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Triangular maximal operators on locally finite trees

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Abstract

We introduce the centred and the uncentred triangular maximal operators $\mathcal{T}$ and $\mathcal{U}$, respectively, on any locally finite tree in which each vertex has at least three neighbours. We prove that both $\mathcal{T}$ and $\mathcal{U}$ are bounded on L^p for every p in $(1, \infty]$, that $\mathcal{T}$ is also bounded on $L^1(\mathfrak{T})$, and that $\mathcal{U}$ is not of weak type (1, 1) on homogeneous trees. Our proof of the L^p boundedness of $\mathcal{U}$ hinges on the geometric approach of Córdoba and Fefferman. We also establish L^p bounds for some related maximal operators. Our results are in sharp contrast with the fact that the centred and the uncentred Hardy–Littlewood maximal operators (on balls) may be unbounded on L^p for every $p < \infty$ even on some trees where the number of neighbours is uniformly bounded.

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1 | INTRODUCTION

The centred and the uncentred Hardy–Littlewood maximal operators on a metric measure space (X, d, μ) are defined by

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| \, \mathrm{d}\mu \quad \text{and} \quad \mathcal{N}f(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \, \mathrm{d}\mu, \tag{1.1}$$

respectively; here $B_r(x)$ denotes the ball with centre x and radius r, and B is any ball in X containing x.

It is well known that if the measure μ is *doubling*, that is, if there exists a constant D such that

$$\mu(B_{2r}(x)) \leqslant D\,\mu(B_r(x)) \tag{1.2}$$

for every x in X and for all $r > 0$, then $\mathcal{M}$ and $\mathcal{N}$ are of weak type (1, 1) and bounded on $L^p(X)$ for every p in $(1, \infty]$ (see, for instance, [13, Chapter 1]).

If, instead, μ is nondoubling, viz. the condition (1.2) fails, then a variety of situations can occur. For instance, on symmetric spaces of the noncompact type Strömberg [14] proved that $\mathcal{M}$ is bounded on L^p for all $p > 1$ and it is of weak type (1, 1), and Ionescu [4] showed that $\mathcal{N}$ is bounded on L^p if and only if $p > 2$.

These results have been complemented by Li [8] who showed that given p_0 in (1, 2), there is a nondoubling Riemannian manifold, which is a generalisation of the hyperbolic space, where $\mathcal{M}$ is bounded on L^p if and only if p belongs to the interval $(p_0, \infty]$. Furthermore there are Riemannian manifolds of the same type where $\mathcal{M}$ is bounded on L^p if and only if $p = \infty$. Similar results for $\mathcal{N}$ are contained in [9]. See also [5] and the references therein for simple examples of nondoubling metric measure spaces where $\mathcal{M}$ and $\mathcal{N}$ have similar boundedness properties on L^p spaces.

In this paper we focus on trees: $\mathfrak{T}$ will denote a tree in which every vertex x has a finite number $\nu(x) \geqslant 3$ of neighbours. We emphasise that the function ν may be unbounded on $\mathfrak{T}$, in which case we say that the locally finite tree $\mathfrak{T}$ has *unbounded geometry*. We endow $\mathfrak{T}$ with the natural graph distance d and the set of its vertices with the counting measure μ. For notational convenience, we write $|E|$ instead of $\mu(E)$ for any subset E of $\mathfrak{T}$.

The metric measure space $(\mathfrak{T}, d, \mu)$ has exponential volume growth. If ν is bounded, then μ is locally, but not globally, doubling; if ν is unbounded, then $\mathfrak{T}$ is not even locally doubling.

In this context, various authors have considered the problem of establishing L^p bounds for $\mathcal{M}$ and $\mathcal{N}$. Note that the definition of $\mathcal{M}$ is usually modified as follows:

$$\mathcal{M}f(x) := \sup_{r\in\mathbb{N}} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f|\,\mathrm{d}\mu;$$

here $B_r(x) := \{y \in \mathfrak{T} : d(x, y) \leqslant r\}$. Examples show that the range of p s where either $\mathcal{M}$ or $\mathcal{N}$ are bounded on $L^p(\mathfrak{T})$ may depend on the bounds of ν.

Here is a brief account of some relevant contributions in the literature concerning the L^p boundedness of $\mathcal{M}$ and $\mathcal{N}$. Recall that a tree where ν is constant is called *homogeneous*: we denote by $\mathfrak{T}_b$ the tree for which $\nu = b + 1$ for some $b \geqslant 2$. Naor and Tao [10, Theorem 1.5] and, independently, Cowling, Meda and Setti [3, Theorem 3.1] proved that $\mathcal{M}$ is bounded on $L^p(\mathfrak{T}_b)$, $1 < p \leqslant \infty$, and of weak type (1, 1) (see also [11]). Veca [15, Theorem 5.1] proved that $\mathcal{N}$ is bounded on $L^p(\mathfrak{T}_b)$, $2 < p \leqslant \infty$, and of restricted weak type (2, 2) (see also the recent work [7] for results concerning related maximal operators).

Generalisations of these results to trees $\mathfrak{T}$ where ν is bounded, but not constant, have been the object of the investigations in [6]. In particular, it is shown that if $3 \leqslant a + 1 \leqslant \nu \leqslant b + 1$ and $b \leqslant a^2$, then the precise form of the Kunze–Stein phenomenon on $\mathfrak{T}_b$ (see [2]) implies that $\mathcal{M}$ is bounded on $L^p(\mathfrak{T})$, $\tau < p \leqslant \infty$, where $\tau = \log_a b$, and it is of restricted weak type (τ, τ), and the result is sharp. If, instead, $b > a^2$, then there are examples of trees in this class for which $\mathcal{M}$ is unbounded on L^p for every $p < \infty$. Even more strikingly, whenever $b > a$ there are trees in this class for which $\mathcal{N}$ is unbounded on L^p for every $p < \infty$.

Extensions of some of these results to graphs are contained in [12]. We refer the interested reader to the introduction of the paper [6] for additional comments on related works in the literature.

The abovementioned results concerning the boundedness of $\mathcal{M}$ and $\mathcal{N}$ on trees raise the question whether there are natural "geometric" maximal operators on locally finite trees with possibly unbounded geometry that possess stable L^p boundedness properties, in the sense that the range of p's for which they are bounded on L^p do not depend on the specific assumptions on ν, besides the condition $\nu \geqslant 3$.

In this paper we answer in the affirmative to this question and propose to investigate the L^p boundedness of the centred and uncentred maximal operators on triangles.

