Conservation laws with regulated fluxes

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January 7, 2019

Abstract

Scalar conservation laws $\partial_t u + \partial_x f(t, x, u) = 0$ where the flux f is discontinuous w.r.t. the time and space variables t, x arise in many applications, related to physical models in rough media. Typical examples include traffic flow with variable road conditions and polymer flooding in porous media. An extensive body of recent literature has dealt with fluxes that are discontinuous along a finite number of curves in the t-x plane. Here we are interested in the existence and uniqueness of solutions obtained via vanishing viscosity approximations i.e. solutions to $\partial_t u + \partial_x f(t, x, u) = \varepsilon \partial_{xx} u$ when $\varepsilon \to 0^+$, for more general discontinuous fluxes.

We first give a definition of regulated functions in two variables. After recalling some results about parabolic equations with discontinuous coefficients, we show how the knowledge of the existence and uniqueness of the vanishing viscosity limit for fluxes with a single discontinuity at $x = 0$ can be used as a building block to prove the existence and uniqueness of the vanishing viscosity limit for regulated fluxes.

Keywords: Nonlinear semigroups of contractions. Conservation law with discontinuous flux, regulated flux function, vanishing viscosity, Hamilton-Jacobi equation, existence and uniqueness of solutions.

2010 MSC: 35L65, 35R05

1 Introduction

We consider the Cauchy problem for a scalar conservation law of the form

$$\begin{cases} u_t + f(t, x, u)_x \;=\; 0, \\ u(0, x) \;=\; \bar{u}(x) \in \mathbf{L}^1(\mathbb{R}), \end{cases} \tag{1.1}$$

where the flux function f is smooth w.r.t. the unknown u but can be discontinuous w.r.t. both variables t and x. Our main concern is the convergence of the viscous approximations u^ε, which solve

$$\begin{cases} u_t + f(t, x, u)_x \;=\; \varepsilon\, u_{xx}, \\ u(0, x) \;=\; \bar{u}(x) \in \mathbf{L}^1(\mathbb{R}), \end{cases} \tag{1.2}$$

to a unique weak solution u to (1.1), as the viscosity parameter $\varepsilon \to 0^+$.

Starting with the works by N. Risebro and collaborators (see [2, 9, 10, 14] and references therein) scalar conservation laws with discontinuous coefficients have now become the subject of an extensive literature also including some multi-dimensional cases (see [1, 2, 7, 12, 13, 17] and references therein).

Results on the uniqueness and stability of vanishing viscosity solutions have been obtained mainly in the case where the flux f is piecewise smooth with discontinuities located on finitely many smooth curves on the (t,x) plane. Aim of this note is to describe an alternative approach, introduced in [3, 11], based on comparison estimates for solutions to the corresponding Hamilton–Jacobi equation. This yields the uniqueness of the vanishing viscosity limit under the more general assumption that $f(t,x,\omega) = F(v(t,x),\omega)$ where F is a smooth function and $v(t,x)$ is a *regulated* function of the two variables t and x, as in Definition 1.1 below.

We recall that a function of a single variable $v : \mathbb{R} \mapsto \mathbb{R}$ is *regulated* if it admits left and right limits at every point. This is true if and only if, for every interval $[x_1, x_2]$ and every $\varepsilon > 0$, there exists a piecewise constant function χ such that $\|\chi - v\|_{\mathbf{L}^\infty([x_1,x_2])} \leq \varepsilon$. We extend this concept to functions of two variables, as follows.

Definition 1.1. *(see Fig. 1) We say that a bounded function $v = v(t,x)$ is* **regulated** *if, for every intervals $[x_1, x_2]$ and $[0,T]$, and any $\varepsilon > 0$, the following holds.*

There exist finitely many disjoint subintervals $[a_i, b_i] \subseteq [0,T]$, Lipschitz continuous curves $\gamma_{i,1}(t) < \cdots < \gamma_{i,N_i}(t)$, $t \in [a_i, b_i]$, and constants $\alpha_{i,0}, \ldots, \alpha_{i,N_i}$ such that

(i) For every $t \in [a_i, b_i]$, the step function

$$\chi_i(t,x) \doteq \begin{cases} \alpha_{i,0}, & \text{if } x < \gamma_{i,1}(t), \\ \alpha_{i,k}, & \text{if } \gamma_{i,k}(t) < x < \gamma_{i,k+1}(t), \qquad k = 1,2,\ldots,N_i - 1, \\ \alpha_{i,N_i}, & \text{if } \gamma_{i,N_i}(t) < x, \end{cases} \tag{1.3}$$

satisfies $\|\chi_i(t,\cdot) - v(t,\cdot)\|_{\mathbf{L}^\infty([x_1,x_2])} \leq \varepsilon$.

