L^{\frac{N}{2}+\epsilon}(O_n)} \|U\|_{L^{q_\epsilon}(O_n)} \\
&\qquad\qquad + \sqrt{\mathcal{S}_{N,q_\epsilon}}\, r_1^{\frac{2\epsilon}{N+2\epsilon}} \|\tilde{f} - f_n\|_{L^{\frac{N}{2}+\epsilon}(B_{r_1})} \|U\|_{L^{q_\epsilon}(B_{r_1})} \Big),
\end{aligned}$$

where q_ϵ is defined in (2.31) and we have used (2.44), (2.29), (2.38), (2.40), and (2.55) in the last inequality. We observe that

$$\lim_{n\to\infty} |O_n| = 0,$$

where $|O_n|$ is the N-dimensional Lebesgue measure of O_n. Then, since $\nabla U \in L^2(B_{r_1}\setminus\tilde\Gamma)$, $U \in L^{q_\epsilon}(B_{r_1})$ by Proposition 2.2.6, and $\tilde f \in L^{\frac{N}{2}+\epsilon}(B_{r_1})$, (2.56) follows by the absolute continuity of the integral and convergence (2.47).

We observe that $f_n U_n \in L^2(\Omega_{n,r_0})$. Indeed, under assumption (2.6), by Remark 2.2.3 we have that $\tilde f \in W^{1,\frac{N}{2}+\epsilon}(B_{r_1}\setminus\tilde\Gamma)$ and then, by Sobolev embeddings and Hölder's inequality, we easily obtain that $f_n U_n = \tilde f U_n \in L^2(\Omega_{n,r_0})$. Under assumption (2.7), f_n is defined in (2.43) and $f_n \in L^\infty(B_{r_1})$, hence $f_n U_n \in L^2(\Omega_{n,r_0})$.

Since Γ_{n,r_0} is C^∞-smooth and $f_n U_n \in L^2(\Omega_{n,r_0})$, by classical elliptic regularity theory, see e.g. [81, Theorem 2.2.2.5], we deduce that $U_n \in H^2(\Omega_{n,r})$ for any $r \in (0,r_0)$. The proof is thereby complete. □

2.3 The Almgren type frequency function

Let $u \in H^1(B_R\setminus\Gamma)$ be a non-trivial weak solution to (2.1) and $U = u\circ F \in H^1(B_{r_1}\setminus\tilde\Gamma)$ be the corresponding solution to (2.23). Let $r_0 \in (0,\min\{1,r_1\})$ be as in (2.48). For any $r \in (0,r_0]$, we define

$$H(r) := \frac{1}{r^{N-1}}\int_{\partial B_r} \mu\, U^2\, dS, \tag{2.58}$$

where μ is the function introduced in (2.16), and

$$D(r) := \frac{1}{r^{N-2}}\int_{B_r\setminus\tilde\Gamma} (A\nabla U\cdot\nabla U - \tilde f\, U^2)\, dy. \tag{2.59}$$

Proposition 2.3.1. *If $r \in (0,r_0]$ then $H(r) > 0$.*

Proof. We suppose by contradiction that there exists $r \in (0,r_0]$ such that $H(r) = 0$. By (2.17), it follows that U weakly solves (2.23) with the extra condition $U = 0$ on ∂B_r. Then by (2.32) we obtain that $U = 0$ on B_r. By classical unique continuation principles for elliptic equations, see e.g. [79], we conclude that $u = 0$ on B_R, which is a contradiction. □

Proposition 2.3.2. *We have that $H \in W^{1,1}_{\rm loc}((0,r_0])$ and*

$$\begin{aligned} H'(r) &= \frac{1}{r^{N-1}}\left(2\int_{\partial B_r} \mu U\frac{\partial U}{\partial\nu}\, dS + \int_{\partial B_r} U^2\nabla\mu\cdot\nu\, dS\right) \\ &= \frac{2}{r^{N-1}}\int_{\partial B_r}\mu U\frac{\partial U}{\partial \nu}\, dS + H(r)O(1) \quad \text{as } r\to 0^+, \end{aligned} \tag{2.60}$$

in a distributional sense and for a.e. $r \in (0,r_0)$.

Remark 2.3.3. To explain in what sense the term $\frac{\partial U}{\partial\nu}$ in (2.60) is meant, we observe that, if ∇U is the distributional gradient of U in $B_{r_1}\setminus\tilde\Gamma$, then $\nabla U \in L^2(B_{r_1},\mathbb{R}^N)$ and $\frac{\partial U}{\partial\nu} := \nabla U\cdot\frac{y}{|y|} \in L^2(B_{r_1})$. By the Coarea Formula it follows that $\nabla U \in L^2(\partial B_r,\mathbb{R}^N)$ and $\frac{\partial U}{\partial\nu} \in L^2(\partial B_r)$ for a.e. $r \in (0,r_1)$.

Proof. For any $\phi \in C_0^\infty(0,r_0)$ we define $v(y) := \phi(|y|)$. Then we have

$$\begin{aligned}\int_0^{r_0} H(r)\phi'(r)\,dy &= \int_0^{r_0} \frac{1}{r^{N-1}}\Big(\int_{\partial B_r} \mu U^2\,dS\Big)\,\phi'(r)\,dr \\ &= \int_{B_{r_0}^+} \frac{1}{|y|^N}\mu(y)U^2(y)\nabla v(y)\cdot y\,dy + \int_{B_{r_0}^-} \frac{1}{|y|^N}\mu(y)U^2(y)\nabla v(y)\cdot y\,dy \\ &= -\int_{B_{r_0}\setminus\tilde\Gamma} \frac{1}{|y|^N}(2\mu(y)v(y)U(y)\nabla U(y)\cdot y + v(y)U^2(y)\nabla\mu(y)\cdot y)\,dy \\ &= -\int_0^{r_0} \frac{2}{r^{N-1}}\Big(\int_{\partial B_r} \mu U\frac{\partial U}{\partial\nu}\,dS\Big)\phi(r)\,dr - \int_0^{r_0} \frac{1}{r^{N-1}}\Big(\int_{\partial B_r} U^2\nabla\mu\cdot\nu\,dS\Big)\phi(r)\,dr,\end{aligned}$$

which proves (2.60) thanks to (2.19). Since r^{-N+1} is bounded in any compact subset of $(0,r_0]$, then, by (2.17), (2.19) and the Coarea Formula, H and H' are locally integrable so that $H \in W^{1,1}_{\rm loc}((0,r_0])$. □

Now we turn our attention to D. Henceforth we let $\{f_n\}$ be as in (2.46), if f satisfies (2.6), or as in (2.43), if f satisfies (2.7), and we consider the sequence $\{U_n\}$ converging to U in $H^1(B_{r_0}\setminus\tilde\Gamma)$ provided by Theorem 2.2.16.

Remark 2.3.4. By Proposition 2.2.6 and (2.31), $U_n \to U$ in $L^{q_\epsilon}(B_{r_0})$. Then, since $f_n \to \tilde f$ in $L^{\frac{N}{2}+\epsilon}(B_{r_0})$ by (2.47), from Hölder's inequality it easily follows that

$$\lim_{n\to\infty}\int_{B_{r_0}} |\tilde f\,U^2 - f_n\,U_n^2|\,dy = 0. \tag{2.61}$$

Moreover, if f satisfies (2.6), $\nabla\tilde f \in L^{\frac{N}{2}+\epsilon}(B_{r_0},\mathbb{R}^N)$ and hence

$$\lim_{n\to\infty}\int_{B_{r_0}\setminus\Gamma} |(\nabla\tilde f\cdot\beta)\,(U^2-U_n^2)|\,dx = 0, \tag{2.62}$$

since the vector field β defined in (2.16) is bounded in view of (2.20).

Lemma 2.3.5. *If $F_n \to F$ in $L^1(B_{r_0})$, then there exists a subsequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ such that, for a.e. $r\in(0,r_0)$,*

$$\lim_{k\to\infty}\int_{\partial B_r} |F-F_{n_k}|\,dS = 0 \quad \textit{and} \quad \lim_{k\to\infty}\int_{S_{n_k,r}} F_{n_k}\,dS = \int_{\partial B_r} F\,dS,$$

where the notation $S_{n,r}$ has been introduced in (2.36).

Proof. Let $h_n(r) := \int_{\partial B_r}|F_n - F|\,dS$. Since, by assumption and the Coarea Formula,

$$\lim_{n\to\infty}\int_{B_{r_0}} |F-F_n|\,dy = \lim_{n\to\infty}\int_0^{r_0} h_n(r)dr = 0,$$

we have that $h_n \to 0$ in $L^1(0,r_0)$. Hence there exists a subsequence $\{h_{n_k}\}_{k\in\mathbb{N}}$ converging to 0 a.e. in $(0,r_0)$. Therefore $F_{n_k}\to F$ in $L^1(\partial B_r)$ for a.e. $r\in(0,r_0)$. It follows that, for a.e. $r\in(0,r_0)$,

$$\int_{S_{n_k,r}} F_{n_k}\,dS - \int_{\partial B_r} F\,dS = \int_{\partial B_r} \chi_{S_{n_k,r}}(F_{n_k}-F)\,dS + \int_{\partial B_r}(\chi_{S_{n_k}}-1)F\,dS \to 0$$

as $k\to\infty$, thus yielding the conclusion. □

Proposition 2.3.6. *We have that* $D \in W^{1,1}_{\mathrm{loc}}((0, r_0])$*,*

$$D(r) = \frac{1}{r^{N-2}} \int_{\partial B_r} U A\nabla U \cdot \nu \, dS = \frac{r}{2} H'(r) + rH(r)O(1) \quad \textit{as } r \to 0^+ \tag{2.63}$$

and

$$D'(r) = (2-N)\frac{1}{r^{N-1}} \int_{B_r \setminus \tilde{\Gamma}} (A\nabla U \cdot \nabla U - \tilde{f}U^2)\, dy + \frac{1}{r^{N-2}} \int_{\partial B_r} (A\nabla U \cdot \nabla U - \tilde{f}U^2)\, dS \tag{2.64}$$

in the sense of distributions and for a.e. $r \in (0, r_0)$*.*

Proof. The fact that $D \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and (2.64) follow from the Coarea Formula and (2.30). To prove (2.63) we consider the sequence $\{U_n\}$ introduced in Theorem 2.2.16. For every $r \in (0, r_0)$ and $n \in \mathbb{N} \setminus \{0\}$,

$$\frac{1}{r^{N-2}} \int_{\Omega_{n,r}} (A\nabla U_n \cdot \nabla U_n - f_n U_n^2)\, dy = \frac{1}{r^{N-2}} \int_{S_{n,r}} U_n A\nabla U_n \cdot \nu \, dS$$

since U_n solve (2.49) and $U_n \in H^2(\Omega_{n,r})$ by Theorem 2.2.16. Thanks to Remark 2.3.4,the Dominated Convergence Theorem, and Lemma 2.3.5, we can pass to the limit, up to a subsequence, as $n \to \infty$ in the above identity for a.e. $r \in (0, r_0)$, thus proving the first equality in (2.63). To prove the second equality in (2.63) we define

$$\zeta(y) := \frac{\mu(y)(\beta(y) - y)}{|y|} = \frac{A(y)y}{|y|} - \frac{A(y)y \cdot y}{|y|^3} y.$$

Then, since $\zeta(y) \cdot y = 0$ and $\zeta \cdot (0, \dots, 0, 1) = 0$ on $\tilde{\Gamma}$, we have that

$$\begin{aligned}\int_{\partial B_r} U A\nabla U \cdot \nu \, dS - \int_{\partial B_r} \mu U \frac{\partial U}{\partial \nu}\, dS &= \frac{1}{2} \int_{\partial B_r} \zeta \cdot \nabla(U^2)\, dS \\ &= -\frac{1}{2} \int_{\partial B_r} \operatorname{div}(\zeta) U^2 \, dS = r^{N-1} H(r) O(1)\end{aligned}$$

as $r \to 0$, where we have used in the last equality the estimate

$$\operatorname{div}(\zeta)(y) = \left(\frac{\nabla \mu(y)}{|y|} - \frac{\mu(y) y}{|y|^3} \right) (\beta(y) - y) + \frac{\mu(y)}{|y|} (\operatorname{div}(\beta)(y) - N) = O(1)$$

which follows from Proposition 2.2.2. Then we conclude by (2.60). □

The approximation procedure developed above also allows us to derive the following integration by parts formula.

Proposition 2.3.7. *There exists a set* $\mathcal{M} \subset [0, r_0]$ *having null* 1*-dimensional Lebesgue measure such that, for all* $r \in (0, r_0] \setminus \mathcal{M}$*,* $A\nabla U \cdot \nu \in L^2(\partial B_r)$ *and*

$$\int_{B_r \setminus \tilde{\Gamma}} A\nabla U \cdot \nabla \phi \, dx = \int_{B_r} \tilde{f} U \phi \, dx + \int_{\partial B_r} (A\nabla U \cdot \nu) \phi \, dS$$

for every $\phi \in H^1(B_{r_0} \setminus \tilde{\Gamma})$*, where* $A\nabla U \cdot \nu$ *on* ∂B_r *is meant in the sense of Remark 2.3.3.*

Proof. Since $U_n \to U$ in $H^1(B_{r_0} \setminus \tilde{\Gamma})$ in view of Theorem 2.2.16, by Lemma 2.3.5 there exist a subsequence $\{U_{n_k}\}$ and a set $\mathcal{M} \subset [0, r_0]$ having null 1-dimensional Lebesgue measure such that $A\nabla U \cdot \nu \in L^2(\partial B_r)$ and $A\nabla U_{n_k} \cdot \nu \to A\nabla U \cdot \nu$ in $L^2(\partial B_r)$ for all $r \in (0, r_0] \setminus \mathcal{M}$. Since $U_n \in H^2(\Omega_{n,r})$ for any $r \in (0, r_0)$ and $n \in \mathbb{N} \setminus \{0\}$ by Theorem 2.2.16, from (2.49) it follows that

$$\int_{\Omega_{n,r}} (A\nabla U_n \cdot \nabla \phi - f_n U_n \phi)\, dy = \int_{S_{n,r}} \phi A\nabla U_n \cdot \nu \, dS.$$

Arguing as in the proof of Proposition 2.3.6, we can pass to the limit along $n = n_k$ as $k \to \infty$ in the above identity for all $r \in (0, r_0] \setminus \mathcal{M}$, thus obtaining the conclusion. □

Theorem 2.3.8. *(Pohozaev type inequality) Under either assumption (2.6) or assumption (2.7), for any $r \in (0, r_0]$ we have that*

$$\begin{aligned} r\int_{\partial B_r} A\nabla U \cdot \nabla U\, dS &\geq 2r \int_{\partial B_r} \frac{|A\nabla U \cdot \nu|^2}{\mu}\, dS + \int_{B_r\setminus\tilde{\Gamma}} (A\nabla U \cdot \nabla U)\operatorname{div}(\beta)\, dy \\ &+ 2\int_{B_r\setminus\tilde{\Gamma}} \frac{A\nabla U \cdot y}{\mu} \tilde{f}\, U\, dy + \int_{B_r\setminus\tilde{\Gamma}} (dA\nabla U \nabla U)\cdot \beta\, dy - 2\int_{B_r\setminus\tilde{\Gamma}} J_\beta(A\nabla U)\cdot \nabla U\, dy, \end{aligned} \tag{2.65}$$

which can be rewritten as

$$\begin{aligned} r\int_{\partial B_r} (A\nabla U \cdot \nabla U - \tilde{f}\, U^2)\, dS &\geq 2r \int_{\partial B_r} \frac{|A\nabla U \cdot \nu|^2}{\mu}\, dS \\ &+ \int_{B_r\setminus\tilde{\Gamma}} (A\nabla U \cdot \nabla U)\operatorname{div}(\beta)\, dy + \int_{B_r\setminus\tilde{\Gamma}} (\tilde{f}\operatorname{div}(\beta) + \nabla\tilde{f}\cdot\beta)\, U^2\, dy \\ &+ \int_{B_r\setminus\tilde{\Gamma}} (dA\nabla U \nabla U)\cdot\beta\, dy - 2\int_{B_r\setminus\tilde{\Gamma}} J_\beta(A\nabla U)\cdot\nabla U\, dy \end{aligned} \tag{2.66}$$

if f satisfies (2.6).

Proof. By Theorem 2.2.16 we have that $U_n \in H^2(\Omega_{n,r})$ for any $r \in (0, r_0)$ and $n \in \mathbb{N} \setminus \{0\}$. Then, since A is symmetric by Proposition 2.2.1, we may write the following Rellich-Nečas identity in a distributional sense in $\Omega_{n,r}$:

$$\begin{aligned} \operatorname{div}((A\nabla U_n \cdot \nabla U_n)\beta - 2(\beta\cdot\nabla U_n)A\nabla U_n) &= (A\nabla U_n \cdot \nabla U_n)\operatorname{div}(\beta) \\ &- 2(\beta\cdot\nabla U_n)\operatorname{div}(A\nabla U_n) + (dA\nabla U_n\nabla U_n)\cdot\beta - 2J_\beta(A\nabla U_n)\cdot\nabla U_n. \end{aligned} \tag{2.67}$$

Since $U_n \in H^2(\Omega_{n,r})$ and the components of A and β are Lipschitz continuous by Propositions 2.2.1 and 2.2.2, then $(A\nabla U_n \nabla U_n)\beta - 2(\beta\cdot\nabla U_n)A\nabla U_n) \in W^{1,1}(\Omega_{n,r})$. Therefore we can integrate both sides of (2.67) on the Lipschitz domain $\Omega_{n,r}$ and apply the Divergence Theorem to obtain, in view of (2.16) and (2.49),

$$\begin{aligned} r\int_{S_{n,r}} \left(A\nabla U_n\cdot\nabla U_n - 2\frac{|A\nabla U_n\cdot\nu|^2}{\mu}\right) dS &+ \int_{\Gamma_{n,r}} (A\nabla U_n\cdot\nabla U_n)\frac{Ay\cdot\nu}{\mu}\, dS \\ = \int_{\Omega_{n,r}} (A\nabla U_n\cdot\nabla U_n)\operatorname{div}(\beta)\, dy &+ 2\int_{\Omega_{n,r}} \frac{A\nabla U_n\cdot y}{\mu} f_n U_n\, dy \\ + \int_{\Omega_{n,r}} (dA\nabla U_n\nabla U_n)\cdot\beta\, dy &- 2\int_{\Omega_{n,r}} J_\beta(A\nabla U_n)\cdot\nabla U_n\, dy. \end{aligned} \tag{2.68}$$

From Proposition 2.2.10, (2.13), and (2.17) it follows that, for all $n \in \mathbb{N} \setminus \{0\}$ and $r \in (0, r_0)$,

$$\int_{\Gamma_{n,r}} (A\nabla U_n \cdot \nabla U_n) \frac{Ay \cdot \nu}{\mu} \, dS \leq 0. \tag{2.69}$$

From Theorem 2.2.16, we recall that $U_n \to U$ strongly in $H^1(B_{r_0} \setminus \tilde{\Gamma})$, while Propositions 2.2.1 and 2.2.2 imply that

$$\begin{aligned} &\mu \in L^\infty(B_{r_0}, \mathbb{R}), \quad \beta \in L^\infty(B_{r_0}, \mathbb{R}^N), \quad \operatorname{div} \beta \in L^\infty(B_{r_0}, \mathbb{R}), \\ &A \in L^\infty(B_{r_0}, \mathbb{R}^{N^2}), \quad \left\{ \frac{\partial a_{i,j}}{\partial y_h} \right\}_{i,j,h=1,\dots,N} \in L^\infty(B_{r_0}, \mathbb{R}^{N^3}). \end{aligned} \tag{2.70}$$

Furthermore, under assumption (2.6), we have that, by Sobolev embeddings (see Proposition 2.2.6), if $N \geq 3$, then $f_n = \tilde{f} \in L^N(B_{r_0})$ and $U_n \to U$ strongly in $L^{2^*}(B_{r_0})$, whereas, if $N = 2$, then $f_n = \tilde{f} \in L^{2(1+\epsilon)/(1-\epsilon)}(B_{r_0})$ and $U_n \to U$ strongly in $L^{(1+\epsilon)/\epsilon}(B_{r_0})$; then, since $\nabla U_n \to \nabla U$ in $L^2(B_{r_0})$, Hölder's inequality ensures that

$$f_n U_n A\nabla U_n \cdot y \to \tilde{f} U A\nabla U \cdot y \quad \text{in } L^1(B_{r_0}). \tag{2.71}$$

Under assumption (2.7), we have that Hardy's inequality (see Proposition 2.2.5), Proposition 2.2.4 and (2.44) yield that

$$\int_{B_{r_0}} |f_n y(U_n - U)|^2 \, dy \leq \text{const}\, r_0^{4\epsilon} \int_{B_{r_0}} |y|^{-2} |U_n - U|^2 \, dy \to 0 \quad \text{as } n \to \infty$$

which, thanks to Proposition 2.2.5 again and the Dominated Convergence Theorem, easily implies that

$$f_n y U_n \to \tilde{f} y U \quad \text{in } L^2(B_{r_0}),$$

thus proving (2.71) also under assumption (2.7).

Then, thanks to the Dominated Convergence Theorem, (2.22), (2.71) and Lemma 2.3.5, we can pass to the limit in (2.68) as $n \to \infty$, up to a subsequence, and, taking into account (2.69), we obtain inequality (2.65).

If assumption (2.6) holds then by (2.16), (2.46) and Proposition 2.2.2 we have that

$$\begin{aligned} 2\int_{\Omega_{n,r}} \frac{A\nabla U_n \cdot y}{\mu} f_n U_n \, dy &= 2\int_{\Omega_{n,r}} (\beta \cdot \nabla U_n) \tilde{f} U_n \, dy \\ &= -\int_{\Omega_{n,r}} (\tilde{f} \operatorname{div}(\beta) + \nabla \tilde{f} \cdot \beta) U_n^2 \, dy + r\int_{S_{n,r}} \tilde{f} U_n^2 \, dS + \int_{\Gamma_{n,r}} \tilde{f} U_n^2 \beta \cdot \nu \, dS. \end{aligned} \tag{2.72}$$

We define

$$\begin{aligned} O_{n,r}^+ &:= O_n \cap B_r^+, \quad O_{n,r}^- := O_n \cap B_r^-, \\ \Gamma_{n,r}^+ &:= \Gamma_{n,r} \cap B_r^+, \quad \Gamma_{n,r}^- := \Gamma_{n,r} \cap B_r^-, \end{aligned}$$

where O_n is defined in (2.57). Taking into account that $\beta \cdot \nu = \frac{Ay}{\mu} \cdot \nu = 0$ on $\partial O_{n,r}^+ \cap \partial \mathbb{R}^N_+$ since $\nu = -(0, \dots, 1)$ and (2.11) holds, the Divergence Theorem yields that

$$\begin{aligned} \int_{\Gamma_{n,r}^+} \tilde{f} U_n^2 \beta \cdot \nu \, dS = -r &\int_{\partial O_{n,r}^+ \cap \partial B_r} \tilde{f} U_n^2 \beta \cdot \nu \, dS \\ &+ \int_{O_{n,r}^+} \left(\tilde{f} U_n^2 \operatorname{div}\beta + U_n^2 \nabla \tilde{f} \cdot \beta + 2\tilde{f} U_n \nabla U_n \cdot \beta \right) dy. \end{aligned} \tag{2.73}$$

By (2.61), (2.70), and Lemma 2.3.5 there exists a subsequence $\{\tilde{f}\, U_{n_k}^2 \beta\cdot\nu\}_{k\in\mathbb{N}}$ converging in $L^1(\partial B_r)$ and hence equi-integrable in ∂B_r for a.e. $r\in(0,r_0)$, hence

$$\lim_{k\to\infty}\int_{\partial O^+_{n_k,r}\cap\partial B_r}\tilde{f}\, U_{n_k}^2\beta\cdot\nu\, dS=0\quad\text{for a.e. } r\in(0,r_0).$$

Since $\nabla U_n\to\nabla U$ in $L^2(B^+_{r_0},\mathbb{R}^N)$, $U_n\to U$ in $L^{q_\epsilon}(B^+_{r_0})$ and $\tilde{f}\in L^{N+2\epsilon}(B^+_{r_0})$ by (2.6) and classical Sobolev embeddings, from (2.70) and Hölder's inequality we deduce that

$$\tilde{f}U_n\nabla U_n\cdot\beta\to\tilde{f}U\nabla U\cdot\beta\quad\text{in } L^1(B^+_{r_0}),$$

so that $\{\tilde{f}U_n\nabla U_n\cdot\beta\}_{n\in\mathbb{N}}$ is equi-integrable in $B^+_{r_0}$. Therefore

$$\lim_{n\to\infty}\int_{O^+_{n,r}}\tilde{f}U_n\nabla U_n\cdot\beta\, dy=0\quad\text{for all } r\in(0,r_0).$$

Moreover, also $\{\mathrm{div}\beta\,\tilde{f}\,U_n^2+U_n^2\,\nabla\tilde{f}\cdot\beta\}_{n\in\mathbb{N}}$ is equi-integrable thanks to (2.61) and (2.62). It follows that

$$\lim_{n\to\infty}\int_{O^+_{n,r}}(\mathrm{div}\beta\,\tilde{f}U_n^2+\nabla\tilde{f}\cdot\beta\, U_n^2)\, dy=0\quad\text{for all } r\in(0,r_0).$$

Then from (2.73) we conclude that

$$\lim_{k\to\infty}\int_{\Gamma^+_{n_k,r}}\tilde{f}U_{n_k}^2\beta\cdot\nu\, dS=0.$$

In a similar way we obtain that $\lim_{k\to\infty}\int_{\Gamma^-_{n_k,r}}\tilde{f}U_{n_k}^2\beta\cdot\nu\, dS=0$ so that

$$\lim_{k\to\infty}\int_{\Gamma_{n_k,r}}\tilde{f}U_{n_k}^2\beta\cdot\nu\, dS=0.$$

Therefore (2.66) follows by passing to the limit in (2.68) and (2.72) as $n\to\infty$ along a subsequence, taking into account Proposition 2.2.10, the Dominated Convergence Theorem, (2.22), Remark 2.3.4 and Lemma 2.3.5. □

Proposition 2.3.9. *For a.e. $r\in(0,r_0)$*

$$\begin{aligned}D'(r)\geq{}& 2r^{2-N}\int_{\partial B_r}\frac{|A\nabla U\cdot\nu|^2}{\mu}\, dS+r^{1-N}\int_{B_r\setminus\tilde{\Gamma}}(\mathrm{div}(\beta)+2-N)A\nabla U\cdot\nabla U\, dy\\ &+r^{1-N}\int_{B_r\setminus\tilde{\Gamma}}(\tilde{f}(\mathrm{div}(\beta)+N-2)+\nabla\tilde{f}\cdot\beta)\, U^2\, dy\\ &+r^{1-N}\int_{B_r\setminus\tilde{\Gamma}}(dA\nabla U\nabla U)\cdot\beta\, dy-2r^{1-N}\int_{B_r\setminus\tilde{\Gamma}}J_\beta(A\nabla U)\cdot\nabla U\, dy,\end{aligned}\tag{2.74}$$

if (2.6) holds, and

$$\begin{aligned}D'(r)\geq{}& 2r^{2-N}\int_{\partial B_r}\frac{|A\nabla U\cdot\nu|^2}{\mu}\, dS-r^{2-N}\int_{\partial B_r}\tilde{f}U^2\, dS+(N-2)r^{1-N}\int_{B_r}\tilde{f}U^2\, dy\\ &+r^{1-N}\int_{B_r\setminus\tilde{\Gamma}}(A\nabla\bar{u}\cdot\nabla U)(\mathrm{div}(\beta)+2-N)\, dy\ +2r^{1-N}\int_{B_r\setminus\tilde{\Gamma}}\frac{A\nabla U\cdot y}{\mu}\tilde{f}U\, dy\\ &+r^{1-N}\int_{B_r\setminus\tilde{\Gamma}}(dA\nabla U\nabla U)\cdot\beta\, dy-2r^{1-N}\int_{B_r\setminus\tilde{\Gamma}}J_\beta(A\nabla U)\cdot\nabla U\, dy\end{aligned}\tag{2.75}$$

if (2.7) holds.

Proof. Estimates (2.74)–(2.75) are direct consequences of (2.64), (2.65), and (2.66). □

We now introduce the Almgren frequency function, defined as

$$\mathcal{N} : (0, r_0] \to \mathbb{R}, \quad \mathcal{N}(r) := \frac{D(r)}{H(r)}. \tag{2.76}$$

The above definition of $\mathcal{N}$ is well posed thanks to Proposition 2.3.1.

Proposition 2.3.10. *If either assumptions* (2.6) *or* (2.7) *hold, then* $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ *and, for any* $r \in (0, r_0]$,

$$\mathcal{N}(r) \geq -2\eta_{\tilde{f}}(r). \tag{2.77}$$

Furthermore, for a.e. $r \in (0, r_0)$,

$$\mathcal{N}'(r) \geq \mathcal{V}(r) + \mathcal{W}(r) \tag{2.78}$$

where

$$\mathcal{V}(r) = \frac{2r\left(\left(\int_{\partial B_r} \frac{|A\nabla U \cdot \nu|^2}{\mu}\, dS\right)\left(\int_{\partial B_r} \mu U^2 dS\right) - \left(\int_{\partial B_r} U A\nabla U \cdot \nu \, dS\right)^2\right)}{\left(\int_{\partial B_r} \mu U^2 \, dS\right)^2} \geq 0 \tag{2.79}$$

and

$$\mathcal{W}(r) = O\left(r^{-1+\frac{4\epsilon}{N+2\epsilon}}\right)(1 + \mathcal{N}(r)) \quad \textit{as } r \to 0^+. \tag{2.80}$$

Proof. Since $1/H, D \in W^{1,1}_{\mathrm{loc}}((0, r_0])$, then $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}((0, r_0])$. Furthermore (2.32) directly implies (2.77).

By(2.63), for a.e. $r \in (0, r_0)$

$$\begin{aligned}
\mathcal{N}'(r) &= \frac{D'(r)H(r) - D(r)H'(r)}{H^2(r)} = \frac{D'(r)H(r) - \frac{2}{r}D^2(r)}{H^2(r)} + \frac{D(r)O(1)}{H(r)} \\
&= \frac{D'(r)H(r) - \frac{2}{r}r^{4-2N}\left(\int_{\partial B_r} U A\nabla U \cdot \nu \, dS\right)^2}{H^2(r)} + O(1)\mathcal{N}(r)
\end{aligned}$$

as $r \to 0^+$. By Proposition 2.2.1, Proposition 2.2.2, (2.31) and (2.32)

$$\begin{aligned}
&\left|\int_{B_r \setminus \tilde{\Gamma}} \Big((A\nabla U \cdot \nabla U)(\operatorname{div}(\beta) + 2 - N) - 2J_\beta(A\nabla U) \cdot \nabla U + (dA\nabla U \nabla U) \cdot \beta\Big)\, dy\right| \\
&\quad \leq O(r) \int_{B_r \setminus \tilde{\Gamma}} |\nabla U|^2 \, dy \\
&\quad \leq O(r) \int_{B_r \setminus \tilde{\Gamma}} (A\nabla U \cdot \nabla U - \tilde{f}U^2)\, dy + O\left(r^{\frac{4\epsilon}{N+2\epsilon}}\right) \int_{\partial B_r} \mu U^2 \, dS \quad \text{as } r \to 0^+.
\end{aligned}$$

By (2.30), (2.32), and (2.17)

$$\begin{aligned}
\int_{B_r} \tilde{f}U^2 \, dy &\leq O\left(r^{\frac{4\epsilon}{N+2\epsilon}}\right) \int_{B_r \setminus \tilde{\Gamma}} |\nabla U|^2 \, dy + O\left(r^{\frac{2\epsilon - N}{N+2\epsilon}}\right) \int_{\partial B_r} U^2 \, dS \\
&\leq O\left(r^{\frac{4\epsilon}{N+2\epsilon}}\right) \int_{B_r \setminus \tilde{\Gamma}} (A\nabla U \cdot \nabla U \, dy - \tilde{f}U^2) + O\left(r^{\frac{2\epsilon - N}{N+2\epsilon}}\right) \int_{\partial B_r} \mu U^2 \, dS
\end{aligned}$$

as $r \to 0^+$ and, by (2.21), the same holds for $\int_{B_r}(\operatorname{div}\beta - N + 2)\tilde{f}U^2\,dy$. In the same way from (2.20) it follows that, if (2.6) holds,

$$\int_{B_r} \nabla\tilde{f}\cdot\beta U^2\,dy \leq O\left(r^{\frac{4\epsilon}{N+2\epsilon}}\right)\int_{B_r\setminus\tilde{\Gamma}}(A\nabla U\cdot\nabla U\,dy - \tilde{f}U^2) + O\left(r^{\frac{2\epsilon-N}{N+2\epsilon}}\right)\int_{\partial B_r}\mu U^2\,dS$$

as $r \to 0^+$.

Under assumption (2.7), by Remark 2.2.3, (2.18), (2.13), (2.31) (2.30), (2.32) and Hölder's inequality,

$$\begin{aligned}
&\int_{B_r\setminus\tilde{\Gamma}}\frac{A\nabla U\cdot y}{\mu}\tilde{f}U\,dy = O(r)\int_{B_r\setminus\tilde{\Gamma}}|\nabla U||\tilde{f}|U\,dy \leq O(r^{\varepsilon})\,\|\nabla U\|_{L^2(B_r\setminus\tilde{\Gamma})}\left(\int_{B_r}|\tilde{f}|U^2\,dx\right)^{\frac{1}{2}}\\
&\leq O\left(r^{\varepsilon+\frac{2\varepsilon}{N+2\varepsilon}}\right)\left(\int_{B_r\setminus\tilde{\Gamma}}(A\nabla U\cdot\nabla U - \tilde{f}U^2)\,dy + \frac{2}{\eta_f(r)}r\int_{\partial B_r}\mu U^2\,dS\right)^{\frac{1}{2}}\times\\
&\qquad\times\left(\int_{B_r\setminus\tilde{\Gamma}}(A\nabla U\cdot\nabla U - \tilde{f}U^2)\,dy + \frac{2}{r}\int_{\partial B_r}\mu U^2\,dS\right)^{\frac{1}{2}}\\
&\leq O\left(r^{\varepsilon+\frac{2\varepsilon}{N+2\varepsilon}}\right)\int_{B_r\setminus\tilde{\Gamma}}(A\nabla U\cdot\nabla U - \tilde{f}U^2)\,dy + O\left(r^{-1+\varepsilon+\frac{2\varepsilon}{N+2\varepsilon}}\right)\int_{\partial B_r}\mu U^2\,dS.
\end{aligned}$$

Under assumptions (2.7), thanks to Remark 2.2.3 and (2.17),

$$\int_{\partial B_r}\tilde{f}U^2\,dS = O\left(r^{2\varepsilon-2}\right)\int_{\partial B_r}\mu U^2\,dS.$$

Collecting the above estimates, we conclude that (2.78), (2.79) and (2.80) follow from (2.74)or (2.75) under hypotheses (2.6) or (2.7) respectively. From the Cauchy–Schwarz inequality we also deduce that $\mathcal{V} \geq 0$ a.e. in $(0, r_0)$. □

We now prove that $\mathcal{N}$ is bounded.

Proposition 2.3.11. *There exists a constant $C > 0$ such that, for every $r \in (0, r_0]$,*

$$\mathcal{N}(r) \leq C. \tag{2.81}$$

Proof. By Proposition 2.3.10 there exists a constant $\kappa > 0$ such that, for a.e. $r \in (0, r_0)$,

$$(\mathcal{N}+1)'(r) \geq \mathcal{W}(r) \geq -\kappa\, r^{-1+\frac{4\epsilon}{N+2\epsilon}}(\mathcal{N}(r)+1).$$

Since $\mathcal{N}+1 > 0$ by (2.77) and the choice of r_0 in (2.48), it follows that

$$(\log(\mathcal{N}+1))' \geq -\kappa r^{-1+\frac{4\epsilon}{N+2\epsilon}}.$$

An integration over (r, r_0) yields

$$\mathcal{N}(r) \leq -1 + \exp\left(\kappa\frac{N+2\epsilon}{4\epsilon}r_0^{\frac{4\epsilon}{2\epsilon+N}}\right)(\mathcal{N}(r_0)+1)$$

and the proof is thereby complete. □

Proposition 2.3.12. *There exists the limit*

$$\gamma := \lim_{r\to 0^+} \mathcal{N}(r). \tag{2.82}$$

Furthermore γ is finite and $\gamma \geq 0$.

Proof. From Proposition 2.3.10 and (2.81) there exists a constant $\kappa > 0$ such that

$$\mathcal{N}'(r) \geq \mathcal{W}(r) \geq -\kappa\, r^{-1+\frac{4\epsilon}{N+2\epsilon}}(\mathcal{N}(r)+1) \geq -\kappa(C+1)r^{-1+\frac{4\epsilon}{N+2\epsilon}} \quad \text{for a.e. } r \in (0, r_0).$$

Then

$$\frac{d}{dr}\left(\mathcal{N}(r) + \frac{\kappa(C+1)(N+2\epsilon)}{4\epsilon} r^{\frac{4\epsilon}{N+2\epsilon}}\right) \geq 0$$

for a.e. $r \in (0, r_0)$. We conclude that $\lim_{r\to 0^+} \mathcal{N}(r)$ exists; moreover such a limit is finite thanks to (2.81) and (2.77). Furthermore from (2.31) and (2.77) we deduce that $\gamma \geq 0$. □

The proofs of Propositions 2.3.13 and 2.3.14 are standard and we omit them, see for example [47, Lemma 3.7, Lemma 4.6], [65, Lemma 5.6, Lemma 6.4], [65, Lemma 5.9, Lemma 6.6] or Propositions 4.4.9 and 4.4.10 in Chapter 4.

Proposition 2.3.13. *There exists a constant $\alpha > 0$ such that, for every $r \in (0, r_0]$,*

$$H(r) \leq \alpha\, r^{2\gamma}. \tag{2.83}$$

Furthermore for every $\sigma > 0$ there exist $\alpha_\sigma > 0$ and $r_\sigma \in (0, r_0)$ such that, for every $r \in (0, r_\sigma]$,

$$H(r) \geq \alpha_\sigma r^{2\gamma+\sigma}. \tag{2.84}$$

Proof. For the proof in a similar situation we refer to [65, Lemma 5.6] and Proposition 4.4.9 in Chapter 4. □

Proposition 2.3.14. *The limit $\lim_{r\to 0^+} r^{-2\gamma} H(r)$ exists and is finite.*

Proof. For the proof in a similar situation we refer to [65, Lemma 6.4] and Proposition 4.4.10 in Chapter 4. □

From the properties of the height function H derived above, in particular from estimate (2.84), we deduce the unique continuation property stated in Theorem 2.1.1.

Proof of Theorem 2.1.1. Let u be a weak solution to (2.1) such that $u(x) = O(|x|^k)$ as $|x| \to 0^+$ for all $k \in \mathbb{N}$. To prove that $u \equiv 0$ in B_R, we argue by contradiction and assume that $u \not\equiv 0$. Then we can define a frequency function for $U = u \circ F$ as in (2.58), (2.59) and (2.76). Choosing $k \in \mathbb{N}$ such that $k > \gamma + \frac{\sigma}{2}$, we would obtain that $H(r) = O(r^{2k}) = o(r^{2\gamma+\sigma})$ as $r \to 0$, contradicting estimate (2.84). □

2.4 The blow-up analysis

In this section we perform a blow-up analysis for scaled solutions to (2.23). To this aim we first study the spectrum of (2.8), which plays a crucial role in the classification of blow-up profiles.

2.4.1 Neumann eigenvalues on cracked sphere

In this section we study the spectrum of (2.8). We recall that $\mu \in \mathbb{R}$ is an eigenvalue of (2.8) if there exists $\psi \in H^1(\mathbb{S}^{N-1} \setminus \Sigma) \setminus \{0\}$ such that

$$\int_{\mathbb{S}^{N-1}\setminus\Sigma} \nabla_{\mathbb{S}^{N-1}\setminus\Sigma}\psi \cdot \nabla_{\mathbb{S}^{N-1}\setminus\Sigma}\phi \, dS = \mu \int_{\mathbb{S}^{N-1}\setminus\Sigma} \psi\phi \, dS \quad \text{for any } \phi \in H^1(\mathbb{S}^{N-1} \setminus \Sigma). \tag{2.85}$$

A Rellich-Kondrakov type theorem is needed to apply the classical Spectral Theorem to problem (2.8).

Proposition 2.4.1. *The embedding $H^1(\mathbb{S}^{N-1} \setminus \Sigma) \hookrightarrow L^2(\mathbb{S}^{N-1})$ is compact.*

Proof. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{S}^{N-1} \setminus \Sigma)$. We observe that $\mathbb{S}^{N-1}_+$ and $\mathbb{S}^{N-1}_-$ are smooth compact manifolds with boundary and that the sequences of restrictions $\{\phi_n|_{\mathbb{S}^{N-1}_+}\}_{n\in\mathbb{N}}$ and $\{\phi_n|_{\mathbb{S}^{N-1}_-}\}_{n\in\mathbb{N}}$ are bounded in $H^1(\mathbb{S}^{N-1}_+)$ and $H^1(\mathbb{S}^{N-1}_-)$ respectively. Then we can extract a subsequence $\{\phi_{n_k}\}_{k\in\mathbb{N}}$ such that $\{\phi_n|_{\mathbb{S}^{N-1}_+}\}_{n\in\mathbb{N}}$ converges in $L^2(\mathbb{S}^{N-1}_+)$ by the classical Rellich-Kondrakov Theorem on compact manifolds with boundary, see [18]. Proceeding in the same way for $\{\phi_{n_k}|_{\mathbb{S}^{N-1}_-}\}_{n\in\mathbb{N}}$ in $H^1(\mathbb{S}^{N-1}_-)$, we conclude that there exists a subsequence $\{\phi_{n_{k_h}}\}_{h\in\mathbb{N}}$ which converges both in $L^2(\mathbb{S}^{N-1}_-)$ and in $L^2(\mathbb{S}^{N-1}_+)$, hence in $L^2(\mathbb{S}^{N-1})$. □

Proposition 2.4.2.

(i) The point spectrum of (2.8) is a diverging and increasing sequence of non-negative eigenvalues $\{\mu_k\}_{k\in\mathbb{N}}$ of finite multiplicity and the eigenvalue $\mu_0 = 0$ is simple. Letting N_k be the multiplicity of μ_k and V_k be the eigenspace associated to μ_k, there exists an orthonormal basis of $L^2(\mathbb{S}^{N-1})$ consisting of eigenfunctions $\{Y_{k,i}\}_{k\in\mathbb{N},i=1,\dots,N_k}$ such that $\{Y_{k,i}\}_{i=1,\dots N_k}$ is a basis of V_k for any $k \in \mathbb{N}$.

(ii) For any $k \in \mathbb{N}$

$$\mu_k = \frac{k(k + 2N - 4)}{4}. \tag{2.86}$$

Moreover any eigenfunction of (2.8) belongs to $L^\infty(\mathbb{S}^{N-1})$.

Proof. The proof of (i) follows from the classical Spectral Theorem for compact self-adjoint operators, taking into account Proposition 2.4.1. We prove now (ii). If μ is an eigenvalue of (2.8) and Ψ an associated eigenfunction, let $\sigma := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu}$ and

$$W(r\theta) := r^\sigma \Psi(\theta), \quad \text{for any } r \in [0, \infty),\, \theta \in \mathbb{S}^{N-1} \setminus \Sigma.$$

Since Ψ is an eigenfunction of (2.8) then W is harmonic on $B_1 \setminus \tilde{\Gamma}$ and $\frac{\partial^+ W}{\partial \nu^+} = \frac{\partial^- W}{\partial \nu^-} = 0$ on $\tilde{\Gamma}$. Therefore we deduce from [41] that there exists $k \in \mathbb{N}$ such that $\sigma = \frac{k}{2}$ and so $\mu = \frac{k(k+2N-4)}{4}$. Moreover from [41] it also follows that $W \in L^\infty(B_1)$ hence $\Psi \in L^\infty(\mathbb{S}^{N-1})$.

Viceversa, if we let $k \in \mathbb{N}$ and define W in cylindrical coordinates as

$$W(x', r\cos(t), r\sin(t)) := r^{\frac{k}{2}} \cos\left(\frac{kt}{2}\right) \quad \text{for any } x' \in \mathbb{R}^{N-2},\ r \in [0, \infty),\ \text{and } t \in [0, 2\pi],$$

then W is harmonic on $B_1 \setminus \tilde{\Gamma}$ and $\frac{\partial^+ W}{\partial \nu^+} = \frac{\partial^- W}{\partial \nu^-} = 0$ on $\tilde{\Gamma}$. Since W is homogeneous of degree $k/2$, then

$$W(r\theta) = r^{\frac{k}{2}}\Psi(\theta), \quad \text{for any } r \in [0,\infty), \text{ and } \theta \in \mathbb{S}^{N-1} \setminus \Sigma,$$

where $\Psi = W_{|_{\mathbb{S}^{N-1}}}$. Then from

$$r^{\frac{k-4}{2}}\left(\frac{k(k-2)}{4}\Psi(\theta) + \frac{k(N-1)}{2}\Psi(\theta) + \Delta_{\mathbb{S}^{N-1}}\Psi(\theta)\right) = 0, \quad r \in [0,\infty), \ \theta \in \mathbb{S}^{N-1} \setminus \Sigma,$$

we deduce that Ψ solves (2.8) with $\mu = \frac{k(k+2N-4)}{4}$. □

Remark 2.4.3. The traces of eigenfunctions of problem (2.8) on both sides of Σ (i.e. the traces of restrictions to $\mathbb{S}^{N-1}_+$ and $\mathbb{S}^{N-1}_+$) cannot vanish identically.

Indeed, if an eigenfunction Ψ associated to the eigenvalue μ_k is such that the trace of $\Psi|_{\mathbb{S}^{N-1}_+}$ on Σ vanishes, then the function $W(x) := |x|^{k/2}\Psi(x/|x|)$ would be a harmonic function in $\mathbb{R}^N \setminus \tilde{\Gamma}$ satisfying both Dirichlet and Neumann homogeneous boundary conditions on the upper side of the crack, thus violating classic unique continuation principles.

2.4.2 The blow-up analysis

Throughout this section we let $u \in H^1(B_R \setminus \Gamma)$ be a non-trivial weak solution to (2.1) with f satisfying either (2.6) or (2.7), $U = u \circ F \in H^1(B_{r_1} \setminus \tilde{\Gamma})$ be the corresponding solution to (2.23), r_0 be as in (2.48) and r_1 be as in Proposition 2.2.1. For all $\lambda \in (0, r_0)$, let

$$W^\lambda(y) := \frac{U(\lambda y)}{\sqrt{H(\lambda)}} \quad \text{for any } y \in B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma}. \tag{2.87}$$

For any $\lambda \in (0, r_0)$ it is easy to verify that $W^\lambda \in H^1(B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma})$ and W^λ satisfies

$$\int_{B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma}} A(\lambda y)\nabla W^\lambda(y) \cdot \nabla\phi(y)\, dy - \lambda^2 \int_{B_{\lambda^{-1}r_1}} \tilde{f}(\lambda y) W^\lambda(y)\phi(y)\, dy = 0$$

for any $\phi \in H^1_{0,\partial B_{\lambda^{-1}r_1}}(B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma})$. In other words W^λ is a weak solution of

$$\begin{cases} -\operatorname{div}(A(\lambda\cdot)\nabla W^\lambda) = \lambda^2 \tilde{f}(\lambda\cdot)W^\lambda, & \text{in } B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma}, \\ A(\lambda\cdot)\nabla^+ W^\lambda \cdot \nu^+ = A(\lambda\cdot)\nabla^- W^\lambda \cdot \nu^- = 0, & \text{on } \tilde{\Gamma}, \end{cases}$$

for any $\lambda \in (0, r_0)$. Since $B_1 \subset B_{\lambda^{-1}r_1}$ for all $\lambda \in (0, r_0)$, it follows that, for any $\lambda \in (0, r_0)$,

$$\int_{B_1 \setminus \tilde{\Gamma}} A(\lambda y)\nabla W^\lambda(y) \cdot \nabla\phi(y)\, dy - \lambda^2 \int_{B_1} \tilde{f}(\lambda y) W^\lambda(y)\phi(y)\, dy = 0, \tag{2.88}$$

for any $\phi \in H^1_{0,\partial B_1}(B_1 \setminus \tilde{\Gamma})$. Furthermore by a change of variables, (2.87) and (2.58),

$$\int_{\mathbb{S}^{N-1}} \mu(\lambda\theta)|W^\lambda(\theta)|^2 dS = 1 \quad \text{for every } \lambda \in (0, r_0). \tag{2.89}$$

Proposition 2.4.4. *Let W^λ be as in (2.87). Then $\{W^\lambda\}_{\lambda \in (0,r_0)}$ is bounded in $H^1(B_1 \setminus \tilde{\Gamma})$.*

Proof. We have

$$\int_{B_1\setminus\tilde\Gamma} |\nabla W^\lambda|^2\,dy = \frac{\lambda^{2-N}}{H(\lambda)}\int_{B_\lambda\setminus\tilde\Gamma}|\nabla U(y)|^2\,dy \le \frac{2}{1-2\eta_{\tilde f}(\lambda)}\mathcal{N}(\lambda)+\frac{4\eta_{\tilde f}(\lambda)}{1-2\eta_{\tilde f}(\lambda)}.$$

by (2.32). Then thanks to (2.81), (2.31), (2.48), (2.29), (2.17), and (2.89) we conclude. □

The following proposition is a doubling type result.

Proposition 2.4.5. *There exists a constant $C_1>0$ such that for any $\lambda\in(0,\frac{r_0}{2})$ and $T\in[1,2]$*

$$\frac{1}{C_1}H(T\lambda)\le H(\lambda)\le C_1H(T\lambda), \tag{2.90}$$

$$\int_{B_T}|W^\lambda(y)|^2dy \le 2^NC_1\int_{B_1}|W^{T\lambda}(y)|^2\,dy, \tag{2.91}$$

and

$$\int_{B_T\setminus\tilde\Gamma}|\nabla W^\lambda(y)|^2dy\le 2^{N-2}C_1\int_{B_1\setminus\tilde\Gamma}|\nabla W^{T\lambda}(y)|^2\,dy. \tag{2.92}$$

Proof. From (2.81), (2.77), (2.63), and (2.48) we deduce that there exist two constants $\kappa_1>0$ and $\kappa_2>0$ such that, for any $r\in(0,r_0)$,

$$-\frac{2}{r}\le -\frac{2\eta_f(r)}{r}\le\frac{H'(r)}{H(r)}\le\frac{2\mathcal{N}(r)+\kappa_1}{r}\le\frac{\kappa_2}{r}.$$

Then (2.90) follows from an integration in $(\lambda,T\lambda)$ of the above inequality. Furthermore from (2.90) we obtain that, for any $\lambda\in(0,\frac{r_0}{2})$ and $T\in[1,2]$,

$$\begin{aligned}\int_{B_T}|W^\lambda(y)|^2\,dy &= \frac{\lambda^{-N}}{H(\lambda)}\int_{B_{\lambda T}}|U(y)|^2\,dy\le\frac{C_12^N}{(\lambda T)^NH(T\lambda)}\int_{B_{\lambda T}}|U(y)|^2\,dy\\ &= C_12^N\int_{B_1}|W^{T\lambda}(y)|^2\,dy.\end{aligned}$$

In the same way (2.92) follows from (2.90). □

Proposition 2.4.6. *Let $\mathcal{M}$ be as in Proposition 2.3.7 and W^λ be defined in (2.87). Then there exist $M>0$ and $\lambda_0>0$ such that for any $\lambda\in(0,\lambda_0)$ there exists $T_\lambda\in[1,2]$ such that $\lambda T_\lambda\notin\mathcal{M}$ and*

$$\int_{\partial B_{T_\lambda}}|\nabla W^\lambda|^2\,dS\le M\int_{B_{T_\lambda}\setminus\tilde\Gamma}(|\nabla W^\lambda|^2+|W^\lambda|^2)\,dy. \tag{2.93}$$

Proof. Since $\{W^\lambda\}_{\lambda\in(0,r_0/2)}$ is bounded in $H^1(B_2\setminus\tilde\Gamma)$ by Proposition 2.4.4, (2.91) and (2.92), then

$$\limsup_{\lambda\to0^+}\int_{B_2\setminus\tilde\Gamma}(|\nabla W^\lambda|^2+|W^\lambda|^2)\,dy<+\infty. \tag{2.94}$$

By the Coarea formula, for any $\lambda\in(0,\frac{r_0}{2})$ the function

$$g_\lambda(r):=\int_{B_r\setminus\tilde\Gamma}(|\nabla W^\lambda|^2+|W^\lambda|^2)\,dy$$

is absolutely continuous in $[1,2]$ with weak derivative

$$g'_\lambda(r) = \int_{\partial B_r} (|\nabla W^\lambda|^2 + |W^\lambda|^2)\, dS \quad \text{for a.e. } r \in [1,2],$$

where the integral $\int_{\partial B_r} |\nabla W^\lambda|^2 dS$ is meant in the sense of Remark 2.3.3. To prove the statement we argue by contradiction. If the conclusion does not hold, for any $M > 0$ there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, r_0/2)$ such that $\lim_{n\to\infty} \lambda_n = 0$ and

$$\int_{\partial B_r} (|\nabla W^{\lambda_n}|^2 + |W^{\lambda_n}|^2)\, dS > M \int_{B_r \setminus \tilde{\Gamma}} (|\nabla W^{\lambda_n}|^2 + |W^{\lambda_n}|^2)\, dy$$

for any $n \in \mathbb{N}$ and $r \in [1,2] \setminus \frac{1}{\lambda_n}\mathcal{M}$, and hence for a.e. $r \in [1,2]$. Hence

$$g'_{\lambda_n}(r) > M g_{\lambda_n}(r) \quad \text{for any } n \in \mathbb{N} \text{ and a.e. } r \in [1,2].$$

An integration in $[1,2]$ yields

$$\limsup_{n\to\infty} g_{\lambda_n}(1) \le e^{-M} \limsup_{n\to\infty} g_{\lambda_n}(2)$$

hence

$$\liminf_{\lambda\to 0^+} g_\lambda(1) \le e^{-M} \limsup_{\lambda\to 0^+} g_\lambda(2).$$

In view of (2.94), letting $M \to \infty$ we conclude that

$$\liminf_{\lambda\to 0^+} \int_{B_1 \setminus \tilde{\Gamma}} (|\nabla W^\lambda|^2 + |W^\lambda|^2)\, dy = 0.$$

Then there exists a sequence $\{\rho_n\}_{k\in\mathbb{N}}$ such that $W^{\rho_n} \to 0$ strongly in $H^1(B_1 \setminus \tilde{\Gamma})$ as $n \to \infty$. Due to the continuity of the trace operator γ_1 defined in Proposition 2.2.4 and (2.18), this is in contradiction with (2.89). □

Proposition 2.4.7. *There exists* $\overline{M} > 0$ *such that*

$$\int_{\mathbb{S}^{N-1}} |\nabla W^{\lambda T_\lambda}|^2\, dS \le \overline{M} \quad \textit{for all } \lambda \in \left(0, \min\left\{\frac{r_0}{2}, \lambda_0\right\}\right).$$

Proof. Since

$$\int_{\mathbb{S}^{N-1}} |\nabla W^{\lambda T_\lambda}|^2\, dS = \frac{\lambda^2 T_\lambda^{3-N}}{H(\lambda T_\lambda)} \int_{\partial B_{T_\lambda}} |\nabla U(\lambda y)|^2\, dS = T_\lambda^{3-N} \frac{H(\lambda)}{H(\lambda T_\lambda)} \int_{\partial B_{T_\lambda}} |\nabla W^\lambda|^2\, dS,$$

then, by (2.90), (2.91), (2.92), (2.93), and since $1 \le T_\lambda \le 2$, for any $\lambda \in (0, \min\{\frac{r_0}{2}, \lambda_0\})$ we have that

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} |\nabla W^{\lambda T_\lambda}|^2\, dS &\le 2C_1 M \int_{B_{T_\lambda} \setminus \tilde{\Gamma}} (|\nabla W^\lambda|^2 + |W^\lambda|^2)\, dy \\ &\le 2^{N+1} C_1^2 M \int_{B_1 \setminus \tilde{\Gamma}} (|\nabla W^{T_\lambda \lambda}|^2 + |W^{T_\lambda \lambda}|^2)\, dy.\end{aligned}$$

Therefore we conclude thanks to Proposition 2.4.4. □

Thanks to the estimates established above, we can now prove a first blow-up result.

Proposition 2.4.8. *Let $u \in H^1(B_R \setminus \Gamma)$, $u \not\equiv 0$, be a non-trivial weak solution to (2.1), with Γ defined in (2.2)–(2.3) and f satisfying either (2.6) or (2.7), and let $U = u \circ F$ be the corresponding solution to (2.23). Let γ be as in (2.82). Then*

$$\text{there exists } k_0 \in \mathbb{N} \text{ such that } \gamma = \frac{k_0}{2}. \tag{2.95}$$

For any sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \lambda_n = 0$ there exists a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and an eigenfunction Ψ of problem (2.8) associated to the eigenvalue μ_{k_0} such that $\|\Psi\|_{L^2(\mathbb{S}^{N-1})} = 1$ and

$$\frac{U(\lambda_{n_k} y)}{\sqrt{H(\lambda_{n_k})}} \to |y|^\gamma \Psi\left(\frac{y}{|y|}\right) \quad \text{strongly in } H^1(B_1 \setminus \tilde{\Gamma}).$$

Proof. Let W^λ be as in (2.87) for any $\lambda \in (0, \min\{\frac{r_0}{2}, \lambda_0\})$ and let us consider a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = 0$. From Proposition 2.4.4 $\{W^{\lambda T_\lambda} : \lambda \in (0, \min\{\frac{r_0}{2}, \lambda_0\})\}$ is bounded in $H^1(B_1 \setminus \tilde{\Gamma})$. Therefore there exists a subsequence $\{W^{\lambda_{n_k} T_{\lambda_{n_k}}}\}_{k\in\mathbb{N}} \subset H^1(B_1 \setminus \tilde{\Gamma})$ and a function $W \in H^1(B_1 \setminus \tilde{\Gamma})$ such that $W^{\lambda_{n_k} T_{\lambda_{n_k}}} \rightharpoonup W$ weakly in $H^1(B_1 \setminus \tilde{\Gamma})$. By compactness of the trace operator γ_1 (see Proposition 2.2.4), (2.18), and (2.89), it follows that

$$\int_{\partial B_1} W^2 \, dS = 1 \tag{2.96}$$

and so $W \not\equiv 0$ on $B_1 \setminus \tilde{\Gamma}$.

By Hölder's inequality and (2.30) we have that, for every $\phi \in H^1(B_1 \setminus \tilde{\Gamma})$,

$$\begin{aligned} \left|\lambda^2 \int_{B_1} \tilde{f}(\lambda y) W^\lambda(y) \phi(y) \, dy\right| &\le \lambda^2 \eta_{\tilde{f}(\lambda\cdot)}(1) \\ &\times \left(\int_{B_1\setminus\tilde{\Gamma}} |\nabla W^\lambda|^2 \, dy + \int_{\partial B_1} |W^\lambda|^2 \, dS\right)^{\frac{1}{2}} \left(\int_{B_1\setminus\tilde{\Gamma}} |\nabla \phi|^2 \, dy + \int_{\partial B_1} \phi^2 \, dS\right)^{\frac{1}{2}}. \end{aligned} \tag{2.97}$$

By (2.31) and a change of variables we have that

$$\begin{aligned} \lambda^2 \eta_{\tilde{f}(\lambda\cdot)}(1) &= S_{N,q_\epsilon} \lambda^2 \left(\int_{B_1} |\tilde{f}(\lambda y)|^{\frac{N}{2}+\epsilon} \, dy\right)^{\frac{2}{N+2\epsilon}} \\ &= S_{N,q_\epsilon} \lambda^{\frac{4\epsilon}{N+2\epsilon}} \|\tilde{f}\|_{L^{\frac{N}{2}+\epsilon}(B_\lambda)} \to 0 \quad \text{as } \lambda \to 0^+. \end{aligned} \tag{2.98}$$

From (2.97), (2.98), the boundedness of $\{W^\lambda\}$ in $H^1(B_1 \setminus \tilde{\Gamma})$ (established in Proposition 2.4.4) and of the traces (following from Proposition 2.2.4), we deduce that

$$\lim_{k\to\infty} \lambda^2_{\lambda_{n_k} T_{\lambda_{n_k}}} \int_{B_1} \tilde{f}(\lambda_{n_k} T_{\lambda_{n_k}} y) W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y) \phi(y) \, dy = 0, \tag{2.99}$$

for every $\phi \in H^1(B_1 \setminus \tilde{\Gamma})$.

Let $\phi \in H^1_{0,\partial B_1}(B_1 \setminus \tilde{\Gamma})$. We can test (2.88) with ϕ to obtain

$$\begin{aligned} &\int_{B_1\setminus\tilde{\Gamma}} A(\lambda_{n_k} T_{\lambda_{n_k}} y) \nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y) \cdot \nabla \phi(y) \, dy \\ &\qquad = (\lambda_{n_k} T_{\lambda_{n_k}})^2 \int_{B_1} \tilde{f}(\lambda_{n_k} T_{\lambda_{n_k}} y) W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y) \phi(y) \, dy, \end{aligned} \tag{2.100}$$

for any $k \in \mathbb{N}$. Since $W^{\lambda_{n_k} T_{\lambda_{n_k}}} \rightharpoonup W$ weakly in $H^1(B_1 \setminus \tilde{\Gamma})$, by (2.15) we have that

$$\lim_{k\to\infty} \int_{B_1\setminus\tilde{\Gamma}} A(\lambda_{n_k} T_{\lambda_{n_k}} y) \nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y) \cdot \nabla \phi(y)\, dy = \int_{B_1\setminus\tilde{\Gamma}} \nabla W \cdot \nabla \phi\, dy. \tag{2.101}$$

Therefore, for any $\phi \in H^1_{0,\partial B_1}(B_1 \setminus \tilde{\Gamma})$ we can pass to the limit as $k \to \infty$ in (2.100) thus obtaining, in view of (2.101) and (2.99),

$$\int_{B_1\setminus\tilde{\Gamma}} \nabla W \cdot \nabla \phi\, dy = 0,$$

i.e. W is a weak solution of

$$\begin{cases} -\Delta W = 0, & \text{on } B_1 \setminus \tilde{\Gamma}, \\ \dfrac{\partial^+ W}{\partial \nu^+} = \dfrac{\partial^- W}{\partial \nu^-} = 0, & \text{on } \tilde{\Gamma}. \end{cases} \tag{2.102}$$

We note that, by classical elliptic regularity theory, W is smooth in $B_1 \setminus \tilde{\Gamma}$.

In view of (2.87) and Propositions 2.4.6 and 2.3.7, by scaling we have that, for every $\phi \in H^1(B_1 \setminus \tilde{\Gamma})$,

$$\begin{aligned} &\int_{B_1\setminus\tilde{\Gamma}} A(\lambda_{n_k} T_{\lambda_{n_k}} y) \nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y) \cdot \nabla \phi(y)\, dy \\ &\qquad - (\lambda_{n_k} T_{\lambda_{n_k}})^2 \int_{B_1} \tilde{f}(\lambda_{n_k} T_{\lambda_{n_k}} y) W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y) \phi(y)\, dy \\ &\qquad\qquad = \int_{\partial B_1} (A(\lambda_{n_k} T_{\lambda_{n_k}} y) \nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y) \cdot \nu)\, \phi(y)\, dS. \end{aligned} \tag{2.103}$$

Thanks to Proposition 2.4.7 and (2.13) there exists a function $h \in L^2(\partial B_1)$ such that

$$(A(\lambda_{n_k} T_{\lambda_{n_k}} y) \nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y) \cdot \nu) \rightharpoonup h \quad \text{weakly in } L^2(\partial B_1), \tag{2.104}$$

up to a subsequence. By the weak convergence $W^{\lambda_{n_k} T_{\lambda_{n_k}}} \rightharpoonup W$ in $H^1(B_1 \setminus \tilde{\Gamma})$, (2.15), (2.99), and (2.104), passing to the limit as $k \to \infty$ in (2.103), we obtain that

$$\int_{B_1\setminus\tilde{\Gamma}} \nabla W \cdot \nabla \phi\, dy = \int_{\partial B_1} h\phi\, dS \tag{2.105}$$

for any $\phi \in H^1(B_1 \setminus \tilde{\Gamma})$. From the compactness of the trace operator γ_1 (see Proposition 2.2.4) and (2.104) it follows that

$$\lim_{k\to\infty} \int_{\partial B_1} (A(\lambda_{n_k} T_{\lambda_{n_k}} y) \nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y) \cdot \nu)\, W^{\lambda_{n_k} T_{\lambda_{n_k}}}(y)\, dS = \int_{\partial B_1} hW\, dS.$$

Therefore, recalling estimates (2.97), (2.98), and the boundedness of $\{W^\lambda\}$ in $H^1(B_1 \setminus \tilde{\Gamma})$, choosing $\phi = W^{\lambda_{n_k} T_{\lambda_{n_k}}}$ in (2.103) and passing to the limit as $k \to \infty$, we obtain that

$$\lim_{k\to\infty} \int_{B_1\setminus\tilde{\Gamma}} A(\lambda_{n_k} T_{\lambda_{n_k}} y) \nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}} \cdot \nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}}\, dy = \int_{\partial B_1} hW\, dS. \tag{2.106}$$

From (2.105) and (2.106) it follows that

$$\lim_{k\to\infty}\int_{B_1\setminus\tilde\Gamma} A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\cdot\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\,dy=\int_{B_1\setminus\tilde\Gamma}|\nabla W|^2\,dy$$

and so, thanks to (2.15),

$$W^{\lambda_{n_k}T_{\lambda_{n_k}}}\to W \quad \text{strongly in } H^1(B_1\setminus\tilde\Gamma). \tag{2.107}$$

For any $k\in\mathbb{N}$ and $r\in(0,1)$ let us define

$$\begin{aligned} D_k(r)&:=r^{2-N}\int_{B_r\setminus\tilde\Gamma}(A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\cdot\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\\ &\qquad\qquad-(\lambda_{n_k}T_{\lambda_{n_k}})^2\tilde f(\lambda_{n_k}T_{\lambda_{n_k}}y)|W^{\lambda_{n_k}T_{\lambda_{n_k}}}|^2)\,dy,\\ H_k(r)&:=r^{1-N}\int_{\partial B_r}\mu(\lambda_{n_k}T_{\lambda_{n_k}}y)|W^{\lambda_{n_k}T_{\lambda_{n_k}}}|^2\,dS,\quad\text{and}\quad \mathcal{N}_k(r):=\frac{D_k(r)}{H_k(r)}.\end{aligned}$$

By a change of variables it is easy to verify that, for any $r\in(0,1)$,

$$\mathcal{N}_k(r)=\frac{D_k(r)}{H_k(r)}=\frac{D(\lambda_{n_k}T_{\lambda_{n_k}}r)}{H(\lambda_{n_k}T_{\lambda_{n_k}}r)}=\mathcal{N}(\lambda_{n_k}T_{\lambda_{n_k}}r). \tag{2.108}$$

For any $r\in(0,1)$, we also define

$$H_W(r):=r^{1-N}\int_{\partial B_r}|W|^2\,dS,\quad D_W(r):=r^{2-N}\int_{B_r\setminus\tilde\Gamma}|\nabla W|^2\,dy\quad\text{and}\quad\mathcal{N}_W(r):=\frac{D_W(r)}{H_W(r)}.$$

The definition of $\mathcal{N}_W$ is well posed. Indeed, if $H_W(r)=0$ for some $r\in(0,1)$, then we may test the equation (2.102) on B_r with W and conclude that $W=0$ in B_r. Thanks to classical unique continuation principles for harmonic functions, this would imply that $W=0$ in B_1, thus contradicting (2.96).

Thanks to (2.107), (2.97)-(2.98) together with the boundedness of $\{W^\lambda\}$ in $H^1(B_1\setminus\tilde\Gamma)$, (2.15), (2.18), and Proposition 2.3.12, passing to the limit as $k\to\infty$ in (2.108) we obtain that

$$\mathcal{N}_W(r)=\lim_{k\to\infty}\mathcal{N}_k(r)=\lim_{k\to\infty}\mathcal{N}(\lambda_{n_k}T_{\lambda_{n_k}}r)=\gamma\quad\text{for any } r\in(0,1). \tag{2.109}$$

Then $\mathcal{N}_W$ is constant in $(0,1)$. Following the proof of Proposition 2.3.10 in the case $f\equiv 0$ and $g\equiv 0$ (where g is the function defined in (2.4)–(2.5)), so that $A=\mathrm{Id}_N$ and $\mu=1$, we obtain that

$$0=\mathcal{N}_W'(r)\ge\frac{2r\left(\left(\int_{\partial B_r}\left|\frac{\partial W}{\partial\nu}\right|^2dS\right)\left(\int_{\partial B_r}W^2dS\right)-\left(\int_{\partial B_r}W\frac{\partial W}{\partial\nu}\,dS\right)^2\right)}{\left(\int_{\partial B_r}W^2\,dS\right)^2}\ge 0$$

for a.e. $r\in(0,1)$. It follows that $\left(\int_{\partial B_r}\left|\frac{\partial W}{\partial\nu}\right|^2dS\right)\left(\int_{\partial B_r}W^2dS\right)=\left(\int_{\partial B_r}W\frac{\partial w}{\partial\nu}\,dS\right)^2$ for a.e. $r\in(0,1)$, i.e. equality holds in the Cauchy-Schwartz inequality for the vectors W and $\frac{\partial W}{\partial\nu}$ in $L^2(\partial B_r)$ for a.e. $r\in(0,1)$. It follows that there exists a function $\zeta(r)$ such that

$$\frac{\partial W}{\partial\nu}(r\theta)=\zeta(r)W(r\theta)\quad\text{for any }\theta\in\mathbb{S}^{N-1}\setminus\Sigma\text{ and a.e. } r\in(0,1]. \tag{2.110}$$

Multiplying by $W(r\theta)$ and integrating on $\mathbb{S}^{N-1}$ we obtain

$$\int_{\mathbb{S}^{N-1}} \frac{\partial W}{\partial \nu}(\theta r) W(r\theta)\, dS = \zeta(r) \int_{\mathbb{S}^{N-1}} W^2(\theta r)\, dS,$$

so that $\zeta(r) = \frac{H_W'(r)}{2H_W(r)} = \frac{\gamma}{r}$ by Proposition 2.3.2 and (2.109). Integrating (2.110) between $r \in (0,1)$ and 1 we obtain that

$$W(r\theta) = r^\gamma W(1\theta) = r^\gamma \Psi(\theta) \qquad \text{for any } \theta \in \mathbb{S}^{N-1} \setminus \Sigma \text{ and any } r \in (0,1],$$

where $\Psi = W|_{\mathbb{S}^{N-1}\setminus\Sigma}$. Then $\Psi \in H^1(\mathbb{S}^{N-1}\setminus\Sigma)$; furthermore, substituting $W(r\theta) = r^\gamma\Psi(\theta)$ in (2.102) we find out that Ψ is an eigenfunction of (2.8) with $(\gamma+N-2)\gamma$ as an associated eigenvalue. Hence by Proposition 2.4.2 there exists $k_0 \in \mathbb{N}$ such that $(\gamma+N-2)\gamma = \frac{k_0(k_0+2N-4)}{4}$. Recalling from Proposition 2.3.12 that $\gamma \geq 0$, we then obtain (2.95).

To conclude the proof it is enough to show that $W^{\lambda_{n_k}} \to W$ strongly in $H^1(B_1 \setminus \tilde{\Gamma})$ (possibly along a subsequence). Since $\{W^{\lambda_{n_k}}\}_{k\in\mathbb{N}}$ is bounded in $H^1(B_1 \setminus \tilde{\Gamma})$ by Proposition 2.4.4, there exists a function $\tilde{W} \in H^1(B_1 \setminus \tilde{\Gamma})$ and $T \in [1,2]$ such that $W^{\lambda_{n_k}} \rightharpoonup \tilde{W}$ weakly in $H^1(B_1 \setminus \tilde{\Gamma})$ and $T_{\lambda_k} \to T$, up to a subsequence.

Moreover, since $\{W^{\lambda_{n_k} T_{\lambda_{n_k}}}\}_{k\in\mathbb{N}}$ and $\{|\nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}}|\}_{k\in\mathbb{N}}$ converge strongly in $L^2(B_1)$ by (2.107), they are dominated by a measurable $L^2(B_1)$-function, up to a subsequence. Similarly, thanks to (2.90), we can suppose that, up to a subsequence, the limit

$$\zeta := \lim_{k\to\infty} \frac{H(\lambda_{n_k} T_{\lambda_{n_k}})}{H(\lambda_{n_k})}$$

exists and it is finite and strictly positive. Then for any $\phi \in C_c^\infty(B_1)$ we have that

$$\begin{aligned}
&\lim_{k\to\infty} \int_{B_1} W^{\lambda_{n_k}}(y)\phi(y)\, dy = \lim_{k\to\infty} T_{\lambda_{n_k}}^N \int_{B_{T^{-1}_{\lambda_{n_k}}}} W^{\lambda_{n_k}}(T_{\lambda_{n_k}} y)\phi(T_{\lambda_{n_k}} y)\, dy \\
&= \lim_{k\to\infty} T_{\lambda_{n_k}}^N \sqrt{\frac{H(\lambda_{n_k} T_{\lambda_{n_k}})}{H(\lambda_{n_k})}} \int_{B_{T^{-1}_{\lambda_{n_k}}}} W^{T_{\lambda_{n_k}}\lambda_{n_k}}(y)\phi(T_{\lambda_{n_k}} y)\, dy \\
&= T^N \sqrt{\zeta} \int_{B_{T^{-1}}} W(y)\phi(Ty)\, dy = \sqrt{\zeta} \int_{B_1} W(y/T)\phi(y)\, dy,
\end{aligned}$$

thanks to the Dominated Convergence Theorem. By density the same holds for any $\phi \in L^2(B_1)$. It follows that $W^{\lambda_{n_k}} \rightharpoonup \sqrt{\zeta} W(\cdot/T)$ weakly in $L^2(B_1)$. Hence, by uniqueness of the weak limit, we have that $\tilde{W}(\cdot) = \sqrt{\zeta} W(\cdot/T)$ and $W^{\lambda_{n_k}} \rightharpoonup \sqrt{\zeta} W(\cdot/T)$ weakly in $H^1(B_1 \setminus \tilde{\Gamma})$. Furthermore

$$\begin{aligned}
&\lim_{k\to\infty} \int_{B_1\setminus\tilde{\Gamma}} |\nabla W^{\lambda_{n_k}}(y)|^2\, dy = \lim_{k\to\infty} T_{\lambda_{n_k}}^N \int_{B_{T^{-1}_{\lambda_{n_k}}}\setminus\tilde{\Gamma}} |\nabla W^{\lambda_{n_k}}(T_{\lambda_{n_k}} y)|^2\, dy \\
&= \lim_{k\to\infty} T_{\lambda_{n_k}}^{N-2} \frac{H(\lambda_{n_k} T_{\lambda_{n_k}})}{H(\lambda_{n_k})} \int_{B_{T^{-1}_{\lambda_{n_k}}}\setminus\tilde{\Gamma}} |\nabla W^{T_{\lambda_{n_k}}\lambda_{n_k}}(y)|^2 dy \\
&= T^{N-2}\zeta \int_{B_{T^{-1}}\setminus\tilde{\Gamma}} |\nabla W(y)|^2 dy = \int_{B_1\setminus\tilde{\Gamma}} |\sqrt{\zeta}\nabla(W(\cdot/T))|^2\, dy.
\end{aligned}$$

Then we can conclude that $W^{\lambda_{n_k}} \to \tilde{W} = \sqrt{\zeta} W(\cdot/T)$ strongly in $H^1(B_1 \setminus \tilde{\Gamma})$. Moreover, by compactness of the trace operator γ_1 (see Proposition 2.2.4), (2.18), and (2.89), we deduce that $\int_{\partial B_1} \tilde{W}^2 \, dS = 1$. Then, since $W(r\theta) = r^{\frac{k_0}{2}} \Psi(\theta)$, we deduce that

$$\tilde{W}(r\theta) = \sqrt{\zeta} W\left(\frac{r}{T}\theta\right) = \left(\frac{\zeta}{T^{k_0}}\right)^{\frac{1}{2}} r^{\frac{k_0}{2}} \Psi(\theta) = \left(\frac{\zeta}{T^{k_0}}\right)^{\frac{1}{2}} W(r\theta)$$

and

$$1 = \int_{\partial B_1} \tilde{W}^2 \, dS = \frac{\zeta}{T^{k_0}} \int_{\partial B_1} W^2 \, dS = \frac{\zeta}{T^{k_0}},$$

thanks to (2.96). Therefore $W = \tilde{W}$ and the proof is complete. □

We are now in position of prove Theorem 2.1.2.

Proof of Theorem 2.1.2. Let us assume that $\operatorname{Tr}^+_\Gamma u(z) = O(|z|^k)$ as $|z| \to 0^+$, $z \in \Gamma$, for all $k \in \mathbb{N}$ (a similar argument works under the assumption $\operatorname{Tr}^-_\Gamma u(z) = O(|z|^k)$). Letting $U = u \circ F$, by the properties of the diffeomorphism F described in Proposition 2.2.1, we have that $\operatorname{Tr}^+_{\tilde{\Gamma}} U(z) = O(|z|^k)$ as $|z| \to 0^+$, so that, for all $k \in \mathbb{N}$,

$$\|\lambda^{-k} \operatorname{Tr}^+_{\tilde{\Gamma}} U(\lambda \cdot)\|_{L^2(B_1 \cap \tilde{\Gamma})} \to 0 \quad \text{as } \lambda \to 0^+. \tag{2.111}$$

On the other hand, if, by contradiction, $u \not\equiv 0$, by Proposition 2.4.8 and classical trace theorems there exist $k_0 \in \mathbb{N}$, a sequence $\lambda_n \to 0^+$, and an eigenfunction Ψ of problem (2.8) such that

$$\lim_{n\to\infty} \frac{\| \operatorname{Tr}^+_{\tilde{\Gamma}} U(\lambda_n \cdot)\|_{L^2(B_1 \cap \tilde{\Gamma})}}{\sqrt{H(\lambda_n)}} = \left\| \operatorname{Tr}^+_{\tilde{\Gamma}} \left(|y|^\gamma \Psi\left(\tfrac{y}{|y|}\right)\right)\right\|_{L^2(B_1 \cap \tilde{\Gamma})} \neq 0, \tag{2.112}$$

where the above limit is nonzero thanks to Remark 2.4.3. Combining (2.111) and (2.112) we obtain that

$$\lim_{n\to\infty} \frac{\sqrt{H(\lambda_n)}}{\lambda_n^k} = 0 \quad \text{for all } k \in \mathbb{N},$$

thus contradicting estimate (2.84). □

2.5 Asymptotics of the height function

In dimension $N \geq 3$, we can further specify the behaviour of $U(\lambda \cdot)$ as $\lambda \to 0^+$, deriving the asymptotics of the function $H(\lambda)$ appearing as a normalization factor in the blowed-up family (2.87). Let $\{Y_{k,i}\}_{k\in\mathbb{N}, i=1,\dots,N_k}$ be the basis of $L^2(\mathbb{S}^{N-1})$ given by Proposition 2.4.2. Let $N \geq 3$, $u \in H^1(B_R \setminus \Gamma)$ be a weak solution to (2.1), with Γ defined in (2.2)–(2.3) and f satisfying either (2.6) or (2.7), and let $U = u \circ F$ be the corresponding solution to (2.23). For any $\lambda \in (0, r_0)$, $k \in \mathbb{N}$ and $i = 1, \dots, N_k$ we define

$$\varphi_{k,i}(\lambda) := \int_{\mathbb{S}^{N-1}} U(\lambda\theta) Y_{k,i}(\theta) \, dS \tag{2.113}$$

and

$$\begin{aligned} \Upsilon_{k,i}(\lambda) &:= -\int_{B_\lambda \setminus \tilde{\Gamma}} (A - \mathrm{Id}_N) \nabla U \cdot \frac{\nabla_{\mathbb{S}^{N-1}} Y_{k,i}(y/|y|)}{|y|} \, dy \\ &+ \int_{B_\lambda} \tilde{f}(y) U(y) Y_{k,i}(y/|y|) \, dy + \int_{\partial B_\lambda} (A - \mathrm{Id}_N) \nabla U \cdot \frac{y}{|y|} Y_{k,i}(y/|y|) \, dS. \end{aligned} \tag{2.114}$$

Proposition 2.5.1. *Let k_0 be as in Proposition 2.4.8. Then, for any $i = 1, \dots, N_{k_0}$ and $r \in (0, r_0]$,*

$$\varphi_{k_0,i}(\lambda) = \lambda^{\frac{k_0}{2}} \Big(r^{-\frac{k_0}{2}} \varphi_{k_0,i}(r) + \frac{2N + k_0 - 4}{2(N + k_0 - 2)} \int_\lambda^r s^{-N+1-\frac{k_0}{2}} \Upsilon_{k_0,i}(s)\, ds \\ + \frac{k_0 r^{-N+2-k_0}}{2(N + k_0 - 2)} \int_0^r s^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(s)\, ds \Big) + o(\lambda^{\frac{k_0}{2}}) \quad as\ \lambda \to 0^+. \tag{2.115}$$

Proof. For any $k \in \mathbb{N}$ and any $i = 1, \dots, N_k$ we consider the distribution $\zeta_{k,i}$ on $(0, r_0)$ defined as

$$ {}_{\mathcal{D}'(0,r_0)}\langle \zeta_{k,i}, \omega \rangle_{\mathcal{D}(0,r_0)} := \int_0^{r_0} \omega(\lambda) \left(\int_{\mathbb{S}^{N-1}} \tilde{f}(\lambda\theta) U(\lambda\theta) Y_{m,k}(\theta)\, dS_\theta \right) d\lambda \\ - \int_{B_{r_0} \setminus \tilde{\Gamma}} (A - \mathrm{Id}_N) \nabla U \cdot \nabla \Big(|y|^{1-N} \omega(|y|) Y_{m,k}(y/|y|) \Big)\, dy,$$

for any $\omega \in \mathcal{D}(0, r_0)$.

Since $\Upsilon_{k,i} \in L^1_{\mathrm{loc}}(0, r_0)$ by (2.114), we may consider its derivative in the sense of distributions. A direct calculation shows that

$$\Upsilon'_{k,i}(\lambda) = \lambda^{N-1} \zeta_{k,i}(\lambda) \tag{2.116}$$

in the sense of distributions on $(0, r_0)$. From the definition of $\zeta_{k,i}$, (2.23), and the fact that $Y_{k,i}$ is a solution of (2.85) we deduce that

$$-\varphi''_{k,i}(\lambda) - \frac{N-1}{\lambda} \varphi'_{k,i}(\lambda) + \frac{\mu_k}{\lambda^2} \varphi_{k,i}(\lambda) = \zeta_{k,i}(\lambda)$$

in the sense of distribution in $(0, r_0)$; the above equation can be rewritten as

$$-(\lambda^{N-1+k} (\lambda^{-\frac{k}{2}} \varphi_{k,i}(\lambda))')' = \lambda^{N-1+\frac{k}{2}} \zeta_{k,i}(\lambda),$$

thanks to (2.86). Integrating the right-hand side of the equation above by parts, since (2.116) holds, we obtain that, for every $r \in (0, r_0)$, $k \in \mathbb{N}$ and $i = 1, \dots, N_k$ there exists a constant $c_{k,i}(r)$ such that

$$(\lambda^{-\frac{k}{2}} \varphi_{k,i}(\lambda))' = -\lambda^{-N+1-\frac{k}{2}} \Upsilon_{k,i}(\lambda) - \frac{k}{2} \lambda^{-N+1-k} \left(c_{k,i}(r) + \int_\lambda^r s^{\frac{k}{2}-1} \Upsilon_{k,i}(s)\, ds \right)$$

in the sense of distribution on $(0, r_0)$. Then $\varphi_{k,i}(\lambda) \in W^{1,1}_{\mathrm{loc}}(0, r_0)$ and a further integration yields

$$\begin{aligned} \varphi_{k,i}(\lambda) &= \lambda^{\frac{k}{2}} \left(r^{-\frac{k}{2}} \varphi_{k,i}(r) + \int_\lambda^r s^{-N+1-\frac{k}{2}} \Upsilon_{k,i}(s)\, ds \right) \\ &\quad + \frac{k}{2} \lambda^{\frac{k}{2}} \left(\int_\lambda^r s^{-N+1-k} \left(c_{k,i}(r) + \int_s^r t^{\frac{k}{2}-1} \Upsilon_{k,i}(t)\, dt \right) ds \right. \\ &= \lambda^{\frac{k}{2}} \left(r^{-\frac{k}{2}} \varphi_{k,i}(r) + \frac{2N+k-4}{2(N+k-2)} \int_\lambda^r s^{-N+1-\frac{k}{2}} \Upsilon_{k,i}(s)\, ds \right) \\ &\quad - \lambda^{\frac{k}{2}} \frac{k c_{k,i}(r) r^{-N+2-k}}{2(N+k-2)} + \frac{k \lambda^{-N+2-\frac{k}{2}}}{2(N+k-2)} \left(c_{k,i}(r) + \int_\lambda^r t^{\frac{k}{2}-1} \Upsilon_{k,i}(t)\, dt \right). \end{aligned} \tag{2.117}$$

Now we claim that, if k_0 is as in Proposition 2.4.8, then

$$\text{the function } s \to s^{-N+1-\frac{k_0}{2}}\Upsilon_{k_0,i}(s) \text{ belongs to } L^1(0,r_0). \tag{2.118}$$

To this end we will estimate each terms in (2.114). Thanks to (2.15), Hölder's inequality, a change of variables and Proposition 2.4.4, we have that

$$\begin{aligned}
&\left|\int_{B_s\setminus\tilde\Gamma}(A-\mathrm{Id}_N)\nabla U\cdot\frac{\nabla_{\mathbb{S}^{N-1}}Y_{k_0,i}(y/|y|)}{|y|}dy\right| \leq \text{const}\int_{B_s\setminus\tilde\Gamma}|y||\nabla U|\frac{|\nabla_{\mathbb{S}^{N-1}}Y_{k_0,i}(y/|y|)|}{|y|}\,dy \\
&\leq \text{const}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla U|^2dy\right)^{\frac12}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla_{\mathbb{S}^{N-1}}Y_{k_0,i}(y/|y|)|^2dy\right)^{\frac12} \\
&\leq \text{const}\, s^{\frac{N-2}{2}}s^{\frac{N}{2}}\sqrt{H(s)}\left(\int_{B_1\setminus\tilde\Gamma}|\nabla W^s(y)|^2\,dy\right)^{\frac12}\leq \text{const}\, s^{N-1}\sqrt{H(s)}.
\end{aligned}$$

From Hölder's inequality, (2.30), (2.17), and Proposition 2.4.4 it follows that

$$\begin{aligned}
&\left|\int_{B_s}\tilde f(y)U(y)Y_{k_0,i}(y/|y|)\,dy\right| \leq \left(\int_{B_s}|\tilde f(y)|U^2(y)\,dy\right)^{\frac12}\left(\int_{B_s}|\tilde f(y)|Y^2_{k_0,i}(y/|y|)\,dy\right)^{\frac12} \\
&\leq \text{const}\, s^{\frac{4\epsilon}{N+2\epsilon}}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla U|^2\,dy+s^{N-2}H(s)\right)^{\frac12}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla Y_{k_0,i}(y/|y|)|^2\,dy+s^{N-2}\right)^{\frac12} \\
&\leq \text{const}\, s^{(N-2)+\frac{4\epsilon}{N+2\epsilon}}\sqrt{H(s)}.
\end{aligned}$$

Furthermore, in view of (2.15), for a.e. $s\in(0,r_0)$ we have that

$$\left|\int_{\partial B_s}(A-\mathrm{Id}_N)\nabla U\cdot\frac{y}{|y|}Y_{k_0,i}(y/|y|)\,dS\right|\leq \text{const}\, s\int_{\partial B_s}|\nabla U||Y_{k_0,i}(y/|y|)|\,dS$$

and an integration by parts and Hölder's inequality yield, for any $r\in(0,r_0]$,

$$\begin{aligned}
&\int_0^r s^{-N+2-\frac{k_0}{2}}\left(\int_{\partial B_s}|\nabla U||Y_{k_0,i}(y/|y|)|\,dS\right)ds = r^{-N+2-\frac{k_0}{2}}\int_{B_r\setminus\tilde\Gamma}|\nabla U||Y_{k_0,i}(y/|y|)| \\
&+\left(N-2+\frac{k_0}{2}\right)\int_0^r s^{-N+1-\frac{k_0}{2}}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla U||Y_{k_0,i}(y/|y|)|\,dS\right)ds \\
&\leq \text{const}\left(r^{1-\frac{k_0}{2}}\sqrt{H(r)}+\int_0^r s^{-\frac{k_0}{2}}\sqrt{H(s)}\,ds\right),
\end{aligned}$$

reasoning as above. In conclusion, combining the above estimates with (2.95) and (2.83), we obtain that, for any $r\in(0,r_0]$,

$$\begin{aligned}
\int_0^r s^{-N+1-\frac{k_0}{2}}|\Upsilon_{k_0,i}(s)|\,ds &\leq \text{const}\left(r^{1-\frac{k_0}{2}}\sqrt{H(r)}+\int_0^r s^{-\frac{k_0}{2}-1+\frac{4\epsilon}{N+2\epsilon}}\sqrt{H(s)}\,ds\right) \quad (2.119)\\
&\leq \text{const}\left(r+\int_0^r s^{\frac{2\epsilon-N}{N+2\epsilon}}\,ds\right)\leq \text{const}\, r^{\frac{4\epsilon}{N+2\epsilon}}
\end{aligned}$$

which in particular implies (2.118). By (2.118), it follows that, for every $r\in(0,r_0]$,

$$\begin{aligned}
&\lambda^{\frac{k_0}{2}}\left(r^{-\frac{k_0}{2}}\varphi_{k_0,i}(r)+\frac{2N+k_0-4}{2(N+k_0-2)}\int_\lambda^r s^{-N+1-\frac{k_0}{2}}\Upsilon_{k_0,i}(s)\,ds-\frac{k_0c_{k_0,i}(r)r^{-N+2-k_0}}{2(N+k_0-2)}\right) \\
&=O\left(\lambda^{\frac{k_0}{2}}\right)=o\left(\lambda^{-N+2-\frac{k_0}{2}}\right)\quad \text{as } \lambda\to0^+ \qquad (2.120)
\end{aligned}$$

and $s \to s^{\frac{k_0}{2}-1}\Upsilon_{k_0,i}(s)$ belongs to $L^1(0,r_0)$.

Next we show that for every $r \in (0,r_0)$

$$c_{k_0,i}(r) + \int_0^r t^{\frac{k_0}{2}-1}\Upsilon_{k_0,i}(t)\,dt = 0. \tag{2.121}$$

We argue by contradiction assuming that there exists $r \in (0,r_0)$ such that (2.121) does not hold. Then by (2.117) and (2.120)

$$\varphi_{k_0,i}(\lambda) \sim \frac{k_0\lambda^{-N+2-\frac{k_0}{2}}}{2(N+k_0-2)}\left(c_{k_0,i}(r) + \int_\lambda^r t^{\frac{k_0}{2}-1}\Upsilon_{k_0,i}(t)\,dt\right) \quad \text{as } \lambda \to 0^+. \tag{2.122}$$

From Hölder's inequality, a change of variables, and (2.28)

$$\int_0^{r_0} \lambda^{N-3}|\varphi_{k_0,i}(\lambda)|^2\,d\lambda \le \int_0^{r_0} \lambda^{N-3}\left(\int_{\mathbb{S}^{N-1}} |U(\lambda\theta)|^2 dS\right) d\lambda = \int_{B_{r_0}} \frac{|U|^2}{|y|^2}dy < +\infty$$

thus contradicting (2.122). Hence (2.121) is proved.

Furthermore from (2.119) and (2.121)

$$\begin{aligned}\left|\lambda^{-N+2-\frac{k_0}{2}}\left(c_{k_0,i}(r) + \int_\lambda^r t^{\frac{k_0}{2}-1}\Upsilon_{k_0,i}(t)\,dt\right)\right| &= \lambda^{-N+2-\frac{k_0}{2}}\left|\int_0^\lambda t^{\frac{k_0}{2}-1}\Upsilon_{k_0,i}(t)\,dt\right| \\ &\le \lambda^{-N+2-\frac{k_0}{2}}\int_0^\lambda t^{N-2+k_0}\left|t^{-N+1-\frac{k_0}{2}}\Upsilon_{k_0,i}(t)\right|\,dt \\ &\le \lambda^{\frac{k_0}{2}}\int_0^\lambda \left|t^{-N+1-\frac{k_0}{2}}\Upsilon_{k_0,i}(t)\right|\,dt = O\left(\lambda^{\frac{4\epsilon}{N+2\epsilon}+\frac{k_0}{2}}\right) \quad \text{as } \lambda \to 0^+.\end{aligned} \tag{2.123}$$

Then the conclusion follows form (2.117), (2.121), and (2.123). □

Proposition 2.5.2. *Let γ be as in (2.82). Then*

$$\lim_{r\to 0^+} r^{-2\gamma}H(r) > 0.$$

Proof. For any $\lambda \in (0,r_0)$ the function $U(\lambda\cdot)$ belongs to $L^2(\mathbb{S}^{N-1})$. Then we can expand it in Fourier series respect to the basis $\{Y_{k,i}\}_{k\in\mathbb{N},i=1,\dots,N_k}$ introduced in Proposition 2.4.2:

$$U(\lambda\,\cdot) = \sum_{k=0}^{\infty}\sum_{i=1}^{N_k}\varphi_{k,i}(\lambda)Y_{k,i} \quad \text{in } L^2(\mathbb{S}^{N-1}),$$

where we have defined $\varphi_{k,i}(\lambda)$ in (2.113) for any $k \in \mathbb{N}$ and any $i = 1,\dots,N_k$. From (2.18), a change of variables and the Parseval identity

$$H(\lambda) = (1+O(\lambda))\int_{\mathbb{S}^{N-1}} U^2(\lambda\theta)\,dS = (1+O(\lambda))\sum_{k=0}^{\infty}\sum_{i=1}^{N_k}|\varphi_{k,i}(\lambda)|^2. \tag{2.124}$$

We argue by contradiction assuming that $\lim_{r\to 0^+} r^{2\gamma}H(r) = 0$. Then by (2.124), letting k_0 be as in (2.95),

$$\lim_{\lambda\to 0^+}\lambda^{-\frac{k_0}{2}}\varphi_{k_0,i}(\lambda) = 0 \quad \text{for any } i = 1,\dots,N_{k_0}.$$

From (2.115) it follows that

$$r^{-\frac{k_0}{2}}\varphi_{k_0,i}(r)+\frac{2N+k_0-4}{2(N+k_0-2)}\int_0^r s^{-N+1-\frac{k_0}{2}}\Upsilon_{k_0,i}(s)\,ds \\ +\frac{k_0 r^{-N+2-k_0}}{2(N+k_0-2)}\int_0^r s^{\frac{k_0}{2}-1}\Upsilon_{k_0,i}(s)\,ds=0 \tag{2.125}$$

for any $r\in(0,r_0)$ and any $i=1,\dots,N_{k_0}$.

In view of (2.87), (2.113), (2.119), and (2.123), (2.125) implies that

$$\sqrt{H(\lambda)}\int_{\mathbb{S}^{N-1}} W^\lambda Y_{k_0,i}\,dS=\varphi_{k_0,i}(\lambda)=O\Big(\lambda^{\frac{4\epsilon}{N+2\epsilon}+\frac{k_0}{2}}\Big)\quad\text{as }\lambda\to0^+ \tag{2.126}$$

for all $i=1,\dots,N_{k_0}$. From (2.84) with $\sigma=\frac{4\epsilon}{N+2\epsilon}$ we have that $\sqrt{H(\lambda)}\geq\sqrt{\alpha_{\frac{4\epsilon}{N+2\epsilon}}}\lambda^{\frac{k_0}{2}+\frac{2\epsilon}{N+2\epsilon}}$ in a neighbourhood of 0, so that (2.126) implies that

$$\int_{\mathbb{S}^{N-1}} W^\lambda Y_{k_0,i}\,dS=O\Big(\lambda^{\frac{2\epsilon}{N+2\epsilon}}\Big)=o(1)\quad\text{as }\lambda\to0^+$$

for all $i=1,\dots,N_{k_0}$.

On the other hand, by Proposition 2.4.8 and continuity of the trace map γ_1 (see Proposition 2.2.4), for every sequence $\lambda_n\to0^+$, there exist a subsequence $\{\lambda_{n_k}\}$ and $\Psi\in\operatorname{span}\{Y_{k_0,i}: m=i,\dots,N_{k_0}\}$ such that

$$\|\Psi\|_{L^2(\mathbb{S}^{N-1})}=1\quad\text{and}\quad W^{\lambda_{n_k}}\to\Psi\quad\text{in }L^2(\mathbb{S}^{N-1}).$$

From (2.5) and (2.5) it follows that

$$0=\lim_{k\to\infty}\int_{\mathbb{S}^{N-1}} W^{\lambda_{n_k}}\Psi\,dS=\|\Psi\|^2_{L^2(\mathbb{S}^{N-1})}=1,$$

thus reaching a contradiction. □

We are now ready to prove he following result, which is a more complete version of Theorem 2.1.3.

Theorem 2.5.3. *Let $N\geq3$ and let $u\in H^1(B_R\setminus\Gamma)$ be a non-trivial weak solution to (2.1), with Γ defined in (2.2)–(2.3) and f satisfying either assumption (2.6) or assumption (2.7). Then there exists $k_0\in\mathbb{N}$ such that, letting $\mathcal{N}$ be as in Section 2.3,*

$$\lim_{r\to0^+}\mathcal{N}(r)=\frac{k_0}{2}. \tag{2.127}$$

Moreover if N_{k_0} is the multiplicity of the eigenvalue μ_{k_0} of problem (2.8) and $\{Y_{k_0,i}\}_{i=1,\dots,N_{k_0}}$ is a $L^2(\mathbb{S}^{N-1})$-orthonormal basis of the eigenspace associated to μ_{k_0}, then

$$\lambda^{-\frac{k_0}{2}}u(\lambda\cdot)\to\Phi\quad\text{and}\quad\lambda^{1-\frac{k_0}{2}}\left(\nabla_{B_R\setminus\Gamma}u\right)(\lambda\cdot)\to\nabla_{\mathbb{R}^N\setminus\tilde\Gamma}\Phi\quad\text{in }L^2(B_1)\quad\text{as }\lambda\to0^+, \tag{2.128}$$

where

$$\Phi=\sum_{i=1}^{N_{k_0}}\alpha_i Y_{k_0,i}\left(\frac{y}{|y|}\right)$$

$(\alpha_1, \dots, \alpha_{N_{k_0}}) \neq (0, \dots, 0)$ and, for all $i \in \{1, 2, \dots, N_{k_0}\}$,

$$\alpha_i = r^{-k_0/2} \int_{\mathbb{S}^{N-1}} u(F(r\theta))Y_{k_0,i}(\theta)\, dS + \frac{1}{2-N-k_0} \int_0^r \left(\frac{2-N-\frac{k_0}{2}}{s^{N+\frac{k_0}{2}-1}} - \frac{k_0 s^{\frac{k_0}{2}-1}}{2r^{N-2+k_0}} \right) \Upsilon_{k_0,i}(s)\, ds \quad (2.129)$$

for any $r \in (0, r_0)$ for some $r_0 > 0$, where we have defined $\Upsilon_{k_0,i}$ in (2.114) and F is the diffeomorphism introduced in Proposition 2.2.1.