Their definition requires a bit of notation, which we now introduce. We fix a geodesic ray $\omega = \{x_m : m \in \mathbb{N}\}$ in $\mathfrak{T}$, and consider the associated *height function* h_ω, which is the discrete analogue of the Busemann function in Riemannian geometry, defined by

$$h_\omega(x) = \lim_{m\to\infty} (m - d(x, x_m)).$$

Note that h_ω is integer valued. Its level sets, called *horocycles* associated to ω, are then defined, for j in $\mathbb{Z}$, by

$$\mathfrak{H}_j^\omega := \{x \in \mathfrak{T} : h_\omega(x) = j\},$$

and $\mathfrak{T} = \bigcup_{j\in\mathbb{Z}} \mathfrak{H}_j^\omega$ (disjoint union). Note that if $x \in \mathfrak{H}_j^\omega$, then $\nu(x) - 1$ neighbours of x, called *successors* of x, belong to $\mathfrak{H}_{j-1}^\omega$. We denote by $s_\omega^1(x)$ the set of *successors* of x, define $s_\omega^0(x) := \{x\}$, and

$$s_\omega^k(x) := \bigcup_{y\in s_\omega^{k-1}(x)} s_\omega^1(y), \qquad k \geqslant 2.$$

For every nonnegative integer R, we call $T_R^\omega(x) := \bigcup_{j=0}^R s_\omega^j(x)$ the triangle with vertex x and height R. The *centred and uncentred triangular maximal operators* $\mathcal{T}^\omega$ and $\mathcal{U}^\omega$ are then defined by

$$\mathcal{T}^\omega f(x) := \sup_{R\geqslant 0} \frac{1}{|T_R^\omega(x)|} \int_{T_R^\omega(x)} |f| \, d\mu \quad \text{and} \quad \mathcal{U}^\omega f(x) := \sup_{T\ni x} \frac{1}{|T|} \int_T |f| \, d\mu,$$

respectively, where T is any triangle in $\mathfrak{T}$. The triangular centred operator may be thought of as "directional" or "one sided" with respect to the height function h_ω. Note that $T_R^\omega(x)$ is the set of the points in $B_R(x)$ that can be reached by geodesics of length $\leqslant R$ starting at x that point "downwards." Our main result states that if $\mathfrak{T}$ is a locally finite tree with $\nu \geqslant 3$, then $\mathcal{T}^\omega$ and $\mathcal{U}^\omega$ are bounded on $L^p(\mathfrak{T})$ for every p in $[1, \infty]$ and for every p in $(1, \infty]$, respectively. Furthermore, $\mathcal{U}^\omega$ is not of weak type $(1, 1)$ on the homogeneous tree $\mathfrak{T}_b$, $b \geqslant 2$.

The operators $\mathcal{T}^\omega$ and $\mathcal{U}^\omega$ depend on ω. However, in all our results either the conclusion is the same for all possible choices of ω or we consider a specific example of tree where ω is clearly specified (see $\mathfrak{S}_{a,b}$ in Section 5). Thus, for simplicity, in the sequel we shall omit the superscripts and write $\mathcal{T}$ and $\mathcal{U}$ instead of $\mathcal{T}^\omega$ and $\mathcal{U}^\omega$.

The proof of the L^p boundedness of $\mathcal{T}$ is not hard, and can be found in Section 3, where we also study the related centred and noncentred maximal functions $\mathcal{B}$ and $\mathcal{B}^u$.

Our approach to the problem of determining the range of p s where $\mathcal{U}$ is bounded is much in the spirit of the work of Córdoba and Fefferman [1]. In Section 4 we show that for every r in $[1, \infty)$ there exists a constant A_r such that for any finite collection $\mathcal{G}$ of triangles in $\mathfrak{T}$ that are maximal with respect to inclusion the following holds:

$$\Big\| \sum_{T \in \mathcal{G}} \mathbf{1}_T \Big\|_r \leqslant A_r \, \|\mathbf{1}_G\|_r, \tag{1.3}$$

where G denotes the union of all T in $\mathcal{G}$. Loosely speaking, this estimate says that the triangles in $\mathcal{G}$ have "finite overlapping in the L^r norm." We mention that Ionescu [4] has used a similar strategy to obtain bounds for the uncentred HL maximal operator on symmetric spaces of the noncompact type and rank $\geqslant 2$.

In Section 5 we show that (1.3) fails for every r in $(1, \infty)$ if we replace the family $\mathcal{G}$ above with a family $\mathcal{G}'$ of *modified* maximal triangles T', where T' is the union of a triangle T of height h and the h^{th} ancestor of the vertex of T. This implies that the uncentred HL maximal operator associated to the family of all modified triangles is unbounded on $L^p(\mathfrak{T})$ for every $p < \infty$. The reason for which (1.3) fails lies in the fact that a point x can be the hth ancestor of the vertices of a lot of mutually disjoint triangles of height h, which makes the left-hand side, but not the right-hand side, of (1.3) big. See the observation after Remark 5.4 for the details.

2 | PRELIMINARIES

Let $\mathfrak{T}$ be a locally finite tree, that is, a connected graph with no loops, in which every vertex x has a finite number $\nu(x) \geqslant 3$ of neighbours; we call $\nu(x)$ the *valence* of x.

Between any two points x and y in $\mathfrak{T}$, such that $d(x, y) = n$, there is a unique *geodesic path* of the form $x_0, x_1, \ldots, x_n$, where $x_0 = x$, $x_n = y$, and $d(x_i, x_j) = |i - j|$ whenever $0 \leqslant i, j \leqslant n$. A *geodesic ray* γ in $\mathfrak{T}$ is a one-sided sequence $\{\gamma_n : n \in \mathbb{N}\}$ of points of $\mathfrak{T}$ such that $d(\gamma_i, \gamma_j) = |i - j|$ for all nonnegative integers i and j. We say that x *lies on* γ, and write $x \in \gamma$, if $x = \gamma_n$ for some n in $\mathbb{N}$. Given a point y, denote by $y \wedge \omega$ the point on ω closest to y (ω is as in the Introduction). Suppose that $y \wedge \omega = x_k$, and denote by γ_y the geodesic ray $[y, x_k] \cup [x_k, x_{k+1}, \ldots]$. Given another point x in $\mathfrak{T}$, we say that x lies above y, and write $x \succeq y$, if $x \in \gamma_y$. If $x \succeq y$ and $x \neq y$, then we write $x \succ y$.

Given a tree $\mathfrak{T}$, we implicitly assume that we have chosen a geodesic ray ω in $\mathfrak{T}$. Many objects on $\mathfrak{T}$ depend on ω. However, in order to simplify the notation, we do not stress this dependence, and write h, $T_R(x)$, $s^k(x)$, $\mathcal{T}$ and $\mathcal{U}$ in place of h_ω, $T_R^\omega(x)$, $s_\omega^k(x)$, $\mathcal{T}^\omega$ and $\mathcal{U}^\omega$.

We agree that the triangle with vertex x and height 0 is just the point x. If T is any triangle, then we denote by $v(T)$, $h(T)$ and $\beta(T)$ its vertex, its height and its base, respectively. Note that $\beta(T) = s^{h(T)}(v(T))$.

Let x be a vertex in $\mathfrak{T}$. We denote by $p(x)$ the *predecessor* of x, viz. the unique neighbour of x with height $h(x) + 1$. Note that $p(x)$ depends on the choice of ω: in order to simplify the notation, we do not stress this dependence. Note that $p\,(p(x))$, also denoted $p^2(x)$, is just a vertex in $\mathfrak{H}_{h(x)+2}$. The kth *ancestor* of x is the point $p^k(x) := p\left(p^{k-1}(x)\right)$. For any subset E of $\mathfrak{T}$ and every positive integer k, $p^k(E)$ will be short for $\bigcup_{y \in E} p^k(y)$.