(ii) For every i,k, the time derivative $\dot\gamma_{i,k}(t) = \frac{d}{dt}\gamma_{i,k}(t)$ coincides a.e. with a regulated function.

(iii) The intervals $[a_i, b_i]$ cover most of $[0,T]$, namely $T - \sum_i (b_i - a_i) \leq \varepsilon$.

We remark that, if $v = v(x)$ is independent of time, then it satisfies Definition 1.1 if and only if it is a regulated function in the usual sense.

Let $T > 0$ be given and consider the open domain $\Omega \doteq]0,T[\times \mathbb{R}$. For future use, we collect here some assumptions that will be imposed on the flux function $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$, at various stages of the analysis.

(F1) The function f satisfies:

(i) For each fixed $\omega \in \mathbb{R}$, the map $(t,x) \mapsto f(t,x,\omega)$ is in $\mathbf{L}^\infty(\Omega)$.

(ii) The map $\omega \mapsto f(t,x,\omega)$ is twice continuously differentiable for any $(t,x) \in \Omega$ and there exists a constant $L \geq 0$ such that

$$\big|f(t,x,\omega_1) - f(t,x,\omega_2)\big| \leq L\,|\omega_1 - \omega_2| \qquad \text{for all } \omega_1, \omega_2 \in \mathbb{R},\ (t,x) \in \Omega. \tag{1.4}$$

(iii) There exists a constant $L_1 \geq 0$ such that, $\int_{\mathbb{R}} \big|f(t,x,0)\big|\,dx \leq L_1$, for all $t \in]0,T[$.

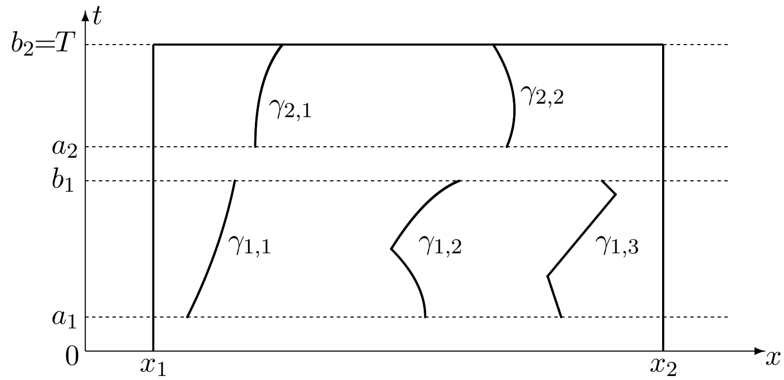

Figure 1: According to Definition 1.1, a **regulated** function of two variables $v = v(t,x)$ can be approximated by a piecewise constant function, with jumps along finitely many Lipschitz curves $\gamma_{i,k}$. The time derivatives $\dot{\gamma}_{i,k}$ are regulated functions.

(F2) For every $(t,x) \in \Omega$, the function f satisfies $f(t,x,0) = 0$ and $f(t,x,1) = h(t)$ for some $h \in \mathbf{L}^\infty\left(]0,T[\,,\mathbb{R}\right)$.

(F3) The flux f has the form $f(t,x,\omega) = F\big(v(t,x),\omega\big)$, where $F(\alpha,\omega)$ is Lipschitz continuous w.r.t. α and twice continuously differentiable w.r.t. ω satisfying $F(\alpha,0) = 0$ and $F(\alpha,1) = h_1 \in \mathbb{R}$ for any $\alpha \in \mathbb{R}$, moreover v is a regulated function.