Proof. (2.127) directly comes from (2.95). Let $U = u \circ F$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} \lambda_n = 0^+$. By Proposition 2.4.8 and Proposition 2.5.2 there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and constants $\alpha_1, \dots, \alpha_{N_{k_0}}$ such that $(\alpha_1, \dots, \alpha_{N_{k_0}}) \neq (0, \dots, 0)$ and

$$\lambda_{n_k}^{-\frac{k_0}{2}} U(\lambda_{n_k} y) \to |y|^{\frac{k_0}{2}} \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i}\left(\frac{y}{|y|}\right) \quad \text{in } H^1(B_1 \setminus \tilde{\Gamma}) \quad \text{as } k \to \infty.$$

Now we show that the coefficients $\alpha_1, \dots, \alpha_{N_{k_0}}$ do not depend on $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Thanks to the continuity of the trace operator γ_1 introduced in Proposition 2.2.4

$$\lambda_{n_k}^{-\frac{k_0}{2}} U(\lambda_{n_k} \cdot) \to \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i} \quad \text{in } L^2(\mathbb{S}^{N-1}) \quad \text{as } k \to \infty$$

and therefore, letting $\varphi_{k_0,i}$ be as in (2.113) for any $i = 1, \dots, N_{k_0}$,

$$\lim_{k\to\infty} \lambda^{-\frac{k_0}{2}} \varphi_{k_0,i}(\lambda_{n_k}) = \lim_{k\to\infty} \int_{\mathbb{S}^{N-1}} \lambda_{n_k}^{-k_0/2} U(\lambda_{n_k}\theta) Y_{k_0,i}(\theta)\, dS = \sum_{j=1}^{N_{k_0}} \alpha_j \int_{\mathbb{S}^{N-1}} Y_{k_0,j} Y_{k_0,i}\, dS = \alpha_i.$$

On the other hand by (2.115)

$$\lim_{k\to\infty} \lambda^{-\frac{k_0}{2}} \varphi_{k_0,i}(\lambda_{n_k}) = r^{-\frac{k_0}{2}} \varphi_{k_0,i}(r) + \frac{2N+k_0-4}{2(N+k_0-2)} \int_0^r s^{-N+1-\frac{k_0}{2}} \Upsilon_{k_0,i}(s)\, ds + \frac{k_0 r^{-N+2-k_0}}{2(N+k_0-2)} \int_0^r s^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(s)\, ds,$$

for all $i = 1, \dots, N_{k_0}$ and $r \in (0, r_0]$, where we have defined $\Upsilon_{k_0,i}$ in (2.114). We deduce that

$$\alpha_i = r^{-\frac{k_0}{2}} \varphi_{k_0,i}(r) + \frac{2N+k_0-4}{2(N+k_0-2)} \int_0^r s^{-N+1-\frac{k_0}{2}} \Upsilon_{k_0,i}(s)\, ds + \frac{k_0 r^{-N+2-k_0}}{2(N+k_0-2)} \int_0^r s^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(s)\, ds \quad (2.130)$$

and so α_i does not depend on $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ thus implying that

$$\lambda^{-\frac{k_0}{2}} U(\lambda y) \to |y|^{\frac{k_0}{2}} \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i}\left(\frac{y}{|y|}\right) \quad \text{in } H^1(B_1 \setminus \tilde{\Gamma}) \quad \text{as } \lambda \to 0^+. \quad (2.131)$$

To prove (2.128) we note that

$$\lambda^{-\frac{k_0}{2}} u(\lambda x) = \lambda^{-\frac{k_0}{2}} U(\lambda G_\lambda(x)), \quad \nabla \left(\lambda^{-\frac{k_0}{2}} u(\lambda x) \right) = \nabla \left(\lambda^{-\frac{k_0}{2}} U(\lambda x) \right) (G_\lambda(x)) J_{G_\lambda}(x),$$

where $G_\lambda(x) = \frac{1}{\lambda} F^{-1}(\lambda x)$ and F is the diffeomorphism introduced in Proposition 2.2.1. We also have by Proposition 2.2.1 that

$$G_\lambda(x) = x + O(\lambda) \quad \text{and} \quad J_G(x) = \mathrm{Id}_N + O(\lambda)$$

as $\lambda \to 0^+$ uniformly respect to $x \in B_1$. Then from (2.131) we deduce (2.128) and (2.129) follows from (2.130) and (2.113). □

Chapter 3

A regularity result for some singular or degenerate elliptic equations

3.1 Weighted Sobolev Spaces

In this section we recall some results about weighted Sobolev Spaces and fix some notations that we will use in Chapters 3, 4, 5, and 6.

Let $s \in (0,1)$, $N \in \mathbb{N}$, $N > 2s$ and $z = (x,y) \in \mathbb{R}^N \times [0,\infty)$. Let

$$\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times (0,\infty),$$

and, for any $r > 0$,

$$\begin{aligned} B_r^+ &:= \{z \in \mathbb{R}^{N+1}_+ : |z| < r\}, \quad B_r' = \{x \in \mathbb{R}^N : |x| < r\}, \\ S_r^+ &:= \{z \in \mathbb{R}^{N+1}_+ : |z| = r\}, \quad S_r' := \{x \in \mathbb{R}^N : |x| = r\}. \end{aligned}$$

For any $p \in [1,\infty)$ and any open set $E \subset \mathbb{R}^{N+1}_+$, let

$$L^p(E, y^{1-2s}) := \left\{ V : E \to \mathbb{R} \text{ measurable } : \int_E y^{1-2s} |V|^p \, dz < +\infty \right\}.$$

For any Lipschitz open set $E \subset \mathbb{R}^{N+1}$ and $\phi \in C^\infty(\overline{E})$ we define

$$\|\phi\|_{H^1(E,y^{1-2s})} := \left(\int_{B_r^+} y^{1-2s} (\phi^2 + |\nabla \phi|^2) \, dz \right)^{\frac{1}{2}} \tag{3.1}$$

and $H^1(E, y^{1-2s})$ as the completion of $C^\infty(\overline{E})$ with respect to the norm defined in (3.1). Thanks to [93, Theorem 11.11, Theorem 11.2, 11.12 Remarks (iii)], for any $r > 0$, the space $H^1(E, y^{1-2s})$ can be explicitly characterized as

$$H^1(E, y^{1-2s}) = \left\{ w \in W^{1,1}_{\text{loc}}(E) : \int_{B_r^+} y^{1-2s} (w^2 + |\nabla w|^2) \, dz < +\infty \right\}.$$

Finally, for any $r > 0$ we define the space

$$H^1_{0,S_r^+}(B_r^+, y^{1-2s}) := \overline{\{\phi \in C^\infty(\overline{B_r^+}) : \phi = 0 \text{ on } S_r^+\}}^{\|\cdot\|_{H^1(B_r^+, y^{1-2s})}}. \tag{3.2}$$

We observe that $H^1(B_r^+, y^{1-2s}) \subset W^{1,1}(B_r^+)$, hence, denoting as Tr the classical trace operator from $W^{1,1}(B_r^+)$ to $L^1(B_r')$, we may consider its restriction to $H^1(B_r^+, y^{1-2s})$; furthermore, for any $r > 0$, such a restriction (still denoted as Tr) turns out to be a linear, continuous trace operator

$$\mathrm{Tr} : H^1(B_r^+, y^{1-2s}) \to H^s(B_r') \tag{3.3}$$

which is onto, see [29, 101], [89, Proposition 2.1], and [106, Theorem 2.8], where $H^s(B_r')$ denotes the usual fractional Sobolev space.

Furthermore, denoting as Tr_1 the classical trace operator from $W^{1,1}(B_r^+)$ to $L^1(S_r^+)$, we can consider its restriction to $H^1(B_r^+, t^{1-2s})$, still denoted as Tr_1; from [111, Theorem 19.7] and the Divergence Theorem one can easily deduce the following proposition.

Proposition 3.1.1. *For any $r > 0$ there exists a linear, continuous, compact trace operator*

$$\mathrm{Tr}_1 : H^1(B_r^+, y^{1-2s}) \to L^2(S_r^+, y^{1-2s}). \tag{3.4}$$

For the sake of simplicity we will always denote $\mathrm{Tr}_1(w)$ with w for any $w \in H^1(B_r^+, y^{1-2s})$.

Proposition 3.1.2. *[60, Lemma 2.6] There exists a constant $\mathcal{S}_{N,s} > 0$ such that, for any $r > 0$ and $w \in H^1(B_r^+, y^{1-2s})$,*

$$\left(\int_{B_r'} |w|^{2_s^*}\, dx\right)^{\frac{2}{2_s^*}} \leq \mathcal{S}_{N,s} \left(\int_{B_r^+} y^{1-2s} |\nabla w|^2\, dz + \frac{N-2s}{2r} \int_{S_r^+} y^{1-2s} w^2\, dS\right), \tag{3.5}$$

where $2_s^* = \frac{2N}{N-2s}$.

We recall the following Hardy-type inequality with boundary terms from [60, Lemma 2.4].

Proposition 3.1.3. *For any $r > 0$ and any $w \in H^1(B_r^+, y^{1-2s})$*

$$\left(\frac{N-2s}{2}\right)^2 \int_{B_r^+} y^{1-2s} \frac{|w(z)|^2}{|z|^2}\, dz \\ \leq \int_{B_r^+} y^{1-2s} \left(\nabla w \cdot \frac{z}{|z|}\right)^2 dz + \left(\frac{N-2s}{2r}\right) \int_{S_r^+} y^{1-2s} w^2\, dS. \tag{3.6}$$

The following Poincaré-type inequality directly follows from (3.6): for all $r > 0$ and $w \in H^1(B_r^+, y^{1-2s})$

$$\int_{B_r^+} y^{1-2s} w^2\, dz \leq \frac{4r}{(N-2s)^2} \left(r \int_{B_r^+} y^{1-2s} |\nabla w|^2\, dz + \frac{N-2s}{2} \int_{S_r^+} y^{1-2s} w^2\, dS\right). \tag{3.7}$$

Remark 3.1.4. As a consequence of (3.7) and by continuity of the trace operator (3.4), for every $r > 0$

$$\left(\int_{S_r^+} y^{1-2s} w^2\, dS + \int_{B_r^+} y^{1-2s} |\nabla w|^2\, dz\right)^{1/2}$$

is an equivalent norm on $H^1(B_r^+, y^{1-2s})$.

For any $i = 1, \dots, N+1$, let $e_i = (\delta_{i,j})_{j=1,\dots,N+1} \in \mathbb{R}^{N+1}$ be the vector with i-th component equal to 1 and all the remaining components equal to 0.

It is well known that, if $w \in W^{1,p}(\Omega)$ with $\Omega \subset \mathbb{R}^{N+1}$ open and $p \in [1, \infty)$, then, for any $i = 1, \dots, N+1$ and $k \in \mathbb{R}$,

$$\int_{\Omega_{k,i}} \frac{|u(x+ke_i) - u(x)|^p}{|k|^p} \leq \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p dx < +\infty,$$

where $\Omega_{k,i} := \{x \in \Omega : x + \tau k e_i \in \Omega \text{ for any } \tau \in [0,1]\}$, see e.g. [98, Theorem 10.55]. We prove below an analogous result for the weighted space $H^1(B_r^+, y^{1-2s})$.

Lemma 3.1.5. *For any $r > 0$, $w \in H^1(B_r^+, y^{1-2s})$, $i = 1, \dots, N$, and $k \in \mathbb{R}$*

$$\int_{B^+_{r,k,i}} y^{1-2s} \frac{|w(z+ke_i) - w(z)|^2}{|k|^2} dz \leq \int_{B_r^+} y^{1-2s} \left| \frac{\partial w}{\partial x_i} \right|^2 dz, \tag{3.8}$$

where $B^+_{r,k,i} := \{z \in B_r^+ : z + \tau k e_i \in B_r^+ \text{ for any } \tau \in [0,1]\}$.

Proof. For a.e. $z \in B^+_{r,k,i}$, by the absolute continuity of Sobolev functions on lines,

$$|w(z+ke_i) - w(z)| = \left| \int_0^1 \frac{d}{d\tau} w(z + \tau k e_i) d\tau \right| \leq \int_0^1 \left| \frac{\partial w}{\partial x_i}(z + \tau k e_i) \right| |k| \, d\tau.$$

Multiplying by y^{1-2s} and integrating on $B^+_{r,k}$ we obtain, by Cauchy-Schwarz's inequality and Fubini-Tonelli's Theorem,

$$\begin{aligned} &\int_{B^+_{r,k,i}} y^{1-2s} \frac{|w(z+ke_i) - w(z)|^2}{|k|^2} dz \\ &\qquad \leq \int_{B^+_{r,k,i}} y^{1-2s} \left(\int_0^1 \left| \frac{\partial w}{\partial x_i}(z + \tau k e_i) \right|^2 d\tau \right) dz \leq \int_{B_r^+} y^{1-2s} \left| \frac{\partial w}{\partial x_i} \right|^2 dz \end{aligned}$$

which proves (3.8). □

We refer to [60] for the following result, which can be deduced from [111, Theorem 19.7].

Lemma 3.1.6. *Let* Tr *be the trace operator introduced in* (3.3). *Then*

(i) *For any $r > 0$, $f \in C^{0,1}(\overline{B_r^+})$ and $w \in H^1(B_r^+, y^{1-2s})$,*

$$\operatorname{Tr}(fw) = f(\cdot, 0) \operatorname{Tr}(w). \tag{3.9}$$

(ii) *For any $r > 0$, $u \in H^1(B_r^+, y^{1-2s})$ and $v \in H^1(B_r^+, y^{2s-1})$, we have that $uv \in W^{1,1}(B_r^+)$ and*

$$\operatorname{Tr}(uv) = \operatorname{Tr}(u) \operatorname{Tr}(v). \tag{3.10}$$

Proof. Let us first prove (i). If $w \in C^\infty(\overline{B_r^+})$ then (3.9) is trivial; if w belongs to $H^1(B_r^+, y^{1-2s})$ there exists $\{\phi_n\}_{n \in \mathbb{N}} \subset C^\infty(\overline{B_r^+})$ such that $\phi_n \to w$ in $H^1(B_r^+, y^{1-2s})$ as $n \to \infty$. Furthermore, for any $f \in C^{0,1}(\overline{B_r^+})$, it is easy to see that $\phi_n f \to wf$ in $H^1(B_r^+, y^{1-2s})$ as $n \to \infty$. Then (3.9) follows from the continuity of the operator Tr.

We now prove (ii). If $u \in H^1(B_r^+, y^{1-2s})$ and $v \in H^1(B_r^+, y^{2s-1})$, the fact that $uv \in W^{1,1}(B_r^+)$ follows easily from Hölder's inequality. Moreover there exist $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{B_r^+})$ such that $u_n \to u$ in $H^1(B_r^+, y^{1-2s})$ and a sequence $\{v_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{B_r^+})$ such that $v_n \to v$ in $H^1(B_r^+, y^{2s-1})$. One can easily verify that $u_n v_n \to uv$ in $W^{1,1}(B_r^+)$, so that $\operatorname{Tr}(u_n v_n) \to \operatorname{Tr}(uv)$ in $L^1(B'_r)$. On the other hand, since by continuity of the operator (3.3) $\operatorname{Tr}(u_n) \to \operatorname{Tr}(u)$ and $\operatorname{Tr}(v_n) \to \operatorname{Tr}(v)$ in $L^2(B'_r)$, we have also that $\operatorname{Tr}(u_n v_n) = \operatorname{Tr}(u_n)\operatorname{Tr}(v_n) \to \operatorname{Tr}(u)\operatorname{Tr}(v)$ in $L^1(B'_r)$, so that necessarily $\operatorname{Tr}(uv) = \operatorname{Tr}(u)\operatorname{Tr}(v)$. □

For any $r > 0$, let

$$H^{1+s}(B'_r) := \left\{ w \in H^1(B'_r) : \frac{\partial w}{\partial x_i} \in H^s(B'_r) \text{ for any } i = 1, \dots, N \right\},$$

see [48] for details on this class of fractional Sobolev spaces. We also consider the space

$$H^2_x(B_r^+, y^{1-2s}) := \left\{ w \in H^1(B_r^+, y^{1-2s}) : \frac{\partial w}{\partial x_i} \in H^1(B_r^+, y^{1-2s}) \text{ for any } i = 1, \dots, N \right\}.$$

Proposition 3.1.7. *Let* Tr *be the trace operator introduced in* (3.3)*. For any* $r > 0$

$$\operatorname{Tr}(H^2_x(B_r^+, y^{1-2s})) \subseteq H^{1+s}(B'_r). \tag{3.11}$$

Furthermore, for any $w \in H^2_x(B_r^+, y^{1-2s})$,

$$\operatorname{Tr}(\nabla_x w) = \nabla \operatorname{Tr}(w), \tag{3.12}$$

where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}\right)$ *denotes the gradient with respect to the first* N *variables.*

Proof. Let $w \in H^2_x(B_r^+, y^{1-2s})$. Let us fix $\phi \in C_c^\infty(B'_r)$; then there exists a function $\tilde{\phi} \in C_c^\infty(B_r^+ \cup B'_r)$ such that $\tilde{\phi}(x, 0) = \phi(x)$ for all $x \in B'_r$. Let $\eta \in C_c^\infty(B_r)$ be a smooth cut-off function such that $\eta \equiv 1$ on $\operatorname{supp}\tilde{\phi}$. Then, denoting as $\hat{w}$ the even reflection of w through the hyperplane $t = 0$, $\tilde{w} = \eta\hat{w} \in H^1(\mathbb{R}^{N+1}, |t|^{1-2s})$ and $\frac{\partial \tilde{w}}{\partial x_i} \in H^1(\mathbb{R}^{N+1}, |t|^{1-2s})$ for all $i \in \{1, \dots, N\}$. Then, letting $\{\rho_n\}$ be a sequence of mollifiers and $w_n = \rho_n * \tilde{w}$, from [91, Lemma 1.5] it follows that $w_n \in C^\infty(\mathbb{R}^{N+1})$ and, for all $i \in \{1, \dots, N\}$,

$$w_n \to \tilde{w} \quad \text{and} \quad \frac{\partial w_n}{\partial x_i} = \rho_n * \frac{\partial \tilde{w}}{\partial x_i} \to \frac{\partial \tilde{w}}{\partial x_i} \quad \text{in } H^1(\mathbb{R}^{N+1}, |t|^{1-2s}).$$

Then, for any $i = 1, \dots, N$,

$$\begin{aligned}\int_{B'_r} \operatorname{Tr}(w)\frac{\partial \phi}{\partial x_i}\,dx &= \int_{B'_r} \operatorname{Tr}(\tilde{w})\frac{\partial \phi}{\partial x_i}\,dx = \lim_{n\to\infty}\int_{B'_r} w_n(x,0)\frac{\partial \phi}{\partial x_i}(x,0)\,dx \\ &= -\lim_{n\to\infty}\int_{B'_r} \frac{\partial w_n}{\partial x_i}(x,0)\phi(x,0)\,dx = -\int_{B'_r} \operatorname{Tr}\left(\frac{\partial \tilde{w}}{\partial x_i}\right)\phi\,dx = -\int_{B'_r} \operatorname{Tr}\left(\frac{\partial w}{\partial x_i}\right)\phi\,dx,\end{aligned}$$

so that the distributional derivative in B'_r of $\operatorname{Tr}(w)$ with respect to x_i is $\operatorname{Tr}\left(\frac{\partial w}{\partial y_i}\right)$ which belongs to $H^s(B'_r)$. Therefore we have proved (3.12), which directly implies (3.11) in view of (3.3). □

3.2 A regularity result and a Pohozaev-type identity

Let $R > 0$ and let ν be the outer normal unit vector to B_R^+ on B'_R, that is $\nu(x) = (0, \dots, 0, -1)$ for any $x \in B'_R$. We are interested in proving Sobolev-type regularity results for a weak solution $U \in H^1(B_R^+, y^{1-2s})$ of the problem

$$\begin{cases} -\operatorname{div}(y^{1-2s} A\nabla U) + y^{1-2s} c = 0, & \text{on } B_R^+, \\ \lim_{y\to 0^+} y^{1-2s} A\nabla U \cdot \nu = h \operatorname{Tr}(U) + g, & \text{on } B'_R, \end{cases} \tag{3.13}$$

under suitable regularity hypotheses on the matrix-valued function A and the functions c, h, g. More precisely we make the following assumptions:

$$A(z) = \left(\begin{array}{c|c} B(z) & 0 \\ \hline 0 & \alpha(z) \end{array}\right) \quad \text{for any } z \in \overline{B_R^+}, \tag{3.14}$$

$$B \in W^{1,\infty}(B_R^+, \mathbb{R}^{N\times N}) \text{ is symmetric}, \quad \alpha \in W^{1,\infty}(B_R^+, \mathbb{R}), \tag{3.15}$$

$$\text{there exist } \lambda_1, \lambda_2 > 0 \text{ s.t. } \lambda_1 |y|^2 \le A(z) y \cdot y \le \lambda_2 |y|^2 \tag{3.16}$$

$$\text{for all } z \in \overline{B_R^+} \text{ and } y \in \mathbb{R}^{N+1},$$

$$g \in W^{1,\frac{2N}{N+2s}}(B'_R), \quad h \in W^{1,\frac{N}{2s}}(B'_R), \tag{3.17}$$

$$c \in L^2(B_R^+, y^{1-2s}). \tag{3.18}$$

Under this conditions, a weak solution of (3.13) is a function $U \in H^1(B_R^+, y^{1-2s})$ such that

$$\int_{B_R^+} y^{1-2s} A\nabla U \cdot \nabla\phi \, dz + \int_{B_R^+} y^{1-2s} c\phi \, dz = \int_{B'_R} [g + h \operatorname{Tr}(U)] \operatorname{Tr}(\phi) \, dx, \tag{3.19}$$

for any $\phi \in H^1_{0,S_R^+}(B_R^+, y^{1-2s})$ see (3.2).

The above definition is well posed since each term in (3.19) is finite, thanks to (3.5).

Our main result is the following theorem.

Theorem 3.2.1. *Let U be a weak solution of (3.13) in the sense of (3.19). If assumptions (3.14), (3.15), (3.16), (3.17), (3.18) are satisfied, then*

$$\nabla_x U \in H^1(B_r^+, y^{1-2s}) \quad \text{and} \quad y^{1-2s}\frac{\partial U}{\partial y} \in H^1(B_r^+, y^{2s-1}) \tag{3.20}$$

for all $r \in (0, R)$. Furthermore

$$\|\nabla_x U\|_{H^1(B_r^+, y^{1-2s})} + \left\| y^{1-2s}\frac{\partial U}{\partial y}\right\|_{H^1(B_r^+, y^{2s-1})} \le C\left(\|U\|_{H^1(B_R^+, y^{1-2s})} + \|c\|_{L^2(B_R^+, y^{1-2s})} + \|g\|_{W^{1,\frac{2N}{N+2s}}(B'_R)}\right) \tag{3.21}$$

for a positive constant $C > 0$ independent of U. More precisely, C depends only on N, s, r, R, $\|h\|_{W^{1,\frac{N}{2s}}(B'_R)}$, λ_1, $\|A\|_{W^{1,\infty}(B_R^+, \mathbb{R}^{(N+1)^2})}$.

As an application of Theorem 3.2.1 we prove a Pohozaev-type identity for weak solutions of (3.13). To this aim we require that the matrix-valued function A satisfies, besides assumptions (3.14), (3.15), and (3.16), also the condition

$$A(0) = \mathrm{Id}_{N+1} \tag{3.22}$$

where Id_{N+1} is the identity $(N+1)\times(N+1)$ matrix.

We first introduce some notation. Let

$$\mu(z) := \frac{A(z)z\cdot z}{|z|^2} \quad \text{and} \quad \beta(z) := \frac{A(z)z}{\mu(z)} \quad \text{for any } z \in \overline{B_R^+}\setminus\{0\}, \tag{3.23}$$
$$\beta'(x) := \beta(x,0) \quad \text{for any } x\in \overline{B_R'}\setminus\{0\}.$$

We also define $dA(z)\xi\xi$, for every $\xi=(\xi_1,\dots,\xi_{N+1})\in\mathbb{R}^{N+1}$ and $z\in\overline{B_R^+}$, as the vector of $\mathbb{R}^{N+1}$ with i-th component given by

$$(dA(z)\xi\xi)_i = \sum_{h,k=1}^{N+1}\frac{\partial a_{kh}}{\partial z_i}(z)\xi_h\xi_k, \quad i=1,\dots,N+1, \tag{3.24}$$

where we have defined the matrix $A=(a_{kh})_{k,h=1,\dots,N+1}$ in (3.14).

Remark 3.2.2. From (3.15), (3.16), and (3.22) it easily follows that

$$\mu\in C^{0,1}(\overline{B_R^+}), \quad \frac{1}{\mu}\in C^{0,1}(\overline{B_R^+}), \quad \beta\in C^{0,1}(\overline{B_R^+},\mathbb{R}^{N+1}), \tag{3.25}$$
$$J_\beta\in L^\infty(B_R^+,\mathbb{R}^{(N+1)^2}), \quad \operatorname{div}(\beta)\in L^\infty(B_R^+),$$
$$\beta'\in L^\infty(B_R',\mathbb{R}^N), \quad \operatorname{div}(\beta')\in L^\infty(B_R'),$$

where J_β is the Jacobian matrix of β.

Proposition 3.2.3. *Under assumptions* (3.14), (3.15), (3.16), (3.17), (3.18), *and* (3.22), *let* U *be a solution of* (3.19). *Then for a.e.* $r\in(0,R)$

$$\begin{aligned}
&\frac{r}{2}\int_{S_r^+} y^{1-2s}A\nabla U\cdot\nabla U\,dS - r\int_{S_r^+} y^{1-2s}\frac{|A\nabla U\cdot\nu|^2}{\mu}\,dS\\
&\quad+\frac12\int_{B_r'}(\operatorname{div}_x(\beta')h+\beta'\cdot\nabla h)|\operatorname{Tr}(U)|^2\,dx - \frac{r}{2}\int_{S_r'} h|\operatorname{Tr}(U)|^2\,dS'\\
&\quad+\int_{B_r'}(\operatorname{div}_x(\beta')g+\beta'\cdot\nabla g)\operatorname{Tr}(U)\,dx - r\int_{S_r'} g\operatorname{Tr}(U)\,dS'\\
&=\frac12\int_{B_r^+} y^{1-2s}A\nabla U\cdot\nabla U\operatorname{div}(\beta)\,dz - \int_{B_r^+} y^{1-2s}c(\nabla U\cdot\beta)\,dz\\
&\quad-\int_{B_r^+} y^{1-2s}J_\beta(A\nabla U)\cdot\nabla U\,dz + \frac12\int_{B_r^+} y^{1-2s}(dA\nabla U\nabla U)\cdot\beta\,dz\\
&\quad+\frac{1-2s}{2}\int_{B_r^+} y^{1-2s}\frac{\alpha}{\mu}A\nabla U\cdot\nabla U\,dz,
\end{aligned}\tag{3.26}$$

where ν *is the outer normal vector to* B_r^+ *on* S_r^+, *that is* $\nu(z)=\frac{z}{|z|}$.

Remark 3.2.4. The two integrals in the first line of (3.26) must be understood for a.e. $r\in(0,R)$ as explained in Remark 3.4.2.

The integrals over S_r' in (3.26) can be instead understood in the classical trace sense. Indeed, $h\in W^{1,\frac{N}{2s}}(B_r')$ by (3.17) and $(\operatorname{Tr}(U))^2\in W^{1,\frac{N}{N-2s}}(B_r')$ thanks to (3.20) and (3.11); then h has a trace on S_r' belonging to $L^{\frac{N}{2s}}(S_r')$ and $(\operatorname{Tr}(U))^2$ has a trace on S_r' belonging to $L^{\frac{N}{N-2s}}(S_r')$, so that $h(\operatorname{Tr}(U))^2$ has a trace on S_r' belonging to $L^1(S_r')$ for all $r\in(0,R)$. Moreover $g\in W^{1,\frac{2N}{N+2s}}(B_r')$ by (3.17) and $\operatorname{Tr}(U)\in W^{1,\frac{2N}{N-2s}}(B_r')$ thanks to (3.20) and (3.11); then, on S_r', g has a trace in $L^{\frac{2N}{N+2s}}(S_r')$ and $\operatorname{Tr}(U)$ has a trace in $L^{\frac{2N}{N-2s}}(S_r')$, so that $g\operatorname{Tr}(U)$ has a trace on S_r' belonging to $L^1(S_r')$ for all $r\in(0,R)$.

3.3 Regularity of weak solutions: proof of Theorem 3.2.1

For any $r > 0$ and $\delta \in (0, r)$, we define

$$B^+_{r,\delta} := \{(x,t) \in B^+_r : y > \delta\}, \quad S^+_{r,\delta} := \{(x,t) \in S^+_r : y > \delta\}. \tag{3.27}$$

We are now ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. For any $r > 0$ we denote

$$C_r := B'_r \times (0, r).$$

Let us fix $0 < r_3 < r_2 < r_1 < R$ with r_2 small enough so that $\overline{C}_{r_2} \subset B^+_{r_1} \cup B'_{r_1}$. We will show that $U \in H^2(B^+_{r_3}, y^{1-2s})$, eventually choosing a smaller r_1.

We start by defining a suitable cut-off function $\eta \in C^\infty_c(B'_{r_2} \times [0, r_2))$. We choose a cut-off function $\rho \in C^\infty_c(B'_{r_2})$ such that $0 \le \rho(x) \le 1$ for any $x \in \mathbb{R}^N$ and $\rho(x) \equiv 1$ on B'_{r_3} and a function $\sigma \in C^\infty_c([0, r_2))$ such that $0 \le \sigma(y) \le 1$ for any $t \in \mathbb{R}$ and $\sigma(t) = 1$ if $t \in [0, r_3]$. Then we define

$$\eta(z) = \eta(x,y) := \rho(x)\sigma(y). \tag{3.28}$$

Then $\eta \in C^\infty_c(B'_{r_2} \times [0, r_2))$ and $0 \le \eta \le 1$. For any $\phi \in H^1(B^+_{r_1}, y^{1-2s})$ we can test (3.19) with $\eta\phi$ obtaining

$$\int_{B^+_{r_1}} [y^{1-2s}\eta A\nabla U \cdot \nabla\phi + y^{1-2s} A\nabla U \cdot \nabla\eta\, \phi]\, dz + \int_{B^+_{r_1}} y^{1-2s} c\eta\phi\, dz = \int_{B'_{r_1}} [h\,\mathrm{Tr}(U) + g]\rho\,\mathrm{Tr}(\phi)\, dx, \tag{3.29}$$

thanks to (3.9) and (3.28). We would like to rewrite (3.29) as an equation for $U_1 := \eta U$. To this end we observe that

$$\mathrm{div}(y^{1-2s} U\phi\, A\nabla\eta) = U\phi\, \mathrm{div}(y^{1-2s} A\nabla\eta) + y^{1-2s}\phi\, A\nabla\eta \cdot \nabla U + y^{1-2s} U\, A\nabla\eta \cdot \nabla\phi \in L^1(B^+_{r_1}). \tag{3.30}$$

Letting $B^+_{r_1,\delta}$ be as in (3.27), the Divergence Theorem yields

$$\int_{B^+_{r_1,\delta}} \mathrm{div}(y^{1-2s} U\phi\, A\nabla\eta) dz = -\delta^{1-2s} \int_{B'_{r_1}} U(x,\delta)\phi(x,\delta)\alpha(x,\delta)\frac{\partial\eta}{\partial y}(x,\delta)\, dx,$$

where α has been defined in (3.14). Since $\frac{\partial\eta}{\partial y}(x,\delta) = 0$ for any $(x,\delta) \in \mathbb{R}^N \times [0, r_3]$, passing to the limit as $\delta \to 0^+$ we conclude that

$$\int_{B^+_{r_1}} \mathrm{div}(y^{1-2s} U\phi\, A\nabla\eta) dz = 0, \tag{3.31}$$

thanks to the Dominated Convergence Theorem and the fact that

$$\mathrm{div}(y^{1-2s} U\phi\, A\nabla\eta) \in L^1(B^+_{r_1})$$

by (3.30). Furthermore

$$\operatorname{div}(y^{1-2s}A\nabla\eta) = y^{1-2s}\left[\operatorname{div}(A\nabla\eta) + \frac{(1-2s)}{y}\alpha\frac{\partial\eta}{\partial y}\right], \tag{3.32}$$

and so, thanks to (3.28) and (3.18),

$$f := U\operatorname{div}(A\nabla\eta) + U\frac{(1-2s)}{y}\alpha\frac{\partial\eta}{\partial y} + 2A\nabla U\cdot\nabla\eta + \eta c \in L^2(B^+_{r_1}, y^{1-2s}). \tag{3.33}$$

In conclusion, combining (3.30), (3.31), and (3.32) we can rewrite (3.29) as

$$\int_{B^+_{r_1}} y^{1-2s}A\nabla U_1\cdot\nabla\phi\,dz + \int_{B^+_{r_1}} y^{1-2s}f\phi\,dz = \int_{B'_{r_1}} [h\operatorname{Tr}(U_1) + \rho g]\operatorname{Tr}(\phi)\,dx \tag{3.34}$$

for any $\phi \in H^1(B^+_{r_1}, y^{1-2s})$, in view of (3.9) and (3.33).

If we show that $\nabla_x U_1 \in H^1(C_{r_2}, y^{1-2s})$ and $y^{1-2s}\frac{\partial U_1}{\partial y} \in H^1(C_{r_2}, y^{2s-1})$, then we obtain that $\nabla_x U \in H^1(B^+_{r_3}, y^{1-2s})$ and $y^{1-2s}\frac{\partial U}{\partial y} \in H^1(B^+_{r_3}, y^{2s-1})$, since $\eta \equiv 1$ on C_{r_3}. To this end we use Nirenberg's tangential difference quotient method [107], proving that the family of the second incremental ratios is L^2-bounded; see also [81] for the difference quotient method for classical elliptic equations.

For any $i = 1,\dots,N$ and $k \in \mathbb{R}\setminus\{0\}$ and for any measurable function w on $\mathbb{R}^{N+1}_+$, we define

$$(\tau_{i,k}w)(x,y) = w(x+ke_i,y) \quad \text{and} \quad (\zeta_{i,k}w)(x,y) = \frac{(\tau_{i,k}w)(x,y) - w(x,y)}{k}.$$

If $\overline{w} = (w_1,\dots,w_{N+1})$ is a vector of measurable functions we set

$$\tau_{i,k}(\overline{w}) := (\tau_{i,k}w_1,\dots,\tau_{i,k}w_{N+1}).$$

We can define $\tau_{i,k}$ similarly for a matrix of measurable functions.

It is easy to see that $\tau_{i,k} : L^2(\mathbb{R}^{N+1}_+, y^{1-2s}) \to L^2(\mathbb{R}^{N+1}_+, y^{1-2s})$ is a well-defined, continuous, linear operator, and the adjoint operator of $\tau_{i,k}$ with respect to the $L^2(\mathbb{R}^{N+1}_+, y^{1-2s})$-scalar product is $\tau_{i,-k}$.

Furthermore $\tau_{i,k} : H^1(\mathbb{R}^{N+1}_+, y^{1-2s}) \to H^1(\mathbb{R}^{N+1}_+, y^{1-2s})$ is a well-defined, continuous, linear operator and, for any $i = 1,\dots,N$ and any $w \in H^1(B^+_r, y^{1-2s})$,

$$\frac{\partial\tau_{i,k}(w)}{\partial x_i} = \tau_{i,k}\left(\frac{\partial w}{\partial x_i}\right),$$

that is, the operator commutes with tangential derivatives. With a slight abuse of notation, for any $i = 1,\dots,N$ and $k \in \mathbb{R}\setminus\{0\}$ we denote as $\tau_{i,k}$, respectively $\zeta_{i,k}$, also the operator $\tau_{i,k}v(x) = v(x+ke_i)$, respectively $\zeta_{i,k}v = \frac{1}{k}(\tau_{i,k}v - v)$, acting on measurable functions $v : \mathbb{R}^N \to \mathbb{R}$ and observe that $\tau_{i,k}, \zeta_{i,k} : W^{1,p}(\mathbb{R}^N) \to W^{1,p}(\mathbb{R}^N)$ are linear and continuous for any $p \in [1,\infty)$; furthermore, the adjoint operator of $\tau_{i,k}$ is $\tau_{i,-k}$.

It is easy to see that, for all measurable functions v, w,

$$\zeta_{i,k}(vw) = \zeta_{i,k}(v)\tau_{i,k}w + v\zeta_{i,k}(w) \tag{3.35}$$

and

$$\frac{w(x+ke_i,y) - 2w(x,t) + w(x-ke_i,y)}{k^2} = (\zeta_{i,k}\circ\zeta_{i,-k})(w)(x,y).$$

We note that the trivial extension of U_1 to $\mathbb{R}^{N+1}_+$ belongs to $H^1(\mathbb{R}^{N+1}_+, y^{1-2s})$ since $U_1 \equiv 0$ on $B^+_{r_1} \setminus C_{r_2}$,; with a slight abuse of notation we will still indicate this extension with U_1.

Let $|k| < \sqrt{r_1^2 - r_2^2} - r_2$ (we note that $\sqrt{r_1^2 - r_2^2} - r_2 > 0$ since $C_{r_2} \subset B^+_{r_1}$). The function $\tilde{\phi} := (\zeta_{i,k} \circ \zeta_{i,-k})(U_1)$ belongs to $H^1_{0,S^+_{r_1}}(B^+_{r_1}, y^{1-2s})$ thanks to (3.28) and so its trivial extension, still denoted as $\tilde{\phi}$, belongs to $H^1(\mathbb{R}^{N+1}_+, y^{1-2s})$. Moreover by (3.2) we have that, for any $i = 1, \dots, N$,

$$\operatorname{Tr}(\zeta_{i,k}(\zeta_{i,-k}(\tilde{\phi}))) = \zeta_{i,k}(\operatorname{Tr}(\zeta_{i,-k}(\tilde{\phi}))) = \zeta_{i,k}(\zeta_{i,-k}(\operatorname{Tr}(\tilde{\phi}))). \tag{3.36}$$

Therefore testing (3.34) with $\tilde{\phi}$ we obtain

$$\begin{aligned}\int_{B^+_{r_1}} y^{1-2s} \zeta_{i,-k}(A\nabla U_1) \cdot \nabla(\zeta_{i,-k}(U_1))\, dz + \int_{B^+_{r_1}} y^{1-2s} f\ (\zeta_{i,k} \circ \zeta_{i,-k})(U_1)\, dz \\ = \int_{B'_{r_1}} \zeta_{i,-k}(\rho g) \operatorname{Tr}(\zeta_{i,-k}(U_1))\, dx + \int_{B'_{r_1}} \zeta_{i,-k}(h \operatorname{Tr}(U_1)) \operatorname{Tr}(\zeta_{i,-k}(U_1))\, dx,\end{aligned} \tag{3.37}$$

thanks to (3.36). From (3.37) it follows that, for any $i = 1, \dots, N$,

$$\begin{aligned}&\int_{B^+_{r_1}} y^{1-2s} A\nabla(\zeta_{i,-k}(U_1)) \cdot \nabla(\zeta_{i,-k}(U_1))\, dz \\ &\le \int_{B^+_{r_1}} y^{1-2s} |\zeta_{i,-k}(A)\nabla(\tau_{i,-k}(U_1)) \cdot \nabla(\zeta_{i,-k}(U_1))|\, dz \\ &\quad + \int_{B^+_{r_1}} y^{1-2s} |f\ (\zeta_{i,k} \circ \zeta_{i,-k})(U_1)|\, dz + \int_{B'_{r_1}} |\zeta_{i,-k}(\rho g) \operatorname{Tr}(\zeta_{i,-k}(U_1))|\, dx \\ &\quad + \int_{B'_{r_1}} |\zeta_{i,-k}(h) \operatorname{Tr}(\tau_{i,-k}(U_1)) \operatorname{Tr}(\zeta_{i,-k}(U_1))|\, dx \\ &\quad + \int_{B'_{r_1}} |h| |\operatorname{Tr}(\zeta_{i,-k}(U_1))|^2\, dx,\end{aligned} \tag{3.38}$$

thanks to (3.35) and (3.36). Now we estimate each term of the right hand side of (3.38). We start by noticing that, thanks to (3.15), there exists a constant $\Lambda > 0$ (depending only on the Lipschitz constants of the entries of A) such that

$$\begin{aligned}\|\zeta_{i,-k}(A)(z)\|_{\mathcal{L}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})} \le \Lambda \quad \text{for all } i = 1, \dots, N,\ z \in B^+_{r_1}, \\ \text{and } k \in \left(r_2 - \sqrt{r_1^2 - r_2^2}, \sqrt{r_1^2 - r_2^2} - r_2\right),\end{aligned} \tag{3.39}$$

where $\|\zeta_{i,-k}(A)(z)\|_{\mathcal{L}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})}$ is the norm of $\zeta_{i,-k}(A)(z)$ as a linear operator from $\mathbb{R}^{N+1}$ to $\mathbb{R}^{N+1}$. Then by (3.39), Hölder's inequality and Cauchy-Schwarz's inequality in $\mathbb{R}^{N+1}$,

$$\begin{aligned}\int_{B^+_{r_1}} y^{1-2s} |\zeta_{i,-k}(A)\nabla(\tau_{i,-k}(U_1)) \cdot \nabla(\zeta_{i,-k}(U_1))|\, dz \\ \le \Lambda \|\nabla(\tau_{i,-k}(U_1))\|_{L^2(B^+_{r_1}, y^{1-2s})} \|\nabla(\zeta_{i,-k}(U_1))\|_{L^2(B^+_{r_1}, y^{1-2s})}.\end{aligned} \tag{3.40}$$

By Hölder's inequality and (3.8),

$$\int_{B^+_{r_1}} y^{1-2s} |f\ (\zeta_{i,k} \circ \zeta_{i,-k})(U_1)|\, dz \le \|f\|_{L^2(B^+_{r_1}, y^{1-2s})} \|\nabla(\zeta_{i,-k}(U_1))\|_{L^2(B^+_{r_1}, y^{1-2s})}. \tag{3.41}$$

Furthermore by (3.5) and Hölder's inequality

$$\int_{B'_{r_1}} |\zeta_{i,-k}(\rho g)\,\mathrm{Tr}(\zeta_{i,-k}(U_1))|\,dx \le \mathcal{S}_{N,s}^{\frac{1}{2}} \|\rho g\|_{W^{1,\frac{2N}{N+2s}}(B'_{r_1})} \|\nabla(\zeta_{i,-k}(U_1))\|_{L^2(B^+_{r_1},y^{1-2s})}, \quad (3.42)$$

$$\int_{B'_{r_1}} |\zeta_{i,-k}(h)\,\mathrm{Tr}(\tau_{i,-k}(U_1))\,\mathrm{Tr}(\zeta_{i,-k}(U_1))|\,dx$$
$$\le \mathcal{S}_{N,s} \|h\|_{W^{1,\frac{N}{2s}}(B'_{r_1})} \|\nabla(\tau_{i,-k}(U_1))\|_{L^2(B^+_{r_1},y^{1-2s})} \|\nabla(\zeta_{i,-k}(U_1))\|_{L^2(B^+_{r_1},y^{1-2s})}, \quad (3.43)$$

and

$$\int_{B'_{r_1}} |h|\,|\mathrm{Tr}(\zeta_{i,-k}(U_1))|^2\,dx \le \mathcal{S}_{N,s} \|h\|_{W^{1,\frac{N}{2s}}(B'_{r_1})} \|\nabla(\zeta_{i,-k}(U_1))\|^2_{L^2(B^+_{r_1},y^{1-2s})}. \quad (3.44)$$

Putting together (3.38), (3.40), (3.41), (3.42), (3.43), (3.44) and (3.16) we obtain that

$$\begin{aligned}
&\left(\lambda_1 - \mathcal{S}_{N,s} \|h\|_{W^{1,\frac{N}{2s}}(B'_{r_1})}\right) \|\nabla(\zeta_{i,-k}(U_1))\|_{L^2(B^+_{r_1},y^{1-2s})} \\
&\quad \le \Lambda \|\nabla(\tau_{i,-k}(U_1))\|_{L^2(B^+_{r_1},y^{1-2s})} + \|f\|_{L^2(B^+_{r_1},y^{1-2s})} + \mathcal{S}_{N,s}^{\frac{1}{2}} \|\rho g\|_{W^{1,\frac{2N}{N+2s}}(B'_{r_1})} \\
&\qquad + \mathcal{S}_{N,s} \|h\|_{W^{1,\frac{N}{2s}}(B'_{r_1})} \|\nabla(\tau_{i,-k}(U_1))\|_{L^2(B^+_{r_1},y^{1-2s})} \\
&\quad = (\Lambda + \mathcal{S}_{N,s} \|h\|_{W^{1,\frac{N}{2s}}(B'_{r_1})}) \|\nabla U_1\|_{L^2(B^+_{r_1},y^{1-2s})} \\
&\qquad + \|f\|_{L^2(B^+_{r_1},y^{1-2s})} + \mathcal{S}_{N,s}^{\frac{1}{2}} C_\rho \|g\|_{W^{1,\frac{2N}{N+2s}}(B'_{r_1})}, \qquad (3.45)
\end{aligned}$$

for some positive constant $C_\rho > 0$ depending only on $\|\nabla\rho\|_{L^\infty(B'_{r_1})}$, where we have used the fact that $\|\nabla(\tau_{i,-k}(U_1))\|_{L^2(B^+_{r_1},y^{1-2s})} = \|\nabla U_1\|_{L^2(B^+_{r_1},y^{1-2s})}$ since $\operatorname{supp}\tau_{i,-k}(U_1) \subset B^+_{r_1} \cup B'_{r_1}$ for all $|k| < \sqrt{r_1^2 - r_2^2} - r_2$.

Eventually choosing r_1 smaller form the beginning, we may suppose that

$$\lambda_1 - \mathcal{S}_{N,s} \|h\|_{W^{1,\frac{N}{2s}}(B'_{r_1})} > 0,$$

by the absolute continuity of the integral. We conclude that for any $i = 1, \dots, N$ and any $j = 1, \dots, N+1$

$$\left\{ \frac{\partial(\zeta_{i,-k}(U_1))}{\partial z_j} : |k| < \sqrt{r_1^2 - r_2^2} - r_2 \right\} \quad \text{is bounded in } L^2(B^+_{r_1}, y^{1-2s}).$$

It follows that, for any $i = 1, \dots, N$ and $j = 1, \dots, N+1$, there exist a function $\psi_{i,j} \in L^2(B^+_{r_1}, y^{1-2s})$ and a sequence $k_n \to 0$ such that $\frac{\partial(\zeta_{i,-k_n}(U_1))}{\partial z_j} \rightharpoonup \psi_{i,j}$ weakly in $L^2(B^+_{r_1}, y^{1-2s})$ as $n \to \infty$. Furthermore, by (3.8), the family of functions $\{(\zeta_{i,-k_n}(U_1)) : n \in \mathbb{N}\}$ is bounded in $L^2(B^+_{r_1}, y^{1-2s})$ and so there exists a function $\varphi_i \in L^2(B^+_{r_1}, y^{1-2s})$ such that $\zeta_{i,-k_n}(U_1) \rightharpoonup \varphi_i$ weakly in $L^2(B^+_{r_1}, y^{1-2s})$ for any $i = 1, \dots, N$, up to a subsequence. For any test function

$\phi \in C_c^\infty(B_{r_1}^+)$, thanks to the Dominated Converge Theorem,

$$\int_{B_{r_1}^+} \varphi_i \phi \, dz = \lim_{n\to\infty} \int_{B_{r_1}^+} \zeta_{i,-k_n}(U_1)\phi \, dz$$
$$= -\lim_{n\to\infty} \int_{B_{r_1}^+} U_1 \zeta_{i,k_n}(\phi)\, dz = -\int_{B_{r_1}^+} U_1 \frac{\partial \phi}{\partial z_i}\, dz = \int_{B_{r_1}^+} \frac{\partial U_1}{\partial z_i}\phi \, dz$$

and hence $\varphi_i = \frac{\partial U_1}{\partial z_i}$, i.e. $\zeta_{i,-k_n}(U_1) \rightharpoonup \frac{\partial U_1}{\partial z_i}$ weakly in $L^2(B_{r_1}^+, y^{1-2s})$, up to a subsequence. Furthermore, for any $\phi \in C_c^\infty(B_{r_1}^+)$, $i = 1, \dots, N$, and $j = 1, \dots, N+1$, we have that

$$\int_{B_{r_1}^+} \psi_{i,j}\phi \, dz = \lim_{n\to\infty} \int_{B_{r_1}^+} \frac{\partial(\zeta_{i,-k_n}(U_1))}{\partial z_j}\phi \, dz$$
$$= -\lim_{n\to\infty} \int_{B_{r_1}^+} \zeta_{i,-k_n}(U_1)\frac{\partial \phi}{\partial z_j} dz = -\int_{B_{r_1}^+} \frac{\partial U_1}{\partial z_i}\frac{\partial \phi}{\partial z_j} dz,$$

that is, $\psi_{i,j} = \frac{\partial}{\partial z_j}\frac{\partial U_1}{\partial z_i}$. Therefore the distributional derivative of $\frac{\partial U_1}{\partial z_i}$ respect to the variable z_j belongs to $L^2(B_{r_1}^+, y^{1-2s})$ for any $j = 1, \dots, N+1$, and $i = 1, \dots, N$, i.e.

$$\nabla_x U_1 \in H^1(B_{r_1}^+, y^{1-2s}). \tag{3.46}$$

Furthermore, estimate (3.45) and weak lower semi-continuity of the $L^2(B_{r_1}^+, y^{1-2s})$-norm imply that

$$\|\nabla_x U_1\|_{H^1(B_{r_1}^+, y^{1-2s})} \le C\left(\|\nabla U_1\|_{L^2(B_{r_1}^+, y^{1-2s})} + \|f\|_{L^2(B_{r_1}^+, y^{1-2s})} + \|g\|_{W^{1,\frac{2N}{N+2s}}(B'_{r_1})}\right) \tag{3.47}$$

for a positive constant $C = C(N, s, \|h\|_{W^{1,\frac{N}{2s}}(B'_{r_1})}, \lambda_1, \Lambda, \|\nabla \rho\|_{L^\infty(B_{r_1}^+)}) > 0$.

This also implies that $\nabla_x(y^{1-2s}\frac{\partial U_1}{\partial y}) \in L^2(B_{r_1}^+, y^{2s-1})$ with norm estimated as above. To conclude, it remains to show that $\frac{\partial}{\partial t}(y^{1-2s}\frac{\partial U_1}{\partial y}) \in L^2(B_{r_1}^+, y^{2s-1})$.

To this aim we observe that, for any $\phi \in C_c^\infty(B_{r_1}^+)$, (3.34), the Divergence Theorem, (3.14), and (3.16) imply that

$$\int_{B_{r_1}^+} y^{1-2s}\frac{\partial U_1}{\partial t}\frac{\partial \phi}{\partial t}\, dz = \int_{B_{r_1}^+} y^{1-2s}\alpha \frac{\partial U_1}{\partial t}\frac{\partial}{\partial t}\left(\frac{\phi}{\alpha}\right) dz + \int_{B_{r_1}^+} y^{1-2s}\frac{\partial \alpha}{\partial y}\frac{\partial U_1}{\partial y}\frac{\phi}{\alpha}\, dz$$
$$= -\int_{B_{r_1}^+} y^{1-2s} B\nabla_x U_1 \cdot \nabla_x\left(\frac{\phi}{\alpha}\right) dz - \int_{B_{r_1}^+} y^{1-2s} f\frac{\phi}{\alpha}\, dz + \int_{B_{r_1}^+} y^{1-2s}\frac{\partial \alpha}{\partial y}\frac{\partial U_1}{\partial t}\frac{\phi}{\alpha}\, dz$$
$$= -\int_{B_{r_1}^+} y^{1-2s}\frac{1}{\alpha}\left(-\operatorname{div}_x(B\nabla_x U_1) + f - \frac{\partial \alpha}{\partial y}\frac{\partial U_1}{\partial y}\right)\phi \, dz.$$

Thanks to (3.15), (3.16), (3.18), (3.33), (3.46), and Hölder's inequality, we then conclude that

$$y^{2s-1}\frac{\partial}{\partial t}\left(y^{1-2s}\frac{\partial U_1}{\partial y}\right) = \frac{1}{\alpha}\left(-\operatorname{div}(B\nabla_x U_1) + f - \frac{\partial \alpha}{\partial y}\frac{\partial U_1}{\partial y}\right) \in L^2(B_{r_1}^+, y^{1-2s})$$

which implies that $\frac{\partial}{\partial t}(y^{1-2s}\frac{\partial U_1}{\partial y}) \in L^2(B_{r_1}^+, y^{2s-1})$ and hence

$$y^{1-2s}\frac{\partial U_1}{\partial y} \in H^1(B_{r_1}^+, y^{2s-1}),$$

with $H^1(B_{r_1}^+, y^{2s-1})$-norm estimated as in (3.47).

Since $\eta \equiv 1$ on $B_{r_3}^+$ we have thereby proved that

$$\nabla_x U \in H^1(B_{r_3}^+, y^{1-2s}) \quad \text{and} \quad y^{1-2s}\frac{\partial U}{\partial y} \in H^1(B_{r_3}^+, y^{2s-1})$$

and, in view of (3.33),

$$\|\nabla_x U\|_{H^1(B_{r_3}^+, y^{1-2s})} + \left\| y^{1-2s}\frac{\partial U}{\partial y}\right\|_{H^1(B_{r_3}^+, y^{2s-1})} \\ \leq C\left(\|U\|_{H^1(B_R^+, y^{1-2s})} + \|c\|_{L^2(B_R^+, y^{1-2s})} + \|g\|_{W^{1,\frac{2N}{N+2s}}(B_R')}\right) \tag{3.48}$$

for a constant $C > 0$ depending only on N, s, r_1, r_3, $\|h\|_{W^{1,\frac{N}{2s}}(B_R')}$, λ_1, $\|A\|_{W^{1,\infty}(B_R^+, \mathbb{R}^{(N+1)^2})}$.

Reasoning in a similar way we can show that, for any $r \in (0, R)$ and any $x \in \overline{B_r'}$, there exists $r_x > 0$ such that $B_{r_x}^+(x) \subset B_R^+$, $\nabla_x U \in H^1(B_{r_x}^+(x), y^{1-2s})$, and $y^{1-2s}\frac{\partial U}{\partial y} \in H^1(B_{r_x}^+(x), y^{2s-1})$, where

$$B_{r_x}^+(x) := \{\xi \in \mathbb{R}_+^{N+1} : |(x, 0) - \xi| < r_x\}.$$

Then we can cover $\overline{B_r'}$ with a finite family of open sets $\{B_{r_{x_i}}^+(x_i)\}_{i\in I}$ such that

$$\nabla_x U \in H^1(B_{r_{x_i}}^+(x_i), y^{1-2s}) \text{ and } y^{1-2s}\frac{\partial U}{\partial y} \in H^1(B_{r_{x_i}}^+(x_i), y^{2s-1}) \quad \text{for all } i \in I$$

and an estimate of type (3.48) is satisfied. Furthermore, letting $B_{R,\delta}^+$ be as in (3.27), it is easy to verify that $y^{1-2s}A \in C^{0,1}(\overline{B_{R,\delta}^+})$ and $y^{1-2s}c \in L^2(B_{R,\delta}^+)$ for any $\delta \in (0, R)$, since the weight y^{1-2s} is Lipschitz continuous on $\overline{B_{R,\delta}^+}$. Then we may conclude that $U \in H^2(B_{r,\delta}^+, y^{1-2s})$ for any $r \in (0, R)$ and $\delta \in (0, R)$ by classical elliptic regularity theory (see e.g. [80, Theorem 8.8]).

Combining the above information we obtain (3.20) and (3.21). □

Remark 3.3.1. The regularity result of Theorem 3.2.1 applies also to problems of the form

$$\begin{cases} -\operatorname{div}(y^{1-2s}A\nabla U) + y^{1-2s}bU + y^{1-2s}c = 0, & \text{on } B_R^+, \\ \lim_{y\to 0^+} y^{1-2s}A\nabla U \cdot \nu = h\operatorname{Tr}(U) + g, & \text{on } B_R', \end{cases}$$

with c, h, g as in assumptions (3.17) and (3.18), and a potential $b \in L^{q_{N,s}}(B_R^+, y^{1-2s})$, where

$$q_{N,s} := \begin{cases} N + 2 - 2s, & \text{if } s \in \left(0, \frac{1}{2}\right), \\ N + 1, & \text{if } s \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Indeed if $b \in L^{q_{N,s}}(B_R^+, y^{1-2s})$ and $U \in H^1(B_R^+, y^{1-2s})$, then $bU \in L^2(B_R^+, y^{1-2s})$ in view of Hölder's inequality and the following Sobolev-type embedding result.

Lemma 3.3.2. *For any $r > 0$, $H^1(B_r^+, y^{1-2s}) \subset L^{2_s^{**}}(B_r^+, y^{1-2s})$, where*

$$2_s^{**} := \min\left\{2\frac{N+2-2s}{N-2s}, 2\frac{N+1}{N-1}\right\} = \begin{cases} 2\frac{N+2-2s}{N-2s}, & \text{if } s \in \left(0, \frac{1}{2}\right), \\ 2\frac{N+1}{N-1}, & \text{if } s \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Furthermore, there exists a constant $K_{N,s} > 0$ *such that, for any* $r > 0$ *and any* $w \in H^1(B_r^+, y^{1-2s})$,

$$\left(\int_{B_r^+} y^{1-2s}|w|^{2_s^{**}}\,dz\right)^{\frac{2}{2_s^{**}}} \le K_{N,s,r}\left(\frac{1}{r^2}\int_{B_r^+} y^{1-2s}w^2\,dz + \int_{B_r^+} y^{1-2s}|\nabla w|^2\,dz\right),$$

where

$$K_{N,s,r} := \begin{cases} K_{N,s}, & \text{if } s \in \left(0, \frac{1}{2}\right), \\ K_{N,s}(2s-1)r^{\frac{2}{N+1}}, & \text{if } s \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Proof. The claim follows from a scaling argument, [64, Appendix A.1] and [111, Theorem 19.20], see also [120, Theorem 2.4]. □

3.4 Proof of Proposition 3.2.3

We start with a useful formula.

Proposition 3.4.1. *Let* U *be a solution of* (3.19)*. For a.e.* $r \in (0, R)$ *and for all* $\phi \in H^1(B_r^+, y^{1-2s})$

$$\int_{B_r^+} y^{1-2s}[A\nabla U\cdot\nabla\phi + c\phi]dz = \frac{1}{r}\int_{S_r^+} y^{1-2s}A\nabla U\cdot z\,\phi\,dS + \int_{B_r'}[h\,\mathrm{Tr}(U) + g]\,\mathrm{Tr}(\phi)\,dx. \quad (3.49)$$

Remark 3.4.2. By Coarea Formula

$$\int_{B_R^+}\left|y^{1-2s}A\nabla U\cdot\frac{z}{|z|}\phi\right|dz = \int_0^R\left(\int_{S_r^+}\left|y^{1-2s}A\nabla U\cdot\frac{z}{r}\,\phi\right|dS\right)dr.$$

It follows that the function $f(r) := \int_{S_r^+} y^{1-2s}A\nabla U\cdot\frac{z}{r}\phi\,dS$ is well-defined as an element of $L^1(0,R)$ and hence a.e. $r \in (0,R)$ is a Lebesgue point of f.

Proof. By density it is enough to prove (3.49) for any $\phi \in C^\infty(\overline{B_r^+})$. Let us consider the following sequence of radial cut-off functions

$$\eta_n(|z|) := \begin{cases} 1, & \text{if} \quad 0 \le |z| \le r - \frac{1}{n}, \\ n(r - |z|), & \text{if} \quad r - \frac{1}{n} \le |z| \le r, \\ 0, & \text{if} \quad |z| \ge r. \end{cases}$$

Testing (3.19) with $\phi\eta_n$ and passing to the limit as $n \to \infty$ we obtain (3.49) thanks to the Dominated Convergence Theorem, (3.9) and Remark 3.4.2. □

Proof of Proposition 3.2.3. The following Rellich-Nečas type identity

$$\begin{aligned}\mathrm{div}\left(y^{1-2s}(A\nabla U\cdot\nabla U)\beta - 2y^{1-2s}(\nabla U\cdot\beta)A\nabla U\right) &= y^{1-2s}A\nabla U\cdot\nabla U\,\mathrm{div}(\beta) \\ &- 2\beta\cdot\nabla U\,\mathrm{div}\left(y^{1-2s}A\nabla U\right) + (d(y^{1-2s}A)\nabla U\nabla U)\cdot\beta - 2J_\beta(y^{1-2s}A\nabla U)\cdot\nabla U\end{aligned}$$

holds in a distributional sense in B_R^+. In view of (3.14) and (3.13) the above equation can be rewritten as

$$\operatorname{div}\left(y^{1-2s}(A\nabla U\cdot\nabla U)\beta - 2y^{1-2s}(\nabla U\cdot\beta)A\nabla U\right)$$
$$= y^{1-2s}A\nabla U\cdot\nabla U\operatorname{div}(\beta) - 2y^{1-2s}c(\beta\cdot\nabla U) + y^{1-2s}dA\nabla U\nabla U\cdot\beta$$
$$+ (1-2s)y^{1-2s}\frac{\alpha}{\mu}A\nabla U\cdot\nabla U - 2J_\beta(y^{1-2s}A\nabla U)\cdot\nabla U \quad (3.50)$$

with dA as in (3.24).

Let $r\in(0,R)$. By Theorem 3.2.1 an Remark 3.2.2, letting $\beta=(\beta_1,\dots,\beta_N,\alpha/\mu)$ (see (3.14) and (3.23)), we have that

$$\nabla U\cdot\beta = \nabla_x U\cdot(\beta_1,\dots,\beta_N) + \frac{\alpha}{\mu}yU_y \in H^1(B_r^+,y^{1-2s}). \quad (3.51)$$

In particular, to prove that $\frac{\partial}{\partial y}(tU_y)\in L^2(B_r^+,y^{1-2s})$, it is useful to observe that

$$\frac{\partial}{\partial y}(yU_y) = y^{2s}\frac{\partial}{\partial y}(y^{1-2s}U_y) + 2sU_y$$

and recall that $\frac{\partial}{\partial y}\left(y^{1-2s}\frac{\partial U}{\partial y}\right)\in L^2(B_r^+,y^{2s-1})$ by (3.20).

We observe that $yU_y = y^{2s}(y^{1-2s}U_y)$, with

$$y^{2s}\in H^1(B_r^+,y^{1-2s}) \quad\text{and}\quad y^{1-2s}U_y\in H^1(B_r^+,y^{2s-1})$$

by (3.20); hence (3.10) implies that $\operatorname{Tr}(tU_t)=\operatorname{Tr}(t^{2s})\operatorname{Tr}(y^{1-2s}U_y)=0$, so that from (3.51), (3.9), and (3.12) we deduce that

$$\operatorname{Tr}(\nabla U\cdot\beta) = \operatorname{Tr}(\nabla_x U\cdot(\beta_1,\dots,\beta_N)) + \operatorname{Tr}\left(\frac{\alpha}{\mu}yU_y\right) = \nabla_x\operatorname{Tr}(U)\cdot\beta'. \quad (3.52)$$

From (3.13), (3.18), and (3.51) it follows that

$$\operatorname{div}(y^{1-2s}(\nabla U\cdot\beta)A\nabla U) = y^{1-2s}c(\nabla U\cdot\beta) + y^{1-2s}A\nabla U\cdot\nabla(\nabla U\cdot\beta)\in L^1(B_r^+) \quad (3.53)$$

so that, in view of (3.50), (3.25), (3.18), and (3.51) we obtain also that

$$\operatorname{div}\left(y^{1-2s}(A\nabla U\cdot\nabla U)\beta\right)\in L^1(B_r^+). \quad (3.54)$$

Applying the Divergence Theorem on the set $B_{r,\delta}^+$ defined in (3.27) (and recalling from Theorem 3.2.1 or classical elliptic regularity theory that $U\in H^2(B_{r,\delta}^+)$), we have that

$$\int_{B_{r,\delta}^+}\operatorname{div}(y^{1-2s}(A\nabla U\cdot\nabla U)\beta)\,dz = r\int_{S_{r,\delta}^+} y^{1-2s}A\nabla U\cdot\nabla U\,dS$$
$$-\delta^{2-2s}\int_{B'_{\sqrt{r^2-\delta^2}}}\frac{\alpha(x,\delta)}{\mu(x,\delta)}(A\nabla U\cdot\nabla U)(x,\delta)\,dx \quad (3.55)$$

with $S_{r,\delta}^+$ as in (3.27). We claim that there exists a sequence $\delta_n\to0^+$ such that

$$\lim_{n\to\infty}\delta_n^{2-2s}\int_{B'_{\sqrt{r^2-\delta_n^2}}}\frac{\alpha(x,\delta_n)}{\mu(x,\delta_n)}(A\nabla U\cdot\nabla U)(x,\delta_n)\,dx=0. \quad (3.56)$$

To prove (3.56) we argue by contradiction. If the claim does not hold, then there exist a constant $C > 0$ and $\bar{r} \in (0, r)$ such that

$$\delta^{1-2s} \int_{B'_r} \frac{\alpha(x,\delta)}{\mu(x,\delta)} (A\nabla U \cdot \nabla U)(x,\delta)\, dx \geq \frac{C}{\delta} \quad \text{for any } \delta \in (0,\bar{r}). \tag{3.57}$$

We may suppose that $B'_{\bar{r}} \times (0,\bar{r}) \subset B^+_r$ and integrating (3.57) in $(0,\bar{r})$ we obtain

$$\begin{aligned}\int_{B^+_R} y^{1-2s} \frac{\alpha}{\mu} A\nabla U \cdot \nabla U\, dz &\geq \int_0^{\bar{r}} y^{1-2s} \left(\int_{B'_r} \frac{\alpha(x,y)}{\mu(x,y)} (A\nabla U \cdot \nabla U)(x,y)\, dx \right) dt \\ &\geq C \int_0^{\bar{r}} \frac{1}{y}\, dt = +\infty,\end{aligned}$$

which is a contradiction since $\frac{\alpha}{\mu} A\nabla U \cdot \nabla U \in L^1(B^+_R, y^{1-2s})$ thanks to (3.25) and Hölder's inequality. Therefore passing to the limit as $n \to \infty$ and $\delta = \delta_n$ in (3.55) and taking into account (3.54) we conclude that

$$\int_{B^+_r} \operatorname{div}(y^{1-2s}(A\nabla U \cdot \nabla U)\beta)\, dz = r \int_{S^+_r} y^{1-2s} A\nabla U \cdot \nabla U\, dS \tag{3.58}$$

for a.e. $r \in (0, R)$. From (3.53) and (3.49) it follows that

$$\begin{aligned}&\int_{B^+_r} \operatorname{div}(y^{1-2s}(\nabla U \cdot \beta) A\nabla U)\, dz \\ &= \int_{B^+_r} y^{1-2s} c(\nabla U \cdot \beta)\, dz + \int_{B^+_r} y^{1-2s} A\nabla U \cdot \nabla(\nabla U \cdot \beta)\, dz \\ &= \frac{1}{r} \int_{S^+_r} y^{1-2s} (A\nabla U \cdot z)(\nabla U \cdot \beta)\, dS + \int_{B'_r} [h \operatorname{Tr}(U) + g] \operatorname{Tr}(\nabla U \cdot \beta)\, dx \\ &= r \int_{S^+_r} y^{1-2s} \frac{|A\nabla U \cdot \nu|^2}{\mu}\, dS + \int_{B'_r} [h \operatorname{Tr}(U) + g](\nabla_x \operatorname{Tr}(U) \cdot \beta')\, dx,\end{aligned} \tag{3.59}$$

thanks to (3.15), (3.23), and (3.52). We observe that $\beta' h \in W^{1,\frac{N}{2s}}(B'_r, \mathbb{R}^N)$ in view of (3.17) and (3.25) and $(\operatorname{Tr}(U))^2 \in W^{1,\frac{N}{N-2s}}(B'_r)$ thanks to (3.20) and (3.11); then an integration by parts on B'_r yields

$$\begin{aligned}&\int_{B'_r} h \operatorname{Tr}(U)(\nabla_x \operatorname{Tr}(U) \cdot \beta')\, dx = \frac{1}{2} \int_{B'_r} \nabla_x (\operatorname{Tr}(U))^2 \cdot (h\beta')\, dx \\ &= \frac{r}{2} \int_{S'_r} h |\operatorname{Tr}(U)|^2\, dS' - \frac{1}{2} \int_{B'_r} (\operatorname{div}_x(\beta') h + \beta' \cdot \nabla h) |\operatorname{Tr}(U)|^2\, dx.\end{aligned} \tag{3.60}$$

Moreover $\beta' g \in W^{1,\frac{2N}{N+2s}}(B'_r, \mathbb{R}^N)$ by (3.17) and $\operatorname{Tr}(U) \in W^{1,\frac{2N}{N-2s}}(B'_r)$ by (3.20) and (3.11), hence, integrating by parts, we obtain that

$$\int_{B'_r} \nabla_x \operatorname{Tr}(U) \cdot (\beta' g)\, dx = r \int_{S'_r} g \operatorname{Tr}(U)\, dS' - \int_{B'_r} (\operatorname{div}_x(\beta') g + \beta' \cdot \nabla g) \operatorname{Tr}(U)\, dx. \tag{3.61}$$

Putting together (3.50), (3.58), (3.59), (3.60), and (3.61), we obtain (3.26). □

Chapter 4

Unique continuation from the boundary for the spectral fractional Laplacian

4.1 Statement of the main results

In this Chapter we establish a unique continuation principle and derive local asymptotics from a point $x_0 \in \partial\Omega$ for the solutions to the following equation

$$(-\Delta)^s u = hu \quad \text{on } \Omega, \tag{4.1}$$

where $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^N$ is a bounded Lipschitz domain whose boundary is $C^{1,1}$ in a neighbourhood of x_0, $N > 2s$, h is a measurable function on Ω satisfying suitable summability properties, (see (4.8)), and $(-\Delta)^s$ is the fractional Laplacian.

In order to introduce a suitable functional setting and give a weak formulation of (4.1), we recall the definition of the spectral fractional Laplacian, which can be given in terms of the Dirichlet eigenvalues of the Laplacian, see e.g. [37], [102] and [9]. From classical spectral theory, the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta\varphi = \mu\varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases}$$

admits an increasing and diverging sequence $\{\mu_k\}_{k\in\mathbb{N}\setminus\{0\}}$ of positive eigenvalues (repeated according to their multiplicity). Furthermore, there exists an orthonormal basis of $L^2(\Omega)$ made of the corresponding eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}\setminus\{0\}}$. Every $v \in L^2(\Omega)$ can be expanded with respect to the basis $\{\varphi_k\}_{k\in\mathbb{N}\setminus\{0\}}$ as

$$v = \sum_{k=1}^{\infty} (v, \varphi_k)_{L^2(\Omega)} \varphi_k \quad \text{in } L^2(\Omega),$$

where $(v, \varphi_k)_{L^2(\Omega)}$ is the L^2-scalar product, i.e. $(v_1, v_2)_{L^2(\Omega)} = \int_\Omega v_1 v_2 \, dx$.

We introduce the functional space

$$\mathbb{H}^s(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{k=1}^{\infty} \mu_k^s (v, \varphi_k)^2_{L^2(\Omega)} < +\infty \right\}$$

which is a Hilbert space with respect to the scalar product

$$(v_1, v_2)_{\mathbb{H}^s(\Omega)} := \sum_{k=0}^{\infty} \mu_k^s (v_1, \varphi_k)_{L^2(\Omega)} (v_2, \varphi_k)_{L^2(\Omega)}, \quad v_1, v_2 \in \mathbb{H}^s(\Omega). \tag{4.2}$$

A more explicit characterization of the space $\mathbb{H}^s(\Omega)$ is provided by the interpolation theory, see [26, Section 3.1.3], and [101] or Proposition 5.1.2:

$$\mathbb{H}^s(\Omega) = [H_0^1(\Omega), L^2(\Omega)]_{1-s} = \begin{cases} H_0^s(\Omega), & \text{if } s \in (0,1) \setminus \{\frac{1}{2}\}, \\ H_{00}^{1/2}(\Omega), & \text{if } s = \frac{1}{2}. \end{cases}$$

Here, denoting as $H^s(\Omega)$ the usual fractional Sobolev space $W^{s,2}(\Omega)$, $H_0^s(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$, and

$$H_{00}^{1/2}(\Omega) := \left\{ u \in H_0^{\frac{1}{2}}(\Omega) : \int_\Omega \frac{u^2(x)}{d(x, \partial\Omega)}\, dx < +\infty \right\}, \tag{4.3}$$

where $d(x, \partial\Omega) := \inf\{|x - y| : y \in \partial\Omega\}$. We recall that $H^s(\Omega) = H_0^s(\Omega)$ if $s \in (0, \frac{1}{2}]$, see [101]. Moreover, if $s \neq \frac{1}{2}$, the trivial extension by 0 outside Ω defines a linear and continuous operator from $H_0^s(\Omega)$ into $H^s(\mathbb{R}^N)$, see [31, Remark 2.5 and Proposition B.1]. On the other hand, the trivial extension defines a linear and continuous operator from $H_{00}^{1/2}(\Omega)$ into $H^{1/2}(\mathbb{R}^N)$, as one can easily deduce from estimate (B.2) in [31]. Then

$$\begin{aligned} \iota : \mathbb{H}^s(\Omega) &\to H^s(\mathbb{R}^N), \\ v \mapsto \tilde{v} &= \begin{cases} v, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \end{aligned} \tag{4.4}$$

is a linear and continuous operator.

It is easy to verify that, if $v \in \mathbb{H}^s(\Omega)$, then the series $\sum_{k=1}^{\infty} \mu_k^s (v, \varphi_k)_{L^2(\Omega)} \varphi_k$ converges in the dual space $(\mathbb{H}^s(\Omega))^*$ to some $F \in (\mathbb{H}^s(\Omega))^*$ such that ${}_{(\mathbb{H}^s(\Omega))^*}\langle F, \varphi_k \rangle_{\mathbb{H}^s(\Omega)} = \mu_k^s (v, \varphi_k)_{L^2(\Omega)}$. Hence, for every $v \in \mathbb{H}^s(\Omega)$, we can define its spectral fractional Laplacian as

$$(-\Delta)^s v = \sum_{k=1}^{\infty} \mu_k^s (v, \varphi_k)_{L^2(\Omega)} \varphi_k \in (\mathbb{H}^s(\Omega))^*. \tag{4.5}$$

Actually, the spectral fractional Laplacian is the Riesz isomorphism between $\mathbb{H}^s(\Omega)$ endowed with the scalar product (4.2) and its dual $(\mathbb{H}^s(\Omega))^*$, i.e.

$${}_{(\mathbb{H}^s(\Omega))^*}\langle (-\Delta)^s v_1, v_2 \rangle_{\mathbb{H}^s(\Omega)} = (v_1, v_2)_{\mathbb{H}^s(\Omega)} \quad \text{for all } v_1, v_2 \in \mathbb{H}^s(\Omega). \tag{4.6}$$

The spectral fractional Laplacian defined in (4.5) is a different operator from the usual fractional Laplacian defined by the Fourier transform as

$$\mathcal{F}((-\Delta)^s v)(\xi) := |\xi|^{2s} \widehat{v}(\xi) \tag{4.7}$$

for any $v \in \mathcal{S}(\mathbb{R}^N)$. Indeed, the spectral fractional Laplacian depends on the domain Ω and it is a global operator in Ω, while the fractional Laplacian is a global operator on the whole $\mathbb{R}^N$. Moreover, the eigenfunctions of the spectral fractional Laplacian coincide with the eigenfunctions of the Dirichlet Laplacian, hence they are smooth up to the boundary if Ω is sufficiently regular; on the other hand, the eigenfunctions of the restricted fractional Laplacian, defined by restricting the operator in (4.7) to act only on functions vanishing outside Ω, are only Hölder continuous, see [117].

Within the functional setting introduced above, we can give the notion of weak solution to (4.1). To this purpose, we assume that

$$h \in W^{1,\frac{N}{2s}+\varepsilon}(\Omega) \tag{4.8}$$

for some $\varepsilon \in (0,1)$. We note that it is not restrictive to assume ε small. In view of (4.6), we say that a function $u \in \mathbb{H}^s(\Omega)$ is a weak solution to (4.1) if

$$(u,\phi)_{\mathbb{H}^s(\Omega)} = \int_\Omega h(x)u(x)\phi(x)\,dx \quad \text{ for any } \phi \in C_c^\infty(\Omega). \tag{4.9}$$

The right hand side in (4.9) is finite in view of (4.8), the Hölder's inequality, and the following fractional Sobolev inequality

$$\|v\|_{L^{2_s^*}(\Omega)} \le \mathcal{K}_{N,s}\,\|v\|_{H^s(\Omega)} \qquad \text{for any } v \in H_0^s(\Omega),$$

where

$$2_s^* := \frac{2N}{N-2s},$$

and $\mathcal{K}_{N,s} > 0$ is a positive constant depending only on N and s, see e.g. [48, Theorem 6.5] and [31, Remark 2.5 and Proposition B.1].

In order to establish a unique continuation property at a fixed point $x_0 \in \partial\Omega$, we need to assume some regularity on the boundary of Ω near x_0; more precisely, we assume that there exist a radius $R > 0$ and a function g such that

$$g \in C^{1,1}(\mathbb{R}^{N-1}, \mathbb{R}) \tag{4.10}$$

and, up to rigid motions, letting $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$,

$$\partial\Omega \cap B'_R(x_0) = \{(x', x_N) \in B'_R(x_0) : x_N = g(x')\}, \tag{4.11}$$

$$\Omega \cap B'_R(x_0) = \{(x', x_N) \in B'_R(x_0) : x_N < g(x')\}, \tag{4.12}$$

where, for any $r > 0$ and $x \in \mathbb{R}^N$,

$$B'_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}. \tag{4.13}$$

The spectral fractional Laplacian defined in (4.5) turns out to be a nonlocal operator on Ω. As we intend to use an approach based on local doubling inequalities, which are deduced from an Almgren-type monotonicity formula in the spirit of [79], it is quite natural to deal with the local realization of the spectral fractional Laplacian. This is obtained by the extension procedure described in [35] (see also [125] and [37]) which transforms (4.1) into a singular or degenerate problem on a cylinder contained in a $N+1$-dimensional space.

We define

$$\mathcal{C}_\Omega := \Omega \times (0, +\infty), \quad \partial_L\mathcal{C}_\Omega := \partial\Omega \times [0, +\infty), \tag{4.14}$$

and

$$H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s}) := \overline{C_c^\infty(\mathcal{C}_\Omega \cup \Omega)}^{\|\cdot\|_{H^1(\mathcal{C}_\Omega, y^{1-2s})}},$$

i.e. $H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s})$ is the closure in $H^1(\mathcal{C}_\Omega, y^{1-2s})$ of $C^\infty_c(\mathcal{C}_\Omega \cup \Omega)$, see also Section 3.1 in Chapter 3. Furthermore there exists a linear and continuous trace operator

$$\mathrm{Tr}_\Omega : H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s}) \to \mathbb{H}^s(\Omega) \tag{4.15}$$

which is also onto (see [37, Proposition 2.1]). Moreover, in [37] it is observed that, for every $v \in \mathbb{H}^s(\Omega)$, the minimization problem

$$\min_{\substack{w\in H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s})\\ \mathrm{Tr}_\Omega(w)=v}} \left\{ \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla w(x,t)|^2 \, dx\, dt \right\}$$

has a unique minimizer $\mathcal{H}(v) = V \in H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s})$ which solves

$$\begin{cases} \mathrm{div}(y^{1-2s}\nabla V) = 0, & \text{in } \mathcal{C}_\Omega, \\ \mathrm{Tr}_\Omega(V) = v, & \text{on } \Omega \times \{0\}, \\ V = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ -\lim_{y\to 0^+} y^{1-2s} \frac{\partial V}{\partial y} = \kappa_{s,N}(-\Delta)^s v, & \text{on } \Omega \times \{0\}, \end{cases} \tag{4.16}$$

where $\kappa_{s,N} > 0$ is a positive constant depending only on N and s. Equation (4.16) has to be interpreted in a weak sense, that is

$$\int_{\mathcal{C}_\Omega} y^{1-2s} \nabla V \cdot \nabla \phi \, dz = \kappa_{s,N} (v, \mathrm{Tr}_\Omega(\phi))_{\mathbb{H}^s(\Omega)} \quad \text{for all } \phi \in H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s}),$$

in view of (4.6). Hence, if $u \in \mathbb{H}^s(\Omega)$ solves (4.1), then its extension $U \in H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s})$ weakly solves

$$\begin{cases} \mathrm{div}(y^{1-2s}\nabla U) = 0, & \text{in } \mathcal{C}_\Omega, \\ \mathrm{Tr}_\Omega(U) = u, & \text{on } \Omega \times \{0\}, \\ U = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ -\lim_{y\to 0^+} y^{1-2s} \frac{\partial U}{\partial y} = \kappa_{s,N} h u, & \text{on } \Omega \times \{0\}, \end{cases} \tag{4.17}$$

according to (4.16), namely

$$\int_{\mathcal{C}_\Omega} y^{1-2s} \nabla U \cdot \nabla \phi \, dz = \kappa_{s,N} \int_\Omega h u \, \mathrm{Tr}_\Omega(\phi)\, dx \quad \text{for all } \phi \in H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s}). \tag{4.18}$$

The asymptotic behavior at $x_0 \in \partial\Omega$ of any solution U of (4.17), and consequently of any solution u of (4.1), turns out to be related to the eigenvalues of the following problem

$$\begin{cases} -\mathrm{div}_{\mathbb{S}}(\theta_{N+1}^{1-2s} \nabla_{\mathbb{S}} Y) = \mu\, \theta_{N+1}^{1-2s}\, Y, & \text{on } \mathbb{S}^+ \\ \lim_{\theta_{N+1}\to 0^+} \theta_{N+1}^{1-2s} \nabla_{\mathbb{S}} Y \cdot \nu = 0, & \text{on } \mathbb{S}', \\ Y \in H^1_{\mathrm{odd}}(\mathbb{S}^+, \theta_{N+1}^{1-2s}), \end{cases} \tag{4.19}$$

where

$$\begin{aligned} &\mathbb{S} := \{\theta = (\theta', \theta_N, \theta_{N+1}) \in \mathbb{R}^{N+1} : |\theta'|^2 + \theta_N^2 + \theta_{N+1}^2 = 1\}, \\ &\mathbb{S}^+ := \{\theta = (\theta', \theta_N, \theta_{N+1}) \in \mathbb{S} : \theta_{N+1} > 0\}, \\ &\mathbb{S}' := \partial\mathbb{S}^+ = \{\theta = (\theta', \theta_N, \theta_{N+1}) \in \mathbb{S} : \theta_{N+1} = 0\}, \end{aligned}$$

and ν is the outer normal vector to $\mathbb{S}^+$ on $\mathbb{S}'$, that is $\nu = -(0,\dots,0,1)$. We consider the weighted space

$$L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s}) := \left\{\Psi : \mathbb{S}^+ \to \mathbb{R} \text{ measurable} : \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}\Psi^2\, dS < +\infty\right\},$$

where dS denotes the volume element on N-dimensional spheres. In order to introduce the space $H^1_{\text{odd}}(\mathbb{S}^+,\theta_{N+1}^{1-2s})$ where problem (4.19) is formulated, we first denote by $H^1(\mathbb{S}^+,\theta_{N+1}^{1-2s})$ the completion of $C^\infty(\overline{\mathbb{S}^+})$ with respect to the norm

$$\|\phi\|_{H^1(\mathbb{S}^+,\theta_{N+1}^{1-2s})} := \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}(\phi^2 + |\nabla_{\mathbb{S}}\phi|^2)\, dS\right)^{1/2}.$$

Then we define

$$H^1_{\text{odd}}(\mathbb{S}^+,\theta_{N+1}^{1-2s}) := \{\Psi \in H^1(\mathbb{S}^+,\theta_{N+1}^{1-2s}) : \Psi(\theta',\theta_N,\theta_{N+1}) = -\Psi(\theta',-\theta_N,\theta_{N+1})\}. \tag{4.20}$$

It is easy to verify that $H^1_{\text{odd}}(\mathbb{S}^+,\theta_{N+1}^{1-2s})$ is a closed subspace of $H^1(\mathbb{S}^+,\theta_{N+1}^{1-2s})$.

A function $Y \in H^1_{\text{odd}}(\mathbb{S}^+,\theta_{N+1}^{1-2s})$ is an eigenfunction of (4.19) if $Y \not\equiv 0$ and

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}\, \nabla_{\mathbb{S}} Y \cdot \nabla_{\mathbb{S}}\Psi\, dS = \mu \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} Y\Psi\, dS \tag{4.21}$$

for all $\Psi \in H^1_{\text{odd}}(\mathbb{S}^+,\theta_{N+1}^{1-2s})$.

By classical spectral theory, the set of the eigenvalues of problem (4.19) is an increasing and diverging sequence of positive real numbers $\{\mu_m\}_{m\in\mathbb{N}\setminus\{0\}}$. In Section 4.7 we explicitly determine the sequence $\{\mu_m\}_{m\in\mathbb{N}\setminus\{0\}}$, obtaining that, for all $m \in \mathbb{N}\setminus\{0\}$,

$$\mu_m = \begin{cases} m^2 + m(N-2s), & \text{if } N > 1, \\ (2m-1)^2 + (2m-1)(N-2s), & \text{if } N = 1. \end{cases} \tag{4.22}$$

Let, for future reference,

V_m be the eigenspace of problem (4.19) associated to the eigenvalue μ_m, (4.23)

M_m be the dimension of V_m, (4.24)

$\{Y_{m,k} : m \in \mathbb{N}\setminus\{0\} \text{ and } k \in \{1,\dots,M_m\}\} L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})$

be an orthonormal basis of such that $\{Y_{m,k} : k = 1,\dots,M_m\}$ is a basis of V_m. (4.25)

Remark 4.1.1. Let Y be an eigenfunction of (4.19) associated to the eigenvalue $m^2+m(N-2s)$. Then Y can not vanish identically on $\mathbb{S}'$.

Indeed, if $Y \equiv 0$ on $\mathbb{S}'$, the function $V(r\theta) := r^m Y(\theta)$ would solve $\text{div}(y^{1-2s}\nabla V) = 0$ on $\mathbb{R}^{N+1}_+$, satisfying both Neumann and Dirichlet boundary condition on $\mathbb{R}^N \times \{0\}$. This would contradict the unique continuation principle for elliptic equations with weights in the Muckenhoupt A_2 class, see [79], [126], and [114, Proposition 2.2].

The main result of this Chapter is a complete classification of asymptotic blow-up profiles at a point $x_0 \in \partial\Omega$ for solutions of (4.16) and, in turn, for the corresponding solutions of (4.1).

Theorem 4.1.2. *Let $N > 2s$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $x_0 \in \partial\Omega$ and assume that there exist $R > 0$ and a function g satisfying (4.10), (4.11), and (4.12). Let u be a non trivial solution of (4.1) in the sense of (4.9), with h satisfying (4.8). Then there exists $m_0 \in \mathbb{N} \setminus \{0\}$ (which is odd in the case $N = 1$) and an eigenfunction Y of (4.19) associated to the eigenvalue $m_0^2 + m_0(N - 2s)$, such that*

$$\lambda^{-m_0} u(\lambda x + x_0) \to |x|^{m_0} \widehat{Y}\left(\frac{x}{|x|}, 0\right) \quad \text{as } \lambda \to 0^+ \quad \text{in } H^s(B_1'),$$

where $B_1' := B_1'(0)$ has been defined in (4.13), u is trivially extended to zero outside Ω as in (4.4), and

$$\widehat{Y}(\theta', \theta_N, \theta_{N+1}) = \begin{cases} Y(\theta', \theta_N, \theta_{N+1}), & \text{if } \theta_N < 0, \\ 0, & \text{if } \theta_N \geq 0. \end{cases} \tag{4.26}$$

Unlike the analogous result for the restricted fractional Laplacian established in [47], the order of homogeneity of limit profiles does not depend on s and it is always an integer. This is a consequence of the regularity of the eigenfunctions of (4.19), see Section 4.7 for further details. In particular, the eigenfunctions of (4.19), after an even reflection through the equator $\theta_{N+1} = 0$, turn out to be smooth thanks to [120, Theorem 1.1]; therefore, they are much more regular than the solutions of the corresponding problem on the half-sphere appearing in [47] and presenting mixed boundary conditions, which are responsible for a lower regularity.

Theorem 4.1.2 is proved by passing to the trace in the following blow-up result for solutions of the extended problem (4.17).

Theorem 4.1.3. *Let $N > 2s$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $x_0 \in \partial\Omega$ and assume that there exist $R > 0$ and a function g satisfying (4.10), (4.11), and (4.12). Let U be a non trivial solution to (4.17) in the sense of (4.18), with h satisfying (4.8). Then there exist $m_0 \in \mathbb{N} \setminus \{0\}$ (which is odd in the case $N = 1$) and eigenfunction Y of (4.19), associated to the eigenvalue $m_0^2 + m_0(N - 2s)$, such that, letting $z_0 = (x_0, 0)$,*

$$\lambda^{-m_0} U(\lambda z + z_0) \to |z|^{m_0} \widehat{Y}\left(\frac{z}{|z|}\right) \quad \text{as } \lambda \to 0^+ \quad \text{in } H^1(B_1^+, y^{1-2s}), \tag{4.27}$$

where $B_1^+ = \{z = (x, y) \in \mathbb{R}^N \times (0, +\infty) : |z| < 1\}$ and U is trivially extended to zero outside $\mathcal{C}_\Omega$.

In Theorem 4.6.1 a more precise characterization of the function $\widehat{Y}$ appearing in (4.26) and (4.27) is given, by writing it as a linear combination of the eigenfunctions $Y_{m_0,k}$ with coefficients computed in (4.134).

From Remark 4.1.1, Theorem 4.1.2 and Theorem 4.1.3 we deduce the following unique continuation principles.

Corollary 4.1.4. *Let $N > 2s$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $x_0 \in \partial\Omega$ and assume that there exist $R > 0$ and a function g satisfying (4.10), (4.11), and (4.12). Let u be a solution to (4.1) in the sense of (4.9) and U be a solution to (4.17) in the sense of (4.18), with h satisfying (4.8).*

(i) If $u(x) = O\left(|x - x_0|^k\right)$ as $x \to x_0$ for any $k \in \mathbb{N}$, then $u \equiv 0$ in Ω.

(ii) If $U(z) = O\left(|z - (x_0, 0)|^k\right)$ as $z \to (x_0, 0)$ for any $k \in \mathbb{N}$, then $U \equiv 0$ on $\mathcal{C}_\Omega$.

This Chapter is organized as follows. In Section 4.2 we recall some preliminary results concerning functional inequalities and trace operators. In Section 4.3 we apply the local diffeomorphism introduced in [12], see also [47, Section 2], to write an equivalent formulation of problem (4.17) on a domain with a straightened lateral boundary in a neighbourhood of x_0, see (4.35). In Section 4.4 we study the Almgren-type frequency function associated to the auxiliary problem (4.35) and prove its boundedness, which is used in Section 4.5 to develop a blow-up analysis. Finally in Section 4.6 we prove our main results and in Section 4.7 we compute the eigenvalues of problem (4.19).

4.2 Preliminaries

In this section prove some preliminary results concerning functional inequalities and trace operators.

Remark 4.2.1. Since $B_r^+ \subset B_r' \times (0,+\infty)$, the trivial extension to 0 is a linear and continuous operator from $H^1_{0,S_r^+}(B_r^+, y^{1-2s})$ to $H^1_{0,L}(\mathcal{C}_{B_r'}, y^{1-2s})$ (see (3.2)).

Proposition 4.2.2. *For every $r > 0$ the restriction to $H^1_{0,S_r^+}(B_r^+, y^{1-2s})$, (see (3.2)) of the trace operator*

$$\mathrm{Tr} : H^1(B_r^+, y^{1-2s}) \to H^s(B_r'),$$

defined in Section 3.1, coincides with the restriction of $\mathrm{Tr}_{B_r'}$ to $H^1_{0,S_r^+}(B_r^+, y^{1-2s})$. In particular, for every $r > 0$,

$$\mathrm{Tr}(H^1_{0,S_r^+}(B_r^+, y^{1-2s})) \subseteq \mathbb{H}^s(B_r').$$

Proof. By Remark 4.2.1, the operator $\mathrm{Tr}_{B_r'}$, see (4.15), is well defined on $H^1_{0,S_r^+}(B_r^+, y^{1-2s})$ and $\mathrm{Tr}_{B_r'}(H^1_{0,S_r^+}(B_r^+, y^{1-2s})) \subseteq \mathbb{H}^s(B_r')$. Furthermore for every $u \in C_c^\infty(B_r^+ \cup B_r')$, we have $\mathrm{Tr}(u) = u_{|_{B_r' \times \{0\}}} = \mathrm{Tr}_{B_r'}(u)$. By density we conclude that Tr and $\mathrm{Tr}_{B_r'}$ are equal on $H^1_{0,S_r^+}(B_r^+, y^{1-2s})$. □

The following inequality will be used to obtain estimates on the Almgren frequency function.

Proposition 4.2.3. *Let ω_N be the N-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$. For any $r > 0$, $v \in H^1(B_r^+, y^{1-2s})$ and $f \in L^{\frac{N}{2s}+\varepsilon}(B_r')$ with $\varepsilon > 0$, we have*

$$\int_{B_r'} f|\mathrm{Tr}(v)|^2\,dx \le \eta_f(r)\left(\int_{B_r^+} y^{1-2s}|\nabla v|^2\,dz + \frac{N-2s}{2r}\int_{S_r^+} y^{1-2s}v^2\,dS\right), \tag{4.28}$$

where

$$\eta_f(r) := \mathcal{S}_{N,s}\,\omega_N^{\frac{4s^2\varepsilon}{N(N+2s\varepsilon)}}\,\|f\|_{L^{\frac{N}{2s}+\varepsilon}(B_r')}\,r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}. \tag{4.29}$$

Proof. By the Hölder inequality

$$\int_{B_r'} f|\mathrm{Tr}(v)|^2\,dx \le \|\mathrm{Tr}(v)\|^2_{L^{2_s^*}(B_r')}\,\|f\|_{L^{\frac{N}{2s}+\varepsilon}(B_r')}\,\omega_N^{\frac{4s^2\varepsilon}{N(N+2s\varepsilon)}}\,r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}.$$

Then (4.28) follows from (3.5). □

4.3 Straightening the boundary

Let $x_0 \in \partial\Omega$, $R > 0$ and g satisfy (4.10), (4.11), and (4.12). Up to a suitable choice of the coordinate system, it is not restrictive to assume that

$$x_0 = 0, \quad g(0) = 0, \quad \nabla g(0) = 0.$$

Proceeding in the same way of 2.2.1, we use the local diffeomorphism F constructed in [47, Section 2] (see also [12]) to straighten the boundary of $\mathcal{C}_\Omega$ in a neighbourhood of 0; for the sake of clarity and completeness we summarize its properties in Propositions 4.3.1 and 4.3.2 below, referring to [47, Section 2] for their proofs. We consider the variable $z = (x, y) \in \mathbb{R}^N \times [0, \infty)$ with $x = (x', x_N) = (x_1, \cdots, x_N)$. For future reference we define

$$M_N := \left(\begin{array}{c|c|c} \mathrm{Id}_{N-1} & 0 & 0 \\ \hline 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \qquad M'_N := \left(\begin{array}{c|c} \mathrm{Id}_{N-1} & 0 \\ \hline 0 & -1 \end{array}\right), \tag{4.30}$$

where Id_{N-1} is the identity $(N-1) \times (N-1)$ matrix.

Proposition 4.3.1. *[47, Section 2] There exist $F = (F_1, \dots, F_{N+1}) \in C^{1,1}(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ and $r_0 > 0$ such that $F\big|_{B_{r_0}} : B_{r_0} \to F(B_{r_0})$ is a diffeomorphism of class $C^{1,1}$,*

$$\begin{aligned} &F(x', 0, 0) = (x', g(x'), 0) \quad \textit{for all } y' \in \mathbb{R}^{N-1}, \\ &F_N(x', x_N, y) = y_N + g(x') \quad \textit{for all } (x', x_N, y) \in \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}, \\ &F_{N+1}(y, t) = t, \quad \textit{for all } (x, y) \in \mathbb{R}^N \times \mathbb{R}, \\ &\alpha(x, y) := \det J_F(x, y) > 0 \quad \textit{in } B_{r_0}, \end{aligned}$$

and

$$F(\{(x', x_N, y) \in B^+_{r_0} : x_N = 0\}) = \partial_L \mathcal{C}_\Omega \cap F(B^+_{r_0}), \tag{4.31}$$

$$F(\{(x', x_N, y) \in B^+_{r_0} : x_N < 0\}) = \mathcal{C}_\Omega \cap F(B^+_{r_0}), \tag{4.32}$$

where $\partial_L \mathcal{C}_\Omega$ is defined in (4.14) and $J_F(x, y)$ is the Jacobian matrix of F. Furthermore the following properties hold:

i) *J_F depends only on the variable y and*

$$J_F(x', x_N) = J_F(y) = \mathrm{Id}_{N+1} + O(|x|) \quad \textit{as } |x| \to 0^+,$$

where Id_{N+1} denotes the identity $(N+1) \times (N+1)$ matrix and $O(|y|)$ denotes a matrix with all entries being $O(|x|)$ as $|x| \to 0^+$;

ii) *$\alpha(y) = \det J_F(x) = 1 + O(|x'|^2) + O(x_N)$ as $|y'| \to 0^+$ and $y_N \to 0$;*

iii) *$\frac{\partial F_i}{\partial y} = \frac{\partial F_{N+1}}{\partial x_i} = 0$ for any $i = 1, \dots, N$ and $\frac{\partial F_{N+1}}{\partial y} = 1$.*

For every $r > 0$, let

$$\mathcal{Q}_r := \{(x', x_N, y) \in B^+_r : x_N < 0\}, \tag{4.33}$$

so that $F(\mathcal{Q}_{r_0}) = \mathcal{C}_\Omega \cap F(B^+_{r_0})$ in view of (4.32). If $U \in H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s})$ solves (4.17), then the function

$$W = U \circ F \in H^1(\mathcal{Q}_{r_0}, y^{1-2s}) \tag{4.34}$$

is a weak solution to

$$\begin{cases} \operatorname{div}(y^{1-2s} A \nabla W) = 0, & \text{in } \mathcal{Q}_{r_0}, \\ -\lim_{y\to 0^+} y^{1-2s} \alpha \frac{\partial W}{\partial y} = \kappa_{s,N} \bar{h} W, & \text{on } \mathcal{Q}'_{r_0}, \end{cases} \tag{4.35}$$

where $\mathcal{Q}'_r := \{(x', x_N) \in B'_r : x_N < 0\}$ for all $r > 0$, $A = A(x)$ is the $(N+1)\times(N+1)$ matrix-valued function given by

$$A(x) := (J_F(x))^{-1} (J_F(x)^{-1})^T |\mathrm{det} J_F(x)|,$$

and

$$\bar{h}(x) = \alpha(y) h(F(x, 0)). \tag{4.36}$$

As observed in [47, Section 2], A has $C^{0,1}$ entries $(a_{ij})_{i,j=1}^{N+1}$ and can be written as

$$A(x) = A(x', x_N) = \left(\begin{array}{c|c} D(x', x_N) & 0 \\ \hline 0 & \alpha(x', x_N) \end{array}\right), \tag{4.37}$$

with

$$D(x', x_N) = \left(\begin{array}{c|c} \mathrm{Id}_{N-1} + O(|x'|^2) + O(y_N) & O(x_N) \\ \hline O(x_N) & 1 + O(|x'|^2) + O(y_N) \end{array}\right), \tag{4.38}$$

where Id_{N-1} is the identity $(N-1)\times(N-1)$ matrix, $O(x_N)$ and $O(|x'|^2)$ denote blocks of matrices with all elements being $O(x_N)$ as $x_N \to 0$ and $O(|x'|^2)$ as $|x'| \to 0$ respectively. In particular, in view of (4.37)-(4.38) we have

$$a_{Nj}(x', 0) = a_{jN}(x', 0) = 0 \quad \text{for all } j = 1, \dots, N-1. \tag{4.39}$$

Having in mind to reflect our problem through the hyperplane $y_N = 0$, we define

$$\widetilde{A}(x', x_N) := \begin{cases} A(x', x_N), & \text{if } x_N \le 0, \\ M_N A(x', -x_N) M_N, & \text{if } x_N > 0, \end{cases} \tag{4.40}$$

$$\widetilde{D}(x', x_N) := \begin{cases} D(x', x_N), & \text{if } x_N \le 0, \\ M'_N D(x', -x_N) M'_N, & \text{if } x_N > 0, \end{cases} \tag{4.41}$$

with M_N, M'_N as in (4.30), and

$$\widetilde{\alpha}(x', x_N) := \begin{cases} \alpha(x', x_N), & \text{if } x_N \le 0, \\ \alpha(x', -x_N), & \text{if } x_N > 0, \end{cases} \tag{4.42}$$

where $\alpha(x) = \det J_F(x)$. We observe that the Lipschitz continuity of A and (4.39) imply that the entries of $\widetilde{A}$ are of class $C^{0,1}$. Furthermore, $\widetilde{A}$ is symmetric and, possibly choosing r_0 smaller from the beginning,

$$\|\widetilde{A}(x)\|_{\mathcal{L}(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})} \le 2 \quad \text{and} \quad \frac{1}{2}|z|^2 \le \widetilde{A}(x) z \cdot z \le 2|z|^2 \quad \text{for all } z \in \mathbb{R}^{N+1},\ y \in \overline{B'_{r_0}}, \tag{4.43}$$

where $\|\cdot\|_{\mathcal{L}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})}$ denotes the operator norm on the space of bounded linear operators from $\mathbb{R}^{N+1}$ into itself. We also observe that (4.37)-(4.38) imply the expansion

$$\widetilde{A}(x) = \mathrm{Id}_{N+1} + O(|x|) \quad \text{as } |x| \to 0^+. \tag{4.44}$$

Letting $\widetilde{A}$ and $\widetilde{D}$ be as in (4.40)-(4.41), we define

$$\mu(z) := \frac{\widetilde{A}(x)z \cdot z}{|z|^2} \quad \text{and} \quad \beta(z) := \frac{\widetilde{A}(x)z}{\mu(z)} \quad \text{for every } z = (x,t) \in \overline{B_{r_0}^+} \setminus \{0\}, \tag{4.45}$$

and

$$\beta'(x) := \frac{\widetilde{D}(x)x}{\mu(x,0)} \quad \text{for every } y \in \overline{B'_{r_0}}. \tag{4.46}$$

For every $z = (z_1, \ldots, z_{N+1}) \in \mathbb{R}^{N+1}$ and $x \in \overline{B'_{r_0}}$, $d\widetilde{A}(x)zz$ is defined as the vector of $\mathbb{R}^{N+1}$ with i-th component given by

$$(d\widetilde{A}(x)zz)_i = \sum_{h,k=1}^{N+1} \frac{\partial \widetilde{a}_{kh}}{\partial z_i}(x) z_h z_k, \quad i = 1, \cdots, N+1, \tag{4.47}$$

where $(\widetilde{a}_{k,h})_{k,h=1}^{N+1}$ are the entries of the matrix $\widetilde{A}$ in (4.40).