The next lemma contains an elementary inequality relating the area of any triangle in $\mathfrak{T}$ and the length of its base. Such inequality can also be deduced from Cheeger's isoperimetric inequality on trees, for which we refer the reader to [11, Lemma 13; 16, Theorem 4.2.2].

Lemma 2.1. *Suppose that $\mathfrak{T}$ is a locally finite tree with $\nu \geqslant 3$, and let T be a triangle in $\mathfrak{T}$ with height h. The following hold:*

(i) $2^k \left|p^k\left(\beta(T)\right)\right| \leqslant |\beta(T)|$ *for every k in $\{0, \dots, h\}$;*
(ii) $|T| \leqslant 2\left|\beta(T)\right|$.

Proof. Since every point in $p^k\left(\beta(T)\right)$ has at least two successors,

$$\left|p^{k-1}(\beta(T))\right| \geqslant 2\left|p^k(\beta(T))\right|.$$

Then (i) follows by iterating this estimate.

Next,

$$|T| = \sum_{k=0}^{h(T)} \left|p^k(\beta(T))\right| \leqslant \sum_{k=0}^{h(T)} 2^{-k}\left|\beta(T)\right| \leqslant 2\left|\beta(T)\right|,$$

and (ii) follows. □

3 | THE CENTRED TRIANGULAR MAXIMAL OPERATOR

In this section we study the centred triangular maximal operator $\mathcal{T}$ defined in the Introduction, and some related maximal operators.

Theorem 3.1 (Centred triangular maximal function). *Suppose that $\mathfrak{T}$ is a tree such that $\nu \geqslant 3$. Then $\mathcal{T}$ is bounded on $L^p(\mathfrak{T})$ for every p in $[1, \infty]$.*

Proof. Define the function $\tau : \mathfrak{T} \times \mathfrak{T} \to [0, \infty)$ by

$$\tau(x, y) := \frac{1}{\left|T_{d(x,y)}(x)\right|}\,\mathbf{1}_E(x, y),$$

where $E := \left\{(x, y) \in \mathfrak{T} \times \mathfrak{T} : x \succeq y\right\}$. Observe that

$$\mathcal{T} f(x) \leqslant \int_{\mathfrak{T}} \sup_{R \in \mathbb{N}} \frac{\mathbf{1}_{T_R(x)}}{\left|T_R(x)\right|}\,|f|\,\mathrm{d}\mu \leqslant \int_{\mathfrak{T}} \tau(x, \cdot)\,|f|\,\mathrm{d}\mu.$$

Therefore,

$$\left\|\mathcal{T} f\right\|_1 \leqslant \int_{\mathfrak{T}} \mathrm{d}\mu(x) \int_{\mathfrak{T}} \tau(x, y)\left|f(y)\right| \mathrm{d}\mu(y) \leqslant A\left\|f\right\|_1,$$

where $A := \sup_{y \in \mathfrak{T}} \int_{\mathfrak{T}} \tau(x, y)\,\mathrm{d}\mu(x)$. Now, given y in $\mathfrak{T}$, the points x for which $\tau(x, y) \neq 0$ are just the points on the geodesic $[y, \omega)$, that is, the points $y, p(y), p^2(y), \dots$ Thus

$$A = \sup_{y \in \mathfrak{T}} \sum_{k=0}^{\infty} \frac{1}{\left|T_k(p^k(y))\right|} \leqslant \sum_{k=0}^{\infty} 2^{-k} = 2.$$

This proves that $|||\mathcal{T}|||_{1;1} \leqslant 2$. Since $\mathcal{T}$ is obviously bounded on $L^\infty(\mathfrak{T})$, the Marcinkiewicz interpolation theorem implies that $\mathcal{T}$ is bounded on $L^p(\mathfrak{T})$ for every p in $[1,\infty]$. □

An examination of the proof above shows that the assumption $\nu \geqslant 3$ can be substantially relaxed. In fact, it suffices to assume that $\nu \geqslant 2$, and that $\sup_{y\in\mathfrak{T}} \sum_{k=0}^{\infty} \frac{1}{|T_k(p^k(y))|}$ is finite.

For each function f on $\mathfrak{T}$, define the *centred* and the *uncentred* maximal functions $\mathcal{B}f$ and $\mathcal{B}^u f$ by

$$\mathcal{B}f(x) := \sup_{r\in\mathbb{N}} \frac{1}{|s^r(x)|} \int_{s^r(x)} |f| \,\mathrm{d}\mu \quad \text{and} \quad \mathcal{B}^u f(x) := \sup_{T\ni x} \frac{1}{|\beta(T)|} \int_{\beta(T)} |f| \,\mathrm{d}\mu.$$

Clearly $\mathcal{B}f \leqslant \mathcal{B}^u f$. By Lemma 2.1(ii), applied to $T_r(x)$, $r \geqslant 0$,

$$\mathcal{B}f(x) \leqslant \sup_{T:v(T)=x} \frac{2}{|T|} \int_T |f| \,\mathrm{d}\mu \leqslant 2\,\mathcal{T}f(x). \tag{3.1}$$

The boundedness properties of $\mathcal{B}$ and $\mathcal{B}^u$ are grouped together in the next result.

Theorem 3.2. *The following hold:*

(i) *if $\mathfrak{T}$ is a tree with $\nu \geqslant 3$, then $\mathcal{B}$ is bounded on $L^p(\mathfrak{T})$ for every p in $[1,\infty]$, and $\mathcal{B}^u$ is bounded on $L^p(\mathfrak{T})$ for every p in $(1,\infty]$, and satisfies the weak type estimate*

$$\big|\{x \in \mathfrak{T} : \mathcal{B}^u f(x) > \alpha\}\big| \leqslant \frac{2}{\alpha}\|f\|_1 \qquad \forall \alpha > 0;$$

(ii) *for every $b \geqslant 2$, the operator $\mathcal{B}^u$ is unbounded on $L^1(\mathfrak{T}_b)$.*

Proof. Suppose that $\alpha > 0$, and consider, for every f in $L^1(\mathfrak{T})$, the level set

$$E_{\mathcal{B}^u f}(\alpha) := \{x \in \mathfrak{T} : \mathcal{B}^u f(x) > \alpha\}.$$

For notational simplicity, for the duration of this proof we write $E(\alpha)$ in place of $E_{\mathcal{B}^u f}(\alpha)$.

First we prove (i). The statement concerning $\mathcal{B}$ follows from Theorem 3.1 and the pointwise bound (3.1).