(F4) The flux f has the following form

$$f(x,\omega) = \begin{cases} f_l(\omega) & \text{if } x \leq 0, \\ f_r(\omega) & \text{if } x > 0, \end{cases}$$

where f_l and f_r are smooth functions satisfying $f_l(0) = f_r(0) = 0$ and $f_l(1) = f_r(1)$.

2 Parabolic equations with discontinuous coefficients

If f is smooth, under mild hypotheses on the growth of the solution, the Cauchy problem (1.2) is equivalent to the integral equation $u = \mathcal{P}^\varepsilon u$, where the transformation $\mathcal{P}^\varepsilon$ is defined by

$$(\mathcal{P}^\varepsilon u)(t,x) \doteq \int_{\mathbb{R}} G^\varepsilon(t,x-y)\,\bar{u}(y)\,dy - \int_0^t \int_{\mathbb{R}} G_x^\varepsilon(t-s,\,x-y) f\big(s,y,u(s,y)\big)\,dy\,ds. \quad (2.1)$$

For $t > 0$, the functions $G(t,x) \doteq \frac{1}{\sqrt{4\pi t}}\,e^{-x^2/4t}$ and $G^\varepsilon(t,x) \doteq \frac{1}{\sqrt{4\varepsilon\pi t}}\,e^{-x^2/4\varepsilon t}$ are the standard Gauss kernels. Observe that the equation $u = \mathcal{P}^\varepsilon u$ is meaningful even when f is discontinuous. Following [16], we say that $u = u(t,x)$ is a **mild solution** to the Cauchy problem (1.2) if it is a fixed point for of the transformation $\mathcal{P}^\varepsilon$. The following facts about mild solutions to (1.2) are proved in [3].

Theorem 2.1. *Consider the Banach space $Y_T \doteq \mathcal{C}^0([0,T],\,\mathbf{L}^1(\mathbb{R}))$ endowed with the supremum norm $\|u\|_T \doteq \sup_{t\in[0,T]} \|u(t)\|_{\mathbf{L}^1(\mathbb{R})}$. Let the flux function f satisfy* **(F1)***. Then there exists a unique mild solution $u \in Y_T$ to the Cauchy problem* (1.2)*. If u and v are two mild solutions of the parabolic equation in* (1.2)*, with initial data $\bar{u}, \bar{v} \in \mathbf{L}^1(\mathbb{R})$. Then the following properties hold.*

(i) The total mass is conserved in time: $\int_{\mathbb{R}} u(t,x)\,dx = \int_{\mathbb{R}} \bar{u}(x)\,dx$, *for all* $t \in [0,T]$.

(ii) A comparison principle holds: $\bar{u} \le \bar{v} \implies u(t,\cdot) \le v(t,\cdot)$, *for all* $t \in [0,T]$.

(iii) The $\mathbf{L}^1$ *distance between the two solutions is non-increasing in time:*

$$\int_{\mathbb{R}} |u(t,x) - v(t,x)|\ dx \ \le \ \int_{\mathbb{R}} |\bar{u}(x) - \bar{v}(x)|\ dx \qquad \text{for all } t \ge 0. \tag{2.2}$$

We now consider a second Cauchy problem with different flux and initial data:

$$\begin{cases} u_t + f^{\sharp}(t,x,u)_x \ = \ \varepsilon u_{xx}\,, \\ u(0,x) \ = \ \bar{u}^{\sharp}(x) \in \mathbf{L}^1(\mathbb{R})\,. \end{cases} \tag{2.3}$$

The following theorem is based on comparison estimates for solutions to the related Hamilton–Jacobi equation. It provides a comparison between two solutions corresponding to not only different initial data, but also possibly different fluxes.

Theorem 2.2. *[3, Theorem 2.3] Let* u *and* $u^{\sharp}$ *be solutions to* (1.2) *and* (2.3), *respectively. Assume that both fluxes* f *and* $f^{\sharp}$ *satisfy* **(F1)**. *Let* U *and* $U^{\sharp}$ *be the integrated functions:*

$$U(t,x) = \int_{-\infty}^{x} u\,(t,\xi)\ d\xi, \qquad U^{\sharp}(t,x) = \int_{-\infty}^{x} u^{\sharp}\,(t,\xi)\ d\xi. \tag{2.4}$$

Then the following comparison property holds.

