Critical sets of elliptic equations

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Abstract

Given a solution u to a linear homogeneous second order elliptic equation with Lipschitz coefficients, we introduce techniques for giving improved estimates of the critical set $\mathcal{C}(u) \equiv \{x : |\nabla u|(x) = 0\}$, as well as the first estimates on the effective critical set $\mathcal{C}_r(u)$, which roughly consists of points x such that the gradient of u is large on $B_r(x)$ compared to the size of u. The results are new even for harmonic functions on $\mathbb{R}^n$. Given such a u, the standard *first order* stratification $\{\mathcal{S}^k\}$ of u separates points x based on the degrees of symmetry of the leading order polynomial of $u - u(x)$. In this paper we give a quantitative stratification $\{\mathcal{S}^k_{\eta,r}\}$ of u, which separates points based on the number of *almost* symmetries of *approximate* leading order polynomials of u at various scales. We prove effective estimates on the volume of the tubular neighborhood of each $\mathcal{S}^k_{\eta,r}$, which lead directly to $(n - 2 + \epsilon)$-Minkowski type estimates for the critical set of u. With some additional regularity assumptions on the coefficients of the equation, we refine the estimate to give new proofs of uniform $(n - 2)$-Hausdorff measure estimate on the critical set and singular sets of u.

Contents

1 Introduction

In this paper, we study solutions u to second order linear homogeneous elliptic equations on subsets of $\mathbb{R}^n$ and on manifolds with both Lipschitz and smooth coefficients. We introduce new quantitative stratification techniques in this context, based on those first introduced in [CN13, CN12]. These techniques allow for new estimates on the critical set

$$\mathcal{C}(u) \equiv \{x : |\nabla u| = 0\} \tag{1.1}$$

and more importantly on the *effective critical set*

$$\mathcal{C}_r(u) \equiv \left\{ x : \inf_{B_r(x)} |\nabla u|^2 < \frac{\epsilon(n)}{r^2} \fint_{\partial B_{2r}(x)} |u - u(x)|^2 \right\}, \tag{1.2}$$

where $\epsilon(n)$ is a small fixed constant. That is, if $x \notin \mathcal{C}_r(u)$ then not only do we have $|\nabla u|(x) \neq 0$, but in fact the gradient has some definite size in a ball of definite size around x.

Though most of our results require only a Lipschitz bound on the coefficients, even when applied to harmonic functions on $\mathbb{R}^n$, the effective estimates are new. The Lipschitz bound is sharp in the sense that the results are false under a Hölder assumption.

Because the techniques are local and do not depend on the underlying space on which the equations are defined, we will often restrict ourselves to the unit ball $B_1(0) \subseteq \mathbb{R}^n$. However, we will point out the appropriate modifications needed in the more general situations. To be specific, we will study equation s of the form

$$\mathcal{L}(u) = \partial_i(a^{ij}(x)\partial_j u) + b^i(x)\partial_i u = 0 \tag{1.3}$$

and

$$\mathcal{L}(u) = \partial_i(a^{ij}(x)\partial_j u) + b^i(x)\partial_i u + c(x)u = 0. \tag{1.4}$$

We will assume that the coefficients a^{ij} are elliptic and uniformly Lipschitz, and that b^i, c are bounded:

$$(1 + \lambda)^{-1}\delta^{ij} \leq a^{ij} \leq (1 + \lambda)\delta^{ij}, \; \text{Lip}(a^{ij}) \leq \lambda, \; |b^i|, |c| \leq \lambda. \tag{1.5}$$

The function u always denotes a weak solution to (1.3) or (1.4). Standard elliptic estimates imply that $u \in C^{1,\alpha}$. Note that if we are interested in studying the critical set $\mathcal{C}(u)$ then Lipschitz continuity of the coefficients is essentially the weakest possible regularity assumption that we can make. Indeed, A. Pliś (see [Pli63]) found counterexamples to the unique continuation principle for solutions of elliptic equations similar to (1.3), where the coefficients a^{ij} are Hölder continuous with any exponent strictly smaller than 1. In such a situation, no reasonable estimates for $\mathcal{C}(u)$ can hold.

Next, we will give some informal statements of our results; see Sections 1.2 and 1.3 for more accurate statements. In the course of doing this, we will also give a brief review of what was previously known.

Harmonic Functions   For simplicity we begin by discussing harmonic functions $\Delta u = 0$ on $B_1(0)$. The standard fact that such a function is analytic implies without difficulty that $H^{n-2}(\mathcal{C}(u) \cap B_{1/2}) < \infty$, if u is not a constant.

Quantitatively, the standard measurement of *nonconstant* behavior of u on a ball $B_r(x)$ is an upper bound on the normalized Almgren frequency defined as:

$$\bar{N}^u(x, r) \equiv \frac{r \int_{B_r(x)} |\nabla u|^2 dV}{\int_{\partial B_r(x)} (u - u(x))^2} \, . \tag{1.6}$$

By unique continuation, if u is not constant then both N^u and $\bar{N}^u$ are well defined for positive r. These definitions suggest that harmonic functions might satisfy an estimate of the form

$$H^{n-2}(\mathcal{C}(u) \cap B_{1/2}) < C(n, \bar{N}^u(0, 1)) \tag{1.7}$$

In other words, if u is bounded away from being a constant by a definite amount, then the critical set can only be so large in the $(n - 2)$-Hausdorff sense. Such an estimate has been proved for the *singular set*, i.e. if one restricts to a level set of u. That is, $H^{n-2}(\mathcal{C}(u) \cap B_{1/2} \cap \{u = const\}) < C(n, \bar{N}^u(0, 1))$; see [HHL98]. The paper, [HL00], gives an estimate of this form for the rank zero sets of harmonic maps. The techniques of [HL00] can be used to treat case of sets for equations of the form (1.3) (although this is not pointed out explicitly in [HL00]). In Theorem 1.21 we give a new proof of this bound based on the quantitative estimates of Theorem 1.10. (For a slightly earlier proof of the *local* finiteness of the $(n - 2)$-dimensional Hausdorff measure of the critical set for equations of the form (1.3), see [HN99].) More generally the results briefly outlined for harmonic functions hold verbatim for solutions of second order equations with sufficiently smooth coefficients.

In this paper, our main focus is on more effective versions of (1.7). The estimate (1.7) is less than optimal in two primary respects. For general subsets, a bound on the $(n - 2)$-dimensional Hausdorff measure does not prevent the subset from being dense. In fact, even if such a subset is closed, it can still be arbitrarily dense. Our first statement's tell us that not only is $\mathcal{C}(u)$ small, but the tube $B_r(\mathcal{C}(u))$ has $(n-2)$-small volume for every r in the form of $\text{Vol}(B_r(\mathcal{C}(u))) < C_\epsilon r^{2-\epsilon}$ for every ϵ, see Theorem 1.17 for a precise statement. This is a much stronger statement, which leads to Minkowski dimension estimates. Secondly, as will be seen in Section 1.3 what we control is not just the critical set but the *effective* critical set. That is, we show in Theorem 1.17 that away from a set of small $(n - 2 - \epsilon)$-volume (for all ϵ), every point has a ball of definite size in which the gradient has some definite size relative the the nonconstancy of the solution. For details, see subsections 1.2 and 1.3.

Lipschitz elliptic equations   In reality, the technical heart of this paper concerns solutions of elliptic equations with Lipschitz coefficients. Most of our results, even in the smooth coefficient cases, are relatively easy consequences of those in the case where only assuming Lipschitz continuity of the coefficients is required. For example, it is known, see [Lin91], that $\mathcal{C}(u) \cap B_{1/2}$ has Hausdorff dimension $\dim_{Haus}(\mathcal{C}(u) \cap B_{1/2}) \leq n-2$. Although we are not able to improve this to an effective finiteness, we do make advances in two directions. First, for all $\epsilon > 0$, we do show effective volume estimates of the form

$$\text{Vol}(B_r(\mathcal{C}(u)) \leq \text{Vol}(B_r(\mathcal{C}_r(u)) \cap B_{1/2}) < C(n, \bar{N}^u(0, 1), \epsilon) r^{2-\epsilon} \, . \tag{1.8}$$

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Among other things this improves $\dim_{Haus} \mathcal{C}(u) = n - 2$ to $\dim_{Min} \mathcal{C}(u) = n - 2$. That is, the Minkowski dimension of the critical set is at most $n - 2$, see Section 1.2 for precise statements. What is more important, this gives effective estimates for the volume of tubes around the critical set, so that even without bounds on $H^{n-2}(\mathcal{C}(u))$ in the Lipschitz case, we still have very definite effective control over the size of the critical set. More than that, the corresponding estimate on the effective critical set tells us that away from a r-tube of definite volume, we have in every ball $B_r(x)$ that u looks close to a linear function after normalization.

The primary technical construction needed to generalize from the harmonic case to the general elliptic case is the *generalized frequency* $\bar{F}(r)$ of Section 3.1. This is an almost monotone quantity, in the sense that $e^{Cr}\bar{F}(r)$ is monotone nondecreasing on some interval $(0, r_0)$; see Theorem 3.8. The function $\bar{F}(r)$ plays the same role as the frequency for harmonic functions. The generalized frequency of Section 3.1 is a variation on a generalized frequency constructed in [GL86, GL87], which is shown there to be almost monotone for operators in divergence form. Although one can use tricks as in [Lin91] to apply this to nondivergence form operators, instead, by modifying the proof in [GL86, GL87], we show directly in Section 3.1 that the frequency $\bar{F}$ is almost monotone for all operators of the form (1.3), which is required for proving (1.8).

Quantitative stratification   More precisely, our primary contribution is the introduction and analysis of a quantitative stratification; see Section 1.2. The standard stratification separates points x in the domain of u, according to the number of independent symmetries of the leading order polynomial of the Taylor expansion of $u - u(x)$; see [HL]. In particular, this stratification does not take into account the degree of the leading order Taylor polynomial at x. More precisely, $\mathcal{S}^k$ consists of those points x such that the leading order polynomial $P(y)$ of $u(y) - u(x)$ is a function of at least $n - k$ variables. For instance, if u has nonvanishing gradient at x, then the leading order polynomial is linear and therefore $x \in \mathcal{S}^{n-1}$.

In a manner similar to [CN13] and [CN12], we will define a quantitative stratification which refines the standard stratification. Very roughly, for a fixed $r, \eta > 0$ this stratification separates points x based on the number of independent η-*almost* symmetries of an approximate leading order polynomials of $u - u(x)$ at scales $\geq r$; for a precise definition, see Section 1.2.

The essential point of this paper is to prove volume estimates on the quantitative stratification, as opposed to the weaker Hausdorff estimates on the standard stratification. As in [CN13, CN12] these estimates require new techniques which provide a quantitative replacement for more traditional blow up arguments. The new techniques work under Lipschitz constraints on the coefficients and, in particular, these arguments give new proofs of the original Hausdorff estimates.

The key ideas involved in proving the estimates for the quantitative stratification are *quantitative differentiation*, the *frequency decomposition* (for the *generalized frequency*) which plays the role the energy played in [CN13, CN12]) and *cone splitting*.

In general, precise cone-splitting is the principle that in the presence of conical structure, nearby symmetries interact to create additional symmetries. In the present context, "0-symmetry" plays the role of conical structure. We say that a function f is 0-symmetric at a point, if for some $d > 0$, it is homogeneous of degree d at that point. If f is homogeneous of degree d with respect to two distinct points, it follows that f is constant on lines parallel to the one joining these points and hence, that f is actually a function of at most

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$n - 1$ variables. [1] In our terminology, we can rephrase this by saying that if a function is 0-symmetric at two distinct points, then the function is actually 1-symmetric. We call cone-splitting (as opposed to precise cone-splitting) a quantitative version of the above statement. (In [CN12] the splitting principle was applied to functions that were simply 0-homogeneous, that is, radially invariant). The frequency decomposition will exploit this by decomposing the space $B_1(0)$ based on which scales u looks almost 0-symmetric. On each such piece of the decomposition, and at every scale, nearby points automatically either force higher order symmetries or a good covering of the space, and thus the estimates of this paper can be proved easily on each piece of the decomposition. The final theorem is obtained by noting that there are far fewer pieces to the decomposition than might *apriori* seem possible, a result which follows from a *quantitative differentiation* argument.

The Hausdorff estimates on the critical sets of solutions of (1.3) with smooth coefficients will be gotten by combining the estimates on the quantitative stratification with an ϵ-regularity type theorem from [HHL98].

1.1 The First-Order Stratification

Even though we will not use the standard stratification in this article, it seems appropriate to recall briefly its definition and main properties. This should help the reader understand the philosophy underlying the quantitative stratification.

The appropriate notion of stratification in our context is based on first order tangent behavior as opposed to the stratifications considered in [CN13, CN12], which were based on zeroth order behavior. Specifically, let us first be more careful about the notion of *tangent behavior* in this context. We will make all definitions on $\mathbb{R}^n$, though the analogous definitions on manifolds are the same up to the use of an exponential map; for example, see [CN12]. We will usually need to work under an assumption of nondegeneracy in order to make sense of the tangential behavior:

Definition 1.1. We call a smooth function u nondegenerate if at every x some derivative of some order is nonzero.