Next we consider $\mathcal{B}^u$. If $z \in E(\alpha)$, then there exists a triangle T_z, containing z, such that

$$\frac{1}{|\beta(T_z)|} \int_{\beta(T_z)} |f| \,\mathrm{d}\mu > \alpha. \tag{3.2}$$

Now, if w and z belong to $E(\alpha)$ and $\beta(T_w) \cap \beta(T_z) \neq \emptyset$, then either $\beta(T_w) \subseteq \beta(T_z)$ or $\beta(T_w) \supseteq \beta(T_z)$. Indeed, $\beta(T_w)$ and $\beta(T_z)$ are both subsets of the same horocycle, and if y belongs to their intersection, then both $v(T_w)$ and $v(T_z)$ (the vertices of T_w and T_z, respectively) must belong to the infinite geodesic $[y,\omega)$. Thus, either $v(T_z) \geq v(T_w)$ or $v(T_w) \geq v(T_z)$.

In the first case $T_z \supseteq T_w$, hence $\beta(T_z) \supseteq \beta(T_w)$, and in the second $T_z \subseteq T_w$, hence $\beta(T_z) \subseteq \beta(T_w)$.

Clearly $E(\alpha)$ is a union of triangles because if $E(\alpha)$ contains x, then it contains T_x, where T_x is such that $\frac{1}{|\beta(T_x)|}\int_{\beta(T_x)}|f|\,\mathrm{d}\mu > \alpha$. Their size is uniformly bounded, for if T is one such triangle, then Lemma 2.1(ii) and (3.2) imply that

$$|T| \leqslant 2\,|\beta(T)| < \frac{2}{\alpha}\int_{\beta(T)}|f|\,\mathrm{d}\mu \leqslant 2\,\frac{\|f\|_1}{\alpha}. \tag{3.3}$$

Thus, $E(\alpha)$ is the union of a finite number of triangles $T_1, \dots, T_N$, where, of course, N depends on α. In view of the observation above, we may assume that $\beta(T_1), \dots, \beta(T_N)$ are mutually disjoint. Then

$$\big|E(\alpha)\big| = \sum_{j=0}^{N}\big|E(\alpha)\cap T_j\big| \leqslant \sum_{j=0}^{N}\big|T_j\big|.$$

These estimates, (3.3) and the disjointness of $\beta(T_1), \dots, \beta(T_N)$, imply that

$$\big|E(\alpha)\big| < \frac{2}{\alpha}\sum_{j=0}^{N}\int_{\beta(T_j)}|f|\,\mathrm{d}\mu \leqslant \frac{2}{\alpha}\,\|f\|_1 \qquad \forall\alpha > 0,$$

as required to prove that $\mathcal{B}^u$ is of weak type (1, 1).

Clearly $\mathcal{B}^u$ is bounded on $L^\infty(\mathfrak{T})$. Then the Marcinkiewicz interpolation theorem implies that $\mathcal{B}^u$ is bounded on $L^p(\mathfrak{T})$ for all p in $(1, \infty)$, as required.

Next we prove (ii). Consider a point o in $\mathfrak{H}_0$, and the function δ_o, which is equal to 1 at o and vanishes elsewhere. For x in $\mathfrak{T}$, denote by $|x|$ the distance between o and x. If $x \in \mathfrak{H}_0$, then the smallest triangle that contains both x and o is $T_{|h(o\wedge x)|}(o\wedge x)$, where $o\wedge x$ denotes the *confluent* of o and x, viz. the point of least height that is a predecessor of both o and x. Note that $2\,h(o\wedge x) = |x|$. Thus,

$$\mathcal{B}^u\delta_o(x) = \frac{1}{\beta\big(T_{|h(o\wedge x)|}(o\wedge x)\big)} = b^{-|x|/2}.$$

Observe that the number of points in $\mathfrak{H}_0$ at distance k from o is equal to 1 if $k = 0$, and to $(b-1)\,b^{k/2-1}$ if k is even. Therefore,

$$\int_{\mathfrak{H}_0}\mathcal{B}^u\delta_o\,\mathrm{d}\mu = \int_{\mathfrak{H}_0} b^{-|x|/2}\,\mathrm{d}\mu(x) = 1 + \frac{b-1}{b}\sum_{k\geqslant 2,\,k\,\text{even}} b^{-k/2}\,b^{k/2} = \infty.$$

This proves (ii), and concludes the proof of the theorem. □

4 | THE UNCENTRED TRIANGULAR MAXIMAL OPERATOR

Suppose that $\mathcal{G}$ is a family of triangles in $\mathfrak{T}$. A triangle T in $\mathcal{G}$ is *maximal* in $\mathcal{G}$ if $T' \in \mathcal{G}$ and $T \neq T'$ imply that $T' \cap T \neq T$. In other words, T is maximal in $\mathcal{G}$ with respect to the partial ordering induced by $\subseteq$.

Our proof of the L^p boundedness of $\mathcal{U}$ for $1 < p < \infty$ is based on the following "geometric" lemma.

Lemma 4.1. *Suppose that $\mathcal{G}$ is a finite collection of maximal triangles in a locally finite tree $\mathfrak{T}$, with $\nu \geqslant 3$, and set $G := \bigcup_{T\in\mathcal{G}} T$. Then for every r in $[1,\infty)$*

$$\Big\| \sum_{T\in\mathcal{G}} \mathbf{1}_T \Big\|_r \leqslant A_r \, \|\mathbf{1}_G\|_r, \tag{4.1}$$

where $A_r^r := 4 \sum_{k=1}^\infty k^r\, 2^{-k}$.

Proof. Define the *overlapping number* Ω of the family $\mathcal{G}$ by

$$\Omega(x) := \sharp\{T \in \mathcal{G} : T \ni x\} \qquad \forall x \in \mathfrak{T}.$$

For x in G, denote by $T_1, \dots, T_{\Omega(x)}$ the (distinct) triangles in $\mathcal{G}$ that contain x, and by $v_1, \dots, v_{\Omega(x)}$ their vertices. By possibly relabelling the triangles, we can assume that the height of the vertices is a nonincreasing sequence, that is, $h(v_j) \geqslant h(v_{j+1})$, $j = 1, \dots, \Omega(x) - 1$. In fact, this sequence is strictly decreasing. Indeed, if $h(v_j) = h(v_{j+1})$ for some j, then either $T_j \subseteq T_{j+1}$ or $T_{j+1} \subseteq T_j$, which would contradict the maximality of either T_j or T_{j+1}. Thus, $v_1 \succ \cdots \succ v_{\Omega(x)}$.

A similar argument shows that $b_1 > \dots > b_{\Omega(x)}$, where b_j denotes the height (with respect to the point at infinity ω) of the points in $\beta(T_j)$. Hence $T_1, \dots, T_{\Omega(x)}$ form a chain of triangles such that

$$h(v_1) > \dots > h(v_{\Omega(x)}) \geqslant h(x) \geqslant b_1 > \dots > b_{\Omega(x)}.$$

A moment's reflection then shows that $d\left(x, \beta(T_{\Omega(x)})\right) \geqslant \Omega(x) - 1$, and that $h(T_j) \geqslant \Omega(x) - 1$, $j = 1, \dots, \Omega(x)$.