Proposition 4.3.2. *Let μ, β, and β' be as in (4.45)-(4.46). Then, possibly choosing r_0 smaller from the beginning, we have*

$$\frac{1}{2} \leq \mu(z) \leq 2 \quad \textit{for any } z \in \overline{B_{r_0}^+} \setminus \{0\}, \tag{4.48}$$

$$\mu(z) = 1 + O(|z|), \quad \nabla\mu(z) = O(1) \quad \textit{as } |z| \to 0^+. \tag{4.49}$$

Moreover β and β' are well-defined and

$$\beta(z) = z + O(|z|^2) = O(|z|) \quad \textit{as } |z| \to 0^+, \tag{4.50}$$

$$J_\beta(z) = \widetilde{A}(x) + O(|z|) = \mathrm{Id}_{N+1} + O(|z|),\ \mathrm{div}(\beta)(z) = N + 1 + O(|z|) \ \textit{as } |z| \to 0^+, \tag{4.51}$$

$$\beta'(x) = x + O(|x|^2) = O(|x|), \quad \mathrm{div}(\beta')(x) = N + O(|x|) \quad \textit{as } |x| \to 0^+. \tag{4.52}$$

Proof. (4.48) easily follows from (4.43). We refer to [47, Lemma 2.1] for the proof of (4.49). As a direct consequence, β and β' are well-defined. From (4.50) and (4.51), whose proof is contained in [47, Lemma 2.2], we derive (4.52), after noting that β' coincides with the first N-components of the vector β. □

Remark 4.3.3. From the Lipschitz continuity of $\widetilde{A}$ observed above and Proposition 4.3.2 we have

$$\widetilde{A} \in C^{0,1}(B_{r_0}^+, \mathbb{R}^{(N+1)^2}),\ \mu \in C^{0,1}(B_{r_0}^+),\ \frac{1}{\mu} \in C^{0,1}(B_{r_0}^+),\ \beta \in C^{0,1}(B_{r_0}^+, \mathbb{R}^{N+1}) \tag{4.53}$$

$$J_\beta \in L^\infty(B_{r_0}^+, \mathbb{R}^{(N+1)^2}),\ \mathrm{div}(\beta) \in L^\infty(B_{r_0}^+),\ \beta' \in L^\infty(B'_{r_0}, \mathbb{R}^N),\ \mathrm{div}(\beta') \in L^\infty(B'_{r_0}).$$

Remark 4.3.4. If $v \in H^1_{0,L}(\mathcal{C}_\Omega, y^{1-2s})$, then $(v \circ F)|_{\mathcal{Q}_{r_0}} \in H^1(\mathcal{Q}_{r_0}, y^{1-2s})$ by Proposition 4.3.1, and

$$(v \circ F)(z) = 0 \quad \text{for any } z \in \{(x', x_N, y) \in B_{r_0}^+ : x_N = 0\} \tag{4.54}$$

in view of (4.31). Equality (4.54) is meant in the sense of the classical theory of traces for Sobolev spaces; this is possible thanks to the fact that $H^1(E, y^{1-2s}) \subset W^{1,1}(E)$ for any bounded open set $E \subseteq \mathbb{R}^N \times (0, \infty)$.

If W is a solution to (4.35), let $\widetilde{W}$ be defined as follows

$$\widetilde{W}(x', x_N, y) := \begin{cases} W(x', x_N, y), & \text{if } (x', x_N, y) \in \mathcal{Q}_{r_0}, \\ -W(x', -x_N, y), & \text{if } (x', x_N, y) \in B_{r_0}^+ \text{ and } y_N > 0. \end{cases} \tag{4.55}$$

For the sake of convenience we will still denote $\widetilde{W}$ with W. Letting $\bar{h}$ be defined in (4.36), we also consider the following function

$$\widetilde{h}(x', x_N) := \begin{cases} \bar{h}(x', x_N), & \text{if } (x', x_N) \in \mathcal{Q}'_{r_0}, \\ \bar{h}(x', -x_N), & \text{if } (x', x_N) \in B'_{r_0}, \text{ and } x_N > 0. \end{cases} \tag{4.56}$$

It is easy to verify that $W \in H^1(B_{r_0}^+, y^{1-2s})$ thanks to Remark 4.3.4 and

$$\widetilde{h} \in W^{1, \frac{N}{2s}+\varepsilon}(B'_{r_0}) \tag{4.57}$$

thanks to (4.8), (4.36) and Proposition 4.3.1. Furthermore W weakly solves

$$\begin{cases} \operatorname{div}(y^{1-2s}\widetilde{A}\nabla W) = 0, & \text{on } B_{r_0}^+, \\ -\lim_{y\to 0^+} y^{1-2s}\widetilde{\alpha}\frac{\partial W}{\partial y} = \kappa_{s,N}\widetilde{h}\operatorname{Tr}(W), & \text{on } B'_{r_0}, \end{cases} \tag{4.58}$$

with $\widetilde{\alpha}$ defined in (4.42), $\widetilde{h}$ in (4.56) and $\widetilde{A}$ in (4.40), namely

$$\int_{B_{r_0}^+} y^{1-2s}\widetilde{A}\nabla W\cdot\nabla\phi\,dz = \kappa_{s,N}\int_{B'_{r_0}} \widetilde{h}\operatorname{Tr}(W)\operatorname{Tr}(\phi)\,dy \quad \text{for all } \phi \in H^1_{0,S_{r_0}^+}(B_{r_1}^+, y^{1-2s}). \tag{4.59}$$

Thanks to Proposition 4.2.2, (4.57) and the Hölder inequality, the second member of (4.59) is well-defined.

Remark 4.3.5. In [75, Theorem 2.1] it is proved that, if $W \in H^1(B_{r_0}^+, y^{1-2s})$ is a weak solution to (4.59) with $\widetilde{A}$ and $\widetilde{h}$ satisfying (4.37), (4.40), (4.53), (4.48), (4.57), then

$$\nabla_x W \in H^1(B_r^+, y^{1-2s}) \quad \text{and} \quad y^{1-2s}\frac{\partial W}{\partial y} \in H^1(B_r^+, y^{2s-1}) \tag{4.60}$$

for all $r \in (0, r_0)$. Furthermore

$$\|\nabla_x W\|_{H^1(B_r^+, y^{1-2s})} + \left\| y^{1-2s}\frac{\partial W}{\partial y}\right\|_{H^1(B_r^+, y^{2s-1})} \le C\,\|W\|_{H^1(B_{r_0}^+, y^{1-2s})}$$

for a positive constant $C > 0$ independent of W. More precisely, C depends only on N, s, r, r_0, $\|\widetilde{h}\|_{W^{1,\frac{N}{2s}}(B'_{r_0})}$, $\|\widetilde{A}\|_{W^{1,\infty}(B_{r_0}^+, \mathbb{R}^{(N+1)^2})}$.

Remark 4.3.6. If $W \in H^1(B_{r_0}^+, y^{1-2s})$ is a weak solution to (4.59), the regularity result (4.60) and (3.4) ensure that, for all $\phi \in H^1(B_{r_0}^+, y^{1-2s})$ and $r \in (0, r_0)$, $y^{1-2s}\operatorname{Tr}_1(\widetilde{D}\nabla_x W\cdot x)\operatorname{Tr}_1\phi \in L^1(S_r^+)$; moreover the function

$$r \mapsto \int_{S_r^+} y^{1-2s}(\widetilde{D}\nabla_x W\cdot x)\phi\,dS$$

is continuous in $(0, r_0)$. Furthermore, since $y^{1-2s}\frac{\partial W}{\partial y} \in H^1(B_r^+, y^{2s-1})$ for all $r \in (0, r_0)$ by (4.60), for all $\phi \in H^1(B_{r_0}^+, y^{1-2s})$ and $r \in (0, r_0)$ we also have $y^{1-2s}\widetilde{\alpha}\frac{\partial W}{\partial y} y\phi \in W^{1,1}(B_r^+)$, so that $\mathrm{Tr}_1(y^{1-2s}\widetilde{\alpha}\frac{\partial W}{\partial y} y\phi) \in L^1(S_r^+)$; moreover the function

$$r \mapsto \int_{S_r^+} y^{1-2s}\widetilde{\alpha}\frac{\partial W}{\partial y} y\phi \, dS$$

is continuous in $(0, r_0)$. We conclude that, for all $\phi \in H^1(B_{r_0}^+, y^{1-2s})$, the function

$$y^{1-2s}(\widetilde{A}\nabla W \cdot z)\phi = y^{1-2s}(\widetilde{D}\nabla_x W \cdot x)\phi + y^{1-2s}\widetilde{\alpha}\frac{\partial W}{\partial y} t\phi$$

has a trace on S_r^+ for all $r \in (0, r_0)$ and the function

$$r \mapsto \int_{S_r^+} y^{1-2s}(\widetilde{A}\nabla W \cdot z)\phi \, dS$$

is continuous in $(0, r_0)$.

The following result provides an integration by parts formula which will be useful in Section 4.5.

Proposition 4.3.7. *Let W be a weak solution to (4.58). For all $r \in (0, r_0)$ and $\phi \in H^1(B_{r_0}^+, y^{1-2s})$*

$$\int_{B_r^+} y^{1-2s}\widetilde{A}\nabla W \cdot \nabla\phi \, dz = \frac{1}{r}\int_{S_r^+} y^{1-2s}(\widetilde{A}\nabla W \cdot z)\phi \, dS + \kappa_{s,N}\int_{B_r'} \widetilde{h}\,\mathrm{Tr}(W)\,\mathrm{Tr}(\phi)\, dx. \quad (4.61)$$

Proof. By density it is enough to prove (4.61) for $\phi \in C^\infty(\overline{B_{r_0}^+})$. Let $r \in (0, r_0)$. For every $n \in \mathbb{N}$, let

$$\eta_n(z) := \begin{cases} 1, & \text{if } \ 0 \le |z| \le r - \frac{1}{n}, \\ n(r - |z|), & \text{if } \ r - \frac{1}{n} \le |z| \le r, \\ 0, & \text{if } \ |z| \ge r. \end{cases}$$

Testing (4.59) with $\phi\eta_n$ and passing to the limit as $n \to \infty$, we obtain (4.61) thanks to the integral mean value theorem and Remark 4.3.6. □

Remark 4.3.8. For all $r \in (0, r_0]$ and any $v \in H^1(B_r^+, y^{1-2s})$, thanks to (4.28), (4.43) and (4.48),

$$\int_{B_r^+} y^{1-2s}|\nabla v|^2 \, dz \le 2\int_{B_r^+} y^{1-2s}\widetilde{A}\nabla v \cdot \nabla v \, dz - 2\kappa_{N,s}\int_{B_r'} \widetilde{h}|\,\mathrm{Tr}(v)|^2 \, dx$$
$$+ 2\kappa_{N,s}\eta_{\tilde{h}}(r)\left(\int_{B_r^+} y^{1-2s}|\nabla v|^2 \, dz + \frac{N-2s}{r}\int_{S_r^+} y^{1-2s}\mu v^2 \, dS\right).$$

Therefore, if $\eta_{\tilde{h}}(r) < \frac{1}{2\kappa_{N,s}}$,

$$\int_{B_r^+} y^{1-2s}|\nabla v|^2 \, dz \le \frac{2}{1 - 2\kappa_{N,s}\eta_{\tilde{h}}(r)}\left(\int_{B_r^+} y^{1-2s}\widetilde{A}\nabla v \cdot \nabla v \, dz - \kappa_{N,s}\int_{B_r'} \widetilde{h}|\,\mathrm{Tr}(v)|^2 \, dx\right)$$
$$+ \frac{2(N-2s)\kappa_{N,s}\eta_{\tilde{h}}(r)}{(1 - 2\kappa_{N,s}\eta_{\tilde{h}}(r))r}\int_{S_r^+} y^{1-2s}\mu v^2 \, dS. \quad (4.62)$$

4.4 The Monotonicity Formula

Let W be a non trivial weak solution of (4.58). For any $r \in (0, r_0]$ we define the height function and the energy function as

$$H(r) := \frac{1}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} \mu W^2 \, dS, \tag{4.63}$$

$$D(r) := \frac{1}{r^{N-2s}} \left(\int_{B_r^+} y^{1-2s} \widetilde{A} \nabla W \cdot \nabla W \, dz - \kappa_{N,s} \int_{B_r'} \widetilde{h} |\operatorname{Tr} W|^2 \, dx \right), \tag{4.64}$$

respectively. Eventually choosing r_0 smaller from the beginning, we may assume that

$$\eta_{\widetilde{h}}(r) < \frac{1}{4\kappa_{N,s}} \quad \text{for all } r \in (0, r_0], \tag{4.65}$$

so that (4.62) holds for every $r \in (0, r_0]$.

Proposition 4.4.1. *Let H and D be as in (4.63) and (4.64). Then $H \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and*

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} \mu W \frac{\partial W}{\partial \nu} \, dS + H(r) O(1) \quad \text{as } r \to 0^+ \tag{4.66}$$

in the sense of distributions and almost everywhere, where ν is the outer normal vector to B_r^+ on S_r^+, i.e. $\nu(z) := \frac{z}{|z|}$. Moreover, almost everywhere we have

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} (\widetilde{A} \nabla W \cdot \nu) W \, dS + H(r) O(1) \quad \text{as } r \to 0^+ \tag{4.67}$$

and

$$H'(r) = \frac{2}{r} D(r) + H(r) O(1) \quad \text{as } r \to 0^+. \tag{4.68}$$

Proof. The proof is similar to that of [47, Lemma 3.1] thus we omit it. □

Proposition 4.4.2. *We have $H(r) > 0$ for every $r \in (0, r_0]$.*

Proof. Let us assume by contradiction that there exists $r \in (0, r_0]$ such that $H(r) = 0$. Then, from (4.63) and (4.48) we deduce that $W \equiv 0$ on S_r^+. Thus we can test (4.59) with W, obtaining that

$$0 = \int_{B_r^+} y^{1-2s} \widetilde{A} \nabla W \cdot \nabla W \, dz - \kappa_{N,s} \int_{B_r'} \widetilde{h} |\operatorname{Tr}(W)|^2 \, dx \geq \left(\frac{1}{2} - \kappa_{N,s} \eta_{\widetilde{h}}(r) \right) \|\nabla W\|^2_{L^2(B_r^+, y^{1-2s})},$$

thanks to (4.62). Then, by (4.65) we can conclude that $W \equiv 0$ on B_r^+; this implies that $W \equiv 0$ on $B_{r_0}^+$ by classical unique continuation principles for second order elliptic operators with Lipschitz coefficients (see e.g. [79]), giving rise to a contradiction. □

The following proposition contains a Pohozaev-type identity for problem (4.58). For its proof we refer to [75, Proposition 2.3], where a more general version is established exploiting some Sobolev-type regularity results.

Proposition 4.4.3. *[75, Proposition 2.3] Let W be a weak solution to equation (4.58). Then, for a.e. $r \in (0, r_0)$,*

$$\begin{aligned}
&\int_{S_r^+} y^{1-2s} \widetilde{A}\nabla W \cdot \nabla W \, dS - \kappa_{N,s} \int_{S_r'} \widetilde{h} |\operatorname{Tr}(W)|^2 \, dS' \\
&= 2 \int_{S_r^+} y^{1-2s} \frac{|\widetilde{A}\nabla W \cdot \nu|^2}{\mu} \, dS - \frac{\kappa_{N,s}}{r} \int_{B_r'} (\operatorname{div}_y(\beta')\widetilde{h} + \beta' \cdot \nabla \widetilde{h}) |\operatorname{Tr}(W)|^2 \, dy \\
&+ \frac{1}{r} \int_{B_r^+} y^{1-2s} \widetilde{A}\nabla W \cdot \nabla W \operatorname{div}(\beta) \, dz - \frac{2}{r} \int_{B_r^+} y^{1-2s} J_\beta(\widetilde{A}\nabla W) \cdot \nabla W \, dz \\
&+ \frac{1}{r} \int_{B_r^+} y^{1-2s} (d\widetilde{A}\, \nabla W \, \nabla W) \cdot \beta \, dz + \frac{1-2s}{r} \int_{B_r^+} y^{1-2s} \frac{\widetilde{\alpha}}{\mu} \widetilde{A}\nabla W \cdot \nabla W \, dz,
\end{aligned} \tag{4.69}$$

where μ and β are defined in (4.45), $\widetilde{\alpha}$ in (4.42), β' in (4.46), ν is the outer normal vector to B_r^+ on S_r^+, i.e. $\nu(z) = \frac{z}{|z|}$, and dS' denotes the volume element on $(N-1)$-dimensional spheres.

Remark 4.4.4. As in Remark 4.3.6, by the Coarea Formula we have

$$\int_{B_{r_0}'} |\widetilde{h}| |\operatorname{Tr}(W)|^2 \, dx = \int_0^{r_0} \left(\int_{S_\rho'} |\widetilde{h}| |\operatorname{Tr}(W)|^2 \, dS' \right) d\rho,$$

hence $\rho \to \int_{S_\rho'} \widetilde{h} |\operatorname{Tr}(W)|^2 \, dS'$ is a well-defined $L^1(0, r_0)$-function, as a consequence of (4.57), (3.5) and the Hölder inequality.

Proposition 4.4.5. *Let D be as in (4.64). Then $D \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and*

$$D'(r) = 2r^{2s-N} \int_{S_r^+} y^{1-2s} \frac{|\widetilde{A}\nabla W \cdot \nu|^2}{\mu} \, dS + O\left(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right) \left[D(r) + \frac{N-2s}{2} H(r) \right] \tag{4.70}$$

as $r \to 0^+$, in the sense of distributions and almost everywhere.

Proof. By the Coarea Formula $D \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and

$$\begin{aligned}
D'(r) = (2s-N) r^{2s-N-1} &\left(\int_{B_r^+} y^{1-2s} \widetilde{A}\nabla W \cdot \nabla W \, dz - \kappa_{N,s} \int_{B_r'} \widetilde{h} |\operatorname{Tr}(W)|^2 \, dx \right) \\
&+ r^{2s-N} \left(\int_{S_r^+} y^{1-2s} \widetilde{A}\nabla W \cdot \nabla W \, dS - \kappa_{N,s} \int_{S_r'} \widetilde{h} |\operatorname{Tr}(W)|^2 \, dS' \right)
\end{aligned} \tag{4.71}$$

a.e. and in the sense of distributions in $(0, r_0)$. Using (4.69) to estimate the second term on the right hand side of (4.71), for a.e. $r \in (0, r_0)$ we have

$$\begin{aligned}
&D'(r) = (2s-N) r^{2s-N-1} \left(\int_{B_r^+} y^{1-2s} \widetilde{A}\nabla W \cdot \nabla W \, dz - \kappa_{N,s} \int_{B_r'} \widetilde{h} |\operatorname{Tr}(W)|^2 \, dx \right) \\
&+ r^{2s-N} \left(2 \int_{S_r^+} y^{1-2s} \frac{|\widetilde{A}\nabla W \cdot \nu|^2}{\mu} \, dS - \frac{\kappa_{N,s}}{r} \int_{B_r'} (\operatorname{div}_y(\beta')\widetilde{h} + \beta' \cdot \nabla \widetilde{h}) |\operatorname{Tr}(W)|^2 \, dy \right) \\
&+ r^{2s-N} \left(\frac{1}{r} \int_{B_r^+} y^{1-2s} \widetilde{A}\nabla W \cdot \nabla W \operatorname{div}(\beta) \, dz - \frac{2}{r} \int_{B_r^+} y^{1-2s} J_\beta(\widetilde{A}\nabla W) \cdot \nabla W \, dz \right) \\
&+ r^{2s-N} \left(\frac{1}{r} \int_{B_r^+} y^{1-2s} (d\widetilde{A}\nabla W \nabla W) \cdot \beta \, dz + \frac{1-2s}{r} \int_{B_r^+} y^{1-2s} \frac{\widetilde{\alpha}}{\mu} \widetilde{A}\nabla W \cdot \nabla W \, dz \right).
\end{aligned} \tag{4.72}$$

Furthermore, thanks to point ii) of Proposition 4.3.1, (4.42), (4.43), (4.48), (4.49), (4.50), (4.51), and (4.62), we deduce that

$$
\begin{aligned}
&r^{2s-N-1}\int_{B_r^+} y^{1-2s}\left[\left(2s-N+\operatorname{div}(\beta)+(1-2s)\tfrac{\widetilde{\alpha}}{\mu}\right)\widetilde{A}\nabla W\cdot\nabla W-2J_\beta(\widetilde{A}\nabla W)\cdot\nabla W\right]dz\\
&+r^{2s-N-1}\int_{B_r^+} y^{1-2s}(d\widetilde{A}\,\nabla W\,\nabla W)\cdot\beta\,dz=O(r)\,r^{2s-N-1}\int_{B_r^+} y^{1-2s}|\nabla W|^2\,dz\\
&=O(1)\left[D(r)+\frac{N-2s}{2}H(r)\right]\quad\text{as } r\to 0^+,
\end{aligned}
\tag{4.73}
$$

where we used also the fact that $d\widetilde{A}\,\nabla W\,\nabla W=O(1)|\nabla W|^2$ as $r\to 0^+$ by (4.47) and (4.53).

In addition, recalling that $\tilde{h}\in W^{1,\frac{N}{2s}+\varepsilon}(B'_{r_1})$, from (4.28), (4.29), (4.53) and (4.62) it follows that

$$
\begin{aligned}
&r^{2s-N-1}\int_{B'_r}[(2s-N+\operatorname{div_y}(\beta'))\widetilde{h}+\beta'\cdot\nabla\widetilde{h}]|\operatorname{Tr}(W)|^2\,dx\\
&\qquad\qquad=O\left(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)\left[D(r)+\frac{N-2s}{2}H(r)\right]
\end{aligned}
\tag{4.74}
$$

as $r\to 0^+$. Combining (4.72), (4.73) and (4.74), we obtain (4.70). □

For every $r\in(0,r_0]$ we define the *frequency function*

$$
\mathcal{N}(r):=\frac{D(r)}{H(r)}. \tag{4.75}
$$

Definition (4.75) is well-posed thanks to Proposition 4.4.2.

Proposition 4.4.6. *We have $\mathcal{N}\in W^{1,1}_{\mathrm{loc}}((0,r_0])$ and*

$$
\mathcal{N}(r)>-\frac{N-2s}{2}\quad\textit{for every } r\in(0,r_0]. \tag{4.76}
$$

Furthermore, if $\nu(z):=\frac{z}{|z|}$ is the outer normal vector to B_r^+ on S_r^+ and

$$
\mathcal{V}(r):=2r\frac{\left(\int_{S_r^+}y^{1-2s}\mu W^2\,dS\right)\left(\int_{S_r^+}y^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}\,dS\right)-\left(\int_{S_r^+}y^{1-2s}W\,A\nabla W\cdot\nu\,dS\right)^2}{\left(\int_{S_r^+}y^{1-2s}\mu W^2\,dS\right)^2},
$$

then

$$
\mathcal{V}(r)\geq 0\quad\textit{for a.e. } r\in(0,r_0) \tag{4.77}
$$

and, for a.e. $r\in(0,r_0)$,

$$
\mathcal{N}'(r)-\mathcal{V}(r)=O\left(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)\left[\mathcal{N}(r)+\frac{N-2s}{2}\right]\quad\textit{as } r\to 0^+. \tag{4.78}
$$

Proof. Since $D\in W^{1,1}_{\mathrm{loc}}((0,r_0])$ and $\frac{1}{H}\in W^{1,1}_{\mathrm{loc}}((0,r_0])$ by Proposition 4.4.1 and Proposition 4.4.2, then $\mathcal{N}\in W^{1,1}_{\mathrm{loc}}((0,r_0])$. Furthermore we recall that (4.62) holds for every $r\in(0,r_1]$, thus

$$
\mathcal{N}(r)\geq-\kappa_{N,s}(N-2s)\eta_{\tilde{h}}(r), \tag{4.79}
$$

for every $r \in (0, r_0]$ and, in virtue of this, (4.76) directly follows from (4.65). Moreover (4.77) is a consequence of the Cauchy-Schwarz inequality in $L^2(S_r^+, y^{1-2s})$. From (4.67), (4.68) and (4.70) we deduce that

$$\begin{aligned}
\mathcal{N}'(r) =& \frac{D'(r)H(r) - D(r)H'(r)}{(H(r))^2} = \frac{D'(r)H(r) - \frac{r}{2}(H'(r))^2 + O(r)H(r)H'(r)}{(H(r))^2} \\
=& \mathcal{V}(r) + O(r) + O(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}) \left[\mathcal{N}(r) + \frac{N-2s}{2}\right] \\
&+ \frac{O(r^{-N+2s})}{H(r)} \int_{S_r^+} y^{1-2s}(A\nabla W \cdot \nu) W \, dS
\end{aligned} \tag{4.80}$$

as $r \to 0^+$. In order to deal with the last term in (4.80), we observe that, for a.e. $r \in (0, r_0)$,

$$\int_{S_r^+} y^{1-2s}(A\nabla W \cdot \nu) W \, dS = r^{N-2s} D(r) + H(r) O(r^{N+1-2s}) \quad \text{as } r \to 0^+,$$

in virtue of (4.67) and (4.68). Thus, substituting into (4.80), we conclude that

$$\mathcal{N}'(r) = \mathcal{V}(r) + O(r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}) \left[\mathcal{N}(r) + \frac{N-2s}{2}\right] \quad \text{as } r \to 0^+,$$

where we have used that $\frac{4s^2\varepsilon}{N+2s\varepsilon} < 1$ since $\varepsilon \in (0,1)$ and $N > 2s$. Estimate (4.78) is thereby proved. □

Proposition 4.4.7. *There exists a constant $C > 0$ such that, for every $r \in (0, r_0]$,*

$$\mathcal{N}(r) \le C. \tag{4.81}$$

Proof. From (4.77) and (4.78) we deduce that there exists a constant $c > 0$ such that

$$\left(\mathcal{N}(r) + \frac{N-2s}{2}\right)' \ge -c\, r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}} \left(\mathcal{N}(r) + \frac{N-2s}{2}\right) \quad \text{for a.e. } r \in (0, r_1), \tag{4.82}$$

for some $r_1 \in (0, r_0)$ sufficiently small. Hence, thanks to (4.76), we are allowed to divide each member of (4.82) by $\mathcal{N}(r) + \frac{N-2s}{2}$, obtaining that

$$\left(\log\left(\mathcal{N}(r) + \frac{N-2s}{2}\right)\right)' \ge -c\, r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}} \quad \text{for a.e. } r \in (0, r_1).$$

Then, integrating over (r, r_1) with $r < r_1$, we have

$$\mathcal{N}(r) \le -\frac{N-2s}{2} + \exp\left(c\, \frac{N+2s\varepsilon}{4s^2\varepsilon} r_1^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right) \left(\mathcal{N}(r_1) + \frac{N-2s}{2}\right) \quad \text{for every } r \in (0, r_1),$$

which proves (4.81), taking into account the continuity of $\mathcal{N}$ in $(0, r_0]$. □

Proposition 4.4.8. *There exists the limit*

$$\gamma := \lim_{r \to 0^+} \mathcal{N}(r). \tag{4.83}$$

Moreover γ is finite and $\gamma \ge 0$.

Proof. Combining (4.81) and (4.82), we infer that

$$\left(\mathcal{N}(r)+\frac{N-2s}{2}\right)' \geq -c\, r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\left(C+\frac{N-2s}{2}\right) \tag{4.84}$$

for a.e. $r\in(0,r_1)$, hence

$$\left(\frac{N-2s}{2}+\mathcal{N}(r)+c\left(\frac{N-2s}{2}+C\right)\frac{N+2s\varepsilon}{4s^2\varepsilon}r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)' \geq 0 \quad \text{for a.e. } r\in(0,r_1).$$

From this, it follows in particular that the limit γ in (4.83) exists. Moreover, by (4.76) and (4.81) γ is finite, whereas (4.79) implies that $\gamma\geq 0$. □

Proposition 4.4.9. *There exist $c_0,\bar{c}>0$ and $\bar{r}\in(0,r_0)$ such that*

$$H(r)\leq c_0\, r^{2\gamma} \quad \textit{for all } r\in(0,r_0] \tag{4.85}$$

and

$$H(Rr)\leq R^{\bar{c}}\, H(r) \quad \textit{for all } R\geq 1 \textit{ and } r\in(0,\tfrac{\bar{r}}{R}]. \tag{4.86}$$

Furthermore, for any $\sigma>0$ there exists a constant $c_\sigma>0$ such that

$$H(r)\geq c_\sigma r^{2\gamma+\sigma} \quad \textit{for all } r\in(0,r_0]. \tag{4.87}$$

Proof. By (4.83) we have $\mathcal{N}(r)=\gamma+\int_0^r\mathcal{N}'(t)\,dt$; hence from (4.68) it follows that

$$\frac{H'(r)}{H(r)}=\frac{2}{r}\mathcal{N}(r)+O(1)=\frac{2}{r}\int_0^r\mathcal{N}'(t)\,dt+\frac{2\gamma}{r}+O(1). \tag{4.88}$$

From (4.84) and up to choosing r_1 smaller, it follows that, for a.e. $r\in(0,r_1)$,

$$\frac{H'(r)}{H(r)}\geq -\kappa r^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}}+\frac{2\gamma}{r}$$

for some positive constant $\kappa>0$. Then an integration over (r,r_1) yields

$$\log\left(\frac{H(r_1)}{H(r)}\right)\geq -\kappa\frac{N+2s\varepsilon}{4s^2\varepsilon}\left(r_1^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}-r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)+\log\left(\frac{r_1}{r}\right)^{2\gamma}$$

and thus

$$H(r)\leq\frac{H(r_1)}{r_1^{2\gamma}}\exp\left(\kappa\frac{N+2s\varepsilon}{4s^2\varepsilon}r_1^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)r^{2\gamma}$$

for all $r\in(0,r_1]$, thus implying (4.85) thanks to the continuity of H in $(0,r_0]$.

To prove (4.86), we observe that (4.88) and (4.81) imply that, for some $\bar{r}\in(0,r_0)$ and $\bar{c}>0$,

$$\frac{H'(r)}{H(r)}\leq\frac{\bar{c}}{r} \quad \text{for all } r\in(0,\bar{r}),$$

whose integration over (r,rR) directly gives (4.86).

In view of Proposition 4.4.8, for any $\sigma>0$ there exists $r_\sigma\in(0,r_0]$ such that

$$\frac{H'(r)}{H(r)}=\frac{2}{r}\mathcal{N}(r)+O(1)\leq\frac{2\gamma+\sigma}{r} \quad \text{for all } r\in(0,r_\sigma].$$

Integrating over (r,r_σ) and recalling that H is continuous in $(0,r_0]$, we deduce (4.87). □

Proposition 4.4.10. *There exists the limit* $\lim_{r\to 0^+} r^{-2\gamma}H(r)$ *and it is finite.*

Proof. By (4.85) it is sufficient to show that the limit does exist. In view of (4.68) we have

$$\left(\frac{H(r)}{r^{2\gamma}}\right)' = \frac{r^{2\gamma}H'(r) - 2\gamma r^{2\gamma-1}H(r)}{r^{4\gamma}} = 2r^{-2\gamma-1}(D(r) - \gamma H(r)) + r^{-2\gamma}O(1)H(r)$$
$$= 2r^{-2\gamma-1}H(r)\left(\mathcal{N}(r) - \gamma + rO(1)\right)$$
$$= 2r^{-2\gamma-1}H(r)\left(\int_0^r \left[\mathcal{N}'(t) - \mathcal{V}(t)\right]\, dt + \int_0^r \mathcal{V}(t)\, dt + rO(1)\right)$$

as $r \to 0^+$. Integrating over $(r, \tilde{r})$ with $\tilde{r} \in (0, r_0)$ small, we obtain that

$$\frac{H(\tilde{r})}{\tilde{r}^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{\tilde{r}} 2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \mathcal{V}(t)\, dt\right) d\rho$$
$$+ \int_r^{\tilde{r}} \left[2\rho^{-2\gamma}H(\rho)O(1) + 2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \left[\mathcal{N}'(t) - \mathcal{V}(t)\right]\, dt\right)\right] d\rho. \tag{4.89}$$

Letting

$$f(\rho) := 2\rho^{-2\gamma}H(\rho)O(1) + 2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \left[\mathcal{N}'(t) - \mathcal{V}(t)\right]\, dt\right),$$

from (4.78), (4.81) and (4.85) it follows that $f \in L^1(0, \tilde{r})$ and hence there exists the limit

$$\lim_{r\to 0^+} \int_r^{\tilde{r}} f(\rho)\, d\rho = \int_0^{\tilde{r}} f(\rho)\, d\rho < +\infty.$$

On the other hand, in view of (4.77), there exists the limit

$$\lim_{r\to 0^+} \int_r^{\tilde{r}} 2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \mathcal{V}(t)\, dt\right) d\rho.$$

Therefore we can conclude thanks to (4.89). □

4.5 The blow-up analysis

In the present section, we aim to classify the possible vanishing orders of solutions to (4.58). To this purpose, let W be a non trivial weak solution to (4.58) and H be defined in (4.63). For any $\lambda \in (0, r_0]$, we consider the function

$$V^\lambda(z) := \frac{W(\lambda z)}{\sqrt{H(\lambda)}}. \tag{4.90}$$

It is easy to verify that V^λ weakly solves

$$\begin{cases} \operatorname{div}(y^{1-2s}\widetilde{A}(\lambda\cdot)\nabla V^\lambda) = 0, & \text{on } B^+_{r_0\lambda^{-1}}, \\ -\lim_{y\to 0^+} y^{1-2s}\widetilde{\alpha}(\lambda\cdot)\frac{\partial V^\lambda}{\partial y} = \kappa_{s,N}\lambda^{2s}\widetilde{h}(\lambda\cdot)\operatorname{Tr}(V^\lambda), & \text{on } B'_{r_0\lambda^{-1}}, \end{cases}$$

where we have defined $\widetilde{\alpha}$ in (4.42). It follows that, for any $\lambda \in (0, r_0]$,

$$\int_{B_1^+} y^{1-2s}\widetilde{A}(\lambda\cdot)\nabla V^\lambda \cdot \nabla\phi\, dz - \kappa_{s,N}\lambda^{2s}\int_{B_1'} \widetilde{h}(\lambda\cdot)\operatorname{Tr}(V^\lambda)\operatorname{Tr}(\phi)\, dy = 0 \tag{4.91}$$

for every $\phi \in H^1_{0,S_1^+}(B_1^+, y^{1-2s})$, (see (3.2)). Furthermore by (4.63) and (4.90)

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}\mu(\lambda\theta)|V^\lambda(\theta)|^2\, dS = 1 \quad \text{for any } \lambda \in (0, r_0]. \tag{4.92}$$

Proposition 4.5.1. *For every $R \geq 1$, the family of functions $\{V^\lambda : \lambda \in (0, \frac{\bar r}{R}]\}$ is bounded in $H^1(B_R^+, y^{1-2s})$.*

Proof. By (4.62) and (4.86), for all $\lambda \in (0, \frac{\bar r}{R}]$ with $\bar r$ as in Lemma 4.4.9, we have

$$\int_{B_R^+} y^{1-2s}|\nabla V^\lambda|^2\, dz = \frac{\lambda^{2s-N}}{H(\lambda)} \int_{B_{\lambda R}^+} y^{1-2s}|\nabla W|^2\, dz \leq \frac{\lambda^{2s-N} R^{\bar c}}{H(\lambda R)} \int_{B_{\lambda R}^+} y^{1-2s}|\nabla W|^2\, dz$$
$$\leq \frac{2R^{\bar c+N-2s}}{1-2\kappa_{N,s}\eta_{\tilde h}(\lambda R)}\mathcal{N}(\lambda R) + \frac{2(N-2s)R^{\bar c+N-2s}\kappa_{N,s}\eta_{\tilde h}(\lambda R)}{1-2\kappa_{N,s}\eta_{\tilde h}(\lambda R)},$$

which, together with (4.65) and (4.81), allows us to deduce that $\{\nabla V^\lambda : \lambda \in (0, \frac{\bar r}{R}]\}$ is uniformly bounded in $L^2(B_R^+, y^{1-2s})$. On the other hand, (4.48), a scaling argument, and (4.86) imply that

$$\int_{S_R^+} y^{1-2s}|V^\lambda|^2 dS = \frac{\lambda^{-N-1+2s}}{H(\lambda)} \int_{S_{R\lambda}^+} y^{1-2s} W^2 dS \leq 2R^{N+1-2s}\frac{H(R\lambda)}{H(\lambda)} \leq 2R^{N+1-2s+\bar c},$$

so that the claim follows from (3.7). ☐

Proposition 4.5.2. *Let W be a non trivial weak solution to (4.58). Let γ be as in Proposition 4.4.8. There exists $m_0 \in \mathbb{N}\setminus\{0\}$ (which is odd in the case $N = 1$) such that*

$$\gamma = m_0. \tag{4.93}$$

Furthermore, for any sequence $\{\lambda_n\}$ such that $\lambda_n \to 0^+$ as $n \to \infty$, there exist a subsequence $\{\lambda_{n_k}\}$ and an eigenfunction Ψ of problem (4.19) associated with the eigenvalue $\mu_{m_0} = m_0^2 + m_0(N-2s)$ such that $\|\Psi\|_{L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})} = 1$ and

$$\frac{W(\lambda_{n_k} z)}{\sqrt{H(\lambda_{n_k})}} \to |z|^\gamma \Psi\left(\frac{z}{|z|}\right) \quad \text{as } k \to +\infty \qquad \text{strongly in } H^1(B_1^+, y^{1-2s}). \tag{4.94}$$

Proof. Let W be a non trivial weak solution to (4.58) and $\{\lambda_n\}$ be a sequence such that $\lambda_n \to 0^+$ as $n \to +\infty$. Thanks to Proposition 4.5.1, there exist a subsequence $\{\lambda_{n_k}\}$ and $V \in H^1(B_1^+, y^{1-2s})$ such that

$$V^{\lambda_{n_k}} \rightharpoonup V \quad \text{weakly in } H^1(B_1^+, y^{1-2s}) \text{ as } k \to +\infty. \tag{4.95}$$

For sufficiently large k we have $\lambda_{n_k} \in (0, r_0)$ and thus $B_1^+ \subset B_{r_0/\lambda_{n_k}}^+$, hence from (4.91) we deduce that

$$\int_{B_1^+} y^{1-2s}\widetilde A(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}} \cdot \nabla\phi\, dz = \kappa_{s,N}\lambda_{n_k}^{2s}\int_{B_1'} \tilde h(\lambda_{n_k}\cdot)\,\mathrm{Tr}(V^{\lambda_{n_k}})\,\mathrm{Tr}(\phi)\, dy \tag{4.96}$$

for every $\phi \in H^1_{0,S_1^+}(B_1^+, y^{1-2s})$ (see (3.2)). In order to study what happens as $k \to +\infty$, we notice that the term on the left hand side of (4.96) can be rewritten as follows

$$\int_{B_1^+} y^{1-2s}\widetilde A(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}} \cdot \nabla\phi\, dz$$
$$= \int_{B_1^+} y^{1-2s}(\widetilde A(\lambda_{n_k}\cdot) - \mathrm{Id}_{N+1})\nabla V^{\lambda_{n_k}} \cdot \nabla\phi\, dz + \int_{B_1^+} y^{1-2s}\nabla V^{\lambda_{n_k}} \cdot \nabla\phi\, dz. \tag{4.97}$$

Therefore, in view of (4.44), Proposition 4.5.1 and (4.95), we conclude that

$$\lim_{k\to+\infty}\int_{B_1^+} y^{1-2s}\widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot\nabla\phi\,dz=\int_{B_1^+} y^{1-2s}\nabla V\cdot\nabla\phi\,dz. \tag{4.98}$$

As for the right hand side in (4.96), we have

$$\left|\lambda_{n_k}^{2s}\int_{B_1'}\widetilde{h}(\lambda_{n_k}\cdot)\,\mathrm{Tr}(V^{\lambda_{n_k}})\,\mathrm{Tr}(\phi)\,dy\right|\le\lambda_{n_k}^{2s}\eta_{\widetilde{h}(\lambda_{n_k}\cdot)}(1)$$
$$\times\left(\int_{B_1^+} y^{1-2s}|\nabla\phi|^2\,dy\right)^{\frac12}\left(\int_{B_1^+} y^{1-2s}|\nabla V^{\lambda_{n_k}}|^2\,dz+\frac{N-2s}{2}\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}|V^{\lambda_{n_k}}|^2\,dS\right)^{\frac12} \tag{4.99}$$

thanks to Hölder's inequality and (4.28). By (4.29) and the change of variable $x\mapsto\lambda_{n_k}x$, we obtain that

$$\lambda_{n_k}^{2s}\eta_{\widetilde{h}(\lambda_{n_k}\cdot)}(1)=\mathcal{S}_{N,s}\omega_N^{\frac{4s^2\varepsilon}{N(N+2s\varepsilon)}}\lambda_{n_k}^{2s}\|\widetilde{h}(\lambda_{n_k}\cdot)\|_{L^{\frac{N}{2s}+\varepsilon}(B_1')}$$
$$=\mathcal{S}_{N,s}\omega_N^{\frac{4s^2\varepsilon}{N(N+2s\varepsilon)}}\|\widetilde{h}\|_{L^{\frac{N}{2s}+\varepsilon}(B'_{\lambda_{n_k}})}\lambda_{n_k}^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}. \tag{4.100}$$

Putting together (4.99) and (4.100), thanks to Proposition 4.5.1, (4.92), and (4.48) we infer that

$$\lim_{k\to+\infty}\lambda_{n_k}^{2s}\int_{B_1'}\widetilde{h}(\lambda_{n_k}\cdot)\,\mathrm{Tr}(V^{\lambda_{n_k}})\,\mathrm{Tr}(\phi)\,dy=0. \tag{4.101}$$

Passing to the limit as $k\to+\infty$ in (4.96) we conclude that V weakly solves the following problem:

$$\begin{cases}\mathrm{div}(y^{1-2s}\nabla V)=0, & \text{in } B_1^+,\\ \lim_{y\to0^+}y^{1-2s}\frac{\partial V}{\partial y}=0, & \text{on } B_1'.\end{cases} \tag{4.102}$$

In particular V is smooth on B_1^+ and $V\not\equiv0$ since, by (4.49), (4.95) and the compactness of the trace operator in (3.4), (4.92) leads to

$$\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}V^2\,dS=1. \tag{4.103}$$

Now we aim to show that, along a further subsequence,

$$V^{\lambda_{n_k}}\to V\quad\text{strongly in } H^1(B_1^+,y^{1-2s}) \text{ as } k\to+\infty. \tag{4.104}$$

To this purpose, we first notice that a change of variables in (4.61) yields

$$\int_{B_1^+} y^{1-2s}\widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot\nabla\phi\,dz-\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}\widetilde{A}(\lambda_{n_k}\cdot)\nabla V^{\lambda_{n_k}}\cdot z\,\phi\,dS$$
$$=\kappa_{s,N}\lambda_{n_k}^{2s}\int_{B_1'}\widetilde{h}(\lambda_{n_k}\cdot)\,\mathrm{Tr}(V^{\lambda_{n_k}})\,\mathrm{Tr}(\phi)\,dy \tag{4.105}$$

for any $\phi\in H^1(B_1^+,y^{1-2s})$ and k sufficiently large.

From Proposition 4.5.1 and the regularity result contained in [75, Theorem 2.1] and recalled in Remark 4.3.5, it follows that $\{\nabla_x V^{\lambda_{n_k}}\}$ and $\{y^{1-2s}\frac{\partial V^{\lambda_{n_k}}}{\partial y}\}$ are uniformly bounded in k in the spaces $H^1(B_1^+, y^{1-2s})$ and $H^1(B_1^+, y^{2s-1})$ respectively. Then, by the continuity of the trace operator Tr_1 from $H^1(B_1^+, y^{1-2s})$ to $L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ and from $H^1(B_1^+, y^{2s-1})$ to $L^2(\mathbb{S}^+, \theta_{N+1}^{2s-1})$, we have that $\{\mathrm{Tr}_1(\nabla_x V^{\lambda_{n_k}})\}$ is bounded in $\left(L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})\right)^N$ and $\{y^{1-2s}\frac{\partial V^{\lambda_{n_k}}}{\partial y}\}$ is bounded in $L^2(\mathbb{S}^+, \theta_{N+1}^{2s-1})$. Therefore

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla V^{\lambda_{n_k}}|^2\, dS = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_x V^{\lambda_{n_k}}|^2\, dS + \int_{\mathbb{S}^+} \theta_{N+1}^{2s-1} \left| \theta_{N+1}^{1-2s} \frac{\partial V^{\lambda_{n_k}}}{\partial t} \right|^2 dS$$

is bounded uniformly with respect to k. Taking into account (4.44), it follows that there exists $f \in L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ such that, up to a further subsequence,

$$\widetilde{A}(\lambda_{n_k} \cdot) \nabla V^{\lambda_{n_k}} \cdot z \rightharpoonup f \quad \text{weakly in } L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s}) \text{ as } k \to +\infty. \tag{4.106}$$

Thus by (4.98) and after proving (4.101) when $\phi \in H^1(B_1^+, y^{1-2s})$ with the same argument (i.e. combining (4.28) with (4.100)), passing to the limit as $k \to +\infty$ in (4.105) we obtain that

$$\int_{B_1^+} y^{1-2s} \nabla V \cdot \nabla \phi \, dz = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} f \phi \, dS \tag{4.107}$$

for any $\phi \in H^1(B_1^+, y^{1-2s})$. Furthermore, by (4.106), combined with (4.95) and compactness of the trace operator in (3.4), we have

$$\lim_{k \to +\infty} \int_{\mathbb{S}^+} y^{1-2s} \widetilde{A}(\lambda_{n_k} \cdot) \nabla V^{\lambda_{n_k}} \cdot z \; V^{\lambda_{n_k}} \, dS = \int_{\mathbb{S}^+} y^{1-2s} f V \, dS. \tag{4.108}$$

Hence, testing (4.105) with $V^{\lambda_{n_k}}$ itself, taking into account (4.108), using (4.101) with $\phi = V^{\lambda_{n_k}}$, and passing to the limit as $k \to +\infty$, we deduce that

$$\lim_{k \to +\infty} \int_{B_1^+} y^{1-2s} \widetilde{A}(\lambda_{n_k} \cdot) \nabla V^{\lambda_{n_k}} \cdot \nabla V^{\lambda_{n_k}} \, dz = \int_{\mathbb{S}^+} y^{1-2s} f V \, dS,$$

which, by (4.107) tested with V, implies that

$$\lim_{k \to +\infty} \int_{B_1^+} y^{1-2s} A(\lambda_{n_k} \cdot) \nabla V^{\lambda_{n_k}} \cdot \nabla V^{\lambda_{n_k}} \, dz = \int_{B_1^+} y^{1-2s} |\nabla V|^2 dz. \tag{4.109}$$

Writing the left hand side in (4.109) as in (4.97), by (4.44) and Proposition 4.5.1 we infer that

$$\lim_{k \to +\infty} \int_{B_1^+} y^{1-2s} |\nabla V^{\lambda_{n_k}}|^2 \, dz = \int_{B_1^+} y^{1-2s} |\nabla V|^2 dz.$$

This convergence and (4.95), allows us to conclude that $\nabla V^{\lambda_{n_k}} \to \nabla V$ in $L^2(B_1^+, y^{1-2s})$. In conclusion, combining this with the compactness of the trace operator given in (3.4), (4.104) easily follows from Remark 3.1.4.

For any $r \in (0, 1]$ and $k \in \mathbb{N}$ we define

$$H_k(r) := \frac{1}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} \mu(\lambda_{n_k} \cdot) |V^{\lambda_{n_k}}|^2 \, dS,$$

$$D_k(r) := \frac{1}{r^{N-2s}} \left(\int_{B_r^+} y^{1-2s} \widetilde{A}(\lambda_{n_k} \cdot) \nabla V^{\lambda_{n_k}} \cdot \nabla V^{\lambda_{n_k}} \, dz - k_{s,N} \lambda_{n_k}^{2s} \int_{B_r'} \widetilde{h}(\lambda_{n_k} \cdot) |\operatorname{Tr}(V^{\lambda_{n_k}})|^2 \, dy \right),$$

and

$$H_V(r) := \frac{1}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} V^2 \, dS, \quad D_V(r) := \frac{1}{r^{N-2s}} \int_{B_r^+} y^{1-2s} |\nabla V|^2 \, dz.$$

By Proposition 4.4.2 in the case $\widetilde{h} = 0$, $\widetilde{A} = \mathrm{Id}_{N+1}$ and $\mu = 1$, it is clear that $H_V(r) > 0$ for any $r \in (0, 1]$. Thus the frequency function

$$\mathcal{N}_V(r) := \frac{D_V(r)}{H_V(r)} \quad r \in (0, 1]$$

is well defined. Furthermore by (4.83), (4.104), a change of variables, and a combination of (4.28) and (4.100), we have

$$\gamma = \lim_{k \to +\infty} \mathcal{N}(\lambda_{n_k} r) = \lim_{k \to +\infty} \frac{D_k(r)}{H_k(r)} = \mathcal{N}_V(r) \quad \text{for any } r \in (0, 1] \tag{4.110}$$

and hence $\mathcal{N}_V'(r) = 0$ for a.e. $r \in (0, 1]$. Arguing as in Proposition 4.4.6 in the case $\widetilde{h} = 0$, $\widetilde{A} = \mathrm{Id}_{N+1}$ and $\mu = 1$, we can prove that

$$\mathcal{N}_V'(r) = 2r \frac{\left(\int_{S_r^+} y^{1-2s} V^2 \, dS\right) \left(\int_{S_r^+} y^{1-2s} |\nabla V \cdot \nu|^2 \, dS\right) - \left(\int_{S_r^+} y^{1-2s} V (\nabla V \cdot \nu) \, dS\right)^2}{\left(\int_{S_r^+} y^{1-2s} V^2 \, dS\right)^2}.$$

Therefore we conclude that

$$\left(\int_{S_r^+} y^{1-2s} V^2 \, dS\right) \left(\int_{S_r^+} y^{1-2s} |\nabla V \cdot \nu|^2 \, dS\right) = \left(\int_{S_r^+} y^{1-2s} V \, (\nabla V \cdot \nu) \, dS\right)^2 \quad \text{a.e. } r \in (0, 1)$$

where $\nu = \frac{z}{|z|}$, i.e. equality holds in the Cauchy-Schwartz inequality for the vectors V and $\nabla V \cdot \nu$ in $L^2(S_r^+, y^{1-2s})$ for a.e. $r \in (0, 1)$. It follows that in polar coordinates

$$\frac{\partial V}{\partial r}(r\theta) = \rho(r) V(r\theta) \quad \text{for a.e. } r \in (0, 1) \text{ and for any } \theta \in \mathbb{S}^+, \tag{4.111}$$

for some function $r \mapsto \rho(r)$. By (4.111) we have

$$\int_{S_r^+} y^{1-2s} V (\nabla V \cdot \nu) \, dS = \rho(r) \int_{S_r^+} y^{1-2s} V^2 \, dS. \tag{4.112}$$

In the case $\widetilde{h} = 0$, $A = \mathrm{Id}_{N+1}$ and $\mu = 1$, (4.66) boils down to $H_V' = \frac{2}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s} V \frac{\partial V}{\partial \nu} \, dS$, since the perturbative term involves $\nabla \mu$, which now trivially equals 0. From this and (4.112) we deduce that $\rho(r) = \frac{H_V'(r)}{2 H_V(r)}$. At this point, we exploit (4.68) which, in the case $\widetilde{h} = 0$, $A = \mathrm{Id}_{N+1}$ and $\mu = 1$, becomes $H_V'(r) = \frac{2}{r} D_V(r)$ and thus implies

$$\rho(r) = \frac{1}{r} \mathcal{N}_V(r) = \frac{\gamma}{r},$$

where we used also (4.110). Then an integration over $(r, 1)$ of (4.111) for any fixed $\theta \in \mathbb{S}^+$ yields

$$V(r\theta) = r^\gamma V(\theta) = r^\gamma \Psi(\theta) \quad \text{for any } (r, \theta) \in (0, 1] \times \mathbb{S}^+, \tag{4.113}$$

where $\Psi := V|_{\mathbb{S}^+}$. In view of [60, Lemma 2.1], (4.102) becomes

$$\gamma(N-2s+\gamma)r^{-1-2s+\gamma}\theta_{N+1}^{1-2s}\Psi(\theta)+r^{-1-2s+\gamma}\operatorname{div}_{\mathbb{S}^+}(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^+}\Psi(\theta))=0$$

for any $(r,\theta)\in(0,1]\times\mathbb{S}^+$, together with the boundary condition $\lim_{\theta_{N+1}\to 0^+}\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}}\Psi\cdot\nu=0$ on $\mathbb{S}'$. Since V^λ is odd with respect to y_N for any $\lambda\in(0,r_0]$ by (4.90) and (4.55), then also V is odd with respect to y_N, so that $\Psi\in H^1_{\mathrm{odd}}(\mathbb{S}^+,\theta_{N+1}^{1-2s})$. By (4.113) and (4.103) we have $\|\Psi\|_{L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})}=1$, so that $\Psi\not\equiv 0$ is an eigenfunction of problem (4.19) associated to the eigenvalue $\gamma(\gamma+N-2s)$. From (4.22) it follows that there exists $m_0\in\mathbb{N}\setminus\{0\}$ (which is odd in the case $N=1$) such that $\gamma(\gamma+N-2s)=m_0(m_0+N-2s)$. Therefore, since $\gamma\geq 0$ by Proposition 4.4.8, we conclude that $\gamma=m_0$ thus proving (4.93). Moreover (4.94) follows from (4.104) and (4.113). □

In Proposition 4.4.10 we have shown that there exists the limit $\lim_{\lambda\to 0^+}\lambda^{-2\gamma}H(\lambda)$ and it is non-negative. Now we prove that $\lim_{\lambda\to 0^+}\lambda^{-2\gamma}H(\lambda)>0$.

To this end we define, for every $\lambda\in(0,r_0]$, $m\in\mathbb{N}\setminus\{0\}$, $k\in\{1,\dots,M_m\}$,

$$\varphi_{m,k}(\lambda):=\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}W(\lambda\theta)Y_{m,k}(\theta)\,dS, \tag{4.114}$$

i.e. $\{\varphi_{m,k}(\lambda)\}_{m,k}$ are the Fourier coefficients of $W(\lambda\cdot)$ with respect to the orthonormal basis $\{Y_{m,k}\}_{m,k}$ introduced in (4.25). For every $\lambda\in(0,r_0]$, $m\in\mathbb{N}\setminus\{0\}$, $k\in\{1,\dots,M_m\}$, we also define

$$\begin{aligned}\Upsilon_{m,k}(\lambda):=&-\int_{B_\lambda^+}y^{1-2s}(\widetilde{A}-\mathrm{Id}_{N+1})\nabla W\cdot\frac{1}{|z|}\nabla_{\mathbb{S}}Y_{m,k}(\tfrac{z}{|z|})\,dz\\&+\int_{S_\lambda^+}y^{1-2s}(\widetilde{A}-\mathrm{Id}_{N+1})\nabla W\cdot\frac{z}{|z|}Y_{m,k}(\tfrac{z}{|z|})\,dS\\&+\kappa_{N,s}\int_{B'_\lambda}\widetilde{h}(y)\operatorname{Tr}(W)\operatorname{Tr}\left(Y_{m,k}(\tfrac{y}{|y|})\right)\,dy,\end{aligned} \tag{4.115}$$

where Id_{N+1} is the identity $(N+1)\times(N+1)$ matrix.

Proposition 4.5.3. *Let γ be as in (4.83) and let $m_0\in\mathbb{N}\setminus\{0\}$ be such that $\gamma=m_0$ according to Proposition 4.5.2. For every $k\in\{1,\dots,M_{m_0}\}$ and $r\in(0,r_0]$*

$$\begin{aligned}\varphi_{m_0,k}(\lambda)=&\lambda^{m_0}\left(\frac{\varphi_{m_0,k}(r)}{r^{m_0}}+\frac{m_0r^{-2m_0-N+2s}}{2m_0+N-2s}\int_0^r\rho^{m_0-1}\Upsilon_{m_0,k}(\rho)\,d\rho\right)\\&+\lambda^{m_0}\frac{m_0+N-2s}{2m_0+N-2s}\int_\lambda^r\rho^{-m_0-N-1+2s}\Upsilon_{m_0,k}(\rho)\,d\rho+O\left(\lambda^{m_0+\frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)\end{aligned} \tag{4.116}$$

as $\lambda\to 0^+$.

Proof. Let $k\in\{1,\dots,M_{m_0}\}$ and $\phi\in\mathcal{D}(0,r_0)$. Testing (4.59) with $|z|^{-N-1+2s}\phi(|z|)Y_{m_0,k}(\frac{z}{|z|})$, since $Y_{m_0,k}$ solves (4.21), we obtain that $\varphi_{m_0,k}$ satisfies

$$-\varphi''_{m_0,k}-\frac{N+1-2s}{\lambda}\varphi'_{m_0,k}+\frac{\mu_{m_0}}{\lambda^2}\varphi_{m_0,k}=\zeta_{m_0,k} \tag{4.117}$$

in the sense of distributions in $(0, r_0)$, where

$$
\begin{aligned}
{}_{\mathcal{D}'(0,r_0)}\langle \zeta_{m_0,k}, \phi\rangle_{\mathcal{D}(0,r_0)} := \kappa_{N,s} \int_0^{r_0} \frac{\phi(\lambda)}{\lambda^{2-2s}} \left(\int_{\mathbb{S}'} \widetilde{h}(\lambda\theta') \operatorname{Tr}(W(\lambda\cdot))(\theta') Y_{m_0,k}(\theta', 0)\, dS' \right) d\lambda \\
- \int_0^{r_0} \left(\int_{S_\lambda^+} y^{1-2s} (A - \mathrm{Id}_{N+1}) \nabla W \cdot \nabla (|z|^{-N-1+2s} \phi(|z|) Y_{m_0,k}(\tfrac{z}{|z|}))\, dS \right) d\lambda.
\end{aligned}
$$

Furthermore, it is easy to verify that $\Upsilon_{m_0,k} \in L^1(0, r_0)$ and

$$
\Upsilon'_{m_0,k}(\lambda) = \lambda^{N+1-2s} \zeta_{m_0,k}(\lambda)
$$

in the sense of distributions in $(0, r_0)$. Then equation (4.117) can be rewritten as follows

$$
-(\lambda^{2m_0+N+1-2s} (\lambda^{-m_0} \varphi_{m_0,k}(\lambda))')' = \lambda^{m_0} \Upsilon'_{m_0,k}(\lambda) \tag{4.118}
$$

in the sense of distributions in $(0, r_0)$. Integrating (4.118) over (λ, r) for any $r \in (0, r_0]$, we obtain that there exists a constant $c_{m_0,k}(r) \in \mathbb{R}$ which depends only on m_0, k, r, such that

$$
\begin{aligned}
(\lambda^{-m_0} \varphi_{m_0,k}(\lambda))' = -\lambda^{-m_0-N-1+2s} \Upsilon_{m_0,k}(\lambda) \\
- m_0 \lambda^{-2m_0-N-1+2s} \left(c_{m_0,k}(r) + \int_\lambda^r \rho^{m_0-1} \Upsilon_{m_0,k}(\rho)\, d\rho \right)
\end{aligned}
$$

in the sense of distributions in $(0, r_0)$. In particular we deduce that $\varphi_{m_0,k} \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and a further integration over (λ, r) gives

$$
\begin{aligned}
\varphi_{m_0,k}(\lambda) =& \lambda^{m_0} \left(\frac{\varphi_{m_0,k}(r)}{r^{m_0}} - \frac{m_0 c_{m_0,k}(r)}{(2m_0 + N - 2s) r^{2m_0+N-2s}} \right) \\
&+ \lambda^{m_0} \frac{m_0 + N - 2s}{2m_0 + N - 2s} \int_\lambda^r \rho^{-m_0-N-1+2s} \Upsilon_{m_0,k}(\rho)\, d\rho \\
&+ \frac{m_0 \lambda^{-m_0-N+2s}}{2m_0 + N - 2s} \left(c_{m_0,k}(r) + \int_\lambda^r \rho^{m_0-1} \Upsilon_{m_0,k}(\rho)\, d\rho \right)
\end{aligned} \tag{4.119}
$$

for every $\lambda, r \in (0, r_0]$. Now we claim that

$$
\int_0^{r_0} \rho^{-m_0-N-1+2s} |\Upsilon_{m_0,k}(\rho)|\, d\rho < +\infty. \tag{4.120}
$$

By the Hölder inequality, a change of variables, (4.44), (4.90), Proposition 4.5.1, and (4.85) we have

$$
\begin{aligned}
&\lambda^{-m_0-N-1+2s} \left| \int_{B_\lambda^+} y^{1-2s} (\widetilde{A} - \mathrm{Id}_{N+1}) \nabla W \cdot \frac{1}{|z|} \nabla_{\mathbb{S}} Y_{m_0,k}(\tfrac{z}{|z|})\, dz \right| \\
&\leq \lambda^{-m_0-N-1+2s} \left(\int_{B_\lambda^+} y^{1-2s} |(\widetilde{A} - \mathrm{Id}_{N+1}) \nabla W|^2\, dz \right)^{\frac{1}{2}} \left(\int_{B_\lambda^+} \frac{y^{1-2s}}{|z|^2} \left| \nabla_{\mathbb{S}} Y_{m_0,k}(\tfrac{z}{|z|}) \right|^2 dz \right)^{\frac{1}{2}} \\
&\leq \lambda^{-m_0-1} O(\lambda) \sqrt{H(\lambda)} \left(\int_{B_1^+} y^{1-2s} |\nabla V^\lambda|^2\, dz \right)^{\frac{1}{2}} \left(\int_{B_1^+} \frac{y^{1-2s}}{|z|^2} \left| \nabla_{\mathbb{S}} Y_{m_0,k}(\tfrac{z}{|z|}) \right|^2 dz \right)^{\frac{1}{2}} \\
&\leq \mathrm{const}\, \lambda^{-m_0} \sqrt{H(\lambda)} \leq \mathrm{const},
\end{aligned} \tag{4.121}
$$

where we used the fact that

$$\int_{B_1^+} \frac{y^{1-2s}}{|z|^2} \left|\nabla_{\mathbb{S}} Y_{m_0,k}(\tfrac{z}{|z|})\right|^2 dz = \int_0^1 \rho^{N-1-2s} \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} Y_{m_0,k}(\theta)|^2 \, dS\right) d\rho$$
$$= \frac{m_0^2 + m_0(N-2s)}{N-2s}.$$

Dealing with the second term of (4.115), from an integration by parts, the Hölder inequality, (4.44) (4.90), Proposition 4.5.1, and (4.85) it follows that, for every $r \in (0, r_0]$,

$$\begin{aligned}
&\int_0^r \lambda^{-m_0-N-1+2s} \left| \int_{S_\lambda^+} y^{1-2s} (\widetilde{A} - \mathrm{Id}_{N+1}) \nabla W \cdot \frac{z}{|z|} Y_{m_0,k}(\tfrac{z}{|z|}) \, dS \right| d\lambda \\
&\le \mathrm{const} \int_0^r \lambda^{-m_0-N+2s} \left(\int_{S_\lambda^+} y^{1-2s} |\nabla W| \left|Y_{m_0,k}(\tfrac{z}{|z|})\right| dS \right) d\lambda \\
&= \mathrm{const} \left(r^{-m_0-N+2s} \int_{B_r^+} y^{1-2s} |\nabla W| \left|Y_{m_0,k}(\tfrac{z}{|z|})\right| dz \right. \\
&\qquad \left. + (m_0 + N - 2s) \int_0^r \lambda^{-m_0-N-1+2s} \left(\int_{B_\lambda^+} y^{1-2s} |\nabla W| \left|Y_{m_0,k}(\tfrac{z}{|z|})\right| dz \right) d\lambda \right) \\
&\le \mathrm{const} \left(r^{-m_0+1} \sqrt{H(r)} + \int_0^r \lambda^{-m_0} \sqrt{H(\lambda)} \, d\lambda \right) \le \mathrm{const}\ r,
\end{aligned} \tag{4.122}$$

taking into account that

$$\int_{B_\lambda^+} y^{1-2s} \left|Y_{m_0,k}(\tfrac{z}{|z|})\right|^2 dz = \frac{\lambda^{N+2-2s}}{N+2-2s}.$$

By the Hölder inequality the third term in (4.115) can be estimated as

$$\begin{aligned}
&\lambda^{-m_0-N-1+2s} \left| \int_{B'_\lambda} \widetilde{h}(y) \, \mathrm{Tr}(W) \, \mathrm{Tr}\left(Y_{m_0,k}(\tfrac{y}{|y|})\right) dy \right| \\
&\le \lambda^{-m_0-N-1+2s} \left(\int_{B'_\lambda} |\widetilde{h}(y)| |\mathrm{Tr}(W)|^2 \, dy \right)^{\frac12} \left(\int_{B'_\lambda} |\tilde{h}(y)| \left|\mathrm{Tr}\left(Y_{m_0,k}(\tfrac{y}{|y|})\right)\right|^2 dy \right)^{\frac12} \\
&\le \lambda^{-m_0-N-1+2s} \eta_{|\tilde{h}|}(\lambda) \left(\int_{B_\lambda^+} y^{1-2s} |\nabla W|^2 \, dz + \frac{N-2s}{2\lambda} \int_{S_\lambda^+} y^{1-2s} W^2 \, dS \right)^{\frac12} \times \\
&\qquad \times \left(\int_{B_\lambda^+} y^{1-2s} \left|\nabla Y_{m_0,k}(\tfrac{z}{|z|})\right|^2 dz + \frac{N-2s}{2\lambda} \int_{S_\lambda^+} y^{1-2s} \left|Y_{m_0,k}(\tfrac{z}{|z|})\right|^2 dS \right)^{\frac12} \\
&\le \lambda^{-m_0-1} \eta_{|\tilde{h}|}(\lambda) \sqrt{H(\lambda)} \left(\int_{B_1^+} y^{1-2s} |\nabla V^\lambda|^2 dz + (N-2s) \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \mu(\lambda\theta) |V^\lambda|^2 \, dS \right)^{\frac12} \times \\
&\qquad \times \left(\lambda^2 \int_{B_1^+} y^{1-2s} \left|\nabla Y_{m_0,k}(\tfrac{z}{|z|})\right|^2 dz + \frac{N-2s}{2} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |Y_{m_0,k}(\theta)|^2 \, dS \right)^{\frac12} \\
&\le \mathrm{const}\, \lambda^{-m_0-1} \eta_{|\tilde{h}|}(\lambda) \sqrt{H(\lambda)} \le \mathrm{const}\, \lambda^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}},
\end{aligned} \tag{4.123}$$

in view of (4.28), (4.29), (4.48), (4.85), (4.90), (4.92) and Proposition 4.5.1. Collecting estimates (4.121), (4.122) and (4.123) we deduce that, for every $r \in (0, r_0]$,

$$\int_0^r \rho^{-m_0-N-1+2s} |\Upsilon_{m_0,k}(\rho)| \, d\rho \leq \text{const} \left(r + \int_0^r \rho^{-1+\frac{4s^2\varepsilon}{N+2s\varepsilon}} \, d\rho \right) \leq \text{const} \, r^{\frac{4s^2\varepsilon}{N+2s\varepsilon}}, \tag{4.124}$$

thus proving (4.120). Moreover we have

$$\int_0^{r_0} \rho^{m_0-1} |\Upsilon_{m_0,k}(\rho)| \, d\rho < +\infty, \tag{4.125}$$

as a consequence of (4.120), since in a neighbourhood of 0, $\rho^{m_0-1} \leq \rho^{-m_0-N-1+2s}$.

Now we claim that, for every $r \in (0, r_0]$,

$$c_{m_0,k}(r) + \int_0^r \rho^{m_0-1} \Upsilon_{m_0,k}(\rho) \, d\rho = 0 \tag{4.126}$$

To prove (4.126) we argue by contradiction. If there exists $r \in (0, r_0]$ such that (4.126) does not hold true, then by (4.119), (4.120) and (4.125)

$$\varphi_{m_0,k}(\lambda) \sim \frac{m_0 \lambda^{-m_0-N+2s}}{2m_0 + N - 2s} \left(c_{m_0,k}(r) + \int_0^r \rho^{m_0-1} \Upsilon_{m_0,k}(\rho) \, d\rho \right) \quad \text{as } \lambda \to 0^+.$$

From this, it follows that

$$\int_0^{r_0} \lambda^{N-1-2s} |\varphi_{m_0,k}(\lambda)|^2 d\lambda = +\infty, \tag{4.127}$$

since $N - 2s + 2m_0 > 0$. On the other hand, from (4.114), the Parseval identity and (3.6) we deduce the following estimate

$$\begin{aligned} \int_0^{r_0} \lambda^{N-1-2s} |\varphi_{m_0,k}(\lambda)|^2 \, d\lambda &\leq \int_0^{r_0} \lambda^{N-1-2s} \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |W(\lambda\theta)|^2 \, dS \right) d\lambda \\ &= \int_0^{r_0} \lambda^{-2} \left(\int_{S_\lambda^+} y^{1-2s} |W|^2 \, dS \right) d\lambda = \int_{B_{r_0}^+} y^{1-2s} \frac{|W(z)|^2}{|z|^2} \, dz < +\infty, \end{aligned}$$

which contradicts (4.127). Hence (4.126) is proved. From (4.126) and (4.124) it follows that, for every $r \in (0, r_0]$,

$$\begin{aligned} \lambda^{-m_0-N+2s} \left| c_{m_0,k}(r) + \int_\lambda^r \rho^{m_0-1} \Upsilon_{m_0,k}(\rho) \, d\rho \right| &= \lambda^{-m_0-N+2s} \left| \int_0^\lambda \rho^{m_0-1} \Upsilon_{m_0,k}(\rho) \, d\rho \right| \\ \leq \lambda^{-m_0-N+2s} \left(\lambda^{2m_0+N-2s} \int_0^\lambda \rho^{-m_0-N-1+2s} |\Upsilon_{m_0,k}(\rho)| \, d\rho \right) &\leq \text{const} \, \lambda^{m_0 + \frac{4s^2\varepsilon}{N+2s\varepsilon}}. \end{aligned} \tag{4.128}$$

We finally deduce (4.116) combining (4.119), (4.126) and (4.128). □

Proposition 4.5.4. *Let γ be as in (4.83). Then*

$$\lim_{\lambda \to 0^+} \lambda^{-2\gamma} H(\lambda) > 0. \tag{4.129}$$

Proof. By (4.49), the Parseval identity and (4.114) we have

$$H(\lambda) = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \mu(\lambda\theta)|W(\lambda\theta)|^2 \, dS = (1 + O(\lambda)) \sum_{m=1}^{\infty} \sum_{k=1}^{M_m} |\varphi_{m,k}(\lambda)|^2. \tag{4.130}$$

Let $m_0 \in \mathbb{N} \setminus \{0\}$ be such that $\gamma = m_0$ according to Proposition 4.5.2. We argue by contradiction and assume that $0 = \lim_{\lambda\to 0^+} \lambda^{-2\gamma} H(\lambda) = \lim_{\lambda\to 0^+} \lambda^{-2m_0} H(\lambda)$. In view of (4.130) this would imply that

$$\lim_{\lambda\to 0^+} \lambda^{-m_0} \varphi_{m_0,k}(\lambda) = 0 \quad \text{for every } k \in \{1, \dots, M_{m_0}\}.$$

Therefore, from (4.116) it follows that, for all $k \in \{1, \dots, M_{m_0}\}$ and $r \in (0, r_0]$,

$$\frac{\varphi_{m_0,k}(r)}{r^{m_0}} + \frac{m_0 r^{-2m_0-N+2s}}{2m_0 + N - 2s} \int_0^r \rho^{m_0-1} \Upsilon_{m_0,k}(\rho) \, d\rho \\ + \frac{m_0 + N - 2s}{2m_0 + N - 2s} \int_0^r \rho^{-m_0-N-1+2s} \Upsilon_{m_0,k}(\rho) \, d\rho = 0,$$

so that, substituting into (4.116), we obtain that

$$\varphi_{m_0,k}(\lambda) = -\frac{m_0 + N - 2s}{2m_0 + N - 2s} \lambda^{m_0} \int_0^\lambda \rho^{-m_0-N-1+2s} \Upsilon_{m_0,k}(\rho) \, d\rho + O\left(\lambda^{m_0 + \frac{4s^2\varepsilon}{N+2s\varepsilon}}\right)$$

as $\lambda \to 0^+$. Hence, from (4.124) we infer that

$$\varphi_{m_0,k}(\lambda) = O\left(\lambda^{m_0 + \frac{4s^2\varepsilon}{N+2s\varepsilon}}\right) \quad \text{as } \lambda \to 0^+ \quad \text{for all } k \in \{1, \dots, M_{m_0}\}. \tag{4.131}$$

Moreover, estimate (4.87) with $\sigma = \frac{2s^2\varepsilon}{N+2s\varepsilon}$ implies that

$$\frac{1}{\sqrt{H(\lambda)}} = O\left(\lambda^{-m_0 - \frac{2s^2\varepsilon}{N+2s\varepsilon}}\right) \quad \text{as } \lambda \to 0^+. \tag{4.132}$$

Since

$$\varphi_{m_0,k}(\lambda) = \sqrt{H(\lambda)} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^\lambda(\theta) Y_{m_0,k}(\theta) \, dS \quad \text{for all } k \in \{1, \dots, M_{m_0}\}$$

by (4.114) and (4.90), from (4.131) and (4.132) we deduce that

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^\lambda(\theta) \Psi(\theta) \, dS = O\left(\lambda^{\frac{2s^2\varepsilon}{N+2s\varepsilon}}\right) \quad \text{as } \lambda \to 0^+, \tag{4.133}$$

for every $\Psi \in \text{Span}\{Y_{m_0,k} : k \in \{1, \dots M_{m_0}\}\}$. By (4.24), (4.25), (3.4) and Proposition 4.5.2, for any sequence $\lambda_n \to 0^+$, there exist a subsequence $\lambda_{n_h} \to 0^+$ and $\Psi \in \text{Span}\{Y_{m_0,k} : k \in \{1, \dots M_{m_0}\}\}$ such that $\|\Psi\|_{L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})} = 1$ and

$$\lim_{h\to+\infty} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^{\lambda_{n_h}}(\theta) \Psi(\theta) \, dS = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\Psi|^2 \, dS = 1,$$

thus contradicting (4.133). □

Theorem 4.5.5. *Let W be a non trivial weak solution to (4.58). Let γ be as in (4.83) and $m_0 \in \mathbb{N} \setminus \{0\}$ be such that $\gamma = m_0$, according to Proposition 4.5.2. Let $\{Y_{m_0,k}\}_{k\in\{1,\dots,M_{m_0}\}}$ be as in (4.25), with V_{m_0} and M_{m_0} defined as in (4.23) and (4.24) respectively. Then*

$$\lambda^{-m_0} W(\lambda z) \to |z|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k Y_{m_0,k}\left(\frac{z}{|z|}\right) \quad \text{as } \lambda \to 0^+ \quad \text{strongly in } H^1(B_1^+, y^{1-2s}),$$

where $(\beta_1, \dots, \beta_{M_{m_0}}) \neq (0, \dots, 0)$ and, for every $k \in \{1, \dots, M_{m_0}\}$,

$$\beta_k = \frac{\varphi_{m_0,k}(r)}{r^{m_0}} + \frac{m_0 r^{-2m_0-N+2s}}{(2m_0+N-2s)} \int_0^r \rho^{m_0-1} \Upsilon_{m_0,k}(\rho)\, d\rho \\ + \frac{m_0+N-2s}{2m_0+N-2s} \int_0^r \rho^{-m_0-N-1+2s} \Upsilon_{m_0,k}(\rho)\, d\rho, \quad (4.134)$$

for all $r \in (0, r_0]$, where $\varphi_{m_0,k}$ is defined in (4.114) and $\Upsilon_{m_0,k}$ in (4.115) .

Proof. From Proposition 4.5.2, (4.25), and (4.129) it follows that, for any sequence $\{\lambda_n\}$ such that $\lambda_n \to 0^+$ as $n \to \infty$, there exist a subsequence $\{\lambda_{n_h}\}$ and real numbers $\beta_1, \dots, \beta_{M_{m_0}}$ such that $(\beta_1, \dots, \beta_{M_{m_0}}) \neq (0, \dots, 0)$ and

$$\lambda_{n_h}^{-m_0} W(\lambda_{n_h} z) \to |z|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k Y_{m_0,k}\left(\frac{z}{|z|}\right) \text{ as } h \to +\infty \text{ strongly in } H^1(B_1^+, y^{1-2s}). \quad (4.135)$$

We claim that the numbers $\beta_1, \dots \beta_{M_{m_0}}$ depend neither on the sequence $\{\lambda_n\}$ nor on its subsequence $\{\lambda_{n_h}\}$. Letting $\varphi_{m_0,k}$ be as (4.114), for every $k \in \{1, \dots, M_{m_0}\}$

$$\lim_{h\to+\infty} \lambda_{n_h}^{-m_0} \varphi_{m_0,k}(\lambda_{n_h}) = \lim_{h\to+\infty} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \lambda_{n_h}^{-m_0} W(\lambda_{n_h}\theta) Y_{m_0,k}(\theta)\, dS = \beta_k, \quad (4.136)$$

thanks to (4.135) and the compactness of the trace operator in (3.4). Combining (4.136) and (4.116) we obtain that, for every $r \in (0, r_0]$, $\beta_k = \lim_{h\to+\infty} \lambda_{n_h}^{-m_0} \varphi_{m_0,k}(\lambda_{n_h})$ is equal to the right hand side in (4.134), thus proving the claim. By Urysohn's subsequence principle we conclude that the convergence in (4.135) holds as $\lambda \to 0^+$, hence the proof is complete. □

4.6 Proofs of the main results

The proof of Theorem 4.1.3 is obtained as a consequence of the following result.

Theorem 4.6.1. *Let $N > 2s$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain such that $0 \in \partial\Omega$ and (4.10)–(4.12) are satisfied with $x_0 = 0$ for some function g and $R > 0$. Let U be a non trivial solution to (4.17) in the sense of (4.18), with h satisfying (4.8), and let*

$$\widehat{U}(z) = \begin{cases} U(z), & \text{if } z \in \mathcal{C}_\Omega \cap F(B_{r_0}^+), \\ 0, & \text{if } z \in F(B_{r_0}^+) \setminus \mathcal{C}_\Omega, \end{cases} \quad (4.137)$$

with F and r_0 being as in Proposition 4.3.1. Then there exist $m_0 \in \mathbb{N} \setminus \{0\}$ (which is odd in the case $N = 1$) such that

$$\lambda^{-m_0} \widehat{U}(\lambda z) \to |z|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k \widehat{Y}_{m_0,k}\left(\frac{z}{|z|}\right) \quad \text{as } \lambda \to 0^+ \quad \text{strongly in } H^1(B_1^+, y^{1-2s}), \quad (4.138)$$

where M_{m_0} *is as in* (4.24),

$$\widehat{Y}_{m_0,k}(\theta', \theta_N, \theta_{N+1}) = \begin{cases} Y_{m_0,k}(\theta', \theta_N, \theta_{N+1}), & \text{if } \theta_N < 0, \\ 0, & \text{if } \theta_N \geq 0, \end{cases} \tag{4.139}$$

with $\{Y_{m_0,k}\}_{k\in\{1,\dots,M_{m_0}\}}$ *being as in* (4.25), *and the coefficients* β_k *satisfy* (4.134).

Proof. If U is a non trivial solution of (4.17), then the function W defined in (4.34) and (4.55) belongs to $H^1(B^+_{r_0}, y^{1-2s})$ and is a non trivial weak solution to (4.58). Letting

$$\widehat{W}(z) = \begin{cases} W(z), & \text{if } z \in \mathcal{Q}_{r_0}, \\ 0, & \text{if } z \in B^+_{r_0} \setminus \mathcal{Q}_{r_0}, \end{cases}$$

where $\mathcal{Q}_{r_0}$ is defined in (4.33), by Remark 4.3.4 we have $\widehat{W} \in H^1(B^+_{r_0}, y^{1-2s})$. Moreover Theorem 4.5.5 implies that

$$\lambda^{-m_0}\widehat{W}(\lambda z) \to \widehat{\Phi}(z) \quad \text{strongly in } H^1(B_1^+, y^{1-2s}) \quad \text{as } \lambda \to 0^+,$$

where

$$\widehat{\Phi}(z) = |z|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k \widehat{Y}_{m_0,k}\left(\frac{z}{|z|}\right)$$

with β_k as in (4.134). Hence, by homogeneity,

$$\lambda^{-m_0}\widehat{W}(\lambda z) \to \widehat{\Phi}(z) \quad \text{strongly in } H^1(B_r^+, y^{1-2s}) \quad \text{as } \lambda \to 0^+ \quad \text{for all } r > 1. \tag{4.140}$$

We note that

$$\lambda^{-m_0}\widehat{U}(\lambda z) = \lambda^{-m_0}\widehat{W}(\lambda G_\lambda(z)) \quad \text{and} \quad \nabla\left(\frac{\widehat{U}(\lambda\cdot)}{\lambda^{m_0}}\right) = \nabla\left(\frac{\widehat{W}(\lambda\cdot)}{\lambda^{m_0}}\right)(G_\lambda(z))J_{G_\lambda}(z) \tag{4.141}$$

where

$$G_\lambda(z) := \frac{1}{\lambda}F^{-1}(\lambda z) \quad \text{for any } \lambda \in (0,1] \text{ and } z \in \frac{1}{\lambda}F(B_{r_0^+}).$$

From Proposition 4.3.1 we deduce that

$$G_\lambda(z) = z + O(\lambda) \quad \text{and} \quad J_{G_\lambda}(z) = \mathrm{Id}_{N+1} + O(\lambda) \quad \text{as } \lambda \to 0^+$$

uniformly respect to $z \in B_1^+$. It follows that, if $f_\lambda \to f$ in $L^2(B_r^+, y^{1-2s})$ as $\lambda \to 0^+$ for some $r > 1$, then $f_\lambda \circ G_\lambda \to f$ in $L^2(B_1^+, y^{1-2s})$ as $\lambda \to 0^+$. Then we conclude in view of (4.140) and (4.141). □

Proof of Theorem 4.1.3 . It follows directly from Theorem 4.6.1 up to a translation. □

Passing to traces in (4.138) we obtain the following blow-up result for solutions to (4.1).

Theorem 4.6.2. *Let* $N > 2s$ *and* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain such that* $0 \in \partial\Omega$ *and* (4.10)–(4.12) *are satisfied with* $x_0 = 0$ *for some function* g *and* $R > 0$. *Let* $u \in \mathbb{H}^s(\Omega)$ *be a non trivial solution of* (4.1) *in the sense of* (4.9), *with* h *satisfying* (4.8), *and let* $\widehat{u}(x) = \iota(u)$

with ι defined in (4.4). Then there exists $m_0 \in \mathbb{N} \setminus \{0\}$ (which is odd in the case $N = 1$) such that

$$\lambda^{-m_0} \widehat{u}(\lambda x) \to |x|^{m_0} \sum_{k=1}^{M_{m_0}} \beta_k \widehat{Y}_{m_0,k} \left(\frac{x}{|x|}, 0 \right) \quad \text{as } \lambda \to 0^+ \quad \text{strongly in } H^s(B_1'),$$

where M_{m_0} is as in (4.24), $\{\widehat{Y}_{m_0,k}\}_{k \in \{1,\dots,M_{m_0}\}}$ are defined in (4.139) and the coefficients β_k satisfy (4.134).

Proof. As observed in [37], if $u \in \mathbb{H}^s(\Omega)$ is a non trivial solution of (4.1), then its extension $\mathcal{H}(u) = U$ is non trivial solution to (4.17). Hence the corresponding function $\widehat{U}$ defined in (4.137) satisfies (4.138) by Theorem 4.6.1. Since $\widehat{u} = \mathrm{Tr}(\widehat{U})$, the conclusion follows from Proposition 4.2.2. □

Proof of Theorem 4.1.2. It follows directly from Theorem 4.6.2 up to a translation. □

4.7 Neumann eigenvalues on the half-sphere under a symmetry condition

In order to determine the eigenvalues of (4.19), we first need the following preliminary lemma.

Lemma 4.7.1. *Let $m, N \in \mathbb{N} \setminus \{0\}$ and let $u \in C^m(\mathbb{R}^N)$ be a positively homogeneous function of degree m, i.e.*

$$u(\lambda x) = \lambda^m u(x) \quad \text{for every } \lambda > 0 \text{ and } x \in \mathbb{R}^N. \tag{4.142}$$

Then u is a homogeneous polynomial of degree m.

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ be a multindex, $|\alpha| := \sum_{i=1}^N \alpha_i$, and $x^\alpha = x_1^{\alpha_1} \dots x_N^{\alpha_N}$ for any $x = (x_1, \dots, x_N) \in \mathbb{R}^N$. By Taylor's Theorem with Lagrange remainder centered at 0, for any $x \in \mathbb{R}^N$ there exists $t \in [0,1]$ such that

$$u(x) = \sum_{|\alpha|<m} c_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha}(0) x^\alpha + \sum_{|\alpha|=m} c_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha}(tx) x^\alpha,$$

where $c_\alpha > 0$ are positive constants depending on α and $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$ stands for $\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$. By (4.142), one can easily prove that $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$ is a positively homogeneous function of degree $m - |\alpha|$ for all α with $|\alpha| \leq m$. Thus, combining this fact with the continuity of $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$, it is clear that $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}(0) = 0$ for every $\alpha \in \mathbb{N}^N$ with $|\alpha| < m$. On the other hand, for every $\alpha \in \mathbb{N}^N$ with $|\alpha| = m$, $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$ is constant and exactly equal to $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}(0)$, being a homogeneous function of degree 0. It follows that

$$u(x) = \sum_{|\alpha|=m} c_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha}(0) x^\alpha \quad \text{for every } x \in \mathbb{R}^N,$$

hence proving the claim. □

Proposition 4.7.2. *All the eigenvalues of problem (4.19) are characterized by formula (4.22).*

Proof. We start by proving that if μ is an eigenvalue of (4.19), then $\mu = m^2 + m(N-2s)$ for some $m \in \mathbb{N} \setminus \{0\}$. If μ is an eigenvalue, then there exists a non trivial solution Y of (4.19). A direct computation shows that Y is a weak solution to (4.19) if and only if the function

$$U(z) := |z|^\gamma Y\left(\frac{z}{|z|}\right), \quad z \in \mathbb{R}^{N+1}_+,$$

with

$$\gamma := -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu}, \tag{4.143}$$

belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^{N+1}_+, y^{1-2s})$, is odd with respect to y_N and weakly solves

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla U) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial \nu} = 0, & \text{on } \mathbb{R}^N. \end{cases} \tag{4.144}$$

Hence, if μ is an eigenvalue of (4.19), there exists a solution U of (4.144) which is odd with respect to y_N and positively homogeneous of degree γ. The regularity result in [120, Theorem 1.1] ensures that $U \in C^\infty(\overline{B_1^+})$. Then there exists $m \in \mathbb{N} \setminus \{0\}$ such that $\gamma = m$ and so $\mu = m^2 + m(N-2s)$ thanks to (4.143). We notice that the case $m = 0$ is excluded since in that case $\mu = 0$ and 0 is not an eigenvalue. Indeed, if by contradiction 0 is an eigenvalue, letting Y be an eigenfunction of (4.19) with associated eigenvalue 0 and choosing in (4.21) $\Psi = Y$, we would have Y constant and $Y \not\equiv 0$, hence $Y \notin H^1_{\mathrm{odd}}(\mathbb{S}^+, \theta^{1-2s}_{N+1})$ which is a contradiction (see (4.20)).

Viceversa, in order to prove that the numbers given in (4.22) are eigenvalues of (4.19), we need to show that, for any fixed $m \in \mathbb{N} \setminus \{0\}$, there actually exist an eigenfunction associated to $m^2 + m(N-2s)$ if $N > 1$ and an eigenfunction associated to $(2m-1)^2 + (2m-1)(N-2s)$ if $N = 1$. Equivalently, for any fixed $m \in \mathbb{N} \setminus \{0\}$ we have to find a non trivial solution to (4.144) which is odd with respect to x_N and positively homogeneous with degree m if $N > 1$ and $2m-1$ if $N = 1$. To this end, we observe that equation $\operatorname{div}(y^{1-2s}\nabla U) = 0$ can be rewritten as

$$\Delta U + \frac{1-2s}{y} U_y = 0. \tag{4.145}$$

We first consider the case $N = 1$. If $n = 2m-1$ with $m \in \mathbb{N} \setminus \{0\}$, we consider the following homogeneous polynomial of degree $2m-1$, odd with respect to x_1,

$$U_{1,m}(x_1, y) := \sum_{k=0}^{m-1} a_k x_1^{2k+1} y^{2m-2k-2}, \tag{4.146}$$

with $a_0, \dots, a_{m-1} \in \mathbb{R}$. A direct computation shows that $U_{1,m}$ is a solution of (4.144), and equivalently of (4.145), if and only if

$$a_k = \frac{-2[(m-k)^2 - s(m-k)]}{k(2k+1)} a_{k-1} \quad \text{for all } k \in \{1, \dots, m-1\}.$$

Thus, for example choosing $a_0 := 1$, we have constructed a non trivial solution to (4.144) which is odd with respect to y_1 and positively homogeneous of degree $2m-1$.

To complete the proof of (4.22) in the case $N = 1$, it remains to show that, if $n = 2m$ with $m \in \mathbb{N} \setminus \{0\}$, then $n^2 + n(N-2s)$ is not an eigenvalue of (4.19). To this aim, we argue by contradiction and assume that $(2m)^2 + 2m(N-2s)$ is an eigenvalue of (4.19) associated to an eigenfunction Ψ. Then the function defined as

$$U(z) = |z|^{\gamma}\Psi\left(\frac{z}{|z|}\right), \quad z = (x_1, y) \in \mathbb{R}^2_+,$$

with

$$\gamma = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + (2m)^2 + 2m(N-2s)} = 2m$$

is a non trivial solution to (4.144), odd with respect to x_1. Hence, if we consider the even reflection of U with respect to y, namely the function $\widetilde{U}(x_1, y) := U(x_1, |y|)$, $\widetilde{U}$ is a solution of $\mathrm{div}(|y|^{1-2s}\nabla\widetilde{U}) = 0$ in $\mathbb{R}^2$. Then, by [120, Theorem 1.1] we deduce that $\widetilde{U} \in C^\infty(\mathbb{R}^2)$. Moreover, $\widetilde{U}$ is positively homogeneous of degree $\gamma = 2m$, therefore from Lemma 4.7.1 it follows that $\widetilde{U}$ is a homogeneous polynomial of degree $2m$, namely

$$\widetilde{U}(x_1, y) = \sum_{k=0}^{2m} a_k x_1^{2m-k} y^k$$

where $a_k = 0$ if k is odd since $\widetilde{U}$ is even with respect to y. In this way $\tilde{U}$ turns out to be even also with respect to x_1 and this contradicts the fact that U is non trivial and odd with respect to x_1.

If $N = 2$ and $m \in \mathbb{N} \setminus \{0\}$ is odd, then we consider $U_2(x_1, x_2, y) := U_{1,n}(x_2, y)$, where $U_{1,n}$ is defined in (4.146) and $n \in \mathbb{N} \setminus \{0\}$ is such that $m = 2n - 1$. Such U_2 is a positively homogeneous solution of (4.144) of degree m, odd with respect to x_2. If $m \in \mathbb{N} \setminus \{0\}$ is even, i.e. $m = 2n$ with $n \in \mathbb{N} \setminus \{0\}$, then we define

$$U_3(x_1, x_2, y) := \sum_{k=0}^{n-1} a_k x_1^{2k+1} x_2^{2n-2k-1},$$

with $a_0, \dots, a_{n-1} \in \mathbb{R}$. A direct computation shows that U_3 is a solution of (4.144), and equivalently of (4.145), if and only if

$$a_{k+1} = \frac{-[2(n-k)^2 - 3n + 3k + 1]}{(2k^2 + 5k + 3)} a_k \quad \text{for all } k \in \{0, \dots, n-2\}.$$

Then, choosing for example again $a_0 = 1$, we obtain that U_3 is a solution of (4.144) which is positively homogeneous of degree m and odd with respect to y_2, as desired.

If $N > 2$, for any $m \in \mathbb{N}\setminus\{0\}$ there exists a harmonic homogeneous polynomial $P \not\equiv 0$ in the variables $y_1, \dots, x_{N-1}$, of degree $m - 1$. Then $U_4(x_1, \dots, x_{N-1}, x_N, t) := P(x_1, \dots, x_{N-1})\, x_N$ is a non trivial solution to (4.144) which is odd with respect to x_N and positively homogeneous of degree m.

□

Chapter 5

Unique continuation for the fractional power of a Schrödinger-type operator

5.1 Statement of the main results

In this Chapter we establish a strong unique continuation principle and classify the asymptotic profiles in the singular point 0 for solutions of the linear equation

$$L^s_{\alpha,k} u = gu \quad \text{in } \Omega \tag{5.1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ is a connected bounded Lipschitz domain such that $0 \in \Omega$ and

$$L_{\alpha,k} u := -\Delta u - \frac{\alpha}{|x|_k^2} u$$

with

$$|x|_k^2 = \sum_{i=1}^k x_i^2 \quad \text{and} \quad \alpha \in \left(-\infty, \left(\frac{k-2}{2}\right)^2\right) \tag{5.2}$$

for any $k \in \{3, \dots, N\}$. The fractional powers $L^s_{\alpha,k}$ of the operator $L_{\alpha,k}$ are rigorously defined in (5.9). The potential g satisfies

$$\begin{cases} g \in W^{1,\infty}_{loc}(\Omega \setminus \{0\}), \\ |g(x)| + |x \cdot \nabla g(x)| \leq C_g |x|^{-2s+\varepsilon}, \quad \text{for a.e. } x \in \Omega, \end{cases} \tag{5.3}$$

for some positive constant $C_g > 0$ and $\varepsilon \in (0, 1)$.

Since we deal with singular potentials of the form $\alpha|x|_k^{-2}$, Hardy-type inequalities with optimal constants are fundamental to study the positivity of $L_{\alpha,k}$ on $H^1_0(\Omega)$. In the case $k = N$ it is well known that

$$\int_{\mathbb{R}^N} \frac{\phi^2}{|x|^2}\, dx \leq \left(\frac{2}{N-2}\right)^2 \int_{\mathbb{R}^N} |\nabla \phi|^2\, dx \quad \text{for any } \phi \in C^\infty_c(\mathbb{R}^N),$$

and that $\left(\frac{2}{N-2}\right)^2$ is the optimal constant. A similar result also holds for cylindrical potential, more precisely for any $k \in \{3, \ldots, N\}$

$$\int_{\mathbb{R}^N} \frac{\phi^2}{|x|_k^2}\, dx \le \left(\frac{2}{k-2}\right)^2 \int_{\mathbb{R}^N} |\nabla \phi^2|\, dx \quad \text{for any } \phi \in C_c^\infty(\mathbb{R}^N), \tag{5.4}$$

see [104, Subsection 2.1.6, Corollary 3] or [20]. Furthermore $\left(\frac{2}{k-2}\right)^2$ is the optimal constant as conjectured in [20] and proved in [116].

Let us consider the eigenvalue problem

$$\begin{cases} L_{\alpha,k} u = \mu u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

We say that μ is an eigenvalue of (5.5) if there exists $Y \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla Y \cdot \nabla v\, dx - \int_\Omega \frac{\alpha}{|x|_k^2} Y v\, dx = \mu \int_\Omega Y v\, dx, \quad \text{for any } v \in H_0^1(\Omega). \tag{5.6}$$

Thanks to (5.2) and (5.4), for any $k \in \{3, \cdots, N\}$ the energy functional

$$J_{\alpha,k}(u) := \int_\Omega |\nabla u|^2\, dx - \int_\Omega \frac{\alpha}{|x|_k^2} u^2\, dx$$

is coercive on $H_0^1(\Omega)$ and so by the Spectral Theorem the set of the eigenvalues of (5.5) is a non-decreasing, positive, diverging sequence $\{\mu_{\alpha,k,n}\}_{n \in \mathbb{N}\setminus\{0\}}$(we repeat each eigenvalue according to its multiplicity). Furthermore there exists an orthonormal basis $\{Y_{\alpha,k,n}\}_{n \in \mathbb{N}\setminus\{0\}}$ of $L^2(\Omega)$ made of corresponding eigenfunctions. Since the first eigenfunction does not change sign, it is not restrictive to suppose that $Y_{\alpha,k,1}$ is positive.

For any Hilbert space X let $(v_1, v_2)_X$ be the scalar product on X. Furthermore let

$$v_n := (v, Y_{\alpha,k,n})_{L^2(\Omega)} \quad \text{for any } v \in L^2(\Omega). \tag{5.7}$$

Remark 5.1.1. In view of (5.4), $\|v\|_{\alpha,k} := (J_{\alpha,k}(v))^{\frac{1}{2}}$ is a norm on $H_0^1(\Omega)$ equivalent to the usual norm $\|v\|_{H_0^1(\Omega)} := (\int_\Omega |\nabla v|^2\, dx)^{\frac{1}{2}}$. The scalar product associated to $\|\cdot\|_{\alpha,k}$ is given by

$$(v, w)_{\alpha,k} := \int_\Omega \nabla v \cdot \nabla w - \frac{\alpha}{|x|_k^2} vw\, dx.$$

By (5.6), $\{Y_{\alpha,k,n}/\sqrt{\mu_{\alpha,k,n}}\}_{n \in \mathbb{N}\setminus\{0\}}$ is an orthonormal basis of $H_0^1(\Omega)$ with respect to the norm $\|\cdot\|_{\alpha,k}$ and for any $v, w \in H_0^1(\Omega)$

$$(v, w)_{\alpha,k} = \sum_{n=1}^\infty \mu_{\alpha,k,n} v_n w_n,$$

where v_n and w_n are as in (5.7).