In particular, according to this definition, a constant function is degenerate. (This is consistent with the fact that this is a first order stratification). On the other hand, any nonconstant analytic function is nondegenerate. We now define our tangent maps:

Definition 1.2. Let $u : B_1(0) \rightarrow \mathbb{R}$ be a smooth nondegenerate function and $r > 0$. Then we make the following definitions

1. For $x \in B_{1-r}(0)$ we define

$$T_{x,r}u(y) = \frac{u(x + ry) - u(x)}{\left(\fint_{\partial B_1(0)}(u(x + ry) - u(x))^2\right)^{1/2}} .\tag{1.9}$$

If the denominator vanishes, we set $T_{x,r} = \infty$.

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[1]To see this, note that if $f(x_1, \ldots x_n)$ is homogeneous of degree d with respect to the points $(0, \ldots, 0)$ and $(a_1, \ldots, a_n)$, then $\sum_i x_i \partial_i(f) = \sum_i (x_i - a_i)\partial_i(f) = d \cdot f$, and so $\sum_i a_i \partial_i(f) = 0$.

2. For $x \in B_1(0)$ we define

$$T_{x,0}u(y) = T_x u(y) = \lim_{r \to 0} T_{x,r}u(y). \qquad (1.10)$$

Note that the limits above exist at x as long as u is nondegenerate at x. In that case, the limit is unique and, up to rescaling, $T_x u$ is just the leading order polynomial of the Taylor expansion of $u - u(x)$ at x. In particular, $T_x u$ is a homogeneous polynomial, and if u satisfies a second order elliptic equation then this polynomial is a homogeneous solution to the constant coefficient equation $a^{ij}(x)\partial_i\partial_j T_x = 0$. Hence, up to a linear change of coordinates is a homogeneous harmonic polynomial.

Remark 1.3. For the sake of simplicity, when studying solutions to (1.3) we will modify the definition of $T_{x,r}$ using this linear change of coordinates (see Definition 3.7). In this way, $T_{x,0}u$ will be a harmonic homogeneous polynomial. Since the change of variables has bi-Lipschitz constant depending only on λ, from the point of view of our results there is no significant difference between these two definitions.

Next, we specify what it means for a function to be symmetric, a key point in the definition of the stratification.

Definition 1.4. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth function:

1. We say u is 0-symmetric if u is a homogeneous polynomial.

2. We say u is k-symmetric if u is 0-symmetric and there exists a k-dimensional subspace V such that for every $x \in \mathbb{R}^n$ and $y \in V$ we have that $u(x + y) = u(x)$.

We can now define the first-order stratification associated to u:

Definition 1.5. Given a smooth nondegenerate function $u : B_1(0) \to \mathbb{R}$ we define the kth-singular stratum of u by

$$S^k(u) \equiv \{x : T_x u \text{ is not k+1-symmetric}\}. \qquad (1.11)$$

Let us make a few remarks about some unusual features of this stratification. They arise from the fact that it is a *first order* stratification. To begin with, it is usually the case in a stratification that S^{n-1} has measure zero, that is, that almost every point has n-degrees of symmetry. The issue in general is that for almost every point of a nondegenerate function u, we have that $T_x u$ is a linear function. Hence, almost every point has $n-1$ degrees of symmetry, and so, S^{n-1} has full measure and $\dim S^{n-1} = n$. Despite this circumstance, for solutions of (1.3) and for $k \leq n - 2$, we will recover the estimate $\dim S^k \leq k$, where dim denotes Hausdorff (or even Minkowski) dimension.

Remark 1.6. The smoothness assumption on u is a sufficient condition to define the standard stratification, but not a necessary one. Indeed, even though solutions to (1.3) with (1.5) are in general only $C^{1,\alpha}$, by unique continuation and the maximum principle it is easy to see that for positive r, $T_{x,r}u$ is still well-defined and finite.

Moreover, by the uniqueness of the tangent maps proved in [Han94, theorem 3.1] [2], also $T_{x,0}u$ is well-defined for all x.

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[2]Note that this theorem requires as an additional assumption that u does not vanish at infinite order at x, which is guaranteed in our context

1.2 The Quantitative Stratification

Notice that for solutions to (1.3) the total singular set $\mathcal{S}^{n-2}$ is precisely the critical points of u, namely the points where $|\nabla u| = 0$. The goal of this paper is to prove refined estimates on $\mathcal{S}$ when u is not only a nondegenerate function, but also satisfies an elliptic equation. To do this, an important step is to quantify the stratification of the last subsection. For solutions of elliptic equations, we will prove effective Minkowski type estimates for this quantitative stratification.

To define the quantitative stratification we begin with the following quantitative version of symmetry. Recall the definition of k-symmetric and $T_{x,r}u$ from the last subsection.

Definition 1.7. Let $u : B_1(0) \to \mathbb{R}$ be an L^2 function. We say that u is (k, ϵ, r, x)-symmetric if there exists a k-symmetric polynomial P with $\fint_{\partial B_1(0)} |P|^2 = 1$ such that

$$\fint_{B_1(0)} |T_{x,r}u - P|^2 < \epsilon. \tag{1.12}$$

Remark 1.8. Note that for harmonic functions and for solutions to (1.3), it would make no significant difference if we added the assumption that the polynomial P is harmonic. Moreover, we can also replace the inequality (1.12) with

$$\fint_{\partial B_1(0)} |T_{x,r}u - P|^2 < \epsilon'. \tag{1.13}$$

Indeed, by the doubling conditions in [HL, Corollary 2.2.7], relation (1.13) implies that u is $(k, \epsilon'/n, r, x)$-symmetric. The converse also holds with the proviso that in this case, ϵ' depends on ϵ, n and also on $\bar{N}^u(0, 1)$. Given the definition of frequency function in (1.6), it is easy to see why this second definition is more convenient to use in case u is harmonic, or more generally a solution to (1.3).

The above gives a quantitative way of stating that u is *almost k-symmetric* on $B_r(x)$. We are now in a position to define the quantitative stratification:

Definition 1.9. Let $u : B_1(0) \to \mathbb{R}$ be an L^2 function. Then we define the (k, η, r)-effective singular stratum by

$$\mathcal{S}^k_{\eta,r} \equiv \{x \in B_1(0) : u \text{ is not } (k + 1, \eta, s, x)\text{-symmetric } \forall s \geq r\}. \tag{1.14}$$

The following properties of the quantitative stratification are immediate. To begin with,

$$\mathcal{S}^k_{\eta,r} \subseteq \mathcal{S}^{k'}_{\eta',r'} \text{ if } (k' \leq k, \eta' \leq \eta, r \leq r'). \tag{1.15}$$

In addition, we can recover the standard stratification by

$$\mathcal{S}^k = \bigcup_\eta \bigcap_r \mathcal{S}^k_{\eta,r}. \tag{1.16}$$

Our first main result is the following effective Minkowski estimate for $\mathcal{S}^k_{\eta,r}$, which holds under the assumption of a frequency bound on u, see (1.6). In particular, we will see that this immediately implies Minkowski dimension control of the critical set for solutions of (1.3).

Theorem 1.10. *Let $u : B_1(0) \to \mathbb{R}$ satisfy (1.3) and (1.5) weakly with $\bar{N}^u(0, 1) \leq \Lambda$. Then*

1. *For every $\eta > 0$ and $k \leq n - 2$ we have*

$$\mathrm{Vol}\left(B_r(\mathcal{S}_{\eta,r}^k) \cap B_{1/2}(0)\right) \leq C(n, \lambda, \Lambda, \eta) r^{n-k-\eta} . \tag{1.17}$$

2. *For every $\epsilon > 0$ and $0 \leq \alpha < 1$ there exists $\bar{\eta}(n, \epsilon, \alpha, \lambda, \Lambda)$ such that if $x \notin \mathcal{S}_{\eta,r}^{n-2}$ with $\eta < \bar{\eta}$ then there exists a linear function $L(x)$ with $\fint_{\partial B_1(0)} |L|^2 = 1$ such that $\|T_{x,r}u - L\|_{C^{1,\alpha}(B_{1/2}(0))} < \epsilon$.*

Remark 1.11. Note that we have only assumed Lipschitz control on the coefficients a^{ij} and L^∞ control over the coefficients b^i.

Remark 1.12. The theorem continues to hold for solutions of (1.4) so long as we only estimate the volume $\mathrm{Vol}\left[B_r\left(\mathcal{S}_{\eta,r}^k \cap u^{-1}(0)\right) \cap B_{1/2}(0)\right]$.

Remark 1.13. The second item in the theorem implies the following important statement: there exists $\eta(n, \lambda, \Lambda)$ such that $B_r(\mathcal{C}(u)) \subseteq \mathcal{S}_{\eta,2r}^{n-2}$. This immediately implies the estimate on tubular neighborhoods of the critical set, which is recorded in Theorem 1.17 below.

Remark 1.14. On a Riemannian manifold the constant C should also depend on the sectional curvature of M and the volume of B_1. In this case one can use local coordinates to immediately deduce the theorem for manifolds from the Euclidean version. The estimates (1.5) are then with respect to the Riemannian geometry on M, where a^{ij} and b^i are now tensors on M and ∂ is the covariant derivative on M.

1.3 The Main Estimates on the Critical Set

Our primary applications of Theorem 1.10 are to the critical sets of solutions of (1.3), or better to the effective critical sets. Indeed, we will not only give estimates on the set of points with vanishing gradient, but also on the set of points where the gradient is small in an appropriate sense.

Given a linear function $L(x) = \langle \vec{L} | x \rangle$, we say that L is normalized if

$$\fint_{\partial B_1(0)} |L(x)|^2 = 1 \quad \Longleftrightarrow \quad |\vec{L}| = \beta(n) . \tag{1.18}$$

Definition 1.15. Given $u \in C^1$ and x in its domain, we define

$$r_x = \sup\left\{ s \geq 0 \text{ s.t. there exists a normalized } L \text{ s.t. } \left\|T_{x,s}u - L\right\|_{C^1(B_{1/2}(0))} \leq \beta(n)/2 \right\} . \tag{1.19}$$

Given the definition, it is immediate to see that $r_x = 0$ if and only if x is a critical point for u. Moreover, we have the estimate

$$\inf_{y \in B_{1/2}(0)} \left\{ |\nabla T_{x,r_x}u(y)| \right\} \geq \beta(n)/2 > 0 . \tag{1.20}$$

We can rephrase the previous estimate in the following form

$$\inf_{y \in B_{r_x/2}(x)} \{ |\nabla u(y)| \} \geq \frac{\beta(n)}{2} \frac{\left(\fint_{\partial B_{r_x}(0)} [u(y) - u(x)]^2 \, dy\right)^{1/2}}{r_x} > 0 . \tag{1.21}$$

Let us give an improved definition of the critical set below. It differs from (1.2) in that for a point $x \notin \mathcal{C}_r(u)$ not only is the gradient a definite size, but in fact u looks almost linear after normalization:

Definition 1.16. Given $r \geq 0$, we define the effective critical set at scale r by

$$\mathcal{C}_r(u) = \{x \text{ s.t. } r_x \leq r\}. \tag{1.22}$$

It is easy to see that for all $0 \leq s \leq r$ we have

$$\mathcal{C}(u) \subseteq \mathcal{C}_s(u) \subseteq \mathcal{C}_r(u). \tag{1.23}$$

Hausdorff measure and Minkowski content   Before stating the results let us quickly recall the notion of Hausdorff measure and Minkowski content. In short, the Hausdorff dimension of a set can be small although the set is dense; if the set is not closed, it can still be arbitrarily dense. On the other hand, Minkowski type estimates bound not only the set in question, but the tubular neighborhood of that set, providing a much more analytically effective notion of *size*. Precisely, given a set $S \subseteq \mathbb{R}^n$ its k-dimensional Hausdorff measure is defined by

$$H^k(S) \equiv \lim_{r \to 0} \sum_{S \subseteq \cup B_{r_i}(x_i): r_i \leq r} w_k r_i^k. \tag{1.24}$$

Hence, the Hausdorff measure is obtained from the most efficient coverings of S by balls of arbitrarily small size. On the other hand, the Minkowski k-content is defined by

$$M^k(S) \equiv \lim_{r \to 0} \sum_{S \subseteq \cup B_r(x_i)} w_k r^k. \tag{1.25}$$

Hence, the Minkowski r-content of S is obtained by covering S with balls of a *the same* size, r, which is then taken to be arbitrarily small. Equivalently in our situation, it is obtained by controlling the volume of tubular neighborhoods of S. The Hausdorff and Minkowski dimensions are then defined as the smallest numbers k such that $H^{k'}(S) = 0$ or $M^{k'}(S) = 0$, respectively, for all $k' > k$. As a simple example note that the Hausdorff dimension of the rationals in $B_1(0)$ is 0, while the Minkowski dimension is n.

Main theorem   Let us begin with the following result which is an immediate consequence of Theorem 1.10 and the remarks following that theorem:

Theorem 1.17. *Let $u : B_1(0) \to \mathbb{R}$ satisfy (1.3) and (1.5) weakly with $\bar{N}^u(0, 1) \leq \Lambda$. Then for every $\eta > 0$ we have*

$$\text{Vol}(B_r(\mathcal{C}_r(u)) \cap B_{1/2}(0)) \leq C(n, \lambda, \Lambda, \eta) r^{2-\eta}. \tag{1.26}$$

Remark 1.18. This immediately gives us the weaker estimate that Minkowski dimension of $\mathcal{C}(u)$ satisfies $\dim_{Min} \mathcal{C}(u) \leq n - 2$.

Thus we really have estimates on an effective version of the critical set.