Let I *be an interval containing the range of* $u^{\sharp}(t,x)$ *and assume that, for some* $\eta \in \mathbf{L}^{\infty}\big([0,T]\big)$ *and some constant* $\bar{\eta} \ge 0$, *one has*

$$\begin{cases} f^{\sharp}(t,x,\omega) \le f(t,x,\omega) + \eta(t) & \text{for all } (t,x,\omega) \in]0,T[\times\mathbb{R}\times I, \\ U(0,x) \le U^{\sharp}(0,x) + \bar{\eta} & \text{for all } x \in \mathbb{R}. \end{cases} \tag{2.5}$$

Then, for all $t \in [0,T]$ *and* $x \in \mathbb{R}$, *one has*

$$U(t,x) \ \le \ U^{\sharp}(t,x) + \bar{\eta} + \int_0^t \eta(s)\,ds\,. \tag{2.6}$$

3 The unique weak vanishing viscosity limit

Without further hypotheses on the flux f, the solution to (1.2) could blow up as $\varepsilon \to 0^+$. Indeed consider the following linear example,

$$\begin{cases} u_t^{\varepsilon} + [\Theta(x)]_x = \varepsilon u_{xx}^{\varepsilon}, \\ u\,(0,\cdot) = 0, \end{cases} \qquad \text{where} \qquad \Theta(x) = \begin{cases} 0 & \text{for } x \le 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Its mild solution is given by

$$u^{\varepsilon}(t,x) = -t\frac{1}{\sqrt{\varepsilon t}}\varPhi\left(\frac{x}{\sqrt{\varepsilon t}}\right), \quad \text{where} \quad \varPhi\,(y) = 2G\,(1,y) - |y|\int_{|y|}^{+\infty} G\,(1,\xi)\,d\xi.$$

Since $u^{\varepsilon} \overset{*}{\rightharpoonup} -t\delta_0(x)$ as $\varepsilon \to 0^+$, it does not converge to any function even in a weak sense.

This motivates the introduction of Hypothesis **(F2)** that allows us to apply the maximum principle (namely, *(ii)* in Theorem 2.1) to the mild solutions to the parabolic equation (1.2). Indeed, let $f = f(t, x, \omega)$ be a flux function satisfying **(F1)**, **(F2)**, and consider the domain

$$\mathcal{D} \doteq \left\{ u \in \mathbf{L}^1(\mathbb{R})\,; \quad u(x) \in [0,1] \quad \text{for all} \;\; x \right\}. \tag{3.1}$$

Let an initial data $\bar{u} \in \mathcal{D}$ be given. Since the constant functions $u^*(t,x) = 1$ and $u_*(t,x) = 0$ are solutions to the parabolic equation in (1.2) for any $\varepsilon > 0$, by a standard comparison argument the solution $u^\varepsilon(t,x)$ to (1.2) satisfies $u(t,\cdot) \in \mathcal{D}$ for all $t \in [0,T]$.

The bound in $\mathbf{L}^\infty$ gives weak* compactness of the sequence of functions, but the uniqueness of the limit as $\varepsilon \to 0^+$ requires additional analysis. Our main goal is to find a general class $\mathcal{F}$ of flux functions for which the vanishing viscosity limits are unique, for any fixed initial data in $\mathcal{D}$. As a starting point, by Theorem 2.2 we know that this class contains all fluxes $f = f(x,u)$ having one single discontinuity at $x = 0$. Next, we prove that this class is closed under certain elementary operations and suitable limits. By repeatedly applying these operations and taking limits, we conclude that all flux functions of the form $f(t,x,u) = F(v(t,x),u)$, with F Lipschitz and v regulated, as in **(F3)**, lie in this class. Hence, for these fluxes the weak solutions obtained as vanishing viscosity limits are unique.