Let us consider the functional space

$$\mathbb{H}^s_{\alpha,k}(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \mu^s_{\alpha,k,n} v_n^2 < +\infty \right\}$$

which is a Hilbert space with respect to the scalar product

$$(v, w)_{\mathbb{H}^s_{\alpha,k}(\Omega)} := \sum_{n=1}^{\infty} \mu^s_{\alpha,k,n} v_n w_n, \qquad \text{for any } v, w \in \mathbb{H}^s_{\alpha,k}(\Omega). \tag{5.8}$$

We can precode as in Chapter 4 to define the fractional powers of the operator $L^s_{\alpha,k}$. Indeed for any $j \in \mathbb{N} \setminus \{0\}$, and $v \in L^2(\Omega)$ it is clear that $\sum_{n=1}^{j} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n} \in L^2(\Omega)$ and that it can be identified with the element of the dual space $(\mathbb{H}^s_{\alpha,k}(\Omega))^*$ acting on $u \in \mathbb{H}^s_{\alpha,k}(\Omega)$ as

$${}_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\left\langle \sum_{n=1}^{j} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n}, u \right\rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} := \left(\sum_{n=1}^{j} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n}, u \right)_{L^2(\Omega)} = \sum_{n=1}^{j} \mu^s_{\alpha,k,n} v_n u_n.$$

It is easy to see that, if $v \in \mathbb{H}^s_{\alpha,k}(\Omega)$, then the series $\sum_{n=1}^{\infty} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n}$ converges in the dual space $(\mathbb{H}^s_{\alpha,k}(\Omega))^*$ to some $F \in (\mathbb{H}^s_{\alpha,k}(\Omega))^*$ such that

$${}_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\langle F, Y_{\alpha,k,n} \rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} = \mu^s_{\alpha,k,n} v_n \qquad \text{for any } n \in \mathbb{N} \setminus \{0\}.$$

It follows that, for every $v \in \mathbb{H}^s_{\alpha,k}(\Omega)$, we can define the fractional s-power of the operator $L_{\alpha,k}$ as

$$L^s_{\alpha,k} v := \sum_{n=1}^{\infty} \mu^s_{\alpha,k,n} v_n Y_{\alpha,k,n} \in (\mathbb{H}^s_{\alpha,k}(\Omega))^*. \tag{5.9}$$

More precisely, the operator $L^s_{\alpha,k}$ is the Rietz isomorphism between $\mathbb{H}^s_{\alpha,k}(\Omega)$ endowed with the scalar product (5.8) and its dual space $(\mathbb{H}^s_{\alpha,k}(\Omega))^*$, that is

$${}_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\left\langle L^s_{\alpha,k} v_1, v_2 \right\rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} = (v_1, v_2)_{\mathbb{H}^s_{\alpha,k}(\Omega)} \qquad \text{for all } v_1, v_2 \in \mathbb{H}^s_{\alpha,k}(\Omega).$$

We would like to characterize the space $\mathbb{H}^s_{\alpha,k}(\Omega)$ more explicitly. To this end, let us define

$$\mathbb{H}^s(\Omega) := \begin{cases} H^s_0(\Omega), & \text{if } s \in (0,1) \setminus \{\frac{1}{2}\}, \\ H^{1/2}_{00}(\Omega), & \text{if } s = \frac{1}{2}. \end{cases}$$

as in Section 4.1 of Chapter 4. In Section 5.7 we will prove the following Proposition by means of Interpolation Theory.

Proposition 5.1.2. *For any $k \in \{3, \dots, N\}$, $s \in (0,1)$ and α as in (5.2)*

$$\mathbb{H}^s_{\alpha,k}(\Omega) = (L^2(\Omega), H^1_0(\Omega))_{s,2} = \mathbb{H}^s(\Omega),$$

with equivalent norms.

Let for any measurable function $v : \Omega \to \mathbb{R}$,

$$\tilde{v}(x) := \begin{cases} v(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then from [31, Proposition B.1] in the case $s \neq \frac{1}{2}$ and from the proof of [31, Proposition B.1] and (4.3) if $s = \frac{1}{2}$ we deduce the following result.

Proposition 5.1.3. *There exists a constant $C_{N,s,\Omega}$ such that*

$$\|\tilde{v}\|_{H^s(\mathbb{R}^n)} \leq C_{N,s,\Omega} \|v\|_{\mathbb{H}^s(\Omega)} \tag{5.10}$$

for any $v \in \mathbb{H}^s(\Omega)$.

Proposition 5.1.4. *There exists a constant $K_{N,s,\Omega}$ such that for any $v \in \mathbb{H}^s(\Omega)$*

$$\int_\Omega \frac{v^2(x)}{|x|^{2s}}\, dx \leq K_{N,s,\Omega} \|v\|^2_{\mathbb{H}^s(\Omega)}. \tag{5.11}$$

Proof. The following Hardy-type inequality due to Herbst [86]

$$2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)} \int_{\mathbb{R}^N} \frac{v^2(x)}{|x|^{2s}}\, dx \leq \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of u, holds for any $v \in H^s(\mathbb{R}^N)$. Then (5.11) follows from (5.10). □

By Proposition 5.1.2, we can define a weak solution to (5.1) as a function $u \in \mathbb{H}^s(\Omega)$ such that

$${}_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\left\langle L^s_{\alpha,k} u, \phi \right\rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} = \int_\Omega g u \phi\, dx, \qquad \text{for any } \phi \in C^\infty_c(\Omega). \tag{5.12}$$

Thanks to (5.3), (5.11) and the Hölder inequality, the right hand side of (5.12) is well defined, that is it belongs to $(\mathbb{H}^s(\Omega))^*$ as a linear functional of ϕ.

Given the local nature of the Almgren monotonicity formula we need to localize the problem by means of an extension procedure in the spirt of [37] or [35], see also [125, Section 3.1]. Let

$$C := \Omega \times (0, +\infty), \quad \partial C_L := \partial\Omega \times (0, +\infty).$$

Proposition 5.1.5. *For any $\phi \in C^\infty_c(\mathbb{R}^N \times [0, +\infty))$ and any $k \in \{3, \dots, N\}$*

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \frac{\phi^2}{|x|^2_k} dz \leq \left(\frac{2}{k-2}\right)^2 \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla_x \phi|^2\, dz, \tag{5.13}$$

where ∇_x is the gradient respect to the first N variables.

Proof. Let $\phi \in C^\infty_c(\Omega \times [0, +\infty))$ and $k \in \{3, \dots, N\}$. Then $\phi(\cdot, y) \in C^\infty_c(\Omega)$ for any $y \in [0, \infty)$ and so multiplying by y^{1-2s} and integrating over $(0, \infty)$ we deduce (5.13) from (5.4). □

Let
$$H^1_{0,L}(C, y^{1-2s}) := \left\{ V \in H^1(C, y^{1-2s}) : V = 0 \text{ on } \partial C_L \right\}.$$
The condition $V = 0$ on ∂C_L is meant in a classical trace sense. Indeed the weight y^{1-2s} is smooth, bounded and strictly positive on $\Omega \times [y_1, y_2]$ for any $0 < y_1 < y_2 < +\infty$, and so we can use classical trace theory for the space $H^1(\Omega \times (y_1, y_2))$ for any $0 < y_1 < y_2 < +\infty$.

From [37, Proposition 2.1] and [34, Proposition 2.1, Lemma 2.6] we deduce the following result.

Proposition 5.1.6. *There exists a linear and continuous trace operator*
$$\mathrm{Tr} : H^1_{0,L}(C, y^{1-2s}) \to \mathbb{H}^s(\Omega)$$
which is also surjective.

See Section 5.2 for a proof of the following next extension theorem,.

Theorem 5.1.7. *Let $v \in \mathbb{H}^s(\Omega)$, $k \in \{3, \dots, N\}$ and α as in (5.2). Then there exists a unique function $V \in H^1_{0,L}(C, y^{1-2s})$ such that V weakly solves the problem*
$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla V) = y^{1-2s}\frac{\alpha}{|x|_k^2}V, & \text{in } C, \\ \mathrm{Tr}(V) = v, & \text{on } \Omega, \\ -\lim_{y\to 0^+} y^{1-2s}\frac{\partial V}{\partial y} = c_{N,s}L^s_{k,\alpha}v, & \text{on } \Omega, \end{cases} \tag{5.14}$$
where $c_{N,s} > 0$ is a constant depending only on N and s, in the sense that
$$\int_C y^{1-2s}\nabla V \cdot \nabla\phi \, dz - \int_C y^{1-2s}\frac{\alpha}{|x|_k^2}V\phi \, dz = c_{N,s} \, {}_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\left\langle L^s_{\alpha,k}v, \phi(\cdot, 0)\right\rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} \tag{5.15}$$
for any $\phi \in C^\infty_c(\Omega \times [0, +\infty))$. Furthermore
$$\int_C y^{1-2s}|\nabla V(x, y)|^2 \, dz - \int_C y^{1-2s}\frac{\alpha}{|x|_k^2}V^2 \, dz = c_{N,s} \, \|v\|^2_{\mathbb{H}^s_{\alpha,k}(\Omega)} \tag{5.16}$$
and V is the only solution to the minimization problem
$$\inf\left\{ \int_C y^{1-2s}\left(|\nabla W|^2 - \frac{\alpha}{|x|_k^2}w^2 \right) dz : W \in H^1_{0,L}(C, y^{1-2s}) \text{ and } \mathrm{Tr}(W) = v \right\}. \tag{5.17}$$

From Theorem 5.1.7 we deduce the following corollary.

Corollary 5.1.8. *Let $u \in \mathbb{H}^s(\Omega)$ be a solution of (5.12). Then there exists a unique $U \in H^1_{0,L}(C, y^{1-2s})$ such that*
$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = y^{1-2s}\frac{\alpha}{|x|_k^2}U, & \text{in } C, \\ \mathrm{Tr}(U) = u, & \text{on } \Omega, \\ -\lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y} = c_{N,s}gu, & \text{on } \Omega, \end{cases} \tag{5.18}$$
where $c_{N,s} > 0$ is the constant depending only on N and s defined in Theorem 5.1.7, in the sense that
$$\int_C y^{1-2s}\nabla U \cdot \nabla\phi \, dz - \int_C y^{1-2s}\frac{\alpha}{|x|_k^2}U\phi \, dz = c_{N,s}\int_\Omega gu\phi(\cdot, 0) \, dx \tag{5.19}$$
for any $\phi \in C^\infty_c(\Omega \times [0, +\infty))$. Furthermore
$$\int_C y^{1-2s}|\nabla U(x, y)|^2 \, dz - \int_C y^{1-2s}\frac{\alpha}{|x|_k^2}U^2 \, dz = c_{N,s} \, \|u\|^2_{\mathbb{H}^s_{\alpha,k}(\Omega)} = c_{N,s}\int_\Omega gu^2 \, dx.$$

Let $\theta = \frac{z}{|z|}$ for any $z \in \mathbb{R}^{N+1}$ and $\theta' = (\theta_1, \dots, \theta_N)$.

The asymptotic profile of a solution U of (5.19) in 0 will turn out to be related to the following eigenvalue problem

$$\begin{cases} -\operatorname{div}_{\mathbb{S}}(\theta_{N+1}^{1-2s} \nabla_{\mathbb{S}} Z) - \theta_{N+1}^{1-2s} \frac{\alpha}{|\theta|_k^2} Z = \gamma \theta_{N+1}^{1-2s} Z, & \text{in } \mathbb{S}^+, \\ -\lim\limits_{\theta_{N+1} \to 0^+} \theta_{N+1}^{1-2s} \nabla_{\mathbb{S}} Z \cdot \nu = 0, & \text{on } \mathbb{S}', \end{cases} \tag{5.20}$$

where ν is the outer normal vector to $\mathbb{S}^+$ on $\mathbb{S}'$, that is $\nu = -(0, \dots, 0, 1)$ and

$$\begin{aligned} &\mathbb{S} := \{\theta \in \mathbb{R}^{N+1} : |\theta|^2 = 1\}, \\ &\mathbb{S}^+ := \{\theta = (\theta', \theta_{N+1}) \in \mathbb{S} : \theta_{N+1} > 0\}, \\ &\mathbb{S}' := \{\theta = (\theta', \theta_{N+1}) \in \mathbb{S} : \theta_{N+1} = 0\}. \end{aligned}$$

We refer to Subsection 5.2.1 for a variational formulation of (5.20). By classical spectral theory, see Subsection 5.2.1 for further details, the eigenvalues of (5.20) are a non-decreasing and diverging sequence $\{\gamma_{\alpha,k,n}\}_{n \in \mathbb{N} \setminus \{0\}}$ (we repeat each eigenvalue according to its multiplicity). We have the following estimate on $\gamma_{\alpha,k,1}$:

$$\gamma_{\alpha,k,1} > -\left(\frac{N-2s}{2}\right)^2$$

for any $k \in \{3, \dots, N\}$ and α as in (5.2), see Proposition 5.2.3. We can actually compute $\gamma_{\alpha,k,1}$ in terms of the first eigenvalue $\eta_{\alpha,k,1}$ of the problem

$$-\Delta_{\mathbb{S}'} \Psi - \frac{\alpha}{|\theta'|_k^2} \Psi = \eta \Psi \quad \text{in } \mathbb{S}' \tag{5.21}$$

as

$$\gamma_{\alpha,k,1} = 2(1-s) \left[\sqrt{\left(\frac{N-2}{2}\right)^2 + \eta_{\alpha,k,1}} - \frac{N-2}{2} \right] + \eta_{\alpha,k,1}, \tag{5.22}$$

see Section 5.6. In particular, if $k = N$ then $\eta_{\alpha,k,1} = -\alpha$ and so

$$\gamma_{\alpha,N,1} = 2(1-s) \left[\sqrt{\left(\frac{N-2}{2}\right)^2 - \alpha} - \frac{N-2}{2} \right] - \alpha. \tag{5.23}$$

If $k = N$, we are able to obtain an explicit expression of $\gamma_{\alpha,N,1}$ for any $\alpha \in \left(-\infty, \frac{N-2}{2}\right)$. For the restricted fractional Laplacian with a Hardy-type potential it is also possible to obtain a formula for the first eigenvalue of the corresponding problem on a hemisphere although with a more implicit expression, see [60, Proposition 2.3].

Theorem 5.1.9. *Let U be a non-trivial solution of (5.19) and suppose that g satisfies (5.3). Then there exist an eigenvalue $\gamma_{\alpha,k,n}$ of (5.20) and a correspondent eigenfunction Z such that*

$$\lambda^{\frac{N-2s}{2} - \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}}} U(\lambda z) \to |z|^{-\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}}} Z(z/|z|) \quad \text{as } \lambda \to 0^+$$

strongly in $H^1(B_1^+, y^{1-2s})$.

From Section 3.1 in Chapter 3 and the previous theorem we obtain the following.

Theorem 5.1.10. *Let u be a non-trivial solution solution of* (5.12) *and suppose that g satisfies* (5.3). *Then there exist an eigenvalue $\gamma_{\alpha,k,n}$ of* (5.20) *and a correspondent eigenfunction Z such that*

$$\lambda^{\frac{N-2s}{2}-\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,n}}}u(\lambda x) \to |x|^{-\frac{N-2s}{2}+\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,n}}}\,\mathrm{Tr}(Z(\cdot/|\cdot|))(x) \quad \text{as } \lambda \to 0^+$$

strongly in $H^s(B_1')$.

We will also prove a more precise and complete version of Theorem 5.1.9 and Theorem 5.1.10 in Section 5.5, computing the coordinates of the eigenfunction Z respect to a basis of the eigenspace corresponding to $\gamma_{\alpha,k,n}$. Furthermore we can deduce the following strong unique continuation properties as corollaries of Theorem 5.1.9 and Theorem 5.1.10 respectively.

Corollary 5.1.11. *Let U be a solution of* (5.19) *and suppose that g satisfies* (5.3). *If*

$$U(z) = o(|z|^n) = o(|(x,y)|^n) \ \text{as } x \to 0,\ y \to 0^+ \quad \text{for any } n \in \mathbb{N} \tag{5.24}$$

then $U \equiv 0$ on $\Omega \times (0,\infty)$.

Corollary 5.1.12. *Let u be a solution of* (5.12) *and suppose that g satisfies* (5.3). *If*

$$u(x) = o(|x|^n) \ \text{as } x \to 0, \quad \text{for any } n \in \mathbb{N}$$

then $u \equiv 0$ on Ω.

Remark 5.1.13. We have considered equation (5.1) with assumption (5.3) on the potential g for the sake of simplicity. With simple modifications to our arguments it is also possible to obtain the same results for a potential $g \in W^{\frac{N}{2s}+\varepsilon}(\Omega)$ for some $\varepsilon \in (0,1)$, see [75, Proposition 2.3] for the corresponding Pohozaev identity. Furthermore we can obtain analogous results for the more general equation

$$L^s_{k,\alpha}u = \frac{\lambda}{|x|^{2s}}u + gu,$$

with $\lambda \in \left(-\infty, 2^{2s}\frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}\right)$ with the same approach, where Γ is the usual Γ-function.

This Chapter is organized as follows. In Section 5.2 we prove the extension Theorem 5.1.7, study an eigenvalue problem on a hemisphere, which will turn out to be correlated to the asymptotic profiles of weak solutions of (5.1), and discuss some useful inequalities. In Section 5.3 we prove a Pohozaev type identity. In Section 5.4 we develop a monotonicity formula for the extend problem while in Section 5.5 we carry out the blow-up argument and prove our main results. Finally in Section 5.6 we compute the first eigenvalue of the problem studied in 5.2 while in Section 5.7 we provide some further details on the functional setting for equation (5.1) which will be introduced in Section 5.1.

5.2 Preliminaries

We start this section by proving Theorem 5.1.7.

Proof of Theorem 5.1.7. We follow the proof of [37, Proposition 2.1]. Let $v \in \mathbb{H}^s(\Omega)$ and consider

$$V(x,y) := \sum_{n=1}^{\infty} v_n Y_{\alpha,k,n}(x) h_n(y), \qquad \text{where } v_n = \int_\Omega v Y_{\alpha,k,n}\, dx \tag{5.25}$$

and $h_n : (0,+\infty) \to \mathbb{R}$ is a solution to the problem

$$\begin{cases} h_n'' + \frac{1-2s}{y} h_n' - \mu_{\alpha,k,n} h_n = 1, \text{ on } (0,+\infty), \\ h_n(0) = 1, \\ \lim_{y\to\infty} h_n(y) = 0. \end{cases} \tag{5.26}$$

From the proof of [37, Proposition 2.1], (5.26) admits a unique solution h_n for any $n \in \mathbb{N}\setminus\{0\}$ and

$$-\lim_{y\to 0^+} y^{1-2s} h_n'(y) = c_{N,s}\mu_{\alpha,k,n}^s, \tag{5.27}$$

for some positive constant $c_{N,s} > 0$ depending only on N and s. Furthermore for any $y \in [0+,\infty)$ by (5.25) and Remark 5.1.1

$$\int_\Omega \left|\frac{\partial V}{\partial y}(x,y)\right|^2 dx + \int_\Omega |\nabla_x V(x,y)|^2\, dx - \int_\Omega \frac{\alpha}{|x|_k^2} V^2(x,y)\, dx \\ = \sum_{n=1}^{\infty} v_n^2 (h_n'(y))^2 + \mu_{\alpha,k,n} v_n^2 h_n(y)^2. \tag{5.28}$$

Proceeding exactly as in [37, Proposition 2.1] we can show that (5.16) holds. Hence, in view of (5.13), $V \in H^1(C, y^{1-2s})$ and $\sum_{n=1}^{j} v_n Y_{\alpha,k,n}(x) h_n(y) \to V$ in $H^1(C, y^{1-2s})$ as $j \to \infty$. In conclusion $V \in H^1_{0,L}(C, y^{1-2s})$ since $\sum_{n=1}^{j} v_n Y_{\alpha,k,n}(x) h_n(y) \in H^1_{0,L}(C, y^{1-2s})$ for any $j \in \mathbb{N}, j \geq 1$.

In contrast to [37, Proposition 2.1], V might not be smooth for $y > 0$ since the functions $Y_{\alpha,k,n}$ might not be smooth on Ω. Then we prove that V satisfies (5.14) in the weak sense given by (5.15). Let $\phi \in C_c^\infty(\Omega \times [0,+\infty))$. Then

$$\phi(x,y) = \sum_{n=1}^{\infty} \phi_n(y) Y_{\alpha,k,n}(x), \qquad \text{where } \phi_n(y) := \int_\Omega \phi(x,y) Y_{\alpha,k,n}(x)\, dx,$$

and similarly to (5.28)

$$\int_\Omega |\nabla\phi(x,y)|^2\, dx - \int_\Omega \frac{\alpha}{|x|_k^2}\phi^2(x,y)\, dx = \sum_{n=1}^{\infty} (\phi_n'(y))^2 + \mu_{\alpha,k,n}\phi_n(y)^2. \tag{5.29}$$

Then by (5.25) and Remark 5.1.1

$$\int_\Omega \nabla V(x,y)\cdot\nabla\phi(x,y)\, dx - \int_\Omega \frac{\alpha}{|x|_k^2} V(x,y)\phi(x,y)\, dx \\ = \sum_{n=1}^{\infty} v_n h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} v_n h_n(y)\phi_n(y). \tag{5.30}$$

Furthermore, for any $j \in \mathbb{N}$, by Hölder's inequality

$$\left| \int_0^{+\infty} y^{1-2s} \left(\sum_{n=j}^{\infty} v_n h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} v_n h_n(y)\phi_n(y) \right) dy \right|$$

$$\leq \frac{1}{2} \int_0^{+\infty} y^{1-2s} \left(\sum_{n=j}^{\infty} v_n^2 (h_n'(y))^2 + \mu_{\alpha,k,n} v_n^2 h_n(y)^2 \right) dy$$

$$+ \frac{1}{2} \int_0^{+\infty} y^{1-2s} \left(\sum_{n=j}^{\infty} (\phi_n'(y))^2 + \mu_{\alpha,k,n} \phi_n(y)^2 \right) dy.$$

By (5.28), (5.29) and the Monotone Convergence Theorem we conclude that

$$\lim_{j\to\infty} \int_0^{\infty} y^{1-2s} \left(\sum_{n=j}^{\infty} v_n h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} v_n h_n(y)\phi_n(y) \right) dy = 0.$$

Hence we may change the order of summation and integration in (5.30) obtaining

$$\int_C y^{1-2s} \left(\nabla V \cdot \nabla \phi - \frac{\alpha}{|x|_k^2} V\phi \right) dz = \sum_{n=1}^{\infty} v_n \int_0^{+\infty} y^{1-2s} (h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} h_n(y)\phi_n(y))\, dy.$$

An integration by parts, in view of (5.26) and (5.27), yields

$$\int_0^{+\infty} y^{1-2s} (h_n'(y)\phi_n'(y) + \mu_{\alpha,k,n} h_n(y)\phi_n(y))\, dy = c_{N,s} \mu_{\alpha,k,n}^s \phi_n(0).$$

It follows that

$$\int_C y^{1-2s} \nabla V \cdot \nabla \phi \, dz - \int_C y^{1-2s} \frac{\alpha}{|x|_k^2} V \phi \, dz = c_{N,s} \sum_{n=1}^{\infty} \mu_{\alpha,k,n}^s v_n \phi_n(0)$$

and so we have proved (5.15). If V_1 and V_2 solve (5.14) then by (5.2), (5.15) and (5.13) we deduce that

$$\int_C y^{1-2s} |\nabla(V_1 - V_2)|^2 \, dz = 0, \qquad \text{and} \qquad \mathrm{Tr}(V_1 - V_2) = 0$$

thus $V_1 = V_2$. Finally V solves the minimizing problem (5.17) in view of (5.15) and a density argument. □

Remark 5.2.1. For any $r > 0$ and any $V, W \in H^1(B_r^+, y^{1-2s})$, thanks to the Coarea Formula,

$$\int_{B_r^+} \left| y^{1-2s} \nabla U \cdot \frac{z}{|z|} W \right| dz = \int_0^r \left(\int_{S_\rho^+} \left| y^{1-2s} \nabla U \cdot \frac{z}{\rho} W \right| dS \right) d\rho$$

hence the function $f(\rho) := \int_{S_\rho^+} \left| y^{1-2s} \nabla U \cdot \frac{z}{\rho} W \right| dS$ is a well-defined element of $L^1(0, r)$. In particular a.e. $\rho \in (0, r)$ is a Lebesgue point of f.

Reasoning as in [60, Lemma 3.1] or [75, Proposition 3.7] we can prove the following.

Proposition 5.2.2. *Let U be a solution of* (5.19)*. For a.e. $r > 0$ such that $\overline{B'_r} \subset \Omega$ and any $W \in H^1(B_r^+, y^{1-2s})$*

$$\int_{B_r^+} y^{1-2s} \left(\nabla U \cdot \nabla W - \frac{\alpha}{|x|_k^2} U W \right) dz \\ = \frac{1}{r} \int_{S_r^+} y^{1-2s} \nabla U \cdot z \, W \, dS + c_{N,s} \int_{B'_r} g \, \mathrm{Tr}(U) \, \mathrm{Tr}(W) \, dx. \tag{5.31}$$

5.2.1 An Eigenvalue Problem on a hemisphere

In this section we provide a variational formulation of problem (5.20). To this end we consider the space

$$L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s}) := \{\Psi : \mathbb{S}^+ \to \mathbb{R} \text{ measurable: } \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \Psi^2 \, dS < +\infty\},$$

and the space $H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ defined as the completion of $C^\infty(\overline{\mathbb{S}^+})$ with respect to the norm

$$\|\phi\|_{H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})} := \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} (\phi^2 + |\nabla_{\mathbb{S}} \phi|^2) \, dS \right)^{1/2},$$

where $\nabla_{\mathbb{S}}$ is the Riemannian gradient respect to the standard metric on $\mathbb{S}$.

Proposition 5.2.3. *For any $k \in \{3, \dots, N\}$*

$$\left(\frac{k-2}{2} \right)^2 \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\Psi^2}{|\theta|_k^2} \, dS \le \left(\frac{N-2s}{2} \right)^2 \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\Psi|^2 \, dS + \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} \Psi|^2 \, dS \tag{5.32}$$

for any $\Psi \in H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})$.

Proof. Let $\phi \in C^\infty(\overline{\mathbb{S}^+})$, $f \in C_c^\infty((0, +\infty))$ with $f \neq 0$, and $V(z) := V(r\theta) = \phi(\theta) f(r)$. From (5.13) we obtain, passing in polar coordinates,

$$\left(\frac{k-2}{2} \right)^2 \left(\int_0^\infty r^{N-1-2s} f^2(r) \, dr \right) \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\phi^2}{|\theta|_k^2} \, dS \right) \\ \le \left(\int_0^\infty r^{N+1-2s} |f'(r)|^2 \, dr \right) \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \phi^2 \, dS \right) \\ + \left(\int_0^\infty r^{N-1-2s} f^2(r) \, dr \right) \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} \phi|^2 \, dS \right)$$

and so, thanks to the optimality of the classical Hardy constant, see [83, Theorem 330],

$$\left(\frac{k-2}{2} \right)^2 \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\phi^2}{|\theta|_k^2} \, dS \right) \\ \le \inf_{f \in C_c^\infty((0,+\infty)), f \neq 0} \frac{\int_0^\infty r^{N+1-2s} |f'(r)|^2 \, dr}{\int_0^\infty r^{N-1-2s} f(r)^2 \, dr} \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \phi^2 \, dS \right) + \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} \phi|^2 \, dS \\ = \left(\frac{N-2s}{2} \right)^2 \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\phi|^2 \, dS + \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} \phi|^2 \, dS.$$

In conclusion (5.32) follows by density. □

For any $k \in \{3, \dots, N\}$ and α as in (5.2), we say that γ is an eigenvalue of (5.20) if there exists a function $Z \in H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s}) \setminus \{0\}$ such that

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \nabla_{\mathbb{S}} Z \cdot \nabla_{\mathbb{S}} \Psi \, dS - \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\alpha}{|\theta|_k^2} Z \Psi \, dS = \gamma \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} Z \Psi \, dS, \tag{5.33}$$

for any $\Psi \in H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s})$. By (5.2), (5.32), the Spectral Theorem, and the compactness of the embedding $H^1(\mathbb{S}^+, \theta_{N+1}^{1-2s}) \hookrightarrow L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ (see [111, Theorem 19.7]) the eigenvalues of (5.20) are a non-decreasing and diverging sequence $\{\gamma_{\alpha,k,n}\}_{n \in \mathbb{N} \setminus \{0\}}$ (we repeat each eigenvalue according to its multiplicity). Let, for future reference,

$V_{\alpha,k,n}$ be the eigenspace of problem (5.20) associated to the eigenvalue $\gamma_{\alpha,k,n}$, (5.34)

$M_{\alpha,k,n}$ be the dimension of $V_{\alpha,k,n}$, (5.35)

$\{Z_{\alpha,k,n,i} : i \in \{1, \dots, M_{\alpha,k,n}\}\}$ be a $L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ orthonormal basis of $V_{\alpha,k,n}$ (5.36)
of eigenfunctions of problem (5.20).

Finally $\{Z_{\alpha,k,n}\}_{n \in \mathbb{N} \setminus \{0\}} := \bigcup_{n=1}^{\infty} \{Z_{\alpha,k,n,i} : i \in \{1, \dots, M_{\alpha,k,n}\}\}$ is an orthonormal basis of $L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$.

Remark 5.2.4. It is worth noticing that $Z_{\alpha,k,n}$ cannot vanish identically on $\mathbb{S}'$. We argue by contradiction. In view of [60, Lemma 2.1], we can show with a direct computation that $V(z) := |z|^{-\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}}} Z_{\alpha,k,n}(z/|z|)$ solves $\operatorname{div}(y^{1-2s} \nabla V) - y^{1-2s} \frac{\alpha}{|x|_k^2} V = 0$ on $\mathbb{R}^{N+1}_+$ and satisfies both zero Dirichlet and zero Neumann condition on $\mathbb{R}^N \times \{0\}$. Let

$$\Sigma_k := \{z \in \mathbb{R}^{N+1} : |x|_k = 0\}. \tag{5.37}$$

Note that Σ_k has Lebesgue measure 0 and that V is a solution to an elliptic equitation with a Muckenhoupt weight and bounded coefficients away from Σ_k. Then by the unique continuation principles proved in [127], we conclude that $V \equiv 0$. Hence $Z_{\alpha,k,n} \equiv 0$ which is a contradiction.

5.2.2 Inequalities in a weighted Sobolev space

In this subsection we prove some useful inequalities.

Proposition 5.2.5. *For any $r > 0$, any $k \in \{0, \dots, N\}$, and any $V \in H^1(B_r^+, y^{1-2s})$*

$$\left(\frac{k-2}{2}\right)^2 \int_{B_r^+} y^{1-2s} \frac{V^2}{|x|_k^2} \, dz \le \int_{B_r^+} y^{1-2s} |\nabla V|^2 \, dz + \frac{N-2s}{2r} \int_{S_r^+} y^{1-2s} V^2 \, dz. \tag{5.38}$$

Proof. By density it is enough to prove (5.38) for any $\phi \in C^\infty(\overline{B_r^+})$. Passing in polar coordinates, by (5.32) and [60, Lemma 2.4], we have that

$$\left(\frac{k-2}{2}\right)^2 \int_{B_r^+} y^{1-2s} \frac{V^2}{|x|_k^2}\, dz = \left(\frac{k-2}{2}\right)^2 \int_0^r \rho^{N-1-2s} \left(\int_{\mathbb{S}^+} \frac{V^2(\rho\theta)}{|\theta|_k^2}\, dS \right) d\rho$$

$$\leq \int_0^r \rho^{N-1-2s} \left(\left(\frac{N-2s}{2}\right)^2 \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |V^2(\rho\theta)|^2\, dS + \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} V(\rho\theta)|^2\, dS \right) d\rho$$

$$= \left(\frac{N-2s}{2}\right)^2 \int_{B_r^+} y^{1-2s} \frac{V^2}{|z|^2}\, dz + \int_0^r \rho^{N-1-2s} \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}} V(\rho\theta)|^2\, dS \right) d\rho$$

$$\leq \frac{N-2s}{2r} \int_{S_r^+} y^{1-2s} V^2\, dS$$

$$+ \int_0^r \rho^{N+1-2s} \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \left(\frac{1}{\rho^2} |\nabla_{\mathbb{S}} V(\rho\theta)|^2 + \left| \frac{\partial V}{\partial \rho}(\rho\theta) \right|^2 \right) dS \right) d\rho$$

$$= \frac{N-2s}{2r} \int_{S_r^+} y^{1-2s} V^2\, dS + \int_{B_r^+} y^{1-2s} |\nabla V|^2\, dz,$$

hence we have proved (5.38). □

Proposition 5.2.6. *Let $r > 0$ and suppose that $h : B_r' :\to \mathbb{R}$ is a measurable function such that*

$$|h(x)| \leq C_h |x|^{-2s+\varepsilon} \quad \text{for a.e. } x \in B_r', \tag{5.39}$$

for some positive constant C_h and some $\varepsilon \in (0,1)$. Then for any $k \in \{3, \ldots, N\}$, any α as in (5.2) and any $V \in H^1(B_r^+, y^{1-2s})$

$$\int_{B_r'} |h| \operatorname{Tr}(V)^2\, dx$$

$$\leq k_{N,s,h} r^\varepsilon \left(\int_{B_r^+} y^{1-2s} |\nabla V|^2\, dz - \int_{B_r^+} y^{1-2s} \frac{\alpha}{|x|_k^2} V^2\, dz + \frac{N-2s}{2r} \int_{S_r^+} y^{1-2s} V^2\, dz \right), \tag{5.40}$$

where $k_{N,s,h}$ is a positive constant depending only on N, s, C_h.

Proof. The claim follows from (5.39), [60, Lemma 2.5], and (5.38). □

In view of (5.2) there exists $r_0 > 0$ such that

$$\overline{B_{r_0}^+} \subset C \quad \text{and} \quad \alpha \left(\frac{2}{k-2}\right)^2 + c_{N,s} k_{N,s,g} r_0^\varepsilon < 1, \tag{5.41}$$

where $k_{N,s,g}$ is as in Proposition 5.2.6, $c_{N,s}$ as in Theorem 5.1.7 and g as in (5.3).

Proposition 5.2.7. *Let $k \in \{3, \ldots, N\}$, α as (5.2), g as in (5.3), $c_{N,s}$ as in Theorem 5.1.7 and r_0 as in (5.41). Then for any $V \in H^1(B_r^+, y^{1-2s})$ and any $r \in (0, r_0]$*

$$\int_{B_r^+} y^{1-2s} \left(|\nabla W|^2 - \frac{\alpha}{|x|_k^2} W^2 \right) dz - c_{N,s} \int_{B_r'} g \operatorname{Tr}(W)^2\, dx + \frac{N-2s}{2r} \int_{S_r^+} y^{1-2s} W^2 dS$$

$$\geq \left(1 - \alpha \left(\frac{2}{k-2}\right)^2 + c_{N,s} k_{N,s,g} r_0^\varepsilon \right)$$

$$\times \left(\int_{B_r^+} y^{1-2s} |\nabla W|^2\, dz + \frac{N-2s}{2r} \int_{S_r^+} y^{1-2s} W^2 dS \right). \tag{5.42}$$

Proof. The claim follows from Proposition 5.2.6, (5.3) and (5.38). □

5.3 Approximated problems and a Pohozaev-type Identity

In order to obtain a Pohozaev type identity for a weak solution of (5.18), we approximate it with a family of solutions to more regular problems. Then we obtain a Pohozaev-type identity for such solutions and pass to the limit.

Remark 5.3.1. Let r_0 be as in (5.41). By (5.42) and (3.7), for any $r \in (0, r_0)$,

$$\|W\|_{g,\alpha,k,0} := \left(\int_{B_r^+} y^{1-2s}\left(|\nabla W|^2 - \frac{\alpha}{|x|_k^2}W^2\right)dz - c_{N,s}\int_{B_r'} g\,\mathrm{Tr}(W)^2\,dx\right)^{\frac{1}{2}}$$

defines a norm on $H^1_{0,S_r^+}(B_r^+, y^{1-2s})$ (see (3.2)) equivalent to (3.1). Furthermore

$$\|W\|_{g,\alpha,k} := \left(\int_{B_r^+} y^{1-2s}\left(|\nabla W|^2 - \frac{\alpha}{|x|_k^2}W^2\right)dz - c_{N,s}\int_{B_r'} g\,\mathrm{Tr}(W)^2\,dx + \int_{S_r^+} y^{1-2s}W^2\,dz\right)^{\frac{1}{2}}$$

defines a norm on $H^1(B_r^+, y^{1-2s})$ equivalent to (3.1).

Theorem 5.3.2. *Let U be a weak solutions of (5.18), and r_0 as in (5.41). Then there exists $\tilde\lambda > 0$ such that for any $\lambda \in (0, \tilde\lambda)$ the problem*

$$\begin{cases} -\,\mathrm{div}(y^{1-2s}\nabla V) = y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}V, & \text{in } B_{r_0}^+,\\ V = U, & \text{on } S_{r_0}^+,\\ -\lim_{y\to 0^+} y^{1-2s}\frac{\partial V}{\partial y} = c_{N,s}g\,\mathrm{Tr}(V), & \text{on } B_{r_0}', \end{cases}$$

where $c_{N,s} > 0$ is as in Theorem 5.1.7, admits a weak solution $U_\lambda \in H^1(B_{r_0}^+, y^{1-2s})$, i.e.

$$\int_{B_{r_0}^+} y^{1-2s}\nabla U_\lambda \cdot \nabla W\,dz - \int_{B_{r_0}^+} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda W\,dz = c_{N,s}\int_{B_{r_0}'} g\,\mathrm{Tr}(V)\,\mathrm{Tr}(W)\,dx \tag{5.43}$$

for any $W \in H^1_{0,S_{r_0}^+}(B_{r_0}^+, y^{1-2s})$ (see (3.2)), and $U_\lambda = U$ on $S_{r_0}^+$. Furthermore

$$U_\lambda \to U \text{ strongly in } H^1(B_{r_0}^+, y^{1-2s}) \quad \text{as } \lambda \to 0^+.$$

Proof. Let us consider the map $\Phi : \mathbb{R} \times H^1_{0,S_r^+}(B_r^+, y^{1-2s}) \to (H^1_{0,S_r^+}(B_r^+, y^{1-2s}))^*$ defined as

$$\begin{aligned}\Phi(\lambda, V)(W) := &\int_{B_{r_0}^+} y^{1-2s}\nabla V \cdot \nabla W\,dz - \int_{B_{r_0}^+} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}VW\,dz\\ &- c_{N,s}\int_{B_{r_0}'} g\,\mathrm{Tr}(V)\,\mathrm{Tr}(W)\,dx + \int_{B_{r_0}^+} y^{1-2s}\left(\frac{\alpha}{|x|_k^2+\lambda^2} - \frac{\alpha}{|x|_k^2}\right)UW\,dz.\end{aligned}$$

for any $W \in H^1_{0,S_{r_0}^+}(B_{r_0}^+, y^{1-2s})$. It is clear that Φ is well defined and that Φ is continuous in $(0,0)$ in view of Hölder's inequality, Proposition 5.2.6, (5.3), and (5.38). Furthermore $\Phi(0,0) = 0$.

Let us prove that $\Phi_V(0,0) \in \mathcal{L}(H^1_{0,S_{r_0}^+}(B_{r_0}^+, y^{1-2s}), (H^1_{0,S_{r_0}^+}(B_{r_0}^+, y^{1-2s})^*)$ is an isomorphism, where Φ_V is the partial derivative with respect to V of Φ. For any $W_1, W_2 \in H^1_{0,S_{r_0}^+}(B_{r_0}^+, y^{1-2s})$

$${}_{(H^1_{0,S_{r_0}^+}(B_{r_0}^+,y^{1-2s}))^*}\langle \Phi_V(0,0)(W_1), W_2\rangle_{H^1_{0,S_{r_0}^+}(B_{r_0}^+,y^{1-2s})} = (W_1, W_2)_{g,\alpha,k,0}\,.$$

Hence, by Remark 5.3.1, $\Phi_V(0,0)$ is the Rietz isomorphism associated to the norm $\|\cdot\|_{g,\alpha,k,0}$.

We are now in position to apply the Implicit Function Theorem to Φ in the point $(0,0)$ and conclude that there exist $\tilde{\lambda}>0$, $\rho>0$ and a function

$$f:(-\tilde{\lambda},\tilde{\lambda})\to B_\rho(0), \tag{5.44}$$

continuous in 0, such that $\Phi(\lambda,V)=0$ if and only if $V=f(\lambda)$ for any $\lambda\in(-\tilde{\lambda},\tilde{\lambda})$ and $V\in B_\rho(0)$. The set $B_\rho(0)$ in (5.44) is defined as $B_\rho(0)=\{V\in H^1_{0,S^+_{r_0}}(B^+_{r_0},y^{1-2s}):\|V\|_{H^1(B^+_{r_0},y^{1-2s})}<\rho\}$.

It follows that $U_\lambda:=U-f(\lambda)$ solves (5.43) for any $\lambda\in(0,\tilde{\lambda})$ since U is a solution of (5.31). Furthermore $U_\lambda\to U$ strongly in $H^1(B^+_{r_0},y^{1-2s})$ as $\lambda\to 0^+$ since f is continuous in 0 and $f(0)=0$. □

Remark 5.3.3. Let U_λ be a solution of (5.43). Then, reasoning in the same way of Proposition 5.2.2, we can prove that for a.e. $r\in(0,r_0)$, a.e. $\rho\in(0,r)$ and any $W\in H^1(B^+_r\setminus B^+_\rho,y^{1-2s})$

$$\int_{B^+_r\setminus B^+_\rho} y^{1-2s}\left(\nabla U_\lambda\cdot\nabla W-\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda W\right)dz=\frac{1}{r}\int_{S^+_r}y^{1-2s}\nabla U_\lambda\cdot z\,W\,dS \\ -\frac{1}{\rho}\int_{S^+_\rho}y^{1-2s}\nabla U_\lambda\cdot z\,W\,dS+c_{N,s}\int_{B'_r\setminus B'_\rho}g\,\mathrm{Tr}(U_\lambda)\,\mathrm{Tr}(W)\,dx. \tag{5.45}$$

Let ν be the outer normal vector to B^+_r on S^+_r, that is $\nu(z)=\frac{z}{|z|}$.

Proposition 5.3.4. *For any $\lambda\in(0,\tilde{\lambda})$, let U_λ be a solution of (5.43). Then for a.e. $r\in(0,r_0)$*

$$\begin{aligned}&\frac{r}{2}\int_{S^+_r}y^{1-2s}|\nabla U_\lambda|^2\,dS-r\int_{S^+_r}y^{1-2s}|\nabla U_\lambda\cdot\nu|^2\,dS\\&+\frac{c_{N,s}}{2}\int_{B'_r}(Ng+x\cdot\nabla g)|\,\mathrm{Tr}(U_\lambda)|^2\,dx-\frac{c_{N,s}r}{2}\int_{S'_r}g|\,\mathrm{Tr}(U_\lambda)|^2\,dS\\&=\frac{N-2s}{2}\int_{B^+_r}y^{1-2s}|\nabla U_\lambda|^2\,dz+\int_{B^+_r}y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z\,dz\end{aligned} \tag{5.46}$$

for a.e. $r\in(0,r_0)$.

Proof. We proceed in the spirit of Proposition 3.2.3, since $(|x|_k^2+\lambda^2)^{-1}U_\lambda\in L^2(B^+_r,y^{1-2s})$ and $g\in W^{1,\infty}_{loc}(\Omega\setminus\{0\})$. Then by Theorem 3.2.1, Proposition 3.1.7 and the proof of Proposition 3.2.3, for any $r\in(0,r_0)$ and $\rho\in(0,r)$,

$$\nabla_x U_\lambda\in H^1(B^+_r\setminus B^+_\rho,y^{1-2s}),\quad\text{and}\quad y^{1-2s}\frac{\partial U_\lambda}{\partial y}\in H^1(B^+_r\setminus B^+_\rho,y^{2s-1}), \tag{5.47}$$

$$\mathrm{Tr}(U_\lambda)\in H^{1+s}(B'_r\setminus B'_\rho),\quad\text{and}\quad \mathrm{Tr}(\nabla_x U_\lambda)=\nabla\,\mathrm{Tr}(U_\lambda), \tag{5.48}$$

$$\nabla U_\lambda\cdot z\in H^1(B^+_r\setminus B^+_\rho,y^{1-2s}),\quad\text{and}\quad \mathrm{Tr}(\nabla U_\lambda\cdot z)=\mathrm{Tr}(\nabla U_\lambda)\cdot x,$$

where $H^{1+s}(B'_r\setminus B'_\rho):=\{w\in H^1(B'_r\setminus B'_\rho):\frac{\partial w}{\partial x_i}\in W^{s,2}(B'_r\setminus B'_\rho)\text{ for any }i=1,\dots,N\}$. We also have, in view of (5.43), the following identity

$$\mathrm{div}(y^{1-2s}|\nabla U_\lambda|^2 z-2y^{1-2s}\nabla U_\lambda\cdot z\nabla U_\lambda)=(N-2s)|\nabla U_\lambda|^2+2\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z \tag{5.49}$$

in a distributional sense in $B_r^+ \setminus B_\rho^+$. Furthermore, thanks to (5.47),

$$\begin{aligned}\operatorname{div}(y^{1-2s}\nabla U_\lambda \cdot z \nabla U_\lambda) \\ = -y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda \nabla U_\lambda \cdot z + y^{1-2s}\nabla U_\lambda \cdot \nabla(\nabla U_\lambda \cdot z) \in L^1(B_r^+ \setminus B_\rho^+)\end{aligned} \tag{5.50}$$

and so by (5.49)

$$\operatorname{div}(y^{1-2s}|\nabla U_\lambda|^2 z) \in L^1(B_r^+ \setminus B_\rho^+).$$

Let, for any $\delta \in (0, r)$,

$$B_{r,\delta}^+ := \{(x,y) \ \in B_r^+ : y > \delta\} \quad \text{and} \quad S_{r,\delta}^+ := \{(x,y) \ \in S_r^+ : y > \delta\}. \tag{5.51}$$

Integrating by part on $B_r^+ \setminus B_\rho^+$ we obtain, for any $\delta \in (0, \rho)$,

$$\begin{aligned}\int_{B_{r,\delta}^+ \setminus B_{\rho,\delta}^+} \operatorname{div}(y^{1-2s}|\nabla U_\lambda|^2 z)\, dz = r\int_{S_{r,\delta}^+} y^{1-2s}|\nabla U_\lambda|^2\, dS - \rho \int_{S_{\rho,\delta}^+} y^{1-2s}|\nabla U_\lambda|^2\, dS \\ - \delta^{2-2s}\int_{B'_{\sqrt{r^2-\delta^2}} \setminus B'_{\sqrt{\rho^2-\delta^2}}} |\nabla U_\lambda|^2(x,\delta)\, dx.\end{aligned} \tag{5.52}$$

We claim that there exists a sequence $\delta_n \to 0^+$ such that

$$\lim_{n\to\infty} \delta^{2-2s}\int_{B'_{\sqrt{r^2-\delta_n^2}} \setminus B'_{\sqrt{\rho^2-\delta_n^2}}} |\nabla U_\lambda|^2(x,\delta)\, dx = 0 \tag{5.53}$$

arguing by contradiction. If the claim does not hold than there exist a constant $C > 0$ and $\delta_0 \in (0, \rho)$ such that $B'_r \times (0, \delta_0) \subseteq B_{r_0}^+$ and

$$\delta^{1-2s}\int_{B'_{\sqrt{r^2-\delta^2}} \setminus B'_{\sqrt{\rho^2-\delta^2}}} |\nabla U_\lambda|^2(x,\delta)\, dx \ge \frac{C}{\delta} \quad \text{for any } \delta \in (0, \delta_0). \tag{5.54}$$

Then integrating (5.54) over $(0, \delta_0)$ we obtain

$$\int_0^{\delta_0}\left(\delta^{1-2s}\int_{B'_r}|\nabla U_\lambda|^2(x,\delta)\, dx\right) d\delta \ge \int_0^{\delta_0}\frac{C}{\delta}d\delta = +\infty,$$

which is a contradiction in view of the Fubini-Tonelli Theorem. Then we can pass to the limit as $\delta = \delta_n$ in (5.52) and conclude that, thanks to the Dominate Convergence Theorem and the Monotone Convergence Theorem,

$$\int_{B_r^+ \setminus B_\rho^+} \operatorname{div}(y^{1-2s}|\nabla U_\lambda|^2 z)\, dz = r\int_{S_r^+} y^{1-2s}|\nabla U_\lambda|^2\, dS - \rho\int_{S_\rho^+} y^{1-2s}|\nabla U_\lambda|^2\, dS \tag{5.55}$$

for a.e $r \in (0, r_0)$ and a.e. $\rho \in (0, r)$. Testing (5.45) with $\nabla U \cdot z$ we obtain, in view of (5.50) and Remark 5.3.3,

$$\begin{aligned}\int_{B_r^+ \setminus B_\rho^+} \operatorname{div}(y^{1-2s}\nabla U_\lambda \cdot z \nabla U_\lambda)\, dz = \int_{B_r^+ \setminus B_\rho^+} y^{1-2s}\nabla U_\lambda \cdot \nabla(\nabla U_\lambda \cdot z)\, dz \\ - \int_{B_r^+ \setminus B_\rho^+} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda \nabla U_\lambda \cdot z\, dz = \frac{1}{r}\int_{S_r^+} y^{1-2s}|\nabla U_\lambda \cdot z|^2\, dS \\ - \frac{1}{\rho}\int_{S_\rho^+} y^{1-2s}|\nabla U_\lambda \cdot z|^2\, dS + c_{N,s}\int_{B'_r \setminus B'_\rho} g\, \mathrm{Tr}(U_\lambda)\, \nabla_x \mathrm{Tr}(U_\lambda) \cdot x\, dx.\end{aligned} \tag{5.56}$$

We note that $g\,\mathrm{Tr}(U_\lambda)^2 x \in W^{1,1}(B'_r \setminus B'_\rho, \mathbb{R}^N)$ by (5.3) and (5.48) hence integrating by part we obtain

$$\int_{B'_r\setminus B'_\rho} g\ \mathrm{Tr}(U_\lambda)\nabla_x \mathrm{Tr}(U_\lambda)\cdot x\,dx = -\frac{1}{2}\int_{B'_r\setminus B'_\rho}(Ng + x\cdot\nabla g)\,\mathrm{Tr}(U_\lambda)^2\,dx$$
$$+\frac{r}{2}\int_{S'_r} g|\,\mathrm{Tr}(U_\lambda)|^2 dS' - \frac{\rho}{2}\int_{S'_\rho} g|\,\mathrm{Tr}(U_\lambda)|^2 dS'. \quad (5.57)$$

Arguing as in the proof of (5.53), we see that there exists a sequence $\rho_n \to 0^+$ such that

$$\lim_{n\to\infty}\rho_n\int_{S^+_{\rho_n}} y^{1-2s}|\nabla U_\lambda|^2\,dS = \lim_{n\to\infty}\rho_n\int_{S^+_{\rho_n}} y^{1-2s}\left|\nabla U_\lambda\cdot\frac{z}{|z|}\right|^2 dS$$
$$= \lim_{n\to\infty}\rho_n\int_{S'_{\rho_n}} g|\,\mathrm{Tr}(U_\lambda)|^2 dS' = 0.$$

Then by the Dominated Convergence Theorem, we can pass to the limit as $\rho = \rho_n$ and $n\to\infty$ in (5.55), (5.56), (5.57) and conclude that (5.46) holds in view of (5.49). □

Proposition 5.3.5. *Let U be a solution of (5.19). Then for a.e. $r \in (0, r_0)$*

$$\frac{r}{2}\int_{S^+_r} y^{1-2s}\left(|\nabla U|^2 - \frac{\alpha}{|x|_k^2}U^2\right)dS - r\int_{S^+_r} y^{1-2s}|\nabla U\cdot\nu|^2\,dS$$
$$+\frac{c_{N,s}}{2}\int_{B'_r}(Ng + x\cdot\nabla g)|\,\mathrm{Tr}(U)|^2\,dx - \frac{c_{N,s}}{2}r\int_{S'_r} g|\,\mathrm{Tr}(U)|^2\,dS'$$
$$=\frac{N-2s}{2}\int_{B^+_r} y^{1-2s}\left(|\nabla U|^2 - \frac{\alpha}{|x|_k^2}U^2\right)dz. \quad (5.58)$$

Proof. Let $r \in (0, r_0)$ and $B^+_{r,\delta}$, $S^+_{r,\delta}$ be as in (5.51) for any $\delta \in (0, r)$. Then, by (5.2),

$$\mathrm{div}\left(y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2\,z\right)$$
$$= y^{1-2s}\left(2\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z + (N+2-2s)\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2 - 2\frac{\alpha|x|_k^2}{(|x|_k^2+\lambda^2)^2}U_\lambda^2\right) \quad (5.59)$$

and $y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2 z \in W^{1,1}(B^+_{r,\delta}, \mathbb{R}^{N+1})$. Integrating (5.59) by part in $B^+_{r,\delta}$ we obtain

$$r\int_{S^+_{r,\delta}} y^{1-2s}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2\,dS - \delta^{2-2s}\int_{B'_{\sqrt{r^2-\delta^2}}}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2(x,\delta)\,dx$$
$$= \int_{B^+_{r,\delta}} y^{1-2s}2\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda\nabla U_\lambda\cdot z\,dz$$
$$+\int_{B^+_{r,\delta}} y^{1-2s}\left((N+2-2s)\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2 - 2\frac{\alpha|x|_k^2}{(|x|_k^2+\lambda^2)^2}U_\lambda^2\right)dz. \quad (5.60)$$

We claim that there exists a sequence $\delta_n \to 0^+$ as $n\to\infty$ such that

$$\lim_{n\to\infty}\delta_n^{2-2s}\int_{B'_{\sqrt{r^2-\delta_n^2}}}\frac{\alpha}{|x|_k^2+\lambda^2}U_\lambda^2(x,\delta_n)\,dx = 0 \quad (5.61)$$

arguing by contradiction. If (5.61) does not hold, then there exists a constant $C > 0$ and $\delta_0 \in (0, r)$ such that $(0, \delta_0) \times B'_r \subseteq B^+_{r_0}$ and

$$\delta^{1-2s} \int_{B'_{\sqrt{r^2-\delta^2}}} \frac{\alpha}{|x|_k^2 + \lambda^2} U_\lambda^2(x, \delta)\, dx \geq \frac{C}{\delta}$$

for any $\delta \in (0, \delta_0)$. Integrating over $(0, \delta_0)$ we obtain

$$+\infty > \int_0^{\delta_0} \delta^{1-2s} \left(\int_{B'_r} \frac{\alpha}{|x|_k^2 + \lambda^2} U_\lambda^2(x, \delta)\, dx \right) d\delta \geq \int_0^{\delta_0} \frac{C}{\delta}\, d\delta,$$

a contradiction in view of the Fubini-Tonelli Theorem. Passing to the limit for $\delta = \delta_n$ as $n \to \infty$ in (5.60) we conclude that

$$\begin{aligned} \int_{B_r^+} y^{1-2s} \frac{\alpha}{|x|_k^2 + \lambda^2} U_\lambda \nabla U_\lambda \cdot z\, dz &= \frac{r}{2} \int_{S_r^+} y^{1-2s} \frac{\alpha}{|x|_k^2 + \lambda^2} U_\lambda^2\, dS \\ &\quad - \frac{1}{2} \int_{B_r^+} y^{1-2s} \left((N + 2 - 2s) \frac{\alpha}{|x|_k^2 + \lambda^2} U_\lambda^2 - 2 \frac{\alpha |x|_k^2}{(|x|_k^2 + \lambda^2)^2} U_\lambda^2 \right) dz. \end{aligned} \tag{5.62}$$

Now we pass to the limit as $\lambda \to 0^+$, eventually along a suitable sequence $\lambda_n \to 0^+$, in each term of (5.46) taking into account (5.62). We recall that, by Theorem 5.3.2, $U_\lambda \to U$ strongly in $H^1(B_r^+, y^{1-2s})$ for any $r \in (0, r_0]$. It is clear that for any $r \in (0, r_0)$

$$\lim_{\lambda \to 0^+} \int_{B_r^+} y^{1-2s} |\nabla U_\lambda|^2\, dz = \int_{B_r^+} y^{1-2s} |\nabla U|^2\, dz.$$

Furthermore there exists a sequence $\lambda_n \to 0$ as $n \to \infty$ and $G \in L^2(B^+_{r_0}, y^{1-2s} |x|_k^{-2})$ such that

$$\begin{aligned} &(N + 2 - 2s) \frac{\alpha}{|x|_k^2 + \lambda_n^2} U_{\lambda_n}^2 - 2 \frac{\alpha |x|_k^2}{(|x|_k^2 + \lambda_n^2)^2} U_{\lambda_n}^2 \to (N - 2s) \frac{\alpha}{|x|_k^2} U^2 \quad \text{for a.e. } z \in B^+_{r_0}, \\ &\frac{\alpha}{|x|_k^2 + \lambda_n^2} U_{\lambda_n} - \frac{\alpha}{|x|_k^2} U \to 0 \quad \text{for a.e. } z \in B^+_{r_0}, \\ &|U_{\lambda_n}| \leq |G| \quad \text{for a.e. } z \in B^+_{r_0} \text{ and any } n \in \mathbb{N}. \end{aligned} \tag{5.63}$$

Then by the Dominated Convergence Theorem we conclude that for any $r \in (0, r_0)$

$$\begin{aligned} \lim_{n \to \infty} \int_{B_r^+} y^{1-2s} \left((N + 2 - 2s) \frac{\alpha}{|x|_k^2 + \lambda_n^2} U_{\lambda_n}^2 - 2 \frac{\alpha |x|_k^2}{(|x|_k^2 + \lambda_n^2)^2} U_{\lambda_n}^2 \right) dz \\ = (N - 2s) \int_{B_r^+} y^{1-2s} \frac{\alpha}{|x|_k^2} U^2\, dz \end{aligned}$$

and

$$\lim_{n \to \infty} \int_{B_r^+} y^{1-2s} \left| \frac{\alpha}{|x|_k^2 + \lambda_n^2} U_{\lambda_n}^2 - \frac{\alpha}{|x|_k^2} U^2 \right| dz = 0. \tag{5.64}$$

By (5.3), (5.40), (5.38) and Proposition 3.4

$$\lim_{\lambda \to 0^+} \int_{B'_r} |Ng + \nabla g \cdot x|\, |\operatorname{Tr}(U_\lambda) - \operatorname{Tr}(U)|^2\, dx = 0 \tag{5.65}$$

hence, for any $r \in (0, r_0)$,

$$\lim_{\lambda\to 0^+} \int_{B'_r} (Ng + x\cdot\nabla g)|\operatorname{Tr}(U_\lambda)|^2\,dx = \int_{B'_r} (Ng + \nabla g\cdot x)|\operatorname{Tr}(U)|^2\,dx.$$

By Fatou's Lemma and the Coarea Formula,

$$\int_0^{r_0} \left(\liminf_{\lambda\to 0^+} \int_{S_r^+} y^{1-2s}|\nabla U_\lambda - \nabla U|^2\,dS\right) dr \le \liminf_{\lambda\to 0^+} \int_{B_{r_0}^+} y^{1-2s}|\nabla U_\lambda - \nabla U|^2\,dS = 0,$$

and so

$$\liminf_{\lambda\to 0^+} \int_{S_r^+} y^{1-2s}|\nabla U_\lambda|^2\,dS = \int_{S_r^+} y^{1-2s}|\nabla U|^2\,dS$$

for a.e. $r \in (0, r_0)$. Similarly, for a.e. $r \in (0, r_0)$

$$\liminf_{\lambda\to 0^+} \int_{S_r^+} y^{1-2s}|\nabla U_\lambda\cdot\nu|^2\,dS = \int_{S_r^+} y^{1-2s}|\nabla U\cdot\nu|^2\,dS,$$

and, by (5.65) and Fatou's Lemma,

$$\liminf_{\lambda\to 0^+} \int_{S'_r} g|\operatorname{Tr}(U_\lambda)|^2\,d'S = \int_{S'_r} g|\operatorname{Tr}(U)|^2\,dS'.$$

Furthermore passing to the limit for $\lambda = \lambda_n$ as $n\to\infty$ and λ_n is as in (5.63), we obtain

$$\lim_{n\to\infty} \int_{S_r^+} y^{1-2s}\frac{\alpha}{|x|_k^2 + \lambda_n^2} U_{\lambda_n}^2\,dS = \int_{S_r^+} y^{1-2s}\frac{\alpha}{|x|_k^2} U^2\,dS$$

for a.e. $r \in (0, r_0)$, thanks to Fatou's Lemma and (5.64). In conclusion (5.58) holds. □

5.4 The Monotonicity Formula

Let U be a non-trivial solution of (5.19), let r_0 be as in (5.41). For any $r \in (0, r_0]$ we define the height and energy functions respectively as

$$H(r) := \frac{1}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s}U^2\,dS, \tag{5.66}$$

$$D(r) := \frac{1}{r^{N-2s}} \left(\int_{B_r^+} y^{1-2s}\left(|\nabla U|^2 - \frac{\alpha}{|x|_k^2}U^2\right)dz - c_{N,s}\int_{B'_r} g|\operatorname{Tr}(U)|^2\,dx\right). \tag{5.67}$$

The proof of the next Proposition is very similar to [47, Lemma 3.1] and we omit it. We also recall that ν is the outer normal vector to B_r^+ on S_r^+, that is $\nu(z) = \frac{z}{|z|}$.

Proposition 5.4.1. *We have that $H \in W^{1,1}_{loc}((0, r_0])$ and*

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{S_r^+} y^{1-2s}\frac{\partial U}{\partial\nu}U\,dS = \frac{2}{r}D(r), \tag{5.68}$$

in a distributional sense and for a.e. $r \in (0, r_0)$.

Proposition 5.4.2. *Let H be as in (5.66). Then $H(r) > 0$ for any $r \in (0, r_0]$.*

Proof. Assume by contradiction that there exists $r \in (0, r_0]$ such that $H(r) = 0$. From (5.31) and Remark 5.3.1 we deduce that $U \equiv 0$ on B_r^+. Let Σ_k be as in (5.37). The function U is a solution of an elliptic equation with bounded coefficients away from Σ_k and $\mathbb{R}^N \times \{0\}$. Then the claim follows from classical unique continuation principles, see for example [134]. □

Proposition 5.4.3. *The function D defined in (5.67) belongs to $W^{1,1}_{loc}((0, r_0])$ and*

$$D'(r) = \frac{2}{r^{N+1-2s}} \left(r \int_{S_r^+} y^{1-2s} |\nabla U \cdot \nu|^2 \, dS - c_{N,s} \int_{B_r'} \left(sg + \frac{1}{2} x \cdot \nabla g \right) |\operatorname{Tr}(U)|^2 \, dx \right) \quad (5.69)$$

in a distributional sense and for a.e. $r \in (0, r_0)$.

Proof. By the Coarea Formula

$$D'(r) = (2s - N) r^{-N+2s-1} \left(\int_{B_r^+} y^{1-2s} \left(|\nabla U|^2 - \frac{\alpha}{|x|_k^2} U^2 \right) dz - c_{N,s} \int_{B_r'} g |\operatorname{Tr}(U)|^2 \, dx \right)$$
$$+ r^{-N+2s} \left(\int_{S_r^+} y^{1-2s} \left(|\nabla U|^2 - \frac{\alpha}{|x|_k^2} U^2 \right) dS - c_{N,s} \int_{S_r'} g |\operatorname{Tr}(U)|^2 \, dS' \right)$$

and so (5.69) follows from (5.58). Furthermore $D \in W^{1,1}_{loc}((0, r_0])$ by (5.69), (5.67) and the Coarea Formula. □

Let us define, for any $r \in (0, r_0]$, the frequency function $\mathcal{N}$ as

$$\mathcal{N}(r) := \frac{D(r)}{H(r)}. \quad (5.70)$$

In view of Proposition 5.4.2 the definition of $\mathcal{N}$ is well-posed.

Proposition 5.4.4. *We have that $\mathcal{N} \in W^{1,1}_{loc}((0, r_0])$ and for any $r \in (0, r_0]$*

$$\mathcal{N}(r) > -\frac{N - 2s}{2}. \quad (5.71)$$

Furthermore

$$\mathcal{N}'(r) = \nu_1(r) + \nu_2(r) \quad (5.72)$$

in a distributional sense and for a.e. $r \in (0, r_0)$, where

$$\nu_1(r) := \frac{2r \left(\left(\int_{S_r^+} y^{1-2s} U^2 \, dS \right) \left(\int_{S_r^+} y^{1-2s} \left| \frac{\partial U}{\partial \nu} \right|^2 dS \right) - \left(\int_{S_r^+} y^{1-2s} U \frac{\partial U}{\partial \nu} \, dS \right)^2 \right)}{\left(\int_{S_r^+} y^{1-2s} U^2 \, dS \right)^2},$$

and

$$\nu_2(r) := -c_{N,s} \frac{\int_{B_r'} (2sg + x \cdot \nabla g) |\operatorname{Tr}(U)|^2 \, dx}{\int_{S_r^+} y^{1-2s} U^2 \, dS}. \quad (5.73)$$

Finally

$$\nu_1(r) \geq 0 \quad \text{for any } r \in (0, r_0]. \quad (5.74)$$

Proof. Since $1/H, D \in W^{1,1}_{loc}((0, r_0])$ it follows that $\mathcal{N} \in W^{1,1}_{loc}((0, r_0])$. We can deduce (5.71) directly from (5.42) and (5.70). Furthermore by (5.68)

$$\frac{d}{dr}\mathcal{N}'(r) = \frac{D'(r)H(r) - D(r)H'(r)}{H^2(r)} = \frac{D'(r)H(r) - \frac{r}{2}(H'(r))^2}{H^2(r)}$$

and so (5.72) follows from (5.66), (5.67) and (5.69). Finally (5.74) is a consequence of the Cauchy-Schwartz inequality in $L^2(S_r^+, y^{1-2s})$ between the vectors U and $\frac{\partial U}{\partial \nu}$. □

Proposition 5.4.5. *There exists a constant $C > 0$ such that*

$$|v_2(r)| \leq Cr^{-1+\varepsilon}\left(\mathcal{N}(r) + \frac{N-2s}{2}\right) \qquad \textit{for any } r \in (0, r_0]. \tag{5.75}$$

Proof. The claim follows from (5.3), (5.40), (5.42) and (5.73). □

Proposition 5.4.6. *There exists a constant $C_1 > 0$ such that*

$$\mathcal{N}(r) \leq C_1 \qquad \textit{for any } r \in (0, r_0]. \tag{5.76}$$

Proof. Thanks to Proposition 5.4.4, for a.e. $r \in (0, r_0)$

$$\left(\mathcal{N} + \frac{N-2s}{2}\right)'(r) \geq v_2(r) \geq -Cr^{-1+\varepsilon}\left(\mathcal{N}(r) + \frac{N-2s}{2}\right).$$

Hence an integration over (r, r_0) yields

$$\mathcal{N}(r) \leq -\frac{N-2s}{2} + \left(\mathcal{N}(r_0) + \frac{N-2s}{2}\right)e^{\frac{C}{\varepsilon}r_0^{\varepsilon}}$$

for any $r \in (0, r_0)$. □

Proposition 5.4.7. *The limit*

$$\gamma := \lim_{r\to 0^+} \mathcal{N}(r) \tag{5.77}$$

exists and it is finite.

Proof. Since $\mathcal{N} \in W^{1,1}_{loc}((0, r_0])$ by Proposition 5.4.4, for any $r \in (0, r_0)$

$$\mathcal{N}(r) = \mathcal{N}(r_0) - \int_r^{r_0} \mathcal{N}'(r)\,dr = \mathcal{N}(r_0) - \int_r^{r_0} v_1(r)\,dr - \int_r^{r_0} v_2(r)\,dr. \tag{5.78}$$

Since $v_1 \geq 0$ by (5.74) and $v_2 \in L^1(0, r_0)$ by (5.75) and (5.76), we can pass to the limit as $r \to 0^+$ in (5.78) and conclude that the limit (5.77) exists. From (5.71) and (5.76) it is finite. □

The proofs of Propositions 5.4.8 and 5.4.9 are standard and we omit them, see for example [47, Lemma 3.7, Lemma 4.6], [65, Lemma 5.6, Lemma 6.4], [65, Lemma 5.9, Lemma 6.6] or Propositions 4.4.9, and 4.4.10 in Chapter 4.

Proposition 5.4.8. *Let γ be as in (5.77). Then there exists a constant $K > 0$ such that*

$$H(r) \leq Kr^{2\gamma} \qquad \textit{for any } r \in (0, r_0). \tag{5.79}$$

Furthermore for any $\sigma > 0$ there exist a constant K_σ such that

$$H(r) \geq K_\sigma r^{2\gamma+\sigma} \qquad \textit{for any } r \in (0, r_0). \tag{5.80}$$

Proposition 5.4.9. *Let γ be as in (5.77). Then there exists the limit*

$$\lim_{r\to 0^+} r^{-2\gamma} H(r) \tag{5.81}$$

and it is finite.

5.5 The Blow-Up Analysis

Let U be a non-trivial solution of (5.19) and let r_0 be as in (5.41). For any $\lambda \in (0, r_0]$ let

$$V^\lambda(z) := \frac{U(\lambda z)}{\sqrt{H(\lambda)}}. \tag{5.82}$$

By a change of variables, it is clear that V^λ weakly solves

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla V^\lambda) = y^{1-2s}\frac{\alpha}{|x|_k^2}V^\lambda, & \text{in } B^+_{r_0/\lambda}, \\ -\lim_{y\to 0^+} y^{1-2s}\frac{\partial V^\lambda}{\partial y} = c_{N,s}\lambda^{2s} g(\lambda\cdot)\operatorname{Tr}(V^\lambda), & \text{on } B'_{r_0/\lambda}, \end{cases}$$

in the sense that

$$\int_{B^+_{r_0/\lambda}} y^{1-2s}\nabla V^\lambda\cdot\nabla W\,dz - \int_{B'_{r_0/\lambda}} y^{1-2s}\frac{\alpha}{|x|_k^2}V^\lambda W\,dz = c_{N,s}\lambda^{2s}\int_{B^+_{r_0/\lambda}} g(\lambda\cdot)\operatorname{Tr}(V^\lambda)\operatorname{Tr}(W)\,dx$$

for any $W \in H^1_{0,S^+_{r_0/\lambda}}(B^+_{r_0/\lambda}, y^{1-2s})$ (see (3.2)). Furthermore by (5.66) and a change of variables

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}|V^\lambda(\theta)|^2 dS = 1 \qquad \text{for any } \lambda \in (0, r_0]. \tag{5.83}$$

Since the frequency function $\mathcal{N}$ is bounded on $[0, r_0]$ (see (5.71) and (5.76)) we can prove the following proposition.

Proposition 5.5.1. *The family of functions $\{V^\lambda\}_{\lambda\in(0,r_0]}$ is bounded in $H^1(B_1^+, y^{1-2s})$.*

Proof. For any $\lambda \in (0, r_0)$, thanks to (5.42), (5.82) and a change of variables,

$$\begin{aligned} \mathcal{N}(\lambda) &= \frac{\lambda^{2s-N}}{H(\lambda)}\left(\int_{B^+_\lambda} y^{1-2s}\left(|\nabla U|^2 - \frac{\alpha}{|x|_k^2}U^2\right)dz - c_{N,s}\int_{B'_\lambda} g|\operatorname{Tr}(U)|^2\,dx\right) \\ &\geq \left(1 - \alpha\left(\frac{2}{k-2}\right)^2 + c_{N,s}k_{N,s,g}r_0^\varepsilon\right)\frac{\lambda^{2s-N}}{H(\lambda)}\left(\int_{B^+_\lambda} y^{1-2s}|\nabla U|^2\,dz\right) - \frac{N-2s}{2} \\ &= \left(1 - \alpha\left(\frac{2}{k-2}\right)^2 + c_{N,s}k_{N,s,g}r_0^\varepsilon\right)\left(\int_{B_1^+} y^{1-2s}|\nabla V^\lambda|^2\,dz\right) - \frac{N-2s}{2}. \end{aligned}$$

Hence the claim follows from (5.76), (5.83) and (5.38). □

Now we establish the following doubling property.

Proposition 5.5.2. *There exists a constant $C_3 > 0$ such that*

$$\frac{1}{C_3} H(\lambda) \le H(R\lambda) \le C_3 H(\lambda), \tag{5.84}$$

$$\int_{B_R^+} y^{1-2s} |V^\lambda|^2 \, dz \le C_3 2^{N+2-2s} \int_{B_1^+} y^{1-2s} |V^{R\lambda}|^2 \, dz, \tag{5.85}$$

$$\int_{B_R^+} y^{1-2s} |\nabla V^\lambda|^2 \, dz \le C_3 2^{N-2s} \int_{B_1^+} y^{1-2s} |\nabla V^{R\lambda}| \, dz, \tag{5.86}$$

for any $\lambda \in (0, r_0)$ and any $R \in [1, 2]$.

Proof. By (5.68), (5.71), and (5.76)

$$-\frac{N-2s}{r} \le \frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} \le \frac{2C_1}{r} \quad \text{for a.e. } r \in (0, r_0).$$

An integration over $(\lambda, R\lambda)$ with $R \in (1, 2]$ yields

$$R^{2s-N} \le \frac{H(R\lambda)}{H(\lambda)} \le R^{2C_1}$$

thus (5.84) holds for $R \in (1, 2]$ while if $R = 1$ it is obvious.

Furthermore for any $\lambda \in (0, r_0)$, by (5.84) and a change of variables,

$$\begin{aligned}\int_{B_R^+} y^{1-2s} |V^\lambda|^2 \, dz &= \frac{\lambda^{-N-2+2s}}{H(\lambda)} \int_{B_{R\lambda}^+} y^{1-2s} |U|^2 \, dz \le C_3 \frac{\lambda^{-N-2+2s}}{H(\lambda R)} \int_{B_{R\lambda}^+} y^{1-2s} |U|^2 \, dz \\ &= C_3 R^{N+2-2s} \int_{B_1^+} y^{1-2s} |V^{\lambda R}|^2 \, dz \le C_3 2^{N+2-2s} \int_{B_1^+} y^{1-2s} |V^{\lambda R}|^2 \, dz,\end{aligned}$$

for any $R \in [1, 2]$. Hence we have proved (5.85) and (5.86) follows from (5.84) in the same way. □

In view of the Coarea Formula, there exists a subset $\mathcal{M} \subset (0, r_0)$ of Lebesgue measure 0 such that $|\nabla U| \in L^2(S_r^+, y^{1-2s})$ and (5.31) holds for any $r \in (0, r_0) \setminus \mathcal{M}$.

Proposition 5.5.3. *There exist $M > 0$ and $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ there exists $R_\lambda \in [1, 2]$ such that $R_\lambda \lambda \notin \mathcal{M}$ and*

$$\int_{S_{R_\lambda}^+} y^{1-2s} |\nabla V^\lambda|^2 \, dS \le M \int_{B_{R_\lambda}^+} y^{1-2s} (|\nabla V^\lambda|^2 + |V^\lambda|^2) \, dz. \tag{5.87}$$

Proof. By Proposition 5.5.1 $\{V^\lambda\}_{\lambda \in (0, \frac{r_0}{2})}$ is bounded in $H^1(B_2^+, y^{1-2s})$. Hence

$$\limsup_{\lambda \to 0^+} \int_{B_2^+} y^{1-2s} (|\nabla V^\lambda|^2 + |V^\lambda|^2) \, dz < +\infty. \tag{5.88}$$

Let, for any $\lambda \in (0, \frac{r_0}{2})$,

$$f_\lambda(R) := \int_{B_R^+} y^{1-2s} (|\nabla V^\lambda|^2 + |V^\lambda|^2) \, dz.$$

The function f is absolutely continuous on $[1, 2]$ and, thanks to the Coarea Formula, its distributional derivative is given by

$$f'_\lambda(R) = \int_{S_R^+} y^{1-2s}(|\nabla V^\lambda|^2 + |V^\lambda|^2)\, dS \quad \text{for a.e. } R \in [1, 2].$$

We argue by contradiction supposing that for any $M > 0$ there exists $\lambda_n \to 0^+$ such that

$$\int_{S_R^+} y^{1-2s}(|\nabla V^{\lambda_n}|^2 + |V^{\lambda_n}|^2)\, dS > M \int_{B_R^+} y^{1-2s}(|\nabla V^{\lambda_n}|^2 + |V^{\lambda_n}|^2)\, dz$$

for any $n \in \mathbb{N}$ and any $R \in [1, 2] \setminus \frac{1}{\lambda_n}\mathcal{M}$, hence for a.e. $R \in [1, 2]$. Therefore

$$f'_{\lambda_n}(R) > M f_{\lambda_n}(R) \quad \text{for a.e. } R \in [1, 2] \text{ and any } n \in \mathbb{N}.$$

An integration over $[1, 2]$ yields $f_{\lambda_n}(2) > e^M f_{\lambda_n}(1)$ for any $n \in \mathbb{N}$. Hence

$$\liminf_{n\to\infty} f_{\lambda_n}(1) \le \limsup_{n\to\infty} f_{\lambda_n}(1) \le e^{-M} \limsup_{n\to\infty} f_{\lambda_n}(2)$$

and so

$$\liminf_{\lambda\to 0^+} f_\lambda(1) \le e^{-M} \limsup_{\lambda\to 0^+} f_\lambda(2)$$

for any $M > 0$. It follows that $\liminf_{\lambda\to 0^+} f_\lambda(1) = 0$ by (5.88). We conclude that there exists a sequence $\lambda_n \to 0^+$ as $n \to \infty$ and $V \in H^1(B_1^+, y^{1-2s})$ such that

$$\lim_{n\to\infty} \int_{B_1^+} y^{1-2s}(|\nabla V^{\lambda_n}|^2 + |V^{\lambda_n}|^2)\, dz = 0$$

and $V_{\lambda_n} \rightharpoonup V$ weakly in $H^1(B_1^+, y^{1-2s})$, taking into account Proposition 5.5.1. By Proposition 3.4, (5.83) and the lower semicontinuity of norms, we obtain

$$\int_{B_1^+} y^{1-2s}(|\nabla V|^2 + |V|^2)\, dz = 0 \quad \text{and} \quad \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} V^2\, dS = 1$$

which is a contradiction. □

Proposition 5.5.4. *Let R_λ be as in Proposition 5.5.3. Then there exists a constant $\overline{M} > 0$ such that*

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla V^{R_\lambda \lambda}|^2 dS \le \overline{M} \quad \text{for any } \lambda \in \left(0, \min\left\{\lambda_0, \frac{r_0}{2}\right\}\right). \tag{5.89}$$

Proof. By a change of variables, the fact that $R_\lambda \in [1, 2]$ and (5.82)

$$\begin{aligned}
\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\nabla V^{R_\lambda \lambda}|^2 dS &= R_\lambda^{-N+1+2s} \frac{H(\lambda)}{H(R_\lambda \lambda)} \int_{S_{R_\lambda}^+} y^{1-2s} |\nabla V^\lambda|^2 dS \\
&\le 2C_3 M \int_{B_{R_\lambda}^+} y^{1-2s}(|\nabla V^\lambda|^2 + |V^\lambda|^2)\, dz \\
&\le 2^{N+3-2s} C_3^2 M \int_{B_1^+} y^{1-2s}(|\nabla V^{R_\lambda \lambda}|^2 + |V^{R_\lambda \lambda}|^2)\, dz \le \overline{M} < +\infty,
\end{aligned}$$

for some $\overline{M} > 0$, in view of Proposition 5.5.1, (5.84), (5.85), (5.86), and (5.87). □

Proposition 5.5.5. *Let U be a non-trivial solution of (5.19) and γ be as in (5.77). Then*

(i) there exists $n \in \mathbb{N} \setminus \{0\}$ such that

$$\gamma = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}}, \tag{5.90}$$

where $\gamma_{\alpha,k,n}$ is an eigenvalue of problem (5.20),

(ii) for any sequence $\lambda_p \to 0^+$ as $p \to \infty$ there exists a subsequence $\lambda_{p_q} \to 0^+$ as $q \to \infty$ and a eigenfunction Z of problem (5.20), corresponding to the eigenvalue $\gamma_{\alpha,k,n}$, such that $\|Z\|_{L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})} = 1$ and

$$\frac{U(\lambda_{p_q} z)}{\sqrt{H(\lambda_{p_q})}} \to |z|^\gamma Z\left(\frac{z}{|z|}\right) \quad \textit{strongly in } H^1(B_1^+, y^{1-2s}) \quad \textit{as } q \to \infty.$$

Proof. Let V^λ be as in (5.82) and R_λ as in Proposition 5.5.3. Then $\{V^{R_\lambda \lambda}\}_{\lambda \in \left(0, \min\left\{\lambda_0, \frac{r_0}{2}\right\}\right)}$ is bounded in $H^1(B_1^+, y^{1-2s})$, thanks to Proposition 5.5.1. Let $\lambda_p \to 0^+$ as $p \to \infty$. Then there exists a subsequence $\lambda_{p_q} \to 0^+$ as $q \to \infty$ and $V \in H^1(B_1^+, y^{1-2s})$ such that $V^{R_{\lambda_{p_q}} \lambda_{p_q}} \rightharpoonup V$ weakly in $H^1(B_1^+, y^{1-2s})$ as $q \to \infty$. By Proposition 3.4 the trace operator $\mathrm{Tr}_{S_1^+}$ is compact and so

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |V|^2 \, dS = 1, \tag{5.91}$$

in view of (5.83). Hence V is non-trivial. We claim that

$$V^{R_{\lambda_{p_q}} \lambda_{p_q}} \rightharpoonup V \quad \text{strongly in } H^1(B_1^+, y^{1-2s}) \text{ as } q \to \infty. \tag{5.92}$$

For q sufficiently large $B_1^+ \subseteq B^+_{r_0/(R_{\lambda_{p_q}} \lambda_{p_q})}$ and since $R_{\lambda_{p_q}} \lambda_{p_q} \notin \mathcal{M}$, where $\mathcal{M}$ is as in Proposition 5.5.3, we have that

$$\begin{aligned}\int_{B_1^+} y^{1-2s} \left(\nabla V^{R_{\lambda_{p_q}} \lambda_{p_q}} \cdot \nabla W - \frac{\alpha}{|x|_k^2} V^{R_{\lambda_{p_q}} \lambda_{p_q}} W\right) dz = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\partial V^{R_{\lambda_{p_q}} \lambda_{p_q}}}{\partial \nu} W \, dS \\ + c_{N,s} (R_{\lambda_{p_q}} \lambda_{p_q})^{2s} \int_{B_1'} g(R_{\lambda_{p_q}} \lambda_{p_q} \cdot) \,\mathrm{Tr}(V^{R_{\lambda_{p_q}} \lambda_{p_q}}) \,\mathrm{Tr}(W) \, dx\end{aligned} \tag{5.93}$$

for any $W \in H^1(B_1^+, y^{1-2s})$, thanks to (5.31) and a change of variables. We will pass to the limit as $q \to \infty$ in (5.93). To this end we observe that, for any $W \in H^1(B_1^+, y^{1-2s})$,

$$\left| \lambda^{2s} \int_{B_1'} g(\lambda \cdot) \operatorname{Tr}(V^\lambda) \operatorname{Tr}(W)\, dx \right| = \left| \frac{\lambda^{2s-N}}{H(\lambda)} \int_{B_\lambda'} g(x) \operatorname{Tr}(U)(x) \operatorname{Tr}(W)(\lambda x)\, dx \right|$$

$$\leq k_{N,s,g} \frac{\lambda^{2s+\varepsilon-N}}{H(\lambda)} \left| \int_{B_\lambda^+} y^{1-2s} |\nabla U|^2 \, dz - \int_{B_\lambda^+} y^{1-2s} \frac{\alpha}{|x|_k^2} |U|^2 \, dz + \frac{N-2s}{2\lambda} \int_{S_\lambda^+} y^{1-2s} |U|^2 \, dS \right|^{\frac{1}{2}}$$

$$\times \left| \int_{B_\lambda^+} y^{1-2s} |\nabla W(\lambda \cdot)|^2 \, dz - \int_{B_\lambda^+} y^{1-2s} \frac{\alpha}{|x|_k^2} W(\lambda \cdot)^2 \, dz + \frac{N-2s}{2\lambda} \int_{S_\lambda^+} y^{1-2s} |W(\lambda \cdot)|^2 \, dS \right|^{\frac{1}{2}}$$

$$= k_{N,s,g} \lambda^\varepsilon \left| \int_{B_1^+} y^{1-2s} |\nabla V^\lambda|^2 \, dz - \int_{B_1^+} y^{1-2s} \frac{\alpha}{|x|_k^2} |V^\lambda|^2 \, dz + \frac{N-2s}{2} \right|^{\frac{1}{2}}$$

$$\times \left| \int_{B_1^+} y^{1-2s} |\nabla W|^2 \, dz - \int_{B_1^+} y^{1-2s} \frac{\alpha}{|x|_k^2} W^2 \, dz + \frac{N-2s}{2} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |W|^2 \, dS \right|^{\frac{1}{2}}$$

by a change of variables, the Hölder inequality, (5.3), (5.40), (5.82) and (5.83). We conclude that

$$\lim_{\lambda \to 0^+} \left| \lambda^{2s} \int_{B_1'} g(\lambda \cdot) \operatorname{Tr}(V^\lambda) \operatorname{Tr}(W)\, dx \right| = 0, \tag{5.94}$$

by Proposition 5.5.1 and (5.38). Thanks to (5.89), there exists a function $f \in L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ such that

$$\frac{\partial V^{R_{\lambda_{p_q}} \lambda_{p_q}}}{\partial \nu} \rightharpoonup f \quad \text{weakly in } L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s}) \text{ as } q \to \infty, \tag{5.95}$$

up to a subsequence. Hence

$$\lim_{q \to \infty} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\partial V^{R_{\lambda_{p_q}} \lambda_{p_q}}}{\partial \nu} W \, dS = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} f W \, dS$$

for any $W \in H^1(B_1^+, y^{1-2s})$. Furthermore

$$\lim_{q \to \infty} \int_{B_1^+} y^{1-2s} \left(\nabla V^{R_{\lambda_{p_q}} \lambda_{p_q}} \cdot \nabla W - \frac{\alpha}{|x|_k^2} V^{R_{\lambda_{p_q}} \lambda_{p_q}} W \right) dz$$
$$= \int_{B_1^+} y^{1-2s} \left(\nabla V \cdot \nabla W - \frac{\alpha}{|x|_k^2} V W \right) dz$$

by Remark 5.3.1. It follows that

$$\int_{B_1^+} y^{1-2s} \left(\nabla V \cdot \nabla W - \frac{\alpha}{|x|_k^2} V W \right) dz = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} f W \, dS$$

for any $W \in H^1(B_1^+, y^{1-2s})$, that is V is a weak solution of the problem

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla V) = \frac{\alpha}{|x|_k^2} V, & \text{in } B_1^+, \\ -\lim_{y \to 0^+} y^{1-2s} \frac{\partial V}{\partial y} = 0, & \text{on } B_1'. \end{cases} \tag{5.96}$$

Furthermore testing (5.93) with $V^{R_{\lambda_{p_q}}\lambda_{p_q}}$,

$$\lim_{q\to\infty}\int_{B_1^+} y^{1-2s}\left(\left|\nabla V^{R_{\lambda_{p_q}}\lambda_{p_q}}\right|^2-\frac{\alpha}{|x|_k^2}\left|V^{R_{\lambda_{p_q}}\lambda_{p_q}}\right|^2\right)dz$$
$$=\lim_{q\to\infty}\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}\frac{\partial V^{R_{\lambda_{p_q}}\lambda_{p_q}}}{\partial\nu}V^{R_{\lambda_{p_q}}\lambda_{p_q}}\,dS=\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}fW\,dS,$$

thanks to (5.95) and the compactness of the trace operator $\mathrm{Tr}_{S_1^+}$, see Proposition 3.4. Hence from Remark 5.3.1 and (5.83) we deduce (5.92). Let for any $r\in(0,1]$

$$D_q(r)=\frac{1}{r^{N-2s}}\Big(\int_{B_r^+} y^{1-2s}\left(|\nabla V^{R_{\lambda_{p_q}}\lambda_{p_q}}|^2-\frac{\alpha}{|x|_k^2}|V^{R_{\lambda_{p_q}}\lambda_{p_q}}|^2\right)dz$$
$$-c_{N,s}(R_{\lambda_{p_q}}\lambda_{p_q})^{2s}\int_{B_r'} g(R_{\lambda_{p_q}}\lambda_{p_q}\cdot)|\operatorname{Tr}(V^{R_{\lambda_{p_q}}\lambda_{p_q}})|^2\,dx\Big)$$

and

$$H_q(r)=\frac{1}{r^{N+1-2s}}\int_{S_r^+} y^{1-2s}|V^{R_{\lambda_{p_q}}\lambda_{p_q}}|^2\,dS.$$

For any $r\in(0,1]$ we also define

$$D_V(r)=\frac{1}{r^{N-2s}}\int_{B_r^+} y^{1-2s}\left(|\nabla V|^2-\frac{\alpha}{|x|_k^2}|V|^2\right)dz$$

and

$$H_V(r)=\frac{1}{r^{N+1-2s}}\int_{S_r^+} y^{1-2s}|V|^2\,dS. \tag{5.97}$$

Thanks to a scaling argument it is easy to see that

$$\mathcal{N}_q(r):=\frac{D_q(r)}{H_q(r)}=\frac{D(R_{\lambda_{p_q}}\lambda_{p_q}r)}{H(R_{\lambda_{p_q}}\lambda_{p_q}r)}=\mathcal{N}(R_{\lambda_{p_q}}\lambda_{p_q}r)\quad\text{for any } r\in(0,1].$$

By (5.92), (5.94) and Remark 5.3.1, it follows that

$$H_q(r)\to H_V(r)\quad\text{and}\quad D_q(r)\to D_V(r),\quad\text{as } q\to\infty,\ \text{for any } r\in(0,1].$$

Furthermore $H_V(r)>0$ for any $r\in(0,1]$ by Proposition 5.4.2 in the case $g\equiv0$ and $\Omega=B_2'$. In particular the function

$$\mathcal{N}:(0,1]\to\mathbb{R},\quad \mathcal{N}_V(r):=\frac{D_V(r)}{H_V(r)}$$

is well defined and $\mathcal{N}_V\in W^{1,1}_{loc}((0,1])$ by Proposition 5.4.4 in the case $g\equiv0$ and $\Omega=B_2'$. In view of (5.97), (5.77)

$$\mathcal{N}_V(r)=\lim_{q\to\infty}\mathcal{N}(R_{\lambda_{p_q}}\lambda_{p_q}r)=\gamma\quad\text{for any } r\in(0,1]. \tag{5.98}$$

Hence $\mathcal{N}_V(r)$ is constant in $[0,1]$ and so

$$\mathcal{N}_V'(r)\equiv0 \text{ for any } r\in(0,1].$$

By Proposition 5.4.4 it follows that

$$\left(\int_{S_r^+} y^{1-2s} V^2\, dS\right)\left(\int_{S_r^+} y^{1-2s}\left|\frac{\partial V}{\partial \nu}\right|^2 dS\right) - \left(\int_{S_r^+} y^{1-2s} V \frac{\partial V}{\partial \nu}\, dS\right)^2 = 0$$

for a.e. $r \in (0,1)$, that is, equality holds in the Cauchy-Schwartz inequality for the vectors V and $\frac{\partial V}{\partial \nu}$ in $L^2(S_r^+, y^{1-2s})$ for a.e. $r \in (0,1)$. Therefore there exists a function $\eta(r)$ defined a.e. in $(0,1)$ such that

$$\frac{\partial V}{\partial \nu}(r\theta) = \eta(r) V(r\theta) \quad \text{for a.e. r } \in (0,1) \text{ and a.e. } \theta \in \mathbb{S}^+.$$

Multiplying by $V(r\theta)$ and integrating over $\mathbb{S}^+$,

$$\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \frac{\partial V}{\partial \nu}(r\theta) V(r\theta)\, dS = \eta(r) \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |V(r\theta)|^2\, dS \quad \text{for a.e. r } \in (0,1)$$

and so $\eta(r) = \frac{H_V'(r)}{2H_V(r)} = \frac{\gamma}{r}$ for a.e. $r \in (0,1)$ by (5.68), (5.68) and (5.98). Since V is smooth away from Σ_k by classical elliptic regularity theory (see (5.37)), an integration over $(r,1)$ yields

$$V(r\theta) = r^\gamma V(1\theta) = r^\gamma Z(\theta) \quad \text{for any r } \in (0,1] \text{ and a.e. } \theta \in \mathbb{S}^+, \tag{5.99}$$

where $Z = V_{|\mathbb{S}^+}$ and $\|Z\|_{L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})} = 1$ by (5.91) . In view of [60, Lemma 1.1], (5.99) and (5.96) the function Z is an eigenfunction of problem (5.20) and the correspondent eigenvalue $\gamma_{\alpha,k,n}$ satisfies the relationship $\gamma(N-2s+\gamma) = \gamma_{\alpha,k,n}$, that is

$$\gamma = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}} \quad \text{or} \quad \gamma = -\frac{N-2s}{2} - \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}}$$

Since $r^\gamma Z(\theta) \in H^1(B_1^+, y^{1-2s})$ by (5.99) then $r^{2\gamma-2} Z^2(\theta) \in L^1(B_1^+, y^{1-2s})$ by (5.38) and so we conclude that (5.90) must hold.

Consider now the sequence $\{V^{\lambda_{p_q}}\}_{q\in\mathbb{N}}$. Up to a further subsequence, $V^{\lambda_{p_q}} \rightharpoonup \tilde{V}$ weakly in $H^1(B_1^+, y^{1-2s})$ as $q \to \infty$, for some $\tilde{V} \in H^1(B_1^+, y^{1-2s})$ and $R_{\lambda_{p_q}} \to \tilde{R}$, for some $\tilde{R} \in [1,2]$ as $q \to \infty$. The strong convergence of $\{V^{R_{\lambda_{p_q}}\lambda_{p_q}}\}_{q\in\mathbb{N}}$ to V in $H^1(B_1^+, y^{1-2s})$ implies that, up to a further subsequence, both $V^{R_{\lambda_{p_q}}\lambda_{p_q}}$ and $\left|\nabla V^{R_{\lambda_{p_q}}\lambda_{p_q}}\right|$ are dominated a.e. by a $L^2(B_1^+, y^{1-2s})$ function, uniformly with respect to $q \in \mathbb{N}$. Up to a further subsequence, we may also assume that the limit

$$\ell = \lim_{q\to\infty} \frac{H(R_{\lambda_{p_q}}\lambda_{p_q})}{H(\lambda_{p_q})}$$

exists, it is finite and strictly positive, taking into account (5.84). Then from the Dominated Convergence Theorem and a change of variables we deduce that

$$\begin{aligned}
\lim_{q\to\infty} \int_{B_1^+} y^{1-2s} V^{\lambda_{p_q}}(z)\phi(z)\, dz &= \lim_{q\to\infty} R_{\lambda_{p_q}}^{N+2-2s} \int_{B_{1/R_{\lambda_{p_q}}}^+} y^{1-2s} V^{\lambda_{p_q}}(R_{\lambda_{p_q}} z)\phi(R_{\lambda_{p_q}} z)\, dz \\
&= \lim_{q\to\infty} R_{\lambda_{p_q}}^{N+2-2s} \sqrt{\frac{H(R_{\lambda_{p_q}}\lambda_{p_q})}{H(\lambda_{p_q})}} \int_{B_1^+} y^{1-2s} \chi_{B_{1/R_{\lambda_{p_q}}}^+}(z) V^{R_{\lambda_{p_q}}\lambda_{p_q}}(z)\phi(R_{\lambda_{p_q}} z)\, dz \\
&= \tilde{R}^{N+2-2s}\sqrt{\ell} \int_{B_{1/\tilde{R}}^+} y^{1-2s} V(z)\phi(\tilde{R}z)\, dz = \sqrt{\ell} \int_{B_1^+} y^{1-2s} V(z/\tilde{R})\phi(z)\, dz
\end{aligned}$$

for any $\phi \in C^\infty(\overline{B_1^+})$. By density we conclude that $V^{\lambda_{p_q}} \rightharpoonup \sqrt{\ell} V(\cdot/\tilde{R})$ weakly in $L^2(B_1^+, y^{1-2s})$ as $q \to \infty$. Since $V^{\lambda_{p_q}} \rightharpoonup \tilde{V}$ weakly in $H^1(B_1^+, y^{1-2s})$ as $q \to \infty$ we conclude that $\tilde{V} = \sqrt{\ell} V(\cdot/\tilde{R})$ and so $V^{\lambda_{p_q}} \rightharpoonup \sqrt{\ell} V(\cdot/\tilde{R})$ weakly in $H^1(B_1^+, y^{1-2s})$ as $q \to \infty$. Furthermore

$$\begin{aligned}\lim_{q\to\infty} \int_{B_1^+} y^{1-2s} |\nabla V^{\lambda_{p_q}}(z)|^2 \, dz &= \lim_{q\to\infty} R_{\lambda_{p_q}}^{N+2-2s} \int_{B^+_{1/R_{\lambda_{p_q}}}} y^{1-2s} |\nabla V^{\lambda_{p_q}}(R_{\lambda_{p_q}} z)|^2 \, dz \\ &= \lim_{q\to\infty} R_{\lambda_{p_q}}^{N-2s} \frac{H(R_{\lambda_{p_q}} \lambda_{p_q})}{H(\lambda_{p_q})} \int_{B_1^+} y^{1-2s} \chi_{B^+_{1/R_{\lambda_{p_q}}}}(z) |\nabla V^{R_{\lambda_{p_q}} \lambda_{p_q}}(z)|^2 \, dz \\ &= \tilde{R}^{N-2s} \ell \int_{B^+_{1/\tilde{R}}} y^{1-2s} |\nabla V|^2 dz = \int_{B_1^+} y^{1-2s} |\sqrt{\ell} \nabla V(\cdot/\tilde{R})|^2 \, dz,\end{aligned}$$

by the Dominated Convergence Theorem and a change of variables. Hence $V^{\lambda_{p_q}} \to \sqrt{\ell} V(\cdot/\tilde{R})$ strongly in $H^1(B_1^+, y^{1-2s})$ as $q \to \infty$.

Thanks to (5.99), V is a homogeneous function of degree γ and so $\tilde{V} = \sqrt{\ell} \tilde{R}^{-\gamma} V$. Moreover, since $V^{\lambda_{p_q}} \to \tilde{V}$ strongly in $L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$ as $q \to \infty$ by Proposition 3.4,

$$1 = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |\tilde{V}(\theta)|^2 dS = \sqrt{\ell} \tilde{R}^{-\gamma} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |V(\theta)|^2 dS = \sqrt{\ell} \tilde{R}^{-\gamma}$$

in view of (5.83) and (5.91). We conclude that $\tilde{V} = V$ thus completing the proof. □

Now we show that the limit (5.81) is strictly positive, by means of a Fourier analysis with respect to the $L^2(\mathbb{S}^+, \theta_{N+1}^{1-2s})$-orthonormal basis $\{Z_{\alpha,k,n}\}_{n \in \mathbb{N} \setminus \{0\}}$ of eigenfunctions of problem (5.20), see Subsection 5.2.1. To this end let us define for any $k \in \{3, \dots, N\}$, α as in (5.2), and $n \in \mathbb{N} \setminus \{0\}$

$$\varphi_{n,i}(\lambda) := \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} U(\lambda\theta) Z_{\alpha,k,n,i}(\theta) \, dS, \qquad \text{for any } \lambda \in (0, r_0], \ i \in 1, \dots, M_{\alpha,k,n}, \tag{5.100}$$

see (5.35) for the definition of $M_{\alpha,k,n}$, and

$$\Upsilon_{n,i}(\lambda) := c_{N,s} \int_{B'_\lambda} g \, \mathrm{Tr}(U) \, \mathrm{Tr}\left(Z_{\alpha,k,n,i}\left(\frac{\cdot}{|\cdot|}\right)\right) dx, . \tag{5.101}$$

for any $\lambda \in (0, r_0]$, $i \in 1, \dots, M_{\alpha,k,n}$. Thanks to Proposition 5.4.7 and Proposition 5.5.5 there exists $n_0 \in \mathbb{N} \setminus \{0\}$ such that

$$\gamma = \lim_{r\to 0^+} \mathcal{N}(r) = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n_0}}. \tag{5.102}$$

For any $i \in \{1, \dots, M_{\alpha,k,n_0}\}$ we need to compute the asymptotics of $\varphi_{n_0,i}(\lambda)$ as $\lambda \to 0^+$.