Remark 1.19. The theorem still holds for solutions u of (1.4), provided we restrict ourself to the zero level set of u. That is, in this case we have $\text{Vol}[B_r(\mathcal{C}(u) \cap u^{-1}(0)) \cap B_{1/2}(0)] \leq C(n, \lambda, \Lambda, \eta) r^{2-\eta}$.

Remark 1.20. On a manifold the constant C should also depend on the sectional curvature of M and the volume of B_1.

$(n-2)$-**Hausdorff estimates**   As an easy application of Theorem 1.10 and an important ϵ-regularity theorem [HHL98, Lemma 3.2], we can show the critical and singular sets have finite $n-2$ measure if we assume the coefficients are sufficiently smooth. Note that this result follows also from the results in [HL00].

Theorem 1.21. *Let $u : B_1(0) \to \mathbb{R}$ satisfy (1.3) and (1.5) weakly with $\bar{N}^u(0,1) \leq \Lambda$, and such that*

$$\|\delta - a\|_{C^M}, \|b\|_{C^M} < \lambda,$$

where $M = M(n, \lambda, \Lambda)$. Then we have that

$$H^{n-2}(\mathcal{C}(u) \cap B_{1/2}(0)) < C(n, \lambda, \Lambda). \tag{1.27}$$

Remark 1.22. On a manifold the constant C should also depend on the sectional curvature of M and the volume of B_1.

Remark 1.23. As mentioned in the introduction, this theorem can also be proved by a simple adaptation of the proof of [HL00, theorem A].

Remark 1.24. The theorem still holds for solutions u of (1.4), provided that we restrict ourselves to the zero level set of u. In this case the result was originally proved in [HHL98, Theorem 3.1] (see also [HL, Theorem 7.2.1]).

For the sake of clarity, in giving the proofs, we will at first restrict our study to harmonic functions on $\mathbb{R}^n$. Technical details aside, all the ideas needed for the proof of the general case are already present in this case. We will then turn our attention to the general elliptic case, pointing out the differences between the two situations.

2   Harmonic functions

Throughout this section, u will denote a harmonic function on the unit ball, i.e., a function $u : B_1(0) \subseteq \mathbb{R}^n \to \mathbb{R}$ which solves

$$\Delta u = 0. \tag{2.1}$$

As in [CN13, CN12] a key tool in the development of a quantitative stratification is the existence of an appropriate monotone quantity. In this context this monotone quantity is the Almgren frequency function and its various generalizations, see Section 3.1. We begin by introducing the standard frequency function.

2.1   Almgren's Frequency and Normalized Frequency

Definition 2.1. If u is a nonzero harmonic function, for $x \in B_1(0)$ and $r \in (0, 1-|x|)$ we define the Almgren's frequency function by:

$$N^u(x, r) = \frac{r \int_{B_r(x)} |\nabla u|^2 \, dV}{\int_{\partial B_r(x)} u^2 \, dS}. \tag{2.2}$$

If u is nonconstant, we define the normalized version of Almgren's frequency function by:

$$\bar{N}^u(x,r) = N^{u-u(x)}(x,r) = \frac{r \int_{B_r(x)} |\nabla u|^2 \, dV}{\int_{\partial B_r(x)} (u - u(x))^2 dS} . \tag{2.3}$$

Remark 2.2. As we will see, the frequency function can be used to control the vanishing order of u at each point. However, since we are interested in the study of the critical set, not just the singular one, we will need information on the vanishing order at x of $u - u(x)$. In this context, the definition of normalized frequency in (2.3) is the natural extension of the standard one.

An essential property of $N(x,r)$ is that it is invariant under rescaling and blow-ups. The normalized frequency $\bar{N}$, has in addition, the property of remaining unchanged if we add a constant to u. More generally, we have the following easily verified lemma.

Lemma 2.3. *Let α, β, γ be real constants, $\alpha, \beta \neq 0$. If $w(x) = \alpha u(\beta x) + \gamma$, then:*

$$\bar{N}^u(0,r) = \bar{N}^w(0,\beta^{-1}r) \tag{2.4}$$

The main property of the frequency function is its monotonicity with respect to r.

Theorem 2.4. *Let u be a nonconstant harmonic function, and $x \in B_1(0)$. Then $\bar{N}(x,r)$ is monotone non-decreasing with respect to r. Moreover, if for some $0 \leq r_1 < r_2$, $\bar{N}(x,r_1) = \bar{N}(x,r_2)$, then $u - u(x)$ is a homogeneous harmonic polynomial of degree $d = N(x,r)$ centered at x.*

Here, by definition we say a polynomial p is homogeneous and centered at x if $p(y) = \sum_{|\beta|=d} c_\beta (y-x)^\beta$, where β is a multi-index and $|\beta| \equiv \sum \beta_i$.

Proof. Since x is fixed, it is evident that the assertions for $\bar{N}$ are equivalent to those for N. In that case, they are well-known (see Section 3.1 for a more general computation). □

Remark 2.5. Using monotonicity, we can define $\bar{N}(x,0) = \lim_{r \to 0} \bar{N}(x,r)$. This quantity has a very concrete interpretation. Indeed, it is easy to see that $\bar{N}(x,0)$ is the degree of the leading polynomial $T_x u$. By assumption, u is not constant, and thus we deduce the important lower bound $\bar{N}(x,r) \geq \bar{N}(x,0) \geq 1$ for all x, r.

Remark 2.6. For positive r, let $H(x,r) = \fint_{\partial B_r(x)} u^2 dS$. A well-known corollary to the monotonicity of N is the following doubling condition on H:

$$H(x,r_2) \leq \left(\frac{r_2}{r_1}\right)^{2N(x,r_2)} H(x,r_1) . \tag{2.5}$$

By replacing u with $u - u(x)$ we obtain an analogous property for the similarly defined quantity $\bar{H}(x,r) = \fint_{\partial B_r(x)} (u - u(x))^2 dS$. Note that this doubling property has as an immediate corollary the unique continuation property for harmonic functions.

The main results in this paper give estimates that rely on $\bar{N}^u(0,1)$. The next lemma proves that an upper bound on this quantity implies uniform upper bounds on $\bar{N}^u(x,r)$, where x and r are chosen in such a way that $B_r(x) \Subset B_1(0)$.

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Lemma 2.7. *Let u be a nonconstant harmonic function in $B_1(0) \subseteq \mathbb{R}^n$ with $\bar{N}(0, 1) \leq \Lambda$. For each positive $\kappa < 1$, there exists a function $C(n, \Lambda, \kappa)$ such that for each $x \in B_\kappa(0)$ and $r \leq \frac{2}{3}(1 - \kappa)$,*

$$\bar{N}(x, r) \leq C(n, \Lambda, \kappa). \tag{2.6}$$

Proof. In [HL, Theorem 2.2.8], a similar lemma is proved with $N(x, r)$ in place of $\bar{N}(x, r)$. Here we only prove the statement for $\kappa = \frac{1}{4}$ and $r = \frac{1}{2}$, a simple covering and compactness argument can be used to prove the general case.

Without loss of generality, we assume $u(0) = 0$, and so $N(0, r) = \bar{N}(0, r) \geq 1$ for all $r \leq 1$. By definition:

$$\bar{N}(x, 1/2) = \frac{r \int_{B_{1/2}(x)} |\nabla u|^2 \, dV}{\int_{\partial B_{1/2}(x)} (u - u(x))^2 dS} = \frac{r^2 \fint_{B_r(x)} |\nabla u|^2 \, dV}{n \fint_{\partial B_{1/2}(x)} (u - u(x))^2 dS} \tag{2.7}$$

The mean value theorem for harmonic functions gives:

$$\fint_{\partial B_r(x)} (u - u(x))^2 dS = \fint_{\partial B_r(x)} u^2 dS - u(x)^2 \geq 0. \tag{2.8}$$

Using the doubling conditions in equation (2.5), we get the estimate

$$u(x)^2 \leq H(x, 1/3) \leq H(x, 1/2)(2/3)^{2N(x, 1/3)}. \tag{2.9}$$

Thus, we have immediately:

$$\bar{N}(x, 1/2) = \frac{(1/2)^2}{n} \frac{\fint_{B_{1/2}(x)} |\nabla u|^2 \, dV}{\left[\fint_{\partial B_{1/2}(x)} (u)^2 dS \right] - u(x)^2} \leq N(x, 1/2) \left(1 - (2/3)^{2N(x, 1/3)} \right)^{-1}. \tag{2.10}$$

By [HL, Theorem 2.2.8], we have that $N(x, 1/2) \leq C(n, \Lambda)$. In order to conclude the proof, we need to show $N(x, 1/3) \geq C(n, \Lambda)$. This follows from simple algebraic manipulations. Indeed, by repeated applications of standard estimates (or the optimal estimate of [HL, Corollary 2.2.7]), we have

$$\int_{\partial B_{1/3}(x)} u^2 dS \leq \frac{1}{3}(n + 2N(x, 1/3)) \int_{B_{1/3}(x)} u^2 dV \leq C(n, \Lambda) \int_{B_1(0)} u^2 dV \leq \frac{C(n, \Lambda)}{n} \int_{\partial B_1(0)} u^2 dS, \tag{2.11}$$

while by using the doubling conditions in equation (2.5), we have

$$\int_{\partial B_1(0)} u^2 dS \leq 12^{n-1+2N(0,1)} \int_{\partial B_{1/12}(0)} u^2 dS. \tag{2.12}$$

Finally, by the inclusion $B_{1/12}(0) \subset B_{1/3}(x)$ we have

$$N(x, 1/3) = \frac{(1/3) \int_{B_{1/3}(x)} |\nabla u|^2}{\int_{\partial B_{1/3}(x)} u^2} \geq C(n, \Lambda) N(0, 1/12) \geq C(n, \Lambda). \tag{2.13}$$

$\square$

2.2 Quantitative Rigidity and Cone-Splitting

In this subsection, we will show that the normalized frequency function can be used to characterize the (k, ϵ, r, x)-symmetric points for u. Then we will prove the cone-splitting theorem for such points.

As we have seen, a function u is a homogeneous harmonic polynomial of degree d if and only if $N(0, r) = d$ for all r, or equivalently for $r \in (r_1, r_2)$. Using a simple compactness argument and the properties of $\bar{N}$, we turn this statement into a quantitative characterization of the almost symmetric points.

Theorem 2.8. *Fix $\eta > 0$ and $0 \leq \gamma < 1$, and let u be a nonconstant harmonic function with $\bar{N}(0, 1) \leq \Lambda$. Then there exists a positive $\epsilon = \epsilon(n, \Lambda, \eta, \gamma)$ such that if*

$$\bar{N}(0, 1) - \bar{N}(0, \gamma) < \epsilon, \tag{2.14}$$

then u is $(0, \eta, 1, 0)$-symmetric.

Proof. Suppose by contradiction that there exists a sequence of harmonic functions u_i with $\bar{N}^{u_i}(0, 1) \leq \Lambda$, $\bar{N}^{u_i}(0, 1) - \bar{N}^{u_i}(0, \gamma) < \frac{1}{i}$ but all the u_i are not $(0, \eta, 1, 0)$-symmetric.

From the invariance under rescaling of the frequency and of the concept of almost symmetry, we can assume without loss of generality that $\fint_{\partial B_1(0)} u_i^2 dS = 1$ and $u_i(0) = 0$ for all i, i.e. $u_i = T_{0,1} u_i$. Thus by compactness, u_i converges weakly in $W^{1,2}(B_1(0))$ to a harmonic function u, and by elliptic estimates, the convergence is also in the local $C^1(B_1)$ sense. Using the theory of traces for Sobolev spaces, it is easily seen that $\fint_{\partial B_1(0)} u^2 dS = 1$ and that $N^u(0, 1) \leq \Lambda$. Moreover, using the monotonicity of $\bar{N}$ and passing to the limit in n we have:

$$\bar{N}^u(0, 1) - \bar{N}^u(0, \gamma) = 0. \tag{2.15}$$

This implies that u is a harmonic homogeneous polynomial, and since

$$\lim_{i \to \infty} \fint_{\partial B_1(0)} (u_i - u)^2 dS = 0, \tag{2.16}$$

we obtain a contradiction. $\qquad \square$

Remark 2.9. By the invariance properties of $\bar{N}$, it is evident that we can replace the hypothesis $\bar{N}(0, 1) - \bar{N}(0, \gamma) < \epsilon$ with $\bar{N}(0, r) - \bar{N}(0, \gamma r) < \epsilon$ and obtain that u is $(0, \eta, r, 0)$-symmetric.

Remark 2.10 (Quantitative Differentiation). Note that the above lemma automatically provides a control on the number of scales at which u is not $(0, \eta, r, x)$-symmetric. Indeed, set $r_i = \gamma^i$ for some $0 < \gamma < 1$. By monotonicity, there can be only a definite number of i's such that $\bar{N}(x, \gamma^i) - \bar{N}(x, \gamma^{i+1}) \geq \epsilon$. Then u is $(0, \eta, \gamma^i, x)$-symmetric, for all the "good" values of i.

In order to describe how two almost symmetric points interact, we briefly recall what happens to homogeneous polynomials.