Definition 3.1. *We denote by $\mathcal{F}_{[a,b]}$ the family of all fluxes $f = f(t,x,u)$ that satisfy* **(F1)**, **(F2)** *for $t \in [a,b]$ (instead of $[0,T]$), and for which the following property holds. For any initial data $\bar{u} \in \mathcal{D}$, calling u^ε the solutions to the viscous Cauchy problem*

$$\begin{cases} u_t + f(t,x,u)_x &= \varepsilon\, u_{xx}, \\ u(a,x) &= \bar{u}(x) \in \mathbf{L}^1(\mathbb{R}), \end{cases} \tag{3.2}$$

as $\varepsilon \to 0^+$ the integrated functions

$$U^\varepsilon(t,x) = \int_{-\infty}^{x} u^\varepsilon(t,y)\; dy$$

converge uniformly in $[a,b] \times \mathbb{R}$ to a unique limit.

Since uniform convergence of the integrated function U^ε corresponds to weak convergence of u^ε (see [3, Lemma 3.1]), if $f \in \mathcal{F}_{[0,T]}$, then as $\varepsilon \to 0^+$ the solutions $u^\varepsilon(t,\cdot)$ to (1.2) converge weakly to a unique limit $u(t,\cdot)$ in the weak topology of $\mathbf{L}^1(\mathbb{R})$ for any fixed $t \in [0,T]$. Our eventual goal is to show that $\mathcal{F}_{[0,T]}$ contains all the flux functions satisfying **(F3)**. The following result, proved in [3] with the help of Theorem 2.2, describes the uniform limit under which $\mathcal{F}_{[a,b]}$ is closed.

Theorem 3.2. *Consider a flux $f = f(t,x,\omega)$ defined in $[0,T] \times \mathbb{R} \times [0,1]$, satisfying* **(F1)** *and* **(F2)**. *Assume that, for any $\delta > 0$, there exists times $0 < a_1 < b_1 < \;\; \cdots \;\; < a_N < b_N < T$ and flux functions $f_i \in \mathcal{F}_{[a_i,b_i]}$ such that $T - \sum_{i=1}^N (b_i - a_i) \; < \; \delta$,*

$$|f(t,x,\omega) - f_i(t,x,\omega)| < \delta, \quad \text{for all } \; (t,x,\omega) \in [a_i,b_i] \times \mathbb{R} \times [0,1],\; i = 1,\ldots,N. \tag{3.3}$$

Then $f \in \mathcal{F}_{[0,T]}$.

The classical result by Kruzhkov [15] implies that the vanishing viscosity limit exists and is unique for conservation law with smooth flux. Consequently, smooth fluxes belong to $\mathcal{F}_{[0,T]}$.

An extensive body of more recent literature has dealt with fluxes satisfying hypothesis **(F4)**. In this case, one can again conclude that $f \in \mathcal{F}_{[0,T]}$, for every $T > 0$.

A detailed proof, based on the theory of nonlinear semigroups [4, 5, 6], can be found in [11]. Our approach avoids the technicalities in previous literature such as traces, Riemann problems, interface conditions, compensated compactness and entropy inequalities etc. , which generally require some additional hypotheses. Consequently the results in [11] holds under the general assumption **(F4)**. Theorems 3.4 and 5.2 in [11] can be restated in the following form.

Theorem 3.3. *Under hypothesis **(F4)**, the parabolic equation in* (1.2) *generates a unique continuous semigroup of contractions* $S_t^\varepsilon : \mathcal{D} \to \mathcal{D}$ *whose trajectories* $S_t^\varepsilon \bar{u}$ *are the unique mild solutions to* (1.2)*. Moreover, as* $\varepsilon \to 0^+$*, for any* $\bar{u} \in \mathcal{D}$*,* $S_t^\varepsilon \bar{u}$ *converges in* $\mathbf{L}^1(\mathbb{R})$ *to* $S_t \bar{u}$ *uniformly on bounded t intervals, where* $S_t : \mathcal{D} \to \mathcal{D}$ *is a continuous semigroup of contractions whose trajectories are weak solutions to the Cauchy problem* (1.1)*. Consequently if the flux f satisfies hypotheses **(F4)**, then* $f \in \mathcal{F}_{[0,T]}$*.*

By a change of variables it can be proved that the existence and uniqueness of the weak limit also holds when the interface between the two fluxes varies in time, under mild regularity assumptions.