For every positive integer k set $\Omega_k := \big\{x \in G : \Omega(x) = k\big\}$. If $x \in \Omega_k$, then x belongs to exactly k triangles in G. By the considerations above, the height of such triangles is $\geqslant k-1$, and there exists at least one of them, T_x say, such that $d\,(x, \beta(T_x)) \geqslant k-1$. In other words, x belongs to

$$\bigcup_{m\geqslant k-1}^{h(T_x)} p^m(\beta(T_x)).$$

Now, we let x vary in Ω_k, and obtain

$$\Omega_k \subseteq \bigcup_{T\in\mathcal{G}:\, h(T)\geqslant k-1} \; \bigcup_{m\geqslant k-1}^{h(T)} p^m(\beta(T)).$$

Note that Lemma 2.1 yields

$$\Big| \bigcup_{m\geqslant k-1}^{h(T)} p^m(\beta(T)) \Big| \leqslant \sum_{m=k-1}^{h(T)} 2^{-m}\, \big|\beta(T)\big| \leqslant 2^{2-k}\, \big|\beta(T)\big|.$$

Hence

$$|\Omega_k| \leqslant 2^{2-k} \sum_{T\in\mathcal{G}} |\beta(T)|.$$

Since the triangles in $\mathcal{G}$ are maximal, their bases are disjoint. Therefore,

$$\sum_{T\in\mathcal{G}} |\beta(T)| = \Big|\bigcup_{T\in\mathcal{G}} \beta(T)\Big| \leqslant |G|.$$

Thus,

$$|\Omega_k| \leqslant 2^{2-k}\, |G|. \tag{4.2}$$

Consequently,

$$\int_G \Omega(x)^r \,\mathrm{d}\mu(x) = \sum_{k=1}^{\infty} k^r\, |\Omega_k| \leqslant 4 \sum_{k=1}^{\infty} k^r\, 2^{-k}\, |G|$$

which is equivalent to the required estimate. □

For notational convenience, for every $\alpha > 0$ we shall denote the level set $E_{\mathcal{V}f}(\alpha)$ also by $E(\alpha)$.

Remark 4.2. Observe that if $x \in E(\alpha)$, then there exists a triangle T containing x such that

$$\frac{1}{|T|} \int_T |f| \,\mathrm{d}\mu > \alpha. \tag{4.3}$$

Then $T \subseteq E(\alpha)$. This entails that $E(\alpha)$ can be written as a union of triangles T for which (5.1) holds. Furthermore, if T is one of these triangles and if $f \in L^p(\mathfrak{T})$ for some p in $(1,\infty)$, then (5.1) and Hölder's inequality imply that

$$|T| < \frac{\|f\|_p^p}{\alpha^p}. \tag{4.4}$$

Now,

$$|T| = \sum_{j=0}^{h(T)} \left|s^j(v(T))\right| \geqslant \sum_{j=0}^{h(T)} 2^j \geqslant 2^{h(T)};$$

the first inequality above follows from the assumption $\nu \geqslant 3$. Therefore,

$$h(T) \leqslant \log_2 |T| \leqslant \log_2 \frac{\|f\|_p^p}{\alpha^p}. \tag{4.5}$$

Note that $\operatorname{diam}(T) = 2h(T)$ for every triangle T; thus, if it has nonempty intersection with $B_R(o)$, then T is contained in the ball $B_{R+2h(T)}(o)$. If, in addition, T satisfies (4.4), then T is contained in the ball with centre o and radius $R(\alpha) := R + 2\log_2\left(\|f\|_p^p/\alpha^p\right)$.

In particular, if f belongs to $L^p(\mathfrak{T})$ for some $p < \infty$, then $E(\alpha)$ can be written as a union of a *finite* number of triangles.

Theorem 4.3 (Uncentred triangular maximal function). *Suppose that $\mathfrak{T}$ is a tree such that $\nu \geqslant 3$. The following hold:*

(i) *the uncentred triangular maximal operator $\mathcal{U}$ is bounded on $L^p(\mathfrak{T})$ for every p in $(1, \infty]$;*
(ii) *if $b \geqslant 2$, then $\mathcal{U}$ is not of weak type (1, 1) on the homogeneous tree $\mathfrak{T}_b$.*

Proof. First we prove (i). We shall show that for every p in $(1, \infty)$

$$\left|\left\{x \in \mathfrak{T} : \mathcal{U}f(x) > \alpha\right\}\right| \leqslant A_{p'}^p \frac{\|f\|_p^p}{\alpha^p} \qquad \forall \alpha > 0 \quad \forall f \in L^p(\mathfrak{T}), \tag{4.6}$$

where $A_{p'} = \sum_{k=1}^{\infty} k^{p'}\, 2^{-k}$. The required result then follows from the Marcinkiewicz interpolation theorem by interpolating (4.6) with the trivial L^∞ bound.

Preliminarily observe that if $\alpha \geqslant \|f\|_p$ and $\emptyset \neq T \subseteq E(\alpha)$, then (4.4) implies $|T| = 0$, which is absurd. Therefore, $E(\alpha)$ is empty for all $\alpha \geqslant \|f\|_p$.

Thus, we assume henceforth that $\alpha < \|f\|_p$. By Remark 4.2, $E(\alpha)$ can be written as a union of a *finite* number of triangles. Denote by $\mathcal{F}(\alpha)$ the collection of all triangles T that are maximal in $E(\alpha)$, that is, that are not properly contained in any larger triangle in $E(\alpha)$; thus, $E(\alpha) = \bigcup_{T \in \mathcal{F}(\alpha)} T$.

We prove (4.6). Much as in the proof of [1, Proposition 1], observe that

$$\left|E(\alpha)\right| \leqslant \sum_{T \in \mathcal{F}(\alpha)} |T| \leqslant \frac{1}{\alpha} \sum_{T \in \mathcal{F}(\alpha)} \int_T |f| \,\mathrm{d}\mu \leqslant \frac{1}{\alpha} \int_{\mathfrak{T}} |f| \sum_{T \in \mathcal{F}(\alpha)} \mathbf{1}_T \,\mathrm{d}\mu.$$

Now Hölder's inequality and (4.1) (with p' in place of r) and Lemma 4.1 (with $\mathcal{F}(\alpha)$ in place of $\mathcal{G}$ and $E(\alpha)$ in place of G) yield

$$\left|E(\alpha)\right| \leqslant \frac{\|f\|_p}{\alpha} \Big\| \sum_{T \in \mathcal{F}(\alpha)} \mathbf{1}_T \Big\|_{p'} \leqslant A_{p'} \frac{\|f\|_p}{\alpha} \left\|\mathbf{1}_{E(\alpha)}\right\|_{p'}.$$

Finally, note that $\left\|\mathbf{1}_{E(\alpha)}\right\|_{p'} = \left|E(\alpha)\right|^{1/p'}$, so that the last inequality may be rewritten as

$$\left|E(\alpha)\right| \leqslant A_{p'}^p\, \alpha^{-p}\, \|f\|_p^p,$$

as claimed.