Proposition 2.11. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a harmonic polynomial of degree d, homogeneous with respect to the origin. Suppose also that P is symmetric with respect to the k dimensional subspace V. Then*

1. P is of degree 1 if and only if it is $n - 1$ symmetric

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2. *if P is not $n-1$ symmetric, and P is also 0-symmetric with respect to $x \notin V$, then P is $k+1$-symmetric with respect to $\mathrm{span}(V, x)$.*

Proof. Since P is supposed to be harmonic, (1) is straightforward to prove. (2) is a standard exercise in algebra. (A similar computation is carried out in the proof of [HL, theorem 4.1.3]). $\qquad\square$

By using a compactness argument similar to the one used for Theorem 2.8, we can turn the previous proposition into a quantitative cone-splitting theorem for almost symmetric harmonic functions. As always, note that this statement is scale invariant.

Theorem 2.12. *Fix some positive ϵ, τ and $0 < r \leq 1$ and let $k \leq n-2$. Let u be a harmonic function with $\bar{N}(0, 1) \leq \Lambda$. There exists a positive $\delta = \delta(n, \Lambda, \tau, \epsilon, r)$ such that if*

1. *u is $(k, \delta, r, 0)$-symmetric with respect to the k-dimensional subspace V,*

2. *for some $x \in B_r(0) \setminus B_\tau(V)$, u is $(0, \delta, r, x)$-symmetric,*

then u is also $(k+1, \epsilon, 1, 0)$-symmetric.

Proof. We set up the usual contradiction argument. In particular, choose a sequence u_i of harmonic functions with $u_i(0) = 0$ and $\fint_{\partial B_1(0)} u_i^2 dS = 1$ which is $(k, i^{-1}, r, 0)$-symmetric with respect to V_i and $(0, i^{-1}, r, 0)$-symmetric with respect to x_i. The bound on the frequency implies that u_i is bounded in $W^{1,2}(B_1(0))$. Thus, after passing to a subsequence if necessary, we can assume that $u_i \to u$, $V_i \to V$ and $x_i \to x \notin V$.

On the other hand, by hypothesis $T_{0,r} u_i$ converges to a k-symmetric normalized homogeneous polynomial P. By the doubling conditions in equation (2.5), we have

$$\fint_{\partial B_r} u_i^2 dS \geq r^{2\Lambda} > 0 , \tag{2.17}$$

so $P = u$. In a similar fashion, u is also a $(0, x)$-symmetric polynomial, and by Proposition 2.11 P is $(k+1, 0)$-symmetric.

Since u_i converges to P in $W^{1,2}(B_1(0))$, we obtain a contradiction. $\qquad\square$

The following equivalent version of Theorem 2.12 will be useful in subsequent sections.

Corollary 2.13. *Fix some positive η, τ and $0 < r \leq 1$ and let $k \leq n-2$. Let u be a harmonic function with $\bar{N}(0, 1) \leq \Lambda$. There exists $\epsilon = \epsilon(n, \Lambda, \tau, \eta, r) > 0$ such that if*

1. *u is $(0, \epsilon, r, 0)$-symmetric,*

2. *for every subspace V of dimension $\leq k$, there exists $x \in B_r(0) \setminus B_\tau(V)$ such that u is $(x, \epsilon, r, 0)$-symmetric,*

then u is also $(k+1, \eta, 1, 0)$-symmetric.

The proof of this corollary is via a simple induction argument which will be omitted. For similar arguments see [CN13, CN12]

We close this subsection with the proof of point (2) in Theorem 1.10. This proposition is essential for turning estimates on the singular strata $\mathcal{S}_{\eta,r}^k$ into estimates on the critical set. In fact, we show the following.

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Proposition 2.14. *Let u be harmonic with $\bar{N}(x, r) \leq \Lambda$. Fix $\epsilon > 0$ and $k \in \mathbb{N}$. There exists $\bar{\eta} = \bar{\eta}(n, k, \epsilon, \Lambda) > 0$ such that if u is $(n - 1, \bar{\eta}, r, x)$-symmetric, then*

$$\left\| T_{x,r}u - L \right\|_{C^k(B_{1/2}(0))} \leq \epsilon, \tag{2.18}$$

where L is a linear polynomial with $\fint_{\partial B_r} |L|^2 \, dS = 1$. In particular, by choosing $k = 1$ and ϵ small enough, there exists $\eta = \eta(n, \Lambda)$ such that if u is $(n - 1, \eta, r, x)$-symmetric then $r_x \geq r$.

Proof. The proof is a simple application of the usual contradiction-compactness argument. Note that, by elliptic estimates, if u_i converges to u in the weak $W^{1,2}(B_1(0))$ sense, then for all $K \Subset B_1(0)$ the convergence is also in the metric of $C^\infty(K)$. Note also that if L is a linear function with $\fint_{\partial B_1(0)} |L|^2 \, dS = 1$, then ∇L is a vector of fixed positive length. Thus the second part of the statement can be proved by choosing $\epsilon = |\nabla L| / 2$. $\square$

2.3 The Frequency Decomposition

We are now ready to prove Theorem 1.10. The proof employs the same techniques that were introduced for corresponding purposes in [CN13, CN12]; the reader may wish to consult these references. Instead of proving the statement for any $r > 0$, we fix a $0 < \gamma < 1$ and restrict ourselves to the case $r = \gamma^j$ for any $j \in \mathbb{N}$. It is evident that the general statement follows. For the reader's convenience we restate Theorem 1.10 under this convention.

Theorem 2.15. *Let $u : B_1(0) \to \mathbb{R}$ be a harmonic function with $\bar{N}^u(0, 1) \leq \Lambda$. Then for every $j \in \mathbb{N}$, $\eta > 0$ and $k \leq n - 2$, there exists $0 < \gamma(n, \eta, \Lambda) < 1$ such that*

$$\mathrm{Vol}\left(B_{\gamma^j}(\mathcal{S}^k_{\eta,\gamma^j}) \cap B_{1/2}(0) \right) \leq C(n, \Lambda, \eta) \left(\gamma^j \right)^{n-k-\eta}. \tag{2.19}$$

The scheme of the proof is the following: for some convenient $0 < \gamma < 1$ we prove that there exists a covering of S^k_{η,γ^j} made of nonempty open sets in the collection $\{\mathcal{C}^k_{\eta,\gamma^j}\}$. Each set $\mathcal{C}^k_{\eta,\gamma^j}$ is the union of a controlled number of balls of radius γ^j. Using Remark 2.10 (Quantitative differentiation) it will follow that the number of nonempty elements in each family has a bound of the form j^D, for some constant $D(n, \eta, \Lambda) > 1$. This will give the desired volume bound. In particular:

Lemma 2.16 (Decomposition Lemma). *There exists $c_0(n), c_1(n) > 0$ and $D(n, \eta, \Lambda) > 1$ such that for every $j \in \mathbb{N}$,*

1. *$\mathcal{S}^k_{\eta,\gamma^j} \cap B_{1/2}(0)$ is contained in the union of at most j^D nonempty open sets C^k_{η,γ^j}.*

2. *Each C^k_{η,γ^j} is the union of at most $(c_1 \gamma^{-n})^D (c_0 \gamma^{-k})^{j-D}$ balls of radius γ^j.*

Once this Lemma is proved, Theorem 2.15 easily follows.

Proof of Theorem 2.15. Let $\gamma = c_0^{-2/\eta} < 1$. Since we have a covering of $\mathcal{S}^k_{\eta,\gamma^j} \cap B_{1/2}(0)$ by balls of radius γ^j, it is easy to get a covering of $B_{\gamma^j}\left(\mathcal{S}^k_{\eta,\gamma^j} \right) \cap B_1(0)$. In fact it is sufficient to double the radius of the original balls. Now it is evident that

$$\mathrm{Vol}\left[B_{\gamma^j}\left(\mathcal{S}^k_{\eta,\gamma^j} \right) \cap B_{1/2}(0) \right] \leq j^D \left((c_1 \gamma^{-n})^D (c_0 \gamma^{-k})^{j-D} \right) \omega_n 2^n \left(\gamma^j \right)^n, \tag{2.20}$$

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where ω_n is the volume of the n-dimensional unit ball. By plugging in the simple rough estimates

$$j^D \le c(n, \Lambda, \eta) \left(\gamma^j\right)^{-\eta/2}, \tag{2.21}$$

$$(c_1 \gamma^{-n})^D (c_0 \gamma^{-k})^{-D} \le c(n, \Lambda, \eta),$$

and using the definition of γ, we obtain the desired result. $\qquad\square$

Proof of the Decomposition Lemma Now we turn to the proof of the Decomposition Lemma. In order to do this, we define a new quantity which measures the non-symmetry of u at a certain scale.

Definition 2.17. Given u as in Theorem 2.15, $x \in B_1(0)$ and $0 < r < 1$, define

$$\mathcal{N}(u, x, r) = \inf\{\alpha \ge 0 \ \ s.t. \ \ u \text{ is } (0, \alpha, r, x)\text{-symmetric}\}. \tag{2.22}$$

Given $\epsilon > 0$, we divide the set $B_{1/2}(0)$ into two subsets according to the behaviour of the points with respect to their quantitative symmetry.

$$H_{r,\epsilon}(u) = \{x \in B_{1/2}(0) \ s.t. \ \mathcal{N}(u, x, r) \ge \epsilon\}, \tag{2.23}$$

$$L_{r,\epsilon}(u) = \{x \in B_{1/2}(0) \ s.t. \ \mathcal{N}(u, x, r) < \epsilon\}.$$

Next, to each point $x \in B_{1/2}(0)$ we associate a j-tuple $T^j(x)$ of numbers $\{0, 1\}$ in such a way that the i-th entry of T^j is 1 if $x \in H_{\gamma^i, \epsilon}(u)$, and zero otherwise. Then, for each fixed j-tuple $\bar{T}^j$, set:

$$E(\bar{T}^j) = \{x \in B_{1/2}(0) \ \ s.t. \ \ T^j(x) = \bar{T}^j\}. \tag{2.24}$$

Also, we denote by T^{j-1}, the $(j-1)$-tuple obtained from T^j by dropping the last entry, and define $\left|T^j\right|$ to be the number of entries that are equal to 1 the j-tuple T^j.

We will build the families $\{C^k_{\eta, \gamma^j}\}$ by induction on j in the following way. For $a = 0$, $\{C^k_{\eta, \gamma^0}\}$ consists of the single ball $B_1(0)$.

Induction step For fixed $a \le j$, consider all the 2^a a-tuples $\bar{T}^a$. Label the sets in the family $\{C^k_{\eta, \gamma^a}\}$ by all the possible $\bar{T}^a$. We will build $C^k_{\eta, \gamma^a}(\bar{T}^a)$ inductively as follows. For each ball $B_{\gamma^{a-1}}(y)$ in $\{C^k_{\eta, \gamma^{a-1}}(\bar{T}^{a-1})\}$ take a minimal covering of $B_{\gamma^{a-1}}(y) \cap \mathcal{S}^k_{\eta, \gamma^j} \cap E(\bar{T}^a)$ by balls of radius γ^a centered at points in $B_{\gamma^{a-1}}(x) \cap \mathcal{S}^k_{\eta, \gamma^j} \cap E(\bar{T}^a)$. Note that it is possible that for some a-tuple $\bar{T}^a$, the set $E(\bar{T}^a)$ is empty, and in this case $\{C^k_{\eta, \gamma^a}(\bar{T}^a)\}$ is the empty set.

Now we need to prove that the minimal covering satisfies points 1 and 2 in Lemma 2.16.

Remark 2.18. The value of $\epsilon > 0$ will be chosen according to Lemma 2.20. For the moment, we take it to be an arbitrary fixed small quantity.

Point 1 in Lemma As we will see below, we can use the monotonicity of $\bar{N}$ to prove that for every $\bar{T}^j$, $E(\bar{T}^j)$ is empty if $\left|\bar{T}^j\right| \ge D$. Since for every j there are at most $\binom{j}{D} \le j^D$ choices of j-tuples with $\left|\bar{T}^j\right| \le D$, the first point will be proved.

Lemma 2.19. *There exists $D = D(\epsilon, \gamma, \Lambda, n)$ such that $E(\bar{T}^j)$ is empty if $\left|\bar{T}^j\right| \geq D$.*

In what follows, we will fix ϵ as a function of η, Λ, n. Thus, D will actually depend only on these three variables.

Proof. Recall that $\bar{N}(x, r)$ is monotone nondecreasing with respect to r, and, by Lemma 2.7, $\bar{N}(x, 1/3)$ is bounded above by a function $C(n, \Lambda)$. For $s < r$, we set

$$\mathcal{W}_{s,r}(x) = \bar{N}(x, r) - \bar{N}(x, s) \geq 0. \tag{2.25}$$

If (s_i, r_i) are *disjoint* intervals with $\max\{r_i\} \leq 1/3$, then by monotonicity of $\bar{N}$:

$$\sum_i \mathcal{W}_{s_i, r_i}(x) \leq \bar{N}(x, 1/3) - \bar{N}(x, 0) \leq C(n, \Lambda) - 1. \tag{2.26}$$

Let $\bar{i}$ be such that $\gamma^{\bar{i}} \leq 1/3$, and consider intervals of the form (γ^{i+1}, γ^i) for $i = \bar{i}, \bar{i} + 1, ...\infty$. By Theorem 2.8 and Lemma 2.7, there exists a $0 < \delta = \delta(\epsilon, \gamma, \Lambda, n)$ independent of x such that

$$\mathcal{W}_{\gamma^{i+1}, \gamma^i}(x) \leq \delta \implies u \text{ is } (0, \epsilon, \gamma^i, x)\text{-symmetric}. \tag{2.27}$$

In particular $x \in L_{\gamma^i, \epsilon}$, so that, if $i \leq j$, the i-th entry of T^j is necessarily zero. By equation (2.26), there can be only a finite number of i's such that $\mathcal{W}_{\gamma^{i+1}, \gamma^i}(x) > \delta$, and this number D is bounded by:

$$D \leq \frac{C(n, \Lambda) - 1}{\delta(\epsilon, \gamma, \Lambda, n)}. \tag{2.28}$$

This completes the proof. $\qquad\square$

Point 2 in Lemma The proof of the second point in Lemma 2.16 is mainly based on Corollary 2.13. In particular, for fixed k and η in the definition of $\mathcal{S}^k_{\eta, \gamma^j}$, choose ϵ in such a way that Corollary 2.13 can be applied with $r = \gamma^{-1}$ and $\tau = 7^{-1}$. Then we can restate the lemma as follows:

Lemma 2.20. *Let $\bar{T}^j_a = 0$. Then the set $A = \mathcal{S}^k_{\eta, \gamma^j} \cap B_{\gamma^{a-1}}(x) \cap E(\bar{T}^j)$ can be covered by $c_0(n)\gamma^{-k}$ balls centered in A of radius γ^a.*

Proof. First of all, note that since $\bar{T}^j_a = 0$, all the points in $E(\bar{T}^j)$ are in $L_{\epsilon, \gamma^a}(u)$.