Proposition 5.5.6. *Let n_0 be as in (5.102). Then for any $i \in \{1, \dots, M_{\alpha,k,n_0}\}$ and any $r \in (0, r_0]$*

$$\begin{aligned}\varphi_{n_0,i}(\lambda) = \lambda^\gamma \Bigg(&\frac{\varphi_{n_0,i}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma} \int_0^r \rho^{-1+\rho} \Upsilon_{n_0,i}(\rho) d\rho \\ &+ \frac{N-2s+\gamma}{N-2s+2\gamma} \int_\lambda^r \rho^{-N-1+2s-\gamma} \Upsilon_{n_0,i}(\rho) \, d\rho \Bigg) + O(\lambda^{\gamma+\varepsilon}) \quad \text{as } \lambda \to 0^+. \end{aligned} \tag{5.103}$$

Proof. Let $n \in \mathbb{N}$ and $i \in \{1, \dots, M_{\alpha,k,n}\}$. Let $f \in C_c^\infty(0, r_0)$. Then testing (5.19) with the function $|z|^{N+1-2s} f(|z|) Z_{\alpha,k,n,i}(z/|z|)$ and passing in polar coordinates, by (5.33), we obtain

$$-\varphi''_{n,i}(\lambda) - \frac{N+1-2s}{\lambda}\varphi'_{n,i}(\lambda) + \frac{\gamma_{\alpha,k,n}}{\lambda^2}\varphi_{n,i}(\lambda) = \zeta_{n,i}(\lambda) \quad \text{in } (0, r_0)$$

in a distributional sense, where the distribution $\zeta_{n,i} \in \mathcal{D}'(0, r_0)$ is define as

$${}_{\mathcal{D}'(0,r_0)}\langle \zeta_{n,i}, f\rangle_{\mathcal{D}(0,r_0)} = \int_0^{r_0} \frac{f(\lambda)}{\lambda^{2-2s}} \left(\int_{\mathbb{S}'} g(\lambda \cdot)\, \mathrm{Tr}(U)(\lambda \cdot)\, \mathrm{Tr}\left(Z_{\alpha,k,n,i}\left(\frac{\cdot}{|\cdot|}\right)\right) dS'\right) d\lambda, \quad (5.104)$$

for any $f \in C_c^\infty(0, r_0)$. In particular $\zeta_{n,i}$ belongs to $L^1_{loc}((0, r_0])$ by the Coarea Formula and a change of variables. If $\Upsilon_{n,i}$ is as in (5.101), a direct computation shows that

$$\Upsilon'_{n,i}(\lambda) = \lambda^{N+1-2s}\zeta_{n,i}(\lambda) \quad \text{in } \mathcal{D}'(0, r_0)$$

hence

$$-\left(\lambda^{N+1-2s+2\sigma_n}\left(\lambda^{-\sigma_n}\varphi_{n,i}(\lambda)\right)'\right)' = \lambda^{\sigma_n}\Upsilon'_{n,i}(\lambda) \quad \text{in } \mathcal{D}'(0, r_0), \quad (5.105)$$

where

$$\sigma_n := -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n.}} \quad (5.106)$$

From (5.105) and (5.104) we deduce that the function $\lambda \mapsto \lambda^{N+1-2s+2\sigma_n}\left(\lambda^{-\sigma_n}\varphi'_{n,i}(\lambda)\right)$ belongs to $W^{1,1}_{loc}((0, r_0])$ hence an integration over (λ, r) yields

$$\begin{aligned}\left(\lambda^{-\sigma_n}\varphi_{n,i}(\lambda)\right)' = &-\lambda^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\lambda)\\ &- \lambda^{-N-1+2s-2\sigma_n}\sigma_n\left(C(r) + \int_\lambda^r \rho^{\sigma_n-1}\Upsilon_{n,i}(\rho)\,d\rho\right) \quad (5.107)\end{aligned}$$

for any $r \in (0, r_0]$, for some real number $C(r)$ depending on r, α, k, n and i. Since in view of (5.107) $\lambda \to \lambda^{-\sigma_n}\varphi_{n,i}(\lambda)$ belongs to $W^{1,1}_{loc}((0, r_0])$, a further integration yields

$$\begin{aligned}
\varphi_{n,i}(\lambda) &= \lambda^{\sigma_n}\Bigg(r^{-\sigma_n}\varphi_{n,i}(r) + \int_\lambda^r \rho^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\rho)\,d\rho\\
&\qquad + \sigma_n\int_\lambda^r \rho^{-N-1+2s-2\sigma_n}\left(C(r) + \int_\rho^r t^{\sigma_n-1}\Upsilon_{n,i}(t)\,dt\right)d\rho\Bigg)\\
&= \lambda^{\sigma_n}\Bigg(r^{-\sigma_n}\varphi_{n,i}(r) + \int_\lambda^r \rho^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\rho)\,d\rho + \frac{\sigma_n C(r) r^{-N+2s-2\sigma_n}}{-N+2s-2\sigma_n}\\
&\qquad - \frac{\sigma_n C(r)\lambda^{-N+2s-2\sigma_n}}{-N+2s-2\sigma_n} - \frac{\sigma_n\lambda^{-N+2s-2\sigma_n}}{-N+2s-2\sigma_n}\int_\lambda^r t^{\sigma_n-1}\Upsilon_{n,i}(t)\,dt\\
&\qquad + \frac{\sigma_n}{-N+2s-2\sigma_n}\int_\lambda^r \rho^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\rho)\,d\rho\Bigg)\\
&= \lambda^{\sigma_n}\left(\frac{\varphi_{n,i}(r)}{r^{\sigma_n}} - \frac{\sigma_n C(r) r^{-N+2s-2\sigma_n}}{N-2s+2\sigma_n} + \frac{N-2s+\sigma_n}{N-2s+2\sigma_n}\int_\lambda^r \rho^{-N-1+2s-\sigma_n}\Upsilon_{n,i}(\rho)\,d\rho\right)\\
&\qquad + \frac{\sigma_n\lambda^{-N+2s-\sigma_n}}{N-2s+2\sigma_n}\left(C(r) + \int_\lambda^r t^{\sigma_n-1}\Upsilon_{n,i}(t)\,dt\right) \quad (5.108)
\end{aligned}$$

for any $\lambda \in (0, r_0]$.

Let n_0 be as in (5.102) and $i \in \{1, \dots, M_{\alpha,k,n_0}\}$. By (5.102) and (5.106), $\gamma = \sigma_{n_0}$ and

$$\lambda^{-N-1+2s-\gamma} |\Upsilon_{n_0,i}(\lambda)| < c_{N,s} \lambda^{-N-1+2s-\gamma} \int_{B'_\lambda} |g| |\operatorname{Tr}(U)| \left| \operatorname{Tr}\left(Z_{\alpha,k,n,i}\left(\frac{\cdot}{|\cdot|}\right)\right)\right| dx$$

$$\leq \lambda^{-N-1+2s-\gamma} \left(\int_{B'_\lambda} |g| |\operatorname{Tr}(U)|^2 dx\right)^{\frac{1}{2}} \left(\int_{B'_\lambda} |g| \left|\operatorname{Tr}\left(Z_{\alpha,k,n,i}\left(\frac{\cdot}{|\cdot|}\right)\right)\right|^2 dx\right)^{\frac{1}{2}}$$

$$\leq k_{N,s,g} \lambda^{-N-1+2s-\gamma+\varepsilon}$$

$$\times \left(\int_{B^+_\lambda} y^{1-2s} |\nabla U|^2 \, dz - \int_{B^+_\lambda} y^{1-2s} \frac{\alpha}{|x|^2_k} U^2 \, dz + \frac{N-2s}{2\lambda} \int_{S^+_\lambda} y^{1-2s} U^2 \, dz\right)^{\frac{1}{2}}$$

$$\times \left(\int_{B^+_\lambda} y^{1-2s} |\nabla Z_{\alpha,k,n,i}(z/|z|)|^2 \, dz - \int_{B^+_\lambda} y^{1-2s} \frac{\alpha}{|x|^2_k} |Z_{\alpha,k,n,i}(z/|z|)|^2 \, dz\right.$$

$$\left. + \frac{N-2s}{2\lambda} \int_{S^+_\lambda} y^{1-2s} |Z_{\alpha,k,n,i}(z/|z|)|^2 \, dz\right)^{\frac{1}{2}}$$

$$= k_{N,s,g} \lambda^{-1-\gamma+\varepsilon} \sqrt{H(\lambda)} \left(\int_{B^+_1} y^{1-2s} |\nabla V^\lambda|^2 \, dz - \int_{B^+_1} y^{1-2s} \frac{\alpha}{|x|^2_k} |V^\lambda|^2 \, dz + \frac{N-2s}{2}\right)^{\frac{1}{2}}$$

$$\times \left(\int_{B^+_1} y^{1-2s} |\nabla Z_{\alpha,k,n,i}(z/|z|)|^2 \, dz - \int_{B^+_1} y^{1-2s} \frac{\alpha}{|x|^2_k} |Z_{\alpha,k,n,i}(z/|z|)|^2 \, dz + \frac{N-2s}{2}\right)^{\frac{1}{2}}$$

$$\leq \text{const } \lambda^{-1+\varepsilon}$$

for any $\lambda \in (0, r_0]$, by Holder inequality, a change of variables, (5.3), (5.40), (5.79), (5.82), (5.83), (5.101). Hence

$$|\Upsilon_{n_0,i}(\lambda)| \leq \text{const } \lambda^{N-2s+\gamma+\varepsilon} \quad \text{for any } \lambda \in (0, r_0]. \tag{5.109}$$

Now we show that for any $r \in (0, r_0]$

$$C(r) + \int_0^r \lambda^{-1+\gamma} \Upsilon_{n_0,i}(\lambda) d\lambda = 0. \tag{5.110}$$

From (5.109) it is clear that $\int_0^{r_0} \lambda^{-1+\gamma} \Upsilon_{n_0,i}(\lambda) d\lambda < +\infty$. We argue by contradiction. Since $\sigma_{n_0} = \gamma > -\frac{N-2s}{2}$ by (5.102) and (5.106), then from (5.108) we deduce that

$$\varphi_{n_0,i}(\lambda) \sim \frac{\gamma \lambda^{-N+2s-\gamma}}{N-2s+2\gamma} \left(C(r) + \int_\lambda^r t^{-1+\gamma} \Upsilon_{n_0,i}(t) \, dt\right) \quad \text{as } \lambda \to 0^+$$

and so by (5.102)

$$\int_0^{r_0} \lambda^{N-1-2s} |\varphi_{n_0,i}(\lambda)|^2 \, d\lambda = +\infty. \tag{5.111}$$

On the other hand by Hölder inequality, a change of variables, (5.100) and [60, Lemma 2.4]

$$\int_0^{r_0} \lambda^{N-1-2s} |\varphi_{n_0,i}(\lambda)|^2 \, d\lambda \leq \int_0^{r_0} \lambda^{N-1-2s} \left(\int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} |U(\lambda\theta)|^2 \, dS\right) d\lambda$$

$$= \int_{B^+_{r_0}} y^{1-2s} \frac{U^2}{|z|^2} \, dz < +\infty,$$

which contradicts (5.111). It follows that

$$\lambda^{-N+2s-\gamma}\left|C(r)+\int_\lambda^r \lambda^{-1+\gamma}\Upsilon_{n_0,i}(\lambda)d\lambda\right| = \lambda^{-N+2s-\gamma}\left|\int_0^\lambda \lambda^{-1+\gamma}\Upsilon_{n_0,i}(\lambda)d\lambda\right|$$
$$= O(\lambda^{\gamma+\varepsilon}), \quad (5.112)$$

in view of (5.109). In conclusion (5.103) follows from (5.108), (5.110), and (5.112). □

Proposition 5.5.7. *Let U be a non-trivial solution of (5.19) and γ be as in (5.77). Then*

$$\lim_{r\to 0^+} r^{-2\gamma} H(r) > 0.$$

Proof. From (5.100), since $\{Z_{\alpha,k,n}\}_{n\in\mathbb{N}\setminus\{0\}}$ is a orthonormal basis of $L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})$, see Subsection 5.2.1, we have that

$$H(\lambda) = \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s}|U(\lambda\theta)|^2\, dS = \sum_{n=1}^{\infty}\sum_{i=1}^{M_{\alpha,k,n}} |\varphi_{n,i}(\lambda)|^2 \quad (5.113)$$

by (5.66) and a change of variables. We argue by contradiction supposing that

$$\lim_{\lambda\to 0^+} \lambda^{-2\gamma} H(\lambda) = 0.$$

Let n_0 be as in (5.102). By (5.113) for any $i \in \{1,\dots,M_{\alpha,k,n_0}\}$,

$$\lim_{\lambda\to 0^+} \lambda^{-2\gamma}|\varphi_{n_0,i}(\lambda)|^2 = 0.$$

By (5.103), for any $i \in \{1,\dots,M_{\alpha,k,n_0}\}$ and any $r \in (0,r_0]$

$$\frac{\varphi_{n,i}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma}\int_0^r \rho^{-1+\rho}\Upsilon_{n_0,i}(\rho)d\rho$$
$$+\frac{N-2s+\gamma}{N-2s+2\gamma}\int_0^r \rho^{-N-1+2s-\gamma}\Upsilon_{n,i}(\rho)\, d\rho = 0. \quad (5.114)$$

Hence by (5.103), (5.109) and (5.114)

$$\varphi_{n,i}(\lambda) = -\lambda^\gamma \frac{N-2s+\gamma}{N-2s+2\gamma}\int_0^\lambda \rho^{-N-1+2s-\gamma}\Upsilon_{n,i}(\rho)\, d\rho + O(\lambda^{\gamma+\varepsilon}) = O(\lambda^{\gamma+\varepsilon})$$

as $\lambda \to 0^+$ for any $i \in \{1,\dots,M_{\alpha,k,n_0}\}$. In view of (5.66) and (5.82), it follows that

$$\sqrt{H(\lambda)}\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}V^\lambda Z\, dS = O(\lambda^{\gamma+\varepsilon}) \quad \text{as } \lambda\to 0^+,$$

for any $Z \in V_{n_0}$, see (5.34). Then, in view of (5.80) with $\sigma = \frac{\varepsilon}{2}$,

$$\int_{\mathbb{S}^+}\theta_{N+1}^{1-2s}V^\lambda Z\, dS = O(\lambda^{\frac{\varepsilon}{2}}) \quad \text{as } \lambda\to 0^+ \quad (5.115)$$

for any $Z \in V_{n_0}$. On the other hand by Proposition 5.5.5 and Proposition 3.4, there exist $Z_0 \in V_{n_0}$ with $\|Z_0\|_{L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})} = 1$ and a sequence $\lambda_q \to 0^+$ as $q\to\infty$ such that

$$V^{\lambda_q} \to Z_0 \quad \text{strongly in } L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s}) \text{ as } q\to\infty. \quad (5.116)$$

Since $Z_0 \in V_{n_0}$, from the Parseval identity, (5.115), and (5.116) we deduce that $Z_0 \equiv 0$ which contradicts the fact that $\|Z_0\|_{L^2(\mathbb{S}^+,\theta_{N+1}^{1-2s})} = 1$. □

We are now in position to state and prove our main results which are a more precise version of Theorem 5.1.9 and Theorem 5.1.10 respectively.

Theorem 5.5.8. *Let U be a solution of* (5.19) *and suppose that g satisfies* (5.3)*. Then there exists $n \in \mathbb{N} \setminus \{0\}$ such that*

$$\gamma = \lim_{r \to 0^+} \mathcal{N}(r) = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,n}}. \tag{5.117}$$

Furthermore let $M_{\alpha,k,n}$ and $\{Z_{\alpha,k,n,i}\}_{i \in \{1,\dots,M_{\alpha,k,n}\}}$ be as in (5.35) *and* (5.36) *respectively. Then for any $i \in \{0,\dots M_{\alpha,k,n}\}$ there exists $\beta_i \in \mathbb{R}$ such that $(\beta_1,\dots,\beta_{M_{\alpha,k,n}}) \neq (0,\dots,0)$ and*

$$\frac{U(\lambda z)}{\lambda^\gamma} \to |z|^\gamma \sum_{i=1}^{M_{\alpha,k,n}} \beta_i Z_{\alpha,k,n,i}(z/|z|) \quad \text{strongly in } H^1(B_1^+, y^{1-2s}) \text{ as } \lambda \to 0^+, \tag{5.118}$$

where

$$\beta_i := \frac{\varphi_{n,i}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma} \int_0^r \rho^{-1+\rho} \Upsilon_{n,i}(\rho) d\rho + \frac{N-2s+\gamma}{N-2s+2\gamma} \int_0^r \rho^{-N-1+2s-\gamma} \Upsilon_{n,i}(\rho)\, d\rho \quad \text{for any } r \in (0,r_0], \tag{5.119}$$

with $\varphi_{n,i}$ and $\Upsilon_{n,i}$ given by (5.100) *and* (5.101) *respectively.*

Proof. In view of (5.77) and Proposition 5.5.5 we know that (5.117) holds for some $n \in \mathbb{N} \setminus \{0\}$. Furthermore for any sequence of strictly positive numbers $\lambda_p \to 0^+$ as $p \to \infty$ there exist a subsequence $\lambda_{p_q} \to 0^+$ as $q \to \infty$ and real numbers $\beta_1,\dots,\beta_{M_{\alpha,k,n}}$ such that

$$\frac{U(\lambda z)}{\lambda^\gamma} \to |z|^\gamma \sum_{i=1}^{M_{\alpha,k,n}} \beta_i Z_{\alpha,k,n,i}(z/|z|) \quad \text{strongly in } H^1(B_1^+, y^{1-2s}) \text{ as } q \to \infty^+, \tag{5.120}$$

taking into account Proposition 5.5.5 and (5.36). We claim that for any $i \in \{1,\dots,M_{\alpha,k,n}\}$ the number β_i does not depend neither on the sequence $\lambda_p \to 0^+$ nor on its subsequence $\lambda_{p_q} \to 0^+$. In view of (5.36), (5.100), (5.120) and Proposition 3.4

$$\lim_{q\to\infty} \lambda_{p_q}^{-\gamma} \varphi_{n,j}(\lambda_{p_q}) = \lim_{q\to\infty} \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} \lambda_{p_q}^{-\gamma} U(\lambda_{p_q}\theta) Z_{\alpha,k,n,j}(\theta)\, dS = \sum_{i=1}^{M_{\alpha,k,n}} \beta_i \int_{\mathbb{S}^+} \theta_{N+1}^{1-2s} Z_{\alpha,k,n,i} Z_{\alpha,k,n,j}\, dS = \beta_j,$$

for any $j \in \{1,\dots,M_{\alpha,k,n}\}$. On the other hand for any $r \in (0,r_0]$

$$\lim_{q\to\infty} \lambda_{p_q}^{-\gamma} \varphi_{n,j}(\lambda_{p_q}) = \frac{\varphi_{n,j}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma} \int_0^r \rho^{-1+\rho} \Upsilon_{n,j}(\rho) d\rho + \frac{N-2s+\gamma}{N-2s+2\gamma} \int_0^r \rho^{-N-1+2s-\gamma} \Upsilon_{n,j}(\rho)\, d\rho$$

by (5.103). Hence

$$\beta_j = \frac{\varphi_{n,j}(r)}{r^\gamma} + \frac{\gamma r^{-N+2s-2\gamma}}{N-2s+2\gamma}\int_0^r \rho^{-1+\rho}\Upsilon_{n,j}(\rho)d\rho + \frac{N-2s+\gamma}{N-2s+2\gamma}\int_0^r \rho^{-N-1+2s-\gamma}\Upsilon_{n,j}(\rho)\,d\rho \quad (5.121)$$

for any $j \in \{1,\dots,M_{\alpha,k,n}\}$ and in particular β_j does not depend neither on the sequence $\lambda_p \to 0^+$ nor on its subsequence $\lambda_{p_q} \to 0^+$. Then by (5.121) and the Urysohn Subsequence Principle we conclude that (5.118) holds, thus completing the proof. □

From Theorem 5.5.8, Proposition 5.1.6 and Section 3.1 in Chapter 3 we can easily deduce the following theorem.

Theorem 5.5.9. *Let u be a solution of (5.12) and suppose that g satisfies (5.3). Let γ, $n \in \mathbb{N}\setminus\{0\}$, $M_{\alpha,k,n}$ and $\{Z_{\alpha,k,n,i}\}_{i\in\{1,\dots,M_{\alpha,k,n}\}}$ be as in Theorem 5.5.8. Then*

$$\frac{u(\lambda x)}{\lambda^\gamma} \to |x|^\gamma \sum_{i=1}^{M_{\alpha,k,n}} \beta_i \,\mathrm{Tr}(Z_{\alpha,k,n,i}((\cdot/|\cdot|))(x) \quad \text{strongly in } H^s(B_1') \text{ as } \lambda \to 0^+,$$

where β_i is as in (5.119) for any $i \in \{1,\cdots,M_{\alpha,k,n}\}$.

Proof of Corollary 5.1.11 and Corollary 5.1.12. We start by proving Corollary 5.1.11. Let U be a solution of (5.19) such that (5.24) holds and assume by contradiction that $U \not\equiv 0$ on $\Omega\times(0,\infty)$. Let γ be as in Theorem 5.5.8. Then there exists a sequence $\lambda_q \to 0^+$ such that

$$\lim_{q\to\infty} \lambda_q^{-\gamma} U(\lambda_q z) = 0 \quad \text{for a.e } z \in B_1^+.$$

On the other hand by Theorem 5.1.9 there exists an eigenfunction Z of (5.20) such that

$$\lim_{q\to\infty} \lambda_q^{-\gamma} U(\lambda_q z) = |z|^\gamma Z(z/|z|) \quad \text{for a.e. } z \in B_1^+,$$

up to a further subsequence, which is a contradiction. Arguing in the same way, we can deduce Corollary 5.1.12 from Theorem 5.1.10, taking into account Remark 5.2.4. □

5.6 Computation of the first eigenvalue on a hemisphere

Proposition 5.6.1. *Equation (5.22) holds for any $k \in \{3,\dots,N\}$. If $k = N$ then (5.23) holds.*

Proof. Let $Y_{\alpha,k,1}$ be the first eigenfunction of (5.5) defined in Section 5.1. In particular $Y_{\alpha,k,1}$ is positive. By [64, Theorem 1.1] there exists an eigenfunction Ψ of problem (5.21), corresponding to the first eigenvalue $\eta_{\alpha,k,1}$, such that

$$\lambda^{\frac{N-2}{2}-\sqrt{\left(\frac{N-2}{2}\right)^2+\eta_{\alpha,k,1}}} Y_{\alpha,k,1}(\lambda x) \to |x|^{-\frac{N-2}{2}+\sqrt{\left(\frac{N-2}{2}\right)^2+\eta_{\alpha,k,1}}}\,\Psi\left(\frac{x}{|x|}\right) \quad (5.122)$$

strongly in $H^1(B_1')$ as $\lambda \to 0^+$, since $Y_{\alpha,k,1}$ is positive. Furthermore for any $\phi \in C_c^\infty(\Omega)$

$$_{(\mathbb{H}^s_{\alpha,k}(\Omega))^*}\left\langle L^s_{\alpha,k} Y_{\alpha,k,1}, \phi \right\rangle_{\mathbb{H}^s_{\alpha,k}(\Omega)} = (Y_{\alpha,k,1},\phi)_{\mathbb{H}^s_{\alpha,k}(\Omega)} = \mu^s_{\alpha,k,1}\int_\Omega Y_{\alpha,k,1}\phi\,dx,$$

in view of (5.8), that is $Y_{\alpha,k,1}$ is weak solution of $L^s_{\alpha,k} Y_{\alpha,k,1} = \mu^s_{\alpha,k,1} Y_{\alpha,k,1}$ in the sense given by (5.12). Let U be the extension of $Y_{\alpha,k,1}$ provided by Theorem 5.1.7. Since $Y_{\alpha,k,1}$ is positive then $|U|$ is the only solution to the minimization problem (5.17) and so we conclude that U is positive. Then, in view of by Theorem 5.5.8 and Theorem 5.5.9,

$$\lambda^{\frac{N-2s}{2}-\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,1}}} Y_{\alpha,k,1}(\lambda x) \\ \to |x|^{-\frac{N-2s}{2}+\sqrt{\left(\frac{N-2s}{2}\right)^2+\gamma_{\alpha,k,1}}} \beta_1 \operatorname{Tr}(Z_{\alpha,k,1}((\cdot/|\cdot|))(x) \tag{5.123}$$

strongly in $H^s(B'_1)$ as $\lambda \to 0^+$. Putting together (5.122) and (5.123) we obtain

$$-\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \gamma_{\alpha,k,1}} = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \eta_{\alpha,k,1}}$$

thus (5.22) follows from a direct computation. Finally, if $k = N$, problem (5.21) reduces to

$$-\Delta_{\mathbb{S}'}\Psi - \alpha\Psi = \eta\Psi \quad \text{ in } \mathbb{S}'$$

which admits $-\alpha$ as first eigenvalue, hence we have proved (5.23) in view of (5.22). □

5.7 A proof of Proposition 5.1.2

In this section we provide, for the sake of completeness, a detailed proof of Proposition 5.1.2 starting with a preliminary lemma. Let us consider, for any positive sequence $\{q_n\}_{n\in\mathbb{N}}$, the weighted $\ell^2(\mathbb{N})$-space defined as

$$\ell^2(\mathbb{N}, \{q_n\}) := \left\{ \{a_n\}_{n\in\mathbb{N}} : \sum_{n=0}^{\infty} q_n a_n^2 < +\infty \right\}$$

endowed with the norm

$$\|\{a_n\}\|_{\ell^2(\mathbb{N},\{q_n\})} := \left(\sum_{n=0}^{\infty} q_n a_n^2 \right)^{\frac{1}{2}}.$$

Lemma 5.7.1. *Let $\ell^2(\mathbb{N}, \{q_n\})$ and $\ell^2(\mathbb{N}, \{p_n\})$ be weighted $\ell^2(\mathbb{N})$-spaces. Then*

$$(\ell^2(\mathbb{N}, \{q_n\}), \ell^2(\mathbb{N}, \{p_n\}))_{s,2} = \ell^2(\mathbb{N}, \{q_n^{1-s} p_n^s\}). \tag{5.124}$$

with equivalent norms.

Proof. We follow the proof of [128, Lemma 23.1]. Let us consider a variant of the standard K function defined as

$$K_2(t, a) := \inf_{b+c=a} \left\{ \left(\|b\|^2_{\ell^2(\mathbb{N},\{q_n\})} + t^2 \|c\|^2_{\ell^2(\mathbb{N},\{p_n\})} \right)^{\frac{1}{2}} : b \in \ell^2(\mathbb{N}, \{q_n\}), c \in \ell^2(\mathbb{N}, \{p_n\}) \right\},$$

for any $t \geq 0$ and any sequence $a \in \ell^2(\mathbb{N}, \{q_n\}) + \ell^2(\mathbb{N}, \{p_n\})$. If $K(t, a)$ is the standard K-function it is clear that $K_2(t, a) \leq K(t, a) \leq \sqrt{2} K_2(t, a)$ for any $t \geq 0$ and any sequence $a \in \ell^2(\mathbb{N}, \{q_n\}) + \ell^2(\mathbb{N}, \{p_n\})$. It follows that we can use K_2 to define a norm on $(\ell^2(\mathbb{N}, \{q_n\}), \ell^2(\mathbb{N}, \{p_n\}))_{s,2}$ equivalent to the standard one.

We can compute $K_2(a,t)$ explicitly. Indeed, fixed $a \in \ell^2(\mathbb{N},\{q_n\}) + \ell^2(\mathbb{N},\{p_n\})$ and $t \geq 0$, we can, for any $n \in \mathbb{N}$, minimize the value of $b_n^2 q_n + t^2(a_n - b_n)^2 p_n$ as a function of b_n choosing

$$b_n := \frac{t^2 p_n}{q_n + t^2 p_n} a_n.$$

With this optimal choice it follows that

$$c_n = a_n - b_n = \frac{q_n}{q_n + t^2 p_n} a_n$$

and so we obtain

$$K_2(t,a)^2 = \sum_{n=0}^{\infty} \frac{t^2 p_n q_n}{q_n + t^2 p_n} a_n^2.$$

Then by the Monotone Convergence Theorem and the change of variables $t = \tau\sqrt{\frac{q_n}{p_n}}$

$$\int_0^{\infty} K_2(t,a)^2 t^{-1-2s}\, dt = \sum_{n=0}^{\infty} a_n^2 \int_0^{\infty} \frac{y^{1-2s} p_n q_n}{q_n + t^2 p_n}\, dt = \left(\int_0^{\infty} \frac{\tau^{1-2s}}{1+\tau^2}\, d\tau\right) \sum_{n=0}^{\infty} a_n^2 q_n^{1-s} p_n^s.$$

Since for any $s \in (0,1)$

$$\int_0^{\infty} \frac{\tau^{1-2s}}{1+\tau^2}\, d\tau < +\infty,$$

we conclude that (5.124) holds. □

Proof of Proposition 5.1.2. Let us start by proving that for any $k \in \{3,\dots,N\}$ and α as in (5.2)

$$\mathbb{H}^1_{\alpha,k}(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \mu_{\alpha,k,n} v_n^2 < +\infty \right\} = H^1_0(\Omega), \tag{5.125}$$

with equivalent norms. If $u \in H^1_0(\Omega)$ then, in view of Remark 5.1.1,

$$u = \sum_{n=1}^{\infty} \left(u, \frac{Y_{\alpha,k,n}}{\sqrt{\mu_{\alpha,k,n}}} \right)_{\alpha,k} \frac{Y_{\alpha,k,n}}{\sqrt{\mu_{\alpha,k,n}}}$$

and so by the Parseval's identity, (5.6), (5.7) and Remark 5.1.1

$$+\infty > \|u\|^2_{\alpha,k} = \sum_{n=1}^{\infty} \mu_{\alpha,k,n} u_n^2. \tag{5.126}$$

On the other hand if $u \in \mathbb{H}^1_{\alpha,k}(\Omega)$ let, in view of (5.6),

$$u^{(j)} := \sum_{n=1}^{j} \left(u, \frac{Y_{\alpha,k,n}}{\sqrt{\mu_{\alpha,k,n}}} \right)_{\alpha,k} \frac{Y_{\alpha,k,n}}{\sqrt{\mu_{\alpha,k,n}}} = \sum_{n=1}^{j} u_n Y_{\alpha,k,n}.$$

For any $j \in \mathbb{N} \setminus \{0\}$ it is clear that $u^{(j)} \in H^1_0(\Omega)$ and if $j > i$

$$\left\| u^{(j)} - u^{(i)} \right\|^2_{\alpha,k} = \sum_{n=i}^{j} \mu_{\alpha,k,n} u_n^2. \tag{5.127}$$

It follows that $\{u^{(j)}\}_{j\in\mathbb{N}\setminus\{0\}}$ converges to u in $H_0^1(\Omega)$ by Remark 5.1.1, and (5.127). In conclusion $u \in H_0^1(\Omega)$. From Remark 5.1.1 and (5.126) we deduce that the norms on $H_0^1(\Omega)$ and $\mathbb{H}^1_{\alpha,k}(\Omega)$ are equivalent.

For any $s \in (0,1]$, since $L^2(\Omega)$ and $\mathbb{H}^s_{\alpha,k}(\Omega)$ are isomorphic to $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{N},\{\mu^s_{\alpha,k,n}\})$ respectively, from Lemma 5.7.1 and (5.125) it follows that

$$\mathbb{H}^s_{\alpha,k}(\Omega) = (L^2(\Omega), \mathbb{H}^1_{\alpha,k}(\Omega))_{s,2} = (L^2(\Omega), H_0^1(\Omega))_{s,2} = \begin{cases} H_0^s(\Omega), & \text{if } s \in (0,1) \setminus \{\frac{1}{2}\}, \\ H_{00}^{1/2}(\Omega), & \text{if } s = \frac{1}{2}, \end{cases}$$

with equivalent norms. The last equality is a classical interpolation result, see for example [101]. □

Part II

Unique continuation for parabolic problems

Chapter 6

Unique continuation for the fractional power of the heat operator

6.1 Statement of the main results

In this Chapter we deal with the singular fractional evolution equation

$$(w_t - \Delta w)^s = \frac{1}{\kappa_s}\left(\frac{\mu}{|x|^{2s}} w + g w\right), \quad \text{in } \mathbb{R}^N \times (t_0 - T, t_0), \tag{6.1}$$

where $T > 0$, and

$$s \in (0,1), \quad N > 2s, \quad \mu < \kappa_s \Lambda_{N,s}, \quad \kappa_s := \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}, \quad \Lambda_{N,s} := 2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}. \tag{6.2}$$

We are interested in studying the asymptotic behaviour of solutions to (6.1) at $(x,t) = (0,t_0)$ along the directions $(\lambda x, t_0 - \lambda^2 t)$ as $\lambda \to 0^+$. On the perturbing potential g we assume the following hypotheses:

$$g, g_t \in L^r((t_0 - T, t_0), L^{\frac{N}{2s}}(\mathbb{R}^N)), \quad g_t \in L^\infty_{loc}((t_0 - T, t_0), L^{\frac{N}{2s}}(\mathbb{R}^N)), \tag{6.3}$$

$$|g(x,t)| + |\nabla g(x,t) \cdot x| \le C_g(1 + |x|^{-2s+\varepsilon}) \text{ for all } t \in (t_0 - T, t_0) \text{ and a.e. } x \in \mathbb{R}^N, \tag{6.4}$$

for some constant $C_g > 0$, $\varepsilon \in (0, 2s)$ and $r > 1$.

To formally introduce the fractional heat operator, let us first set some notations. For any real Hilbert space X we denote with X^* its dual space and with ${}_{X^*}\langle \cdot, \cdot \rangle_X$ the duality between X^* and X; $(\cdot, \cdot)_X$ denotes the scalar product in X.

The operator H^s can be defined by means of the Fourier transform as follows: for any function $w \in \mathcal{S}(\mathbb{R}^{N+1})$,

$$\widehat{H^s(w)}(\xi, \theta) := (i\theta + |\xi|^2)^s \widehat{w}(\xi, \theta),$$

where the Fourier transform of w is defined as

$$\mathcal{F}(w)(\xi, \theta) = \widehat{w}(\xi, \theta) := \frac{1}{(2\pi)^{\frac{N+1}{2}}} \int_{\mathbb{R}^{N+1}} e^{-i(x\cdot\xi + t\theta)} w(x,t)\, dx\, dt.$$

Furthermore, we can extend H^s to its natural domain; more precisely, we can define H^s on

$$\mathrm{Dom}(H^s) := \left\{ w \in L^2(\mathbb{R}^{N+1}) : \int_{\mathbb{R}^{N+1}} |i\theta + |\xi|^2|^s |\widehat{w}(\xi,\theta)|^2 \, d\xi \, d\theta < +\infty \right\},$$

endowed with the norm

$$\|w\|_{\mathrm{Dom}(H^s)} := \left(\int_{\mathbb{R}^{N+1}} w^2(x,t) \, dx \, dt + \int_{\mathbb{R}^{N+1}} |i\theta + |\xi|^2|^s |\widehat{w}(\xi,\theta)|^2 d\xi \, d\theta \right)^{\frac{1}{2}},$$

as the map from $\mathrm{Dom}(H^s)$ into its dual space $(\mathrm{Dom}(H^s))^*$, defined as

$$_{(\mathrm{Dom}(H^s))^*}\langle H^s(w), v \rangle_{\mathrm{Dom}(H^s)} := \int_{\mathbb{R}^{N+1}} (i\theta + |\xi|^2)^s \widehat{w}(\xi,\theta) \overline{\widehat{v}(\xi,\theta)} d\xi d\eta, \tag{6.5}$$

for any $w, v \in \mathrm{Dom}(H^s)$.

It is worth noticing that, since $|\xi|^{2s} \le |i\theta + |\xi|^2|^s$ for any $(\theta,\xi) \in \mathbb{R}^{N+1}$,

$$\|v\|_{L^2(\mathbb{R}, W^{s,2}(\mathbb{R}^N))} \le \|v\|_{\mathrm{Dom}(H^s)}$$

for any $v \in \mathrm{Dom}(H^s)$. Hence the natural embedding

$$\mathrm{Dom}(H^s) \hookrightarrow L^2(\mathbb{R}, W^{s,2}(\mathbb{R}^N)) \tag{6.6}$$

is linear and continuous. In this Chapter we will denote with $W^{s,2}(\mathbb{R}^N)$ the usual fractional Sobolev space $H^s(\mathbb{R}^N)$ to avoid any confusions with the fractional power of the heat operator H. Furthermore, since we are dealing with a Hardy-type potential, the weighted L^2-space

$$L^2(\mathbb{R}^N, |x|^{-2s}) := \left\{ v : \mathbb{R}^N \to \mathbb{R} \text{ measurable: } \int_{\mathbb{R}^N} \frac{v^2}{|x|^{2s}} \, dx < +\infty \right\}$$

will play a role in our analysis, together with the following Hardy-type inequality due to Herbst [86]:

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{\phi}|^2 \, d\xi \ge \Lambda_{N,s} \int_{\mathbb{R}^N} |x|^{-2s} \phi^2 \, dx \tag{6.7}$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, where $\Lambda_{N,s} > 0$, defined in (6.2), is optimal and not attained.

In view of (6.5), we define a weak solution of (6.1) as a function $w \in \mathrm{Dom}(H^s)$ such that

$$_{(\mathrm{Dom}(H^s))^*}\langle H^s(w), \phi \rangle_{\mathrm{Dom}(H^s)} = \frac{1}{\kappa_s} \int_{t_0 - T}^{t_0} \left(\int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} w\phi + gw\phi \right) \, dx \right) dt, \tag{6.8}$$

for any $\phi \in C_c^\infty(\mathbb{R}^N \times (t_0 - T, t_0))$. In view of (6.4), (6.6), (6.7), and the Hölder inequality, the above definition of weak solution is well-posed, that is the right hand side, as a function of ϕ, belongs to $(\mathrm{Dom}(H^s))^*$.

In order to develop an Almgren-Poon type monotonicity formula, we apply the extension procedure of [24] (see also [21, 110, 125]) to localize the problem.

We use the symbols ∇ and div to denote the gradient, respectively the divergence, with respect to the space variable $z = (x, y)$.

We also note that by [101] there exists a linear and continuous trace operator

$$\mathrm{Tr} : H^1(\mathbb{R}^{N+1}_+, y^{1-2s}) \to W^{2,s}(\mathbb{R}^N), \tag{6.9}$$

see also Section 3.1 in Chapter 3.

The following theorem is a particular case of a very general extension result proved in [24]. See also [110, Theorem], [125, Theorem 1.7] and [21, Section 3, Section 4].

Theorem 6.1.1. *[24, Theorem 4.1, Remark 4.3] If $w \in \mathrm{Dom}(H^s)$, then there exists a function $W \in L^2(\mathbb{R}, H^1(\mathbb{R}^{N+1}_+, y^{1-2s}))$ that weakly solves*

$$\begin{cases} y^{1-2s}W_t - \mathrm{div}(y^{1-2s}\nabla W) = 0, & \text{in } \mathbb{R}^{N+1}_+ \times \mathbb{R}, \\ \mathrm{Tr}(W(\cdot,t)) = w(\cdot,t), & \text{on } \mathbb{R}^N, \text{ for a.e. } t \in \mathbb{R}, \\ -\lim\limits_{y\to 0^+} y^{1-2s}\frac{\partial W}{\partial y} = \kappa_s H^s(w), & \text{on } \mathbb{R}^N \times \mathbb{R}, \end{cases}$$

in the sense that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} W \phi_t \, dz \right) dt$$
$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla W \cdot \nabla \phi \, dz \right) dt - \kappa_s \, {}_{(\mathrm{Dom}(H^s))^*}\langle H^s(w), \phi(\cdot, 0, \cdot)\rangle_{\mathrm{Dom}(H^s)}$$

for any $\phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \times \mathbb{R})$.

The following corollary is an easy consequence of Theorem 6.1.1 and (6.8).

Corollary 6.1.2. *If $w \in \mathrm{Dom}(H^s)$ is a solution of (6.8), then there exists a function $W \in L^2(\mathbb{R}, H^1(\mathbb{R}^{N+1}_+, y^{1-2s}))$ that weakly solves*

$$\begin{cases} y^{1-2s}W_t - \mathrm{div}(y^{1-2s}\nabla W) = 0, & \text{in } \mathbb{R}^{N+1}_+ \times (t_0 - T, t_0), \\ \mathrm{Tr}(W(\cdot,t)) = w(\cdot,t), & \text{on } \mathbb{R}^N, \text{ for a.e. } t \in (t_0 - T, t_0), \\ -\lim\limits_{y\to 0^+} y^{1-2s}\frac{\partial W}{\partial y} = \frac{\mu}{|x|^{2s}} w + gw, & \text{on } \mathbb{R}^N \times (t_0 - T, t_0), \end{cases} \tag{6.10}$$

in the sense that

$$\int_{t_0 - T}^{t_0} \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} W \phi_t \, dz \right) dt$$
$$= \int_{t_0 - T}^{t_0} \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla W \cdot \nabla \phi \, dz \right) dt - \int_{t_0 - T}^{t_0} \left(\int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} w\phi + gw\phi \right) dx \right) dt, \tag{6.11}$$

for any $\phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \times (t_0 - T, t_0))$.

The asymptotic behavior at $(0, t_0)$ of a solution W of (6.10), and consequently of a solution w of (6.8), will turn out to be related to the following eigenvalue problem for a weighted Ornstein-Uhlenbeck operator:

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla Y) + y^{1-2s}\frac{z}{2} \cdot \nabla Y = \gamma y^{1-2s} Y, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{y\to 0^+} y^{1-2s}\frac{\partial Y}{\partial y} = \frac{\mu}{|x|^{2s}} \mathrm{Tr}(Y), & \text{on } \mathbb{R}^N, \end{cases} \tag{6.12}$$

with $\mu < \kappa_s \Lambda_{N,s}$, see (6.2) for the definition of κ_s and $\Lambda_{N,s}$. To introduce a suitable functional setting for problem (6.12), we define

$$G_s(z,t) := t^{-\frac{N+2-2s}{2}} e^{-\frac{|z|^2}{4t}} \quad \text{for any } (z,t) \in \mathbb{R}^{N+1}_+ \times (0, \infty).$$

It is easy to verify that $G_s \in C^\infty(\mathbb{R}^{N+1}_+ \times (0,\infty))$ solves the problem

$$\begin{cases} y^{1-2s}\frac{\partial G_s}{\partial t} - \operatorname{div}(y^{1-2s}\nabla G_s) = 0, & \text{in } \mathbb{R}^{N+1}_+ \times (0,\infty), \\ \lim\limits_{y\to 0^+} y^{1-2s}\frac{\partial G_s}{\partial y} = 0, & \text{on } \mathbb{R}^N \times (0,\infty), \end{cases}$$

in a classical sense. Furthermore

$$\nabla G_s(z,t) = -\frac{z}{2t} G_s(z,t) \quad \text{for any } (z,t) \in \mathbb{R}^{N+1}_+ \times (0,\infty). \tag{6.13}$$

Letting

$$G(z) := G_s(z,1) = e^{-\frac{|z|^2}{4}}, \qquad \text{for any } z \in \mathbb{R}^{N+1},$$

we define

$$\mathcal{L} := \left\{ V : \mathbb{R}^{N+1}_+ \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V^2 G \, dz < +\infty \right\}$$

and $\mathcal{H}$ as the completion of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|\phi\|_{\mathcal{H}} := \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}(\phi^2 + |\nabla\phi|^2) G \, dz \right)^{\frac{1}{2}}.$$

It is clear that both $\mathcal{L}$ and $\mathcal{H}$ are Hilbert spaces with respect to the natural scalar product associated to the $\|\cdot\|_{\mathcal{L}}$-norm and the $\|\cdot\|_{\mathcal{H}}$-norm respectively. We observe that

$$\|W\|_{\mathcal{H}} \le \|W\|_{H^1(\mathbb{R}^{N+1}_+, y^{1-2s})} \qquad \text{for any } W \in H^1(\mathbb{R}^{N+1}_+, y^{1-2s}),$$

hence the embedding

$$H^1(\mathbb{R}^{N+1}_+, y^{1-2s}) \hookrightarrow \mathcal{H}$$

is linear and continuous. Furthermore, we consider the weighted L^2-spaces

$$L^2(\mathbb{R}^N, G(x,0)) := \left\{ v : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} v^2(x) G(x,0) \, dx < +\infty \right\}$$

and

$$L^2(\mathbb{R}^N, |x|^{-2s}G(x,0)) := \left\{ v : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} \frac{v^2(x)}{|x|^{2s}} G(x,0) \, dx < +\infty \right\}.$$

The trace operator Tr introduced in (6.9) can be extended to a continuous linear trace operator, still denoted as Tr, from $\mathcal{H}$ to $L^2(\mathbb{R}^N, G(x,0))$, see Proposition 6.2.3 in Section 6.2. Furthermore Tr takes values in $L^2(\mathbb{R}^N, |x|^{-2s}G(x,0))$ and

$$\operatorname{Tr} : \mathcal{H} \to L^2(\mathbb{R}^N, |x|^{-2s}G(x,0)),$$

is linear and continuous, see Proposition 6.2.5 in Section 6.2.

We say that γ is an eigenvalue of problem (6.12) if there exists an eigenfunction $Y \in \mathcal{H} \setminus \{0\}$ weakly satisfying (6.12), i.e.

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla Y \cdot \nabla V \, G \, dz - \int_{\mathbb{R}^N} \frac{\mu}{|x|^{2s}} \operatorname{Tr}(Y) \operatorname{Tr}(V) \, G(0,\cdot) \, dx = \gamma \int_{\mathbb{R}^{N+1}_+} y^{1-2s} Y V \, G \, dz \tag{6.14}$$

for any $V \in \mathcal{H}$. Proposition 6.2.5 in Section 6.2 ensures that the above definition of weak solution is well posed.

In order to compute the eigenvalues of (6.12) we separate the variable z in radial and angular parts. Henceforward we denote

$$\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}, \quad \mathbb{S}^N_+ := \{z \in \mathbb{R}^{N+1}_+ : |z| = 1\},$$

identifying $\partial \mathbb{S}^N_+$ with $\mathbb{S}^{N-1}$. Writing as $\theta = (\theta_1, \dots, \theta_{N+1})$ the coordinates on $\mathbb{S}^N$, we define

$$L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}) := \left\{ v : \mathbb{S}^N_+ \to \mathbb{R} \text{ measurable} : \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} |v|^2 \, dz < +\infty \right\}$$

and $H^1(\mathbb{S}^N_+, \theta_{N+1}^{1-2s})$ as the completion of $C_c^\infty(\overline{\mathbb{S}^N_+})$ with respect to the norm

$$\|\phi\|_{H^1(\mathbb{S}^N_+, \theta_{N+1}^{1-2s})} := \left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} (|\phi|^2 + |\nabla_{\mathbb{S}^N} \phi|^2) \, dS \right)^{\frac{1}{2}},$$

where $\nabla_{\mathbb{S}^N}$ and dS denote the Riemannian gradient and the volume element, respectively, with respect to the standard metric on the unit N-dimensional sphere $\mathbb{S}^N$.

We refer to [60] for the following proposition.

Proposition 6.1.3. *[60, Lemma 2.2] There exists a linear and continuous trace operator*

$$\mathcal{T} : H^1(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}) \to L^2(\mathbb{S}^{N-1}) = L^2(\partial \mathbb{S}^N_+).$$

Furthermore, letting κ_s and $\Lambda_{N,s}$ be as in (6.2),

$$\kappa_s \Lambda_{N,s} \int_{\mathbb{S}^{N-1}} |\mathcal{T}(V)|^2 dS' \le \left(\frac{N-2s}{2} \right)^2 \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} |V|^2 dS + \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} |\nabla_{\mathbb{S}^N} V|^2 dS \qquad (6.15)$$

for any $v \in H^1(\mathbb{S}^N, \theta_{N+1}^{1-2s})$, where dS' denotes the volume element on $\mathbb{S}^{N-1}$.

Let us consider the following eigenvalue problem

$$\begin{cases} -\operatorname{div}_{\mathbb{S}^N}(\theta_{N+1}^{1-2s} \nabla_{\mathbb{S}^N} \psi) = \nu \theta_{N+1}^{1-2s} \psi, & \text{in } \mathbb{S}^N_+, \\ -\lim\limits_{\theta_{N+1} \to 0^+} \theta_{N+1}^{1-2s} \nabla_{\mathbb{S}^N} \psi \cdot e_{N+1} = \mu \mathcal{T}(\psi), & \text{on } \mathbb{S}^{N-1}, \end{cases} \qquad (6.16)$$

where $e_{N+1} := (0, \dots, 1) \in \mathbb{R}^{N+1}$ and $\mu < \kappa_s \Lambda_{N,s}$ as in (6.2). We say that $\nu \in \mathbb{R}$ is an eigenvalue of (6.16) if there exists $\psi \in H^1(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}) \setminus \{0\}$, called eigenfunction, such that

$$\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \nabla_{\mathbb{S}^N} \psi \cdot \nabla_{\mathbb{S}^N} V \, dS - \mu \int_{\mathbb{S}^{N-1}} \mathcal{T}(\psi) \mathcal{T}(V) \, dS' = \nu \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \psi V \, dS$$

for any $V \in H^1(\mathbb{S}^N_+, \theta_{N+1}^{1-2s})$. Since the natural embedding $H^1(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}) \hookrightarrow L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s})$ is compact, see [58] and [111], by classical spectral theory the eigenvalues of (6.16) are a non-decreasing and diverging sequence $\{\nu_k(\mu)\}_{k \in \mathbb{N} \setminus \{0\}}$. In the sequence $\{\nu_k(\mu)\}_{k \in \mathbb{N} \setminus \{0\}}$ we repeat each eigenvalue as many times as the dimension of the associated eigenspace. Inequality (6.15) implies the following estimate on the first eigenvalue:

$$\nu_1(\mu) > -\left(\frac{N-2s}{2} \right)^2. \qquad (6.17)$$

Furthermore there exists an orthonormal basis $\{\psi_k\}_{k \in \mathbb{N} \setminus \{0\}}$ of $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s})$ such that, for any $k \in \mathbb{N} \setminus \{0\}$, the function ψ_k is an eigenfunction of problem (6.16) associated to $\nu_k(\mu)$.

Remark 6.1.4. If $\mu = 0$, then a combination of the regularity result of [120, Theorem 1.1] with the blow-up analysis done in [60] for the Caffarelli-Silvestre extended problem implies that the set of eigenvalues of (6.16) is $\{k^2 + k(N-2s) : k \in \mathbb{N}\}$.

Let, for any $n \in \mathbb{N}$ and $j \in \mathbb{N} \setminus \{0\}$,

$$Y_{n,j}(z) := |z|^{-\alpha_j} P_{j,n}\left(\frac{|z|^2}{4}\right) \psi_j\left(\frac{z}{|z|}\right) \tag{6.18}$$

where

$$\alpha_j := \frac{N-2s}{2} - \sqrt{\left(\frac{N-2s}{2}\right)^2 + \nu_j(\mu)}, \tag{6.19}$$

$$P_{j,n}(t) := \sum_{i=0}^{n} \frac{(-n)_i}{\left(\frac{N+2-2s}{2} - \alpha_j\right)_i} \frac{t^i}{i!}, \quad \text{with} \quad \begin{cases} (s)_i = \prod_{j=0}^{i-1}(s+j), \\ (s)_0 = 1. \end{cases} \tag{6.20}$$

Let us also consider the $\mathcal{L}$-normalized functions

$$\widetilde{Y}_{n,j} := \frac{Y_{n,j}}{\|Y_{n,j}\|_{\mathcal{L}}} \quad \text{for any } (n,j) \in \mathbb{N} \times \mathbb{N} \setminus \{0\}. \tag{6.21}$$

The following result is proved in Section 6.4 and provides a complete description of the spectrum of problem (6.12).

Proposition 6.1.5. *The set of eigenvalues of problem* (6.12) *is*

$$\left\{\gamma_{m,k} := m - \frac{\alpha_k}{2} : k \in \mathbb{N} \setminus \{0\}, m \in \mathbb{N}\right\}, \tag{6.22}$$

where $\{\nu_k(\mu)\}_{k \in \mathbb{N}\setminus\{0\}}$ are the eigenvalues of problem (6.16) *and α_k is defined in* (6.19)*. The multiplicity of each eigenvalue $\gamma_{m,k}$ is finite and equal to*

$$\#\left\{j \in \mathbb{N} \setminus \{0\} : \gamma_{m,k} + \frac{\alpha_j}{2} \in \mathbb{N}\right\}.$$

Furthermore, for any $(m,k) \in \mathbb{N} \times \mathbb{N} \setminus \{0\}$,

$$E_{m,k} = \left\{\widetilde{Y}_{n,j} : (n,j) \in \mathbb{N} \times \mathbb{N} \setminus \{0\} \text{ and } \gamma_{m,k} = n - \frac{\alpha_j}{2}\right\}$$

is an $\mathcal{L}$-orthonormal basis of the eigenspace associated to the eigenvalue $\gamma_{m,k}$, where $\widetilde{Y}_{n,j}$ has been defined in (6.21)*. Finally*

$$\bigcup_{(m,k)\in\mathbb{N}\times\mathbb{N}\setminus\{0\}} E_{m,k} \tag{6.23}$$

is a orthonormal basis of $\mathcal{L}$.

The main result of this Chapter is the following classification of the asymptotic behaviour near $(0, t_0)$ of any solution W of (6.10), based on the limit as $t \to t_0^-$ of the following Almgren-Poon type frequency function

$$\mathcal{N}(t) := \frac{(t_0 - t)\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla W|^2 G_s(z, t_0 - t)\, dz}{\int_{\mathbb{R}^{N+1}_+} y^{1-2s} W^2 G_s(\cdot, t_0 - t)\, dz} - \frac{(t_0 - t)\int_{\mathbb{R}^N}\left(\frac{\mu}{|x|^{2s}} w^2 + g w^2\right) G_s(x, 0, t_0 - t)\, dx}{\int_{\mathbb{R}^{N+1}_+} y^{1-2s} W^2 G_s(\cdot, t_0 - t)\, dz}. \tag{6.24}$$

Theorem 6.1.6. *Let $W \not\equiv 0$ be a weak solution to (6.10). Then there exist $m_0 \in \mathbb{N}$ and $k_0 \in \mathbb{N} \setminus \{0\}$ such that*

$$\lim_{t \to t_0^-} \mathcal{N}(t) = \gamma_{m_0,k_0}, \tag{6.25}$$

where $\mathcal{N}$ has been defined in (6.24) and γ_{m_0,k_0} in (6.22). Furthermore, letting

$$J_0 := \left\{ (m,k) \in \mathbb{N} \times \mathbb{N} \setminus \{0\} : \gamma_{m_0,k_0} = m - \frac{\alpha_k}{2} \right\}, \tag{6.26}$$

for any $\tau \in (0,1)$

$$\lim_{\lambda \to 0^+} \int_\tau^1 \left\| \lambda^{-2\gamma_{m_0,k_0}} W(\lambda z \sqrt{t}, t_0 - \lambda^2 t) - t^{\gamma_{m_0,k_0}} \sum_{(m,k) \in J_0} \beta_{m,k} \widetilde{Y}_{m,k}(z) \right\|_{\mathcal{H}}^2 dt = 0$$

and

$$\lim_{\lambda \to 0^+} \sup_{t \in [\tau,1]} \left\| \lambda^{-2\gamma_{m_0,k_0}} W(\lambda z \sqrt{t}, t_0 - \lambda^2 t) - t^{\gamma_{m_0,k_0}} \sum_{(m,k) \in J_0} \beta_{m,k} \widetilde{Y}_{m,k}(z) \right\|_{\mathcal{L}}^2 = 0$$

where $\widetilde{Y}_{m,k}$ has been defined in (6.21),

$$\beta_{m,k} = \Lambda^{-2\gamma_{m_0,k_0}} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} W(\Lambda z, t_0 - \Lambda^2) \widetilde{Y}_{m,k}(z) G(z)\, dz + 2$$
$$\times \int_0^\Lambda \tau^{2s-1-2\gamma_{m_0,k_0}} \left(\int_{\mathbb{R}^N} g(\tau x, t_0 - \tau^2) \operatorname{Tr}(W)(\tau x, t_0 - \tau^2) \operatorname{Tr}(\widetilde{Y}_{m,k})(x) e^{-\frac{|x|^2}{4}}\, dx \right) d\tau, \tag{6.27}$$

for any $\Lambda \in (0, \Lambda_0)$ and for some $\Lambda_0 \in (0, \sqrt{T})$, and Tr *has been defined in (6.9). Finally $\beta_{m,k} \neq 0$ for some $(m,k) \in J_0$.*

From Theorem 6.1.6 and the relationship between problems (6.8) and (6.10) given by Corollary 6.1.2 we can easily deduce a similar result for solutions to (6.8).

Theorem 6.1.7. *Let $w \not\equiv 0$ be a solution to (6.8). Then there exist $m_0 \in \mathbb{N}$ and $k_0 \in \mathbb{N} \setminus \{0\}$ such that, for any $\tau \in (0,1)$,*

$$\lim_{\lambda \to 0^+} \int_\tau^1 \left\| \lambda^{-2\gamma_{m_0,k_0}} w(\lambda x \sqrt{t}, t_0 - \lambda^2 t) - t^{\gamma_{m_0,k_0}} \sum_{(m,k) \in J_0} \beta_{m,k} \operatorname{Tr}(\widetilde{Y}_{m,k})(x) \right\|_{L^2(\mathbb{R}^N, G(\cdot,0))}^2 dt = 0,$$

where $\widetilde{Y}_{m,k}$, $\beta_{m,k}$, and Tr *are defined in (6.21), (6.27), and (6.9), respectively.*

Thanks to Theorem 6.1.6 and Theorem 6.1.7, we can prove that a strong unique continuation principle holds for solutions of equations (6.8) and (6.11).

Corollary 6.1.8. *Let W be a weak solution of problem (6.10) such that*

$$W(z,t) = O\left((|z|^2 + (t_0 - t))^k \right) \quad \text{as } z \to 0 \text{ and } t \to t_0^- \quad \text{for all } k \in \mathbb{N}. \tag{6.28}$$

Then $W \equiv 0$ on $\mathbb{R}^{N+1}_+ \times (t_0 - T, t_0)$.

Corollary 6.1.9. *Let w be a solution of (6.8) such that*

$$w(x,t) = O\left((|x|^2 + (t_0 - t))^k\right) \qquad \text{as } x \to 0 \text{ and } t \to t_0^- \qquad \text{for all } k \in \mathbb{N}.$$

Then $w \equiv 0$ in $\mathbb{R}^N \times (t_0 - T, t_0)$.

The next theorem is a backward uniqueness result for the Cauchy problem associated with (6.10). Its proof relies exclusively on the monotonicity argument developed in Section 6.5 and does not require the blow-up argument which is instead needed to obtain the above space-like unique continuation properties.

Theorem 6.1.10. *If W is a solution of (6.10) and there exists $t_1 \in (t_0 - T, t_0)$ such that*

$$W(z, t_1) = 0 \qquad \text{for a.e. } z \in \mathbb{R}^{N+1}_+,$$

then $W \equiv 0$ in $\mathbb{R}^{N+1}_+ \times (t_0 - T, t_0)$.

This Chapter is organized as follows. In Section 6.2 we prove some functional inequalities and trace results in Gaussian spaces. In Section 6.3 we give an alternative weak formulation of the extended problem in Gaussian spaces and prove a regularity result. In Section 6.4 we describe the eigenvalues of a weighted Ornstein-Uhlenbeck operator which turn out to be related to the classification of the asymptotic behaviour of weak solutions to (6.1) at $(0, t_0)$. In Section 6.5 we derive an Almgren-Poon type monotonicity formula for the extended problem, which is combined with a blow-up analysis in Section 6.6 to obtain our main results, i.e. the asymptotic of solutions and the strong space-like unique continuation property.

6.2 Inequalities and Traces in Gaussian spaces

In this section we prove some inequalities and trace results for Gaussian spaces. We start with a Hardy-type inequality.

Proposition 6.2.1. *For any $V \in \mathcal{H}$*

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \frac{V^2}{|z|^2} G\, dz + \frac{1}{4(N-2s)^2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |z|^2 V^2 G\, dz \\ \leq \frac{4}{(N-2s)^2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla V|^2 G\, dz + \frac{N+2-2s}{(N-2s)^2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V^2 G\, dz. \tag{6.29}$$

Proof. By density, it is enough to prove (6.29) for any $\phi \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$. Thanks to [60, Lemma 2.4]

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \frac{\phi^2}{|z|^2} G\, dz \leq \frac{4}{(N-2s)^2} \int_{\mathbb{R}^{N+1}} y^{1-2s} \left| \nabla \left(\phi e^{-\frac{|z|^2}{8}} \right) \right|^2 dz \\ = \frac{4}{(N-2s)^2} \int_{\mathbb{R}^{N+1}} y^{1-2s} \left(|\nabla \phi|^2 - \frac{1}{4} \nabla \left(\phi^2\right) \cdot z + \frac{1}{16} |z|^2 \phi^2 \right) e^{-\frac{|z|^2}{4}}\, dz. \tag{6.30}$$

Let, for any $\delta > 0$,

$$\mathbb{R}^{N+1}_\delta := \{(x,y) \in \mathbb{R}^{N+1} : y > \delta\}. \tag{6.31}$$

Since on $\mathbb{R}^{N+1}_+$

$$\operatorname{div}(y^{1-2s}\phi^2 e^{-\frac{|z|^2}{4}} z) = y^{1-2s}\left[(1-2s)\phi^2 + \nabla\left(\phi^2\right)\cdot z - \frac{1}{2}|z|^2\phi^2 + (N+1)\phi^2\right] e^{-\frac{|z|^2}{4}},$$

then

$$\int_{\mathbb{R}^{N+1}_\delta} y^{1-2s}\nabla\left(\phi^2\right)\cdot z\, e^{-\frac{|z|^2}{4}} dz = \int_{\mathbb{R}^{N+1}_\delta} y^{1-2s}\left[(-N-2+2s)\phi^2 + \frac{1}{2}|z|^2\phi^2\right] e^{-\frac{|z|^2}{4}} dz \\ - \delta^{2-2s}\int_{\mathbb{R}^N}\phi^2(x,\delta)e^{-\frac{|x|^2+\delta^2}{4}}\, dx.$$

Since $(2-2s)>0$, we can pass to the limit as $\delta\to 0^+$ and conclude that

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla\left(\phi^2\right)\cdot z\, e^{-\frac{|z|^2}{4}} dz = \int_{\mathbb{R}^{N+1}_+} y^{1-2s}\left[(-N-2+2s)\phi^2 + \frac{1}{2}|z|^2\phi^2\right] e^{-\frac{|z|^2}{4}} dz.$$

Then from (6.30) we deduce that

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\frac{\phi^2}{|z|^2} G\, dz \\ \le \frac{4}{(N-2s)^2}\int_{\mathbb{R}^{N+1}} y^{1-2s}\left(|\nabla\phi|^2 + \frac{1}{16}|z|^2\phi^2 + \frac{1}{4}(N+2-2s)\phi^2 - \frac{1}{8}|z|^2\phi^2\right) e^{-\frac{|z|^2}{4}}\, dz.$$

which proves (6.29). □

Proposition 6.2.2. *Let $V\in\mathcal{H}$. Then $V\sqrt{G}\in H^1(\mathbb{R}^{N+1}_+, y^{1-2s})$ and*

$$\left\|\nabla\left(V\sqrt{G}\right)\right\|^2_{L^2(\mathbb{R}^{N+1}_+, y^{1-2s})} \\ \le \frac{(N+2-2s)}{2}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}V^2 G\, dz + 4\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V|^2 G\, dz. \quad (6.32)$$

Proof. If $V\in\mathcal{H}$ then, in view of (6.13),

$$|\nabla(V\sqrt{G})|^2 = \left|\nabla V\sqrt{G} + \frac{1}{2}VG^{-\frac{1}{2}}\nabla G\right|^2 \le 2|\nabla V|^2 G + \frac{1}{8}V^2|z|^2 G$$

and so by Proposition 6.2.1 it is a clear that $V\sqrt{G}\in H^1(\mathbb{R}^{N+1}_+, y^{1-2s})$ and (6.32) holds. □

Proposition 6.2.3. *The trace operator* Tr *introduced in* (6.9) *can be extended to a linear and continuous trace operator, still denoted as* Tr,

$$\mathrm{Tr}:\mathcal{H}\to L^2(\mathbb{R}^N, G(x,0)).$$

In particular there exists a constant $K_{N,s}>0$, which depends only on N and s, such that, for any $V\in\mathcal{H}$,

$$\int_{\mathbb{R}^N}|\operatorname{Tr}(V)|^2 G(\cdot,0)\, dx \le K_{N,s}\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V|^2 G\, dz + \int_{\mathbb{R}^{N+1}_+} y^{1-2s}V^2 G\, dz\right). \quad (6.33)$$

Proof. There exists a constant $C_{N,s} > 0$, which depends only on N and s, such that, for any $\phi \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$,

$$\int_{\mathbb{R}^N} |\phi(x,0)|^2 \, dx \le C_{N,s} \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla \phi|^2 \, dz + \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \phi^2 \, dz \right), \tag{6.34}$$

see for example [101]. Testing (6.34) with $\phi\sqrt{G} \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$, by Proposition 6.2.2 we obtain (6.33) for any $\phi \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$. Then the operator Tr is densely defined on $\mathcal{H}$ and it is continuous. Hence it can be extended to a continuous trace operator on $\mathcal{H}$ satisfying (6.33). □

Proposition 6.2.4. *Letting κ_s and $\Lambda_{N,s}$ be as in (6.2), for any function $V \in \mathcal{H}$*

$$\kappa_s \Lambda_{N,s} \int_{\mathbb{R}^N} \frac{|\operatorname{Tr}(V)|^2}{|x|^{2s}} G(\cdot,0) \, dx + \frac{1}{16} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |z|^2 V^2 G \, dz \\ \le \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla V|^2 G \, dz + \frac{N+2-2s}{4} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} V^2 G \, dz. \tag{6.35}$$

Proof. It is enough to prove (6.35) for any $\phi \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$. Thanks to [60, Lemma 2.5], for any $\phi \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$

$$\int_{\mathbb{R}^N} \frac{\phi(\cdot,0)^2}{|x|^{2s}} G \, dx \le \kappa_s^{-1} \Lambda_{N,s}^{-1} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \left| \nabla \left(\phi e^{-\frac{|z|^2}{8}} \right) \right|^2 dz.$$

Then we can follow the proof of Proposition 6.2.1 to conclude that (6.35) holds. □

Proposition 6.2.4 directly implies the following trace result.

Proposition 6.2.5. *Let* Tr *be the trace operator introduced in (6.9). Then*

$$\operatorname{Tr}(\mathcal{H}) \subseteq L^2(\mathbb{R}^N, |x|^{-2s} G(x,0))$$

and $\operatorname{Tr} : \mathcal{H} \to L^2(\mathbb{R}^N, |x|^{-2s} G(x,0))$ *is a well defined, linear and continuous operator.*

Proposition 6.2.6. *For μ being as in (6.2), let us consider the quadratic form*

$$B(V) := \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla V|^2 G \, dz - \mu \int_{\mathbb{R}^N} \frac{|\operatorname{Tr}(V)|^2}{|x|^{2s}} G(\cdot,0) \, dx,$$

for any $V \in \mathcal{H}$. Then

$$\inf_{V \in \mathcal{H}\setminus\{0\}} \frac{B(V) + \frac{N+2-2s}{4} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} V^2 G \, dz}{\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla V|^2 G \, dz + \frac{N+2-2s}{4} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} V^2 G \, dz} > 0.$$

Proof. We argue by contradiction assuming that, for any $\epsilon \in (0,1)$, there exists $V_\epsilon \in \mathcal{H}$ such that

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla V_\epsilon|^2 G \, dz - \mu \int_{\mathbb{R}^N} \frac{|\operatorname{Tr}(V_\epsilon)|^2}{|x|^{2s}} G \, dx + \frac{N+2-2s}{4} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} V_\epsilon^2 G \, dz \\ < \epsilon \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla V_\epsilon|^2 G \, dz + \frac{N+2-2s}{4} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} V_\epsilon^2 G \, dz \right), \tag{6.36}$$

i.e.

$$-\frac{\mu}{1-\epsilon}\int_{\mathbb{R}^N}\frac{|\operatorname{Tr}(V_\epsilon)|^2}{|x|^{2s}}G\,dx < -\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V_\epsilon|^2 G\,dz - \frac{N+2-2s}{4}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}V_\epsilon^2 G\,dz.$$

Hence by (6.35)

$$\left(\kappa_s\Lambda_{N,s}-\frac{\mu}{1-\epsilon}\right)\int_{\mathbb{R}^N}\frac{|\operatorname{Tr}(V_\epsilon)|^2}{|x|^{2s}}G\,dx<0.$$

By (6.2), we conclude that, choosing $\epsilon<1$ small enough, $\operatorname{Tr}(V_\epsilon)=0$ thus contradicting (6.36). □

Proposition 6.2.7. *Let $K>0$. Then there exist a constant $C_{N,s,\mu}>0$ depending only on N, s, μ and $\bar{T}\in(0,\min\{T,1\})$, depending only on N, s, K, μ, such that, for every $\tilde{T}\in(0,\bar{T}]$ and any measurable function $f:\mathbb{R}^N\times(0,\tilde{T})\to\mathbb{R}$ satisfying*

$$|f(x,t)|\le K\left(1+|x|^{-2s+\varepsilon}\right)\qquad \textit{for a.e. } t\in(0,\tilde{T}) \textit{ and a.e. } x\in\mathbb{R}^N, \tag{6.37}$$

the following inequality

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V|^2 G\,dz - \int_{\mathbb{R}^N}\left(\frac{\mu}{|x|^{2s}}|\operatorname{Tr}(V)|^2 + t^s f(\sqrt{t}x,t)|\operatorname{Tr}(V)|^2\right)G(\cdot,0)\,dx\\ +\frac{N+2-2s}{4}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}V^2 G\,dz\\ \ge C_{N,s,\mu}\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V|^2 G\,dz+\int_{\mathbb{R}^{N+1}_+} y^{1-2s}V^2 G\,dz\right)\end{aligned} \tag{6.38}$$

is satisfied for a.e. $t\in(0,\tilde{T})$ and for any $V\in\mathcal{H}$. Furthermore, there exists a constant $C'_{N,s}>0$, depending only on N,s, such that, for a.e. $t\in(0,\tilde{T})$ and any $V\in\mathcal{H}$,

$$\begin{aligned}\int_{\mathbb{R}^N} t^s|f(\sqrt{t}x,t)||\operatorname{Tr}(V)|^2 G(\cdot,0)\,dx\\ \le K\,C'_{N,s}(t^s+t^{\frac{\varepsilon}{2}})\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V|^2 G\,dz+\int_{\mathbb{R}^{N+1}_+} y^{1-2s}V^2 G\,dz\right).\end{aligned} \tag{6.39}$$

Proof. Thanks to (6.37), for any $V\in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ and a.e. $t\in(0,\tilde{T})$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} f(\sqrt{t}x,t)|\phi(x,0)|^2 G(x,0)\,dx\right|\\ &\le K\int_{\mathbb{R}^N}|\phi(x,0)|^2G(x,0)\,dx + Kt^{-s+\frac{\varepsilon}{2}}\int_{\mathbb{R}^N}|x|^{-2s+\varepsilon}|\phi(x,0)|^2G(x,0)\,dx\\ &\le K\int_{\mathbb{R}^N}|\phi(x,0)|^2G(x,0)\,dx + Kt^{-s+\frac{\varepsilon}{2}}\int_{\{|x|\ge1\}}|\phi(x,0)|^2G(x,0)\,dx\\ &\quad + Kt^{-s+\frac{\varepsilon}{2}}\int_{\{|x|\le1\}}\frac{|\phi(x,0)|^2}{|x|^{2s}}G(x,0)\,dx\\ &\le \frac{K}{t^s}\left(t^s+t^{\frac{\varepsilon}{2}}\right)\int_{\mathbb{R}^N}|\phi(x,0)|^2G(x,0)\,dx + Kt^{-s+\frac{\varepsilon}{2}}\int_{\mathbb{R}^N}\frac{|\phi(x,0)|^2}{|x|^{2s}}G(x,0)\,dx.\end{aligned}$$

Then, in view of (6.33), (6.35), a density argument implies (6.39). From Proposition 6.2.6 and (6.39), choosing $\bar{T}$ small enough, we deduce (6.38). □

Proposition 6.2.8. *There exists a constant $C''_{N,s} > 0$, depending only on N and s, such that, for any $\rho \in L^{\frac{N}{2s}}(\mathbb{R}^N)$ and $V \in \mathcal{H}$,*

$$\int_{\mathbb{R}^N} |\rho| |\operatorname{Tr}(V)|^2 G(\cdot, 0)\, dx \le C''_{N,s} \|\rho\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} \|V\|^2_{\mathcal{H}}. \tag{6.40}$$

Proof. By Proposition 6.2.2 and (6.9), $\operatorname{Tr}(V)\sqrt{G(\cdot,0)} \in W^{s,2}(\mathbb{R}^N)$. Hence, thanks to the Fractional Sobolev Embedding Theorem, $\operatorname{Tr}(V)\sqrt{G(\cdot,0)} \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$ and there exists a constant $S_{N,s} > 0$, which depends only on N and s, such that

$$\left\| \operatorname{Tr}(V)\sqrt{G(\cdot,0)} \right\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)} \le S_{N,s} \left\| \operatorname{Tr}(V)\sqrt{G(\cdot,0)} \right\|_{W^{s,2}(\mathbb{R}^N)}.$$

Furthermore, by the Holdër inequality,

$$\int_{\mathbb{R}^N} |\rho| |\operatorname{Tr}(V)|^2 G(\cdot, 0)\, dx \le \|\rho\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} \left\| \operatorname{Tr}(V)\sqrt{G(\cdot,0)} \right\|^2_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)}$$

and so (6.40) follows from (6.9) and (6.32). □

6.3 An alternative formulation in Gaussian spaces

In this section we present an alternative formulation of (6.11) and a regularity result. Henceforth, for the sake of simplicity, we will assume that $t_0 = 0$; this is not restrictive up to a translation. We deal with the backward version of (6.10) which is completely equivalent to (6.10). Let

$$h(x,t) := g(x,-t) \quad \text{for any } t \in (0,T) \text{ and a.e. } x \in \mathbb{R}^N, \tag{6.41}$$

$$U(z,t) := W(z,-t) \text{ for a.e.} t \in (0,T) \text{ and } z \in \mathbb{R}^{N+1}_+, \tag{6.42}$$

$$u := \operatorname{Tr}(U). \tag{6.43}$$

Then $U \in L^2(\mathbb{R}, H^1(\mathbb{R}^{N+1}_+, y^{1-2s}))$ and U weakly solves

$$\begin{cases} y^{1-2s} U_t + \operatorname{div}(y^{1-2s}\nabla U) = 0, & \text{in } \mathbb{R}^{N+1}_+ \times (0,T), \\ \operatorname{Tr}(U(\cdot,t)) = u(\cdot,t), & \text{on } \mathbb{R}^N, \text{ for a.e. } t \in (0,T), \\ -\lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y} = \frac{\mu}{|x|^{2s}} u + hu & \text{on } \mathbb{R}^N \times (0,T), \end{cases} \tag{6.44}$$

in the sense that

$$\int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} U \phi_t \, dz \right) dt$$
$$= -\int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla U \cdot \nabla \phi \, dz \right) dt + \int_0^T \left(\int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} u\phi + hu\phi \right) dx \right) dt, \tag{6.45}$$

for any $\phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \times (0,T))$, if and only if W is a weak solution to problem (6.10).

Definition 6.3.1. Let X be a Hilbert space and $[T_1, T_2] \subset \mathbb{R}$. A function $U \in L^2((T_1, T_2), X)$ has a weak derivative $\Psi \in L^2((T_1, T_2), X)$ if, for any $\phi \in C_c^\infty((T_1, T_2), X)$,

$$\int_{T_1}^{T_2} (U, \phi_t)_X \, dt = -\int_{T_1}^{T_2} (\Psi, \phi)_X \, dt.$$

Furthermore we define

$$H^1((T_1, T_2), X) := \{U \in L^2((T_1, T_2), X) : U \text{ has a weak derivative } \Psi \in L^2((T_1, T_2), X)\}.$$

Let (X, L, X^*) be a Hilbert triplet. Thanks to the Riesz isomorphism, the property that $U \in L^2((T_1, T_2), X^*)$ has a weak derivative $\Psi \in L^2((T_1, T_2), X^*)$ can be rephrased equivalently as

$$\int_{T_1}^{T_2} {}_{X^*}\langle U(t), \phi_t(t)\rangle_X \, dt = -\int_{T_1}^{T_2} {}_{X^*}\langle \Psi(t), \phi(t)\rangle_X \, dt$$

for any $\phi \in C_c^\infty((T_1, T_2), X)$. Then the property that a function $U \in L^2((T_1, T_2), X)$ has a weak derivative $\Psi \in L^2((T_1, T_2), X^*)$ is equivalent to the fact that

$$\int_{T_1}^{T_2} (U(t), \phi_t(t))_L \, dt = -\int_{T_1}^{T_2} {}_{X^*}\langle \Psi(t), \phi(t)\rangle_X \, dt \tag{6.46}$$

for any $\phi \in C_c^\infty((T_1, T_2), X)$.

For any $U \in L^2(\mathbb{R}, H^1(\mathbb{R}^{N+1}_+, y^{1-2s}))$ satisfying (6.45), let us consider the function

$$V(z, t) := U(\sqrt{t}z, t). \tag{6.47}$$

By a density argument, we can easily verify that

$$v(\cdot, t) := \mathrm{Tr}(U(\sqrt{t}\cdot, t)) = u(\sqrt{t}\cdot, t).$$

In Proposition 6.3.2 below, we derive the weak formulation of the problem solved by V.

Proposition 6.3.2. *Let $U \in L^2(\mathbb{R}, H^1(\mathbb{R}^{N+1}_+, y^{1-2s}))$ be a solution of (6.45). Then, letting V be as in (6.47),*

$$V \in L^2((\tau, T), \mathcal{H}), \quad V_t \in L^2((\tau, T), \mathcal{H}^*) \quad \text{for any } \tau \in (0, T), \tag{6.48}$$

and

$$\begin{aligned} {}_{\mathcal{H}^*}\langle V_t, \phi\rangle_{\mathcal{H}} &= \frac{1}{t}\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla V \cdot \nabla \phi \, G \, dz \\ &\quad - \frac{1}{t}\int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} v(x,t)\phi(x,0) + t^s h(\sqrt{t}x, t) v(x,t)\phi(x,0)\right) G(x,0)\, dx, \end{aligned} \tag{6.49}$$

for any $\phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ and for a.e. $t \in (0, T)$.

Proof. Let $\phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \times (0, T))$. Testing (6.45) with ϕG_s we obtain

$$\begin{aligned} &\int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} U\phi \left[-\frac{N+2-2s}{2t} + \frac{|z|^2}{4t^2}\right] G_s \, dz\right) dt + \int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} U \phi_t G_s \, dz\right) dt \\ &= \int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla U \cdot \frac{z}{2t}\phi \, G_s dz\right) dt - \int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla U \cdot \nabla \phi \, G_s dz\right) dt \\ &\quad + \int_0^T \left(\int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} u(x)\phi(x,0,t) + h(x,t)u(x)\phi(x,0,t)\right) G_s(x,0,t)\, dx\right) dt. \end{aligned}$$

Let $\tilde{\phi}(z,t) := \phi(\sqrt{t}z,t)$. Then the change of variables $z=\sqrt{t}z'$ yields

$$\begin{aligned}\int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} V\tilde{\phi}\left[-\frac{N+2-2s}{2t}+\frac{|z|^2}{4t}\right] G\,dz\right) dt \\ +\int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} V\left[\tilde{\phi}_t-\nabla\tilde{\phi}\cdot\frac{z}{2t}\right] G\,dz\right) dt \\ =\int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla V\cdot\frac{z}{2t}\tilde{\phi}\,G dz\right) dt-\int_0^T \frac{1}{t}\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla V\cdot\nabla\tilde{\phi}\,G dz\right) dt \\ +\int_0^T \frac{1}{t}\left(\int_{\mathbb{R}^N}\left(\frac{\mu}{|x|^{2s}}v\tilde{\phi}(x,0,t)+t^s h(\sqrt{t}x,t)v\tilde{\phi}(x,0,t)\right) G\,dx\right) dt \end{aligned} \tag{6.50}$$

for any $\tilde{\phi}\in C_c^\infty(\overline{\mathbb{R}^{N+1}_+}\times(0,T))$, where V is defined in (6.47).

Let $\mathbb{R}^{N+1}_\delta$ be as in (6.31) for any $\delta>0$. Then, by the Dominated Convergence Theorem,

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}V(\nabla\tilde{\phi}\cdot z)G\,dz=\lim_{\delta\to 0^+}\int_{\mathbb{R}^{N+1}_\delta} y^{1-2s}V(\nabla\tilde{\phi}\cdot z)G\,dz.$$

Furthermore, since

$$\operatorname{div}\left(y^{1-2s}V\tilde{\phi}Gz\right)=y^{1-2s}\left[(N+2-2s)V\tilde{\phi}+(\nabla V\cdot z)\tilde{\phi}+V(\nabla\tilde{\phi}\cdot z)-V\tilde{\phi}\,\frac{|z|^2}{2}\right]G,$$

an integration by parts on $\mathbb{R}^{N+1}_\delta$ yields

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_\delta} y^{1-2s}V(\nabla\tilde{\phi}\cdot z)G\,dz=-(N+2-2s)\int_{\mathbb{R}^{N+1}_\delta} y^{1-2s}V\tilde{\phi}G\,dz \\ -\int_{\mathbb{R}^{N+1}_\delta} y^{1-2s}(\nabla V\cdot z)\tilde{\phi}G\,dz+\int_{\mathbb{R}^{N+1}_\delta} y^{1-2s}V\tilde{\phi}\,\frac{|z|^2}{2}G\,dz \\ -\delta^{2-2s}\int_{\mathbb{R}^N} V(x,\delta,t)\tilde{\phi}(x,\delta,t)G(x,\delta)\,dx.\end{aligned} \tag{6.51}$$

We claim that

$$\liminf_{\delta\to 0^+}\delta^{2-2s}\int_{\mathbb{R}^N} V(x,\delta,t)\tilde{\phi}(x,\delta,t)G(x,\delta)\,dx=0. \tag{6.52}$$

To prove (6.52) we argue by contradiction. If (6.52) does not hold, then there exists a constant $C>0$ and $\bar{\delta}\in(0,+\infty)$ such that

$$\delta^{1-2s}\int_{\mathbb{R}^N} V(x,\delta,t)\tilde{\phi}(x,\delta,t)G(x\delta)\,dx>\frac{C}{\delta}$$

for any $\delta\in(0,\bar{\delta})$. Integrating on $(0,\bar{\delta})$, we obtain, thanks to the Fubini-Tonelli Theorem,

$$+\infty\ >\ \int_{\mathbb{R}^{N+1}_+} y^{1-2s}V\tilde{\phi}G\,dz\ \geq\ \int_0^{\bar{\delta}}\left(\int_{\mathbb{R}^N} y^{1-2s}V\tilde{\phi}G\,dx\right)dy\ \geq\ C\int_0^{\bar{\delta}}\frac{1}{y}\,dy\ =\ +\infty,$$

which is a contradiction. Hence there exists a sequence $\delta_n \to 0^+$ such that, passing to the limit as $\delta = \delta_n$ and $n \to \infty$ in (6.51), we obtain that, for a.e. $t \in (0,T)$,

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s} V(\nabla\tilde{\phi}\cdot z) G\,dz = -(N+2-2s)\int_{\mathbb{R}^{N+1}_+} y^{1-2s} V\tilde{\phi} G\,dz \\ - \int_{\mathbb{R}^{N+1}_+} y^{1-2s}(\nabla V\cdot z)\tilde{\phi} G\,dz + \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V\tilde{\phi}\frac{|z|^2}{2} G\,dz. \quad (6.53)$$

Putting together (6.50) and (6.53), we conclude that

$$\int_0^T \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} V\tilde{\phi}_t\,dz\right) dt = -\int_0^T \left(\frac{1}{t}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla V\cdot\nabla\tilde{\phi}\, G\,dz\right) dt \\ + \int_0^T \left(\frac{1}{t}\int_{\mathbb{R}^N}\left(\frac{\mu}{|x|^{2s}} v(x,t)\tilde{\phi}(x,0,t) + t^s h(\sqrt{t}x,t) v(x,t)\tilde{\phi}(x,0,t)\right) G(x,0)\,dx\right) dt.$$

The integrand at the right hand side of the above equation belongs to $\mathcal{H}^*$ as a function of $\tilde{\phi}$ for a.e. $t \in (\tau,T)$ in view of (6.4), (6.33), (6.35), (6.41) and the Hölder inequality. Hence, in view of (6.46), we conclude that (6.48) and (6.49) are satisfied. □

Remark 6.3.3. From the theory of abstract parabolic equations, see for example [99, Theorem 8.60] and [44, Theorem 1, p. 473, Theorem 2, p. 477], if V satisfies (6.48), then

$$V \in C^0([\tau,T],\mathcal{L}), \text{ for any } \tau \in (0,T), \\ t \to \|V(\cdot,t)\|^2_{\mathcal{L}} \text{ is absolutely continuous on } [\tau,T] \text{ for any } \tau \in (0,T), \\ {}_{\mathcal{H}^*}\langle V_t(\cdot,t), V(\cdot,t)\rangle_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\|V(\cdot,t)\|^2_{\mathcal{L}} = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^{N+1}_+} y^{1-2s} V^2 G\,dz$$

in a distributional sense and for a.e. $t \in (0,T)$. More in general, if V, W satisfies (6.48), then

$$t \to (V(\cdot,t), W(\cdot,t))_{\mathcal{L}} \text{ is absolutely continuous on } [\tau,T] \text{ for any } \tau \in (0,T), \\ {}_{\mathcal{H}^*}\langle V_t(\cdot,t), W(\cdot,t)\rangle_{\mathcal{H}} + {}_{\mathcal{H}^*}\langle W_t(\cdot,t), V(\cdot,t)\rangle_{\mathcal{H}} = \frac{d}{dt}(V(\cdot,t), W(\cdot,t))_{\mathcal{L}}$$

in a distributional sense and for a.e. $t \in (0,T)$.

Proposition 6.3.4. *Let* $(\widetilde{\mathrm{Tr}}\, U)(\cdot,t) := \mathrm{Tr}(U(\cdot,t))$ *for any* $U \in H^1((0,T),\mathcal{H})$. *Then*

$$\widetilde{\mathrm{Tr}} : H^1((0,T),\mathcal{H}) \to H^1((0,T), L^2(\mathbb{R}^N, |x|^{-2s}G))$$

is a linear and continuous trace operator such that

$$(\widetilde{\mathrm{Tr}}(U))_t(\cdot,t) = \mathrm{Tr}(U_t(\cdot,t)), \text{ for any } U \in H^1((0,T),\mathcal{H}) \text{ and a.e. } t \in (0,T).$$

Proof. In view of (6.35) we have that $\widetilde{\mathrm{Tr}}(U) \in L^2((0,T), L^2(\mathbb{R}^N, |x|^{-2s}G(x,0)))$. Furthermore, there exists a sequence $\{U_n\}_{n\in\mathbb{N}} \subset C^\infty([0,T],\mathcal{H})$ such that $U_n \to U$ as $n \to \infty$ in $H^1((0,T),\mathcal{H})$ thanks to [87, Lemma 2.5.6.].

Let us prove that $\widetilde{\mathrm{Tr}}(U_n) \in C^1([0,T], L^2(\mathbb{R}^N, |x|^{-2s}G(x,0)))$ and that

$$(\widetilde{\mathrm{Tr}}(U_n))_t(\cdot,t) = \mathrm{Tr}((U_n)_t(\cdot,t)) \quad \text{for any } t \in (0,T).$$

We start by showing that the incremental ratio of $\widetilde{\mathrm{Tr}}(U_n)(\cdot,t)$ tends to $\mathrm{Tr}((U_n)_t(\cdot,t))$ strongly in $L^2(\mathbb{R}^N,|x|^{-2s}G(x,0))$ for any $t\in(0,T)$. Let $t\in(0,T)$ and $h\in\mathbb{R}$ be such that $|h|\le\min\{t,T-t\}$. Then, by definition of $\widetilde{\mathrm{Tr}}$ and linearity,

$$\int_{\mathbb{R}^N}\frac{G(x,0)}{|x|^{2s}}\left|\left(\frac{\widetilde{\mathrm{Tr}}(U_n)(\cdot,t+h)-\widetilde{\mathrm{Tr}}(U_n)(\cdot,t)}{h}\right)-\widetilde{\mathrm{Tr}}((U_n)_t(\cdot,t))\right|^2dx$$
$$=\int_{\mathbb{R}^N}\frac{G(x,0)}{|x|^{2s}}\left|\mathrm{Tr}\left(\frac{U_n(\cdot,t+h)-U_n(\cdot,t)}{h}-(U_n)_t(\cdot,t)\right)\right|^2dx$$
$$\le\mathrm{const}\left\|\frac{U_n(\cdot,t+h)-U_n(\cdot,t)}{h}-(U_n)_t(\cdot,t)\right\|_{\mathcal{H}}^2\to0$$

as $h\to0^+$, in view of (6.35).

In the same way we can show that $(\widetilde{\mathrm{Tr}}(U_n))_t\in C^0([0,T],L^2(\mathbb{R}^N,|x|^{-2s}G(x,0)))$. It follows that, taking the limit of the incremental ratio,

$$\frac{d}{dt}\left(\widetilde{\mathrm{Tr}}(U_n),\phi\right)_{L^2(\mathbb{R}^N,|x|^{-2s}G(x,0))}$$
$$=(\mathrm{Tr}((U_n)_t),\phi)_{L^2(\mathbb{R}^N,|x|^{-2s}G(x,0))}+\left(\widetilde{\mathrm{Tr}}(U_n),\phi_t\right)_{L^2(\mathbb{R}^N,|x|^{-2s}G(x,0))}$$

for any function $\phi\in C_c^\infty((0,T),L^2(\mathbb{R}^N,|x|^{-2s}G(x,0)))$.

Then, for any test function $\phi\in C_c^\infty((0,T),L^2(\mathbb{R}^N,|x|^{-2s})G(x,0)))$,

$$\int_0^T\left(\int_{\mathbb{R}^N}\frac{\widetilde{\mathrm{Tr}}(U)}{|x|^{2s}}\phi_tG(x,0)\,dx\right)dt=\lim_{n\to\infty}\int_0^T\left(\int_{\mathbb{R}^N}\frac{\widetilde{\mathrm{Tr}}(U_n)}{|x|^{2s}}\phi_tG(x,0)\,dx\right)dt$$
$$=-\lim_{n\to\infty}\int_0^T\left(\int_{\mathbb{R}^N}\frac{\mathrm{Tr}((U_n)_t)}{|x|^{2s}}\phi G(x,0)\,dx\right)dt=\int_0^T\left(\int_{\mathbb{R}^N}\frac{\mathrm{Tr}(U_t)}{|x|^{2s}}\phi G(x,0)\,dx\right)dt.$$

We conclude that there exists the weak derivative with respect to t of $\widetilde{\mathrm{Tr}}(U)$ and that

$$(\widetilde{\mathrm{Tr}}(U))_t(\cdot,t)=\mathrm{Tr}(U_t(\cdot,t))\text{ for a.e. }t\in(0,T).$$

The continuity of the operator follows from (6.35). □

Remark 6.3.5. The natural embedding

$$I:\mathcal{L}\to\mathcal{H}^*,\quad I(V)(W):=\int_{\mathbb{R}^{N+1}_+}y^{1-2s}VWG\,dz$$

is linear, continuous and injective. With a slight abuse of notation, we will identify $\mathcal{L}$ and $I(\mathcal{L})$.

Proposition 6.3.6. *For $K>0$, let $\bar T\in(0,\min\{T,1\})$ depending on K be as in Proposition 6.2.7. Let $\tilde T\in(0,\bar T]$ and $f\in L^1_{loc}(\mathbb{R}^N\times(0,\tilde T))$ be such that*

$$|f(x,t)|+|\nabla f(x,t)\cdot x|\le K(1+|x|^{-2s+\varepsilon})\text{ for a.e. }t\in(0,\tilde T)\text{ and a.e. }x\in\mathbb{R}^N,\quad(6.54)$$
$$f_t\in L^\infty_{loc}((0,\tilde T),L^{\frac{N}{2s}}(\mathbb{R}^N)).$$

If $\tau\in(0,\tilde T)$, $V\in L^2((\tau,\tilde T),\mathcal{H})$, $V_t\in L^2((\tau,\tilde T),\mathcal{H}^*)$, $V(\cdot,\tilde T)\in\mathcal{H}$, and V is a solution of (6.49) with $h=f$, then $V_t\in L^2((\tau,\tilde T),\mathcal{L})$ in the sense of Remark 6.3.5.

Proof. For all $t \in (0, \tilde{T})$, let us consider the linear map

$$\begin{aligned}
&A_t : \mathcal{H} \to \mathcal{H}^*, \\
&{}_{\mathcal{H}*}\langle A_t(V), \phi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla V \cdot \nabla \phi \, G \, dz \\
&- \int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} \operatorname{Tr}(V) \operatorname{Tr}(\phi) + t^s f(\sqrt{t}x, t) \operatorname{Tr}(V) \operatorname{Tr}(\phi) \right) G \, dx \quad \text{ for any } \phi, V \in \mathcal{H}.
\end{aligned}$$

In view of (6.54), the Hölder inequality, (6.33) and (6.35), A_t is well defined and continuous. From standard techniques in the theory of parabolic equations, see for example [57], the Faedo-Galerkin method provides a sequence of functions $\{V_n\}_{n \in \mathbb{N}}$ such that:

$$V_n \in L^2((\tau, \tilde{T}), \mathcal{H}) \text{ for any } n \in \mathbb{N},$$

$$V_n \rightharpoonup V \text{ weakly in } L^2((\tau, \tilde{T}), \mathcal{H}) \text{ as } n \to \infty, \tag{6.55}$$

$$(V_n)_t \in L^2((\tau, \tilde{T}), \mathcal{H}) \text{ for any } n \in \mathbb{N},$$

$$(V_n)_t \rightharpoonup V_t \text{ weakly in } L^2((\tau, \tilde{T}), \mathcal{H}^*) \text{ as } n \to \infty, \tag{6.56}$$

$$V_n(\cdot, \tilde{T}) \to V(\cdot, \tilde{T}) \text{ strongly in } \mathcal{H} \text{ as } n \to \infty,$$

$$\{V_n\}_{n \in \mathbb{N}} \text{ is bounded in } C([\tau, \tilde{T}], \mathcal{L}). \tag{6.57}$$

For any $n \in \mathbb{N}$, the function V_n belongs to $H^1((\tau, \tilde{T}), W_n)$ and solves, for a.e. $t \in (0, \tilde{T})$,

$$\begin{aligned}
t \int_{\mathbb{R}^{N+1}_+} y^{1-2s} (V_n)_t \phi \, G \, dz &= \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla V_n \cdot \nabla \phi \, G \, dz \\
&- \int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} \operatorname{Tr}(V_n) \operatorname{Tr}(\phi) + t^s f(\sqrt{t}x, t) \operatorname{Tr}(V_n) \operatorname{Tr}(\phi) \right) G(x, 0) \, dx,
\end{aligned} \tag{6.58}$$

for any $\phi \in W_n \subset \mathcal{H}$, where W_n is a suitable finite dimensional subspace of $\mathcal{H}$. Testing (6.58) with $(V_n)_t$ and integrating with respects to t on $(\tau, \tilde{T})$, we obtain that

$$\begin{aligned}
\int_\tau^{\tilde{T}} t \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |(V_n)_t|^2 \, G \, dz \right) dt &= \int_\tau^{\tilde{T}} {}_{\mathcal{H}*}\langle A_t(V_n), (V_n)_t \rangle_{\mathcal{H}} \, dt \\
&= \frac{1}{2} {}_{\mathcal{H}*}\left\langle A_{\tilde{T}}(V_n)(\tilde{T}), V_n(\tilde{T}) \right\rangle_{\mathcal{H}} - \frac{1}{2} {}_{\mathcal{H}*}\langle A_\tau(V_n)(\tau), V_n(\tau) \rangle_{\mathcal{H}} \\
+ \frac{1}{2} \int_\tau^{\tilde{T}} \left(\int_{\mathbb{R}^N} \left[s t^{s-1} f(\sqrt{t}x, t) + \frac{1}{2} t^{s-\frac{1}{2}} \nabla f(\sqrt{t}x, t) \cdot x + t^s f_t(\sqrt{t}x, t) \right] \right. & \left. |\operatorname{Tr}(V_n)|^2 \, G(\cdot, 0) dx \right) dt
\end{aligned}$$

thanks to Proposition 6.3.4. For a.e. $t \in (\tau, \tilde{T})$,

$$\int_{\mathbb{R}^N} \left| s t^{s-1} f(\sqrt{t}x, t) + \frac{1}{2} t^{s-\frac{1}{2}} \nabla f(\sqrt{t}x, t) \cdot x \right| |\operatorname{Tr}(V_n)|^2 \, G(\cdot, 0) dx \le \text{const} \, \|V_n\|^2_{\mathcal{H}} \tag{6.59}$$

in view of (6.33), (6.35), and (6.54).

By (6.38), (6.40), (6.55), (6.57), and (6.59), and we conclude that $\{(V_n)_t\}_{n \in \mathbb{N}}$ is bounded in $L^2((\tau, \tilde{T}), \mathcal{L})$. Then, up to a subsequence, there exists $W \in L^2((\tau, \tilde{T}), \mathcal{L})$ such that

$$(V_n)_t \rightharpoonup W \text{ weakly in } L^2((\tau, \tilde{T}), \mathcal{L}).$$

By (6.56) we conclude that $W = V_t$, hence $V_t \in L^2((\tau, \tilde{T}), \mathcal{L})$. □

6.4 Spectrum of a weighted Ornstein-Uhlenbeck operator

In this section we prove Proposition 6.1.5. The following compactness result ensures that the point spectrum of (6.12) is discrete.

Proposition 6.4.1. *The embedding $i : \mathcal{H} \to \mathcal{L}$, $i(V) = V$, is compact.*

Proof. Let $\{V_n\}$ be a sequence converging to some $V \in \mathcal{H}$ weakly in $\mathcal{H}$ as $n \to \infty$. Then by [111, Theorem 19.7], for any $R > 0$

$$\lim_{n\to\infty} \int_{B_R^+} y^{1-2s}|V - V_n|^2 G\, dz = 0. \tag{6.60}$$

Moreover, for any $n \in \mathbb{N}$ and $R > 0$, by (6.29),

$$\begin{aligned} \int_{\mathbb{R}^{N+1}_+ \setminus B_R^+} y^{1-2s}|V - V_n|^2 G dz &\le \frac{1}{R^2} \int_{\mathbb{R}^{N+1}_+ \setminus B_R^+} y^{1-2s}|z|^2|V - V_n|^2 G dz \\ &\le \frac{16}{R^2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla(V - V_n)|^2 G\, dz + \frac{4(N+2-2s)}{R^2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|V - V_n|^2 G\, dz. \end{aligned} \tag{6.61}$$

Since $\{V_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$ we conclude by (6.61) that

$$\int_{\mathbb{R}^{N+1}_+ \setminus B_R^+} y^{1-2s}|V - V_n|^2 G dz \le \frac{\text{const}}{R^2} \quad \text{for any } R > 0 \text{ and for any } n \in \mathbb{N}. \tag{6.62}$$

Putting together (6.60) and (6.62) we obtain that $V_n \to V$ strongly in $\mathcal{L}$ as $n \to +\infty$, thus completing the proof. □

Proposition 6.4.2. *The eigenvalues of (6.12) form a non-decreasing, diverging sequence $\{\gamma_k\}_{k\in\mathbb{N}\setminus\{0\}}$. Furthermore there exists an orthonormal basis of $\mathcal{L}$ of eigenfunctions of (6.12) whose elements belong to $\mathcal{H}$.*

Proof. Let $L : \mathcal{H} \to \mathcal{H}^*$ be defined as

$$\begin{aligned} L(V)(\phi) := &\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla U \cdot \nabla\phi\, G\, dz \\ &- \int_{\mathbb{R}^N} \frac{\mu}{|x|^{2s}} \operatorname{Tr}(U) \operatorname{Tr}(\phi)\, G\, dx + \frac{N+2-2s}{4} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V \phi\, G\, dz \end{aligned}$$

for any $V, \phi \in \mathcal{H}$. In view of (6.38) in the case $f \equiv 0$, the operator L is coercive. It follows that the operator $T : \mathcal{L} \to \mathcal{L}$ defined as $T := L^{-1}$ is well-defined. Since T is also compact in view of Proposition 6.4.1, the conclusion follows from by the Spectral Theorem. □

Remark 6.4.3. For any $r > 0$, there exists a linear, continuous and compact trace operator

$$\operatorname{Tr}_{S_r^+} : \mathcal{H} \to L^2(S_r^+, y^{1-2s}).$$

Indeed $\mathcal{H} \hookrightarrow H^1(B_r^+, y^{1-2s})$ since $G > \text{const} > 0$ on B_r^+; moreover, in view of [111, Theorem 19.7] and the Divergence Theorem, one can easily verify that the trace operator from $H^1(B_r^+, y^{1-2s})$ to $L^2(S_r^+, y^{1-2s})$ is compact.