Lemma 3.4. *([3, Lemma 3.5]) Let f satisfy **(F4)**. Let* $\gamma : [0,T] \mapsto \mathbb{R}$ *be a Lipschitz function whose derivative* $\dot{\gamma}$ *coincides a.e. with a regulated function. Then the flux function* $\tilde{f}$ *defined by* $\tilde{f}(t,x) = \tilde{f}(x - \gamma(t))$ *belongs to* $\mathcal{F}_{[0,T]}$*.*

Thanks to the finite speed of propagation, the functions in $\mathcal{F}_{[0,T]}$ can be patched together horizontally, provided that they coincide on an intermediate domain.

Lemma 3.5. *([3, Lemma 3.6]) Consider two flux functions* f_1*,* f_2*, both satisfying* **(F1)** *and* **(F2)***. Assume that* $f_1, f_2 \in \mathcal{F}_{[0,T]}$ *and that there exists* $\alpha < \beta$ *such that* $f_1(t,x,\omega) = f_2(t,x,\omega)$ *for all* $t \in [0,T]$*,* $x \in]\alpha, \beta[$ *, and* $\omega \in [0,1]$*. Then the flux f defined by*

$$f(t,x,\omega) \doteq \begin{cases} f_1(t,x,\omega) & \text{if } x < \beta \\ f_2(t,x,\omega) & \text{if } x > \alpha \end{cases} \tag{3.4}$$

belongs to $\mathcal{F}_{[0,T]}$*.*

Lemma 3.6. *([3, Lemma 3.8]) Let* $f = f(t,x,\omega)$ *be a flux function satisfying **(F1)**, **(F2)**. Assume that, for every bounded interval* $[x_1, x_2]$ *the function*

$$\hat{f}(t,x,\omega) = \begin{cases} f(t,x_1,\omega) & \text{if } x < x_1, \\ f(t,x,\omega) & \text{if } x \in [x_1,x_2], \\ f(t,x_2,\omega) & \text{if } x > x_2, \end{cases} \tag{3.5}$$

lies in $\mathcal{F}_{[0,T]}$*. Then* $f \in \mathcal{F}_{[0,T]}$ *as well.*

Combining the previous results, the main theorem can be proved.

Theorem 3.7. *Let* $f = f(t,x,\omega)$ *be a flux function satisfying **(F3)**. Then* $f \in \mathcal{F}_{[0,T]}$*.*

Proof. By the assumption **(F3)**, the flux function f satisfies **(F1)** and **(F2)**.

Fix an interval $[x_1, x_2]$. Let $\delta > 0$ be given. Since v is regulated we can find disjoint intervals $[a_i, b_i]$, Lipschitz continuous curves $\gamma_{i,k}$ and constants $\alpha_{i,k}$ such that all conditions *(i)–(iii)* in Definition 1.1 hold.

For each i, let the piecewise constant function $\chi_i(t,x)$ be as in (1.3). Applying Lemma 3.5 and Lemma 3.4, by induction we show that the flux function

$$\begin{aligned} f_i(t,x,\omega) \doteq F(\chi_i(t,x),\omega) &= F(\alpha_{i,0},\omega)\,\chi_{\{x<\gamma_{i,1}(t)\}} \\ &\quad + \sum_{k=1}^{N_i-1} F(\alpha_{i,k},\omega)\,\chi_{\{\gamma_{i,k}(t)<x<\gamma_{k+1,1}(t)\}} \\ &\quad + F(\alpha_{i,N_i},\omega)\,\chi_{\{x>\gamma_{i,N_i}(t)\}} \end{aligned}$$

lies in $\mathcal{F}_{[a_i,b_i]}$. In turn, an application of Theorem 3.2 shows that the function $\hat{f}$ in (3.5) lies in $\mathcal{F}_{[0,T]}$. Since the interval $[x_1, x_2]$ is arbitrary, by Lemma 3.6, the flux function f lies in $\mathcal{F}_{[0,T]}$ as well. □

4 The strong vanishing viscosity limit

In this section, we assume **(F3)**. Moreover we consider the following additional hypotheses.

(V1) $v(t,x)$ is a bounded measurable function whose total variation w.r.t. x is integrable. More precisely, for every rectangular domain of the form $[0,T] \times [x_1,x_2]$ one has

$$\int_0^T \left(\text{Tot.Var.}\left\{v(t,\cdot);[x_1,x_2]\right\}\right)\,dt < +\infty. \tag{4.1}$$

(V2) For each $\alpha \in \mathbb{R}$ the partial derivative $\omega \mapsto F_\omega(\alpha,\omega)$ is not constant on any open interval.