The set A is contained in $B_{7^{-1}\gamma^a}(V^k) \cap B_{\gamma^{a-1}}(x)$ for some k-dimensional subspace V^k. Indeed, if there were a point $x \in A$, such that $x \notin B_{7^{-1}\gamma^a}(V^k) \cap B_{\gamma^{a-1}}(x)$, then by Corollary 2.13 and Lemma 2.7, u would be $(k + 1, \eta, \gamma^{a-1}, x)$-symmetric. This contradicts $x \in \mathcal{S}^k_{\eta, \gamma^j}$. By standard geometry, it follows that $V^k \cap B_{\gamma^{a-1}}(x)$ can be covered by $c_0(n)\gamma^{-k}$ balls of radius $\frac{6}{7}\gamma^a$, and by the triangle inequality it is evident that the same balls with radius γ^a cover the whole set A. $\qquad\square$

If instead $\bar{T}^j_a = 1$, then without any effort we can say that $A = \mathcal{S}^k_{\eta, \gamma^j} \cap B_{a-1}(x) \cap E(\bar{T}^j)$ can be covered by $c_0(n)\gamma^{-n}$ balls of radius γ^a. Now by a simple induction argument the proof is complete.

Lemma 2.21. *Each (nonempty) C^k_{η, γ^j} is the union of at most $(c_1\gamma^{-n})^D \cdot (c_0\gamma^{-k})^{j-D}$ balls of radius γ^j.*

Proof. Fix a sequence $\bar{T}^j$ and consider the set $C^k_{\eta,\gamma^j}(\bar{T}^j)$. By Lemma 2.19, we can assume that $\left|\bar{T}^j\right| \leq D$, otherwise there is nothing to prove since $C^k_{\eta,\gamma^j}(\bar{T}^j)$ would be empty.

Consider that for each step a, in order to get a (minimal) covering of $B_{\gamma^{a-1}}(x) \cap S^k_{\eta,\gamma^i} \cap E(\bar{T}^j)$ for $B_{\gamma^{a-1}}(x) \in C^k_{\eta,\gamma^{a-1}}(\bar{T}^j)$, we require at most $(c_0\gamma^{-k})$ balls of radius γ^a if $\bar{T}^j_a = 0$ or $(c_0\gamma^n)$ otherwise. Since the latter situation can occur at most D times, the proof is complete. □

2.4 Minkowski Type Estimates on the Critical Set

Apart from the volume estimate, Theorem 1.10 has a useful corollary for measuring the size of the critical set. Indeed, by Proposition 2.14, the critical set of u is contained in $S^{n-2}_{\epsilon,r}$, thus we have proved Theorem 1.17 for harmonic functions:

Corollary 2.22. *Let $u : B_1(0) \to \mathbb{R}$ be a harmonic function with $\bar{N}^u(0,1) \leq \Lambda$. Then, for every $\eta > 0$,*

$$\mathrm{Vol}(B_r(\mathcal{C}_r(u)) \cap B_{1/2}(0)) \leq C(n, \Lambda, \eta)r^{2-\eta}. \tag{2.29}$$

Proof. By Proposition 2.14, for $\eta > 0$ small enough, we have the inclusion

$$B_{r/2}(\mathcal{C}_r(u)) \subseteq S^{n-2}_{\eta,r}. \tag{2.30}$$

Using Theorem 1.10, we obtain the desired volume estimate for η sufficiently small. However, since

$$\mathrm{Vol}(B_r(\mathcal{C}_r(u)) \cap B_{1/2}(0)) \leq \mathrm{Vol}(B_{1/2}(0)), \tag{2.31}$$

it is evident that if (2.29) holds for some η, then a similar statement holds also for any $\eta' \geq \eta$. □

Remark 2.23. As already mentioned in the introduction, this volume estimate on the critical set and its tubular neighborhoods immediately implies that $\dim_{Mink}(\mathcal{C}(u)) \leq n - 2$. This result is clearly optimal.

2.5 The Uniform $(n-2)$-Hausdorff Bound for the Critical Set

By combining the results of the previous sections with an ϵ-regularity theorem from [HHL98], in this subsection we give a new proof of an effective uniform bound on the $(n-2)$-dimensional Hausdorff measure of $\mathcal{C}(u)$. The bound will not depend on u itself, but only on the normalized frequency $\bar{N}^u(0,1)$. Specifically, the proof will be obtained by combining the $(n-3+\eta)$-Minkowski type estimates available for $S^{n-3}_{\eta,r}$ with the following ϵ-regularity lemma. The lemma states that if a harmonic function u is sufficiently close to a homogeneous harmonic polynomial of only 2 variables, then the whole critical set of u has a definite upper bound on its $(n-2)$-dimensional Hausdorff measure.

As noted in the introduction, these results also follow from an adaptation of the techniques used in [HL00]

Lemma 2.24. *[HHL98, Lemma 3.2] Let P be a homogeneous harmonic polynomial with exactly $n-2$ symmetries in $\mathbb{R}^n$. Then there exist positive constants ϵ and $\bar{r}$ depending on P, such that for any $u \in C^{2d^2}(B_1(0))$, if*

$$\|u - P\|_{C^{2d^2}(B_1)} < \epsilon, \tag{2.32}$$

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then for all $r \le \bar{r}$:

$$H^{n-2}(\nabla u^{-1}(0) \cap B_r(0)) \le c(n)(d-1)^2 r^{n-2}\,. \tag{2.33}$$

It is not difficult to see that, if we assume u harmonic in B_1 with $\bar{N}^u(0,1) \le \Lambda$, then ϵ and $\bar{r}$ can be chosen to be independent of P, but dependent only on Λ. Indeed, up to rotations and rescaling, all polynomials with $n-2$ symmetries in $\mathbb{R}^n$ of degree d look like $P(r,\theta,z) = r^d \cos(d\theta)$, where we used cylindrical coordinates for $\mathbb{R}^n$. Combining this with elliptic estimates yields the following corollary.

Corollary 2.25. *Let $u : B_1 \to \mathbb{R}$ be a harmonic function with $\bar{N}(0,1) \le \Lambda$. Then there exist positive constants $\epsilon(\Lambda, n)$ and $\bar{r}(\Lambda, n)$ such that if there exists a normalized homogeneous harmonic polynomial P with $n-2$ symmetries such that*

$$\left\| T^u_{0,1} - P \right\|_{L^2(\partial B_1)} < \epsilon\,, \quad \fint_{\partial B_1(0)} P^2 = 1\,, \tag{2.34}$$

then for all $r \le \bar{r}$:

$$H^{n-2}(\nabla u^{-1}(0) \cap B_r(0)) \le c(\Lambda, n) r^{n-2}\,. \tag{2.35}$$

To prove the effective bound on the $(n-2)$-dimensional Hausdorff measure, we combine the Minkowski type estimates of Theorem 1.10 with the above corollary. Using the quantitative stratification, we will use an inductive construction to split the critical set at different scales into a good part, the points where the function is close to an $(n-2)$-symmetric polynomial, and a bad part, whose tubular neighborhoods have definite bounds. Since we have estimates on the whole critical set in the good part, we do not have to worry any longer when we pass to a smaller scale. As for the bad part, by induction, we start the process over and split it again into a good and a bad part. By summing the various contributions to the $(n-2)$-dimensional Hausdorff measure given by the good parts, we prove the following theorem:

Theorem 2.26. *Let u be a harmonic function in $B_1(0)$ with $\bar{N}(0,1) \le \Lambda$. There exists a constant $C(\Lambda, n)$ such that*

$$H^{n-2}(\mathcal{C}(u) \cap B_{1/2}(0)) \le C(n, \Lambda)\,. \tag{2.36}$$

Proof. Note that by Lemma 2.7, for every $r \le 1/3$ and $x \in B_{1/2}(0)$, the functions $T_{x,r}u$ have frequency uniformly bounded by $N^{T_{x,r}u}(0,1) \le C(\Lambda, n)$. This will allow us to apply Corollary 2.25 to each $T_{x,r}u$ and obtain uniform constants $\epsilon(\Lambda, n)$ and $\bar{r}(\Lambda, n)$ such that the conclusion of the Corollary holds for all $x \in B_{1/2}(0)$ and $r \le \bar{r}$.

Now fix $\eta > 0$ to be the minimum of $\eta(n, \Lambda)$ from Proposition 2.14 and $\epsilon(n, \Lambda)$ from Corollary 2.25. Let $0 < \gamma \le 1/3$ and define the following sets:

$$\mathcal{C}^{(0)}(u) = \mathcal{C}(u) \cap \left(S^{n-2}_{\eta,1} \setminus S^{n-3}_{\eta,1} \right) \cap B_{1/2}(0)\,. \tag{2.37}$$

$$\mathcal{C}^{(j)}(u) = \mathcal{C}(u) \cap \left(S^{n-2}_{\eta,\gamma^j} \setminus S^{n-3}_{\eta,\gamma^j} \right) \cap S^{n-3}_{\eta,\gamma^{j-1}} \cap B_{1/2}(0)\,. \tag{2.38}$$

We decompose the critical set as follows:

$$\mathcal{C}(u) \cap B_{1/2}(0) = \bigcup_{j=0}^{\infty} \mathcal{C}^{(j)}(u) \bigcup \left(\mathcal{C}(u) \bigcap_{j=1}^{\infty} .S_{\eta,\gamma^j}^{n-3} \right). \tag{2.39}$$

It is evident from Theorem 2.15 that

$$H^{n-2} \left(\mathcal{C}(u) \bigcap_{j=1}^{\infty} S_{\eta,\gamma^j}^{n-3} \cap B_{1/2}(0) \right) = 0. \tag{2.40}$$

As for the other set, we will prove that

$$H^{n-2} \left(\bigcup_{j=0}^{k} \mathcal{C}^{(j)}(u) \right) \leq C(\Lambda, n, \eta) \sum_{j=0}^{k} \gamma^{(1-\eta)j}. \tag{2.41}$$

Using Corollary 2.25 and a simple covering argument, it is easy to see that this statement is valid for $k = 0$.

Choose a covering of the set $\mathcal{C}^{(k)}(u)$ by balls centered at $x_i \in \mathcal{C}^{(k)}(u)$ of radius $\gamma^k \bar{r}$, such that the same balls with half the radius are disjoint. Let $m(k)$ be the number of such balls. By the volume estimates in Theorem 1.10, we have

$$m(k) \leq C(\eta, \Lambda, n) \gamma^{(3-\eta-n)k}. \tag{2.42}$$

By construction of the set $\mathcal{C}^{(k)}(u)$, for each x_i there exists a scale $s \in [\gamma^k, \gamma^{k-1}]$ such that for some normalized homogeneous polynomial of two variables P, we have

$$\left\| T_{x_i,s} u - P \right\|_{L^2(\partial B_1)} < \eta. \tag{2.43}$$

Note that since u is harmonic, we can assume without loss of generality that P is harmonic as well. Indeed, if η is small enough, we can find a homogeneous harmonic polynomial P' such that $\|P - P'\|_{L^2(\partial B_1)} < \eta$.

Using Corollary 2.25 we can deduce that

$$H^{n-2} \left(\nabla u^{-1}(0) \cap B_{\gamma^k \bar{r}}(x_i) \right) \leq C(\Lambda, n) \gamma^{(n-2)k}. \tag{2.44}$$

Therefore,

$$H^{n-2} \left(\mathcal{C}^{(k)}(u) \right) \leq C(\Lambda, n, \eta) \gamma^{(1-\eta)k}. \tag{2.45}$$

Since $0 < \gamma, \eta < 1$, the proof is complete. □

3 Elliptic equations

With appropriate modifications, the results proved for harmonic functions are valid for solutions to elliptic equations of the form (1.3) with conditions (1.5). Indeed, a Minkowski type estimate of the form given in Theorem 2.15 and Corollary 2.22 (in which there is an arbitrarily small positive loss in the exponent) remains valid without any further regularity assumption on the coefficients a^{ij} and b^i. However, in order

to get an effective bound on the $(n-2)$-dimensional Hausdorff measure of the critical set, we will assume some additional control on the higher order derivatives of the coefficients of the PDE.