Next we prove (ii). Suppose that T is a triangle in $\mathfrak{T}_b$ with vertex x and height h. Note the following relation between h and the volume of T:

$$\left|T\right| = \sum_{j=0}^{h} \left|s^j(x)\right| = \frac{b^{h+1} - 1}{b - 1}. \tag{4.7}$$

Consider the unit point mass δ_o at the point o. We shall show that $\mathcal{U}\delta_o$ does not belong to weak $L^1(\mathfrak{T}_b)$. Let $\alpha > 0$. Clearly $E_{\mathcal{U}_{\delta_o}}(\alpha)$ can be written as the union of maximal triangles on which the average of δ_o exceeds α. Each such triangle T satisfies

$$\frac{1}{(b+1)\alpha} \leqslant |T| < \frac{1}{\alpha}. \tag{4.8}$$

Indeed, the right-hand inequality is a direct consequence of the fact that the average of δ_o on T exceeds α. As to the left inequality, let x and h be the vertex and the height of T, respectively, and consider the triangle $\widetilde{T}$ with vertex x and height $h+1$. Since T is maximal, $\widetilde{T}$ is not contained in $E_{\mathcal{U}\delta_o}(\alpha)$, when $1/|\widetilde{T}| \leqslant \alpha$. Furthermore, (4.7) implies that $|\widetilde{T}| \leqslant (b+1)|T|$. The left-hand inequality in (4.8) follows by combining these two inequalities.

Denote by h_α the largest integer such that a triangle T in $\mathfrak{T}_b$ with height h_α satisfies the right-hand inequality in (4.8). If T contains o, then T is a maximal triangle in $E_{\mathcal{U}\delta_o}(\alpha)$ and therefore it satisfies also the left-hand inequality in (4.8). A simple calculation then shows that $b^{h_\alpha} \geqslant 1/(3b\alpha)$.

It is straightforward to see that the triangles with vertices $o, p(o), \dots, p^{h_\alpha}(o)$ and of height h_α are contained in $E_{\mathcal{U}\delta_o}(\alpha)$. Thus,

$$E_{\mathcal{U}\delta_o}(\alpha) \supset T_{h_\alpha}(o) \cup \bigcup_{k=1}^{h_\alpha} \Big(T_{h_\alpha}(p^k(o)) \setminus T_{h_\alpha}(p^{k-1}(o))\Big).$$

Note that

$$\big|T_{h_\alpha}(p^k(o)) \setminus T_{h_\alpha}(p^{k-1}(o))\big| = 1 + (b-1)\sum_{j=0}^{h_\alpha - 1} b^j = b^{h_\alpha}.$$

Therefore, if α belongs to $(0, 1/(3b))$, then

$$\big|E_{\mathcal{U}\delta_o}(\alpha)\big| \geqslant \frac{b^{h_\alpha+1}-1}{b-1} + \sum_{k=1}^{h_\alpha} b^{h_\alpha} \geqslant h_\alpha\, b^{h_\alpha} \geqslant \frac{1}{3b\alpha}\log_b \frac{1}{3b\alpha}. \tag{4.9}$$

Letting $\alpha \to 0$, we see that $\mathcal{U}\delta_o$ does not belong to weak L^1, as required. □

5 | FURTHER COMMENTS AND EXOTIC MAXIMAL OPERATORS

Theorem 4.3 raises the question of finding an endpoint result for $\mathcal{U}$ when $p=1$. We can prove the following estimate on the homogeneous tree $\mathfrak{T}_b$, $b \geqslant 2$.

Theorem 5.1. *There exists a constant C such that*

$$\big|E_{\mathcal{U}f}(\alpha)\big| \leqslant C\,\frac{\|f\|_1}{\alpha}\log_b\left(1+\frac{\|f\|_1}{\alpha}\right) \qquad \forall \alpha > 0 \quad \forall f \in L^1(\mathfrak{T}_b).$$

Proof. For simplicity we write $E(\alpha)$ instead of $E_{\mathcal{U}f}(\alpha)$ for short.

Much as in the proof of Theorem 4.3(i), observe that if $\alpha \geqslant \|f\|_1$, then $E(\alpha)$ is empty, so that we can assume that $\alpha < \|f\|_1$.

A slight variant of the argument in Remark 4.2 shows that $E(\alpha)$ can be written as union of a finite number of triangles T such that

$$\frac{1}{|T|}\int_T |f|\,\mathrm{d}\mu > \alpha. \tag{5.1}$$

Denote by $\mathcal{F}(\alpha)$ the collection of all triangles T that are maximal in $E(\alpha)$, that is, that are not properly contained in any larger triangle in $E(\alpha)$; thus, $E(\alpha) = \bigcup_{T\in\mathcal{F}(\alpha)} T$. Observe that

$$\big|E(\alpha)\big| \leqslant \sum_{T\in\mathcal{F}(\alpha)} |T| \leqslant \frac{1}{\alpha}\sum_{T\in\mathcal{F}(\alpha)}\int_T |f|\,\mathrm{d}\mu \leqslant \frac{1}{\alpha}\int_{\mathfrak{T}} |f| \sum_{T\in\mathcal{F}(\alpha)} \mathbf{1}_T\,\mathrm{d}\mu. \tag{5.2}$$

We adopt the notation introduced in the proof of Lemma 4.1, and for each x in $\bigcup_{T\in\mathcal{F}(\alpha)} T$ we denote by $\Omega(x)$ the overlapping number at x of the family $\mathcal{F}(\alpha)$. In the proof of Lemma 4.1 it is shown that $\Omega(x) \leqslant h(T)+1$ for each triangle T in $\mathcal{F}(\alpha)$ that contains x. Now,

$$|T| = \sum_{j=0}^{h(T)} \big|s^j(v(T))\big| = \sum_{j=0}^{h(T)} b^j \geqslant b^{h(T)}.$$

Therefore,

$$h(T) \leqslant \log_b |T| \leqslant \log_b \frac{\|f\|_1}{\alpha}. \tag{5.3}$$

Hence $\Omega(x) \leqslant 1 + \log_b \frac{\|f\|_1}{\alpha}$. By combining this and (5.2) we see that

$$\big|E(\alpha)\big| \leqslant \frac{\|f\|_1}{\alpha}\left(1 + \log_b \frac{\|f\|_1}{\alpha}\right) \qquad \forall \alpha < \|f\|_1.$$

Since $1 + \log_b s \leqslant C\,\log_b(1+s)$ for all $s \geqslant 1$ and C large enough, we conclude that

$$\big|E(\alpha)\big| \leqslant C\,\frac{\|f\|_1}{\alpha}\,\log_b\left(1 + \frac{\|f\|_1}{\alpha}\right) \qquad \forall \alpha > 0,$$

as required. □

We believe that this estimate is not very interesting, for it seems not strong enough to imply the boundedness of $\mathcal{U}$ on $L^p(\mathfrak{T}_b)$ for $p > 1$.