Proof of Proposition 6.1.5. Let γ be an eigenvalue of problem (6.12) and Y an associated eigenfunction. Let $\{\psi_k\}_{k\in\mathbb{N}\setminus\{0\}}$ be the orthonormal basis of $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s})$ introduced in Section 6.1. By Remark 6.4.3, for any $r > 0$ the function Y admits a trace $\mathrm{Tr}_{S_r^+}(Y) \in L^2(S_r^+, y^{1-2s})$. By a change of variables $\mathrm{Tr}_{\mathbb{S^N}_+}(Y(r\cdot)) \in L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s})$, hence

$$\mathrm{Tr}_{\mathbb{S^N}_+}(Y(r\cdot)) = \sum_{k=1}^{\infty} \varphi_k(r)\psi_k, \qquad \text{with } \varphi_k(r) := \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}\, \mathrm{Tr}_{\mathbb{S^N}_+}(Y(r\theta))\, \psi_k(\theta)\, dS.$$

Since, by classical elliptical regularity theory, $Y \in C^\infty_{loc}(\mathbb{R}^{N+1}_+)$, so writing $z = r\theta$ where $r = |z|$ and $\theta = \frac{z}{|z|}$, we have that $Y(z) = Y(r\theta) = \mathrm{Tr}_{\mathbb{S^N}_+}(Y(r\cdot))|_\theta = \sum_{k=1}^{\infty} \varphi_k(r)\psi_k(\theta)$ for any $z = r\theta \in \mathbb{R}^{N+1}_+$. Then thanks to [60, Lemma 2.1], (6.12), and (6.16), a direct computation shows that

$$\varphi_k''(r) + \left(\frac{N+1-2s}{r} - \frac{r}{2}\right)\varphi_k'(r) + \left(\gamma - \frac{\nu_k}{r^2}\right)\varphi_k(r) = 0, \quad \text{in } (0,+\infty) \tag{6.63}$$

for any $k \in \mathbb{N}\setminus\{0\}$. Since $Y \in \mathcal{H}$

$$\begin{aligned} +\infty &> \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \frac{Y^2}{|z|^2} e^{-\frac{|z|^2}{4}}\, dz = \int_0^\infty r^{N-1-2s} e^{-\frac{r^2}{4}} \left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} Y^2(r\theta)\, dS\right) dr \\ &= \int_0^\infty r^{N-1-2s} e^{-\frac{r^2}{4}} \left(\sum_{k=1}^{\infty} \varphi_k^2(r)\right) dr \ge \int_0^\infty r^{N-1-2s} e^{-\frac{r^2}{4}} \varphi_k^2(r)\, dr, \end{aligned} \tag{6.64}$$

for any $k \in \mathbb{N}\setminus\{0\}$, thanks to Plancherel's identity and (6.29). Analogously

$$+\infty > \int_{\mathbb{R}^{N+1}_+} y^{1-2s} Y^2 e^{-\frac{|z|^2}{4}}\, dz \ge \int_0^\infty r^{N+1-2s} e^{-\frac{r^2}{4}} \varphi_k^2(r)\, dr, \tag{6.65}$$

for any $k \in \mathbb{N}\setminus\{0\}$. Furthermore, letting

$$w_k(t) := (4t)^{\frac{\alpha_k}{2}} \varphi_k(2\sqrt{t}) \qquad \text{for any } t \in (0,\infty),$$

(6.63) and a direct computation imply that w_k solves

$$t w_k''(t) + \left(\frac{N+2-2s}{2} - \alpha_k - t\right) w_k'(t) + \left(\frac{\alpha_k}{2} + \gamma\right) w_k(t) \quad \text{in } (0,\infty). \tag{6.66}$$

Equation (6.66) is the well-known Kummer Confluent Hypergeometric Equation,

$$t w_k''(t) + (b-t)\, w_k'(t) - c w_k(t) \quad \text{in } (0,\infty), \tag{6.67}$$

with parameters $b = \left(\frac{N+2-2s}{2} - \alpha_k\right) > 1$, by (6.17) and (6.19), and $c = -\left(\frac{\alpha_k}{2} + \gamma\right)$, see [11] or [103]. Then the solution w_k can be written as

$$w_k(t) = A_k M\left(-\frac{\alpha_k}{2} - \gamma, \frac{N+2-2s}{2} - \alpha_k, t\right) + B_k T\left(-\frac{\alpha_k}{2} - \gamma, \frac{N+2-2s}{2} - \alpha_k, t\right)$$

with $A_k, B_k \in \mathbb{R}$, where $M(c,b,t)$ denotes the Kummer function and $T(c,b,t)$ denotes the Tricomi function; $M(c,b,t)$ and $T(c,b,t)$ are linearly independent solutions of (6.67) (see [11] or [103]). Furthermore from [11]

$$T\left(-\frac{\alpha_k}{2} - \gamma, \frac{N+2-2s}{2} - \alpha_k, t\right) \sim \mathrm{const}\, t^{1-\frac{N+2-2s}{2}+\alpha_k} \quad \text{as } t \to 0^+, \tag{6.68}$$

where the constant in (6.68) depends only on s, α_k, N, γ and is different from 0. We recall the following expression for the Kummer function:

$$M(c,b,t) = \sum_{n=0}^{\infty} \frac{(c)_n}{(b)_n} \frac{t^n}{n!},$$

where $(\cdot)_n$ is the Pochhammer's symbol defined in (6.20). It is clear that $M(c,b,t)$ has a finite limit as $t \to 0^+$, while its asymptotic behaviour at $+\infty$ depends on the parameter c. Then, for any $k \in \mathbb{N} \setminus \{0\}$, if $B_k \neq 0$

$$w_k(t) \sim \text{const}\, B_k t^{1-\frac{N+2-2s}{2}+\alpha_k} \quad \text{as } t \to 0^+,$$

for some const $\neq 0$, and so

$$\varphi_k(r) \sim B_k \text{const}\, r^{-N+2s+\alpha_k} \quad \text{as } r \to 0^+.$$

From (6.64) we deduce that necessarily $B_k = 0$ for any $k \in \mathbb{N} \setminus \{0\}$. Hence

$$w_k(t) = A_k M\left(-\frac{\alpha_k}{2} - \gamma, \frac{N+2-2s}{2} - \alpha_k, t\right). \tag{6.69}$$

Moreover, if $(\frac{\alpha_k}{2} + \gamma) \notin \mathbb{N}$, then

$$M\left(-\frac{\alpha_k}{2} - \gamma, \frac{N+2-2s}{2} - \alpha_k, t\right) \sim \text{const}\, e^t t^{\frac{\alpha_k}{2}-\gamma-\frac{N}{2}-1+s} \quad \text{as } t \to +\infty, \tag{6.70}$$

for some const $\neq 0$, see [11]. From (6.69) and (6.70) it follows that

$$\varphi_k(r) \sim A_k \text{const}\, e^{\frac{r^2}{4}} r^{-2\gamma-N-2+2s} \quad \text{as } r \to +\infty$$

and hence necessarily $A_k = 0$ for any $k \in \mathbb{N} \setminus \{0\}$ in view of (6.65). In conclusion, if γ is an eigenvalue of (6.12), then there exists $k \in \mathbb{N} \setminus \{0\}$ such that $(\frac{\alpha_k}{2} + \gamma) \in \mathbb{N}$.

On the other hand, for any $m \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$, letting $Y_{m,k}$ be as in (6.18), a direct computation shows that $Y_{m,k}$ is a solution of (6.12) with $\gamma := m - \frac{\alpha_k}{2}$, by [60, Lemma 2.1], and (6.16). Hence $(m - \frac{\alpha_k}{2})$ is an eigenvalue of (6.12).

From the well-known correspondence between Kummer functions and the generalized Laguerre polynomials L_n^a, we have that $P_{j,n}(t) = \binom{n+a_j}{n}^{-1} L_n^{a_j}(t)$, where $a_j = ((\frac{N-2s}{2})^2 + \nu_j(\mu))^{1/2}$. Then, recalling that $\{\psi_k\}_{k\in\mathbb{N}\setminus\{0\}}$ is an orthonormal basis of $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s})$ and using the orthogonality relation for Laguerre polynomials, it is easy to verify that Y_{m_1,j_1} is orthogonal to Y_{m_2,j_2} in $\mathcal{L}$ whenever $(m_1,j_1) \neq (m_2,j_2)$. Then we conclude that (6.23) is an orthonormal basis of $\mathcal{L}$. □

Proposition 6.4.4. *Let Y be a solution of* (6.12) *in the sense of* (6.14) *such that* $\mathrm{Tr}(Y) = 0$. *Then $Y \equiv 0$ on $\mathbb{R}^{N+1}_+$.*

Proof. If $\mathrm{Tr}(Y) = 0$, by (6.12) we have that $(-\lim_{y\to 0^+} y^{1-2s}\frac{\partial Y}{\partial y}) = 0$ on $\mathbb{R}^N$. Hence the function

$$\widehat{Y}(x,y) = \begin{cases} Y(x,y), & \text{if } y \geq 0, \\ 0, & \text{if } y < 0, \end{cases}$$

belongs to $H^1_{loc}(\mathbb{R}^{N+1}, |y|^{1-2s})$ and weakly solves

$$-\operatorname{div}(|y|^{1-2s} G\, \nabla \widehat{Y}) = \gamma |y|^{1-2s} G\, \widehat{Y} \quad \text{in } \mathbb{R}^{N+1}.$$

The unique continuation principle for elliptic equations with Muckenhoupt weights proved in [127] then implies that $\widehat{Y} \equiv 0$ in $\mathbb{R}^{N+1}$, so that $Y \equiv 0$ on $\mathbb{R}^{N+1}_+$. □

6.5 An Almgren-Poon type monotonicity formula

In this section we develop an Almgren-Poon type monotonicity formula for solutions of (6.49). Let $\bar{T}$ be as in Proposition 6.2.7 with $K = C_g$ and C_g as in (6.4), and

$$\alpha := \frac{T}{2(\lfloor T/\bar{T} \rfloor + 1)}$$

where $\lfloor \cdot \rfloor$ denotes the floor function, i.e. $\lfloor x \rfloor = \max\{j \in \mathbb{Z} : j \le x\}$. It follows that

$$(0, T) = \bigcup_{i=1}^{k} (a_i, b_i)$$

where

$$k = 2(\lfloor T/\bar{T} \rfloor + 1) - 1, \quad a_i = (i-1)\alpha, \quad \text{and} \quad b_i = (i+1)\alpha.$$

It is clear that $2\alpha \in (0, \bar{T})$ and $(a_i, b_i) \cap (a_{i+1}, b_{i+1}) \neq \emptyset$. For every $i \in \{1, \dots, k\}$ we define

$$V_i(z, t) = U(\sqrt{t} z, t + a_i), \quad z \in \mathbb{R}^{N+1}_+, \quad t \in (0, 2\alpha),$$
$$v_i(x, t) = u(\sqrt{t} x, t + a_i), \quad x \in \mathbb{R}^N, \quad t \in (0, 2\alpha),$$

see (6.42) and (6.43). Then $\mathrm{Tr}(V_i(\cdot, t)) = v_i(\cdot, t)$ for every $i = 1, \dots, k$ and a.e. $t \in (0, 2\alpha)$.

Remark 6.5.1. Reasoning as in Section 6.3, it is easy to see that, for any $i = 1 \dots, k$, the function V_i solves

$$\begin{aligned} {}_{\mathcal{H}^*}\langle (V_i)_t, \phi \rangle_{\mathcal{H}} &= \frac{1}{t} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla V_i \cdot \nabla \phi \, G \, dz \\ &\quad - \frac{1}{t} \int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} v_i(x,t) \phi(x,0) + t^s h(\sqrt{t}x, t + a_i) v_i(x,t) \phi(x,0) \right) G(x,0) \, dx, \end{aligned} \tag{6.71}$$

for any $\phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}})$ and a.e. $t \in (0, 2\alpha)$. Furthermore, by Proposition 6.3.2 $V_i \in L^2((\tau, 2\alpha), \mathcal{H})$ and $(V_i)_t \in L^2((\tau, 2\alpha), \mathcal{H}^*)$ for any $\tau \in (0, 2\alpha)$.

For any $i = i, \dots, k$ and $t \in (0, 2\alpha)$, let

$$H_i(t) := \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V_i^2 G \, dz$$

and

$$D_i(t) := \frac{1}{t} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla V_i|^2 G \, dz - \frac{1}{t} \int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} v_i^2 + t^s h(\sqrt{t}x, t + a_i) v_i^2 \right) G(x,0) \, dx. \tag{6.72}$$

Proposition 6.5.2. *For any $i = 1, \dots, k$, we have that $H_i \in W^{1,1}_{loc}(0, 2\alpha)$ and*

$$H_i'(t) = 2 \, {}_{\mathcal{H}_t^*}\langle (V_i)_t, V_i \rangle_{\mathcal{H}_t} = 2 D_i(t) \tag{6.73}$$

in a distributional sense and a.e. in $(0, 2\alpha)$.

Proof. The claim follows from Remark 6.3.3 and (6.72). □

Proposition 6.5.3. *Let $C_{N,s,\mu}$ be as in Proposition 6.2.7 with $K := C_g$ and C_g as in (6.4). Then the function*

$$t \to t^{-2C_{N,s,\mu}+\frac{N-2+2s}{2}} H_i(t)$$

is non-decreasing in $(0, 2\alpha)$.

Proof. In view of (6.38) and (6.73)

$$H_i'(t) \geq \frac{1}{t}\left(2C_{N,s,\mu} - \frac{N-2+2s}{2}\right) H_i(t) \quad \text{for a.e. } t \in (0, 2\alpha),$$

hence

$$\frac{d}{dt}\left(t^{-2C_{N,s,\mu}+\frac{N-2+2s}{2}} H_i(t)\right) \geq 0 \quad \text{for a.e. } t \in (0, 2\alpha).$$

We conclude that $t \to t^{-2C_{N,s,\mu}+\frac{N-2+2s}{2}} H_i(t)$ is non-decreasing in $(0, 2\alpha)$. □

Corollary 6.5.4. *If $1 \leq i \leq k$ and $H_i(\bar{t}) = 0$ for some $\bar{t} \in (0, 2\alpha)$, then $H_i(t) = 0$ for any $t \in (0, \bar{t})$.*

Proof. Since $t \to t^{-2C_{N,s,\mu}+\frac{N-2+2s}{2}} H_i(t)$ is non-decreasing in $(0, 2\alpha)$ by Proposition 6.5.3 and it is non-negative, from the assumption $H(\bar{t}) = 0$ it follows that $H_i(t) = 0$ for any $t \in (0, \bar{t})$. □

The regularity of the function $tD_i(t)$ is discussed in the following proposition.

Proposition 6.5.5. *If $1 \leq i \leq k$ and $T_i \in (0, 2\alpha)$ is such that $V_i(\cdot, T_i) \in \mathcal{H}$ then*

(i) $(V_i)_t \in L^2((\tau, T_i), \mathcal{L})$ for any $\tau \in (0, T_i)$,

(ii) the function $t \to tD_i(t)$ belongs to $W^{1,1}_{loc}(0, T_i)$ and its weak derivative is as follows:

$$\frac{d}{dt}(tD_i(t)) = 2t\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\,|(V_i)_t|^2\, G dz - \int_{\mathbb{R}^N} st^{s-1}h(\sqrt{t}x, t+a_i)v_i^2(x,t)G(x,0)\,dx$$
$$- \int_{\mathbb{R}^N}\left(t^{s-\frac{1}{2}}\nabla h(\sqrt{t}x, t+a_i)\cdot\frac{x}{2} + t^s h_t(\sqrt{t}x, t+a_i)\right) v_i^2(x,t)G(x,0)\,dx. \quad (6.74)$$

Proof. Let $1 \leq i \leq k$. Then (i) follows from Proposition 6.3.6 and Remark 6.5.1.

With an approximating procedure similar to Proposition 6.3.6, formally testing (6.71) with $(V_i)_t$ yields, for a.e. $\tau \in (0, T_i)$,

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V_i(\cdot,\tau)|^2 G\,dz - \int_{\mathbb{R}^N}\left(\frac{\mu}{|x|^{2s}}v_i^2(\cdot,\tau) + \tau^s h(\sqrt{\tau}x, \tau+a_i)v_i^2(\cdot,\tau)\right)G(x,0)\,dx$$
$$= \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V_i(\cdot,T_i)|^2 G\,dz$$
$$- \int_{\mathbb{R}^N}\left(\frac{\mu}{|x|^{2s}}v_i^2(\cdot,T_i) + T_i{}^s h(\sqrt{T_i}x, T_i+a_i)v_i^2(\cdot,T_i)\right)G(x,0)\,dx$$
$$- 2\int_\tau^{T_i} t\left(\int_{\mathbb{R}^{N+1}} y^{1-2s}(V_i)_t^2 G\,dz\right)dt$$
$$+ \int_\tau^{T_i}\left(\int_{\mathbb{R}^N}\left(st^{s-1}h(\sqrt{t}x, t+a_i) + t^s h_t(\sqrt{t}x, t+a_i)\right)v_i^2(x,t)G(x,0)\,dx\right)dt$$
$$+ \int_\tau^{T_i}\left(\int_{\mathbb{R}^N} t^{s-\frac{1}{2}}\nabla h(\sqrt{t}x, t+a_i)\cdot\frac{x}{2}v_i^2(x,t)G(x,0)\,dx\right)dt,$$

hence, thanks to (6.3), (6.4) and (6.41), the function

$$\tau \mapsto \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V_i(\cdot,\tau)|^2 G\,dz - \int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} v_i^2(\cdot,\tau) + \tau^s h(\sqrt{\tau}x,\tau) v^2(\cdot,\tau)\right) G(x,0)\,dx$$

is absolute continuous on $[T_1,T_2]$ for any $[T_1,T_2] \subset (0,T_i)$ and, for a.e. $\tau \in (0,T_i)$,

$$\begin{aligned}
&\frac{d}{d\tau}\Bigg(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V_i(\cdot,\tau)|^2 G\,dz \\
&\qquad - \int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} v_i^2(\cdot,\tau) + \tau^s h(\sqrt{\tau}x,\tau+a_i) v_i^2(\cdot,\tau)\right) G(x,0)\,dx \Bigg) \\
&= 2\tau \int_{\mathbb{R}^{N+1}_+} y^{1-2s}(V_i)_t^2(z,\tau) G(z)\,dz - \int_{\mathbb{R}^N} \tau^{s-\frac{1}{2}} \nabla h(\sqrt{\tau}x,\tau+a_i)\cdot\frac{x}{2} v_i^2(\tau,x) G(x,0)\,dx \\
&\qquad - \int_{\mathbb{R}^N} \left(s\tau^{s-1} h(\sqrt{\tau}x,\tau+a_i) + \tau^s h_t(\sqrt{\tau}x,\tau+a_i)\right) v_i^2(\tau,x) G(x,0)\,dx.
\end{aligned}$$

The proof is thereby complete. □

For any $i = 1\dots k$, let us define the Almgren-Poon frequency function

$$\mathcal{N}_i : (0,2\alpha) \to \mathbb{R}\cup\{-\infty,+\infty\}, \quad \mathcal{N}(t) := \frac{tD_i(t)}{H_i(t)}.$$

Proposition 6.5.6. *If there exists $\beta_i, T_i \in (0,2\alpha)$ such that*

$$\beta_i < T_i, \quad H_i(t) > 0 \text{ for all } t \in (\beta_i,T_i), \text{ and } V_i(\cdot,T_i) \in \mathcal{H}, \tag{6.75}$$

then $\mathcal{N}_i \in W^{1,1}_{loc}(\beta_i,T_i)$ and the weak derivative of $\mathcal{N}_i$ can be written as

$$\mathcal{N}_i'(t) = \nu_{1,i}(t) + \nu_{2,i}(t) \quad \text{for a.e. } t \in (\beta_i,T_i) \tag{6.76}$$

where

$$\begin{aligned}
\nu_{1,i}(t) := \frac{2t}{H_i^2(t)}\Bigg[&\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\left|(V_i)_t\right|^2 G dz\right)\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} V_i^2 G\,dz\right) \\
&- \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}(V_i)_t V_i G dz\right)^2 \Bigg]
\end{aligned} \tag{6.77}$$

and

$$\begin{aligned}
\nu_{2,i}(t) := -\frac{1}{H_i(t)} \times \Bigg(\int_{\mathbb{R}^N} \Big(& s t^{s-1} h(\sqrt{t}x,t+a_i) + t^{s-\frac{1}{2}} \nabla h(\sqrt{t}x,t+a_i)\cdot\frac{x}{2} \\
&+ t^s h_t(\sqrt{t}x,t+a_i)\Big) v_i^2(x,t) G(x,0)\,dx\Bigg).
\end{aligned} \tag{6.78}$$

Furthermore $\nu_{1,i}(t) \geq 0$ for a.e. $t \in (\beta_i,T_i)$ and

$$\mathcal{N}_i(t) > -\frac{N+2-2s}{4} \quad \text{for any } t \in (\beta_i,T_i). \tag{6.79}$$

Proof. Since $H_i(t) > 0$ for any $t \in (\beta_i, T_i)$, then $1/H_i, tD_i \in W^{1,1}_{loc}(\beta_i, T_i)$ by Proposition 6.5.2 and Proposition 6.5.5. Hence $\mathcal{N}_i \in W^{1,1}_{loc}(\beta_i, T_i)$. Furthermore

$$\mathcal{N}_i'(t) = \frac{(tD_i)' H_i - tD_i H_i'}{H_i^2}$$

and so, thanks to (6.73) and (6.74), we conclude that (6.76) holds with $\nu_{1,i}$ and $\nu_{2,i}$ as in (6.77) and (6.78) respectively. By the Cauchy-Schwarz inequality in $\mathcal{L}$ we have $\nu_{1,i}(t) \geq 0$ for a.e. $t \in (\beta_i, T_i)$. Finally, (6.79) follows directly from (6.38) with the function $f(x,t) := h(x, t + a_i)$. □

The remainder term $\nu_{2,i}$ can be estimated in terms of the frequency function as follows.

Proposition 6.5.7. *Let $\nu_{2,i}$ be as in (6.78). Then there exists a constant $C_1 > 0$ such that, if $i \in \{i, \dots, k\}$ and $\beta_i, T_i \in (0, 2\alpha)$ are as in (6.75), then*

$$|\nu_{2,i}(t)| \leq C_1 \left(t^{-1+\frac{\varepsilon}{2}} + \|h_t(\cdot, t + a_i)\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} \right) \left(\mathcal{N}_i(t) + \frac{N+2-2s}{4} \right) \tag{6.80}$$

for a.e. $t \in (\beta_i, T_i)$.

Proof. Estimate (6.80) follows from (6.38), (6.39), and (6.40), taking into account that, by a change of variables,

$$\left\| t^s h_t(\sqrt{t}\cdot, t + a_i) \right\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} = \|h_t(\cdot, t + a_i)\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)}$$

for any $i \in \{i, \dots, k\}$. □

Proposition 6.5.8. *There exists a constant $C_2 > 0$ such that, if $i \in \{i, \dots, k\}$ and $\beta_i, T_i \in (0, 2\alpha)$ are as in (6.75), then*

$$\mathcal{N}_i(t) \leq -\frac{N+2-2s}{4} + C_2 \left(\mathcal{N}_i(T_i) + \frac{N+2-2s}{4} \right) \tag{6.81}$$

for any $t \in (\beta_i, T_i)$.

Proof. Since $\nu_{1,i} \geq 0$ by Proposition 6.5.6, from (6.80) it follows that, a.e. in (β_i, T_i),

$$\mathcal{N}_i'(t) \geq -C_1 \left(t^{-1+\frac{\varepsilon}{2}} + \|h_t(\cdot, t + a_i)\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} \right) \left(\mathcal{N}_i(t) + \frac{N+2-2s}{4} \right).$$

By integration we obtain the estimate

$$\mathcal{N}_i(t) \leq -\frac{N+2-2s}{4} + \left(\mathcal{N}_i(T_i) + \frac{N+2-2s}{4} \right) e^{\left(\frac{2C_1}{\epsilon} T_i^{\varepsilon/2} + C_1 \|h_t(\cdot, t+a_i)\|_{L^1((0,2\alpha), L^{N/(2s)}(\mathbb{R}^N))} \right)}$$

for any $t \in (\beta_i, T_i)$, which implies (6.81) in view of (6.3) and (6.41). □

Proposition 6.5.9. *For any $i \in \{i, \dots, k\}$, if $H_i(t) \not\equiv 0$ then*

$$H_i(t) > 0 \quad \textit{for all } t \in (0, 2\alpha). \tag{6.82}$$

Proof. Since $H_i(t) \not\equiv 0$ and H_i is continuous by Remark 6.3.3, there exists $T_i \in (0, 2\alpha)$ such that

$$H_i(T_i) > 0 \quad \text{ and } \quad V_i(\cdot, T_i) \in \mathcal{H}. \tag{6.83}$$

By Proposition 6.5.3 it follows that $H_i(t) > 0$ for any $t \in [T_i, 2\alpha)$. If we define

$$t_i := \inf\{\tau \in (0, T_i) : H_i(t) > 0 \text{ for all } t \in (\tau, 2\alpha)\},$$

then either

$$t_i = 0 \quad \text{ and } \quad H_i(t) > 0 \text{ for all } t \in (0, 2\alpha)$$

or

$$0 < t_i < T_i \quad \text{ and } \quad \begin{cases} H_i(t) = 0, \text{ for any } & t \in (0, t_i], \\ H_i(t) > 0, \text{ for any } & t \in (t_i, 2\alpha). \end{cases} \tag{6.84}$$

Now we prove that the second case can not occur arguing by contradiction. If (6.84) holds, then, thanks to Proposition 6.5.8 and (6.73),

$$\frac{t}{2} H_i'(t) \leq \left(-\frac{N+2-2s}{4} + C_2 \left(\mathcal{N}_i(T_i) + \frac{N+2-2s}{4} \right) \right) H_i(t)$$

for a.e. $t \in (t_i, T_i)$. Integrating the above inequality we obtain

$$H_i(t) \geq \frac{t^{2\left(-\frac{N+2-2s}{4} + C_2\left(\mathcal{N}_i(T_i) + \frac{N+2-2s}{4}\right)\right)}}{T_i^{2\left(-\frac{N+2-2s}{4} + C_2\left(\mathcal{N}_i(T_i) + \frac{N+2-2s}{4}\right)\right)}} H_i(T_i)$$

for all $t \in [t_i, T_i)$. Since $H_i(t_i) = 0$, we have reached a contradiction in view of (6.83). In conclusion, (6.82) must hold. $\square$

Proposition 6.5.10. *For any* $i \in \{1, \ldots, k-1\}$

$$H_i(t) \equiv 0 \textit{ in } (0, 2\alpha) \textit{ if and only if } H_{i+1}(t) \equiv 0 \textit{ in } (0, 2\alpha).$$

Proof. We start by proving that if $H_i(t) \equiv 0$ in $(0, 2\alpha)$ then $H_{i+1}(t) \equiv 0$ in $(0, 2\alpha)$. By contradiction, if there exists $\bar{t} \in (0, 2\alpha)$ such that $H_{i+1}(\bar{t}) > 0$, then $H_{i+1}(t) > 0$ for all $t \in (0, 2\alpha)$ by Proposition 6.5.9. It follows that $V_{i+1}(\cdot, t) \not\equiv 0$ for all $t \in (0, 2\alpha)$ and $V(\cdot, t) \not\equiv 0$ for all $t \in (i\alpha, (i+1)\alpha)$. Therefore $V_i(\cdot, t) \not\equiv 0$ for some $t \in (0, 2\alpha)$, which is a contradiction.

Now let us prove that, if $H_{i+1}(t) \equiv 0$ in $(0, 2\alpha)$, then $H_i(t) \equiv 0$ in $(0, 2\alpha)$. By contradiction, let us assume that $H_i(t) \not\equiv 0$. Then $H_i(t) > 0$ for any $t \in (0, 2\alpha)$ by Proposition 6.5.9. It follows that $V_i(\cdot, t) \not\equiv 0$ for all $t \in (0, 2\alpha)$ and so $V_{i+1}(\cdot, t) \not\equiv 0$ for all $t \in (0, \alpha)$, hence $H_{i+1}(t) \not\equiv 0$ for all $t \in (0, \alpha)$, which is a contradiction. $\square$

Proposition 6.5.11. *If* U *is a weak solution of* (6.44) *such that* $U \not\equiv 0$ *in* $\mathbb{R}^{N+1}_+ \times (0, T)$, *then*

$$H_i(t) > 0 \quad \textit{for any } t \in (0, 2\alpha) \textit{ and } i \in \{1, \ldots, k\}.$$

Proof. If $U \not\equiv 0$ in $\mathbb{R}^{N+1}_+ \times (0, T)$, then there exists some $i_0 \in \{1, \ldots, k\}$ such that $V_i \not\equiv 0$ in $(0, 2\alpha)$. Then $H_{i_0}(t) \not\equiv 0$ in $(0, 2\alpha)$. Thanks to Proposition 6.5.10, $H_i(t) \not\equiv 0$ in $(0, 2\alpha)$ for any $i \in \{1, \ldots, k\}$. In view of Proposition 6.5.9, we can therefore conclude that $H_i(t) > 0$ for any $t \in (0, 2\alpha)$ and $i \in \{1, \ldots, k\}$. $\square$

Proof of Theorem 6.1.10. It is not restrictive to assume that $t_0 = 0$. Let W be a solution of (6.11). Let $\bar{t} \in (0, T)$ be such that $W(z, -\bar{t}) \equiv 0$ in $\mathbb{R}^{N+1}_+$, so that, letting U be as in (6.42), $U(z, \bar{t}) \equiv 0$ in $\mathbb{R}^{N+1}_+$. Then $\bar{t} \in (a_i, b_i)$ for some $i \in \{1, \dots, k\}$ and $H_i(\bar{t} - a_i) = 0$. By Proposition 6.5.11 it follows that $U \equiv 0$ in $\mathbb{R}^{N+1}_+ \times (0, T)$ and hence, by (6.42), $W \equiv 0$ in $\mathbb{R}^{N+1}_+ \times (-T, 0)$. □

From now on, we assume that $U \not\equiv 0$ in $\mathbb{R}^{N+1}_+ \times (0, T)$ and, defining V as in (6.47), we denote, for all $t \in (0, 2\alpha)$,

$$H(t) := H_1(t) = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V^2(z,t) G(z)\, dz,$$

$$D(t) := D_1(t) = \frac{1}{t}\bigg(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla V|^2 G\, dz - \int_{\mathbb{R}^N} \Big(\frac{\mu}{|x|^{2s}} v^2 + t^s h(\sqrt{t}x, t) v^2 \Big) G(x,0)\, dx \bigg).$$

Since we are assuming that $U \not\equiv 0$ in $\mathbb{R}^{N+1}_+ \times (0, T)$, thanks to Proposition 6.5.11 the Almgren-Poon type frequency function

$$\mathcal{N} : (0, 2\alpha) \to \mathbb{R}, \quad \mathcal{N}(t) := \frac{tD(t)}{H(t)}.$$

is well-defined. Furthermore, in view of Proposition 6.5.6 $\mathcal{N} \in W^{1,1}_{loc}(0, 2\alpha)$ and

$$\mathcal{N}'(t) = \nu_1(t) + \nu_2(t) \quad \text{for a.e. } t \in (0, 2\alpha),$$

where we have defined

$$\nu_1(t) := \nu_{1,1}(t), \quad \nu_2(t) := \nu_{2,1}(t), \tag{6.85}$$

according to notation (6.77)–(6.78). Since $V(\cdot, t) \in \mathcal{H}$ for a.e. $t \in (0, T)$, there exists

$$T_0 \in (0, 2\alpha) \quad \text{such that} \quad V(\cdot, T_0) \in \mathcal{H}. \tag{6.86}$$

Proposition 6.5.12. *The limit*

$$\gamma := \lim_{t \to 0^+} \mathcal{N}(t) \tag{6.87}$$

exists and it is finite.

Proof. From Propositions 6.5.6 and 6.5.8 it follows that $\mathcal{N}$ is bounded. Hence the limit (6.87), if it exists, is finite. Furthermore, from Proposition 6.5.6 we have that $\nu_1 \geq 0$ a.e. in $(0, 2\alpha)$, whereas $\nu_2 \in L^1(0, T_0)$ by Proposition 6.5.7, Proposition 6.5.8, (6.41), and (6.3), where T_0 is as in (6.86). Then, from

$$\mathcal{N}(t) = \mathcal{N}(T_0) - \int_t^{T_0} \mathcal{N}'(\tau)\, d\tau = \mathcal{N}(T_0) - \int_t^{T_0} \nu_1(\tau)\, d\tau - \int_t^{T_0} \nu_2(\tau)\, d\tau,$$

we conclude that the limit (6.87) exists. □

Proposition 6.5.13. *Let T_0 be as in (6.86) and γ as in (6.87). Then there exists a constant $K_1 > 0$ such that*

$$H(t) \leq K_1 t^{2\gamma} \quad \textit{for all } t \in (0, T_0). \tag{6.88}$$

Moreover, for any $\sigma > 0$, there exists a constant $K_2(\sigma) > 0$ such that

$$H(t) \geq K_2(\sigma) t^{2\gamma + \sigma} \quad \textit{for all } t \in (0, T_0). \tag{6.89}$$

Proof. Thanks to the Hölder inequality, (6.41), (6.3), Proposition 6.5.6, Proposition 6.5.7, and Proposition 6.5.8, for any $t \in (0, T_0)$ we have that

$$\mathcal{N}(t) - \gamma = \int_0^t (\nu_1(\tau) + \nu_2(\tau))\, d\tau \geq \int_0^t \nu_2(\tau)\, d\tau$$
$$\geq -C_1 C_2 \left(\mathcal{N}_i(T_i) + \frac{N+2-2s}{4}\right) \int_0^t \left(\tau^{-1+\frac{\varepsilon}{2}} + \|h_t(\cdot,\tau)\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)}\right) d\tau \geq -C_3 t^{\delta},$$

for some constant $C_3 > 0$, where $\delta := \min\left\{\frac{\varepsilon}{2}, 1 - \frac{1}{r}\right\}$. It follows that, taking into account (6.73),

$$\frac{H'(t)}{H(t)} = \frac{2}{t}\mathcal{N}(t) \geq \frac{2\gamma}{t} - 2C_3 t^{-1+\delta} \quad \text{for a.e. } t \in (0, T_0).$$

An integration over (t, T_0) yields

$$H(t) \leq \frac{H(T_0)}{T_0^{2\gamma}} e^{\frac{2C_3 T_0^{\delta}}{\delta}} t^{2\gamma} \quad \text{for all } t \in (0, T_0),$$

so that (6.88) is proved.

Furthermore, since $\gamma := \lim_{t\to 0^+} \mathcal{N}(t)$, for any $\sigma > 0$ there exists $T_\sigma \in (0, T_0)$ such that $\mathcal{N}(t) < \gamma + \sigma/2$ for any $t \in (0, T_\sigma)$, and hence

$$\frac{H'(t)}{H(t)} = \frac{2}{t}\mathcal{N}(t) < \frac{2\gamma + \sigma}{t} \quad \text{for any } t \in (0, T_\sigma).$$

An integration of the above estimate over (t, T_σ), together with continuity and positivity of H in $[T_\sigma, T_0]$, yields (6.89) for some constant $K_2(\sigma)$. □

6.6 The Blow-up Analysis

If V is a solution of (6.49), then it is easy to see that, for any $\lambda > 0$, the function

$$V_\lambda(z,t) := V(z, \lambda^2 t) \tag{6.90}$$

belongs to $L^2((\tau, T/\lambda^2), \mathcal{H})$ for all $0 < \tau < T/\lambda^2$ and solves

$$\begin{aligned} {}_{\mathcal{H}^*}\langle (V_\lambda)_t, \phi \rangle_{\mathcal{H}} &= \frac{1}{t} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla V_\lambda \cdot \nabla \phi \, G \, dz \\ &- \frac{1}{t} \int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} v_\lambda(x,t)\phi(x,0) + \lambda^{2s} t^s h(\lambda\sqrt{t}x, \lambda^2 t) v_\lambda(x,t)\phi(x,0) \right) G(x,0)\, dx, \end{aligned} \tag{6.91}$$

for a.e. $t \in (0, T/\lambda^2)$ and any $\phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$, where

$$v_\lambda(x,t) := v(x, \lambda^2 t) = \mathrm{Tr}(V_\lambda(\cdot,t))(x).$$

We can also define the height and energy functions associated to the scaled equation (6.91) as

$$H_\lambda(t) = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V_\lambda^2 G \, dz,$$
$$D_\lambda(t) = \frac{1}{t} \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla V_\lambda|^2 G \, dz - \int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} v_\lambda^2 + \lambda^{2s} t^s h(\lambda\sqrt{t}x, \lambda^2 t) v_\lambda^2 \right) G(x,0)\, dx \right)$$

and the Almgren-Poon frequency function as

$$\mathcal{N}_\lambda(t) := \frac{tD_\lambda(t)}{H_\lambda(t)}. \tag{6.92}$$

For any $\lambda > 0$, we have the following scaling properties:

$$D_\lambda(t) = \lambda^2 D(\lambda^2 t), \quad H_\lambda(t) = H(\lambda^2 t) \quad \text{and} \quad \mathcal{N}_\lambda(t) = \frac{\lambda^2 t D(\lambda^2 t)}{H(\lambda^2 t)} = \mathcal{N}(\lambda^2 t), \tag{6.93}$$

on $(0, 2\alpha/\lambda^2)$. Let, for any $\lambda \in (0, \sqrt{T_0})$ and for any $t \in (0, T/T_0)$,

$$W_\lambda(z,t) := \frac{V(z, \lambda^2 t)}{\sqrt{H(\lambda^2)}}. \tag{6.94}$$

In particular we note that $1 \in (0, T/T_0)$. Similarly, we define

$$w_\lambda(z,t) := \frac{v(z, \lambda^2 t)}{\sqrt{H(\lambda^2)}}$$

for any $t \in (0, T/T_0)$. From (6.90) and (6.94) we deduce that W_λ belongs to $L^2((\tau, T/\lambda^2), \mathcal{H})$ for all $0 < \tau < T/\lambda^2$ and solves (6.91).

Proposition 6.6.1. *Let T_0 be as in (6.86). Then*

$$\{W_\lambda\}_{\lambda \in (0,\sqrt{T_0})} \text{ is bounded in } L^\infty((\tau,1), \mathcal{H}) \quad \text{for any } \tau \in (0,1), \tag{6.95}$$

and

$$\{(W_\lambda)_t\}_{\lambda \in (0,\sqrt{T_0})} \text{ is bounded in } L^\infty((\tau,1), \mathcal{H}^*) \quad \text{for any } \tau \in (0,1). \tag{6.96}$$

Moreover

$$\{W_\lambda\}_{\lambda \in (0,\sqrt{T_0})} \text{ is relatively compact in } C^0([\tau,1], \mathcal{L}) \quad \text{for any } \tau \in (0,1). \tag{6.97}$$

Proof. From Proposition 6.5.3, for any $t \in (0,1)$,

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s} W_\lambda^2(z,t)\, G(z)\, dz = \frac{H(\lambda^2 t)}{H(\lambda^2)} \le t^{2C_{N,s,+\mu} - \frac{N-2+2s}{2}}. \tag{6.98}$$

Furthermore, by (6.38), (6.81), and (6.93)

$$\begin{aligned}
&\frac{1}{t}\left(-\frac{N+2-2s}{4} + C_2\left(\mathcal{N}(T_0) + \frac{N+2-2s}{4}\right)\right) H_\lambda(t) \ge \lambda^2 D(\lambda^2 t) \\
&\qquad \ge \frac{1}{t}\left(-\frac{N+2-2s}{4} + C_{N,s,\mu}\right) H_\lambda(t) + \frac{1}{t} C_{N,s,\mu} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla V_\lambda(z,t)|^2 G(z)\, dz,
\end{aligned}$$

for all $\lambda \in (0, \sqrt{T_0})$ and a.e. $t \in (0,1)$. It follows that, taking into account (6.98),

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla V_\lambda(z,t)|^2 G(z)\, dz \\
&\qquad \le C_{N,s,h}^{-1}\left(C_2\left(\mathcal{N}(T_0) + \frac{N+2-2s}{4}\right) - C_{N,s,\mu}\right) \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V_\lambda^2(z,t) G(z)\, dz \\
&\qquad \le C_{N,s,h}^{-1}\left(C_2\left(\mathcal{N}(T_0) + \frac{N+2-2s}{4}\right) - C_{N,s,\mu}\right) t^{2C_{N,s,\mu} - \frac{N-2+2s}{2}} H(\lambda^2)
\end{aligned} \tag{6.99}$$

for a.e. $t \in (0,1)$. Hence we have proved (6.95) in view of (6.98) and (6.99).

To prove (6.96), we note that, from (6.94),

$$(W_\lambda)_t = \frac{\lambda^2}{\sqrt{H(\lambda^2)}} V_t(z, \lambda^2 t).$$

To estimate $\|(W_\lambda)_t(\cdot,t)\|_{\mathcal{H}^*}$, we observe that, for any $\phi \in \mathcal{H}$,

$$\begin{aligned}
&\lambda^{2s} t^{-1+s} \left| \int_{\mathbb{R}^N} h(\lambda\sqrt{t}x, \lambda^2 t) w_\lambda(x,t) \operatorname{Tr}\phi(x) G(x,0)\, dx \right| \\
&\le C_g \lambda^{2s} t^{-1+s} \int_{\mathbb{R}^N} |w_\lambda(x,t)| |\operatorname{Tr}\phi(x)| G(x,0)\, dx \\
&\quad + C_g \lambda^{\varepsilon} t^{-1+\varepsilon/2} \int_{\mathbb{R}^N} |x|^{-2s+\varepsilon} |w_\lambda(x,t)| |\operatorname{Tr}\phi(x)| G(x,0)\, dx \\
&\le C_g K_{N,s} \lambda^{2s} t^{-1+s} \|W_\lambda(\cdot,t)\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}} \\
&\quad + C_g \lambda^{\varepsilon} t^{-1+\varepsilon/2} \int_{\{|x|\le 1\}} |x|^{-2s} |w_\lambda(x,t)| |\operatorname{Tr}\phi(x)| G(x,0)\, dx \\
&\quad + C_g \lambda^{\varepsilon} t^{-1+\varepsilon/2} \int_{\{|x|\ge 1\}} |w_\lambda(x,t)| |\operatorname{Tr}\phi(x)| G(x,0)\, dx \\
&\le \frac{C_g\, \lambda^{\varepsilon}}{t^{1-\varepsilon/2}} K_{N,s} t^{s-\varepsilon/2} \lambda^{2s-\varepsilon} \|W_\lambda(\cdot,t)\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}} \\
&+ \frac{C_g\, \lambda^{\varepsilon}}{t^{1-\varepsilon/2}} \left(\kappa_s^{-1} \Lambda_{N,s}^{-1} \max\left\{1, \tfrac{N+2-2s}{4}\right\} + K_{N,s} \right) \|W_\lambda(\cdot,t)\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}}, \qquad (6.100)
\end{aligned}$$

for any $\lambda \in (0, \sqrt{T_0})$ and a.e. $t \in (0,1)$, thanks to (6.4), (6.33), (6.35), (6.41) and the Hölder inequality. Then, by (6.35) and (6.100),

$$\begin{aligned}
|{}_{\mathcal{H}^*}\langle (W_\lambda)_t(\cdot,t), \phi\rangle_{\mathcal{H}}| \le \Bigg(& 1 + \mu\, \kappa_s^{-1} \Lambda_{N,s}^{-1} \max\left\{1, \tfrac{N+2-2s}{4}\right\} \\
& + C_g T_0^{\varepsilon/2} \left(K_{N,s} T_0^{\frac{2s-\varepsilon}{2}} + \kappa_s^{-1} \Lambda_{N,s}^{-1} \max\left\{1, \tfrac{N+2-2s}{4}\right\} + K_{N,s} \right) \Bigg) \frac{\|W_\lambda(\cdot,t)\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}}}{t}.
\end{aligned}$$

Hence

$$\|(W_\lambda)_t(\cdot,t)\|_{\mathcal{H}^*} \le \frac{\text{const}}{t} \|W_\lambda(\cdot,t)\|_{\mathcal{H}},$$

so that (6.96) follows from (6.95).

Finally, in view of (6.95) and (6.96), we can apply [119, Corollary 8] to obtain (6.97). □

Proposition 6.6.2. *Let V be a solution to (6.49) such that $V \not\equiv 0$ in $\mathbb{R}^{N+1}_+ \times (0,T)$ and let γ be as in (6.87). Then γ is an eigenvalue of problem (6.12). Furthermore, for any sequence $\lambda_n \to 0^+$, there exist a subsequence $\lambda_{n_k} \to 0^+$ and an eigenfunction Y of problem (6.12) associated to γ such that $\|Y\|_{\mathcal{L}} = 1$ and, for any $\tau \in (0,1)$,*

$$\lim_{k\to\infty} \int_\tau^1 \left\| \frac{V(z, \lambda_{n_k}^2 t)}{\sqrt{H(\lambda_{n_k}^2)}} - t^{\gamma} Y(z) \right\|_{\mathcal{H}}^2 dt = 0,$$

$$\lim_{k\to\infty} \sup_{t\in[\tau,1]} \left\| \frac{V(z, \lambda_{n_k}^2 t)}{\sqrt{H(\lambda_{n_k}^2)}} - t^{\gamma} Y(z) \right\|_{\mathcal{L}} = 0.$$

Proof. Let $\lambda_n \to 0^+$. By Proposition 6.6.1, there exists a subsequence $\lambda_{n_k} \to 0^+$ and $\widetilde{W} \in \cap_{\tau\in(0,1)}(C^0([\tau,1],\mathcal{L}) \cap L^2((\tau,1),\mathcal{H}))$ such that $\widetilde{W}_t \in \cap_{\tau\in(0,1)} L^2((\tau,1),\mathcal{H}^*)$,

$$W_{\lambda_{n_k}} \rightharpoonup \widetilde{W} \text{ weakly in } L^2((\tau,1),\mathcal{H}), \quad (W_{\lambda_{n_k}})_t \rightharpoonup \widetilde{W}_t \text{ weakly in } L^2((\tau,1),\mathcal{H}^*), \tag{6.101}$$

and

$$W_{\lambda_{n_k}} \to \widetilde{W} \text{ strongly in } C^0([\tau,1],\mathcal{L}), \tag{6.102}$$

as $k \to +\infty$, for any $\tau \in (0,1)$. Since, by (6.94),

$$\left\| W_{\lambda_{n_k}}(\cdot,1) \right\|_{\mathcal{L}} = 1,$$

from (6.102) we obtain that

$$\left\| \widetilde{W}(\cdot,1) \right\|_{\mathcal{L}} = 1, \tag{6.103}$$

hence $\widetilde{W} \not\equiv 0$. Now we claim that

$$W_{\lambda_{n_k}} \to \widetilde{W} \text{ strongly in } L^2((\tau,1),\mathcal{H}) \quad \text{for any } \tau \in (0,1). \tag{6.104}$$

Thanks to (6.100) and (6.101), we can pass to the limit in (6.91) thus obtaining, for any $\phi \in \mathcal{H}$ and a.e. $t \in (0,1)$,

$$\begin{aligned} {}_{\mathcal{H}^*}\left\langle (\widetilde{W})_t(\cdot,t), \phi \right\rangle_{\mathcal{H}} = \frac{1}{t} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla \widetilde{W}(z,t) \cdot \nabla \phi(z)\, G(z)\, dz \\ - \frac{1}{t} \int_{\mathbb{R}^N} \frac{\mu}{|x|^{2s}} \widetilde{w}(x,t) \operatorname{Tr} \phi(x) G(x,0)\, dx, \end{aligned} \tag{6.105}$$

where $\widetilde{w}(\cdot,t) := \operatorname{Tr}(\widetilde{W}(\cdot,t))$. Testing the difference between (6.91) and (6.105) with $(W_{\lambda_{n_k}} - \widetilde{W})$, integrating between τ and 1, and taking into account Remark 6.3.3, we obtain that

$$\begin{aligned} &\int_\tau^1 \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \widetilde{W} - \nabla W_{\lambda_{n_k}}|^2\, G\, dz \right) dt \\ &\qquad - \int_\tau^1 \left(\int_{\mathbb{R}^N} \frac{\mu}{|x|^{2s}} |\widetilde{w}(x,t) - w_{\lambda_{n_k}}(x,t)|^2 G(x,0)\, dx \right) dt \\ &= \frac{1}{2} \left\| \widetilde{W}(\cdot,1) - W_{\lambda_{n_k}}(\cdot,1) \right\|_{\mathcal{L}}^2 - \frac{\tau}{2} \left\| \widetilde{W}(\cdot,\tau) - W_{\lambda_{n_k}}(\cdot,\tau) \right\|_{\mathcal{L}}^2 \\ &\qquad - \frac{1}{2} \int_\tau^1 \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\widetilde{W} - W_{\lambda_{n_k}}|^2 G\, dz \right) dt \\ &\qquad + \lambda^{2s} \int_\tau^1 \left(\int_{\mathbb{R}^N} t^s h(\lambda\sqrt{t}x, \lambda^2 t) w_{\lambda_{n_k}}(x,t)(w_{\lambda_{n_k}}(x,t) - \tilde{w}(x,t)) G(x,0)\, dx \right) dt. \end{aligned}$$

Then by (6.100), (6.101) and (6.102) we conclude that

$$\begin{aligned} \lim_{k\to 0^+} \Bigg(&\int_\tau^1 \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \left(|\nabla \widetilde{W} - \nabla W_{\lambda_{n_k}}|^2 + \frac{1}{2} |\widetilde{W} - W_{\lambda_{n_k}}|^2 \right) G\, dz \right) dt \\ &- \int_\tau^1 \left(\int_{\mathbb{R}^N} \frac{\mu}{|x|^{2s}} |\widetilde{w}(x,t) - w_{\lambda_{n_k}}(x,t)|^2 G(x,0)\, dx \right) dt \Bigg) = 0. \end{aligned}$$

Thanks to Proposition 6.2.7 with $f \equiv 0$ and (6.102), we conclude that (6.104) holds. Therefore, for any $\tau \in (0,1)$,

$$\lim_{k\to\infty} \int_\tau^1 \left\| W_{\lambda_{n_k}}(\cdot,t) - \widetilde{W}(\cdot,t) \right\|_{\mathcal{H}}^2 dt = 0 \tag{6.106}$$

and, by (6.102),

$$\lim_{k\to\infty} \sup_{t\in[\tau,1]} \left\| W_{\lambda_{n_k}}(\cdot,t) - \widetilde{W}(\cdot,t) \right\|_{\mathcal{L}}^2 dt = 0.$$

Let us define, for any $t \in (0,1)$,

$$H_{\widetilde{W}}(t) := \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \widetilde{W}^2 G\, dz,$$

$$D_{\widetilde{W}}(t) := \frac{1}{t}\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \widetilde{W}|^2 G\, dz - \int_{\mathbb{R}^N} \frac{\mu}{|x|^{2s}} \widetilde{w}^2(x,t) G(x,0)\, dx \right).$$

Since $H_{\widetilde{W}}(1) = \|\widetilde{W}(\cdot,1)\|_{\mathcal{L}}^2 = 1$ by (6.103), then

$$H_{\widetilde{W}}(t) > 0 \text{ for any } t \in (0,1)$$

as we can prove arguing as in Proposition 6.5.9. Hence the function

$$\mathcal{N}_{\widetilde{W}} : (0,1) \to \mathbb{R}, \quad \mathcal{N}_{\widetilde{W}}(t) := \frac{t D_{\widetilde{W}}(t)}{H_{\widetilde{W}}(t)}$$

is well defined. Furthermore, from (6.92) and (6.94) , it follows that

$$\mathcal{N}_\lambda(t) = \frac{\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla W_\lambda|^2 G\, dz - \int_{\mathbb{R}^N} \left(\frac{\mu}{|x|^{2s}} w_\lambda^2(x,t) + \lambda^{2s} t^s h(\lambda\sqrt{t}x, \lambda^2 t) w_\lambda^2(x,t) \right) G(x,0)\, dx}{\int_{\mathbb{R}^{N+1}_+} y^{1-2s} W_\lambda^2(z,t) G\, dz}.$$

Then, from (6.35), (6.100), and (6.106) we deduce that

$$\lim_{k\to\infty} \mathcal{N}_{\lambda_{n_k}}(t) = \mathcal{N}_{\widetilde{W}}(t) \quad \text{ for a.e. } t \in (0,1).$$

On the other hand, $\mathcal{N}_{\lambda_{n_k}}(t) = \mathcal{N}(\lambda_{n_k}^2 t)$ for any $t \in (0,1)$ by (6.93) and hence

$$\mathcal{N}_{\widetilde{W}}(t) = \lim_{k\to\infty} \mathcal{N}_{\lambda_{n_k}}(t) = \lim_{k\to\infty} \mathcal{N}(\lambda_{n_k}^2 t) = \gamma \quad \text{ for any } t \in (0,1),$$

with γ as in (6.87). It follows that $\mathcal{N}'_{\widetilde{W}}(t) \equiv 0$ in $(0,1)$. In view of Proposition 6.5.6 in the case $h \equiv 0$, we deduce that

$$\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \widetilde{W}_t^2 G\, dz \right) \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \widetilde{W}^2 G\, dz \right) = \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \widetilde{W}_t \widetilde{W} G dz \right)^2$$

for a.e. $t \in (0,1)$. In particular, for the vectors $\widetilde{W}_t(\cdot,t)$ and $\widetilde{W}(\cdot,t)$ in $\mathcal{L}$, equality holds in the Cauchy-Schwarz inequality for a.e. $t \in (0,1)$. Hence there exists a function $\beta : (0,1) \to \mathbb{R}$ such that

$$\widetilde{W}_t(z,t) = \beta(t) \widetilde{W}(z,t) \quad \text{ for a.e. } z \in \mathbb{R}^{N+1}_+ \text{ and a.e. } t \in (0,1). \tag{6.107}$$

Thanks to (6.73) and Remark 6.3.5,

$$D_{\widetilde{W}}(t) = {}_{\mathcal{H}^*}\Big\langle (\widetilde{W})_t(\cdot,t), \widetilde{W}(\cdot,t)\Big\rangle_{\mathcal{H}} = \beta(t) H_{\widetilde{W}}(t),$$

and so

$$\beta(t) = \frac{\gamma}{t} \quad \text{for a.e. } t \in (0,1). \tag{6.108}$$

Combining (6.105), (6.107) and (6.108), we conclude that $\widetilde{W}$ satisfies

$$\gamma \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \widetilde{W} \phi \, G \, dz = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla \widetilde{W} \cdot \nabla \phi \, G \, dz - \int_{\mathbb{R}^N} \frac{\mu}{|x|^{2s}} \widetilde{w} \phi \, G(x,0) \, dx, \tag{6.109}$$

for all $\phi \in \mathcal{H}$ and a.e. $t \in (0,1)$.

Furthermore from (6.107) it follows that, letting $\widetilde{W}^\eta(z,t) := \widetilde{W}(z, \eta^2 t)$ for any $\eta > 0$,

$$\frac{d\widetilde{W}^\eta}{d\eta}(z,t) = \frac{2\gamma}{\eta} \widetilde{W}^\eta(z,t) \quad \text{in a distributional sense and a.e. in } \mathbb{R}^{N+1}_+ \times (0,1).$$

An integration yields

$$\widetilde{W}^\eta(z,t) = \eta^{2\gamma} \widetilde{W}(z,t) \quad \text{for all } \eta > 0 \text{ and a.e. in } \mathbb{R}^{N+1}_+ \times (0,1).$$

Let $Y(z) := \widetilde{W}(z,1)$. Then $\|Y\|_{\mathcal{L}} = 1$ and

$$\widetilde{W}(z,t) = \widetilde{W}^{\sqrt{t}}(z,1) = t^\gamma Y(z) \ \text{ for a.e. } z \in \mathbb{R}^{N+1}_+ \text{ and a.e. } t \in (0,1). \tag{6.110}$$

Moreover, from (6.109) and (6.110), $Y \in \mathcal{H}$ and Y satisfies

$$\gamma \int_{\mathbb{R}^{N+1}_+} y^{1-2s} Y \phi G \, dz = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla Y \cdot \nabla \phi \, G \, dz - \int_{\mathbb{R}^N} \frac{\mu}{|x|^{2s}} \operatorname{Tr}(Y) \operatorname{Tr}(\phi) G(x,0) \, dx$$

for any $\phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$, i.e. γ is an eigenvalue of problem (6.12) and Y is an associated eigenfunction. The proof is then complete. □

Now we study the asymptotic behavior of $H(t)$ as $t \to 0^+$.

Proposition 6.6.3. *Let γ be as in (6.87). Then the limit $\lim_{t\to 0^+} t^{-2\gamma} H(t)$ exists and it is finite.*

Proof. Thanks to (6.88), we only need to show that the limit exists. By (6.73), Proposition 6.5.6, and (6.87),

$$\begin{aligned}\frac{d}{dt}(t^{-2\gamma} H(t)) &= -2\gamma t^{-2\gamma-1} H(t) + t^{-2\gamma} H'(t) = 2t^{-2\gamma-1}(tD(t) - \gamma H(t)) \\ &= 2t^{-2\gamma-1} H(t) \int_0^t (\nu_1(\tau) + \nu_2(\tau)) \, d\tau\end{aligned}$$

for a.e. $t \in (0, T_0)$, where ν_1 and ν_2 have been defined in (6.85). An integration over (t, T_0) yields

$$\frac{H(T_0)}{T_0^{2\gamma}} - \frac{H(t)}{t^{2\gamma}} = \int_t^{T_0} 2\rho^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_1(\tau) d\tau \right) d\rho + \int_t^{T_0} 2\rho^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_2(\tau) d\tau \right) d\rho.$$

Since by Proposition 6.5.6 it follows that $\nu_1 \geq 0$, then the function

$$t \to \int_t^{T_0} 2\rho^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_1(\tau) d\tau \right) d\rho$$

is non-increasing on $(0, T_0)$ and it has a limit as $t \to 0^+$. From (6.3), (6.41), Proposition 6.5.7 and Proposition 6.5.8 we deduce that

$$\left| \int_0^\rho \nu_2(\tau) d\tau \right| \leq \text{const}\, \rho^\delta,$$

where $\delta := \min\{\frac{\varepsilon}{2}, 1 - \frac{1}{r}\}$. Then by (6.88)

$$\left| 2\rho^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_2(\tau) d\tau \right) d\rho \right| \leq \text{const}\, \rho^{-1+\delta}$$

hence the function

$$\rho \to 2\rho^{-2\gamma-1} H(\rho) \left(\int_0^\rho \nu_2(\tau) d\tau \right) dr$$

belongs to $L^1(0, T_0)$. We conclude that limit $\lim_{t\to 0^+} t^{-2\gamma} H(t)$ exists. □

Proposition 6.6.4. *Let γ be as in (6.87). Then $\lim_{t\to 0^+} t^{-2\gamma} H(t) > 0$.*

Proof. We argue by contradiction assuming that $\lim_{t\to 0^+} t^{-2\gamma} H(t) = 0$.

Since $V_\lambda(z, 1) = V(z, \lambda^2) \in \mathcal{H}$ and $h(\lambda x, \lambda^2)\,\mathrm{Tr}(V_\lambda)(x, 1) \in \mathcal{H}^*$ for a.e. $\lambda \in (0, \sqrt{T_0})$, by Proposition 6.1.5 we can expand them in $\mathcal{L}$ and $\mathcal{H}^*$ respectively as

$$V_\lambda(z, 1) = \sum_{(n,j)\in\mathbb{N}\times(\mathbb{N}\setminus\{0\})} V_{n,j}(\lambda) \widetilde{Y}_{n,j}(z) \quad \text{in } \mathcal{L},$$
$$h(\lambda x, \lambda^2)\,\mathrm{Tr}(V_\lambda)(x, 1) = \sum_{(n,j)\in\mathbb{N}\times(\mathbb{N}\setminus\{0\})} \xi_{n,j}(\lambda) \widetilde{Y}_{n,j}(z) \quad \text{in } \mathcal{H}^*,$$

where

$$V_{n,j}(\lambda) := \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V_\lambda(z, 1) \widetilde{Y}_{n,j}(z) G(z)\, dz, \tag{6.111}$$

$$\xi_{n,j}(\lambda) := {}_{\mathcal{H}^*}\left\langle h(\lambda\cdot, \lambda^2)\,\mathrm{Tr}(V_\lambda)(\cdot, 1), \widetilde{Y}_{n,j} \right\rangle_{\mathcal{H}}$$
$$= \int_{\mathbb{R}^N} h(\lambda x, \lambda^2) v_\lambda(x, 1)\,\mathrm{Tr}(\widetilde{Y}_{n,j}) G(x, 0)\, dx, \tag{6.112}$$

for a.e. $\lambda \in (0, \sqrt{T_0})$. By Parseval's identity

$$H(\lambda^2) = \sum_{(m,i)\in\mathbb{N}\times(\mathbb{N}\setminus\{0\})} (V_{m,i}(\lambda))^2 \geq (V_{n,j}(\lambda))^2$$

for any $\lambda \in (0, T_0)$ and $(n, j) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\})$. Hence, from $\lim_{t\to 0^+} t^{-2\gamma} H(t) = 0$ we deduce that

$$\lim_{\lambda\to 0^+} \lambda^{-2\gamma} V_{n,j}(\lambda) = 0 \quad \text{for any } (n, j) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\}). \tag{6.113}$$

By (6.90), Remarks 6.3.3, 6.3.5, and Proposition 6.3.6, it is easy to see that, for any $(n, j) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\})$, $V_{n,j}$ is absolute continuous on any closed sub-interval of $(0, \sqrt{T_0})$ and

$$\frac{d}{d\lambda} V_{n,j}(\lambda) = {}_{\mathcal{H}^*}\left\langle \frac{d}{d\lambda} V_\lambda(\cdot, 1), \widetilde{Y}_{n,j} \right\rangle_{\mathcal{H}} = \left(\frac{d}{d\lambda} V_\lambda(\cdot, 1), \widetilde{Y}_{n,j} \right)_{\mathcal{L}} = 2\lambda \left(V_t(\cdot, \lambda^2), \widetilde{Y}_{n,j} \right)_{\mathcal{L}}$$

a.e. and in a distributional sense in $(0, \sqrt{T_0})$. Furthermore, for any $(n,j) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\})$, since $\widetilde{Y}_{n,j}$ is an eigenfunction of problem (6.12) we have that

$$
\begin{aligned}
2\lambda \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V_t(z,\lambda^2)\widetilde{Y}_{n,j}(z)G(z)\,dz &= \frac{2}{\lambda}\bigg(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla V(\cdot,\lambda^2)\cdot \nabla \widetilde{Y}_{n,j}\, G\,dz \\
&\quad - \int_{\mathbb{R}^N} \Big(\frac{\mu}{|x|^{2s}} v(x,\lambda^2)\operatorname{Tr}(\widetilde{Y}_{n,j}) + \lambda^{2s} h(\lambda x,\lambda^2) v(x,\lambda^2)\operatorname{Tr}(\widetilde{Y}_{n,j})\Big) G(x,0)\,dx \bigg) \\
&= \frac{2}{\lambda}\gamma_{n,j} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V(\cdot,\lambda^2)\widetilde{Y}_{n,j} G\,dz \\
&\quad - 2\lambda^{2s-1}\int_{\mathbb{R}^N} h(\lambda x,\lambda^2)v(x,\lambda^2)\operatorname{Tr}(\widetilde{Y}_{n,j})G(x,0)\,dx = \frac{2}{\lambda}\gamma_{n,j}V_{n,j}(\lambda) - 2\lambda^{2s-1}\xi_{n,j}(\lambda),
\end{aligned}
$$

for a.e. $\lambda \in (0, \sqrt{T_0})$, by (6.49), (6.111) and (6.112). In conclusion we have proved that, for any $(n,j) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\})$,

$$
\frac{d}{d\lambda} V_{n,j}(\lambda) = \frac{2}{\lambda}\gamma_{n,j}V_{n,j}(\lambda) - 2\lambda^{2s-1}\xi_{n,j}(\lambda)
$$

for a.e. $\lambda \in (0, \sqrt{T_0})$ and in a distributional sense. An integration yields

$$
V_{n,j}(\bar\lambda) = \bar\lambda^{2\gamma_{n,j}} \left(\lambda^{-2\gamma_{n,j}} V_{n,j}(\lambda) + 2\int_{\bar\lambda}^{\lambda} \tau^{2s-1-2\gamma_{n,j}} \xi_{n,j}(\tau)\,d\tau \right) \tag{6.114}
$$

for any $\bar\lambda, \lambda \in (0, \sqrt{T_0})$.

Thanks to Proposition 6.6.2, there exists an eigenvalue γ_{m_0,k_0} of (6.12) such that $\gamma = \gamma_{m_0,k_0}$. Then, for any $(n,j) \in J_0$ (see (6.26)), we can estimate $\xi_{n,j}$ as follows. From the Hölder inequality, the fact that $\widetilde{Y}_{n,j} \in \mathcal{H}$, (6.38), and (6.39) it follows that

$$
\begin{aligned}
\lambda^{2s}|\xi_{n,j}(\lambda)| &= \lambda^{2s}\left| \int_{\mathbb{R}^N} h(\lambda x,\lambda^2)v(x,\lambda^2)\operatorname{Tr}(\widetilde{Y}_{n,j})(x)G(x,0)\,dx \right| \qquad (6.115)\\
&\le \left(\int_{\mathbb{R}^N} \lambda^{2s}|h(\lambda x,\lambda^2)||v(x,\lambda^2)|^2 G(x,0)dx \right)^{\frac12} \left(\int_{\mathbb{R}^N} \lambda^{2s}|h(\lambda x,\lambda^2)||\operatorname{Tr}(\widetilde{Y}_{n,j})|^2 G(x,0)dx \right)^{\frac12} \\
&\le \text{const}\,\lambda^{\varepsilon+2\gamma}
\end{aligned}
$$

for any $\lambda \in (0, \sqrt{T_0})$, where Proposition 6.5.8 and (6.88) have been used.

It follows that $\tau \to \tau^{2s-1-2\gamma}\xi_{n,j}(\tau)$ belongs to $L^1(0, \sqrt{T_0})$ for any $(n,j) \in J_0$. Passing to the limit as $\bar\lambda \to 0^+$ in (6.114), from (6.113) we deduce that

$$
V_{n,j}(\lambda) = -2\lambda^{2\gamma}\int_0^{\lambda} \tau^{2s-1-2\gamma}\xi_{n,j}(\tau)\,d\tau \quad \text{for any } (n,j) \in J_0. \tag{6.116}
$$

Combining (6.115) and (6.116) we obtain that

$$
|V_{n,j}(\lambda)| \le \text{const}\,\lambda^{\varepsilon+2\gamma} \quad \text{for any } \lambda \in (0, \sqrt{T_0}) \text{ and some const} > 0 \text{ independent of } \lambda.
$$

Fixing some $\sigma \in (0, \varepsilon)$, by (6.89) there exists a constant $K(\sigma) > 0$ such that, for any $\lambda \in (0, \sqrt{T_0})$,

$$
H(\lambda^2) \ge K(\sigma)\lambda^{2(2\gamma+\sigma)}.
$$

We conclude that
$$\frac{|V_{n,j}(\lambda)|}{\sqrt{H(\lambda^2)}} = O\left(\lambda^{\varepsilon-\sigma}\right) = o(1) \quad \text{as } \lambda \to 0^+. \tag{6.117}$$
On the other hand, for any sequence $\lambda_i \to 0^+$, Proposition 6.6.2 provides a subsequence $\lambda_{i_k} \to 0^+$ and an eigenfunction Y of (6.12) associated to the eigenvalue γ such that
$$\frac{V_{\lambda_{i_k}}(\cdot,1)}{\sqrt{H(\lambda_{i_k}^2)}} \to Y \quad \text{strongly in } \mathcal{L} \text{ as } k \to \infty.$$
In particular, for any $(n,j) \in J_0$,
$$\frac{V_{n,j}(\lambda_{i_k})}{\sqrt{H(\lambda_{i_k}^2)}} = \left(\frac{V_{\lambda_{i_k}}(\cdot,1)}{\sqrt{H(\lambda_{i_k}^2)}}, \widetilde{Y}_{n,j}\right)_{\mathcal{L}} \to \left(Y, \widetilde{Y}_{n,j}\right)_{\mathcal{L}} \quad \text{as } k \to \infty. \tag{6.118}$$
From (6.117) and (6.118) we deduce that $(Y, \widetilde{Y}_{n,j})_{\mathcal{L}} = 0$ for any $(n,j) \in J_0$. We conclude that $Y \equiv 0$, a contradiction. □

Proof of Theorems 6.1.6 and 6.1.7. In view of Proposition 6.6.2, there exists an eigenvalue γ_{m_0,k_0} of problem (6.12) such that (6.25) holds. Let J_0 be as in (6.26) and $\lambda_i \to 0^+$ as $i \to +\infty$. Thanks to Proposition 6.6.2 and Proposition 6.6.4 there exists a subsequence $\{\lambda_{i_k}\}_{k\in\mathbb{N}}$ and real numbers $\{\beta_{n,j} : (n,j) \in J_0\}$ such that $\beta_{\tilde{n},\tilde{j}} \neq 0$ for some $(\tilde{n},\tilde{j}) \in J_0$ and, for any $\tau \in (0,1)$,
$$\lim_{k\to\infty} \int_\tau^1 \left\| \lambda_{i_k}^{-2\gamma_{m_0,k_0}} V(z, \lambda_{i_k}^2 t) - t^{\gamma_{m_0,k_0}} \sum_{(n,j)\in J_0} \beta_{n,j} \widetilde{Y}_{n,j}(z) \right\|_{\mathcal{H}}^2 dt = 0 \tag{6.119}$$
and
$$\lim_{k\to\infty} \sup_{t\in[\tau,1]} \left\| \lambda_{i_k}^{-2\gamma_{m_0,k_0}} V(z, \lambda_{i_k}^2 t) - t^{\gamma_{m_0,k_0}} \sum_{(n,j)\in J_0} \beta_{n,j} \widetilde{Y}_{n,j}(z) \right\|_{\mathcal{L}}^2 = 0. \tag{6.120}$$
It follows that
$$\lambda_{i_k}^{-2\gamma_{m_0,k_0}} V(z, \lambda_{i_k}^2) \to \sum_{(n,j)\in J_0} \beta_{n,j} \widetilde{Y}_{n,j}(z) \quad \text{strongly in } \mathcal{L} \text{ as } k \to \infty. \tag{6.121}$$
Let us prove that $\{\beta_{n,j} : (n,j) \in J_0\}$ depends neither on the sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{i_k}\}_{k\in\mathbb{N}}$. Let $\Lambda \in (0, \sqrt{T_0})$ and let $V_{n,j}$, $\xi_{n,j}$ be as in (6.111) and (6.112) respectively. From (6.121) we obtain that, for any $(n,j) \in J_0$,
$$\lambda_{i_k}^{-2\gamma_{m_0,k_0}} V_{n,j}(\lambda_{i_k}) \to \beta_{n,j} \quad \text{as } k \to \infty.$$
By (6.114), for any $(n,j) \in J_0$ and $\lambda \in (0,\Lambda)$,
$$V_{n,j}(\lambda) = \lambda^{2\gamma_{m_0,k_0}} \left(\Lambda^{-2\gamma_{m_0,k_0}} V_{n,j}(\Lambda) + 2\int_\lambda^\Lambda \tau^{2s-1-2\gamma_{m_0,j_0}} \xi_{n,j}(\tau)\, d\tau \right).$$

Furthermore, proceeding as in Proposition 6.6.4, we can prove that $\tau \to \tau^{2s-1-2\gamma_{m_0,j_0}}\xi_{n,j}(\tau)$ belongs to $L^1(0,\sqrt{T_0})$. Hence

$$\begin{aligned}\beta_{n,j} &= \Lambda^{-2\gamma_{m_0,k_0}} V_{n,j}(\Lambda) + 2\int_0^\Lambda \tau^{2s-1-2\gamma_{m_0,j_0}}\xi_{n,j}(\tau)\,d\tau \\ &= \Lambda^{-2\gamma_{m_0,k_0}} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} V(z,\Lambda^2)\widetilde{Y}_{n,j}(z)G(z)\,dz \\ &\quad + 2\int_0^\Lambda \tau^{2s-1-2\gamma_{m_0,k_0}} \left(\int_{\mathbb{R}^N} h(\tau x,\tau^2)v(x,\tau^2)\,\mathrm{Tr}(\widetilde{Y}_{n,j})(x)G(x,0)\,dx\right)d\tau,\end{aligned}$$

so that $\beta_{n,j}$ depends neither on the sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{i_k}\}_{k\in\mathbb{N}}$ for any $(n,j)\in J_0$. Then, by the Urysohn subsequence principle, we conclude that the convergences in (6.119) and (6.120) actually hold as $\lambda \to 0^+$, thus proving Theorem 6.1.6. Theorem 6.1.7 follows from Theorem 6.1.6 and the continuity of the trace operator Tr from $\mathcal{H}$ into $L^2(\mathbb{R}^N, G(x,0))$, see Proposition 6.2.3. □

The strong unique continuation principles stated in Corollaries 6.1.8 and 6.1.9 easily follow from Theorem 6.1.6 and Theorem 6.1.7.

Proof of Corollaries 6.1.8 and 6.1.9. We start by proving Corollary 6.1.8. Let us assume by contradiction that $W \not\equiv 0$ on $\mathbb{R}^{N+1}_+ \times (-T,0)$ and let γ_{m_0,k_0} be as in Theorem 6.1.6. In view of (6.28) we have that

$$\lim_{\lambda\to 0^+} \lambda^{-2\gamma_{m_0,k_0}} t^{-\gamma_{m_0,k_0}} W(\lambda z\sqrt{t}, -\lambda^2 t) = 0 \quad \text{for a.e. } (z,t)\in\mathbb{R}^{N+1}_+\times(0,1).$$

On the other hand, by Theorem 6.1.6 there exists $Y \in \mathcal{H}\setminus\{0\}$ such that Y is an eigenfunction of problem (6.12) and, for a.e. $z\in\mathbb{R}^{N+1}_+$ and $t\in(0,1)$,

$$\lim_{n\to\infty} \lambda_n^{-2\gamma_{m_0,k_0}} t^{-\gamma_{m_0,k_0}} W(\lambda_n z\sqrt{t}, -\lambda_n^2 t) = Y(z),$$

along a sequence $\lambda_n \to 0^+$. We conclude that $Y \equiv 0$, thus reaching a contradiction. In the same way, we can deduce Corollary 6.1.9 from Theorem 6.1.7, in view of Proposition 6.4.4. □

Part III

Spectral Stability for Aharonov-Bohm operators

Chapter 7

Quantitative spectral stability for Aharonov-Bohm operators with many coalescing poles

7.1 Statement of the main results

To give a variational formulation of problem (1.9), we introduce the space $H^{1,\varepsilon}(\Omega,\mathbb{C})$, defined as the completion of

$$\{\phi \in H^1(\Omega,\mathbb{C}) \cap C^\infty(\Omega,\mathbb{C}) : \phi \equiv 0 \text{ in a neighbourhood of } a_\varepsilon^j \text{ for all } j = 1,\dots,k\}$$

with respect to the norm

$$\|w\|_{H^{1,\varepsilon}(\Omega,\mathbb{C})} := \left(\|w\|^2_{L^2(\Omega,\mathbb{C})} + \|\nabla w\|^2_{L^2(\Omega,\mathbb{C}^2)} + \sum_{j=1}^k \left\| \tfrac{w}{|\cdot - a_\varepsilon^j|} \right\|^2_{L^2(\Omega,\mathbb{C})} \right)^{1/2}. \tag{7.1}$$

We observe that $H^{1,\varepsilon}(\Omega,\mathbb{C}) = \left\{ u \in H^1(\Omega,\mathbb{C}) : \frac{u}{|\cdot - a_\varepsilon^j|} \in L^2(\Omega,\mathbb{C}) \text{ for all } j = 1,\dots,k \right\}$.

In [95] (see also [15] and [66, Lemma 3.1, Remark 3.2]), the following local magnetic Hardy-type inequality

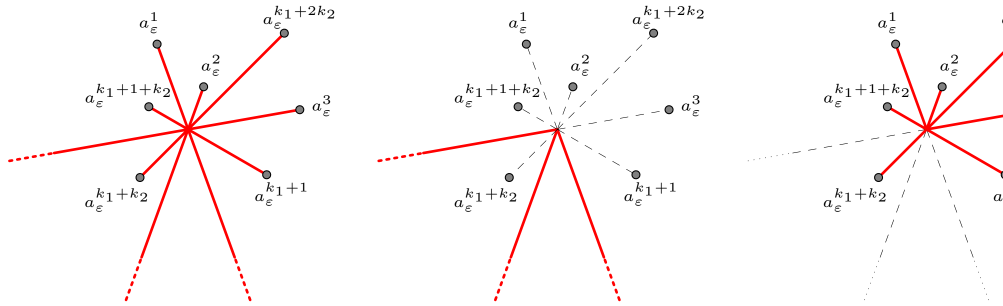
$$\int_{B_r(b)} |i\nabla w + A_b^\rho w|^2 \, dx \geq \left(\min_{j\in\mathbb{Z}} |j-\rho| \right)^2 \int_{B_r(b)} \frac{|w(x)|^2}{|x-b|^2} \, dx$$

is proved for every $b \in \mathbb{R}^2$ and $w \in C^\infty_{\rm c}(\overline{B_r(b)} \setminus \{b\},\mathbb{C})$. It follows that the norm (7.1) is equivalent to the norm

$$\left(\|(i\nabla + \mathcal{A}_\varepsilon)\, u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} \right)^{1/2}.$$

To deal with homogeneous Dirichlet boundary conditions, we introduce the space $H^{1,\varepsilon}_0(\Omega,\mathbb{C})$ defined as the closure of $C^\infty_{\rm c}(\Omega \setminus \{a_\varepsilon^1,\dots,a_\varepsilon^k\})$ in $H^{1,\varepsilon}(\Omega,\mathbb{C})$. The space $H^{1,\varepsilon}_0(\Omega,\mathbb{C})$ can be explicitly characterized as

$$H^{1,\varepsilon}_0(\Omega,\mathbb{C}) = \left\{ w \in H^1_0(\Omega,\mathbb{C}) : \tfrac{w}{|\cdot - a_\varepsilon^j|} \in L^2(\Omega,\mathbb{C}) \text{ for all } j = 1,\dots,k \right\}.$$



(a) The set Γ_ε. (b) The set Γ_0. (c) The sets S_ε^j.

Figure 7.1: The sets Γ_ε, Γ_0, S_ε^j $(1 \le j \le k_1 + k_2)$.

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of (1.9) if there exists $u \in H_0^{1,\varepsilon}(\Omega, \mathbb{C}) \setminus \{0\}$ (called *eigenfunction*) such that

$$\int_\Omega (i\nabla + \mathcal{A}_\varepsilon)\, u \cdot \overline{(i\nabla + \mathcal{A}_\varepsilon)\, w}\, dx = \lambda \int_\Omega u\overline{w}\, dx \quad \text{for all } w \in H_0^{1,\varepsilon}(\Omega, \mathbb{C}). \tag{7.2}$$

We recall from the introduction that the eigenvalue problem (1.9) (and hence (7.2)) admits a diverging sequence of real positive eigenvalues

$$\lambda_{\varepsilon,1} \le \lambda_{\varepsilon,2} \le \lambda_{\varepsilon,3} \le \cdots,$$

repeated in the enumeration according to their multiplicity.

In a similar way, the variational formulation of (1.10) in the case k odd (corresponding to a problem of type (1.9) with only one pole located at 0) can be be given in the functional space $\{w \in H_0^1(\Omega, \mathbb{C}) : \frac{w}{|x|} \in L^2(\Omega, \mathbb{C})\}$. In the case k even, instead, (1.10) takes the form of the classical eigenvalue problem for the Dirichlet Laplacian, whose variational formulation is well known. In both cases, (1.10) admits a diverging sequence of real positive eigenvalues

$$\lambda_{0,1} \le \lambda_{0,2} \le \lambda_{0,3} \le \cdots,$$

repeated according to their multiplicity.

A suitable gauge transformation allows us to obtain equivalent formulations of (1.9) and (1.10) as eigenvalue problems for the Laplacian in domains with straight cracks. For every $\varepsilon \in [0, 1]$ we define

$$\begin{aligned}
&\Sigma^j := \{ta^j : t \in \mathbb{R}\} \quad \text{for all } j = 1, \dots, k_1 + k_2,\\
&\Gamma_\varepsilon^j := \{ta^j : t \in (-\infty, \varepsilon]\}, \quad S_\varepsilon^j := \{ta^j : t \in [0, \varepsilon]\} \quad \text{for all } j = 1, \dots, k_1,\\
&S_\varepsilon^j := \{ta^j + (\varepsilon - t)a^{j+k_2} : t \in [0, \varepsilon]\} \quad \text{for all } j = k_1 + 1, \dots, k_1 + k_2,\\
&\Gamma_\varepsilon := \left(\bigcup_{j=1}^{k_1} \Gamma_\varepsilon^j\right) \cup \left(\bigcup_{j=k_1+1}^{k_1+k_2} S_\varepsilon^j\right),
\end{aligned}$$

see 7.1. We note that, for every $j = 1, \dots, k_1$, $\Gamma_0^j = \Gamma_\varepsilon^j \setminus S_\varepsilon^j$ is the straight half-line starting at 0 with slope $\alpha_j + \pi$. For every $\varepsilon \in [0, 1]$, we consider the functional space $\mathcal{H}_\varepsilon$ defined as the closure of

$$\Big\{w \in H^1(\Omega \setminus \Gamma_\varepsilon) = H^1(\Omega \setminus \Gamma_\varepsilon, \mathbb{R}) : w = 0 \text{ on a neighbourhood of } \partial\Omega\Big\}$$

in $H^1(\Omega \setminus \Gamma_\varepsilon)$ endowed with the norm $\|w\|_{H^1(\Omega\setminus\Gamma_\varepsilon)} = \|\nabla w\|_{L^2(\Omega\setminus\Gamma_\varepsilon)} + \|w\|_{L^2(\Omega)}$. From the Poincaré type inequality stated in Proposition 7.2.2, it follows that

$$\|w\|_{\mathcal{H}_\varepsilon} := \left(\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla w|^2 \, dx\right)^{1/2}$$

is a norm on $\mathcal{H}_\varepsilon$ equivalent to $\|w\|_{H^1(\Omega\setminus\Gamma_\varepsilon)}$. The corresponding scalar product is denoted as $(\cdot,\cdot)_{\mathcal{H}_\varepsilon}$.

For every $j = 1, \dots, k_1 + k_2$, with the notation $\nu^j := \big(-\sin(\alpha^j), \cos(\alpha^j)\big)$ we consider the half-planes

$$\pi^j_+ := \{x \in \mathbb{R}^2 : x \cdot \nu^j > 0\} \quad \text{and} \quad \pi^j_- := \{x \in \mathbb{R}^2 : x \cdot \nu^j < 0\}.$$

We observe that ν^j is the unit outer normal vector to π^j_- on $\partial\pi^j_-$. In view of classical trace results and embedding theorems for fractional Sobolev spaces in dimension 1, for every $j = 1, \dots, k_1 + k_2$ and $p \in [2, +\infty)$ there exist continuous trace operators

$$\gamma^j_+ : H^1(\pi^j_+ \setminus \Gamma_1) \to L^p(\Sigma^j) \quad \text{and} \quad \gamma^j_- : H^1(\pi^j_- \setminus \Gamma_1) \to L^p(\Sigma^j). \tag{7.3}$$

We also define the trace operators

$$T^j : H^1(\mathbb{R}^2 \setminus \Gamma_1) \to L^p(\Sigma^j), \quad T^j(w) := \gamma^j_+(w|_{\pi^j_+}) + \gamma^j_-(w|_{\pi^j_-}), \tag{7.4}$$

for every $j = 1, \dots, k_1 + k_2$ and $p \in [2, +\infty)$. For every $\varepsilon \in [0, 1]$, the restrictions to $\mathcal{H}_\varepsilon$ of the operators γ^j_+, γ^j_- and T^j are linear and continuous, since any element of $\mathcal{H}_\varepsilon$ can be trivially extended by 0 to an element of $H^1(\mathbb{R}^2 \setminus \Gamma_1)$; furthermore, due to the boundedness of Ω, such restrictions are continuous and compact from $\mathcal{H}_\varepsilon$ into $L^p(\Sigma^j \cap \Omega)$ for all $p \in [1, +\infty)$.

For every $\varepsilon \in (0, 1]$, we define the space

$$\widetilde{\mathcal{H}}_\varepsilon := \left\{ w \in \mathcal{H}_\varepsilon : \begin{array}{l} T^j(w) = 0 \text{ on } \Gamma^j_\varepsilon \text{ for all } j = 1, \dots, k_1, \\ T^j(w) = 0 \text{ on } S^j_\varepsilon \text{ for all } j = k_1 + 1, \dots, k_1 + k_2 \end{array} \right\}, \tag{7.5}$$

and, for $\varepsilon = 0$,

$$\widetilde{\mathcal{H}}_0 := \{w \in \mathcal{H}_0 : T^j(w) = 0 \text{ on } \Gamma^j_0 \text{ for all } j = 1, \dots, k_1\}. \tag{7.6}$$

In Section 7.2.3 we construct a function

$$\Theta_\varepsilon : \mathbb{R}^2 \setminus \{a^j_\varepsilon : j = 1, \dots, k\} \to \mathbb{R} \tag{7.7}$$

such that

$$\begin{cases} \Theta_\varepsilon \in C^\infty(\mathbb{R}^2 \setminus \Gamma_\varepsilon) \\ \nabla\Theta_\varepsilon \text{ can be extended to be in } C^\infty(\mathbb{R}^2 \setminus \{a^j_\varepsilon : j = 1, \dots, k\}) \text{ with } \nabla\Theta_\varepsilon = \mathcal{A}_\varepsilon, \end{cases} \tag{7.8}$$

see (7.46) for the definition of Θ_ε. The phase multiplication

$$u(x) \mapsto v(x) := e^{-i\Theta_\varepsilon(x)} u(x), \quad x \in \Omega \setminus \Gamma_\varepsilon, \tag{7.9}$$

transforms any solution u to problem (1.9) into a solution v to

$$\begin{cases} -\Delta v = \lambda v, & \text{in } \Omega \setminus \Gamma_\varepsilon, \\ v = 0, & \text{on } \partial\Omega, \\ T^j(v) = 0, & \text{on } \Gamma_\varepsilon^j \text{ for all } j = 1, \dots, k_1, \\ T^j(\nabla v \cdot \nu^j) = 0, & \text{on } \Gamma_\varepsilon^j \text{ for all } j = 1, \dots, k_1, \\ T^j(v) = 0, & \text{on } S_\varepsilon^j \text{ for all } j = k_1+1, \dots, k_1+k_2, \\ T^j(\nabla v \cdot \nu^j) = 0, & \text{on } S_\varepsilon^j \text{ for all } j = k_1+1, \dots, k_1+k_2, \end{cases} \tag{7.10}$$

In (7.48) we also define a function

$$\Theta_0 : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \tag{7.11}$$

satisfying

$$\begin{cases} \Theta_0 \in C^\infty(\mathbb{R}^2 \setminus \Gamma_0) \\ \nabla\Theta_0 \text{ can be extended to be in } C^\infty(\mathbb{R}^2 \setminus \{0\}) \text{ with } \nabla\Theta_0 = \frac{1+(-1)^{k+1}}{2} A_0. \end{cases} \tag{7.12}$$

The gauge transformation

$$u(x) \mapsto v(x) := e^{-i\Theta_0(x)} u(x), \quad x \in \Omega \setminus \Gamma_0, \tag{7.13}$$

shows that the limit eigenvalue problem (1.10) is equivalent to

$$\begin{cases} -\Delta v = \lambda v, & \text{in } \Omega \setminus \Gamma_0, \\ v = 0, & \text{on } \partial\Omega, \\ T^j(v) = 0, & \text{on } \Gamma_0^j \text{ for all } j = 1, \dots, k_1, \\ T^j(\nabla v \cdot \nu^j) = 0, & \text{on } \Gamma_0^j \text{ for all } j = 1, \dots, k_1, \end{cases} \tag{7.14}$$

in the sense that the two problems have the same eigenvalues and their eigenfunctions match each other via the phase multiplication (7.13), see Section 7.2.3 for details. Therefore, under assumption (1.15), λ_{0,n_0} is also a simple eigenvalue of (7.14). Let

$$v_0 \text{ be an eigenfunction of (7.14) associated to } \lambda_{0,n_0} \text{ such that } \|v_0\|_{L^2(\Omega)} = 1; \tag{7.15}$$

it is not restrictive to assume that v_0 is real-valued, see Remark 7.2.5. Once v_0 is fixed as above, for every $\varepsilon \in (0,1]$ we define

$$L_\varepsilon : \mathcal{H}_1 \to \mathbb{R}, \quad L_\varepsilon(w) := 2 \sum_{j=1}^{k_1+k_2} \int_{S_\varepsilon^j} \nabla v_0 \cdot \nu^j \gamma_+^j(w)\, dS \tag{7.16}$$

and

$$J_\varepsilon : \mathcal{H}_\varepsilon \to \mathbb{R}, \quad J_\varepsilon(w) := \frac{1}{2} \int_{\Omega\setminus\Gamma_\varepsilon} |\nabla w|^2 \, dx + L_\varepsilon(w). \tag{7.17}$$

As proved in Proposition 7.3.2, for every $\varepsilon \in (0,1]$ there exists a unique $V_\varepsilon \in \mathcal{H}_\varepsilon$ such that

$$V_\varepsilon - v_0 \in \widetilde{\mathcal{H}}_\varepsilon \quad \text{and} \quad J_\varepsilon(V_\varepsilon) = \min\left\{ J_\varepsilon(w) : w \in \mathcal{H}_\varepsilon \text{ and } w - v_0 \in \widetilde{\mathcal{H}}_\varepsilon \right\}. \tag{7.18}$$

Our first main result is the following expansion of the eigenvalue variation $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0}$ in terms of

$$\mathcal{E}_\varepsilon = J_\varepsilon(V_\varepsilon) \tag{7.19}$$

and $L_\varepsilon(v_0)$.

Theorem 7.1.1. *Under assumption* (1.15), *let* v_0 *be as in* (7.15). *Then*

$$\lambda_{\varepsilon,n_0} - \lambda_{0,n_0} = 2(\mathcal{E}_\varepsilon - L_\varepsilon(v_0)) + o(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}) \quad \text{as } \varepsilon \to 0^+, \tag{7.20}$$

where $\mathcal{E}_\varepsilon$ *and* V_ε *are defined in* (7.19) *and* (7.18), *respectively.*

7.1.1 The case k odd

For k odd, the asymptotic behaviour of $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0^+$ can be quantified in terms of the vanishing order of v_0 at the collision point 0. Indeed, as detailed in Proposition 7.2.6, if k is odd, there exists $\beta \in \mathbb{R} \setminus \{0\}$ such that, as $\varepsilon \to 0^+$,

$$\varepsilon^{-\frac{m}{2}} v_0(\varepsilon \cos t, \varepsilon \sin t) \to \beta\, f(t)\, \sin\left(\tfrac{m}{2}(t - \alpha_0)\right) \tag{7.21}$$

in $C^{1,\tau}([0, 2\pi] \setminus \{\alpha^j + \pi\}_{j=1}^{k_1}, \mathbb{R})$ for all $\tau \in (0, 1)$, where $m \in \mathbb{N}$ is odd and corresponds to the number of nodal lines of v_0 meeting at 0 (which equals the number of nodal lines of eigenfunctions of (1.10) associated to λ_{0,n_0}), $\alpha_0 \in [0, \frac{2\pi}{m})$ is the minimal slope of such nodal lines, and

$$f : [0, 2\pi] \to \{-1, 1\}, \quad f(t) := \prod_{j=1}^{k_1} (-1)^{\chi_{[\alpha^j+\pi,2\pi)}(t)}, \tag{7.22}$$

where

$$\chi_{[\alpha^j+\pi,2\pi)}(t) := \begin{cases} 0, & \text{if } t \in [0, \alpha^j + \pi), \\ 1, & \text{if } t \in [\alpha^j + \pi, 2\pi). \end{cases} \tag{7.23}$$

From (7.21) we realize that the m nodal lines of v_0 which meet at 0 are tangent to the m straight half-lines

$$\mathcal{R}_j = \left\{\left(\cos\left(\alpha_0 + j\tfrac{2\pi}{m}\right), \sin\left(\alpha_0 + j\tfrac{2\pi}{m}\right)\right)r : r \geq 0\right\}, \quad j = 0, 1, \ldots, m-1,$$

which divide the whole 2π-angle into m equal sectors. We define the functional space

$$\widetilde{\mathcal{X}} := \left\{ \begin{aligned} & w \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \ w \in H^1(B_r \setminus \Gamma_1) \text{ for all } r > 0, \\ & \qquad \nabla w \in L^2(\mathbb{R}^2 \setminus \Gamma_1, \mathbb{R}^2),\ T^j(w) = 0 \text{ on } \Gamma_0^j \text{ for } j = 1, \ldots, k_1 \end{aligned} \right\}, \tag{7.24}$$

and consider its closed subspace

$$\widetilde{\mathcal{H}} := \{w \in \widetilde{\mathcal{X}} : T^j(w) = 0 \text{ on } S_1^j \text{ for any } j = 1, \ldots, k_1 + k_2\}. \tag{7.25}$$

Letting

$$\Psi_0(x) = \Psi_0(r \cos t, r \sin t) = \beta\, r^{\frac{m}{2}}\, f(t)\, \sin\left(\tfrac{m}{2}(t - \alpha_0)\right) \tag{7.26}$$

with f, m, β, and α_0 as in (7.21), we observe that the nodal set of Ψ_0 is given by $\bigcup_{j=0}^{m-1} \mathcal{R}_j$. We define

$$L : \widetilde{\mathcal{X}} \to \mathbb{R}, \quad L(w) := 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} \nabla\Psi_0 \cdot \nu^j \gamma_+^j(w)\, dS \tag{7.27}$$

and

$$J : \widetilde{\mathcal{X}} \to \mathbb{R}, \quad J(w) := \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Gamma_1} |\nabla w|^2\, dx + L(w). \tag{7.28}$$

We observe that $L(w)$ is well-defined also for any function $w \in H^1(D_1 \setminus \Gamma_1)$.

Let $\eta \in C_c^\infty(\mathbb{R}^2)$ be a radial cut-off function such that

$$\begin{cases} 0 \le \eta(x) \le 1 \text{ for all } x \in \mathbb{R}^2, \\ \eta(x) = 1 \text{ if } x \in D_1, \quad \eta(x) = 0 \text{ if } x \in \mathbb{R}^2 \setminus D_2, \\ |\nabla \eta| \le 2 \text{ in } D_2 \setminus D_1. \end{cases} \tag{7.29}$$

As proved in Proposition 7.5.4, there exists a unique $\widetilde{V} \in \widetilde{\mathcal{X}}$ such that

$$\widetilde{V} - \eta\Psi_0 \in \widetilde{\mathcal{H}} \quad \text{and} \quad J(\widetilde{V}) = \min\left\{J(w) : w \in \widetilde{\mathcal{X}} \text{ and } w - \eta\Psi_0 \in \widetilde{\mathcal{H}}\right\}. \tag{7.30}$$

Theorem 7.1.2. *Let k be odd. Under assumption (1.15), let v_0 be as in (7.15). Then*

(i) $\lim_{\varepsilon\to 0^+} \varepsilon^{-m}\mathcal{E}_\varepsilon = \mathcal{E}$, *where m is the vanishing order of v_0 at 0 as in (7.21) and*

$$\mathcal{E} = J(\widetilde{V}) = \min_{\eta\Psi_0 + \widetilde{\mathcal{H}}} J; \tag{7.31}$$

(ii) $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0} = 2\,\varepsilon^m(\mathcal{E} - L(\Psi_0)) + o(\varepsilon^m)$ *as* $\varepsilon \to 0^+$.

The expansion proved in Theorem 7.1.2-(ii) identifies the sharp asymptotic behaviour of the eigenvalue variation $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0}$ if $\mathcal{E} - L(\Psi_0) \neq 0$; if instead $\mathcal{E} - L(\Psi_0) = 0$, Theorem 7.1.2-(ii) only provides the information that $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0}$ is an infinitesimal of higher order than m. It is therefore natural to ask whether there are configurations of poles $\{a^j\}$ for which the quantity $\mathcal{E} - L(\Psi_0)$ does or does not vanish. The following proposition gives an answer in this sense, also providing precise information on the sign of the eigenvalue variation in two remarkable cases: the case in which each pole moves along the tangent to a nodal line of the limit eigenfunction and the case in which each pole moves along the bisector between two nodal lines.

Proposition 7.1.3. *Let $k = k_1 \le m$ be odd and $k_2 = 0$. Under assumption (1.15), let v_0 be as in (7.15) and α_0 as in (7.21). For every $j \in \{1, \dots, k_1\}$ let α^j be as in (1.6).*

(i) *If $\alpha^j \in \{\alpha_0 + \ell\frac{2\pi}{m} : \ell = 0, 1, 2, \dots, m-1\}$ for all $j \in \{1, \dots, k_1\}$, then*

$$\mathcal{E} < 0 \quad \textit{and} \quad L(\Psi_0) = 0;$$

furthermore, $\lambda_{\varepsilon,n_0} < \lambda_{0,n_0}$ provided that $\varepsilon > 0$ is sufficiently small.

(ii) *If $\alpha^j \in \{\alpha_0 + (1+2\ell)\frac{\pi}{m} : \ell = 0, 1, 2, \dots, m-1\}$ for all $j \in \{1, \dots, k_1\}$, then*

$$\mathcal{E} > 0 \quad \textit{and} \quad L(\Psi_0) = 0;$$

furthermore, $\lambda_{\varepsilon,n_0} > \lambda_{0,n_0}$ provided that $\varepsilon > 0$ is sufficiently small.

(iii) *There exists a choice of $\{\alpha^j : j = 1, \dots, k\}$ such that $\mathcal{E} - L(\Psi_0) = 0$ and $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0} = o(\varepsilon^m)$ as $\varepsilon \to 0^+$.*

The proof of claim (iii) in Proposition 7.1.3 is based on a continuity argument. Indeed, the function $\mathcal{E} - L(\Psi_0)$ varies continuously under rotations of the configuration of poles, see Theorem 7.5.8. Hence (i) and (ii), together with Bolzano's Theorem, guarantee the existence of intermediate configurations for which $\mathcal{E} - L(\Psi_0)$ vanishes. The proof of claims (i) and (ii) highlights the fact that, analogously to $\mathcal{E}_\varepsilon$, also $\mathcal{E}$ represents an intermediate notion between the capacity and the torsional rigidity of the set $\cup_{j=1}^{k_1} S_1^j$. Indeed, in case (i) it occurs that

$$\mathcal{E} = \min_{w \in \widetilde{\mathcal{H}}} \left\{ \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Gamma_1} |\nabla w|^2 \, dx + L(w) \right\} < 0,$$

see (7.146), i.e. $\mathcal{E}$ is the minimum of a functional containing a (quadratic) energy term and a linear one, over a linear space: this makes it somehow behaving like a torsional rigidity of the set $\cup_{j=1}^{k_1} S_1^j$. On the other hand, in case (ii) we have the characterization

$$\mathcal{E} = \min \left\{ \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Gamma_1} |\nabla w|^2 \, dx \colon w - \eta \Psi_0 \in \widetilde{\mathcal{H}} \right\} > 0,$$

see (7.147), which yields a notion resembling that of Ψ_0-capacity of the set $\cup_{j=1}^{k_1} S_1^j$.