The basic ideas needed to estimate the critical sets of solutions to elliptic equations are exactly the same as in the harmonic case. The primary new technical ingredient is a *generalized frequency* function, $\bar{F}(r)$ which is an almost monotone quantity, i.e., for r effectively small the function $e^{Cr}\bar{F}(r)$ is monotone nondecreasing; see Theorem 3.8. The function $\bar{F}(r)$ will replace the frequency function of the harmonic case. It is constructed by a generalizing a constructions of [GL86, GL87]. Their function however, is only almost monotone for operators of divergence form on $\mathbb{R}^n$ for $n \geq 3$. Our construction will take up most of the next subsection. Though the proofs of many points involve standard techniques, we will include them for convenience and completeness.

3.1 The Generalized Frequency Function

In this section we define a generalized version of Almgren's frequency, denoted by $\bar{F}$, suitable to study the properties of solutions to (1.3). Even though the ideas in the construction are the same as in [GL86, GL87] [3], some of the details are different. This allow us to prove the almost monotonicity for a wider class of operators, and in particular, for those dealt with in this paper. For the reader's convenience, we include the proof of almost monotonicity of $\bar{F}$.

As a first step towards the definition, we introduce a new metric related to the coefficients a_{ij}, which is closely related to the constructions in [HL]. For the sake of simplicity, we will occasionally use the terms and notations typical of Riemannian manifolds. For instance, we denote by a_{ij} the elements of the inverse matrix of a^{ij} and by a the determinant of a_{ij}. The metric g_{ij} (also denoted by g) will be defined on $B_1(0) \subseteq \mathbb{R}^n$ and e_{ij} will denote the standard Euclidean metric. For ease of notation, we define $B(g, x, r)$ to be the geodesic ball centered at x with radius r with respect to the metric g.

It would seem natural to define a metric $g_{ij} = a_{ij}$ and use this metric in the definition of the frequency function. However, for such a metric the geodesic polar coordinates at a point x are well defined only in a small ball centered at x whose radius is not easily bounded from below with only Lipschitz control on the a_{ij}. To avoid this problem, we define a similar but slightly different metric which has been introduced in [AKS62, eq. (2.6)], and later used also in [GL86, GL87]; see also the nice survey paper [HL, Section 3.2]. In these papers, the authors use this metric to define a frequency function which turns out to be almost monotone at small scales for elliptic equations in divergence form on $\mathbb{R}^n$ with $n \geq 3$, and only bounded at small enough scales for more general equations.

We will introduce a modified frequency function which we will prove to be almost monotone at small scales for all solutions of equation (1.3), with neither a restriction on the dimension n, nor a divergence form assumption.

To begin with, we recall from [AKS62], the definition and some properties of the new metric g_{ij}. Fix an origin $\bar{x}$, and define the function r^2 on the Euclidean ball $B_1(0)$ by

$$r^2 = r^2(\bar{x}, x) = a_{ij}(\bar{x})(x - \bar{x})^i(x - \bar{x})^j, \tag{3.1}$$

---

[3]see also the survey [HL]

where $x = x^i e_i$ is the usual decomposition in the canonical basis of $\mathbb{R}^n$. Note that the level sets of r are Euclidean ellipsoids centered at $\bar{x}$, and the assumptions on the coefficients a_{ij} lead to the estimate

$$\lambda^{-1} |x - \bar{x}|^2 \le r^2(\bar{x}, x) \le \lambda |x - \bar{x}|^2 . \tag{3.2}$$

Proposition 3.1. *With the definitions above, set*

$$\eta(\bar{x}, x) = a^{kl}(x) \frac{\partial r(\bar{x}, x)}{\partial x^k} \frac{\partial r(\bar{x}, x)}{\partial x^l} = a^{kl}(x) \frac{a_{ks}(\bar{x}) a_{lt}(\bar{x})(x - \bar{x})^s (x - \bar{x})^t}{r^2} , \tag{3.3}$$

$$g_{ij}(\bar{x}, x) = \eta(\bar{x}, x) a_{ij}(x) . \tag{3.4}$$

Then for each $\bar{x} \in B_1(0)$, the geodesic distance $d_{\bar{x}}(\bar{x}, x)$ in the metric $g_{ij}(\bar{x}, x)$ is equal to $r(\bar{x}, x)$. In particular, geodesic polar coordinates with respect to $\bar{x}$ are well-defined on the Euclidean ball of radius $\lambda^{-1/2}(1 - |\bar{x}|)$. Moreover in these coordinates the metric assumes the form

$$g_{ij}(\bar{x}, (r, \theta)) = dr^2 + r^2 b_{st}(\bar{x}, (r, \theta)) d\theta^s d\theta^t , \tag{3.5}$$

where the $b_{st}(\bar{x}, r, \theta)$ can be extended to Lipschitz functions in $[0, \lambda^{-1/2}(1 - |\bar{x}|)] \times \partial B_1$ with

$$\left| \frac{\partial b_{st}}{\partial r} \right| \le C(\lambda) , \tag{3.6}$$

and $b_{st}(\bar{x}, 0, \theta)$ is the standard Euclidean metric on ∂B_1.

Remark 3.2. For the time being, let $\bar{x} = 0$ be fixed. As seen in the proposition, if a^{ij} is Lipschitz, then so is also the metric g_{ij}. However, if the coefficients a^{ij} are assumed to have higher regularity, for example C^1 or C^m, it easily seen that g_{ij} is of higher regularity away from the origin. But at the origin, in general, g_{ij} is only Lipschitz.

Before giving the formula for the generalized frequency, we rewrite equation (1.3) in a Riemannian form with respect to the metric g_{ij}. Using the Riemannian scalar product and Laplace operator, relation (1.3) is equivalent to

$$\Delta_g(u) = \langle B \,|\, \nabla u \rangle_g , \tag{3.7}$$

where B is the vector field which in the standard Euclidean coordinates has components

$$B_i = -\eta^{-1} b_i + \frac{\partial}{\partial x^i} \log \left(g^{1/2} \eta^{-1} \right) . \tag{3.8}$$

Given conditions (1.5), it is easy to prove the bound

$$\langle B \,|\, B \rangle_g = |B|_g^2 \le C(\lambda) .$$

Now we are ready to define the generalized frequency function for a (weak) solution u to (1.3). For convenience of notation, we will denote this new frequency $\bar{F}$.

Definition 3.3. For a solution u to equation (1.3), for each $\bar{x} \in B_1(0)$ and $r \leq \lambda^{-1/2}(1 - |\bar{x}|)$, define

$$D(u, \bar{x}, g, r) = \int_{B(g(\bar{x}), \bar{x}, r)} \|\nabla u\|_{g(\bar{x})}^2 \, dV_{g(\bar{x})} = \int_{r(\bar{x}, x) \leq r} \eta^{-1}(\bar{x}, x) a^{ij}(x) \partial_i u \partial_j u \sqrt{\eta^n(\bar{x}, x) a(x)} dx \,. \tag{3.9}$$

$$I(u, \bar{x}, g, r) = \int_{B(g(\bar{x}), \bar{x}, r)} \|\nabla u\|_{g(\bar{x})}^2 + (u - u(\bar{x})) \Delta_{g(\bar{x})}(u) dV_{g(\bar{x})} = \tag{3.10}$$

$$= \int_{r(\bar{x}, x) \leq r} \|\nabla u\|_{g(\bar{x})}^2 + (u - u(\bar{x})) \langle B \,|\, \nabla u \rangle_{g(\bar{x})} \, dV_{g(\bar{x})} \,.$$

$$H(u, \bar{x}, g, r) = \int_{\partial B(g(\bar{x}), \bar{x}, r)} [u - u(\bar{x})]^2 \, dS_{g(\bar{x})} = r^{n-1} \int_{\partial B_1} [u(r, \theta) - u(\bar{x})]^2 \sqrt{b(\bar{x}, r, \theta)} d\theta \,. \tag{3.11}$$

$$\bar{F}(u, \bar{x}, g, r) = \frac{rI(u, \bar{x}, g, r)}{H(u, \bar{x}, g, r)} \,. \tag{3.12}$$

Note that, by elliptic regularity, $\bar{F}$ is a locally Lipschitz function for $r > 0$. Moreover, since u is not constant, by unique continuation and the maximum principle, $H(r) > 0$ for all positive r. So $\bar{F}$ is well-defined. Note also that if the operator $\mathcal{L}$ in (1.3) is the usual Laplace operator, then it is easily seen that $\bar{F}(u, x, g, r) = \bar{N}^u(x, r)$.

For t sufficiently small, we can bound D in terms of I and vice versa. Moreover, by using the Poincaré inequality, we can bound $\bar{F}$ away from zero.

Proposition 3.4. *Fix u, x and the relative metric g. There exists a constant $C(\lambda)$ and $r_0 = r_0(n, \lambda) > 0$ such that for all admissible r,*

$$I(r) \leq CD(r) \,,$$

while for $r \leq r_0$,

$$D(r) \leq CI(r) \,.$$

Moreover, there exits $c(n, \lambda) > 0$ for which

$$\bar{F}(r) \geq c(n, \lambda) \,,$$

for all $r \leq r_0$.

Proof. Assume for simplicity that $x = 0$ and $u(0) = 0$. By definition, we have

$$I(r) = D(r) + \int_{B(r)} u \langle B \,|\, \nabla u \rangle \, dV \,. \tag{3.13}$$

Using Hölder and Poincaré's inequalities, it is easy to see that there exists a constant $C(\lambda)$ for which

$$\left| \int_{B(r)} u \langle B \,|\, \nabla u \rangle \, dV \right| \leq C(\lambda) \sqrt{\int_{B(r)} u^2 dV} \cdot D(r)^{1/2} \leq C(\lambda) r D(r) \,. \tag{3.14}$$

Thus, the estimates follow easily.

For the lower bound on $\bar{F}$, note that

$$\int_{\partial B(r)} u^2 dS = \frac{1}{r} \int_{\partial B(r)} u^2 \langle \vec{v} \,|\, \hat{n} \rangle \, dS = \frac{1}{r} \int_{B(r)} 2u \langle \nabla u \,|\, \vec{x} \rangle \, dV + \frac{1}{r} \int_{B(r)} u^2 \, \text{div}(\vec{v}) dV \,, \tag{3.15}$$

where $\vec{v}$ is the Lipschitz vector field $r\partial_r$. By conditions (1.5), div $(\vec{v}) \leq C(n, \lambda)$, and a simple application of Poincaré's inequality leads to

$$H(r) \leq c^{-1}(n, \lambda)rD(r) \leq c^{-1}(n, \lambda)rI(r). \tag{3.16}$$

$\square$

The frequency function $\bar{F}$ has invariance properties similar to those which hold for harmonic functions. For instance, it is invariant under blow-ups, as long as they are redefined in a geodesic sense. The following lemma is the counterpart of Lemma 2.3.

Lemma 3.5. *Let u be a nonconstant solution to (1.3). Fix $x \in B_1(0)$ and the relative metric g_{ij} as in Proposition 3.1. Consider the blow up given in geodesic polar coordinates centered at x by $(r, \theta) \to (tr, \theta)$. If we define $w(r, \theta) = \alpha u(tr, \theta) + \beta$ and $g_{ij}^t(r, \theta) = g_{ij}(tr, \theta)$, then*

$$\bar{F}(u, x, g, r) = \bar{F}(w, x, g^t, t^{-1}r). \tag{3.17}$$

Blow-up function $U_{x,r}$ Here we define two auxiliary functions $T_{x,r}$ and $U_{x,r}$ which are generalizations of the blow-up function $T_{x,r}$ for harmonic functions.

Using the geodesic blow-up given in the previous lemma, we introduce the function $U_{x,t}u(y)$ as follows.

Definition 3.6. We define

$$U_{x,t}u(r, \theta) \equiv \frac{u(tr, \theta) - u(0)}{\left(\fint_{\partial B(g(0),0,t)} [u(r, \theta) - u(0)]^2 dS(g) \right)^{1/2}} \qquad U_{x,t}u(0) = 0. \tag{3.18}$$

Note that elliptic regularity ensures that for all t, $U_{x,t}u \in W^{2,p}(B_1(0)) \cap C^{1,\alpha}(B_1(0))$. Moreover, $U_{x,t}$ is normalized in the sense that:

$$\fint_{\partial B(g(x)^t,0,1)} |U_{x,t}|^2 dS(g(x)^t) = 1. \tag{3.19}$$

Using a simple change of variables, it is easy to see that U_t satisfies (in the weak sense) the equation

$$\Delta_{g(x)^t} U_{x,t} = t \langle B | \nabla U_{x,t}u \rangle_{g(x)^t}, \tag{3.20}$$

where B is defined by equation (3.8).

Blow-up function $T_{x,r}$ For a fixed x, let $q^{ij}(x)$ be the square root of the matrix $a^{ij}(x)$, and define the linear operator Q_x by

$$Q_x(y) = q_{ij}(y - x)^i e_j. \tag{3.21}$$

It is evident that, independently of x, Q_x is a bi-Lipschitz equivalence from $\mathbb{R}^n$ to itself with Lipschitz constant $(1 + \lambda)^{1/2}$. Moreover, note that the ellipsoid $\|Q_x(y)\| \leq r$ is exactly the geodesic ball $B(g(x), x, r)$, where $g(x)$ is the metric introduced in Proposition 3.1.