Under **(V1)**, the *unique* weak limit found in the previous section is a solution to the conservation law

$$u_t + f(t,x,u)_x = 0. \tag{4.2}$$

Moreover, if we assume **(V2)** as well, the convergence of u^ε is in $\mathbf{L}^1([0,T]\times\mathbb{R})$. These results can be obtained using a well established compensated compactness argument [8, 18].

Theorem 4.1. *([3, Theorem 4.2]) Let the flux f satisfy* **(F1)**, **(F2)**, **(F3)** *and **(V1)**, and choose an initial data $\bar{u} \in \mathcal{D}$. Let u^ε be the solution to the Cauchy problem* (1.2). *Then the unique weak viscosity limit $u(t,\cdot) = \lim_{\varepsilon\to 0} u^\varepsilon(t,\cdot)$ is a weak solution to the conservation law* (4.2).

*Moreover if the flux satisfies **(V2)** as well, then the convergence $u^\varepsilon \to u$ is in $\mathbf{L}^1(\Omega)$ endowed with its strong topology.*

References

[1] B. Andreianov, K. H. Karlsen, and N. H. Risebro. On vanishing viscosity approximation of conservation laws with discontinuous flux. *Netw. Heterog. Media*, 5(3):617–633, 2010.

[2] B. Andreianov, K. H. Karlsen, and N. H. Risebro. A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux. *Arch. Ration. Mech. Anal.*, 201(1):27–86, 2011.

[3] A. Bressan, G. Guerra, and W. Shen. Vanishing viscosity solutions for conservation laws with regulated flux. *Journal of Differential Equations*, 266(1):312 – 351, 2019.

[4] H. Brézis and A. Pazy. Convergence and approximation of semigroups of nonlinear operators in Banach spaces. *J. Functional Analysis*, 9:63–74, 1972.

[5] M. G. Crandall. The semigroup approach to first order quasilinear equations in several space variables. *Israel J. Math.*, 12:108–132, 1972.

[6] M. G. Crandall and T. M. Liggett. Generation of semi-groups of nonlinear transformations on general Banach spaces. *Amer. J. Math.*, 93:265–298, 1971.

[7] G. Crasta, V. De Cicco, and G. De Philippis. Kinetic formulation and uniqueness for scalar conservation laws with discontinuous flux. *Comm. Partial Differential Equations*, 40(4):694–726, 2015.

[8] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010.

[9] T. Gimse and N. H. Risebro. Riemann problems with a discontinuous flux function. In *Third International Conference on Hyperbolic Problems, Vol. I, II (Uppsala, 1990)*, pages 488–502. Studentlitteratur, Lund, 1991.

[10] T. Gimse and N. H. Risebro. Solution of the Cauchy problem for a conservation law with a discontinuous flux function. *SIAM J. Math. Anal.*, 23(3):635–648, 1992.

[11] G. Guerra and W. Shen. Backward euler approximations for conservation laws with discontinuous flux. Preprint: https://arxiv.org/abs/1803.00493, 2018.

[12] P. Gwiazda, A. Świerczewska-Gwiazda, P. Wittbold, and A. Zimmermann. Multi-dimensional scalar balance laws with discontinuous flux. *J. Funct. Anal.*, 267(8):2846–2883, 2014.

[13] K. H. Karlsen, M. Rascle, and E. Tadmor. On the existence and compactness of a two-dimensional resonant system of conservation laws. *Commun. Math. Sci.*, 5(2):253–265, 2007.

[14] R. A. Klausen and N. H. Risebro. Stability of conservation laws with discontinuous coefficients. *J. Differential Equations*, 157(1):41–60, 1999.

[15] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[16] R. H. Martin, Jr. *Nonlinear operators and differential equations in Banach spaces.* Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976. Pure and Applied Mathematics.

[17] E. Y. Panov. Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux. *Arch. Ration. Mech. Anal.*, 195(2):643–673, 2010.

[18] D. Serre. *Systems of conservation laws. 1, 2.* Cambridge University Press, Cambridge, 2000.