The proof of Theorem 7.1.2 is based on a blow-up analysis, which also provides the following result on the behavior of eigenfunctions, characterizing their blow-up profile and quantifying the convergence speed of the eigenfunctions of problem (1.9) towards the corresponding eigenfunction of the limit problem (1.10).

Theorem 7.1.4. *Let k be odd and $n_0 \in \mathbb{N} \setminus \{0\}$ be such that (1.15) is satisfied. Let u_0 be an eigenfunction of (1.10) associated to λ_{0,n_0} such that $\int_\Omega |u_0|^2 \, dx = 1$. For every $\varepsilon \in (0,1]$, let $u_\varepsilon \in H_0^{1,\varepsilon}(\Omega, \mathbb{C})$ be the eigenfunction of (1.9) associated to the eigenvalue $\lambda_{n_0,\varepsilon}$ such that*

$$\int_\Omega |u_\varepsilon|^2 \, dx = 1 \quad \textit{and} \quad \int_\Omega e^{-i(\Theta_\varepsilon - \Theta_0)} u_\varepsilon \overline{u_0} \, dx \ \textit{is a positive real number,} \tag{7.32}$$

where Θ_ε and Θ_0 are as in (7.7)–(7.8) and (7.11)–(7.12), respectively. Then

$$\varepsilon^{-m/2} u_\varepsilon(\varepsilon \, \cdot) \to e^{i\Theta_1}(\Psi_0 - \widetilde{V}) \quad \textit{as } \varepsilon \to 0^+ \tag{7.33}$$

in $H^{1,1}(B_R, \mathbb{C})$ for all $R > 0$, where $\widetilde{V}$ and Ψ_0 are as in (7.30) and (7.26), respectively. Moreover,

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-m} \int_{\mathbb{R}^2 \setminus \Gamma_1} \left| e^{-i(\Theta_\varepsilon - \Theta_0)} (i\nabla + \mathcal{A}_\varepsilon) u_\varepsilon - (i\nabla + A_0) u_0 \right|^2 dx = \|\nabla \widetilde{V}\|^2_{L^2(\mathbb{R}^2 \setminus \Gamma_1)}. \tag{7.34}$$

We observe that condition (7.32) allows us to identify, among all the eigenfunctions of (1.9) associated to the eigenvalue $\lambda_{n_0,\varepsilon}$ (that are multiples of a given one due to the simplicity of $\lambda_{n_0,\varepsilon}$), the one that converges to u_0 as $\varepsilon \to 0^+$.

7.1.2 The case of two opposite poles ($k_1 = 0$, $k_2 = 1$)

In the case of two opposite poles a_ε^1, $a_\varepsilon^2 = -a_\varepsilon^1$ colliding to 0 from the two sides of the same straight line, we can rewrite the terms appearing in (7.20) in elliptic coordinates in the spirit of [3, Subsection 2.2], thus determining the dominant term in the asymptotic expansion. This allows us to generalize [5, Theorem 2.6, Theorem 2.8], see also [3, Theorem 1.16], removing any symmetry assumption on the domain Ω. Let us assume that

$$\text{the } n_0\text{-th eigenvalue } \lambda_{n_0} \text{ of the Dirichlet Laplacian in } \Omega \text{ is simple.}$$

We recall that, since k is even in this case, $\lambda_{n_0} = \lambda_{0,n_0}$ coincides with the n_0-th eigenvalue of the limit problem (1.10). Let

$$u_0 \text{ be an eigenfunction of (1.11) associated to } \lambda_{n_0} = \lambda_{0,n_0} \text{ such that } \int_\Omega u_0^2\,dx = 1. \tag{7.35}$$

If $u_0(0) \neq 0$, then, for any bounded simply connected domain Ω, a sharp expansion of the variation $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0}$ has already been obtained in [4, Theorem 1.2], see Remark 7.6.7. Hence we assume that $u_0(0) = 0$. Up to a suitable choice of the coordinate system, according to the notation introduced in (1.6), it is not restrictive to consider the case $\alpha^1 = 0$, $\alpha^2 = \pi$, so that, for some $r_1 \in (0, R)$, the configuration of the two opposite poles is given by

$$a_\varepsilon^1 = r_1(\varepsilon, 0) \quad \text{and} \quad a_\varepsilon^2 = r_1(-\varepsilon, 0), \tag{7.36}$$

and

$$S_\varepsilon := S_\varepsilon^1 = [-r_1\varepsilon, r_1\varepsilon] \times \{0\}, \tag{7.37}$$

see 7.2. Furthermore, since $u_0(0) = 0$, it is well known that there exists $m \in \mathbb{N} \setminus \{0\}$,

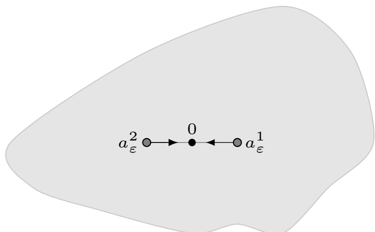


Figure 7.2: Two opposite poles colliding at 0 ($k_1 = 0$, $k_2 = 1$).

$\beta \in \mathbb{R} \setminus \{0\}$ and $\alpha_0 \in [0, \frac{\pi}{m})$ such that

$$r^{-m} u_0(r\cos t, r\sin t) \to \beta \sin(m(t - \alpha_0)) \quad \text{in } C^{1,\tau}([0, 2\pi], \mathbb{C}) \quad \text{as } r \to 0^+, \tag{7.38}$$

for any $\tau \in (0, 1)$. In particular, the $2m$ half-lines with slopes $\alpha_0 + j\frac{\pi}{m}$, $j = 0, \dots, 2m-1$, are tangent to the nodal lines of u_0 meeting at 0.

Remark 7.1.5. By standard regularity theory, u_0 is analytic in Ω. Let T_m be the Taylor polynomial of u_0 centered at 0 of order m, with $m \in \mathbb{N} \setminus \{0\}$ being as in (7.38). Then, in view of (7.38),

$$T_m(x_1, x_2) = \sum_{j=0}^{m} \frac{1}{(m-j)!j!} \frac{\partial^m u_0}{\partial x_1^{m-j} \partial x_2^j}(0)\, x_1^{m-j} x_2^j. \tag{7.39}$$

For every $t \in [0, 2\pi]$, we have

$$T_m(\cos t, \sin t) = \beta \sin(m(t - \alpha_0)),$$
$$(\nabla T_m)(\cos t, \sin t) \cdot (-\sin t, \cos t) = m\beta \cos(m(t - \alpha_0)).$$

Hence

$$\frac{1}{m!}\frac{\partial^m u_0}{\partial x_1^m}(0) = -\beta\sin(m\alpha_0) \quad \text{and} \quad \frac{1}{(m-1)!}\frac{\partial^m u_0}{\partial x_1^{m-1}\partial x_2^1}(0) = m\beta\cos(m\alpha_0),$$

so that, in particular,

$$\beta = \frac{(-1)^j}{m!}\frac{\partial^m u_0}{\partial x_1^{m-1}\partial x_2^1}(0) \quad \text{if } \alpha_0 = \frac{j\pi}{m} \quad \text{for some } j\in\{0,1,\dots,2m-1\}, \tag{7.40}$$

and

$$\beta = \frac{(-1)^{j+1}}{m!}\frac{\partial^m u_0}{\partial x_1^m}(0) \quad \text{if } \alpha_0 = \frac{\pi}{2m}+\frac{j\pi}{m} \quad \text{for some } j\in\{0,1,\dots,2m-1\}. \tag{7.41}$$

If the segment S_ε is tangent to a nodal line of u_0, i.e. if $\alpha_0 = \frac{j\pi}{m}$ for some $j\in\{0,1,\dots,2m-1\}$, we have the following result which generalizes [5, Theorem 2.8] dropping any symmetry assumption on Ω.

Theorem 7.1.6. *Let λ_{n_0} be a simple eigenvalue of (1.11) and let u_0 be as in (7.35). Assume that $u_0(0)=0$ and let $m\in\mathbb{N}\setminus\{0\}$ and α_0 be as in (7.38). Let $k_1=0$ and $k_2=1$ with the configuration of poles as in assumption (7.36). If $\alpha_0=\frac{j\pi}{m}$ for some $j\in\{0,1,\dots,2m-1\}$, then*

$$\lambda_{\varepsilon,n_0}-\lambda_{n_0} = -\frac{m\pi\beta^2 r_1^{2m}}{4^{m-1}}\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}^2\varepsilon^{2m}+o(\varepsilon^{2m}) \quad \text{as } \varepsilon\to0^+,$$

with β as in (7.40).

On the other hand, if S_ε lays on the bisector of the angle between the tangents to nodal lines, i.e. if $\alpha_0=\frac{\pi}{2m}+\frac{j\pi}{m}$ for some $j\in\{0,1,\dots,2m-1\}$, then we prove the following expansion.

Theorem 7.1.7. *Let λ_{n_0} be a simple eigenvalue of (1.11) and let u_0 be as in (7.35). Assume that $u_0(0)=0$ and let $m\in\mathbb{N}\setminus\{0\}$ and α_0 be as in (7.38). Let $k_1=0$ and $k_2=1$ with the configuration of poles as in assumption (7.36). If $\alpha_0=\frac{\pi}{2m}+\frac{j\pi}{m}$ for some $j\in\{0,1,\dots,2m-1\}$, then*

$$\lambda_{\varepsilon,n_0}-\lambda_{n_0} = \frac{m\pi\beta^2 r_1^{2m}}{4^{m-1}}\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}^2\varepsilon^{2m}+o(\varepsilon^{2m}) \quad \text{as } \varepsilon\to0^+,$$

with β as in (7.41).

We observe that Theorem 7.1.7 is a generalization of [5, Theorem 2.6] and [3, Theorem 1.16].

The rest of Chapter 7 is organized as follows. In Section 7.2 we collect some basic facts, such as the gauge invariance property of the problem, useful features of the functional spaces involved, and some known results that will be used in the rest of the chapter. In Section 7.3 we provide some preliminary estimates on the quantities $\mathcal{E}_\varepsilon$ and L_ε that appear in formula (7.20); such estimates are used in Section 7.4, where the proof of Theorem 7.1.1 is completed. In Section 7.5 we perform a blow-up analysis of the potential V_ε appearing in (7.18), in the case k odd; this is the key ingredient in the proof of Theorems 7.1.2 and 7.1.4. In the same section we also complete the proof of 7.1.3. Finally, in Section 7.6 we consider the case of two poles colliding to 0 from opposite sides of the same straight line, thus proving Theorems 7.1.6 and 7.1.7.

7.2 Preliminaries

7.2.1 Scalar potential functions for A_b outside half-lines

The construction of the gauge transformation, which makes problems (1.9) and (1.10) equivalent to eigenvalue problems for the Laplacian in domains with straight cracks, is based on the remark that, since Aharonov-Bohm vector fields are irrotational, they are gradients of some scalar potential functions in simply connected domains, such as the complement of straight half-lines starting at the pole.

For every $b = (b_1, b_2) \in \mathbb{R}^2$, let $\theta_b : \mathbb{R}^2 \setminus \{b\} \to [0, 2\pi)$ be defined as

$$\theta_b(x_1, x_2) := \begin{cases} \arctan\left(\frac{x_2 - b_2}{x_1 - b_1}\right), & \text{if } x_1 > b_1,\ x_2 \geq b_2, \\ \frac{\pi}{2}, & \text{if } x_1 = b_1,\ x_2 > b_2, \\ \pi + \arctan\left(\frac{x_2 - b_2}{x_1 - b_1}\right), & \text{if } x_1 < b_1, \\ \frac{3}{2}\pi, & \text{if } x_1 = b_1,\ x_2 < b_2, \\ 2\pi + \arctan\left(\frac{x_2 - b_2}{x_1 - b_1}\right), & \text{if } x_1 > b_1,\ x_2 < b_2, \end{cases}$$

i.e.,

$$\theta_b(b + r(\cos t, \sin t)) = t \quad \text{for all } t \in [0, 2\pi) \text{ and } r > 0.$$

We observe that $\theta_b \in C^\infty(\mathbb{R}^2 \setminus \{(x_1, b_2) : x_1 \geq b_1\})$ and $\nabla\theta_b$ can be extended to be in $C^\infty(\mathbb{R}^2 \setminus \{b\})$, with $\nabla\left(\frac{\theta_b}{2}\right) = A_b$ in $\mathbb{R}^2 \setminus \{b\}$. For every $b \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$, and $x = (x_1, x_2) \in \mathbb{R}^2$, we define

$$R_{b,\alpha}(x) := \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + M_\alpha \begin{bmatrix} x_1 - b_1 \\ x_2 - b_2 \end{bmatrix}, \tag{7.42}$$

with

$$M_\alpha := \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \tag{7.43}$$

i.e., $R_{b,\alpha}$ is a rotation about b by an angle α. Let

$$\theta_{b,\alpha} := \theta_b \circ R_{b,\alpha}, \tag{7.44}$$

so that $\theta_{b,\alpha}(b + r(\cos t, \sin t)) = \alpha + t$ for every $r > 0$ and $t \in [-\alpha, -\alpha + 2\pi)$. We observe that $\theta_{b,\alpha}$ is smooth in $\mathbb{R}^2 \setminus \{b + r(\cos\alpha, -\sin\alpha) : r \geq 0\}$ and $\nabla\theta_{b,\alpha}$ can be extended to be in $C^\infty(\mathbb{R}^2 \setminus \{b\})$, with $\nabla\left(\frac{\theta_{b,\alpha}}{2}\right) = A_b$, see 7.3.

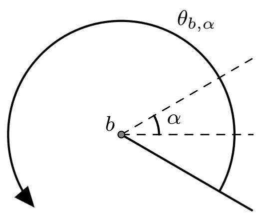


Figure 7.3: $\theta_{b,\alpha}$ is smooth in $\mathbb{R}^2 \setminus \{b + r(\cos\alpha, -\sin\alpha) : r \geq 0\}$.

7.2.2 Some remarks on functional spaces

In this subsection we describe some properties of the functional spaces $\mathcal{H}_\varepsilon$ introduced in Section 7.1.

Remark 7.2.1. The natural embedding $I : \mathcal{H}_\varepsilon \to L^2(\Omega)$ is compact. Indeed, we can cut Ω along the lines Σ^j for $j = 1 \dots, k_1 + k_2$, where Σ^j are defined in Section 7.1. Then we can use classical compact embedding results for each resulting subset, see for example [99, Theorem 12.30].

Arguing as in 7.2.1, from the Poincaré inequality for functions vanishing on a portion of the boundary we can deduce the following Poincaré inequality in $\mathcal{H}_1$, and hence in $\mathcal{H}_\varepsilon$ for any $\varepsilon \in [0, 1]$.

Proposition 7.2.2. *There exists a constant $C_P > 0$ such that, for every $\varepsilon \in [0, 1]$ and $w \in \mathcal{H}_\varepsilon$,*

$$\int_\Omega w^2 dx \le C_P \int_{\Omega \setminus \Gamma_\varepsilon} |\nabla w|^2 \, dx.$$

Since $\Omega \setminus \Gamma_1 \subseteq \Omega \setminus \Gamma_{\varepsilon_1} \subseteq \Omega \setminus \Gamma_{\varepsilon_2}$, we have $\mathcal{H}_{\varepsilon_2} \subseteq \mathcal{H}_{\varepsilon_1} \subseteq \mathcal{H}_1$ for all $0 \le \varepsilon_2 \le \varepsilon_1 \le 1$. Proposition 7.2.3 below establishes a Mosco-type convergence result for the spaces $\mathcal{H}_\varepsilon$ as $\varepsilon \to 0^+$.

Proposition 7.2.3. *Let $\{\varepsilon_n\} \subset (0, 1)$ be such that $\lim_{n\to\infty} \varepsilon_n = 0$. If $\{v_n\}_n \subset \mathcal{H}_1$ and $v \in \mathcal{H}_1$ are such that $v_n \in \mathcal{H}_{\varepsilon_n}$ for all $n \in \mathbb{N}$ and $v_n \rightharpoonup v$ in $\mathcal{H}_1$ as $n \to \infty$, then $v \in \mathcal{H}_0$.*

Proof. For every $\varepsilon \in (0, 1]$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $v_n \in \mathcal{H}_\varepsilon$ for all $n > n(\varepsilon)$. The weak convergence $v_n \rightharpoonup v$ in $\mathcal{H}_1$ then implies that $v \in \mathcal{H}_\varepsilon$ for all $\varepsilon \in (0, 1]$. It follows that there exists $\mathbf{f} \in L^2(\Omega, \mathbb{R}^N)$ such that $\nabla v = \mathbf{f}$ in $\mathcal{D}'(\Omega \setminus \Gamma_\varepsilon)$ for all $\varepsilon \in (0, 1]$. Actually, $\nabla v = \mathbf{f}$ in $\mathcal{D}'(\Omega \setminus \Gamma_0)$, since, for every $\varphi \in C^\infty_c(\Omega \setminus \Gamma_0)$, $\operatorname{supp} \varphi \subset \Omega \setminus \Gamma_\varepsilon$ for ε sufficiently small. Therefore, $v \in H^1(\Omega \setminus \Gamma_0)$. From the fact that $v \in \mathcal{H}_1 \cap H^1(\Omega \setminus \Gamma_0)$ it follows that $v \in \mathcal{H}_0$. □

Since the singleton $\{0\}$ has null capacity in Ω, functions in $\mathcal{H}_0$, respectively in $\widetilde{\mathcal{H}}_0$, can be approximated by functions vanishing in a neighbourhood of 0, as stated in Lemma 7.2.4.

Lemma 7.2.4.

(i) *The set $\mathcal{H}_{0,0} := \{v \in \mathcal{H}_0 : v \equiv 0$ in a neighbourhood of $0\}$ is dense in $\mathcal{H}_0$.*

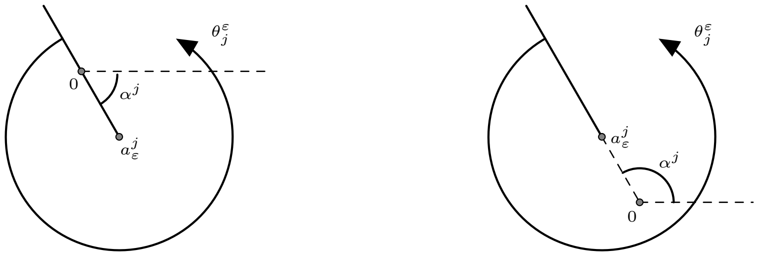
(ii) *The set $\widetilde{\mathcal{H}}_{0,0} := \{v \in \widetilde{\mathcal{H}}_0 : v \equiv 0$ in a neighbourhood of $0\}$ is dense in $\widetilde{\mathcal{H}}_0$.*

Proof. To prove (i) we first notice that, if $v \in \mathcal{H}_0$, then, defining v_n as

$$v_n(x) = \begin{cases} v(x), & \text{if } |v(x)| < n, \\ -n, & \text{if } v(x) < -n, \\ n, & \text{if } v(x) > n, \end{cases}$$

$v_n \in \mathcal{H}_0 \cap L^\infty(\Omega)$ for all $n \in \mathbb{N}$ and $v_n \to v$ in $\mathcal{H}_0$. Therefore it is enough to prove that $\mathcal{H}_{0,0} \cap L^\infty(\Omega)$ is dense in $\mathcal{H}_0 \cap L^\infty(\Omega)$. To this aim, let us fix some $v \in \mathcal{H}_0 \cap L^\infty(\Omega)$. For every $\varepsilon \in (0, 1)$ we consider the cut-off function $\omega_\varepsilon \in W^{1,\infty}(\mathbb{R}^2)$ defined as

$$\omega_\varepsilon(x) := \begin{cases} 1, & \text{if } x \in D_\varepsilon, \\ \frac{2\log|x| - \log\varepsilon}{\log\varepsilon}, & \text{if } x \in D_{\sqrt{\varepsilon}} \setminus D_\varepsilon, \\ 0, & \text{if } x \in \Omega \setminus D_{\sqrt{\varepsilon}}. \end{cases} \tag{7.45}$$



(a) θ_j^ε for $j \leq k_1 + k_2$.

(b) θ_j^ε for $j \geq k_1 + k_2 + 1$.

Figure 7.4: The angles θ_j^ε for $1 \leq j \leq k_1 + 2k_2$. The half-lines represent the singular set of the function θ_j^ε.

One may directly verify that $(1-\omega_\varepsilon)v \in \mathcal{H}_{0,0} \cap L^\infty(\Omega)$ for all $\varepsilon \in (0,1)$ and $(1-\omega_\varepsilon)v \to v$ in $\mathcal{H}_0$ as $\varepsilon \to 0$. The proof of (i) is thereby complete. We can proceed in a similar way to obtain (ii). □

7.2.3 An equivalent eigenvalue problem by gauge transformation

For every $\varepsilon \in (0,1]$, using the notation introduced in (7.44), we define

$$\theta_\varepsilon^j := \begin{cases} \theta_{a_\varepsilon^j, \pi - \alpha^j}, & \text{if } j = 1, \dots, k_1 + k_2, \\ \theta_{a_\varepsilon^j, -\alpha^j}, & \text{if } j = k_1 + k_2 + 1, \dots, k_1 + 2k_2, \end{cases}$$

with α^j as in (1.6), see 7.4, and

$$\Theta_\varepsilon : \mathbb{R}^2 \setminus \{a_\varepsilon^j : j = 1, \dots, k\} \to \mathbb{R}, \quad \Theta_\varepsilon := \frac{1}{2} \sum_{j=1}^{k} (-1)^{j+1} \theta_\varepsilon^j. \tag{7.46}$$

We observe that Θ_ε verifies (7.8).

For any $\varepsilon \in (0,1]$, let $\lambda \in \mathbb{R}$ be an eigenvalue of problem (1.9) associated to the eigenfunction $u \in H_0^{1,\varepsilon}(\Omega, \mathbb{C}) \setminus \{0\}$. Then the function

$$v(x) := e^{-i\Theta_\varepsilon(x)} u(x), \quad x \in \Omega \setminus \Gamma_\varepsilon,$$

belongs to $\mathcal{H}_\varepsilon$ and weakly solves (7.10), in the sense that $v \in \widetilde{\mathcal{H}}_\varepsilon$ and

$$\int_{\Omega \setminus \Gamma_\varepsilon} \nabla v \cdot \nabla w \, dx = \lambda \int_\Omega vw \, dx \quad \text{for all } w \in \widetilde{\mathcal{H}}_\varepsilon, \tag{7.47}$$

where $\widetilde{\mathcal{H}}_\varepsilon$ is defined in (7.5). On the other hand, if $v \in \widetilde{\mathcal{H}}_\varepsilon$ solves (7.47), then $u = e^{i\Theta_\varepsilon} v$ solves (1.9). Therefore the eigenvalue problems (1.9) and (7.10) have the same eigenvalues and their eigenfunctions match each other via the phase $e^{-i\Theta_\varepsilon}$.

A similar gauge transformation can be made for solutions to (1.10). For every $j = 1, \dots, k_1$, let α^j be as in (1.6) and

$$\theta_0^j := \theta_0 \circ R_{0,\pi-\alpha^j}.$$

We define

$$\Theta_0 : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}, \quad \Theta_0(x) = \frac{1}{2} \sum_{j=1}^{k_1} (-1)^{j+1} \theta_0^j(x). \tag{7.48}$$

If $k_1 = 0$ we just take $\Theta_0 \equiv 0$. We observe that Θ_0 satisfies (7.12). Furthermore, if $t \in [0, 2\pi)$,

$$\begin{aligned} \theta_0^j(\cos t, \sin t) &= \begin{cases} t - \alpha^j + \pi, & \text{if } t \in [0, \alpha^j + \pi), \\ t - \alpha^j - \pi, & \text{if } t \in [\alpha^j + \pi, 2\pi), \end{cases} \\ &= -\alpha^j + t + \pi(1 - 2\chi_{[\alpha^j+\pi,2\pi)}), \end{aligned} \tag{7.49}$$

where χ is defined in (7.23). We have that u is an eigenfunction of problem (1.10), associated to the eigenvalue λ, if and only if the function

$$v(x) := e^{-i\Theta_0(x)} u(x), \quad x \in \Omega \setminus \Gamma_0, \tag{7.50}$$

is a non-zero weak solution of (7.14) in the sense that $v \in \widetilde{\mathcal{H}}_0$ and

$$\int_{\Omega\setminus\Gamma_0} \nabla v \cdot \nabla w \, dx = \lambda \int_\Omega vw \, dx \quad \text{for all } w \in \widetilde{\mathcal{H}}_0, \tag{7.51}$$

where $\widetilde{\mathcal{H}}_0$ is defined in (7.6). We recall that, if k_1 is even, then, letting v as in (7.50), the function $ve^{i\Theta_0} = u$ is an eigenfunction of the Dirichlet Laplacian in Ω, hence it is smooth in Ω.

Remark 7.2.5. We may treat eigenfunctions of problems (7.10) and (7.14) as real-valued functions (thus justifying the choice to consider $\mathcal{H}_\varepsilon$ as a space of real functions). Indeed, since all the coefficients in (7.10) and (7.14) are real, both the real and the imaginary part of any eigenfunction are eigenfunctions, if not trivial. Hence, any eigenspace of (7.10) and (7.14) admits a basis made of reals eigenfunctions. See also [1, Subsection 2.3].

7.2.4 Asymptotics of solutions to the limit eigenvalue problem

Let $\{\alpha^j\}_{j=1}^{k_1}$ and $\chi_{[\alpha^j+\pi,2\pi)}$ be as in (1.6) and (7.23), respectively. Let f be the function defined in (7.22).

Proposition 7.2.6. *Let k_1 be odd. If v is a non-trivial solution to (7.14), in the sense that $v \in \widetilde{\mathcal{H}}_0$ satisfies (7.51), then there exist an odd number $m \in \mathbb{N}$, $\beta \in \mathbb{R} \setminus \{0\}$, and $\alpha_0 \in [0, \frac{2\pi}{m})$ such that*

$$\varepsilon^{-\frac{m}{2}} v(\varepsilon \cos t, \varepsilon \sin t) \to \beta \, f(t) \, \sin\left(\tfrac{m}{2}(t - \alpha_0)\right) \tag{7.52}$$

in $C^{1,\tau}([0, 2\pi] \setminus \{\alpha^j + \pi\}_{j=1}^{k_1}, \mathbb{R})$ as $\varepsilon \to 0^+$, for all $\tau \in (0, 1)$. Moreover, there exists a constant $C > 0$ such that

$$|v(x)| \le C|x|^{\frac{m}{2}} \quad \text{and} \quad |\nabla v(x)| \le C|x|^{\frac{m}{2}-1} \quad \text{for all } x \in \Omega \setminus \Gamma_0. \tag{7.53}$$

Furthermore, letting

$$\Psi(x) = \Psi(r\cos t, r\sin t) = \beta\, r^{\frac{m}{2}}\, f(t)\, \sin\left(\tfrac{m}{2}(t-\alpha_0)\right),$$

with m, β, and α_0 as in (7.52) and f as in (7.22), we have that, as $\varepsilon \to 0^+$,

$$\varepsilon^{-\frac{m}{2}} v(\varepsilon \cdot) \to \Psi \quad \textit{in } H^1(D_\rho \setminus \Gamma_0) \textit{ for all } \rho > 0. \tag{7.54}$$

Proof. As observed above, the function $u := e^{i\Theta_0} v$ is an eigenfunction of (1.10) with k odd, i.e.

$$\begin{cases} (i\nabla + A_0)^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with A_0 defined in (1.7). From [66, Theorem 1.3, Section 7] it follows that there exist an odd $m \in \mathbb{N}$ and $\beta_1, \beta_2 \in \mathbb{C}$ such that $(\beta_1, \beta_2) \neq (0,0)$ and, as $\varepsilon \to 0^+$,

$$\varepsilon^{-\frac{m}{2}} u(\varepsilon\cos t, \varepsilon\sin t) \to e^{\frac{i}{2}t}\left(\beta_1 \cos\left(\tfrac{m}{2}t\right) + \beta_2 \sin\left(\tfrac{m}{2}t\right)\right) \quad \text{in } C^{1,\tau}([0,2\pi],\mathbb{C}) \tag{7.55}$$

and

$$\begin{aligned} \varepsilon^{1-\frac{m}{2}} \nabla u(\varepsilon\cos t, \varepsilon\sin t) \to{}& \tfrac{m}{2} e^{\frac{i}{2}t}\left(\beta_1 \cos\left(\tfrac{m}{2}t\right) + \beta_2 \sin\left(\tfrac{m}{2}t\right)\right) \boldsymbol{\theta}(t) \\ &+ \tfrac{d}{dt}\left(e^{\frac{i}{2}t}\left(\beta_1 \cos\left(\tfrac{m}{2}t\right) + \beta_2 \sin\left(\tfrac{m}{2}t\right)\right)\right) \boldsymbol{\tau}(t) \quad \text{in } C^{0,\tau}([0,2\pi],\mathbb{C}) \end{aligned} \tag{7.56}$$

for all $\tau \in (0,1)$, where $\boldsymbol{\theta}(t) = (\cos t, \sin t)$ and $\boldsymbol{\tau}(t) = (-\sin t, \cos t)$. Furthermore, by (7.49), for all $t \in [0,2\pi]$ we have

$$\sum_{j=1}^{k_1} (-1)^{j+1} \theta_0^j(\varepsilon\cos t, \varepsilon\sin t) = t + \sum_{j=1}^{k_1} (-1)^j \alpha^j + \pi - 2\pi \sum_{j=1}^{k_1} (-1)^{j+1} \chi_{[\alpha^j+\pi, 2\pi)}(t). \tag{7.57}$$

From (7.55), the definition of u, and (7.57) it follows that

$$\varepsilon^{-\frac{m}{2}} v(\varepsilon\cos t, \varepsilon\sin t) \to f(t) e^{-\frac{i}{2}\left(\pi + \sum_{j=1}^{k_1} (-1)^j \alpha^j\right)} \Big(\beta_1 \cos\left(\tfrac{m}{2}t\right) + \beta_2 \sin\left(\tfrac{m}{2}t\right)\Big)$$

in $C^{1,\tau}([0,2\pi] \setminus \{\alpha^j + \pi\}_{j=1}^{k_1}, \mathbb{C})$ as $\varepsilon \to 0^+$, for all $\tau \in (0,1)$. Then, since v is real-valued (see Remark 7.2.5), we have proved that there exist $c_1, c_2 \in \mathbb{R}$ such that $(c_1, c_2) \neq (0,0)$ and

$$\varepsilon^{-\frac{m}{2}} v(\varepsilon\cos t, \varepsilon\sin t) \to f(t)\Big(c_1 \cos\left(\tfrac{m}{2}t\right) + c_2 \sin\left(\tfrac{m}{2}t\right)\Big). \tag{7.58}$$

Letting

$$\alpha_0 = \begin{cases} \frac{2}{m} \operatorname{arccot}\left(-\frac{c_2}{c_1}\right), & \text{if } c_1 \neq 0, \\ 0, & \text{if } c_1 = 0, \end{cases}$$

we can rewrite (7.58) as (7.52). Estimate (7.53) is a consequence of (7.55) and (7.56).

Finally, to prove (7.54), we define

$$\tilde{u}_\varepsilon(x) := \varepsilon^{-\frac{m}{2}} u(\varepsilon x), \qquad \Phi(x) = \Phi(r\cos t, r\sin t) = r^{\frac{m}{2}} e^{\frac{i}{2}t}\left(\beta_1 \cos\left(\tfrac{m}{2}t\right) + \beta_2 \sin\left(\tfrac{m}{2}t\right)\right).$$

We observe that (7.55), (7.56), and the Dominated Convergence Theorem imply that

$$\nabla \tilde{u}_\varepsilon \to \nabla\Phi \quad \text{and} \quad \frac{\tilde{u}_\varepsilon}{|x|} \to \frac{\Phi}{|x|} \quad \text{in } L^2(D_\rho) \quad \text{for all } \rho > 0,$$

which easily provides (7.54). ∎

In the case k even, solutions to (7.14) are more regular.

Proposition 7.2.7. *Let k_1 be even. If v is a non-trivial solution to (7.14), then there exist $m \in \mathbb{N}$, $\beta \in \mathbb{R} \setminus \{0\}$, and $\alpha_0 \in [0, \frac{\pi}{m})$ such that*

$$\varepsilon^{-m} v(\varepsilon \cos t, \varepsilon \sin t) \to \beta f(t) \sin(m(t - \alpha_0)) \tag{7.59}$$

in $C^{1,\tau}([0, 2\pi] \setminus \{\pi + \alpha^j\}_{j=1}^{k_1}, \mathbb{R})$ as $\varepsilon \to 0^+$, for all $\tau \in (0, 1)$. Moreover, there exists a constant $C > 0$ such that

$$|v(x)| \leq C|x|^m \quad \text{and} \quad |\nabla v(x)| \leq \begin{cases} C|x|^{m-1}, & \text{if } m \geq 1, \\ C, & \text{if } m = 0, \end{cases} \tag{7.60}$$

for all $x \in \Omega \setminus \Gamma_0$.

Proof. The function $u := e^{i\Theta_0} v$ is an eigenfunction of (1.10) with k even, i.e. u is an eigenfunction of the Dirichlet Laplacian. From (7.49) we deduce the analogue of (7.57) in the even case:

$$\sum_{j=1}^{k_1} (-1)^{j+1} \theta_0^j(\varepsilon \cos t, \varepsilon \sin t) = \sum_{j=1}^{k_1} (-1)^j \alpha^j - 2\pi \sum_{j=1}^{k_1} (-1)^{j+1} \chi_{[\alpha^j + \pi, 2\pi)}(t) \tag{7.61}$$

for all $t \in [0, 2\pi]$. Claims (7.59) and (7.60) follow from the fact that u is analytic, the definition of u and (7.61), observing that, since k_1 is even, $|\nabla u| = |\nabla v|$. □

Remark 7.2.8. For the sake of simplicity, for any $w \in \mathcal{H}_0$ we simply write w instead of $\gamma_+^j(w)$ on S_ε^j, since $\gamma_+^j(w) = \gamma_-^j(w)$ on S_ε^j for any $j = 1, \ldots, k_1 + k_2$. We also simply write v_0, ∇v_0 and $\nabla v_0 \cdot \nu^j$ when considering their traces on S_ε^j.

7.3 Definition and properties of $\mathcal{E}_\varepsilon$

For some $n_0 \in \mathbb{N} \setminus \{0\}$, let u_0 be an eigenfunction of (1.10) associated to the eigenvalue $\lambda_0 = \lambda_{0,n_0}$ and v_0 be as in (7.50), so that v_0 is a non-zero weak solution of (7.14) with $\lambda = \lambda_0$. By Remark 7.2.5 it is not restrictive to assume that v_0 is real-valued and $\|u_0\|_{L^2(\Omega,\mathbb{C})} = \|v_0\|_{L^2(\Omega)} = 1$.

Let L_ε be the functional introduced in (7.16). We observe that L_ε is well-defined; indeed, for every $j = 1, \ldots, k_1 + k_2$, we have $\nabla v_0 \in L^p(S_\varepsilon^j)$ for all $p \in [1, 2)$ in view of (7.53) and (7.60), whereas $\gamma_+^j(w) \in L^q(S_\varepsilon^j)$ for all $w \in \mathcal{H}_1$ and $q \in [2, +\infty)$ by (7.3). We provide below an estimate of the norm of L_ε in $\mathcal{H}_1^*$, where $\mathcal{H}_1^*$ is the dual space of $\mathcal{H}_1$.

Proposition 7.3.1. *Let $m \in \mathbb{N}$ be as in Proposition 7.2.6 for $v = v_0$, if k is odd, or as in Proposition 7.2.7, if k is even. For every $\varepsilon \in (0, 1]$, the map L_ε defined in (7.16) belongs to $\mathcal{H}_1^*$ and, as $\varepsilon \to 0^+$,*

$$\|L_\varepsilon\|_{\mathcal{H}_1^*} = \begin{cases} O(\varepsilon^{\frac{m}{2} - 1 + \frac{1}{p}}), & \text{if } k \text{ is odd}, \\ O(\varepsilon^{\frac{1}{p}}), & \text{if } k \text{ is even and } m = 0, \\ O(\varepsilon^{m - 1 + \frac{1}{p}}), & \text{if } k \text{ is even and } m > 0, \end{cases} \tag{7.62}$$

for every $p \in (1, 2)$. In particular, $L_\varepsilon \to 0$ in $\mathcal{H}_1^*$ as $\varepsilon \to 0^+$.

Proof. If k is odd, for every $p \in (1,2)$ and $w \in \mathcal{H}_1$, from the Hölder inequality, (7.3) and (7.53) it follows that, letting $p' = \frac{p}{p-1}$,

$$|L_\varepsilon(w)| \leq 2 \sum_{j=1}^{k_1+k_2} \|\nabla v_0\|_{L^p(S_\varepsilon^j)} \|\gamma_+^j(w)\|_{L^{p'}(S_\varepsilon^j)} \leq C\varepsilon^{\frac{m}{2}-1+\frac{1}{p}} \|w\|_{\mathcal{H}_1},$$

for some constant $C > 0$ independent of ε. If k is even, the proof is similar due to (7.60). $\square$

For every $\varepsilon \in (0,1]$, we now consider the functional J_ε defined in (7.17) and, recalling the definition of $\widetilde{\mathcal{H}}_\varepsilon$ in (7.5), the minimization problem

$$\inf \left\{ J_\varepsilon(w) : w \in \mathcal{H}_\varepsilon \text{ and } w - v_0 \in \widetilde{\mathcal{H}}_\varepsilon \right\}. \tag{7.63}$$

Note that, since $v_0 \in \widetilde{\mathcal{H}}_0$, the condition $w - v_0 \in \widetilde{\mathcal{H}}_\varepsilon$ is equivalent to

$$T^j(w) = \begin{cases} 0, & \text{on } \Gamma_0^j \text{ for all } j = 1, \dots, k_1, \\ 2v_0, & \text{on } S_\varepsilon^j \text{ for all } j = 1, \dots, k_1 + k_2. \end{cases} \tag{7.64}$$

Proposition 7.3.2. *The infimum in* (7.63) *is achieved by a unique* $V_\varepsilon \in \mathcal{H}_\varepsilon$*. Furthermore,* V_ε *weakly solves the problem*

$$\begin{cases} -\Delta V_\varepsilon = 0, & \text{in } \Omega \setminus \Gamma_\varepsilon, \\ V_\varepsilon = 0, & \text{on } \partial\Omega, \\ T^j(V_\varepsilon - v_0) = 0, & \text{on } \Gamma_\varepsilon^j \quad \text{for all } j = 1, \dots, k_1, \\ T^j(\nabla V_\varepsilon \cdot \nu^j - \nabla v_0 \cdot \nu^j) = 0, & \text{on } \Gamma_\varepsilon^j \quad \text{for all } j = 1, \dots, k_1, \\ T^j(V_\varepsilon - v_0) = 0, & \text{on } S_\varepsilon^j \quad \text{for all } j = k_1 + 1, \dots, k_1 + k_2, \\ T^j(\nabla V_\varepsilon \cdot \nu^j - \nabla v_0 \cdot \nu^j) = 0, & \text{on } S_\varepsilon^j \quad \text{for all } j = k_1 + 1, \dots, k_1 + k_2, \end{cases} \tag{7.65}$$

in the sense that $V_\varepsilon \in \mathcal{H}_\varepsilon$, $V_\varepsilon - v_0 \in \widetilde{\mathcal{H}}_\varepsilon$, *and*

$$\int_{\Omega\setminus\Gamma_\varepsilon} \nabla V_\varepsilon \cdot \nabla w \, dx = -L_\varepsilon(w) \quad \text{for all } w \in \widetilde{\mathcal{H}}_\varepsilon. \tag{7.66}$$

Proof. In view of (7.17), the continuity of the linear operator L_ε, and Proposition 7.2.2, we can easily verify that J_ε is continuous and coercive on the set $v_0 + \widetilde{\mathcal{H}}_\varepsilon = \{w \in \mathcal{H}_\varepsilon : w - v_0 \in \widetilde{\mathcal{H}}_\varepsilon\}$, which is closed and convex. Moreover, J_ε is convex. Therefore, the infimum in (7.63) is achieved by some V_ε, which weakly solves (7.65) in the sense of (7.66). If $V_{\varepsilon,1}, V_{\varepsilon,2} \in v_0 + \widetilde{\mathcal{H}}_\varepsilon$ are weak solutions of (7.65), then $V_{\varepsilon,1} - V_{\varepsilon,2} \in \widetilde{\mathcal{H}}_\varepsilon$ and

$$\int_{\Omega\setminus\Gamma_\varepsilon} (\nabla V_{\varepsilon,1} - \nabla V_{\varepsilon,2}) \cdot \nabla w \, dx = 0 \quad \text{for all } w \in \widetilde{\mathcal{H}}_\varepsilon. \tag{7.67}$$

Testing (7.67) with $w = V_{\varepsilon,1} - V_{\varepsilon,2}$ we obtain $\nabla(V_{\varepsilon,1} - V_{\varepsilon,2}) = 0$ and hence, by Proposition 7.2.2, we conclude that $V_{\varepsilon,1} = V_{\varepsilon,2}$. $\square$

For every $\varepsilon \in (0,1]$, let J_ε and V_ε be as (7.17) and Proposition 7.3.2, respectively. We consider the quantity $\mathcal{E}_\varepsilon := J_\varepsilon(V_\varepsilon)$ as in (7.19). $\mathcal{E}_\varepsilon$ plays a significant role in the asymptotic expansion of the eigenvalue variation $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0}$, as the poles a_ε^j move towards the collision at 0.

To derive a first upper and lower bound for $\mathcal{E}_\varepsilon$, we consider, for every $r > 0$, the radial cut-off function $\eta_r \in C^\infty_{\rm c}(\mathbb{R}^2)$ defined as

$$\eta_r(x) := \eta\left(\frac{x}{r}\right) \tag{7.68}$$

with η as in (7.29).

Proposition 7.3.3. *Let $m \in \mathbb{N}$ be as in Proposition 7.2.6 for $v = v_0$, if k is odd, or as in Proposition 7.2.7, if k is even. Then there exists a constant $C_1 > 0$ such that, for all $\varepsilon \in (0,1]$,*

$$\mathcal{E}_\varepsilon \le \begin{cases} C_1\,\varepsilon^m, & \text{if } k \text{ is odd},\\ C_1\,\frac{1}{|\log\varepsilon|}, & \text{if } k \text{ is even and } m = 0,\\ C_1\,\varepsilon^{2m}, & \text{if } k \text{ is even and } m > 0. \end{cases} \tag{7.69}$$

Moreover, for every $p \in (1,2)$ there exists $C_2 = C_2(p) > 0$ such that

$$\mathcal{E}_\varepsilon \ge \begin{cases} -C_2\,\varepsilon^{m-2+\frac{2}{p}}, & \text{if } k \text{ is odd},\\ -C_2\,\varepsilon^{\frac{2}{p}}, & \text{if } k \text{ is even and } m = 0,\\ -C_2\,\varepsilon^{2m-2+\frac{2}{p}}, & \text{if } k \text{ is even and } m > 0. \end{cases} \tag{7.70}$$

In particular, $\mathcal{E}_\varepsilon \to 0$ as $\varepsilon \to 0^+$.

Proof. If k is odd, let $\eta_\varepsilon \in C^\infty_{\rm c}(\mathbb{R}^2)$ be a cut-off function as in (7.68) with $r = \varepsilon$. From (7.16), (7.17), (7.63), (7.19), and (7.53) it follows that

$$\begin{aligned} J_\varepsilon(V_\varepsilon) \le J_\varepsilon(\eta_\varepsilon v_0) &\le \frac{1}{2}\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla(\eta_\varepsilon v_0)|^2\,dx + 2\sum_{j=1}^{k_1+k_2}\int_{S^j_\varepsilon} |\nabla v_0|\,|v_0|\;dS \\ &\le \int_{(\Omega\cap D_{2\varepsilon}(0))\setminus\Gamma_\varepsilon} |\nabla v_0|^2\,dx + \int_{\Omega\cap D_{2\varepsilon}(0)} |\nabla\eta_\varepsilon|^2 v_0^2\,dx + 2\sum_{j=1}^{k_1+k_2}\int_{S^j_\varepsilon} |\nabla v_0|\,|v_0|\;dS \le C_1\varepsilon^m \end{aligned}$$

for some constant $C_1 > 0$ independent of ε. If k is even and $m \in \mathbb{N}\setminus\{0\}$, (7.69) can be proved arguing in a similar way and using (7.60) instead of (7.53).

If k is even and $m = 0$, for every $\varepsilon \in (0,1]$ we consider the cut-off function $\omega_\varepsilon \in W^{1,\infty}(\mathbb{R}^2)$ defined in (7.45). We have $0 \le \omega_\varepsilon \le 1$ and, thanks to (7.16), (7.17), (7.63), (7.19), and (7.60) with $m = 0$,

$$\begin{aligned} J_\varepsilon(V_\varepsilon) \le J_\varepsilon(\omega_\varepsilon v_0) &\le \frac{1}{2}\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla(\omega_\varepsilon v_0)|^2\,dx + 2\sum_{j=1}^{k_1+k_2}\int_{S^j_\varepsilon} |\nabla v_0|\,|v_0|\;dS \\ &\le \int_{(\Omega\cap D_{\sqrt{\varepsilon}}(0))\setminus\Gamma_\varepsilon} |\nabla v_0|^2\,dx + \int_{\Omega\cap D_{\sqrt{\varepsilon}}(0)} |\nabla\omega_\varepsilon|^2 v_0^2\,dx + 2\sum_{j=1}^{k_1+k_2}\int_{S^j_\varepsilon} |\nabla v_0|\,|v_0|\;dS \le C_1\frac{1}{|\log\varepsilon|} \end{aligned}$$

for some constant $C_1 > 0$ independent of ε. Estimate (7.69) is thereby proved.

To prove (7.70), we observe that

$$\begin{aligned} \|V_\varepsilon\|^2_{\mathcal{H}_1} = \|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon} &= 2\mathcal{E}_\varepsilon - 2L_\varepsilon(V_\varepsilon) \le 2\mathcal{E}_\varepsilon + 2|L_\varepsilon(V_\varepsilon)| \\ &\le 2\mathcal{E}_\varepsilon + 2\,\|L_\varepsilon\|_{\mathcal{H}_1^*}\,\|V_\varepsilon\|_{\mathcal{H}_1} \le 2\mathcal{E}_\varepsilon + 2\,\|L_\varepsilon\|^2_{\mathcal{H}_1^*} + \frac{1}{2}\,\|V_\varepsilon\|^2_{\mathcal{H}_1}, \end{aligned}$$

and hence

$$\mathcal{E}_\varepsilon + \|L_\varepsilon\|^2_{\mathcal{H}_1^*} \geq \frac{1}{4}\|V_\varepsilon\|^2_{\mathcal{H}_1} \geq 0, \tag{7.71}$$

which, together with (7.62), implies (7.70). □

Proposition 7.3.4. *We have $V_\varepsilon \to 0$ as $\varepsilon \to 0^+$ strongly in $\mathcal{H}_1$.*

Proof. From Proposition 7.3.3 we have $\lim_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon = 0$, whereas Proposition 7.3.1 implies that $\lim_{\varepsilon\to 0^+} \|L_\varepsilon\|_{\mathcal{H}_1^*} = 0$. The conclusion then follows from (7.71). □

Proposition 7.3.5. *We have $\mathcal{E}_\varepsilon = o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right)$ as $\varepsilon \to 0^+$.*

Proof. Proceeding similarly to the previous proof, we have

$$|\mathcal{E}_\varepsilon| \leq \frac{\|V_\varepsilon\|^2_{\mathcal{H}_1}}{2} + \|L_\varepsilon\|_{\mathcal{H}_1^*}\|V_\varepsilon\|_{\mathcal{H}_1}$$

and we can conclude thanks to (7.62) and Proposition 7.3.4. □

Proposition 7.3.6. *We have*

$$\int_\Omega V_\varepsilon^2\,dx = o(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}) \quad \text{as } \varepsilon \to 0^+.$$

Proof. Let us assume by contradiction that there exist a positive constant $C > 0$ and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$\int_\Omega V_{\varepsilon_n}^2\,dx \geq C\int_{\Omega\setminus\Gamma_{\varepsilon_n}} |\nabla V_{\varepsilon_n}|^2\,dx \quad \text{for all } n \in \mathbb{N}. \tag{7.72}$$

For every $n \in \mathbb{N}$, we define $W_n := \frac{V_{\varepsilon_n}}{\|V_{\varepsilon_n}\|_{L^2(\Omega)}}$. Then $\|W_n\|_{L^2(\Omega)} = 1$ for every $n \in \mathbb{N}$ and $\{W_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_1$ thanks to (7.72). It follows that there exists $W \in \mathcal{H}_1$ such that $W_n \rightharpoonup W$ weakly in $\mathcal{H}_1$ as $n \to \infty$, up to a subsequence. Since $W_n \in \mathcal{H}_{\varepsilon_n}$ for every n, from Proposition 7.2.3 we deduce that $W \in \mathcal{H}_0$, while Remark 7.2.1 ensures that

$$\|W\|_{L^2(\Omega)} = 1. \tag{7.73}$$

Since $W_n - \|V_{\varepsilon_n}\|^{-1}_{L^2(\Omega)} v_0 \in \widetilde{\mathcal{H}}_{\varepsilon_n}$, we have $T^j(W_n) = 0$ on Γ_0^j for all $j = 1,\dots,k_1$, see (7.64). By continuity of the trace operator (7.4), we deduce that $T^j(W) = 0$ on Γ_0^j for all $j = 1,\dots,k_1$, hence $W \in \widetilde{\mathcal{H}}_0$.

Let $w \in \widetilde{\mathcal{H}}_{0,0}$, where $\widetilde{\mathcal{H}}_{0,0}$ is defined in Lemma 7.2.4. For n sufficiently large, $w \in \widetilde{\mathcal{H}}_{\varepsilon_n}$ and $L_{\varepsilon_n}(w) = 0$, hence we can test (7.66) with w, thus obtaining

$$\int_{\Omega\setminus\Gamma_1} \nabla W_n \cdot \nabla w\,dx = \int_{\Omega\setminus\Gamma_{\varepsilon_n}} \nabla W_n \cdot \nabla w\,dx = -L_{\varepsilon_n}(w) = 0.$$

Letting $n \to \infty$ in the above identity, we obtain $\int_{\Omega\setminus\Gamma_0} \nabla W \cdot \nabla w\,dx = 0$ for all $w \in \widetilde{\mathcal{H}}_{0,0}$ and hence, by the density of $\widetilde{\mathcal{H}}_{0,0}$ in $\widetilde{\mathcal{H}}_0$ established in Lemma 7.2.4,

$$\int_{\Omega\setminus\Gamma_0} \nabla W \cdot \nabla w\,dx = 0 \quad \text{for all } w \in \widetilde{\mathcal{H}}_0. \tag{7.74}$$

Choosing $w = W$ in (7.74), we conclude that $W = 0$, thus contradicting (7.73). □

7.4 Asymptotic expansion of the eigenvalue variation

For every $\varepsilon \in [0,1]$, we consider the bilinear form $q_\varepsilon : \widetilde{\mathcal{H}}_\varepsilon \times \widetilde{\mathcal{H}}_\varepsilon \to \mathbb{R}$ defined as

$$q_\varepsilon(w_1, w_2) := \int_{\Omega\setminus\Gamma_\varepsilon} \nabla w_1 \cdot \nabla w_2 \, dx, \tag{7.75}$$

where $\widetilde{\mathcal{H}}_\varepsilon$ is as in (7.5). To simplify notation, we denote by q_ε both the bilinear form defined above and the associated quadratic form

$$q_\varepsilon(w) = \int_{\Omega\setminus\Gamma_\varepsilon} |\nabla w|^2 \, dx = \|w\|^2_{\mathcal{H}_\varepsilon}, \quad w \in \widetilde{\mathcal{H}}_\varepsilon.$$

The following preliminary result can be obtained in a standard way from the compactness properties pointed out in Remark 7.2.1 and abstract spectral theory, see for example [84, Theorems 6.16 and 6.21, Proposition 8.20].

Proposition 7.4.1. *Let $\varepsilon \in [0,1]$ and $\mathcal{F}_\varepsilon : \widetilde{\mathcal{H}}_\varepsilon \to \widetilde{\mathcal{H}}_\varepsilon$ be the linear operator defined as*

$$q_\varepsilon(\mathcal{F}_\varepsilon(w_1), w_2) = (w_1, w_2)_{L^2(\Omega)}. \tag{7.76}$$

Then

(i) $\mathcal{F}_\varepsilon$ is symmetric, non-negative and compact; in particular 0 belongs to its spectrum $\sigma(\mathcal{F}_\varepsilon)$.

(ii) $\sigma(\mathcal{F}_\varepsilon) \setminus \{0\} = \{\mu_{n,\varepsilon}\}_{n\in\mathbb{N}\setminus\{0\}}$, where $\mu_{n,\varepsilon} := 1/\lambda_{\varepsilon,n}$ for every $n \in \mathbb{N}\setminus\{0\}$.

(iii) For every $\mu \in \mathbb{R}$ and $w \in \widetilde{\mathcal{H}}_\varepsilon$,

$$\big(\operatorname{dist}(\mu, \sigma(\mathcal{F}_\varepsilon))\big)^2 \le \frac{q_\varepsilon(\mathcal{F}_\varepsilon(w) - \mu w)}{q_\varepsilon(w)}.$$

Letting $n_0 \in \mathbb{N}\setminus\{0\}$, v_0 and $\lambda_0 = \lambda_{0,n_0}$ be as in Section 7.3, to prove an asymptotic expansion of the eigenvalue variation we further assume that

$$\lambda_0 \text{ is simple as an eigenvalue of (1.10)}, \tag{7.77}$$

and, consequently, as an eigenvalue of (7.14). Therefore, the continuity result of [97, Theorem 1.2], see (1.12), implies that also $\lambda_{\varepsilon,n_0}$ is simple for ε sufficiently small. From now on, we denote

$$\lambda_\varepsilon = \lambda_{\varepsilon,n_0}.$$

For ε small, let $v_\varepsilon \in \widetilde{\mathcal{H}}_\varepsilon$ be the unique eigenfunction of (7.10) associated to the eigenvalue $\lambda_\varepsilon = \lambda_{\varepsilon,n_0}$ satisfying

$$\int_\Omega v_\varepsilon^2 \, dx = 1 \quad \text{and} \quad \int_\Omega v_\varepsilon v_0 \, dx > 0. \tag{7.78}$$

We denote as Π_ε the projection onto the one-dimensional space spanned by v_ε, i.e.

$$\begin{aligned} \Pi_\varepsilon : L^2(\Omega) &\to \widetilde{\mathcal{H}}_\varepsilon, \\ w &\mapsto (w, v_\varepsilon)_{L^2(\Omega)} \, v_\varepsilon. \end{aligned} \tag{7.79}$$

Theorem 7.1.1 is contained in the following result, the proof of which is inspired by [3, Appendix A].

Theorem 7.4.2. *Under assumption (7.77), the following asymptotic expansion holds:*

$$\lambda_\varepsilon - \lambda_0 = 2\mathcal{E}_\varepsilon - 2L_\varepsilon(v_0) + o(L_\varepsilon(v_0)) + o(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}) \quad \text{as } \varepsilon \to 0^+, \tag{7.80}$$

where V_ε is as Proposition 7.3.2. Furthermore,

$$\|v_0 - V_\varepsilon - \Pi_\varepsilon(v_0 - V_\varepsilon)\|_{\mathcal{H}_\varepsilon} = o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+, \tag{7.81}$$

$$\|v_0 - \Pi_\varepsilon(v_0 - V_\varepsilon)\|_{L^2(\Omega)} = o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+, \tag{7.82}$$

$$\|\Pi_\varepsilon(v_0 - V_\varepsilon)\|^2_{L^2(\Omega)} = 1 + o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+. \tag{7.83}$$

Proof. Let $\psi_\varepsilon := v_0 - V_\varepsilon$. We recall that we are assuming that v_0 is real-valued and $\|v_0\|_{L^2(\Omega)} = 1$. From (7.65) and (7.14) it follows that $\psi_\varepsilon \in \widetilde{\mathcal{H}}_\varepsilon$ is a weak solution of the problem

$$\begin{cases} -\Delta\psi_\varepsilon = \lambda_0 v_0, & \text{in } \Omega \setminus \Gamma_\varepsilon, \\ \psi_\varepsilon = 0, & \text{on } \partial\Omega, \\ T^j(\psi_\varepsilon) = 0, & \text{on } \Gamma^j_\varepsilon \text{ for all } j = 1, \dots, k_1, \\ T^j(\nabla\psi_\varepsilon \cdot \nu^j) = 0, & \text{on } \Gamma^j_\varepsilon \text{ for all } j = 1, \dots, k_1, \\ T^j(\psi_\varepsilon) = 0, & \text{on } S^j_\varepsilon \text{ for all } j = k_1 + 1, \dots, k_1 + k_2, \\ T^j(\nabla\psi_\varepsilon \cdot \nu^j) = 0, & \text{on } S^j_\varepsilon \text{ for all } j = k_1 + 1, \dots, k_1 + k_2, \end{cases}$$

in the sense that, letting q_ε be as in (7.75),

$$q_\varepsilon(\psi_\varepsilon, w) = \lambda_0 \left(v_0, w\right)_{L^2(\Omega)} \quad \text{for all } w \in \widetilde{\mathcal{H}}_\varepsilon. \tag{7.84}$$

Let v_ε be an eigenfunction of (7.10) associated to λ_ε chosen as in (7.78). Let Π_ε be the projection operator onto the one-dimensional space spanned by v_ε defined in (7.79). Moreover, we define

$$\hat{v}_\varepsilon := \frac{\Pi_\varepsilon(\psi_\varepsilon)}{\|\Pi_\varepsilon(\psi_\varepsilon)\|_{L^2(\Omega)}}. \tag{7.85}$$

From (7.84) we deduce that

$$q_\varepsilon(\psi_\varepsilon, w) - \lambda_0 \left(\psi_\varepsilon, w\right)_{L^2(\Omega)} = \lambda_0 \left(V_\varepsilon, w\right)_{L^2(\Omega)} \quad \text{for all } w \in \widetilde{\mathcal{H}}_\varepsilon. \tag{7.86}$$

Choosing $w = \hat{v}_\varepsilon$ in (7.86), by (7.47) and (7.85) we obtain

$$(\lambda_\varepsilon - \lambda_0)\left(\psi_\varepsilon, \hat{v}_\varepsilon\right)_{L^2(\Omega)} = \lambda_0 \left(V_\varepsilon, v_0\right)_{L^2(\Omega)} + \lambda_0 \left(V_\varepsilon, \hat{v}_\varepsilon - v_0\right)_{L^2(\Omega)}. \tag{7.87}$$

We claim that

$$\lambda_0 \int_\Omega V_\varepsilon v_0 \, dx = 2\mathcal{E}_\varepsilon - 2L_\varepsilon(v_0). \tag{7.88}$$

Indeed, an integration by parts yields

$$\begin{aligned} &\int_{\Omega\setminus\Gamma_\varepsilon} \nabla v_0 \cdot \nabla V_\varepsilon \, dx - \lambda_0 \int_\Omega v_0 V_\varepsilon \, dx \qquad (7.89) \\ &= \sum_{j=1}^{k_1} \int_{\Gamma^j_0} \Big(-\gamma^j_+(V_\varepsilon)\gamma^j_+(\nabla v_0 \cdot \nu^j) + \gamma^j_-(V_\varepsilon)\gamma^j_-(\nabla v_0 \cdot \nu^j)\Big) \, dS \\ &\quad + \sum_{j=1}^{k_1+k_2} \int_{S^j_\varepsilon} \Big(-\gamma^j_+(V_\varepsilon)\nabla v_0 \cdot \nu^j + \gamma^j_-(V_\varepsilon)\nabla v_0 \cdot \nu^j\Big) \, dS \\ &= -2\sum_{j=1}^{k_1+k_2} \int_{S^j_\varepsilon} \gamma^j_+(V_\varepsilon)\nabla v_0 \cdot \nu^j \, dS + 2\sum_{j=1}^{k_1+k_2} \int_{S^j_\varepsilon} v_0 \nabla v_0 \cdot \nu^j \, dS, \end{aligned}$$

thanks to (7.65). Testing (7.84) with $V_\varepsilon - v_0$ we obtain

$$\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla(V_\varepsilon - v_0)|^2 = -\lambda_0 \int_\Omega v_0(V_\varepsilon - v_0)\, dx,$$

and hence, in view of (7.51),

$$\int_{\Omega\setminus\Gamma_\varepsilon} \nabla V_\varepsilon \cdot \nabla v_0 \, dx = \frac{1}{2}\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla V_\varepsilon|^2 \, dx + \frac{\lambda_0}{2}\int_\Omega v_0 V_\varepsilon \, dx. \tag{7.90}$$

Combining (7.16), (7.17), (7.19), (7.89) and (7.90), we derive (7.88).

From (7.87) and (7.88) we deduce that, for all $\varepsilon \in (0,1)$,

$$(\lambda_\varepsilon - \lambda_0)\,(\psi_\varepsilon, \hat{v}_\varepsilon)_{L^2(\Omega)} = 2\mathcal{E}_\varepsilon - 2L_\varepsilon(v_0) + \lambda_0\,(V_\varepsilon, \hat{v}_\varepsilon - v_0)_{L^2(\Omega)}\,. \tag{7.91}$$

Now we study the asymptotics, as $\varepsilon \to 0^+$, of each term in (7.91). For the sake of clarity, we divide the rest of the proof into several steps.

Step 1. We claim that

$$|\lambda_\varepsilon - \lambda_0| = o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+. \tag{7.92}$$

Letting $\mu_0 := \lambda_0^{-1}$ and $\mu_\varepsilon := \lambda_\varepsilon^{-1}$, since λ_0 is simple and $\lambda_\varepsilon \to \lambda_0$ by (1.12), we have

$$|\lambda_\varepsilon - \lambda_0| = \lambda_\varepsilon\lambda_0|\mu_\varepsilon - \mu_0| \le 2\lambda_0^2 \,\mathrm{dist}(\mu_0, \sigma(\mathcal{F}_\varepsilon)) \le 2\lambda_0^2 \left(\frac{q_\varepsilon(\mathcal{F}_\varepsilon(\psi_\varepsilon) - \mu_0\psi_\varepsilon)}{q_\varepsilon(\psi_\varepsilon)}\right)^{1/2}, \tag{7.93}$$

where the last inequality is justified by Proposition 7.4.1. Since $\|v_0\|_{L^2(\Omega)} = 1$, Proposition 7.3.4 and the Cauchy-Schwarz inequality imply that

$$q_\varepsilon(\psi_\varepsilon) = \lambda_0 + \int_{\Omega\setminus\Gamma_\varepsilon} |\nabla V_\varepsilon|^2 \, dx - 2\int_{\Omega\setminus\Gamma_\varepsilon} \nabla V_\varepsilon \cdot \nabla v_0 \, dx = \lambda_0 + o(1). \tag{7.94}$$

Furthermore, in view of (7.76) and (7.84) tested with $\mathcal{F}_\varepsilon(\psi_\varepsilon) - \mu_0\psi_\varepsilon$,

$$\begin{aligned} q_\varepsilon(\mathcal{F}_\varepsilon(\psi_\varepsilon) - \mu_0\psi_\varepsilon) &= -\,(V_\varepsilon, \mathcal{F}_\varepsilon(\psi_\varepsilon) - \mu_0\psi_\varepsilon)_{L^2(\Omega)} + (v_0, \mathcal{F}_\varepsilon(\psi_\varepsilon) - \mu_0\psi_\varepsilon)_{L^2(\Omega)} \\ &\quad - q_\varepsilon(\mu_0\psi_\varepsilon, \mathcal{F}_\varepsilon(\psi_\varepsilon) - \mu_0\psi_\varepsilon) = -\,(V_\varepsilon, \mathcal{F}_\varepsilon(\psi_\varepsilon) - \mu_0\psi_\varepsilon)_{L^2(\Omega)}\,. \end{aligned}$$

Hence, by Proposition 7.2.2, Proposition 7.3.6 and the Cauchy-Schwarz inequality we conclude that

$$(q_\varepsilon(\mathcal{F}_\varepsilon(\psi_\varepsilon) - \mu_0\psi_\varepsilon))^{1/2} = o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+. \tag{7.95}$$

Claim (7.92) is proved by combining (7.93), (7.94), and (7.95).

Step 2. We claim that

$$q_\varepsilon(\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon) = o(\,\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}\,) \quad \text{as } \varepsilon \to 0^+. \tag{7.96}$$

Let

$$\chi_\varepsilon := \psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon \quad \text{and} \quad \xi_\varepsilon := \mathcal{F}_\varepsilon(\chi_\varepsilon) - \mu_\varepsilon\chi_\varepsilon. \tag{7.97}$$

By definition we have

$$\chi_\varepsilon \in N_\varepsilon := \{w \in \widetilde{\mathcal{H}}_\varepsilon : (w, v_\varepsilon)_{L^2(\Omega)} = 0\}$$

and, since v_ε is an eigenfunction of (7.10), from (7.76) it follows that $\mathcal{F}_\varepsilon(w) \in N_\varepsilon$ for all $w \in N_\varepsilon$. Hence the operator

$$\widetilde{\mathcal{F}}_\varepsilon := \mathcal{F}_\varepsilon\Big|_{N_\varepsilon} : N_\varepsilon \to N_\varepsilon$$

is well-defined. Furthermore, it is easy to verify that $\widetilde{\mathcal{F}}_\varepsilon$ satisfies properties (i)-(iii) of Proposition 7.4.1 and $\sigma(\widetilde{\mathcal{F}}_\varepsilon) = \sigma(\mathcal{F}_\varepsilon) \setminus \{\mu_\varepsilon\}$. In particular, there exists a constant $K > 0$, which does not depends on ε, such that $\left(\operatorname{dist}(\mu_\varepsilon, \sigma(\widetilde{F}_\varepsilon))\right)^2 \geq K$. Then, by (7.97),

$$q_\varepsilon(\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon) = q(\chi_\varepsilon) \leq \frac{1}{K}\left(\operatorname{dist}(\mu_\varepsilon, \sigma(\widetilde{\mathcal{F}}_\varepsilon))\right)^2 q_\varepsilon(\chi_\varepsilon) \leq \frac{1}{K}\, q_\varepsilon(\widetilde{\mathcal{F}}_\varepsilon(\chi_\varepsilon) - \mu_\varepsilon\chi_\varepsilon)$$
$$= \frac{1}{K}\, q_\varepsilon(\xi_\varepsilon). \quad (7.98)$$

To estimate $q_\varepsilon(\xi_\varepsilon)$ we use (7.86) and (7.47) tested with ξ_ε, thus obtaining

$$q_\varepsilon(\chi_\varepsilon, \xi_\varepsilon) - \lambda_\varepsilon\,(\chi_\varepsilon, \xi_\varepsilon)_{L^2(\Omega)} = \lambda_0\,(V_\varepsilon, \xi_\varepsilon)_{L^2(\Omega)} + (\lambda_0 - \lambda_\varepsilon)\,(\psi_\varepsilon, \xi_\varepsilon)_{L^2(\Omega)}\,. \quad (7.99)$$

From (7.76) and (7.99) we deduce that

$$q_\varepsilon(\xi_\varepsilon) = q_\varepsilon(\mathcal{F}_\varepsilon(\chi_\varepsilon), \xi_\varepsilon) - \mu_\varepsilon q_\varepsilon(\chi_\varepsilon, \xi_\varepsilon) = -\mu_\varepsilon[q_\varepsilon(\chi_\varepsilon, \xi_\varepsilon) - \lambda_\varepsilon q_\varepsilon(\mathcal{F}_\varepsilon(\chi_\varepsilon), \xi_\varepsilon)]$$
$$= -\frac{\lambda_0}{\lambda_\varepsilon}\,(V_\varepsilon, \xi_\varepsilon)_{L^2(\Omega)} - \frac{(\lambda_0 - \lambda_\varepsilon)}{\lambda_\varepsilon}\,(\psi_\varepsilon, \xi_\varepsilon)_{L^2(\Omega)}\,.$$

From the Cauchy-Schwarz inequality, Proposition 7.2.2, and (1.12) it follows that

$$(q_\varepsilon(\xi_\varepsilon))^{1/2} \leq C\left(\|V_\varepsilon\|_{L^2(\Omega)} + |\lambda_\varepsilon - \lambda_0|\,\|\psi_\varepsilon\|_{L^2(\Omega)}\right) \quad (7.100)$$

for some constant $C > 0$ which does not depend on ε. Furthermore, (7.86) tested with ψ_ε, (7.94), Proposition 7.2.2, and Proposition 7.3.4 yield

$$\|\psi_\varepsilon\|^2_{L^2(\Omega)} - 1 = -\,(V_\varepsilon, \psi_\varepsilon)_{L^2(\Omega)} + o(1) = o(1) \quad \text{as } \varepsilon \to 0^+.$$

Then (7.96) follows from Proposition 7.3.6, (7.92), (7.98), and (7.100). Estimate (7.81) is thereby proved.

Step 3. We claim that

$$\|v_0 - \hat{v}_\varepsilon\|_{L^2(\Omega)} = o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+. \quad (7.101)$$

By (7.85)

$$v_0 - \hat{v}_\varepsilon = v_0 - \frac{\Pi_\varepsilon\psi_\varepsilon}{\|\Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)}} = \frac{1}{\|\Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)}}\left(\left(\|\Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)} - 1\right)v_0 + v_0 - \Pi_\varepsilon\psi_\varepsilon\right). \quad (7.102)$$

Furthermore, from the definition of ψ_ε, Proposition 7.3.6, Proposition 7.2.2 and (7.96) it follows that

$$\|v_0 - \Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)} \leq \|v_0 - \psi_\varepsilon\|_{L^2(\Omega)} + \|\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)} = o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+, \quad (7.103)$$

thus proving (7.82). Since $\|v_0\|_{L^2(\Omega)} = 1$, (7.103) and the Cauchy-Schwarz inequality imply that

$$\|\Pi_\varepsilon\psi_\varepsilon\|^2_{L^2(\Omega)} = \|v_0 - \Pi_\varepsilon\psi_\varepsilon\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega)} - 2\left(v_0 - \Pi_\varepsilon\psi_\varepsilon, v_0\right)_{L^2(\Omega)}$$
$$= 1 + o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \quad (7.104)$$

as $\varepsilon \to 0^+$, thus proving estimate (7.83). Combining (7.102), (7.103) and (7.104) we obtain (7.101).

Step 4. We claim that

$$(\psi_\varepsilon, \hat{v}_\varepsilon)_{L^2(\Omega)} = 1 + o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \qquad \text{as } \varepsilon \to 0^+. \tag{7.105}$$

Indeed, by (7.85) we have

$$(\psi_\varepsilon, \hat{v}_\varepsilon)_{L^2(\Omega)} = \frac{(\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon, \Pi_\varepsilon\psi_\varepsilon)_{L^2(\Omega)} + \|\Pi_\varepsilon\psi_\varepsilon\|^2_{L^2(\Omega)}}{\|\Pi_\varepsilon\psi_\varepsilon\|_{L^2(\Omega)}}.$$

Hence claim (7.105) follows from (7.96) and (7.104).

Putting together Proposition 7.3.6, (7.88), (7.91), (7.101), and (7.105), we finally obtain

$$\begin{aligned}\lambda_\varepsilon - \lambda_0 &= (1 + o(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}))\left(2\mathcal{E}_\varepsilon - 2L_\varepsilon(v_0) + o(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon})\right)\\ &= 2\mathcal{E}_\varepsilon - 2L_\varepsilon(v_0) + o(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}) \quad \text{as } \varepsilon \to 0^+,\end{aligned}$$

thus proving (7.80). □

7.5 Blow-up Analysis for k odd

In this section we assume that k, and consequently k_1, are odd and we perform a blow-up analysis for the solution V_ε of problem (7.65). In order to characterize the functional space containing the limit profile, we first need a Hardy-type inequality, for the validity of which the assumption that k is odd is crucial.

7.5.1 A Hardy type inequality for functions jumping on an odd number of lines.

Let $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{H}}$ be the functional spaces defined in (7.24) and (7.25), respectively. To prove a Hardy-type inequality in $\mathbb{R}^2 \setminus D_1$ for functions in $\widetilde{\mathcal{X}}$, we first need the following Hardy inequality on annuli for functions jumping on an odd number of lines. For every $r > 0$, we define

$$\widetilde{\mathcal{X}}_r := \{w \in H^1((D_{2r} \setminus B_r) \setminus \Gamma_0) : T^j(w) = 0 \text{ on } \Gamma_0^j \text{ for all } j = 1, \dots, k_1\}.$$

Lemma 7.5.1. *Let k and k_1 be odd. There exists a constant $C_H > 0$ such that, for every $r > 0$ and $w \in \widetilde{\mathcal{X}}_r$,*

$$r^{-2}\int_{D_{2r}\setminus B_r} w^2\,dx \le C_H \int_{(D_{2r}\setminus B_r)\setminus\Gamma_0} |\nabla w|^2\,dx. \tag{7.106}$$

and

$$\int_{D_{2r}\setminus B_r} \frac{w^2}{|x|^2}\,dx \le C_H \int_{(D_{2r}\setminus B_r)\setminus\Gamma_0} |\nabla w|^2\,dx. \tag{7.107}$$

Proof. Inequality (7.107) is a direct consequence of (7.106).

Let us first prove (7.106) for $r = 1$. We argue by contradiction and assume that there exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{X}}_1$ such that, for all $n \in \mathbb{N}$,

$$\int_{D_2\setminus D_1} w_n^2\,dx = 1 \quad \text{and} \quad \int_{(D_2\setminus D_1)\setminus\Gamma_0)} |\nabla w_n|^2\,dx < \frac{1}{n}. \tag{7.108}$$

Hence $\{w_n\}_{n\in\mathbb{N}}$ is bounded in $\widetilde{\mathcal{X}}_1$ and, up to a subsequence, $w_n \rightharpoonup w$ weakly in $\widetilde{\mathcal{X}}_1$ for some $w \in \widetilde{\mathcal{X}}_1$. From (7.108) and weak lower semi-continuity of the L^2-norm, we have $\nabla w \equiv 0$ in $(D_2 \setminus D_1) \setminus \Gamma_0$; furthermore, reasoning as in Remark 7.2.1, the natural embedding of $H^1((D_2\setminus D_1)\setminus\Gamma_0))$ into $L^2(D_2\setminus D_1)$ is compact, hence $\|w\|_{L^2(D_2\setminus D_1)} = 1$. It follows that w is constant on each connected component of $(D_2\setminus D_1)\setminus\Gamma_0$ and $w \not\equiv 0$. Since $(D_2\setminus D_1)\setminus\Gamma_0$ has k_1 connected components and k_1 is odd, a contradiction arises from the condition $T^j(w) = 0$, which is satisfied on Γ_0^j for all $j = 1,\dots,k_1$.

For every $r > 0$ and $w \in \widetilde{\mathcal{X}}_r$, it is enough to write the proved inequality for the scaled function $w(rx)$ to obtain (7.106). □

We draw attention to the fact that the constant C_H in Lemma 7.5.1 does not depend on r. Hence, summing over annuli that fill $\mathbb{R}^2 \setminus D_1$, we obtain the following result.

Proposition 7.5.2. *Let k and k_1 be odd. Let $C_H > 0$ be as in Lemma 7.5.1. Then, for every $w \in \widetilde{\mathcal{X}}$,*

$$\int_{\mathbb{R}^2\setminus D_1} \frac{w^2}{|x|^2}\,dx \le C_H \int_{(\mathbb{R}^2\setminus D_1)\setminus\Gamma_1} |\nabla w|^2\,dx. \tag{7.109}$$

Furthermore, there exists a constant $C'_H > 0$ such that, for all $w \in \widetilde{\mathcal{X}}$,

$$\int_{D_1} w^2\,dx \le C'_H \int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla w|^2\,dx. \tag{7.110}$$

Proof. If $w \in \widetilde{\mathcal{X}}$, then $w \in \widetilde{\mathcal{X}}_r$ for all $r > 1$. Hence, by (7.107),

$$\begin{aligned}\int_{\mathbb{R}^2\setminus D_1} \frac{w^2}{|x|^2}\,dx &= \sum_{h=0}^{\infty} \int_{D_{2^{h+1}}\setminus D_{2^h}} \frac{w^2}{|x|^2}\,dx \\ &\le C_H \sum_{h=0}^{\infty} \int_{(D_{2^{h+1}}\setminus D_{2^h})\setminus\Gamma_0} |\nabla w|^2\,dx = C_H \int_{(\mathbb{R}^2\setminus D_1)\setminus\Gamma_1} |\nabla w|^2\,dx,\end{aligned}$$

thus proving (7.109).

By integrating the identity $\operatorname{div}(u^2x) = 2u\nabla u\cdot x + 2u^2$ on each subset of D_1 obtained by cutting along the lines Σ^j, $j = 1\dots,k_1+k_2$, and using the Divergence Theorem, we can prove that, for all $w \in \widetilde{\mathcal{X}}$,

$$\int_{D_1} w^2\,dx \le \int_{\partial D_1} w^2\,dS + \int_{D_1\setminus\Gamma_1} |\nabla w|^2\,dx.$$

Then, by continuity of the trace operator from $H^1((D_2\setminus D_1)\setminus\Gamma_0)$ into $L^2(\partial D_1)$ and (7.109), there exists a positive constant $C > 0$ such that

$$\begin{aligned}\int_{D_1} w^2\,dx &\le C\left(\int_{D_2\setminus D_1} w^2\,dx + \int_{(D_2\setminus D_1)\setminus\Gamma_1} |\nabla w|^2\,dx\right) + \int_{D_1\setminus\Gamma_1} |\nabla w|^2\,dx \\ &\le 4C\int_{D_2\setminus D_1} \frac{w^2}{|x|^2}\,dx + (C+1)\int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla w|^2\,dx \le (4CC_H + C + 1)\int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla w|^2\,dx,\end{aligned}$$

this proving (7.110). □

From Proposition 7.5.2 it follows that

$$\|w\|_{\widetilde{\mathcal{X}}} := \left(\int_{\mathbb{R}^2 \setminus \Gamma_1} |\nabla w|^2 \, dx \right)^{1/2} \tag{7.111}$$

is a norm on $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{X}}$ is a Hilbert space with respect to the corresponding scalar product. Proposition 7.5.2 also ensures that the restriction operator

$$\widetilde{\mathcal{X}} \to H^1(D_\rho \setminus \Gamma_1) \tag{7.112}$$

is continuous with respect to the norm defined in (7.111) for every $\rho > 0$. Hence, for every $p \in [1, +\infty)$, the trace operators

$$\gamma_+^j : \widetilde{\mathcal{X}} \to L^p(S_1^j) \quad \text{and} \quad \gamma_-^j : \widetilde{\mathcal{X}} \to L^p(S_1^j) \tag{7.113}$$

are well-defined and continuous with respect to the norm $\| \cdot \|_{\widetilde{\mathcal{X}}}$. In particular, since $\widetilde{\mathcal{H}} \subset \widetilde{\mathcal{X}}$,

$$\sup_{w \in \widetilde{\mathcal{H}} \setminus \{0\}} \frac{\|\gamma_+^j(w)\|^2_{L^p(S_1^j)}}{\|w\|^2_{\widetilde{\mathcal{X}}}} < +\infty \quad \text{for every } p \in [1, +\infty) \text{ and } j = 1, \dots, k_1 + k_2. \tag{7.114}$$

Using (7.109), we prove now that functions in $\widetilde{\mathcal{H}}$ can be approximated with functions with compact support. To this aim, we define

$$\widetilde{\mathcal{H}}_c := \{ w \in \widetilde{\mathcal{H}} : \text{ there exists } r > 0 \text{ such that } w \equiv 0 \text{ on } \mathbb{R}^2 \setminus B_r \}.$$

Proposition 7.5.3. *$\widetilde{\mathcal{H}}_c$ is dense in $\widetilde{\mathcal{H}}$.*

Proof. For every $r > 1$, let η_r be a cut-off function as in (7.68). If $w \in \widetilde{\mathcal{H}}$, it is clear that $\{\eta_r w\}_{r>1} \subset \widetilde{\mathcal{H}}_c$; moreover, by (7.109) we have $\frac{w}{|x|} \in L^2(\mathbb{R}^2 \setminus D_1)$ and hence

$$\int_{\mathbb{R}^2 \setminus \Gamma_1} |\nabla \eta_r|^2 w^2 \, dx \le 16 \int_{D_{2r} \setminus B_r} \frac{w^2}{|x|^2} \, dx \to 0^+ \quad \text{as } r \to \infty.$$

This implies that $\nabla(\eta_r w) \to \nabla w$ in $L^2(\mathbb{R}^2 \setminus \Gamma_1)$ and hence $\eta_r w \to w$ in $\widetilde{\mathcal{H}}$. $\square$

7.5.2 Limit profile for blown-up potentials

In this subsection, we introduce and characterize the function $\widetilde{V}$ appearing as limit profile in a blow-up analysis for the potentials V_ε.

Proposition 7.5.4. *There exists a unique solution $\widetilde{V} \in \widetilde{\mathcal{X}}$ to the minimization problem (7.30). Furthermore, $\widetilde{V}$ satisfies*

$$\begin{cases} \widetilde{V} - \eta \Psi_0 \in \widetilde{\mathcal{H}} \\ \displaystyle\int_{\mathbb{R}^2 \setminus \Gamma_1} \nabla \widetilde{V} \cdot \nabla w \, dx = -2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} \nabla \Psi_0 \cdot \nu^j \gamma_+^j(w) \, dS \quad \textit{for all } w \in \widetilde{\mathcal{H}}. \end{cases} \tag{7.115}$$

Proof. Since $\nabla\Psi_0 \in L^p(S_1^j)$ for all $p \in [1,2)$, by continuity of the trace operators in (7.113) we have that the linear functional L defined in (7.27) is well-defined and continuous. Then the convex functional J defined in (7.28) is continuous and coercive on the closed and convex set $\eta\Psi_0 + \widetilde{\mathcal{H}} = \{w \in \widetilde{\mathcal{X}} : w - \eta\Psi_0 \in \widetilde{\mathcal{H}}\}$. Therefore (7.30) admits a solution $\widetilde{V}$, which satisfies (7.115).

If $\widetilde{V}_1$ and $\widetilde{V}_2$ are solutions of (7.115), then we may take the difference between (7.115) for $\widetilde{V}_1$ and (7.115) for $\widetilde{V}_2$, both tested with $\widetilde{V}_1 - \widetilde{V}_2 \in \widetilde{\mathcal{H}}$, and conclude that $\widetilde{V}_1 = \widetilde{V}_2$ thanks to (7.109). Hence $\widetilde{V}$ is the unique solution to (7.115). □

7.5.3 An equivalent characterization of the energy functional

In this subsection, we obtain an equivalent characterization of the energy $\mathcal{E}_\varepsilon$ introduced in (7.19), which will be used to improve (7.70) and obtain an optimal estimate for $|\mathcal{E}_\varepsilon|$ in the case k odd.

Proposition 7.5.5. *Let $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^2)$ be a cut-off function as in (7.68) with $r = \varepsilon$. Then, for every $\varepsilon \in (0,1]$,*

$$\mathcal{E}_\varepsilon = -\frac{1}{2} \sup_{w \in \widetilde{\mathcal{H}}_\varepsilon \setminus \{0\}} \frac{\left(\int_{\Omega\setminus\Gamma_\varepsilon} \nabla w \cdot \nabla(\eta_\varepsilon v_0) + L_\varepsilon(w)\right)^2}{\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla w|^2\, dx} + \frac{1}{2}\int_{\Omega\setminus\Gamma_0} |\nabla(\eta_\varepsilon v_0)|^2\, dx + L_\varepsilon(v_0). \tag{7.116}$$

Proof. Since $\mathcal{E}_\varepsilon$ is the infimum in (7.63) and $\varphi - v_0 \in \widetilde{\mathcal{H}}_\varepsilon$ if and only if $\varphi - \eta_\varepsilon v_0 \in \widetilde{\mathcal{H}}_\varepsilon$, we have

$$\mathcal{E}_\varepsilon = \inf_{w \in \widetilde{\mathcal{H}}_\varepsilon} J_\varepsilon(w + \eta_\varepsilon v_0) = \inf_{w \in \widetilde{\mathcal{H}}_\varepsilon \setminus \{0\}} \left(\inf_{t \in [0,+\infty)} J_\varepsilon(tw + \eta_\varepsilon v_0)\right). \tag{7.117}$$

Moreover, by (7.17)

$$\begin{aligned} J_\varepsilon(tw + \eta_\varepsilon v_0) = \frac{t^2}{2}\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla w|^2\, dx + t\left(\int_{\Omega\setminus\Gamma_\varepsilon} \nabla w \cdot \nabla(\eta_\varepsilon v_0)\, dx + L_\varepsilon(w)\right) \\ + \frac{1}{2}\int_{\Omega\setminus\Gamma_0} |\nabla(\eta_\varepsilon v_0)|^2\, dx + L_\varepsilon(v_0). \end{aligned}$$

Hence, for every $w \in \widetilde{\mathcal{H}}_\varepsilon \setminus \{0\}$,

$$\inf_{t\in[0,+\infty)} J_\varepsilon(tw+\eta_\varepsilon v_0) = -\frac{1}{2}\frac{\left(\int_{\Omega\setminus\Gamma_\varepsilon} \nabla w \cdot \nabla(\eta_\varepsilon v_0)\, dx + L_\varepsilon(w)\right)^2}{\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla w|^2\, dx} + \frac{1}{2}\int_{\Omega\setminus\Gamma_0} |\nabla(\eta_\varepsilon v_0)|^2\, dx + L_\varepsilon(v_0),$$

which implies (7.116) in view of (7.117). □

Proposition 7.5.6. *Let k and k_1 be odd and $m \in \mathbb{N}$ be as in Proposition 7.2.6 for $v = v_0$. Then*

$$\mathcal{E}_\varepsilon = O(\varepsilon^m) \quad \text{as } \varepsilon \to 0^+.$$

Proof. From Proposition 7.5.5 and the Cauchy-Schwarz inequality it follows that

$$\begin{aligned} |\mathcal{E}_\varepsilon| &\le \frac{1}{2} \sup_{w \in \widetilde{\mathcal{H}}_\varepsilon \setminus \{0\}} \frac{\left(\int_{\Omega\setminus\Gamma_\varepsilon} \nabla w \cdot \nabla(\eta_\varepsilon v_0) + L_\varepsilon(w)\right)^2}{\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla w|^2\, dx} + \frac{1}{2}\int_{\Omega\setminus\Gamma_0} |\nabla(\eta_\varepsilon v_0)|^2\, dx + |L_\varepsilon(v_0)| \\ &\le \sup_{w \in \widetilde{\mathcal{H}}_\varepsilon \setminus \{0\}} \frac{|L_\varepsilon(w)|^2}{\int_{\Omega\setminus\Gamma_\varepsilon} |\nabla w|^2\, dx} + \frac{3}{2}\int_{\Omega\setminus\Gamma_0} |\nabla(\eta_\varepsilon v_0)|^2\, dx + |L_\varepsilon(v_0)|. \end{aligned} \tag{7.118}$$

From (7.53) and (7.16) it follows that

$$\int_{\Omega\setminus\Gamma_0} |\nabla(\eta_\varepsilon v_0)|^2\,dx \le 2\int_{D_{2\varepsilon}} |\nabla\eta_\varepsilon|^2 v_0^2\,dx + 2\int_{D_{2\varepsilon}\setminus\Gamma_0} |\nabla v_0|^2\,dx = O(\varepsilon^m) \quad \text{as } \varepsilon\to 0^+ \tag{7.119}$$

and

$$|L_\varepsilon(v_0)| = O(\varepsilon^m) \quad \text{as } \varepsilon\to 0^+. \tag{7.120}$$

By (7.16), the Hölder inequality, and (7.53), for every $p\in(1,2)$ and $p' = \frac{p}{p-1}$ we have

$$\begin{aligned}\sup_{w\in\widetilde{\mathcal{H}}_\varepsilon\setminus\{0\}} \frac{|L_\varepsilon(w)|^2}{\int_{\Omega\setminus\Gamma_\varepsilon}|\nabla w|^2\,dx} &\le 4(k_1+k_2)\sup_{w\in\widetilde{\mathcal{H}}_\varepsilon\setminus\{0\}} \frac{\sum_{j=1}^{k_1+k_2}\left(\int_{S^j_\varepsilon}|\nabla v_0||\gamma^j_+(w)|\,dS\right)^2}{\int_{\Omega\setminus\Gamma_\varepsilon}|\nabla w|^2\,dx} \\ &\le 4(k_1+k_2)\sum_{j=1}^{k_1+k_2}\left(\int_{S^j_\varepsilon}|\nabla v_0|^p\,dS\right)^{2/p}\sup_{w\in\widetilde{\mathcal{H}}_\varepsilon\setminus\{0\}}\frac{\|\gamma^j_+(w)\|^2_{L^{p'}(S^j_\varepsilon)}}{\int_{\Omega\setminus\Gamma_\varepsilon}|\nabla w|^2\,dx} \\ &= O(\varepsilon^{m-2+\frac{2}{p}})\sum_{j=1}^{k_1+k_2}\sup_{w\in\widetilde{\mathcal{H}}_\varepsilon\setminus\{0\}}\frac{\|\gamma^j_+(w)\|^2_{L^{p'}(S^j_\varepsilon)}}{\int_{\Omega\setminus\Gamma_\varepsilon}|\nabla w|^2\,dx}.\end{aligned} \tag{7.121}$$

A change of variables and (7.114) yield

$$\sup_{w\in\widetilde{\mathcal{H}}_\varepsilon\setminus\{0\}}\frac{\|\gamma^j_+(w)\|^2_{L^{p'}(S^j_\varepsilon)}}{\int_{\Omega\setminus\Gamma_\varepsilon}|\nabla w|^2\,dx} \le \varepsilon^{2/p'}\sup_{v\in\widetilde{\mathcal{H}}\setminus\{0\}}\frac{\|\gamma^j_+(v)\|^2_{L^{p'}(S^j_1)}}{\|v\|^2_{\widetilde{\mathcal{X}}}} = O(\varepsilon^{2/p'}) \quad \text{as } \varepsilon\to 0^+,$$

hence from (7.121) we deduce that

$$\sup_{w\in\widetilde{\mathcal{H}}_\varepsilon\setminus\{0\}} \frac{|L_\varepsilon(w)|^2}{\int_{\Omega\setminus\Gamma_\varepsilon}|\nabla w|^2\,dx} = O(\varepsilon^m) \quad \text{as } \varepsilon\to 0^+. \tag{7.122}$$

The conclusion follows by combining estimates (7.118), (7.119), (7.120), and (7.122). □

7.5.4 Blow-up analysis

Let k and k_1 be odd and $m\in\mathbb{N}$ be as in Proposition 7.2.6 for $v = v_0$. For every $\varepsilon\in(0,1]$, letting V_ε be as Proposition 7.3.2, we define

$$\widetilde{V}_\varepsilon(x) := \varepsilon^{-\frac{m}{2}} V_\varepsilon(\varepsilon x) \quad \text{and} \quad \widetilde{V}_{0,\varepsilon}(x) := \varepsilon^{-\frac{m}{2}} v_0(\varepsilon x). \tag{7.123}$$

Extending trivially $\widetilde{V}_\varepsilon$ and $\widetilde{V}_{0,\varepsilon}$ in $\mathbb{R}^2\setminus\Omega$, we have $\widetilde{V}_\varepsilon, \widetilde{V}_{0,\varepsilon}\in\widetilde{\mathcal{X}}$. Moreover

$$\widetilde{V}_\varepsilon - \widetilde{V}_{0,\varepsilon} \in \widetilde{\mathcal{H}} \tag{7.124}$$

and, by (7.66) and Proposition 7.5.3,

$$\int_{\mathbb{R}^2\setminus\Gamma_1} \nabla\widetilde{V}_\varepsilon\cdot\nabla w\,dx = -2\sum_{j=1}^{k_1+k_2}\int_{S^j_1}\nabla\widetilde{V}_{0,\varepsilon}\cdot\nu^j\gamma^j_+(w)\,dS \quad \text{for all } w\in\widetilde{\mathcal{H}}. \tag{7.125}$$

Let Ψ_0 be as in (7.26). From (7.52) it follows that, for every $j=1,\dots,k_1+k_2$,

$$\nabla\widetilde{V}_{0,\varepsilon}(x)\cdot\nu^j \to \nabla\Psi_0(x)\cdot\nu^j \tag{7.126}$$

as $\varepsilon \to 0^+$ for every $x \in S_1^j$, with

$$\nabla\Psi_0(x)\cdot\nu^j \tag{7.127}$$
$$= \begin{cases} \beta\frac{m}{2}|x|^{\frac{m}{2}-1}f(\alpha^j)\cos\left(\frac{m}{2}(\alpha^j-\alpha_0)\right), & \text{if } j=1,\dots,k_1,\\ \beta\frac{m}{2}|x|^{\frac{m}{2}-1}f(\alpha^j)\cos\left(\frac{m}{2}(\alpha^j-\alpha_0)\right), & \text{if } x\in(S_1^j)',\ j=k_1+1,\dots,k_1+k_2,\\ -\beta\frac{m}{2}|x|^{\frac{m}{2}-1}f(\alpha^j+\pi)\cos\left(\frac{m}{2}(\alpha^j+\pi-\alpha_0)\right), & \text{if } x\in(S_1^j)'',\ j=k_1+1,\dots,k_1+k_2,\end{cases}$$

where, for every $j \in k_1,\dots,k_1+k_2$,

$$(S_1^j)' := \{ta^j : t\in[0,1]\}, \quad (S_1^j)'' := \{ta^{j+k_2} : t\in[0,1]\}.$$

On the other hand, (7.53) implies that

$$|\nabla\widetilde{V}_{0,\varepsilon}(x)| \le C|x|^{\frac{m}{2}-1} \quad \text{in } \mathbb{R}^2\setminus\Gamma_0. \tag{7.128}$$

From (7.126) and (7.128) we deduce that, for every $j=1,\dots,k_1+k_2$ and $p\in[1,2)$,

$$\nabla\Psi_0\cdot\nu^j \in L^p(S_1^j) \quad \text{and} \quad \nabla\widetilde{V}_{0,\varepsilon}\cdot\nu^j \to \nabla\Psi_0\cdot\nu^j \quad \text{in } L^p(S_1^j) \text{ as } \varepsilon\to 0^+. \tag{7.129}$$

Furthermore, by (7.54) we know that

$$\widetilde{V}_{0,\varepsilon} \to \Psi_0 \quad \text{in } H^1(D_\rho\setminus\Gamma_0) \text{ for all } \rho>0. \tag{7.130}$$

Proposition 7.5.7. *Let k and k_1 be odd and $m\in\mathbb{N}$ be as in Proposition 7.2.6 for $v=v_0$. For every $\varepsilon\in(0,1]$, let V_ε be as Proposition 7.3.2 and $\widetilde{V}_\varepsilon$ as in (7.123). Then*

$$\widetilde{V}_\varepsilon \to \widetilde{V} \quad \text{strongly in } \widetilde{\mathcal{X}} \text{ as } \varepsilon\to 0^+, \tag{7.131}$$

where $\widetilde{V}\in\widetilde{\mathcal{X}}$ is the unique solution to the minimization problem (7.30) (and then to (7.115), see Proposition 7.5.4).

Proof. Taking into account (7.53), (7.19), and (7.123), a change of variables, (7.114), the Hölder inequality, and Proposition 7.5.6 imply that

$$\begin{aligned}\|\widetilde{V}_\varepsilon\|^2_{\widetilde{\mathcal{X}}} &= \int_{\mathbb{R}^2\setminus\Gamma_1}|\nabla\widetilde{V}_\varepsilon|^2\,dx = \varepsilon^{-m}\,\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon} = \varepsilon^{-m}(2\mathcal{E}_\varepsilon - 2L_\varepsilon(V_\varepsilon))\\ &\le O(1) + 4\varepsilon^{-m}\sum_{j=1}^{k_1+k_2}\int_{S_\varepsilon^j}|\nabla v_0||\gamma_+^j(V_\varepsilon)|\,dS = O(1)+O(1)\sum_{j=1}^{k_1+k_2}\int_{S_1^j}|x|^{\frac{m}{2}-1}|\gamma_+^j(\widetilde{V}_\varepsilon)|\,dS\\ &= O(1)+O(1)\|\widetilde{V}_\varepsilon\|_{\widetilde{\mathcal{X}}}, \text{ as } \varepsilon\to 0^+.\end{aligned} \tag{7.132}$$

Hence $\{\widetilde{V}_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $\widetilde{\mathcal{X}}$. It follows that, for any sequence $\{\varepsilon_n\}_n$ such that $\varepsilon_n\to 0$ as $n\to\infty$, there exist a subsequence, still denoted by $\{\varepsilon_n\}_n$, and $V\in\widetilde{\mathcal{X}}$ such that $\widetilde{V}_{\varepsilon_n}\rightharpoonup V$ weakly in $\widetilde{\mathcal{X}}$ as $n\to\infty$. Therefore, from (7.125), (7.114), and (7.129) we deduce that V solves the variational equation in (7.115). Furthermore, by (7.124) we have $\widetilde{V}_\varepsilon - \eta\widetilde{V}_{0,\varepsilon}\in\widetilde{\mathcal{H}}$, hence (7.130) ensures that V satisfies the condition $V-\eta\Psi_0\in\widetilde{\mathcal{H}}$. By the uniqueness part of Proposition 7.5.4 we conclude that $V=\widetilde{V}$.

Since $\widetilde{V}-\eta\Psi_0\in\widetilde{\mathcal{H}}$, we may test (7.115) with $\widetilde{V}-\eta\Psi_0$, thus obtaining

$$\int_{\mathbb{R}^2\setminus\Gamma_1}|\nabla\widetilde{V}|^2\,dx = \int_{\mathbb{R}^2\setminus\Gamma_1}\nabla\widetilde{V}\cdot\nabla(\eta\Psi_0)\,dx - 2\sum_{j=1}^{k_1+k_2}\int_{S_1^j}\nabla\Psi_0\cdot\nu^j\gamma_+^j(\widetilde{V}-\eta\Psi_0)\,dS. \tag{7.133}$$

On the other hand, testing (7.125) with $\widetilde{V}_{\varepsilon_n} - \eta \widetilde{V}_{0,\varepsilon_n} \in \widetilde{\mathcal{H}}$ we obtain

$$\int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla \widetilde{V}_{\varepsilon_n}|^2 \, dx = \int_{\mathbb{R}^2\setminus\Gamma_1} \nabla \widetilde{V}_{\varepsilon_n} \cdot \nabla(\eta \widetilde{V}_{0,\varepsilon_n}) \, dx \\ - 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} \nabla \widetilde{V}_{0,\varepsilon_n} \cdot \nu^j \gamma_+^j (\widetilde{V}_{\varepsilon_n} - \eta \widetilde{V}_{0,\varepsilon_n}) \, dS. \quad (7.134)$$

In view of the weak convergence $\widetilde{V}_{\varepsilon_n} \rightharpoonup \widetilde{V}$ in $\widetilde{\mathcal{X}}$, (7.130), (7.129), and the continuity of the trace operators (7.113), the limit of the right hand side of (7.134) as $n \to \infty$ is equal to the right hand side of (7.133), thus proving that $\widetilde{V}_{\varepsilon_n} \to \widetilde{V}$ strongly in $\widetilde{\mathcal{X}}$ as $n \to \infty$ by. Since $\widetilde{V}$ is the unique solution of (7.115), (7.131) follows from the Urysohn Subsequence Principle. □

In view of the blow-up analysis performed above, we are in position to prove Theorem 7.1.2.