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Definition 3.7. Define the function $T_{x,t} : B_1(0) \to \mathbb{R}$ by

$$T_{x,t}(y) = \frac{u(x + tQ_x^{-1}(y)) - u(x)}{\left(\int_{\partial B_1} [u(x + tQ_x^{-1}(y)) - u(x)]^2 dS\right)^{1/2}}. \tag{3.22}$$

Using a simple change of variables, it is easy to see that the function T satisfies an elliptic PDE of the form:

$$\tilde{\mathcal{L}}(u) = \partial_i \left(\tilde{a}^{ij} \partial_j T\right) + \tilde{b}^i \partial_i T = 0, \tag{3.23}$$

with $\tilde{a}^{ij}(x) = \delta^{ij}$. Moreover, as long as $t \leq 1$, condition (3.52) implies a similar estimate for the coefficients $\tilde{a}^{ij}$, $\tilde{b}^i$:

$$\left\|\tilde{a}^{ij}\right\|_{C^M(B_1)}, \ \left\|\tilde{b}^i\right\|_{C^M(B_1)} \leq C(n, \lambda, L). \tag{3.24}$$

Thus, on $B_1(0)$ we have uniform elliptic estimates on $T_{x,t}u(y)$ for $x \in B_{1/2}(0)$ and $t \leq (1 + \lambda)^{-1}/3$.

Note in addition that as t converges to 0, $U_{x,t}$ converges to $T_{x,t}$ in $C^{0,1}(B_1(0))$.

Almost monotonicity   By an argument that is philosophically identical to the one for harmonic functions, although technically more complicated, we show that this modified frequency is almost monotone in the following sense.

Theorem 3.8. *Let $u : B_1(0) \to \mathbb{R}$ be a nonconstant solution to equation* (1.3) *with* (1.5) *and let $x \in B_{1/2}(0)$. Then there exists a positive $r_0 = r_0(\lambda)$ and a constant $C = C(n, \lambda)$ such that*

$$e^{Cr}\bar{F}(r) \equiv e^{Cr}\bar{F}(u, x, g(x), r) \tag{3.25}$$

is monotone nondecreasing on $(0, r_0)$.

Proof. By a standard $C^{1,\alpha}$ density argument, we can assume that a^{ij} and b^i are smooth. Indeed, there exists a sequence of smooth solutions to elliptic pdes with smooth coefficients that converge in the $C^{1,\alpha}$ sense to u. Moreover, for simplicity we assume $x = 0$ and $u(0) = 0$. We will prove that, for $r \in (0, r_0)$:

$$\frac{\bar{F}'(r)}{\bar{F}(r)} \geq -C(n, \lambda). \tag{3.26}$$

Define $U_t u = U_{0,t} u$ as in (3.18). Using Lemma 3.5, the last statement is equivalent to

$$\frac{\bar{F}_t'(1)}{\bar{F}_t(1)} \equiv \frac{\bar{F}'(U_t u, g^t, 0, 1)}{\bar{F}(U_t u, g^t, 0, 1)} \geq -C(n, \lambda)t. \tag{3.27}$$

For the moment, fix t and set $U = U_t u$. We begin by computing the derivative of H. We have,

$$H(r) = H(U, g^t, 0, r) = r^{n-1} \int_{\partial B_1} U^2(r, \theta) \sqrt{b(tr, \theta)} d\theta, \tag{3.28}$$

$$H'|_{r=1} = (n - 1)H(1) + 2 \int_{\partial B_1} U \langle \nabla U \,|\, \nabla r \rangle \sqrt{b(t, \theta)} d\theta + \int_{\partial B_1} \left(\frac{t}{2} \frac{\partial \log(b)}{\partial r}\right)\bigg|_{(tr,\theta)} U^2(1, \theta) \sqrt{b(t, \theta)} d\theta.$$

By using equation (3.6), we obtain the estimate

$$\left| H'(1) - (n-1)H(1) - 2 \int_{\partial B(g^t,0,1)} U U_n dS(g^t) \right| \le C(n,\lambda) t \, H(1),$$ (3.29)

where $U_n = \langle \nabla U | \partial_r \rangle$ is the normal derivative of U on $\partial B(g^t, 0, r)$. As for the derivative of I, we split it into two parts:

$$\begin{aligned}
I' &= \frac{d}{dr} I(U, g^t, r) = \int_{\partial B(g^t,0,r)} \left(\|\nabla U\|_{g^t}^2 + U \Delta_{g^t}(U) \right) dS(g^t) \\
&= \int_{\partial B(g^t,0,r)} \|\nabla U\|_{g^t}^2 \, dS(g^t) + \int_{\partial B(g^t,0,r)} U \Delta_{g^t}(U) dS(g^t) \\
&\equiv I'_\alpha + I'_\beta .
\end{aligned}$$ (3.30)

Using geodesic polar coordinates relative to g^t, set $\vec{v} = r \nabla r$. By the divergence theorem we get

$$\begin{aligned}
I'_\alpha &= \frac{1}{r} \int_{\partial B(g^t,0,r)} \|\nabla U\|_{g^t}^2 \left\langle \vec{v} \, \middle| \, r^{-1}\vec{v} \right\rangle dS(g^t) = \frac{1}{r} \int_{B(g^t,0,r)} \operatorname{div} \left(\|\nabla U\|_{g^t}^2 \, \vec{v} \right) dV(g^t) \\
&= \frac{1}{r} \int_{B(g^t,0,r)} \|\nabla U\|_{g^t}^2 \operatorname{div}(\vec{v}) \, dV(g^t) + \frac{2}{r} \int_{B(g^t,0,r)} \nabla^i \nabla^j U \, \nabla_i U \, \vec{v}_j \, dV(g^t) \\
&= \frac{1}{r} \int_{B(g^t,0,r)} \|\nabla U\|_{g^t}^2 \operatorname{div}(\vec{v}) \, dV(g^t) + \frac{2}{r} \int_{B(g^t,0,r)} \left\langle \nabla \left\langle \nabla U \, \middle| \, \vec{v} \right\rangle \middle| \nabla U \right\rangle dV(g^t) - \frac{2}{r} \int_{B(g^t,0,r)} \nabla^j U \nabla_i U \left(\nabla^i \vec{v} \right)_j dV(g^t) \\
&= \frac{1}{r} \int_{B(g^t,0,r)} \|\nabla U\|_{g^t}^2 \operatorname{div}(\vec{v}) \, dV(g^t) + 2 \int_{\partial B(g^t,0,r)} (U_n)^2 \, dS(g^t) \\
&\quad - \frac{2}{r} \int_{B(g^t,0,r)} t \left\langle \nabla U \, \middle| \, \vec{v} \right\rangle \langle B | \nabla U \rangle dV(g^t) - \frac{2}{r} \int_{B(g^t,0,r)} \nabla^j U \nabla_i U \left(\nabla^i \vec{v} \right)_j dV(g^t) .
\end{aligned}$$ (3.31)

Using geodesic polar coordinates, it is easy to see that

$$\left\| \left(\nabla^i \vec{v} \right)_j - \delta^i_j \right\|_{(r,\theta)} \le rt C(\lambda).$$ (3.32)

Therefore, we have the estimate

$$\left| I'_\alpha(1) - (n-2)D(1) - 2 \int_{\partial B(g^t,0,1)} (U_n)^2 \, dS(g^t) \right| \le t C(n,\lambda) D(1).$$ (3.33)

Using Proposition 3.4 we conclude that for $t \le r_0 = r_0(\lambda)$,

$$\left| I'_\alpha(1) - (n-2)I(1) - 2 \int_{\partial B(g^t,0,1)} (U_n)^2 \, dS(g^t) \right| \le t C(n,\lambda) I(1).$$ (3.34)

To estimate I'_β, we use the divergence theorem to write

$$I(r) = \int_{\partial B(g^t,0,r)} U U_n dS(g^t).$$ (3.35)

Note that for $tr \le r_0$, $I(r) > 0$. From Cauchy's inequality and Proposition 3.4, we get

$$I^2(r) \le H(r) \int_{\partial B(g^t,0,r)} U_n^2 dS(g^t) \le \frac{rI(r)}{c(n,\lambda)} \int_{\partial B(g^t,0,r)} U_n^2 dS(g^t),$$

$$I(r) \le \frac{r}{c(n,\lambda)} \int_{\partial B(g^t,0,r)} U_n^2 dS(g^t),$$ (3.36)

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and so, using equation (3.34), we get

$$\int_{\partial B(g^t,0,1)} \|\nabla U\|_{g^t}^2 \, dS(g^t) = I'_\alpha(1) \le C(n,\lambda) \int_{\partial B(g^t,0,1)} U_n^2 dS(g^t). \tag{3.37}$$

Following [HL, pag 56], we divide the rest of the proof in two cases.

Case 1. Suppose

$$\int_{\partial B(g^t,0,1)} U^2 dS(g^t) \int_{\partial B(g^t,0,1)} U_n^2 dS(g^t) \le 2 \left(\int_{\partial B(g^t,0,1)} U U_n dS(g^t) \right)^2 = 2I^2(1). \tag{3.38}$$

In this case, using Cauchy's inequality and (3.37), we have the estimate

$$\left| I'_\beta(1) \right| = \left| \int_{\partial B(g^t,0,1)} tU \langle B|\nabla U \rangle dS(g^t) \right| \le tC(n,\lambda)I(1). \tag{3.39}$$

So, from equations (3.29), (3.30), (3.34) and (3.39), we get for $t \le r_0$,

$$\frac{\bar{F}'_t(1)}{\bar{F}_t(1)} = 1 + \frac{I'(1)}{I(1)} - \frac{H'(1)}{H(1)} \ge 0 + 2 \left(\frac{\int_{\partial B(g^t,0,1)} U_n^2 dS(g^t)}{\int_{\partial B(g^t,0,1)} U U_n dS(g^t)} - \frac{\int_{\partial B(g^t,0,1)} U U_n dS(g^t)}{\int_{\partial B(g^t,0,1)} U^2 dS(g^t)} \right) - tC(n,\lambda) \ge -tC(n,\lambda),$$

where the last inequality comes from a simple application of Cauchy's inequality.

Case 2. To complete the proof, suppose

$$\int_{\partial B(g^t,0,1)} U^2 dS(g^t) \int_{\partial B(g^t,0,1)} U_n^2 dS(g^t) > 2 \left(\int_{\partial B(g^t,0,1)} U U_n dS(g^t) \right)^2 = 2I^2(1). \tag{3.40}$$

Then we have the following estimate for estimate I'_β.

$$\left| I'_\beta(1) \right| = \left| \int_{\partial B(g^t,0,1)} tU \langle B|\nabla U \rangle dS(g^t) \right| \le t \left(\int_{\partial B(g^t,0,1)} U^2 dS(g^t) \int_{\partial B(g^t,0,1)} \|\nabla U\|_{g^t}^2 \, dS(g^t) \right)^{1/2} \le \tag{3.41}$$

$$\le C(n,\lambda) t \left(\int_{\partial B(g^t,0,1)} U^2 dS(g^t) \int_{\partial B(g^t,0,1)} U_n^2 dS(g^t) \right)^{1/2}.$$

Applying Young's inequality with the right constant and Proposition 3.4, we obtain that for $t \le r_0$,

$$\left| I'_\beta(1) \right| \le \int_{\partial B(g^t,0,1)} U_n^2 dS(g^t) + C(n,\lambda)t^2 \int_{\partial B(g^t,0,1)} U^2 dS(g^t) \le \int_{\partial B(g^t,0,1)} U_n^2 dS(g^t) + C(n,\lambda)t^2 I(1). \tag{3.42}$$

Using equations (3.29), (3.30), (3.34) and (3.42), we get for $t \le r_0$,

$$\frac{\bar{F}'_t(1)}{\bar{F}_t(1)} = 1 + \frac{I'(1)}{I(1)} - \frac{H'(1)}{H(1)} \ge 0 + \frac{\int_{\partial B(g^t,0,1)} U_n^2 dS(g^t)}{\int_{\partial B(g^t,0,1)} U U_n dS(g^t)} - \frac{2 \int_{\partial B(g^t,0,1)} U U_n dS(g^t)}{\int_{\partial B(g^t,0,1)} U^2 dS(g^t)} - tC(n,\lambda) \ge -tC(n,\lambda),$$

$$\tag{3.43}$$

where the last inequality follows directly from the assumption (3.40). □

For the proof of Theorem 2.15, Lemma 2.7 is crucial. It states that a bound on $\bar{N}^u(0, 1)$ gives a bound also on $\bar{N}^u(x, r)$, for well-chosen x and r. A similar statement holds for solutions to (1.3). However this statement is valid only for $r \le r_0(n, \lambda, \Lambda)$.

Lemma 3.9. *There exists $r_0 = r_0(n, \lambda, \Lambda)$ and $C = C(n, \lambda, \Lambda)$ such that if u is a solution to (1.3) with (1.5) on $B_{\lambda^{-1/2}r}(0)$, $0 < r \le r_0$ and $\bar{F}(0, r) \le \Lambda$, then for all $x \in B_{r/3}(0)$,*

$$\bar{F}(x, r/3) \le C. \tag{3.44}$$

Remark 3.10. Even though it might be possible to prove this lemma using doubling conditions for $H(r)$ and mean value theorems, it is much more convenient to set up a contradiction/compactness argument. Such an argument does not give explicit quantitative control on the constants C and r_0. Rather, it only proves their existence. For our purposes, this is sufficient.