Observe that an estimate of the form

$$\big|E_{\mathcal{U}f}(\alpha)\big| \leqslant C\,\frac{\|f\|_1}{\alpha}\,\log_b\left(1 + \frac{1}{\alpha}\right) \qquad \forall \alpha > 0 \quad \forall f \in L^1(\mathfrak{T}_b),$$

which would imply the boundedness of $\mathcal{U}$ on $L^p(\mathfrak{T}_b)$ for $p > 1$, fails.

Indeed, let o be a point in $\mathfrak{T}_b$, and consider $n\delta_o$, where n is a positive integer. Observe that $E_{\mathcal{U}(n\delta_o)}(\alpha) = E_{\mathcal{U}(\delta_o)}(\alpha/n)$. If the above estimate held, we would have

$$\big|E_{\mathcal{U}\delta_o}(\alpha/n)\big| \leqslant C\,\frac{n}{\alpha}\,\log_b\left(1 + \frac{1}{\alpha}\right).$$

By (4.9), the left-hand side is bounded below by $c\ (n/\alpha)\ \log_b(n/\alpha)$, at least for α small and fixed, which is clearly incompatible with the upper bound above when n tends to infinity.

Recall that a fairly common strategy to prove weak type (1, 1) estimates for the "global part" of the HL maximal operator on manifolds with exponential volume growth is to majorise the maximal function with an appropriate integral operator, and prove that the latter is of weak type (1, 1). See, for instance, [14], where this strategy is shown to be effective in the study of the centred HL maximal function on symmetric spaces of the noncompact type, and [3] for the case of homogeneous trees.

We shall prove that a similar approach fails for the uncentred triangular maximal operator $\mathcal{U}$ on the homogeneous tree $\mathfrak{T}_b$, $b \geqslant 2$. Consider the kernel

$$\kappa(x,y) := \sup_{T\ni x} \frac{\mathbf{1}_T(y)}{|T|} \qquad \forall x,y \in \mathfrak{T}_b, \tag{5.4}$$

and denote by $\mathcal{K}$ the corresponding integral operator, defined by

$$\mathcal{K}f(x) := \int_{\mathfrak{T}_b} \kappa(x,y)\, f(y)\, \mathrm{d}\mu(y) \qquad \forall x \in \mathfrak{T}_b,$$

where f is any reasonable function on $\mathfrak{T}_b$. Note that $\mathcal{U}f \leqslant \mathcal{K}|f|$. The following result implies that $\mathcal{U}$ and $\mathcal{K}$ have a quite different boundedness properties as operators acting on $L^p(\mathfrak{T}_b)$.

Proposition 5.2. *The operator $\mathcal{K}$ is unbounded on $L^p(\mathfrak{T}_b)$ for every p in $[1,\infty]$ and for every $b \geqslant 2$.*

Proof. It is straightforward to check that the smallest triangle that contains two points x and y is the triangle with vertex $x \wedge y$ (see the proof of Theorem 3.2(ii) for the notation) and height

$$\eta(x,y) := \max\big(d(x, x\wedge y), d(y, x\wedge y)\big).$$

Clearly $\eta(x,y) = \frac{1}{2}\,[d(x,y) + |h(x) - h(y)|]$. From the definition of κ (see (5.4)) and Lemma 2.1(ii) we deduce that

$$\kappa(x,y) = \frac{1}{|T_{x,y}|} \geqslant \frac{1}{2\,|\beta(T_{x,y})|} \geqslant \frac{1}{2}\, b^{-\eta(x,y)} \qquad \forall x,y \in \mathfrak{T}_b.$$

Suppose that o is a point in $\mathfrak{H}_0$, and consider, for each positive integer n, the set $E_n := s^n(p^n(o))$, which is the base of the triangle with vertex $p^n(o)$ and height n. Observe that for every x and y in E_n we have $\eta(x,y) = d(x,y)/2$, so that

$$\mathcal{K}\mathbf{1}_{E_n}(x) \geqslant \frac{1}{2} \int_{E_n} b^{-d(x,y)/2}\, \mathrm{d}\mu(y).$$

Note that for every positive integer $j \leqslant n$ there are exactly $(b-1)\, b^{j-1}$ points in E_n at distance $2j$ from x. Therefore, the last integral can be rewritten as

$$1 + \frac{b-1}{b} \sum_{j=1}^{n} b^{-j}\, b^{j} = 1 + \frac{b-1}{b}\, n.$$

Altogether

$$\mathcal{K}\mathbf{1}_{E_n}(x) \geqslant \frac{b-1}{2b}\, n \qquad \forall x \in E_n,$$

from which the desired result for $p = \infty$ follows directly.

Now, set $C_b := (b-1)/(2b)$, and observe that if $p < \infty$, then for every positive integer n the previous inequality yields

$$\left\|\mathcal{K}\mathbf{1}_{E_n}\right\|_p^p \geqslant C_b^p\, n^p\, |E_n| = C_b^p\, n^p \left\|\mathbf{1}_{E_n}\right\|_p^p,$$

which implies that $\mathcal{K}$ is unbounded on $L^p(\mathfrak{T}_b)$, as required. □

It is worth observing that replacing triangles with appropriate slightly larger sets in the definition of $\mathcal{T}$ and $\mathcal{U}$ may yield significant modifications of the boundedness properties of the corresponding maximal operators, as we presently show. This is a further example that illustrates how sensitive are maximal operators to the shape of the sets with respect to which we take averages.

For every nonnegative integer r consider the *modified triangle* $T_r'(x) := T_r(x) \cup p^r(x)$, and the corresponding centred and uncentred maximal operators

$$\mathcal{T}' f(x) = \sup_{r\in\mathbb{N}} \frac{1}{|T_r'(x)|} \int_{T_r'(x)} |f|\, \mathrm{d}\mu \quad \text{and} \quad \mathcal{U}' f(x) = \sup_{T'\ni x} \frac{1}{|T'|} \int_{T'} |f|\, \mathrm{d}\mu,$$

where T' is any modified triangle containing x. We emphasise that T' is obtained from a triangle T by adjoining just a point at distance $h(T)$ from the vertex of T. Observe that if there exists a positive constant C such that $\left|T_r(x)\right| \geqslant C\left|B_r(x)\right|$ for every triangle $T_r(x)$ in $\mathfrak{T}$, then

$$\mathcal{T}' \leqslant C\,\mathcal{M} \qquad \text{and} \qquad \mathcal{U}' \leqslant C\,\mathcal{N}. \tag{5.5}$$

For instance, this happens if $\mathfrak{T} = \mathfrak{T}_b$, or $\mathfrak{T} = \mathfrak{S}_{a,b}$ and $a \leqslant b < a^2$: here $\mathfrak{S}_{a,b}$ denotes the tree such that each vertex has either $a+1$ or $b+1$ neighbours according to the fact that its height is < 1 or $\geqslant 1$. We refer the reader to [6] for more on $\mathfrak{S}_{a,b}$. For each pair a, b of positive integers, we denote the number $\log_a b$ by τ. In the next proposition we show that there are trees where $\mathcal{T}'$ and $\mathcal{U}'$ have different boundedness properties than $\mathcal{T}$ and $\mathcal{U}$, respectively.