Proof of Theorem 7.1.2. From (7.17), (7.19), (7.123), and a change of variables it follows that

$$\varepsilon^{-m} \mathcal{E}_\varepsilon = \frac{1}{2} \int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla \widetilde{V_\varepsilon}|^2 \, dx + 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} \nabla \widetilde{V}_{0,\varepsilon} \cdot \nu^j \gamma_+^j (\widetilde{V_\varepsilon}) \, dS. \quad (7.135)$$

The convergences (7.131) and (7.129), together with the continuity of the trace operators in (7.113), allow us to pass to the limit in the right hand side of (7.135), thus yielding

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-m} \mathcal{E}_\varepsilon = \frac{1}{2} \int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla \widetilde{V}|^2 \, dx + 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} \nabla \Psi_0 \cdot \nu^j \gamma_+^j (\widetilde{V}) \, dS = J(\widetilde{V}) = \mathcal{E} \quad (7.136)$$

and proving claim (i). Furthermore, by (7.123), a change of variable, (7.129), and (7.130), we have

$$\varepsilon^{-m} L_\varepsilon(v_0) = 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} \nabla \widetilde{V}_{0,\varepsilon} \cdot \nu^j \gamma_+^j (\widetilde{V}_{0,\varepsilon}) \, dS \quad (7.137)$$
$$= 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} \nabla \Psi_0 \cdot \nu^j \gamma_+^j (\Psi_0) \, dS + o(1) = L(\Psi_0) + o(1) \quad \text{as } \varepsilon \to 0^+.$$

Claim (ii) follows from (7.20), (7.136), (7.137), and estimate (7.132), which in particular ensures that $\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon} = O(\varepsilon^m)$ as $\varepsilon \to 0^+$. □

7.5.5 Continuity of $\mathcal{E} - L(\Psi_0)$ with respect to rotations of poles

In this subsection we prove the continuity of $\mathcal{E} - L(\Psi_0)$ with respect to rotations of the configuration of poles. We fix a configuration of poles $\{a^j\}$ as in (1.6). Then, for every $\zeta \in [-\pi, \pi)$, we define $\Psi_0^{(\zeta)}$, $L^{(\zeta)}(\Psi_0^{(\zeta)})$ and $\mathcal{E}^{(\zeta)}$ as in (7.26), (7.27), and (7.31), respectively, for a rotated configuration of poles $\{a_\zeta^j\}$, where a_ζ^j are defined as in (1.6) with angles $\alpha^j + \zeta$ instead of α^j, i.e.

$$a_\zeta^j = \mathcal{R}_\zeta(a^j),$$

being $\mathcal{R}_\zeta := R_{0,\zeta}$ with $R_{0,\zeta}$ as in (7.42), see 7.5.

In the next theorem we prove that the function $\zeta \mapsto \mathcal{E}^{(\zeta)} - L^{(\zeta)}(\Psi_0^{(\zeta)})$ is continuous.

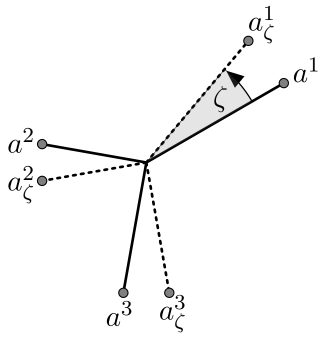


Figure 7.5: The rotated configuration $\{a_\zeta^j\}$.

Theorem 7.5.8. *The function* $G : [-\pi, \pi) \to \mathbb{R}$*,* $G(\zeta) := \mathcal{E}^{(\zeta)} - L^{(\zeta)}(\Psi_0^{(\zeta)})$ *is continuous.*

Proof. Through a rotation, the problem of continuity at any $\zeta \in [-\pi, \pi)$ can be reduced to the problem of continuity at $\zeta = 0$. Hence, it is enough to prove that $\lim_{\zeta \to 0} G(\zeta) = \mathcal{E} - L(\Psi_0)$.

We have

$$\Psi_0^{(\zeta)}(r \cos t, r \sin t) = f(t - \zeta)\phi_0(r \cos t, r \sin t),$$

where f is defined in (7.22) and

$$\phi_0(r \cos t, r \sin t) := \beta\, r^{\frac{m}{2}} \sin\left(\tfrac{m}{2}(t - \alpha_0)\right).$$

With a slight abuse of notation, henceforth we denote by f also the function $(r \cos t, r \sin t) \mapsto f(t)$ defined on $\mathbb{R}^2 \setminus \{0\}$.

A change of variables yields

$$\mathcal{E}^{(\zeta)} = \min\left\{ I_\zeta(w) : w \in \widetilde{\mathcal{X}} \text{ and } w - \eta f(\phi_0 \circ \mathcal{R}_\zeta) \in \widetilde{\mathcal{H}} \right\},$$

where $\eta \in C_c^\infty(\mathbb{R}^2)$ is a radial cut-off function as in (7.29) and

$$I_\zeta(w) = \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Gamma_1} |\nabla w|^2 \, dx + 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} f(\nabla \phi_0 \circ \mathcal{R}_\zeta) M_\zeta \cdot \nu^j \gamma_+^j(w) \, dS,$$

being M_ζ the matrix defined in (7.43). Moreover

$$L^{(\zeta)}(\Psi_0^{(\zeta)}) = 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} (\nabla \phi_0 \circ \mathcal{R}_\zeta) M_\zeta \cdot \nu^j (\phi_0 \circ \mathcal{R}_\zeta) \, dS.$$

Since, in a neighbourhood of 0,

$$|\nabla \phi_0(\mathcal{R}_\zeta(x))| \le C|x|^{\frac{m}{2}-1} \quad \text{and} \quad |\phi_0(\mathcal{R}_\zeta(x))| \le C|x|^{\frac{m}{2}} \tag{7.138}$$

for some $C > 0$ independent of ζ, from the Dominated Convergence Theorem we deduce that

$$\lim_{\zeta \to 0} L^{(\zeta)}(\Psi_0^{(\zeta)}) = L(\Psi_0). \tag{7.139}$$

By Proposition 7.5.4, for every ζ there exists a unique $\widetilde{V}_\zeta \in \widetilde{\mathcal{X}}$ such that $\widetilde{V}_\zeta - \eta f(\phi_0 \circ \mathcal{R}_\zeta) \in \widetilde{\mathcal{H}}$ and $\mathcal{E}^{(\zeta)} = I_\zeta(\widetilde{V}_\zeta)$; furthermore, $\widetilde{V}_\zeta$ satisfies

$$\int_{\mathbb{R}^2\setminus\Gamma_1} \nabla \widetilde{V}_\zeta \cdot \nabla w \, dx = -2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} f(\nabla\phi_0 \circ \mathcal{R}_\zeta) M_\zeta \cdot \nu^j \gamma_+^j(w) \, dS \quad \text{for all } w \in \widetilde{\mathcal{H}}. \tag{7.140}$$

Choosing $w = \widetilde{V}_\zeta - \eta f(\phi_0 \circ \mathcal{R}_\zeta)$ in (7.140) we obtain

$$\begin{aligned}\int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla \widetilde{V}_\zeta|^2 \, dx = &\int_{\mathbb{R}^2\setminus\Gamma_1} \nabla \widetilde{V}_\zeta \cdot \nabla(\eta f(\phi_0 \circ \mathcal{R}_\zeta)) \, dx \\ &- 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} f(\nabla\phi_0 \circ \mathcal{R}_\zeta) M_\zeta \cdot \nu^j \gamma_+^j(\widetilde{V}_\zeta) \, dS \\ &+ 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} f(\nabla\phi_0 \circ \mathcal{R}_\zeta) M_\zeta \cdot \nu^j \gamma_+^j(f(\phi_0 \circ \mathcal{R}_\zeta)) \, dS. \end{aligned} \tag{7.141}$$

Using Young's inequality, estimate (7.138), and the continuity of the trace operators (7.113), from the above identity we deduce that

$$\|\widetilde{V}_\zeta\|_{\widetilde{\mathcal{X}}} \leq C$$

for some $C > 0$ independent of ζ. It follows that every sequence $\zeta_n \to 0$ admits a subsequence $\{\zeta_{n_\ell}\}_\ell$ such that $\widetilde{V}_{\zeta_{n_\ell}} \rightharpoonup W$ weakly in $\widetilde{\mathcal{X}}$ as $\ell \to \infty$, for some $W \in \widetilde{\mathcal{X}}$. On account of (7.138) and (7.113), the Dominated Convergence Theorem yields

$$\int_{S_1^j} f(\nabla\phi_0 \circ \mathcal{R}_{\zeta_{n_\ell}}) M_{\zeta_{n_\ell}} \cdot \nu^j \gamma_+^j(w) \, dS \to \int_{S_1^j} f \nabla\phi_0 \cdot \nu^j \gamma_+^j(w) \, dS = \int_{S_1^j} \nabla\Psi_0 \cdot \nu^j \gamma_+^j(w) \, dS$$

as $\ell \to \infty$, for every $j = 1, \dots k_1 + k_2$ and $w \in \widetilde{\mathcal{H}}$. By choosing $\zeta = \zeta_{n_\ell}$ in (7.140) and letting $\ell \to \infty$ we obtain that

$$\int_{\mathbb{R}^2\setminus\Gamma_1} \nabla W \cdot \nabla w \, dx = -2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} \nabla\Psi_0 \cdot \nu^j \gamma_+^j(w) \, dS \quad \text{for all } w \in \widetilde{\mathcal{H}}. \tag{7.142}$$

Furthermore, since $\widetilde{V}_\zeta - \eta f(\phi_0 \circ \mathcal{R}_\zeta) \in \widetilde{\mathcal{H}}$, $\widetilde{\mathcal{H}}$ is a closed subspace of $\widetilde{\mathcal{X}}$, and $\eta f(\phi_0 \circ \mathcal{R}_\zeta) \to \eta\Psi_0$ as $\zeta \to 0$ in $\widetilde{\mathcal{X}}$ by the Dominated Convergence Theorem, we have

$$W - \eta\Psi_0 \in \widetilde{\mathcal{H}}. \tag{7.143}$$

From (7.142)-(7.143) and the uniqueness part of Proposition 7.5.4 we deduce that $W = \widetilde{V}$. Having uniquely identified the weak limit independently of the subsequence, by the Urysohn subsequence principle we conclude that

$$\widetilde{V}_\zeta \rightharpoonup \widetilde{V} \quad \text{weakly in } \widetilde{\mathcal{X}} \quad \text{as } \zeta \to 0. \tag{7.144}$$

The weak convergence (7.144) allows us to pass to the limit as $\zeta \to 0$ on the right hand side of (7.141), thus proving that

$$\begin{aligned}\lim_{\zeta\to 0} \int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla \widetilde{V}_\zeta|^2 \, dx &= \int_{\mathbb{R}^2\setminus\Gamma_1} \nabla \widetilde{V} \cdot \nabla(\eta\Psi_0) \, dx - 2 \sum_{j=1}^{k_1+k_2} \int_{S_1^j} \nabla\Psi_0 \cdot \nu^j \gamma_+^j(\widetilde{V} - \Psi_0) \, dS \\ &= \int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla \widetilde{V}|^2 \, dx, \end{aligned} \tag{7.145}$$

the last equality being a consequence of (7.115) tested with $w = \widetilde{V} - \eta\Psi_0$. From (7.145) it follows that $\lim_{\zeta\to 0} \mathcal{E}^{(\zeta)} = \lim_{\zeta\to 0} I_\zeta(\widetilde{V}_\zeta) = J(\widetilde{V}) = \mathcal{E}$, which, together with (7.139), yields the conclusion. □

When $k_2 = 0$ and the poles $\{a^j\}_{j=1,\dots,k_1}$ are on the tangents to nodal lines of v_0 (i.e. on the nodal set of Ψ_0), we have $\Psi_0 = 0$ on S_1^j for all $j = 1,\dots,k_1$; on the other hand, if the poles are on the bisectors between nodal lines, then $\nabla\Psi_0 \cdot \nu^j = 0$ on S_1^j for all $j = 1,\dots,k_1$. This leads to Proposition 7.1.3, which determines, in these particular cases, the sign of the dominant term in the asymptotic expansion obtained in Theorem 7.1.2, and, consequently, exploits the continuity result established in Theorem 7.5.8 to find configurations of poles for which the eigenvalue variation is an infinitesimal of higher order.

Proof of Proposition 7.1.3. (i) If $\alpha^j \in \{\alpha_0 + \ell\frac{2\pi}{m} : \ell = 0,1,2,\dots,m-1\}$ for all $j \in \{1,\dots,k_1\}$, then $\Psi_0 = 0$ on S_1^j for all $j \in \{1,\dots,k_1\}$, so that $L(\Psi_0) = 0$ and $\eta\Psi_0 + \widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}$. It follows that

$$\mathcal{E} - L(\Psi_0) = \mathcal{E} = \min_{\widetilde{\mathcal{H}}} J. \tag{7.146}$$

Furthermore, $\nabla\Psi_0 \cdot \nu^j \not\equiv 0$ on S_1^j for all $j \in \{1,\dots,k_1\}$, see (7.127), hence $L \not\equiv 0$ in $\widetilde{\mathcal{H}}$. Fixing some $w \in \widetilde{\mathcal{H}}$ such that $L(w) \neq 0$, we have then $J(tw) = \frac{t^2}{2}\int_{\mathbb{R}^2\setminus\Gamma_1} |\nabla w|^2\,dx + tL(w) < 0$ for some small t, thus implying that $\mathcal{E} = \min_{\widetilde{\mathcal{H}}} J < 0$. Once we have established that $\mathcal{E} - L(\Psi_0) = \mathcal{E} < 0$, from the asymptotic expansion of Theorem 7.1.2-(ii) we deduce that $\lambda_{\varepsilon,n_0} < \lambda_{0,n_0}$ for sufficiently small $\varepsilon > 0$.

(ii) If $\alpha^j \in \{\alpha_0 + (1+2\ell)\frac{\pi}{m} : \ell = 0,1,2,\dots,m-1\}$ for all $j \in \{1,\dots,k_1\}$, then $\nabla\Psi_0 \cdot \nu^j \equiv 0$ on S_1^j for all $j \in \{1,\dots,k_1\}$, see (7.127). It follows that $L \equiv 0$, and hence $J(w) = \frac{1}{2}\|w\|^2_{\widetilde{\mathcal{X}}}$. Since, in this case, $\Psi_0 \not\equiv 0$ on S_1^j for all $j \in \{1,\dots,k_1\}$, we have $w \not\equiv 0$ for every $w \in \eta\Psi_0 + \widetilde{\mathcal{H}}$. Therefore

$$\mathcal{E} - L(\Psi_0) = \mathcal{E} = \min_{\eta\Psi_0 + \widetilde{\mathcal{H}}} J = \frac{1}{2}\min_{w\in\eta\Psi_0+\widetilde{\mathcal{H}}} \|w\|^2_{\widetilde{\mathcal{X}}} > 0. \tag{7.147}$$

From the asymptotic expansion of Theorem 7.1.2-(ii) we finally deduce that $\lambda_{\varepsilon,n_0} > \lambda_{0,n_0}$ for sufficiently small $\varepsilon > 0$.

(iii) Let us fix a configuration $\{a^j\}_{j=1}^k$ with $k = k_1 \le m$ odd and $\alpha^j \in \{\alpha_0 + \ell\frac{2\pi}{m} : 0 \le \ell \le m-1\}$ for all $j \in \{1,\dots,k_1\}$ as in (i). Then the rotated configuration $\{a^j_{\pi/m}\}$ is as in (ii). By (i)-(ii) we have $G(0) < 0$ and $G(\frac{\pi}{m}) > 0$. Since G is continuous by Theorem 7.5.8, Bolzano's Theorem ensures the existence of some $\zeta_0 \in (0, \frac{\pi}{m})$ such that $G(\zeta_0) = 0$, so that the angles $\{\alpha^j + \zeta_0 : j = 1,\dots,k\}$ are as we are looking for. □

Remark 7.5.9. 7.6 and 7.7 provide an example that helps to better visualize the result in 7.1.3. In 7.6 we zoom in near a point (the origin) where the limit eigenfunction v_0 vanishes of order $3/2$, namely (7.21) holds with $m = 3$. We consider the case $\alpha_0 = 0$. The function Ψ_0 as in (7.26) is the $3/2$-homogeneous limit profile describing the local behavior of v_0. In the image on the left, the black lines are the nodal lines of v_0, which are tangent to the nodal lines of Ψ_0 (in green). The dotted lines denote the bisectors of the nodal lines of Ψ_0. In the image on the right, we fix an admissible configuration of poles $\{a^j\}_{j=1,2,3}$ with $k = 3$ and $\alpha_j = 2\pi(j-1)/3$

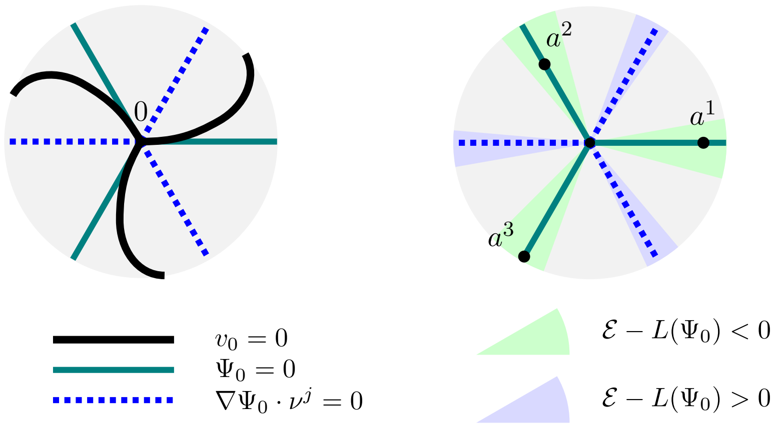


Figure 7.6: Nodal set of Ψ_0 and sign of $\mathcal{E} - L(\Psi_0)$ ($m = k = 3$, $\alpha_0 = 0$).

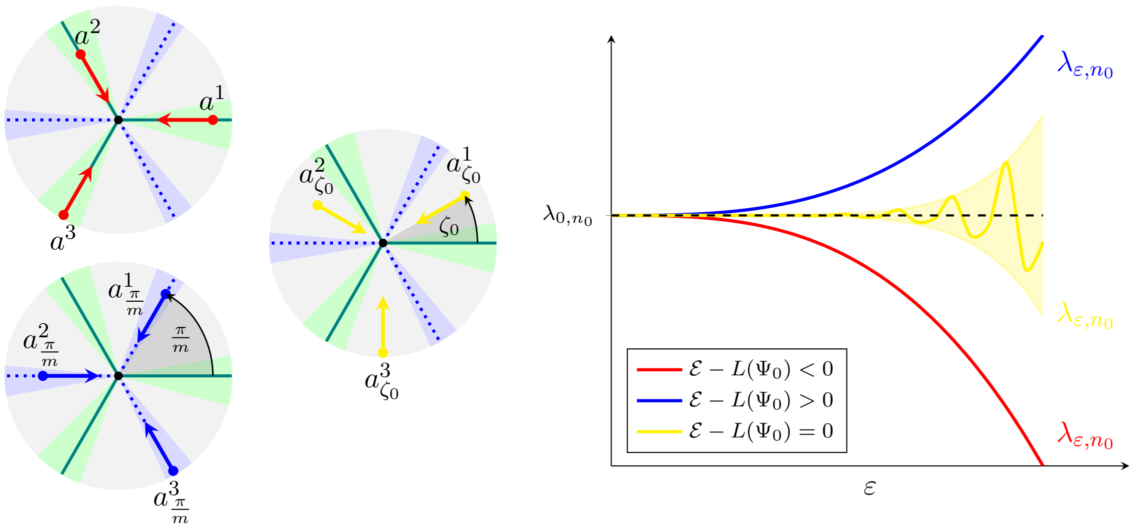


Figure 7.7: A visualization of Proposition 2.3.

for $j = 1, 2, 3$. From 7.1.3 we know that, if all the poles lie on the nodal set of Ψ_0, then the coefficient $\mathcal{E} - L(\Psi_0)$ of the leading term in the asymptotic expansion stated in 7.1.2 is strictly negative. On the other hand, if all the poles lie on the bisectors of the nodal lines, then the coefficient $\mathcal{E} - L(\Psi_0)$ is strictly positive. In 7.7 on the left, in the first picture (red arrows) we have our initial fixed configuration, which then provides a negative coefficient. In the second picture (blue arrows) we consider a rotation about the origin by an angle $\pi/m = \pi/3$: the rotated configuration ends up with all the poles lying on the bisectors, thus giving a positive coefficient $\mathcal{E} - L(\Psi_0)$. Furthermore, the continuity result in 7.5.8 ensures the existence of some $\zeta_0 \in (0, \pi/3)$ such that, if we rotate the initial configuration by an angle ζ_0, we find a configuration of poles for which $\mathcal{E} - L(\Psi_0) = 0$: this is represented in the third picture on the left (yellow arrows). Finally, the right picture in 7.7 presents the behavior of the perturbed eigenvalue in the three cases previously described. We point out that, when $\mathcal{E} - L(\Psi_0) = 0$ (yellow graph), it is currently not known what is the vanishing order of $\lambda_{\varepsilon,n_0} - \lambda_{0,n_0}$.

7.5.6 Blow-up and convergence rate for eigenfunctions

From the blow-up analysis for the potential V_ε performed in Subsection 7.5.4 and the energy estimate given in (7.81), we derive the following blow-up result for scaled eigenfunctions, together with a sharp estimate for their rate of convergence in the $\mathcal{H}_1$-norm.

Proposition 7.5.10. *Under assumption (7.77), let k be odd and v_0 be an eigenfunction of (7.14) associated to the eigenvalue $\lambda_0 = \lambda_{0,n_0}$ with $\|v_0\|_{L^2(\Omega)} = 1$. For $\varepsilon > 0$ small, let $\lambda_\varepsilon = \lambda_{\varepsilon,n_0}$ and v_ε be an eigenfunction of (7.10) associated to λ_ε and chosen as in (7.78). Let $m \in \mathbb{N}$ be given in Proposition 7.2.6 for $v = v_0$. Then*

$$\varepsilon^{-\frac{m}{2}} v_\varepsilon(\varepsilon x) \to \Psi_0 - \widetilde{V} \quad \text{as } \varepsilon \to 0^+ \text{ in } H^1(D_\rho \setminus \Gamma_1) \text{ for all } \rho > 0, \tag{7.148}$$

where Ψ_0 is defined in (7.26) and $\widetilde{V}$ is the unique solution to (7.115). Furthermore,

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-\frac{m}{2}} \|v_\varepsilon - v_0\|_{\mathcal{H}_1} = \|\widetilde{V}\|_{\widetilde{\mathcal{X}}}. \tag{7.149}$$

Proof. Using the same notation as in the proof of Theorem 7.4.2, let $\psi_\varepsilon = v_0 - V_\varepsilon$, where V_ε is defined as in Proposition 7.3.2. From (7.81) it follows that

$$\|\Pi_\varepsilon \psi_\varepsilon - \psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} = o\left(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+.$$

Therefore, defining

$$W_\varepsilon(x) := \varepsilon^{-\frac{m}{2}} (\Pi_\varepsilon \psi_\varepsilon - \psi_\varepsilon)(\varepsilon x), \quad x \in \tfrac{1}{\varepsilon}\Omega,$$

and extending trivially W_ε in $\mathbb{R}^2 \setminus \frac{1}{\varepsilon}\Omega$, we have $W_\varepsilon \in \widetilde{\mathcal{H}}$ and, in view of Proposition 7.5.7,

$$\|W_\varepsilon\|^2_{\widetilde{\mathcal{X}}} = \varepsilon^{-m} \|\Pi_\varepsilon \psi_\varepsilon - \psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} = \varepsilon^{-m} o\left(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}\right) = \|\widetilde{V_\varepsilon}\|^2_{\widetilde{\mathcal{X}}}\, o(1) = o(1)$$

as $\varepsilon \to 0^+$. By continuity of the restriction operator in (7.112) we deduce that

$$W_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ in } H^1(D_\rho \setminus \Gamma_1) \text{ for all } \rho > 0. \tag{7.150}$$

Let us define

$$U_\varepsilon(x) := \varepsilon^{-\frac{m}{2}} (\Pi_\varepsilon \psi_\varepsilon)(\varepsilon x), \quad x \in \tfrac{1}{\varepsilon}\Omega, \tag{7.151}$$

and extend trivially U_ε in $\mathbb{R}^2 \setminus \frac{1}{\varepsilon}\Omega$. We have

$$U_\varepsilon = \widetilde{V}_{0,\varepsilon}(x) - \widetilde{V}_\varepsilon + W_\varepsilon,$$

where $\widetilde{V}_{0,\varepsilon}$ and $\widetilde{V}_\varepsilon$ are defined in (7.123). Combining (7.130), (7.131), and (7.150), we conclude that

$$U_\varepsilon \to \Psi_0 - \widetilde{V} \quad \text{as } \varepsilon \to 0^+ \text{ in } H^1(D_\rho \setminus \Gamma_1) \text{ for all } \rho > 0. \tag{7.152}$$

From (7.82) it follows that

$$\int_\Omega v_0 \Pi_\varepsilon \psi_\varepsilon \, dx = 1 + o\left(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+,$$

and hence, for $\varepsilon > 0$ small enough,

$$\int_\Omega \frac{\Pi_\varepsilon \psi_\varepsilon}{\|\Pi_\varepsilon \psi_\varepsilon\|_{L^2(\Omega)}} v_0 \, dx > 0.$$

Since v_ε is the unique eigenfunction of (7.10) associated to λ_ε satisfying (7.78), we conclude that necessarily

$$v_\varepsilon = \frac{\Pi_\varepsilon \psi_\varepsilon}{\|\Pi_\varepsilon \psi_\varepsilon\|_{L^2(\Omega)}}. \tag{7.153}$$

The convergence stated in (7.148) follows from (7.153), (7.151), (7.152), and (7.83).

Moreover, (7.83) implies that

$$\begin{aligned}\|v_\varepsilon - \Pi_\varepsilon \psi_\varepsilon\|_{\mathcal{H}_1} &= \frac{|1 - \|\Pi_\varepsilon \psi_\varepsilon\|_{L^2(\Omega)}|}{\|\Pi_\varepsilon \psi_\varepsilon\|_{L^2(\Omega)}} \|\Pi_\varepsilon \psi_\varepsilon\|_{\mathcal{H}_1} \\ &= |1 - \|\Pi_\varepsilon \psi_\varepsilon\|_{L^2(\Omega)}| \|v_\varepsilon\|_{\mathcal{H}_\varepsilon} = o(\|V_\varepsilon\|_{\mathcal{H}_\varepsilon}) \quad \text{as } \varepsilon \to 0^+,\end{aligned} \tag{7.154}$$

whereas (7.81) yields that

$$\|\Pi_\varepsilon \psi_\varepsilon - v_0\|^2_{\mathcal{H}_1} = \|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + \|\Pi_\varepsilon \psi_\varepsilon - \psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} - 2(V_\varepsilon, \Pi_\varepsilon \psi_\varepsilon - \psi_\varepsilon)_{\mathcal{H}_\varepsilon} = \|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + o(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}) \tag{7.155}$$

as $\varepsilon \to 0^+$. Combining (7.154) and (7.155) we deduce that

$$\|v_\varepsilon - v_0\|^2_{\mathcal{H}_1} = \|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + o(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}) \quad \text{as } \varepsilon \to 0^+. \tag{7.156}$$

Letting $\widetilde{V}_\varepsilon$ be as in (7.123), from (7.156) and (7.131) we deduce that

$$\varepsilon^{-m} \|v_\varepsilon - v_0\|^2_{\mathcal{H}_1} = \|\widetilde{V}_\varepsilon\|^2_{\widetilde{\mathcal{X}}}(1 + o(1)) = \|\widetilde{V}\|^2_{\widetilde{\mathcal{X}}}(1 + o(1)) \quad \text{as } \varepsilon \to 0^+,$$

thus proving (7.149). □

Going back to the eigenfunctions of the original magnetic problem via the inverse of transformation (7.9), we deduce Theorem 7.1.4 from Proposition 7.5.10.

Proof of Theorem 7.1.4. If u_0 is an eigenfunction of (1.10) associated to the eigenvalue λ_{0,n_0} such that $\int_\Omega |u_0|^2\,dx = 1$, and u_ε is the eigenfunction of (1.9) associated to $\lambda_{n_0,\varepsilon}$ satisfying (7.32), then $v_\varepsilon := e^{-i\Theta_\varepsilon} u_\varepsilon$ is an eigenfunction of (7.10) associated to $\lambda_{n_0,\varepsilon}$ and $v_0 := e^{-i\Theta_0} u_0$ is an eigenfunction of (7.14) associated to $\lambda_{n_0,0}$ such that condition (7.78) is satisfied. From Proposition 7.5.10 it follows that v_ε satisfies (7.148) and (7.149), in which we replace v_ε with $e^{-i\Theta_\varepsilon} u_\varepsilon$ and v_0 with $e^{-i\Theta_0} u_0$ to get exactly (7.33) and (7.34), taking into account that $\Theta_\varepsilon(\varepsilon x) = \Theta_1(x)$ for all $x \in \mathbb{R}^2 \setminus \{a^j : j = 1, \dots, k\}$. □

7.6 The case of two poles

The purpose of this section is to prove Theorems 7.1.6 and 7.1.7. We consider the case $k_1 = 0$ and $k_2 = 1$, with the configuration of poles as in assumption (7.36), being $r_1 \in (0, R)$ and $\varepsilon \in (0, 1]$. For the sake of simplicity, let us denote

$$T := T^1, \quad \gamma_+ := \gamma_+^1, \quad \gamma_- := \gamma_-^1, \quad \text{and} \quad \nu := \nu^1 = (0, 1),$$

see (7.4). We first consider a linear functional $L_{\varepsilon,h,\Lambda}$ more general than the one introduced in (7.16), defined for a generic domain Λ and with the limit eigenfunction v_0 replaced by a generic function h; the corresponding minimal energy $\mathcal{E}_{\varepsilon,h,\Lambda}$ thus generalizes the energy

$\mathcal{E}_\varepsilon$ defined in (7.19). For every simply connected open bounded domain $\Lambda \subset \mathbb{R}^2$ such that $B_R \subseteq \Lambda$ and every $h \in H^1_0(\Lambda) \cap C^\infty(\Lambda)$, let

$$L_{\varepsilon,h,\Lambda} : \mathcal{H}_{1,\Lambda} \to \mathbb{R}, \quad L_{\varepsilon,h,\Lambda}(w) := 2\int_{S_\varepsilon} \frac{\partial h}{\partial x_2}\gamma_+(w)\, dS$$

and

$$J_{\varepsilon,h,\Lambda} : \mathcal{H}_{\varepsilon,\Lambda} \to \mathbb{R}, \quad J_{\varepsilon,h,\Lambda}(w) := \frac{1}{2}\int_{\Lambda\setminus S_\varepsilon} |\nabla w|^2\, dx + L_{\varepsilon,h,\Lambda}(w),$$

where, for all $\varepsilon \in (0,1]$, S_ε is defined in (7.37) and the functional space $\mathcal{H}_{\varepsilon,\Lambda}$ is the closure of

$$\left\{w \in H^1(\Lambda \setminus S_\varepsilon) : w = 0 \text{ on a neighbourhood of } \partial\Lambda\right\}$$

with respect to the norm $\|w\|_{H^1(\Omega\setminus S_\varepsilon)}$. Then the minimization problem

$$\inf\left\{J_{\varepsilon,h,\Lambda}(w) : w \in \mathcal{H}_{\varepsilon,\Lambda} \text{ and } w - h \in \widetilde{\mathcal{H}}_{\varepsilon,\Lambda}\right\} \tag{7.157}$$

with $\widetilde{\mathcal{H}}_{\varepsilon,\Lambda} := \{w \in \mathcal{H}_{\varepsilon,\Lambda} : T(w) = 0 \text{ on } S_\varepsilon\}$, is uniquely achieved, as stated in the following proposition. We omit the proof, being similar to the one of Proposition 7.3.2.

Proposition 7.6.1. *The infimum in (7.157) is achieved by a unique $V_{\varepsilon,h,\Lambda} \in \mathcal{H}_{\varepsilon,\Lambda}$. Furthermore, $V_{\varepsilon,h,\Lambda}$ weakly solves the problem*

$$\begin{cases} -\Delta V_{\varepsilon,h,\Lambda} = 0, & \text{in } \Lambda \setminus S_\varepsilon,\\ V_{\varepsilon,h,\Lambda} = 0, & \text{on } \partial\Lambda,\\ T(V_{\varepsilon,h,\Lambda} - h) = 0, & \text{on } S_\varepsilon,\\ T\left(\frac{\partial V_{\varepsilon,h,\Lambda}}{\partial x_2} - \frac{\partial h}{\partial x_2}\right) = 0, & \text{on } S_\varepsilon, \end{cases} \tag{7.158}$$

in the sense that $V_{\varepsilon,h,\Lambda} \in \mathcal{H}_{\varepsilon,\Lambda}$, $V_{\varepsilon,h,\Lambda} - h \in \widetilde{\mathcal{H}}_{\varepsilon,\Lambda}$, and

$$\int_{\Lambda\setminus S_\varepsilon} \nabla V_{\varepsilon,h,\Lambda} \cdot \nabla w\, dx = -L_{\varepsilon,h,\Lambda}(w) \quad \text{for all } w \in \widetilde{\mathcal{H}}_{\varepsilon,\Lambda}. \tag{7.159}$$

For every Λ, h as above and $\varepsilon \in (0,1]$, let

$$\mathcal{E}_{\varepsilon,h,\Lambda} := J_{\varepsilon,h,\Lambda}(V_{\varepsilon,h,\Lambda}). \tag{7.160}$$

For every $L > 0$ and $\varepsilon > 0$, let $E_\varepsilon(L)$ be the ellipse defined as

$$E_\varepsilon(L) := \left\{(x_1,x_2) \in \mathbb{R}^2 : \frac{x_1^2}{L^2 + r_1^2\varepsilon^2} + \frac{x_2^2}{L^2} < 1\right\}.$$

We are going to compute $\mathcal{E}_{\varepsilon,P_m,E_\varepsilon(L)}$, where P_m is a homogeneous polynomial of degree $m \geq 1$. We shall later apply such estimate with P_m being the Taylor polynomial of u_0 centered at 0 of order m, with u_0 and m as in 7.1.2.

Proposition 7.6.2. *Let $m \in \mathbb{N}$, $m \geq 1$, and let P_m be a homogeneous polynomial of degree m, i.e.*

$$P_m(x_1,x_2) := \sum_{j=0}^{m} \ell_j x_1^{m-j} x_2^j, \tag{7.161}$$

for some $\ell_0, \ell_1, \dots, \ell_m \in \mathbb{R}$. *Then, for every* $L > 0$, *we have*

$$\int_{E_\varepsilon(L)\setminus S_\varepsilon} |\nabla V_{\varepsilon,P_m,E_\varepsilon(L)}|^2\,dx = \pi(\varepsilon r_1)^{2m}\left(\ell_0^2\sum_{j=1}^m j|c_j|^2 + \ell_1^2\sum_{j=1}^m \frac{|d_j|^2}{j}\right) + o(\varepsilon^{2m}) \tag{7.162}$$

as $\varepsilon \to 0^+$, *where*

$$c_j = \frac{1}{\pi}\int_0^{2\pi} (\cos\eta)^m \cos(j\eta)\,d\eta \quad \textit{for every } j \in \mathbb{N}, \tag{7.163}$$

$$d_j = \frac{1}{\pi}\int_0^{2\pi} (\cos\eta)^{m-1} \sin\eta\sin(j\eta)\,d\eta \quad \textit{for every } j \in \mathbb{N}\setminus\{0\}. \tag{7.164}$$

Proof. We consider elliptic coordinates (ξ, η) defined as

$$\begin{cases} x_1 = \varepsilon r_1 \cosh(\xi)\cos(\eta), \\ x_2 = \varepsilon r_1 \sinh(\xi)\sin(\eta), \end{cases} \qquad \xi \geq 0,\ \eta \in [0, 2\pi), \tag{7.165}$$

see e.g. [3, Section 2.2]. In this coordinates S_ε is described by the conditions

$$\xi = 0, \quad \eta \in [0, 2\pi),$$

whereas $E_\varepsilon(L)$ is described by

$$\xi \in [0, \xi_\varepsilon), \quad \eta \in [0, 2\pi),$$

where ξ_ε is such that $r_1\varepsilon\sinh(\xi_\varepsilon) = L$, that is

$$\xi_\varepsilon = \operatorname{arcsinh}\left(\frac{L}{r_1\varepsilon}\right) = \log\left(\frac{L}{r_1\varepsilon} + \sqrt{1 + \frac{L^2}{r_1^2\varepsilon^2}}\right). \tag{7.166}$$

In particular $\partial E_\varepsilon(L)$ is described by the conditions

$$\xi = \xi_\varepsilon, \quad \eta \in [0, 2\pi).$$

The map

$$F_\varepsilon : [0, \xi_\varepsilon) \times [0, 2\pi) \to E_\varepsilon(L), \quad F_\varepsilon(\xi, \eta) = (x_1, x_2),$$

defined by (7.165), has a Jacobian matrix of the form

$$J_{F_\varepsilon}(\xi, \eta) = \varepsilon r_1 \sqrt{\cosh^2\xi - \cos^2\eta}\; O(\xi, \eta)$$

for some orthogonal matrix $O(\xi, \eta)$, and $\det J_{F_\varepsilon}(\xi, \eta) = \varepsilon^2 r_1^2(\cosh^2\xi - \cos^2\eta)$. In particular F_ε is a conform mapping and $J_{F_\varepsilon}(\xi, \eta)$ is an invertible matrix if $(\xi, \eta) \neq (0, 0)$ and $(\xi, \eta) \neq (0, \pi)$.

Let $\widehat{V}_{\varepsilon,P_m,L} := V_{\varepsilon,P_m,E_\varepsilon(L)} \circ F_\varepsilon$, where $V_{\varepsilon,P_m,E_\varepsilon(L)}$ is the solution of (7.158) in the case $\Lambda = E_\varepsilon(L)$ and $h = P_m$. We observe that, since $F_\varepsilon(\xi, \eta) \in \mathbb{R}^2_+$ if $\eta \in (0, \pi)$ and $F_\varepsilon(\xi, \eta) \in \mathbb{R}^2_-$ if $\eta \in (\pi, 2\pi)$,

$$\widehat{V}_{\varepsilon,P_m,L}(0, \eta) = \begin{cases} \gamma_+(V_{\varepsilon,P_m,E_\varepsilon(L)})(\varepsilon r_1\cos\eta, 0), & \text{if } \eta \in (0, \pi), \\ \gamma_-(V_{\varepsilon,P_m,E_\varepsilon(L)})(\varepsilon r_1\cos\eta, 0), & \text{if } \eta \in (\pi, 2\pi). \end{cases}$$

Furthermore,

$$\frac{\partial \widehat{V}_{\varepsilon,P_m,L}}{\partial \xi}(0,\eta) = \begin{cases} \varepsilon r_1 (\sin\eta)\, \gamma_+ \left(\dfrac{\partial V_{\varepsilon,P_m,E_\varepsilon(L)}}{\partial x_2}\right)(\varepsilon r_1 \cos\eta, 0), & \text{if } \eta \in (0,\pi), \\ \varepsilon r_1 (\sin\eta)\, \gamma_- \left(\dfrac{\partial V_{\varepsilon,P_m,E_\varepsilon(L)}}{\partial x_2}\right)(\varepsilon r_1 \cos\eta, 0), & \text{if } \eta \in (\pi,2\pi). \end{cases}$$

We also note that, for every $\eta \in [0,2\pi)$,

$$P_m(F_\varepsilon(0,\eta)) = (\varepsilon r_1)^m \ell_0 (\cos\eta)^m \quad \text{and} \quad \frac{\partial P_m}{\partial x_2}(F_\varepsilon(0,\eta)) = \ell_1 (\varepsilon r_1)^{m-1} (\cos\eta)^{m-1}.$$

Therefore, $\widehat{V}_{\varepsilon,P_m,L}$ solves the problem

$$\begin{cases} -\Delta \widehat{V}_{\varepsilon,P_m,L} = 0, & \text{in } (0,\xi_\varepsilon)\times(0,2\pi), \\ \widehat{V}_{\varepsilon,P_m,L}(\xi_\varepsilon,\eta) = 0, & \text{for all } \eta \in [0,2\pi), \\ \widehat{V}_{\varepsilon,P_m,L}(\xi,0) = \widehat{V}_{\varepsilon,P_m,L}(\xi,2\pi), & \text{for all } \xi \in (0,\xi_\varepsilon), \\ \widehat{V}_{\varepsilon,P_m,L}(0,\eta) + \widehat{V}_{\varepsilon,P_m,L}(0,2\pi-\eta) = 2\ell_0 (\varepsilon r_1)^m (\cos\eta)^m, & \text{for all } \eta \in (0,\pi), \\ \dfrac{\partial \widehat{V}_{\varepsilon,P_m,L}}{\partial \xi}(0,\eta) - \dfrac{\partial \widehat{V}_{\varepsilon,P_m,L}}{\partial \xi}(0,2\pi-\eta) & \\ = 2\ell_1 (\varepsilon r_1)^m (\cos\eta)^{m-1} \sin\eta, & \text{for all } \eta \in (0,\pi). \end{cases} \tag{7.167}$$

Let us consider the Fourier expansion of $(\varepsilon r_1)^{-m}\widehat{V}_{\varepsilon,P_m,L}$ with respect to the variable η

$$\frac{1}{(\varepsilon r_1)^m}\widehat{V}_{\varepsilon,P_m,L}(\xi,\eta) = \frac{a_{0,\varepsilon}(\xi)}{2} + \sum_{j=1}^{\infty}\Big(a_{j,\varepsilon}(\xi)\cos(j\eta) + b_{j,\varepsilon}(\xi)\sin(j\eta)\Big),$$

where

$$a_{j,\varepsilon}(\xi) := \frac{(\varepsilon r_1)^{-m}}{\pi}\int_0^{2\pi} \widehat{V}_{\varepsilon,P_m,L}(\xi,\eta)\cos(j\eta)\,d\eta \quad \text{for all } j \in \mathbb{N},$$
$$b_{j,\varepsilon}(\xi) := \frac{(\varepsilon r_1)^{-m}}{\pi}\int_0^{2\pi} \widehat{V}_{\varepsilon,P_m,L}(\xi,\eta)\sin(j\eta)\,d\eta \quad \text{for all } j \in \mathbb{N}\setminus\{0\}.$$

Since $\cos(2\pi-\eta) = \cos\eta$ for any $\eta \in (0,\pi)$, from (7.167) it follows that

$$a_{0,\varepsilon}(0) + 2\sum_{j=1}^{\infty} a_{j,\varepsilon}(0)\cos(j\eta) = 2\ell_0(\cos\eta)^m \qquad \text{for all } \eta \in (0,2\pi),$$

hence $\{a_{j,\varepsilon}(0)\}_{j\in\mathbb{N}}$ are the Fourier coefficients of $\ell_0(\cos\eta)^m$ with respect to the orthonormal basis $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(j\eta), \frac{1}{\sqrt{\pi}}\sin(j\eta)\}_{j\in\mathbb{N}\setminus\{0\}}$ of $L^2(0,2\pi)$, i.e.

$$a_{j,\varepsilon}(0) = \ell_0 c_j \quad \text{for all } j \in \mathbb{N},$$

with c_j as in (7.163). In particular

$$a_{j,\varepsilon}(0) = \ell_0 c_j = 0 \quad \text{if } j > m. \tag{7.168}$$

On the other hand, the last condition in (7.167) reads as

$$\sum_{j=1}^{\infty} b'_{j,\varepsilon}(0)\sin(j\eta) = \ell_1(\cos\eta)^{m-1}\sin\eta \quad \text{for all } \eta \in (0,2\pi).$$

It follows that $b'_{j,\varepsilon}(0)$ are independent of ε and

$$b'_{j,\varepsilon}(0) = \ell_1 d_j \quad \text{for all } j \in \mathbb{N}\setminus\{0\},$$

with d_j as in (7.164); hence

$$b'_{j,\varepsilon}(0) = \ell_1 d_j = 0 \quad \text{if } j > m. \tag{7.169}$$

From the equation in (7.167) it follows that

$$\begin{aligned} 0 &= \frac{1}{(\varepsilon r_1)^m}\Delta\widehat{V}_{\varepsilon,P_m,L} \\ &= \frac{a''_{0,\varepsilon}(\xi)}{2} + \sum_{j=1}^{\infty}\Big((a''_{j,\varepsilon}(\xi) - j^2 a_{j,\varepsilon}(\xi))\cos(j\eta) + (b''_{j,\varepsilon}(\xi) - j^2 b_{j,\varepsilon}(\xi))\sin(j\eta)\Big), \end{aligned}$$

hence

$$\begin{aligned} a_{0,\varepsilon}(\xi) &= -\frac{a_{0,\varepsilon}(0)}{\xi_\varepsilon}\xi + a_{0,\varepsilon}(0) = -\frac{\ell_0 c_0}{\xi_\varepsilon}\xi + \ell_0 c_0 \quad \text{for all } \xi \in (0,\xi_\varepsilon), \\ a_{j,\varepsilon}(\xi) &= \ell_0 c_j \left(\frac{e^{j\xi}}{1-e^{2j\xi_\varepsilon}} + \frac{e^{-j\xi}}{1-e^{-2j\xi_\varepsilon}}\right) \quad \text{for all } \xi \in (0,\xi_\varepsilon) \text{ and } j \in \mathbb{N}\setminus\{0\}, \\ b_{j,\varepsilon}(\xi) &= \frac{\ell_1 d_j}{j}\left(\frac{e^{j\xi}}{1+e^{2j\xi_\varepsilon}} - \frac{e^{-j\xi}}{1+e^{-2j\xi_\varepsilon}}\right) \quad \text{for all } \xi \in (0,\xi_\varepsilon) \text{ and } j \in \mathbb{N}\setminus\{0\}, \end{aligned} \tag{7.170}$$

with ξ_ε as in (7.166). Then, by (7.168) and (7.169), $a_{j,\varepsilon} \equiv b_{j,\varepsilon} \equiv 0$ for all $j > m$, so that

$$\frac{1}{(\varepsilon r_1)^m}\widehat{V}_{\varepsilon,P_m,L}(\xi,\eta) = \frac{a_{0,\varepsilon}(\xi)}{2} + \sum_{j=1}^{m}\Big(a_{j,\varepsilon}(\xi)\cos(j\eta) + b_{j,\varepsilon}(\xi)\sin(j\eta)\Big).$$

By a change of variables and the Parseval identity,

$$\begin{aligned} \int_{E_\varepsilon(L)\setminus S_\varepsilon} |\nabla V_{\varepsilon,P_m,E_\varepsilon(L)}|^2\,dx &= \int_0^{\xi_\varepsilon}\int_0^{2\pi} |\nabla \widehat{V}_{\varepsilon,P_m,L}|^2\,d\eta\,d\xi \\ &= (\varepsilon r_1)^{2m}\frac{\pi}{2}\int_0^{\xi_\varepsilon} |a'_{0,\varepsilon}(\xi)|^2\,d\xi \\ &\quad + (\varepsilon r_1)^{2m}\pi\sum_{j=1}^{m}\int_0^{\xi_\varepsilon}\Big(|a'_{j,\varepsilon}(\xi)|^2 + j^2|b_{j,\varepsilon}(\xi)|^2 + |b'_{j,\varepsilon}(\xi)|^2 + j^2|a_{j,\varepsilon}(\xi)|^2\Big)\,d\xi. \end{aligned} \tag{7.171}$$

Let us compute each integral in the above formula. In view of (7.170) and (7.166), it is clear that

$$\int_0^{\xi_\varepsilon} |a'_{0,\varepsilon}(\xi)|^2\,d\eta = \frac{\ell_0^2 c_0^2}{\xi_\varepsilon} = \frac{\ell_0^2 c_0^2}{|\log\varepsilon|} + O\left(\frac{1}{|\log\varepsilon|^2}\right) \quad \text{as } \varepsilon \to 0^+.$$

Furthermore, for every $j \in \mathbb{N} \setminus \{0\}$,

$$
\begin{aligned}
j^2 \int_0^{\xi_\varepsilon} |b_{j,\varepsilon}(\xi)|^2 \, d\xi &= \ell_1^2 d_j^2 \int_0^{\xi_\varepsilon} \left(\frac{e^{j\xi}}{1+e^{2j\xi_\varepsilon}} - \frac{e^{-j\xi}}{1+e^{-2j\xi_\varepsilon}} \right)^2 d\xi \\
&= \frac{\ell_1^2 d_j^2}{(1+e^{2j\xi_\varepsilon})^2} \int_0^{\xi_\varepsilon} e^{2j\xi} \, d\xi + \frac{\ell_1^2 d_j^2}{(1+e^{-2j\xi_\varepsilon})^2} \int_0^{\xi_\varepsilon} e^{-2j\xi} \, d\xi - \frac{2\ell_1^2 d_j^2 \xi_\varepsilon}{2+e^{-2j\xi_\varepsilon}+e^{2j\xi_\varepsilon}} \\
&= \frac{\ell_1^2 d_j^2}{2j} \left(\frac{1}{(1+e^{2j\xi_\varepsilon})^2}(e^{2j\xi_\varepsilon}-1) + \frac{1}{(1+e^{-2j\xi_\varepsilon})^2}(1-e^{-2j\xi_\varepsilon}) \right) - \frac{2\ell_1^2 d_j^2 \xi_\varepsilon}{2+e^{-2j\xi_\varepsilon}+e^{2j\xi_\varepsilon}} \\
&= \frac{\ell_1^2 d_j^2}{2j}(1+o(1)) \quad \text{as } \varepsilon \to 0^+
\end{aligned}
$$

and similarly

$$
\int_0^{\xi_\varepsilon} |b'_{j,\varepsilon}(\xi)|^2 \, d\xi = \ell_1^2 d_j^2 \int_0^{\xi_\varepsilon} \left(\frac{e^{j\xi}}{1+e^{2j\xi_\varepsilon}} + \frac{e^{-j\xi}}{1+e^{-2j\xi_\varepsilon}} \right)^2 d\xi = \frac{\ell_1^2 d_j^2}{2j}(1+o(1)) \quad \text{as } \varepsilon \to 0^+.
$$

Finally, for every $j \in \mathbb{N} \setminus \{0\}$,

$$
\begin{aligned}
\int_0^{\xi_\varepsilon} |a'_{j,\varepsilon}(\xi)|^2 \, d\xi &= j^2 \ell_0^2 c_j^2 \int_0^{\xi_\varepsilon} \left(\frac{e^{j\xi}}{1-e^{2j\xi_\varepsilon}} - \frac{e^{-j\xi}}{1-e^{-2j\xi_\varepsilon}} \right)^2 d\xi = \ell_0^2 c_j^2 \frac{j}{2}(1+o(1)), \\
j^2 \int_0^{\xi_\varepsilon} |a_{j,\varepsilon}(\xi)|^2 \, d\xi &= j^2 \ell_0^2 c_j^2 \int_0^{\xi_\varepsilon} \left(\frac{e^{j\xi}}{1-e^{2j\xi_\varepsilon}} + \frac{e^{-j\xi}}{1-e^{-2j\xi_\varepsilon}} \right)^2 d\xi = \ell_0^2 c_j^2 \frac{j}{2}(1+o(1)),
\end{aligned}
$$

as $\varepsilon \to 0^+$, as shown in the proof of [3, Lemma 2.3]. Replacing the above estimates in (7.171) we obtain (7.162). □

Proposition 7.6.3. *Let $m \in \mathbb{N} \setminus \{0\}$. For every $j \in \mathbb{N} \setminus \{0\}$, let c_j and d_j be as in (7.163) and (7.164), respectively. Then*

$$
\sum_{j=1}^{m} j|c_j|^2 = \frac{m}{4^{m-1}} \binom{m-1}{\lfloor \frac{m-1}{2} \rfloor}^2, \tag{7.172}
$$

$$
\sum_{j=1}^{m} \frac{1}{j}|d_j|^2 = \frac{1}{m4^{m-1}} \binom{m-1}{\lfloor \frac{m-1}{2} \rfloor}^2. \tag{7.173}
$$

Proof. For the proof of (7.172) we refer to [4, Proposition A.3]. To prove (7.173), we observe that, in view of (7.163),

$$
(\cos\eta)^m = \frac{c_0}{2} + \sum_{j=1}^{m} c_j \cos(j\eta) \quad \text{for all } \eta \in [0, 2\pi].
$$

Deriving the previous identity with respect to η, we obtain

$$
(\cos\eta)^{m-1} \sin\eta = \frac{1}{m} \sum_{j=1}^{m} j c_j \sin(j\eta) = \sum_{j=1}^{m} d_j \sin(j\eta) \quad \text{for all } \eta \in [0, 2\pi],
$$

in view of (7.164). It follows that $d_j = \frac{j}{m} c_j$ for all $j = 1, \dots, m$, hence (7.173) follows from (7.172). □

Remark 7.6.4. Let $m \in \mathbb{N}$, $m \geq 1$, and let P_m be a homogeneous polynomial of degree m as in (7.161). Let $\Lambda \subset \mathbb{R}^2$ be a simply connected open bounded domain such that $B_R \subseteq \Lambda$.

(i) If the coefficient ℓ_0 in (7.161) is zero, then $P_m \equiv 0$ on S_ε for all $\varepsilon \in (0,1]$. Hence $V_{\varepsilon,P_m,\Lambda} \in \widetilde{\mathcal{H}}_{\varepsilon,\Lambda}$ and, in view of (7.159), $\int_{\Lambda\setminus S_\varepsilon} |\nabla V_{\varepsilon,P_m,\Lambda}|^2\,dx = -L_{\varepsilon,P_m,\Lambda}(V_{\varepsilon,P_m,\Lambda})$, so that
$$\mathcal{E}_{\varepsilon,P_m,\Lambda} = -\frac{1}{2}\int_{\Lambda\setminus S_\varepsilon} |\nabla V_{\varepsilon,P_m,\Lambda}|^2\,dx.$$

(ii) If the coefficient ℓ_1 in (7.161) is zero, then $\frac{\partial P_m}{\partial x_2} \equiv 0$ on S_ε for all $\varepsilon \in (0,1]$. Hence $L_{\varepsilon,P_m,\Lambda} \equiv 0$ and
$$\mathcal{E}_{\varepsilon,P_m,\Lambda} = \frac{1}{2}\int_{\Lambda\setminus S_\varepsilon} |\nabla V_{\varepsilon,P_m,\Lambda}|^2\,dx.$$

Proposition 7.6.5. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected open bounded domain with $0 \in B_R \subseteq \Omega$. For every $\varepsilon \in (0,1)$, let S_ε be defined in (7.37). Let P_m be a homogeneous polynomial of degree m as in (7.161) and $\mathcal{E}_{\varepsilon,P_m,\Omega}$ be defined in (7.160) with $\Lambda = \Omega$ and $h = P_m$. Then, letting ℓ_0 and ℓ_1 be as in (7.161), we have*

(i) *if $\ell_0 = 0$, then*
$$\mathcal{E}_{\varepsilon,P_m,\Omega} = -\frac{\pi}{2} r_1^{2m}\ell_1^2\varepsilon^{2m}\frac{1}{m4^{m-1}}\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}^2 + o(\varepsilon^{2m}) \quad \textit{as } \varepsilon \to 0^+;$$

(ii) *if $\ell_1 = 0$, then*
$$\mathcal{E}_{\varepsilon,P_m,\Omega} = \frac{\pi}{2} r_1^{2m}\ell_0^2\varepsilon^{2m}\frac{m}{4^{m-1}}\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}^2 + o(\varepsilon^{2m}) \quad \textit{as } \varepsilon \to 0^+.$$

Proof. The set Ω is open and $0 \in \Omega$, hence there exist $L_1, L_2 > 0$ such that, for every $\varepsilon \in (0,1]$, $S_\varepsilon \subset E_\varepsilon(L_1) \subset \Omega \subset E_\varepsilon(L_2)$ (e.g. we can choose any $0 < L_1 < \sqrt{R^2 - r_1^2}$ and $L_2 = \operatorname{diam}\Omega$). From (7.157), (7.160), and the space inclusions $\mathcal{H}_{\varepsilon,E_\varepsilon(L_1)} \subset \mathcal{H}_{\varepsilon,\Omega} \subset \mathcal{H}_{\varepsilon,E_\varepsilon(L_2)}$, $\widetilde{\mathcal{H}}_{\varepsilon,E_\varepsilon(L_1)} \subset \widetilde{\mathcal{H}}_{\varepsilon,\Omega} \subset \widetilde{\mathcal{H}}_{\varepsilon,E_\varepsilon(L_2)}$ obtained by trivial extension, we deduce that, for every $\varepsilon \in (0,1]$,
$$\mathcal{E}_{\varepsilon,P_m,E_\varepsilon(L_2)} \leq \mathcal{E}_{\varepsilon,P_m,\Omega} \leq \mathcal{E}_{\varepsilon,P_m,E_\varepsilon(L_1)}. \tag{7.174}$$
If $\ell_0 = 0$, from Remark 7.6.4, (7.162), and (7.173) it follows that, for $i = 1,2$,
$$\begin{aligned}\mathcal{E}_{\varepsilon,P_m,E_\varepsilon(L_i)} &= -\frac{1}{2}\int_{E_\varepsilon(L_i)\setminus S_\varepsilon} |\nabla V_{\varepsilon,P_m,E_\varepsilon(L_i)}|^2\,dx \\ &= -\frac{\pi}{2}(\varepsilon r_1)^{2m}\ell_1^2\Big(\sum_{j=1}^m \frac{|d_j|^2}{j}\Big) + o(\varepsilon^{2m}) = -\frac{\pi}{2} r_1^{2m}\ell_1^2\varepsilon^{2m}\frac{1}{m4^{m-1}}\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}^2 + o(\varepsilon^{2m})\end{aligned}$$
as $\varepsilon \to 0^+$, thus proving (i) in view of (7.174).

On the other hand, if $\ell_1 = 0$, then Remark 7.6.4, (7.162), and (7.172) imply that, for $i = 1,2$,
$$\begin{aligned}\mathcal{E}_{\varepsilon,P_m,E_\varepsilon(L_i)} &= \frac{1}{2}\int_{E_\varepsilon(L_i)\setminus S_\varepsilon} |\nabla V_{\varepsilon,P_m,E_\varepsilon(L_i)}|^2\,dx \\ &= \frac{\pi}{2}(\varepsilon r_1)^{2m}\ell_0^2\Big(\sum_{j=1}^m j|c_j|^2\Big) + o(\varepsilon^{2m}) = \frac{\pi}{2} r_1^{2m}\ell_0^2\varepsilon^{2m}\frac{m}{4^{m-1}}\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}^2 + o(\varepsilon^{2m})\end{aligned}$$

as $\varepsilon \to 0^+$, thus proving (ii) in view of (7.174). □

Let u_0 be as in (7.35) with $u_0(0) = 0$ and m, β, α_0 be as in (7.38). Let T_m be the Taylor polynomial of u_0 centered at 0 of order m written in (7.39). In particular T_m is of the form (7.161) with

$$\ell_j = \frac{1}{(m-j)!j!} \frac{\partial^m u_0}{\partial x_1^{m-j} \partial x_2^j}(0).$$

If $\alpha_0 = \frac{j\pi}{m}$ for some $j \in \{0, 1, \dots, 2m-1\}$, then, by Remark 7.1.5,

$$\ell_0 = T_m(1, 0) = 0$$

and

$$\ell_1 = \frac{\partial T_m}{\partial x_2}(1, 0) = \nabla T_m(1, 0) \cdot (0, 1) = m\beta \cos(j\pi) = (-1)^j m\beta.$$

Hence, by Proposition 7.6.5, in this case we have

$$\mathcal{E}_{\varepsilon, T_m, \Omega} = -\frac{\pi}{2} r_1^{2m} \varepsilon^{2m} \frac{m\beta^2}{4^{m-1}} \binom{m-1}{\lfloor \frac{m-1}{2} \rfloor}^2 + o(\varepsilon^{2m}) \quad \text{as } \varepsilon \to 0^+. \tag{7.175}$$

On the other hand, if $\alpha_0 = \frac{\pi}{2m} + \frac{j\pi}{m}$ for some $j \in \{0, 1, \dots, 2m-1\}$, then, by Remark 7.1.5,

$$\ell_0 = T_m(1, 0) = -\beta \sin\left(\tfrac{\pi}{2} + j\pi\right) = (-1)^{j+1}\beta$$

and

$$\ell_1 = \frac{\partial T_m}{\partial x_2}(1, 0) = m\beta \cos\left(\tfrac{\pi}{2} + j\pi\right) = 0.$$

In this case, Proposition 7.6.5 then provides the expansion

$$\mathcal{E}_{\varepsilon, T_m, \Omega} = \frac{\pi}{2} r_1^{2m} \varepsilon^{2m} \frac{m\beta^2}{4^{m-1}} \binom{m-1}{\lfloor \frac{m-1}{2} \rfloor}^2 + o(\varepsilon^{2m}) \quad \text{as } \varepsilon \to 0^+. \tag{7.176}$$

Let $g := u_0 - T_m$. Since u_0 is smooth and T_m is its Taylor polynomial at 0 of order m, then

$$g(x) = O(|x|^{m+1}) \quad \text{and} \quad |\nabla g(x)| = O(|x|^m) \quad \text{as } x \to 0. \tag{7.177}$$

Proposition 7.6.6. *Let m and α_0 be as in (7.38). For every $\varepsilon \in (0, 1]$, let $V_{\varepsilon, T_m, \Omega}$ and $\mathcal{E}_{\varepsilon, T_m, \Omega}$ be as in (7.157) and (7.160), with $\Lambda = \Omega$ and $h = T_m$, and let $V_\varepsilon = V_{\varepsilon, u_0, \Omega}$ and $\mathcal{E}_\varepsilon = \mathcal{E}_{\varepsilon, u_0, \Omega}$ be as in (7.18) and (7.19), respectively. Then*

$$\|V_\varepsilon - V_{\varepsilon, T_m, \Omega}\|^2_{\mathcal{H}_\varepsilon} = O(\varepsilon^{2m+1}) = o(\varepsilon^{2m}) \quad \text{as } \varepsilon \to 0^+ \tag{7.178}$$

and, if either $\alpha_0 = \frac{j\pi}{m}$ or $\alpha_0 = \frac{\pi}{2m} + \frac{j\pi}{m}$ for some $j \in \{0, 1, \dots, 2m-1\}$,

$$\|V_\varepsilon\|_{\mathcal{H}_\varepsilon} = O(\varepsilon^m) \quad \text{as } \varepsilon \to 0^+, \tag{7.179}$$

$$\mathcal{E}_\varepsilon - \mathcal{E}_{\varepsilon, T_m, \Omega} = o(\varepsilon^{2m}) \quad \text{as } \varepsilon \to 0^+. \tag{7.180}$$

Proof. Let $W_\varepsilon := V_\varepsilon - V_{\varepsilon,T_m,\Omega}$. Then W_ε satisfies (7.159) with $h := g$. Let η_ε be as in (7.68). Testing (7.159) with $w = W_\varepsilon - \eta_\varepsilon g$, by Young's Inequality and (7.3) we obtain

$$\begin{aligned}\|W_\varepsilon\|^2_{\mathcal{H}_\varepsilon} &= \int_{\Omega\setminus S_\varepsilon} \eta_\varepsilon \nabla W_\varepsilon \cdot \nabla g\, dx + \int_{\Omega\setminus S_\varepsilon} g \nabla W_\varepsilon \cdot \nabla \eta_\varepsilon\, dx - 2\int_{S_\varepsilon} \frac{\partial g}{\partial x_2}\gamma^+(W_\varepsilon)\, dS + 2\int_{S_\varepsilon}\frac{\partial g}{\partial x_2} g\, dS \\ &\le \frac{1}{2}\|W_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + C\left(\int_\Omega \eta_\varepsilon^2 |\nabla g|^2\, dx + \int_\Omega g^2 |\nabla \eta_\varepsilon|^2\, dx + \int_{S_\varepsilon}\left|\frac{\partial g}{\partial x_2}\right|^2 dS\right) + 2\int_{S_\varepsilon}\left|\frac{\partial g}{\partial x_2}\right| |g|\, dS,\end{aligned}$$

for some positive constant $C > 0$. Hence (7.178) follows from (7.68) and (7.177).

We have

$$\begin{aligned}\mathcal{E}_\varepsilon - \mathcal{E}_{\varepsilon,T_m,\Omega} = \frac{1}{2}&\left(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon} - \|V_{\varepsilon,T_m,\Omega}\|^2_{\mathcal{H}_\varepsilon}\right) \\ &+ 2\int_{S_\varepsilon}\left(\frac{\partial u_0}{\partial x_2}\gamma_+(V_\varepsilon) - \frac{\partial T_m}{\partial x_2}\gamma_+(V_{\varepsilon,T_m,\Omega})\right) dS. \qquad (7.181)\end{aligned}$$

By Remark 7.6.4 and Proposition 7.6.5 we have that, if either $\alpha_0 = \frac{j\pi}{m}$ or $\alpha_0 = \frac{\pi}{2m} + \frac{j\pi}{m}$ for some $j \in \{0, 1, \dots, 2m-1\}$, then $\|V_{\varepsilon,T_m,\Omega}\|_{\mathcal{H}_\varepsilon} = \sqrt{2|\mathcal{E}_{\varepsilon,T_m,\Omega}|} = O(\varepsilon^m)$ as $\varepsilon \to 0^+$. Then, (7.179) follows from (7.178). Using again (7.178) we conclude that

$$\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon} - \|V_{\varepsilon,T_m,\Omega}\|^2_{\mathcal{H}_\varepsilon} = (V_\varepsilon - V_{\varepsilon,T_m,\Omega}, V_\varepsilon + V_{\varepsilon,T_m,\Omega})_{\mathcal{H}_\varepsilon} = o(\varepsilon^{2m}) \quad \text{as } \varepsilon \to 0^+. \qquad (7.182)$$

Furthermore, fixing some $p > 2$ and letting $p' = \frac{p}{p-1}$, Hölder's inequality, (7.177), and the continuity of the trace operators (7.3) imply that

$$\begin{aligned}&\left|\int_{S_\varepsilon}\left(\frac{\partial u_0}{\partial x_2}\gamma_+(V_\varepsilon) - \frac{\partial T_m}{\partial x_2}\gamma_+(V_{\varepsilon,T_m,\Omega})\right) dS\right| \\ &\quad = \left|\int_{S_\varepsilon}\left(\frac{\partial g}{\partial x_2}\gamma_+(V_\varepsilon) + \frac{\partial T_m}{\partial x_2}(\gamma_+(V_\varepsilon) - \gamma_+(V_{\varepsilon,T_m,\Omega}))\right) dS\right| \\ &\quad \le \int_{S_\varepsilon}\left|\frac{\partial g}{\partial x_2}\right| |\gamma_+(V_\varepsilon)|\, dS + \int_{S_\varepsilon}\left|\frac{\partial T_m}{\partial x_2}\right| |\gamma_+(W_\varepsilon)|\, dS \\ &\quad \le \text{const}\left(\varepsilon^{m+\frac{1}{p'}}\left(\int_{S_\varepsilon}|\gamma_+(V_\varepsilon)|^p\, dS\right)^{1/p} + \varepsilon^{m-1+\frac{1}{p'}}\left(\int_{S_\varepsilon}|\gamma_+(W_\varepsilon)|^p\, dS\right)^{1/p}\right) \\ &\quad \le \text{const}\left(\varepsilon^{m+\frac{1}{p'}}\|V_\varepsilon\|_{\mathcal{H}_\varepsilon} + \varepsilon^{m-1+\frac{1}{p'}}\|W_\varepsilon\|_{\mathcal{H}_\varepsilon}\right) \\ &\quad = O\left(\varepsilon^{2m+\frac{1}{p'}}\right) + O\left(\varepsilon^{2m-\frac{1}{2}+\frac{1}{p'}}\right) = o(\varepsilon^{2m}) \quad \text{as } \varepsilon \to 0^+, \qquad (7.183)\end{aligned}$$

where we used estimates (7.179) and (7.178). Combining (7.181), (7.182), and (7.183) we finally obtain (7.180). □

Proof of Theorems 7.1.6 and 7.1.7. Since we are considering only two opposite poles on the same line, we have $v_0 = e^{-i\Theta_0}u_0 = u_0$. Let $m \in \mathbb{N}\setminus\{0\}$ and $\alpha_0 \in [0, \frac{\pi}{m})$ be as in (7.38).

If $\alpha_0 = \frac{j\pi}{m}$ or $\alpha_0 = \frac{\pi}{2m} + \frac{j\pi}{m}$ for some $j \in \{0, 1, \dots, 2m-1\}$, then, by (7.38) (see Remark 7.1.5), $u_0(x) = T_m(x) + O(|x|^{m+1})$ and $\frac{\partial u_0}{\partial x_2}(x) = \frac{\partial T_m}{\partial x_2}(x) + O(|x|^m)$ as $x \to 0$, so that

$$L_\varepsilon(u_0) = 2\int_{S_\varepsilon}\frac{\partial T_m}{\partial x_2} T_m\, dS + O(\varepsilon^{2m+1}) = O(\varepsilon^{2m+1}) \quad \text{as } \varepsilon \to 0^+ \qquad (7.184)$$

since, in this case, either $T_m|_{S_\varepsilon} \equiv 0$ or $\frac{\partial T_m}{\partial x_2}|_{S_\varepsilon} \equiv 0$.

From Theorem 7.1.1, (7.184), (7.179), and (7.180), it follows that

$$\lambda_{\varepsilon,n_0} - \lambda_{0,n_0} = 2\mathcal{E}_\varepsilon - 2L_\varepsilon(u_0) + o(L_\varepsilon(u_0)) + o(\|V_\varepsilon\|^2_{\mathcal{H}_\varepsilon}) = 2\mathcal{E}_{\varepsilon,T_m,\Omega} + o(\varepsilon^{2m}) \quad \text{as } \varepsilon \to 0^+. \tag{7.185}$$

Theorem 7.1.6 follows from (7.185) and (7.175), while Theorem 7.1.7 is a consequence of (7.185) and (7.176). □

Remark 7.6.7. The case $m = 0$ has been omitted in the present section as, for $u_0(0) \neq 0$ the sharp expansion is already contained in [5] even without symmetry assumptions on the domain; however, the above argument could also apply in such a case, providing an alternative proof of the result of [5].

7.7 Dealing with more general configurations of poles

In this section, we give a hint on how our approach could be extended to treat other possible configurations of poles, which are not covered in detail for the sake of simplicity of exposition. By 7.1.1, the quantity that sharply measures the eigenvalue variation is $\mathcal{E}_\varepsilon - L_\varepsilon(v_0)$, where $\mathcal{E}_\varepsilon$ is as in (7.19), L_ε as in (7.16) and v_0 is the limit eigenfunction after a gauge transformation, thus solving (7.14). As explained in the introduction, $\mathcal{E}_\varepsilon$ is essentially an intermediate quantity between a capacity and a torsional rigidity, *measuring* the set $\cup_{j=1}^{k_1+k_2} S_\varepsilon^j$. For the success of our method it is important that the limit eigenfunction v_0 is regular on the sets S_ε^j, while the perturbed eigenfunction v_ε jumps on them, together with $\nabla v_\varepsilon \cdot \nu^j$. Our approach can be applied to all configurations of poles for which, after a gauge transformation as in 7.2.3, the origin belongs to the half-lines on which the perturbed eigenfunction v_ε jumps.

We provide below some examples. Since the gauge transformation for a configuration of poles is the composition of the gauge transformations of the families of poles lying on the same straight line, we now focus on a single set of k collinear poles. Hence, for sake of simplicity, we assume

$$\{a^j\}_{j=1,\dots,k} \subset B_R(0) \cap \{(x_1, 0) \colon x_1 \in \mathbb{R}\} \subset \Omega.$$

More precisely, we assume that $k = n_1 + n_2$, where $n_1, n_2 \in \mathbb{N}$ denote, respectively, the number of poles which lie on the left and on the right side with respect to the origin (either n_1 or n_2 might be zero). Namely,

$$a^j = \begin{cases} (-\delta_j, 0), & \text{for } j = 1, \dots, n_1, \\ (\delta_j, 0), & \text{for } j = n_1 + 1, \dots, n_1 + n_2, \end{cases}$$

where $\delta_j > 0$ are such that

$$-\delta_1 < -\delta_2 < \dots < -\delta_{n_1} < 0 < \delta_{n_1+1} < \dots < \delta_{n_1+n_2}.$$

For the above configuration, we consider problem (1.9). One of the following cases occurs:

(i) n_1 and n_2 are both even;

(ii) n_1 and n_2 are both odd;

(iii) n_1 is odd and n_2 is even (or vice versa).

The procedure developed to prove our main result 7.1.1 can be reproduced in cases (i) and (ii), as well as in case (iii) if $n_2 = 0$.

Let us now briefly describe, in these cases, how problem (1.9) becomes after a tailored gauge transformation. Hereafter, we denote by $\Sigma := \mathbb{R} \times \{0\}$ the x_1 axis, by $T\colon H^1(\mathbb{R}^2 \setminus \Sigma) \to L^p(\Sigma)$ the jump trace operator defined as in (7.4) with Σ instead of Σ^j, and by $\nu := (0, 1)$.

Case (i): even number of poles evenly distributed, i.e. $n_1 = 2N$ and $n_2 = 2M$ for some $N, M \in \mathbb{N}$ (see 7.8a). In this case, reasoning as in 7.2.3, it is possible to find a gauge transformation such that problem (1.9) is equivalent to

$$\begin{cases} -\Delta v = \lambda v, & \text{in } \Omega \setminus \bigcup_{j=1}^{N+M} S_\varepsilon^j, \\ v = 0, & \text{on } \partial\Omega, \\ T(v) = T(\nabla v \cdot \nu) = 0, & \text{on } \bigcup_{j=1}^{N+M} S_\varepsilon^j, \end{cases}$$

where

$$S_\varepsilon^j := \begin{cases} [-\varepsilon\delta_{2j-1}, -\varepsilon\delta_{2j}] \times \{0\}, & \text{if } j = 1, \dots, N, \\ [\varepsilon\delta_{2j-1}, \varepsilon\delta_{2j}] \times \{0\}, & \text{if } j = N+1, \dots, N+M. \end{cases}$$

Case (ii): even number of poles oddly distributed, i.e. $n_1 = 2N + 1$ and $n_2 = 2M + 1$ for some $N, M \in \mathbb{N}$ (see 7.8b). Once again, reasoning as in (7.2.3), one can find a gauge transformation such that problem (1.9) is equivalent to

$$\begin{cases} -\Delta v = \lambda v, & \text{in } \Omega \setminus \bigcup_{j=1}^{N+M+1} S_\varepsilon^j, \\ v = 0, & \text{on } \partial\Omega, \\ T(v) = T(\nabla v \cdot \nu) = 0, & \text{on } \bigcup_{j=1}^{N+M+1} S_\varepsilon^j, \end{cases}$$

where

$$S_\varepsilon^j := \begin{cases} [-\varepsilon\delta_{2j-1}, -\varepsilon\delta_{2j}] \times \{0\}, & \text{for } j = 1, \dots, N, \\ [-\varepsilon\delta_{2N+1}, \varepsilon\delta_{2N+2}] \times \{0\}, & \text{for } j = N+1, \\ [\varepsilon\delta_{2j-1}, \varepsilon\delta_{2j}] \times \{0\}, & \text{for } j = N+2, \dots, N+M+1. \end{cases}$$

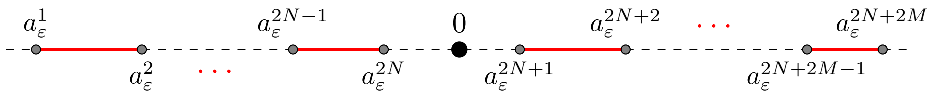
(iii): odd number of poles all on the same side, i.e. $n_1 = 2N + 1$ and $n_2 = 0$ (see 7.8c). In this case, problem (1.9) is equivalent to

$$\begin{cases} -\Delta v = \lambda v, & \text{in } \Omega \setminus \left[\Gamma_0 \cup \left(\bigcup_{j=1}^{N+1} S_\varepsilon^j\right)\right], \\ v = 0, & \text{on } \partial\Omega, \\ T(v) = T(\nabla v \cdot \nu) = 0, & \text{on } \Gamma_0, \\ T(v) = T(\nabla v \cdot \nu) = 0, & \text{on } \bigcup_{j=1}^{N+1} S_\varepsilon^j, \end{cases}$$

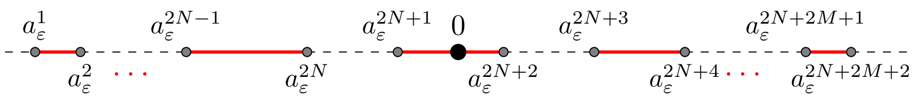
where

$$S_\varepsilon^j := \begin{cases} [-\varepsilon\delta_{2j-1}, -\varepsilon\delta_{2j}] \times \{0\}, & \text{for } j = 1, \dots, N, \\ [-\varepsilon\delta_{2N+1}, 0] \times \{0\}, & \text{for } j = N+1. \end{cases}$$

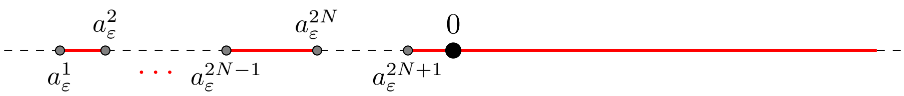
To conclude, the only case left open is case (iii) with $n_2 \neq 0$. This requires non-trivial technical adaptations and will be the object of future investigation.



(a) Case (i)



(b) Case (ii)



(c) Case (iii) with $n_2 = 0$

Figure 7.8: The jumping set after gauge transformation in cases (i), (ii), and (iii).

Bibliography

[1] Abatangelo, L., and Felli, V. Sharp asymptotic estimates for eigenvalues of Aharonov-Bohm operators with varying poles. *Calc. Var. Partial Differential Equations 54*, 4 (2015), 3857–3903.

[2] Abatangelo, L., and Felli, V. On the leading term of the eigenvalue variation for Aharonov-Bohm operators with a moving pole. *SIAM J. Math. Anal. 48*, 4 (2016), 2843–2868.

[3] Abatangelo, L., Felli, V., Hillairet, L., and Léna, C. Spectral stability under removal of small capacity sets and applications to Aharonov-Bohm operators. *J. Spectr. Theory 9*, 2 (2019), 379–427.

[4] Abatangelo, L., Felli, V., and Léna, C. On Aharonov-Bohm operators with two colliding poles. *Adv. Nonlinear Stud. 17*, 2 (2017), 283–296.

[5] Abatangelo, L., Felli, V., and Léna, C. Eigenvalue variation under moving mixed Dirichlet-Neumann boundary conditions and applications. *ESAIM Control Optim. Calc. Var. 26* (2020), Paper No. 39, 47.

[6] Abatangelo, L., Felli, V., Noris, B., and Nys, M. Sharp boundary behavior of eigenvalues for Aharonov-Bohm operators with varying poles. *J. Funct. Anal. 273*, 7 (2017), 2428–2487.

[7] Abatangelo, L., and Nys, M. On multiple eigenvalues for Aharonov-Bohm operators in planar domains. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods 169* (2018), 1–37.

[8] Abatangelo, L., and Ognibene, R. Sharp behavior of Dirichlet–Laplacian eigenvalues for a class of singularly perturbed problems. *Preprint* (2023), `arxiv.org/abs/2301.11729`.

[9] Abatangelo, N., and Dupaigne, L. Nonhomogeneous boundary conditions for the spectral fractional Laplacian. *Ann. Inst. H. Poincaré C Anal. Non Linéaire 34*, 2 (2017), 439–467.

[10] Abdellaoui, B., Medina, M., Peral, I., and Primo, A. The effect of the Hardy potential in some Calderón-Zygmund properties for the fractional Laplacian. *J. Differential Equations 260*, 11 (2016), 8160–8206.

[11] Abramowitz, M., and Stegun, I. A. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* National Bureau of Standards Applied Mathematics Series, No. 55. U. S. Government Printing Office, Washington, D.C., 1964. For sale by the Superintendent of Documents.

[12] Adolfsson, V., and Escauriaza, L. $C^{1,\alpha}$ domains and unique continuation at the boundary. *Comm. Pure Appl. Math. 50*, 10 (1997), 935–969.

[13] Adolfsson, V., Escauriaza, L., and Kenig, C. Convex domains and unique continuation at the boundary. *Rev. Mat. Iberoamericana 11*, 3 (1995), 513–525.

[14] Almgren, Jr., F. J. Q valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two. *Bull. Amer. Math. Soc. (N.S.) 8*, 2 (1983), 327–328.

[15] Alziary, B., Fleckinger-Pellé, J., and Takáč, P. Eigenfunctions and Hardy inequalities for a magnetic Schrödinger operator in $\mathbb{R}^2$. *Math. Methods Appl. Sci. 26*, 13 (2003), 1093–1136.

[16] Aronszajn, N. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. *J. Math. Pures Appl. (9) 36* (1957), 235–249.

[17] Arya, V., Banerjee, A., Danielli, D., and Garofalo, N. Space-like strong unique continuation for some fractional parabolic equations. *J. Funct. Anal. 284*, 1 (2023), Paper No. 109723, 38.

[18] Aubin, T. *Nonlinear analysis on manifolds. Monge-Ampère equations*, vol. 252 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982.

[19] Audrito, A., and Terracini, S. On the nodal set of solutions to a class of nonlocal parabolic equations. *To appear on Journal of the American Mathematical Society*.

[20] Badiale, M., and Tarantello, G. A Sobolev-Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics. *Arch. Ration. Mech. Anal. 163*, 4 (2002), 259–293.

[21] Banerjee, A., and Garofalo, N. Monotonicity of generalized frequencies and the strong unique continuation property for fractional parabolic equations. *Adv. Math. 336* (2018), 149–241.

[22] Banerjee, A., Garofalo, N., and Manna, R. A strong unique continuation property for the heat operator with Hardy type potential. *J. Geom. Anal. 31*, 6 (2021), 5480–5504.

[23] Bernard, J.-M. E. Density results in Sobolev spaces whose elements vanish on a part of the boundary. *Chin. Ann. Math. Ser. B 32*, 6 (2011), 823–846.

[24] Biswas, A., De León-Contreras, M., and Raúl Stinga, P. Harnack inequalities and Hölder estimates for master equations. *SIAM J. Math. Anal. 53*, 2 (2021), 2319–2348.

[25] Bogdan, K., Grzywny, T., Jakubowski, T., and Pilarczyk, D. Fractional Laplacian with Hardy potential. *Comm. Partial Differential Equations 44*, 1 (2019), 20–50.

[26] Bonforte, M., Sire, Y., and Vázquez, J. L. Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains. *Discrete Contin. Dyn. Syst. 35*, 12 (2015), 5725–5767.

[27] Bonnaillie-Noël, V., Helffer, B., and Hoffmann-Ostenhof, T. Aharonov-Bohm Hamiltonians, isospectrality and minimal partitions. *J. Phys. A 42*, 18 (2009), 185203, 20.

[28] Bonnaillie-Noël, V., Noris, B., Nys, M., and Terracini, S. On the eigenvalues of Aharonov-Bohm operators with varying poles. *Anal. PDE 7*, 6 (2014), 1365–1395.

[29] Brändle, C., Colorado, E., de Pablo, A., and Sánchez, U. A concave-convex elliptic problem involving the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A 143*, 1 (2013), 39–71.

[30] Brasco, L., del Mar González, M., and Ispizua, M. A Steklov version of the torsional rigidity. *Preprint* (2022).

[31] Brasco, L., Lindgren, E., and Parini, E. The fractional Cheeger problem. *Interfaces Free Bound. 16*, 3 (2014), 419–458.

[32] Brasco, L., and Mazzoleni, D. On principal frequencies, volume and inradius in convex sets. *NoDEA, Nonlinear Differ. Equ. Appl. 27*, 2 (2020), 26. Id/No 12.

[33] Cabré, X., and Sire, Y. Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates. *Ann. Inst. H. Poincaré Anal. Non Linéaire 31*, 1 (2014), 23–53.

[34] Cabré, X., and Tan, J. Positive solutions of nonlinear problems involving the square root of the Laplacian. *Adv. Math. 224*, 5 (2010), 2052–2093.

[35] Caffarelli, L., and Silvestre, L. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations 32*, 7-9 (2007), 1245–1260.

[36] Calderón, A.-P. Lebesgue spaces of differentiable functions and distributions. In *Proc. Sympos. Pure Math., Vol. IV* (1961), American Mathematical Society, Providence, R.I., pp. 33–49.

[37] Capella, A., Dávila, J., Dupaigne, L., and Sire, Y. Regularity of radial extremal solutions for some non-local semilinear equations. *Comm. Partial Differential Equations 36*, 8 (2011), 1353–1384.

[38] Carleman, T. Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat., Astr. Fys. 26*, 17 (1939), 9.

[39] Chen, X.-Y. A strong unique continuation theorem for parabolic equations. *Math. Ann. 311*, 4 (1998), 603–630.

[40] Chkadua, O., and Duduchava, R. Asymptotics of functions represented by potentials. *Russ. J. Math. Phys. 7*, 1 (2000), 15–47.

[41] Costabel, M., Dauge, M., and Duduchava, R. Asymptotics without logarithmic terms for crack problems. *Comm. Partial Differential Equations 28*, 5-6 (2003), 869–926.

[42] Courtois, G. Spectrum of manifolds with holes. *J. Funct. Anal. 134*, 1 (1995), 194–221.

[43] Dal Maso, G., Orlando, G., and Toader, R. Laplace equation in a domain with a rectilinear crack: higher order derivatives of the energy with respect to the crack length. *NoDEA Nonlinear Differential Equations Appl. 22*, 3 (2015), 449–476.

[44] Dautray, R., and Lions, J.-L. *Mathematical analysis and numerical methods for science and technology. Vol. 5*. Springer-Verlag, Berlin, 1992. Evolution problems. I, With the collaboration of Michel Artola, Michel Cessenat and Hélène Lanchon, Translated from the French by Alan Craig.

[45] De Luca, A., and Felli, V. Unique continuation from the edge of a crack. *Math. Eng. 3*, 3 (2021), Paper No. 023, 40.

[46] De Luca, A., Felli, V., and Siclari, G. Strong unique continuation from the boundary for the spectral fractional Laplacian. *ESAIM Control Optim. Calc. Var. 29* (2023), Paper No. 50, 37.

[47] De Luca, A., Felli, V., and Vita, S. Strong unique continuation and local asymptotics at the boundary for fractional elliptic equations. *Adv. Math. 400* (2022), Paper No. 108279, 67.

[48] Di Nezza, E., Palatucci, G., and Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math. 136*, 5 (2012), 521–573.

[49] Dipierro, S., Felli, V., and Valdinoci, E. Unique continuation principles in cones under nonzero Neumann boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire 37*, 4 (2020), 785–815.

[50] Dong, H., and Phan, T. On parabolic and elliptic equations with singular or degenerate coefficients. *arXiv preprint arXiv:2007.04385* (2020).

[51] Dong, H., and Phan, T. Regularity for parabolic equations with singular or degenerate coefficients. *Calc. Var. Partial Differential Equations 60*, 1 (2021), Paper No. 44, 39.

[52] Duduchava, R., and Wendland, W. L. The Wiener-Hopf method for systems of pseudodifferential equations with an application to crack problems. *Integral Equations Operator Theory 23*, 3 (1995), 294–335.

[53] Escauriaza, L. Carleman inequalities and the heat operator. *Duke Math. J. 104*, 1 (2000), 113–127.

[54] Escauriaza, L., and Fernández, F. J. Unique continuation for parabolic operators. *Ark. Mat. 41*, 1 (2003), 35–60.

[55] Escauriaza, L., Kenig, C. E., Ponce, G., and Vega, L. Decay at infinity of caloric functions within characteristic hyperplanes. *Math. Res. Lett. 13*, 2-3 (2006), 441–453.

[56] Escauriaza, L., and Vega, L. Carleman inequalities and the heat operator. II. *Indiana Univ. Math. J. 50*, 3 (2001), 1149–1169.

[57] Evans, L. C. *Partial differential equations*, second ed., vol. 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010.

[58] Fabes, E. B., Kenig, C. E., and Serapioni, R. P. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations 7*, 1 (1982), 77–116.

[59] Fall, M. M. Semilinear elliptic equations for the fractional Laplacian with Hardy potential. *Nonlinear Anal. 193* (2020), 111311, 29.

[60] Fall, M. M., and Felli, V. Unique continuation property and local asymptotics of solutions to fractional elliptic equations. *Comm. Partial Differential Equations 39*, 2 (2014), 354–397.

[61] Fall, M. M., Felli, V., Ferrero, A., and Niang, A. Asymptotic expansions and unique continuation at Dirichlet-Neumann boundary junctions for planar elliptic equations. *Math. Eng. 1*, 1 (2019), 84–117.

[62] Fall, M. M., and Weth, T. Nonexistence results for a class of fractional elliptic boundary value problems. *J. Funct. Anal. 263*, 8 (2012), 2205–2227.

[63] Felli, V., and Ferrero, A. Almgren-type monotonicity methods for the classification of behaviour at corners of solutions to semilinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A 143*, 5 (2013), 957–1019.

[64] Felli, V., and Ferrero, A. Unique continuation principles for a higher order fractional Laplace equation. *Nonlinearity 33*, 8 (2020), 4133–4190.

[65] Felli, V., Ferrero, A., and Terracini, S. Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential. *J. Eur. Math. Soc. (JEMS) 13*, 1 (2011), 119–174.

[66] Felli, V., Ferrero, A., and Terracini, S. Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential. *J. Eur. Math. Soc. (JEMS) 13*, 1 (2011), 119–174.

[67] Felli, V., Ferrero, A., and Terracini, S. A note on local asymptotics of solutions to singular elliptic equations via monotonicity methods. *Milan J. Math. 80*, 1 (2012), 203–226.

[68] Felli, V., Ferrero, A., and Terracini, S. On the behavior at collisions of solutions to Schrödinger equations with many-particle and cylindrical potentials. *Discrete Contin. Dyn. Syst. 32*, 11 (2012), 3895–3956.

[69] Felli, V., Mukherjee, D., and Ognibene, R. On fractional multi-singular Schrödinger operators: positivity and localization of binding. *J. Funct. Anal. 278*, 4 (2020), 108389, 47.

[70] Felli, V., Noris, B., Ognibene, R., and Siclari, G. Quantitative spectral stability for Aharonov-Bohm operators with many coalescing poles. *arXiv preprint arXiv:2306.05008* (2023).

[71] Felli, V., Noris, B., and Siclari, G. Quantitative spectral stability for Aharonov-Bohm operators with real circulation. *In preparation* (2023).

[72] Felli, V., and Primo, A. Classification of local asymptotics for solutions to heat equations with inverse-square potentials. *Discrete Contin. Dyn. Syst. 31*, 1 (2011), 65–107.

[73] Felli, V., and Primo, A. On parabolic equations with critical electromagnetic potentials. *J. Differential Equations 268*, 2 (2020), 738–783.

[74] Felli, V., Primo, A., and Siclari, G. On fractional parabolic equations with Hardy-type potentials. *To appear on Communications in Contemporary Mathematics* (2022).

[75] Felli, V., and Siclari, G. Sobolev-type regularity and Pohozaev-type identities for some degenerate and singular problems. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 33*, 3 (2022), 553–574.

[76] Felli, V., and Siclari, G. Unique continuation from a crack's tip under Neumann boundary conditions. *Nonlinear Anal. 222* (2022), Paper No. 113002, 36.

[77] Fernandez, F. J. Unique continuation for parabolic operators. II. *Comm. Partial Differential Equations 28*, 9-10 (2003), 1597–1604.

[78] Frank, R. L., Merz, K., and Siedentop, H. Equivalence of Sobolev norms involving generalized Hardy operators. *Int. Math. Res. Not. IMRN*, 3 (2021), 2284–2303.

[79] Garofalo, N., and Lin, F.-H. Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J. 35*, 2 (1986), 245–268.

[80] Gilbarg, D., and Trudinger, N. S. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[81] Grisvard, P. *Elliptic problems in nonsmooth domains*, vol. 69 of *Classics in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683], With a foreword by Susanne C. Brenner.

[82] Grubb, G. Regularity of spectral fractional Dirichlet and Neumann problems. *Math. Nachr. 289*, 7 (2016), 831–844.

[83] Hardy, G. H., Littlewood, J. E., and Pólya, G. *Inequalities.* Cambridge, at the University Press, 1952. 2d ed.

[84] Helffer, B. *Spectral theory and its applications*, vol. 139 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.

[85] Henrot, A., and Pierre, M. *Shape variation and optimization. A geometrical analysis*, vol. 28 of *EMS Tracts Math.* Zürich: European Mathematical Society (EMS), 2018.

[86] Herbst, I. W. Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$. *Comm. Math. Phys. 53*, 3 (1977), 285–294.

[87] Hytönen, T., van Neerven, J., Veraar, M., and Weis, L. *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory*, vol. 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2016.

[88] Jerison, D., and Kenig, C. E. Unique continuation and absence of positive eigenvalues for Schrödinger operators. *Ann. of Math. (2) 121*, 3 (1985), 463–494. With an appendix by E. M. Stein.

[89] Jin, T., Li, Y., and Xiong, J. On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions. *J. Eur. Math. Soc. (JEMS) 16*, 6 (2014), 1111–1171.

[90] Khludnev, A., Leontiev, A., and Herskovits, J. Nonsmooth domain optimization for elliptic equations with unilateral conditions. *J. Math. Pures Appl. (9) 82*, 2 (2003), 197–212.

[91] Kilpeläinen, T. Weighted Sobolev spaces and capacity. *Ann. Acad. Sci. Fenn. Ser. A I Math. 19*, 1 (1994), 95–113.

[92] Koch, H., and Tataru, D. Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients. *Communications on Pure and Applied Mathematics 54*, 3 (2001), 339–360.

[93] Kufner, A. *Weighted Sobolev spaces.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985. Translated from the Czech.

[94] Kukavica, I., and Nyström, K. Unique continuation on the boundary for Dini domains. *Proc. Amer. Math. Soc. 126*, 2 (1998), 441–446.

[95] Laptev, A., and Weidl, T. Hardy inequalities for magnetic Dirichlet forms. In *Mathematical results in quantum mechanics (Prague, 1998)*, vol. 108 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 1999, pp. 299–305.

[96] Leinfelder, H. Gauge invariance of Schrödinger operators and related spectral properties. *J. Operator Theory 9*, 1 (1983), 163–179.

[97] Léna, C. Eigenvalues variations for Aharonov-Bohm operators. *J. Math. Phys. 56*, 1 (2015), 011502, 18.

[98] Leoni, G. *A first course in Sobolev spaces*, vol. 105 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[99] Leoni, G. *A first course in Sobolev spaces*, second ed., vol. 181 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2017.

[100] Lin, F.-H. A uniqueness theorem for parabolic equations. *Comm. Pure Appl. Math. 43*, 1 (1990), 127–136.

[101] Lions, J.-L., and Magenes, E. *Non-homogeneous boundary value problems and applications. Vol. I.* Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth.

[102] Lischke, A., Pang, G., Gulian, M., and et al. What is the fractional Laplacian? A comparative review with new results. *J. Comput. Phys. 404* (2020), 109009, 62.

[103] MacDonald, A. D. Properties of the confluent hypergeometric function. *J. Math. Physics 28* (1949), 183–191.

[104] Maz'ja, V. G. *Sobolev spaces.* Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

[105] Medková, D., and Krutitskii, P. Neumann and Robin problems in a cracked domain with jump conditions on cracks. *J. Math. Anal. Appl. 301*, 1 (2005), 99–114.

[106] Nekvinda, A. Characterization of traces of the weighted Sobolev space $W^{1,p}(\Omega, d_M^\epsilon)$ on M. *Czechoslovak Math. J. 43(118)*, 4 (1993), 695–711.

[107] Nirenberg, L. Remarks on strongly elliptic partial differential equations. *Comm. Pure Appl. Math. 8* (1955), 649–675.

[108] Noris, B., Nys, M., and Terracini, S. On the Aharonov-Bohm operators with varying poles: the boundary behavior of eigenvalues. *Comm. Math. Phys. 339*, 3 (2015), 1101–1146.

[109] Noris, B., and Terracini, S. Nodal sets of magnetic Schrödinger operators of Aharonov-Bohm type and energy minimizing partitions. *Indiana Univ. Math. J. 59*, 4 (2010), 1361–1403.

[110] Nyström, K., and Sande, O. Extension properties and boundary estimates for a fractional heat operator. *Nonlinear Anal. 140* (2016), 29–37.

[111] Opic, B., and Kufner, A. *Hardy-type inequalities*, vol. 219 of *Pitman Research Notes in Mathematics Series.* Longman Scientific & Technical, Harlow, 1990.

[112] Pólya, G., and Szegö, G. *Isoperimetric inequalities in mathematical physics*, vol. 27 of *Ann. Math. Stud.* Princeton University Press, Princeton, NJ, 1951.

[113] Poon, C.-C. Unique continuation for parabolic equations. *Comm. Partial Differential Equations 21*, 3-4 (1996), 521–539.

[114] Rüland, A. Unique continuation for fractional Schrödinger equations with rough potentials. *Comm. Partial Differential Equations 40*, 1 (2015), 77–114.

[115] Saut, J.-C., and Scheurer, B. Unique continuation for some evolution equations. *J. Differential Equations 66*, 1 (1987), 118–139.

[116] Secchi, S., Smets, D., and Willem, M. Remarks on a Hardy-Sobolev inequality. *C. R. Math. Acad. Sci. Paris 336*, 10 (2003), 811–815.

[117] Servadei, R., and Valdinoci, E. On the spectrum of two different fractional operators. *Proc. Roy. Soc. Edinburgh Sect. A 144*, 4 (2014), 831–855.

[118] Siclari, G. On the fractional powers of a Schrödinger operator with a Hardy-type potential. *arXiv preprint arXiv:2302.08259* (2023).

[119] Simon, J. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4) 146* (1987), 65–96.

[120] Sire, Y., Terracini, S., and Vita, S. Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions. *Comm. Partial Differential Equations 46*, 2 (2021), 310–361.

[121] Sire, Y., Terracini, S., and Vita, S. Liouville type theorems and regularity of solutions to degenerate or singular problems part II: odd solutions. *Math. Eng. 3*, 1 (2021), Paper No. 5, 50.

[122] Sogge, C. D. Oscillatory integrals and unique continuation for second order elliptic differential equations. *J. Amer. Math. Soc. 2*, 3 (1989), 491–515.

[123] Sogge, C. D. A unique continuation theorem for second order parabolic differential operators. *Ark. Mat. 28*, 1 (1990), 159–182.

[124] Stein, E. M. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[125] Stinga, P. R., and Torrea, J. L. Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation. *SIAM J. Math. Anal. 49*, 5 (2017), 3893–3924.

[126] Tao, X., and Zhang, S. Boundary unique continuation theorems under zero Neumann boundary conditions. *Bull. Austral. Math. Soc. 72*, 1 (2005), 67–85.

[127] Tao, X., and Zhang, S. Weighted doubling properties and unique continuation theorems for the degenerate Schrödinger equations with singular potentials. *J. Math. Anal. Appl. 339*, 1 (2008), 70–84.

[128] Tartar, L. *An introduction to Sobolev spaces and interpolation spaces*, vol. 3 of *Lecture Notes of the Unione Matematica Italiana.* Springer, Berlin; UMI, Bologna, 2007.

[129] Teytel, M. How rare are multiple eigenvalues? *Commun. Pure Appl. Math. 52*, 8 (1999), 0917–0934.

[130] Tolsa, X. Unique continuation at the boundary for harmonic functions in C^1 domains and Lipschitz domains with small constant. *Comm. Pure Appl. Math. 76*, 2 (2023), 305–336.

[131] van den Berg, M., Buttazzo, G., and Velichkov, B. Optimization problems involving the first Dirichlet eigenvalue and the torsional rigidity. In *New trends in shape optimization*, vol. 166 of *Internat. Ser. Numer. Math.* Birkhäuser/Springer, Cham, 2015, pp. 19–41.

[132] Wang, Z.-Q., and Zhu, M. Hardy inequalities with boundary terms. *Electron. J. Differential Equations* (2003), No. 43, 8.

[133] Wolff, T. H. Unique continuation for $|\Delta u| \leq V|\nabla u|$ and related problems. *Rev. Mat. Iberoamericana 6*, 3-4 (1990), 155–200.

[134] Wolff, T. H. A property of measures in $\mathbf{R}^N$ and an application to unique continuation. *Geom. Funct. Anal. 2*, 2 (1992), 225–284.

[135] Yu, H. Unique continuation for fractional orders of elliptic equations. *Ann. PDE 3*, 2 (2017), Paper No. 16, 21.