Proof. Assume, by contradiction, that there exists a sequence of solutions u_i to $\mathcal{L}_i(u_i) = 0$, where the operators $\mathcal{L}_i$ satisfy conditions (1.5). Assume also that $\bar{F}(u_i, 0, , g^i(0), i^{-1}) \le \Lambda$, but for some $x_i \in B_{i^{-1}/3}(0)$, $\bar{F}(u_i, x_i, g^i(x_i), i^{-1}/3) \ge i$. For each operator $\mathcal{L}_i$, consider the associated metric g at the origin and define $g^i(r, \theta) = g(i^{-1}r, \theta)$. An easy consequence of the conditions (1.5) is that $g^i(r, \theta)$ converges in the Lipschitz sense on $B_1(0)$ to the Euclidean metric.

For simplicity, set $U_i(r, \theta) = U_{0,i^{-1}}u_i(r, \theta)$, where the latter is defined in equation (3.18).

The bound on the frequency $\bar{F}$ together with Lemma 3.4 implies that, for i large enough,

$$\int_{B_1} |\nabla U_i|^2 \, dV \le \lambda^{\frac{n-2}{2}} \int_{B_1(0)} \|\nabla U_i\|^2_{g^i} \, dV(g^i) \le C(n, \lambda)\bar{F}(0, i^{-1}) \le C(n, \lambda)\Lambda. \tag{3.45}$$

Since $U_i(0) = 0$, U_i have uniform bound in the $W^{1,2}(B_1(0))$ norm and, by elliptic estimates, also in the $C^{1,\alpha}(B_{2/3})$ norm.

Consider a subsequence U_i which converges in the weak $W^{1,2}$ sense to some U, and a subsequence of x_i converging to some $x \in \bar{B}_{1/3}$. It is easy to see that U is a nonconstant harmonic function, and, by the convergence properties of the sequence U_i, we also have

$$\lim_{i \to \infty} \bar{F}(U_i, 0, g^i(0), 1) = \bar{F}(U, 0, e, 1) = \bar{N}^U(0, 1), \tag{3.46}$$

$$\lim_{i \to \infty} \bar{F}(U_i, x_i, g^i(x_i), 1/3) = \bar{F}(U, x, e, 1/3) = \bar{N}^U(x, 1/3). \tag{3.47}$$

Recall that e is the standard Euclidean metric on $\mathbb{R}^n$. The contradiction is a consequence of Lemma 2.7. □

With a standard compactness argument, we can turn the previous lemma into the following statement.

Lemma 3.11. *Let $u : B_1(0) \to \mathbb{R}$ be a nonconstant solution to (1.3) with (1.5). Then there exist constants $r_1(n, \lambda, \Lambda)$ and $C(n, \lambda, \Lambda)$ such that if $\bar{N}^u(0, 1) \le \Lambda$, then for all $x \in B_{1/2}(0)$ and $r \le r_1$*

$$\bar{F}(u, x, r) \le C(n, \lambda, \Lambda). \tag{3.48}$$

3.2 The Frequency Decomposition and Cone-Splitting

Similar properties to the one proved for harmonic function in Section 2.2 are available also for solutions to (1.3), although it is necessary to restrict the result to scale smaller than some $r_0(n, \lambda, \Lambda)$. In some sense, the smaller scales the closer the solutions to (1.3) are to harmonic functions, so if we choose the scale small enough we can replace "harmonic" with "elliptic" without changing the final result.

The proofs of the following theorems are obtained using arguments similar to the proof of Proposition 3.9 and contradiction/compactness arguments like the ones in Section 2.2. For this reason, we omit them.

Theorem 3.12. *Fix $\eta > 0$ and $0 \leq \gamma < 1$, and let $u : B_1(0) \to \mathbb{R}$ be a nonconstant solution to (1.3) with $\bar{N}^u(0, 1) \leq \Lambda$. Then there exist positive $\epsilon = \epsilon(n, \lambda, \eta, \gamma, \Lambda)$ and $r_2 = r_2(n, \lambda, \eta, \gamma, \Lambda)$ such that if $r \leq r_2$ and*

$$\bar{F}(0, r) - \bar{F}(0, \gamma r) < \epsilon, \tag{3.49}$$

then u is $(0, \eta, r, 0)$-symmetric.

In a similar way, we can also prove a generalization of Corollary 2.13:

Corollary 3.13. *Fix $\eta > 0$, $\tau > 0$, $0 < \chi \leq 1$ and $k \leq n - 2$. There exist $\epsilon(\lambda, \eta, \tau, \chi, \Lambda)$ and $r_3 = r_3(\lambda, \eta, \tau, \chi, \Lambda)$ with the following property. Assume u solves (1.3) with $\bar{N}^u(0, 1) \leq \Lambda$ and for some $x \in B_{1/2}(0)$ we have*

1. *u is $(0, \epsilon, \chi r_3, x)$-symmetric,*

2. *for every affine subspace V passing through x of dimension $\leq k$, there exists $y \in B_{\chi r_3}(x) \setminus B_\tau V$ such that u is $(0, \epsilon, \chi r_3, y)$-symmetric.*

Then u is $(k + 1, \epsilon, r_3, x)$-symmetric.

By (1.5), we have uniform $C^{1,\alpha}$ estimates on the solutions to (1.3) (see [GT01] for details). For this reason, it is straightforward to prove the following proposition, which is a generalization of Proposition (2.14).

Proposition 3.14. *Let $u : B_1(0) \to \mathbb{R}$ be a solution to (1.3) with (1.5) such that $\bar{F}^u(0, 1) \leq \Lambda$. For every $\epsilon > 0$ and $0 \leq \alpha < 1$, there exists positive $\bar{\eta}$ and r_0 depending on $(n, \epsilon, \alpha, \lambda, \Lambda)$ such that if for some $x \in B_{1/2}(0)$ and $r \leq r_0$ u is $(n - 1, \bar{\eta}, r, x)$-symmetric, then*

$$\left\| U_{x,r} u - L \right\|_{C^{1,\alpha}(B_{1/2})} \leq \epsilon, \tag{3.50}$$

where L denotes a linear function satisfying $\fint_{\partial B_1} |L|^2 \, dS = 1$. In particular, by choosing $\alpha = 0$ and ϵ sufficiently small, there exist positive η and r_0 depending on n, λ, Λ, such that if u $(n - 1, \eta, r, x)$-symmetric, then u does not have critical points in $B_{r/2}(x)$.

3.3 Minkowski Type Estimates and the Proof of Theorem 1.10

Now we are ready to prove Theorem 1.10. As in the harmonic case, we prove the theorem only for some $r = \gamma^j$ for a suitable value of $0 < \gamma < 1$ and every j, the general case follows easily from this. For the reader's convenience, here we restate the theorem in this context.

Theorem 3.15. *Let $u : B_1(0) \to \mathbb{R}$ be a solution to (1.3) with (1.5) and such $\bar{N}^u(0, 1) \leq \Lambda$. Then for some $0 < \gamma(n, \eta, \lambda, \Lambda) < 1$, for every $j \in \mathbb{N}$, $\eta > 0$ and $k \leq n - 2$ we have*

$$\mathrm{Vol}\left(B_{\gamma^j}(S^k_{\eta, \gamma^j}) \cap B_{1/2}(0)\right) \leq C(n, \lambda, \Lambda, \eta)\left(\gamma^j\right)^{n-k-\eta}. \tag{3.51}$$

Proof. Since the proof is almost identical to that of Theorem 2.15, we simply mention how to adapt the proof from the harmonic case.

Fix $\eta > 0$ and let $\gamma = c_0^{-2/\eta} < 1$, $\chi = \gamma$. Let $\tau > 0$. Take r_0 to be the minimum of r_1 given by Lemma 3.11, r_2 given by Corollary 3.12 and let r_3 be given by Corollary 3.13. Then, if i is large enough so that $\gamma^i \leq r_0$, then the same proof as in the harmonic case applies also to this more general case with Lemma 2.7 replaced by Lemma 3.11, Theorem 2.8 by 3.12 and Corollary 2.13 by Corollary 3.13.

Note that $\gamma^i > r_0$ for only a finite number of exponents i, and that the number of such exponents is bounded by a uniform constant $D' = D'(n, \lambda, \eta, \Lambda)$. Finally, even though in the elliptic case $\bar{F}$ not monotone, but rather, only almost monotone, it is straightforward to see that an estimate of the form given in equation (2.28) still holds. □

Remark 3.16. The main application for this theorem is the volume estimate on the tubular neighborhoods of the critical set (Theorem 1.17). As in the harmonic case, this theorem is a simple corollary of Theorem 1.10 and Proposition 3.14.

3.4 Estimates on $(n-2)$-dimensional Hausdorff Measure, for Solutions of Elliptic Equations

As for the Minkowski type estimates, it is also possible to generalize the effective estimates for the critical set involving $(n - 2)$-dimensional Hausdorff measure, to solutions to elliptic equations of the form (1.3). However for this estimate, we require higher order regularity assumptions on the coefficients a^{ij} and b^i.

The following lemma is the generalization of Corollary 2.25 for solutions to (1.3).

Lemma 3.17. *Let P be an $(n-2)$-symmetric homogeneous harmonic polynomial normalized with $\fint_{\partial B_1} P^2 dS = 1$. Let $u : B_1(0) \to \mathbb{R}$ be a solution to (1.3) with conditions (1.5) and such that $\bar{N}^u(0, 1) \leq \Lambda$. There exists a positive integer $M = M(n, \lambda, \Lambda)$ such that if*

$$\left\|a^{ij}\right\|_{C^M(B_1(0))}, \left\|b^i\right\|_{C^M(B_1(0))} \leq L, \tag{3.52}$$

then there exist positive $C = C(n, L, \Lambda)$, $\bar{r} = \bar{r}(n, L, \Lambda)$, $\epsilon = \epsilon(n, L, \Lambda)$ and $\chi = \chi(n, L, \Lambda)$ such that if for some $x \in B_{1/2}(0)$ and $r \leq \bar{r}$ we have

$$\fint_{\partial B_1(0)} \left|U_{x,r} u - P\right|^2 dS < \epsilon, \tag{3.53}$$

then for all $s \leq \chi r$,

$$H^{n-2}\left(\nabla u^{-1}(0) \cap B_s(x)\right) \leq C s^{n-2}. \tag{3.54}$$

Proof. As in the harmonic case, this lemma is a corollary of Lemma 2.24. The only delicate aspect is the generalization of the elliptic estimates.

Recall that the metric $g(\bar{x})$ defined in Proposition 3.1 is only Lipschitz at the origin, no matter the regularity of a^{ij}. Thus, it is not possible to obtain bounds on the higher order derivatives of $U_{x,r}u$. However, we do have uniform higher order elliptic estimates on $T_{x,r}$. As t approaches zero, $U_{x,t}$ converges in the Lipschitz sense to $T_{x,t}$. So, for t small enough, condition (3.53) implies

$$\fint_{\partial B_1(0)} \left| T_{x,r}u - P \right|^2 dS < \epsilon. \tag{3.55}$$

By a simple application of Lemma 2.24 (the ϵ-regularity lemma) the conclusion follows just as in the harmonic case. $\qquad\square$

Remark 3.18. Following the same scheme as in the harmonic case it is now easy to prove Theorem 1.21 for solutions to (1.3).

4 The Singular Set

With simple modifications, the quantitative stratification technique can also be used to derive estimates on the singular sets of solutions to (1.4) with (1.5).

Since constant functions do not solve (1.4), we cannot use the normalized frequency function. For solutions to homogeneous elliptic equations with a zero order term, we can define the generalized frequency function $F(x, r)$ by

$$F(u, \bar{x}, g, r) = \frac{r \int_{B(g(\bar{x}), \bar{x}, r)} \|\nabla u\|_{g(\bar{x})}^2 + u\Delta_{g(\bar{x})}(u) dV_{g(\bar{x})}}{\int_{\partial B(g(\bar{x}), \bar{x}, r)} u^2 dS_{g(\bar{x})}}. \tag{4.1}$$

This function turns out to be almost monotone as a function of r on $(0, r_0(\lambda))$ if $u(\bar{x}) = 0$.

Once this is proved, it is not difficult to see that a theorem similar to 1.17 holds for solutions to this kind of elliptic equation, although in this case, the $(n - 2 + \eta)$-Minkowski type estimate holds on the *singular* set, not the *critical* set.

Theorem 4.1. *Let $u : B_1(0) \to \mathbb{R}$ be a solution to (1.3) with (1.5) and such that $\bar{N}^u(0, 1) \leq \Lambda$. For every $\eta > 0$, there exists a positive $C = C(n, \lambda, \Lambda, \eta)$ such that*

$$\mathrm{Vol}\left[B_r\left(\mathcal{C}(u) \cap u^{-1}(0) \right) \cap B_{1/2}(0) \right] \leq Cr^{2-\eta}. \tag{4.2}$$

We also point our that the effective $(n - 2)$-dimensional Hausdorff measure estimate is easily generalized to the singular set in this context, although even in this case, we need to add some regularity requirements on the coefficients of the equation. With different techniques, the $(n - 2)$-dimensional Hausdorff measure result has already been proved in [HHL98, Theorem 1.1]; see also [HL, Theorem 7.2.1].

Remark 4.2. As noted in [HN99, Remark at page 362], it is not possible to get effective bounds on the critical sets of solutions to (1.4) with (1.5). Indeed, every closed subset of $\mathbb{R}^n$ can be the critical set of such a function.

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