Proposition 5.3. *The following hold:*

(i) *the operator $\mathcal{T}'$ is bounded on $L^p(\mathfrak{T}_b)$ for every p in $(1,\infty]$, and unbounded on $L^1(\mathfrak{T}_b)$;*
(ii) *if $a < b < a^2$, then $\mathcal{T}'$ is bounded $L^p(\mathfrak{S}_{a,b})$ for $p > \tau$ and it is unbounded on $L^p(\mathfrak{S}_{a,b})$ for $p < \tau$;*
(ii) *the operator $\mathcal{U}'$ is bounded on $L^p(\mathfrak{T}_b)$ if and only if $p > 2$.*

Proof. The L^p boundedness of $\mathcal{T}'$ and $\mathcal{U}'$ in the ranges described in (i)–(iii) above follow from the bounds (5.5) and the positive results for $\mathcal{M}$ and $\mathcal{N}$ proved in [3, 6, 10, 15].

Next we prove that $\mathcal{T}'$ is unbounded on $L^1(\mathfrak{T}_b)$. Fix a point o in $\mathfrak{H}_0$, and consider the set $E := \{x \in \mathfrak{T}_b : o \succeq x\}$. Clearly E is the infinite triangle with vertex o. It is straightforward to check that for each x in E

$$\mathcal{T}'\delta_o(x) = \frac{1}{|T'_{|x|}(x)|}.$$

By Lemma 2.1(ii), $|T'_{|x|}(x)| = |T_{|x|}(x)| + 1 \leqslant 2\,b^{|x|} + 1$, so that

$$\|\mathcal{T}'\delta_o\|_{L^1(\mathfrak{T}_b)} \geqslant \int_E \mathcal{T}'\delta_o \,\mathrm{d}\mu \geqslant \sum_{j=0}^{\infty} \int_{E\cap\mathfrak{H}_{-j}} \frac{1}{2b^j+1}\,\mathrm{d}\mu.$$

Since $|E \cap \mathfrak{H}_{-j}| = b^j$, the series above is not convergent, and the unboundedness of $\mathcal{T}'$ on $L^1(\mathfrak{T}_b)$ follows, thereby completing the proof of (i).

To complete the proof of (ii), fix a point o in $\mathfrak{H}_0$, and for each positive integer n consider the set $E_n := s^n(p^n(o))$, which is a subset of the horocycle $\mathfrak{H}_0$ in $\mathfrak{S}_{a,b}$. By Lemma 2.1 (ii), and the fact that each vertex with nonpositive height has exactly a successors,

$$\mathcal{T}'\delta_{p^n(o)}(x) = \frac{1}{|T'_n(x)|} \geqslant \frac{1}{2\,a^n+1} \qquad \forall x \in E_n,$$

when, much as above,

$$\|\mathcal{T}'\delta_{p^n(o)}\|^p_{L^p(\mathfrak{S}_{a,b})} \geqslant \int_{E_n} \left(\mathcal{T}'\delta_{p^n(o)}\right)^p \mathrm{d}\mu \geqslant \frac{|E_n|}{(2a^n+1)^p}.$$

Observe that $|E_n| = b^n = a^{\tau n}$. Altogether, we see that

$$\|\mathcal{T}'\delta_{p^n(o)}\|^p_{L^p(\mathfrak{S}_{a,b})} \geqslant \frac{a^{\tau n}}{(2a^n+1)^p}.$$

Since, by assumption, $p < \tau$, the right-hand side above cannot be bounded with respect to n, and the desired result follows.

Finally we complete the proof of (iii) by showing that $\mathcal{U}'$ is unbounded on $L^p(\mathfrak{T}_b)$ for every $p \leqslant 2$. Let $E := \{x \in \mathfrak{T}_b : o \succeq x\}$. If $x \in E$ and $d(o,x)$ is even, then

$$\mathcal{U}'\delta_o(x) = \frac{1}{\left|T'_{|x|/2}\left(p^{|x|/2}(x)\right)\right|},$$

and Lemma 2.1(ii) implies that $\left|T'_{|x|/2}\left(p^{|x|/2}(x)\right)\right| \leqslant 2\,b^{|x|/2} + 1$. Thus,

$$\|\mathcal{U}'\delta_o\|^p_{L^p(\mathfrak{T}_b)} \geqslant \sum_{j=0}^{\infty} \int_{E_n\cap\mathfrak{H}_{-2j}} (2b^j+1)^{-p}\,\mathrm{d}\mu = \sum_{j=0}^{\infty} b^{2j}\,(2b^j+1)^{-p}.$$

The required conclusion follows from the fact that for every $p \leqslant 2$ the series above is not convergent. □

Remark 5.4. Finally, we present an example of a tree $\mathfrak{T}$ with unbounded geometry where $\mathcal{T}'$, and *a fortiori* $\mathcal{U}'$, is unbounded on L^p for every $p < \infty$.

Let $\mathfrak{T}$ be the tree characterised by the property that each vertex off $\mathfrak{H}_0$ has three neighbours, and $\nu(x_j) = j + 2$ where $\{x_j : j \geqslant 1\}$ is an enumeration of the points of $\mathfrak{H}_0$.

Note that for every $j \geqslant 1$

$$\mathcal{T}'\delta_{x_j}(y) = \frac{1}{|T_1'(y)|} = \frac{1}{4} \qquad \forall y \in s^1(x_j).$$

Therefore,

$$\|\mathcal{T}'\delta_{x_j}\|_p^p \geqslant \sum_{y \in s^1(x_j)} \mathcal{T}'\delta_{x_j}(y)^p = \frac{1}{4^p}\,|s^1(x_j)| = \frac{j+2}{4^p}.$$

Since $\|\delta_{x_j}\|_p = 1$, the operator norm of $\mathcal{T}'$ on $L^p(\mathfrak{T})$ is at least $(j+2)^{1/p}/4$. By letting j vary we obtain the required conclusion.

Since $\mathcal{U}' \geqslant \mathcal{T}'$ pointwise, $\mathcal{U}'$ is unbounded on $L^p(\mathfrak{T})$ for every $p \in [1, \infty)$.

It is straightforward to check that for each $r > 1$ there is no constant C such that

$$\Big\| \sum_{T' \in \mathcal{G}'} \mathbf{1}_{T'} \Big\|_r \leqslant C\, \|\mathbf{1}_{G'}\|_r \tag{5.6}$$

for every finite family $\mathcal{G}'$ of maximal modified triangles in $\mathfrak{T}$. Here G' is the union of the modified triangles in $\mathcal{G}'$.

Indeed, it suffices to consider, for every positive integer j, the family $\mathcal{G}_j'$ of the modified triangles $\{T_1'(y) : y \in s^1(x_j)\}$. Then the r^{th} power of the right-hand side of (5.6) is equal to $C^r\ (3(j+2)+1)$, whereas the rth power of left-hand side is equal to $3(j+2) + (j+2)^r$. Thus, (5.6) fails for large values of j.

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