Joint PhD Program in Mathematics
Milano Bicocca-Pavia-INdAM

New results on effective non-vanishing and boundedness of foliations

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Academic year 2022/23 - XXXVI Cycle

Contents

Introduction

This thesis is naturally divided in two parts, each aimed at studying a different problem in algebraic geometry.

The first part, which is covered in Chapter 1, focuses on proving a particular case of the following conjecture, usually called the *Ambro-Kawamata conjecture*.

Conjecture 1 ([Kaw00, Conjecture 1]). *Let (X, Δ) be a klt pair, H an ample Cartier divisor such that $H - (K_X - \Delta)$ is ample. Then, $H^0(X, H) \neq 0$.*

The problem of deciding the vanishing or non-vanishing of some of the cohomology groups of sheaves is ubiquitous in algebraic and birational geometry. The 0-th degree cohomology of a divisor D (or of a multiple of it) relates to a wide range of problems: for example, when working with divisors it can be used to understand whether some multiple mD of D is linearly equivalent to an effective divisor (when $H^0(X, mD)$ is non-trivial), or to show that mD is big, if $H^0(X, mD)$ is large enough. Instead, the vanishing of higher degree cohomology is often used, given a morphism between sheaves, to understand when global sections of the target sheaf can be lifted, that is, when the induced morphism on global sections is surjective: starting with a short exact sequence and looking at the induced long exact sequence on cohomology, the vanishing of higher degree cohomology groups implies the surjectivity of the last morphism between the 0-th degree cohomology groups.

Understanding the vanishing or non-vanishing of the cohomology groups of a divisor is the content of fundamental theorems such as the Kawamata-Viehweg theorem and Shokurov's non-vanishing theorem, to which the conjecture is closely related. The former states the vanishing of higher degree cohomology of line bundles of the form $L \otimes K_X$, where L is big and nef; the conjecture then aims to describe the only cohomology group of $H = L \otimes K_X$ which might be non-trivial, that is $H^0(X, H)$. While the conjecture requires H to be ample, if it is true then it extends to the case of H big and nef as well, by means of the reductions of [Kaw00]. On a related note, the conjecture would also work as a partial improvement to Shokurov's non-vanishing, which states the non-vanishing of the degree zero cohomology of divisors of the form $bD + \lceil A \rceil$ for $b > 0$ sufficiently large, if $aD + A - K_X$ is big and nef for some $a > 0$ and under some assumptions on the structure of A: when $A = 0$, the conjecture would imply that $b = 1$.

Before the conjecture is first stated in [Kaw00], a particular case was proved by Ambro [Amb99] in order to study properties of Fano varieties. Inspired by this, and motivated by the relation to the Kawamata-Viehweg theorem, Kawamata is the first to study the conjecture in its general setting as an independent problem. In particular, he proves it in the case that X is a surface or a minimal threefold, together with partial improvements to Ambro's results. Since then, only few more cases of the conjecture, which remains for the most part open, have been proved to be true. The present work focuses on further understanding the case of quasi-smooth weighted complete intersections, improving on previous work of Pizzato, Sano and Tasin [PST17]. There are several reasons to consider this setting: first of all, weighted complete intersections work as generalisations of complete intersections, whose properties are usually easier to study. In many cases, this happens with quasi-smooth weighted complete intersections as well, so that smooth complete intersections and quasi-smooth weighted complete intersections behave similarly (some examples of such properties will be mentioned in Chapter 1). At the same time, there is enough difference between classical complete intersections and weighted complete intersections to make the latter worth of independent study: for one, weighted complete intersections are almost never smooth, which is the reason why the weaker notion of quasi-smoothness is often used instead; this amounts to asking that the only singularities appearing are cyclic quotient singularities arising from the quotient action defining the ambient weighted projective space. While at first glance this might seem an undesirable feature, the fact that many properties of quasi-smooth weighted complete intersections are generally well known and appear to be numerical in nature is useful to construct singular varieties satisfying given properties. This feature is even used in the present thesis to construct Example 2.4.7 of Chapter 2.

In [PST17], the conjecture is shown to hold for Fano and Calabi-Yau well-formed quasi-smooth weighted complete intersections of any dimension; the general type case, which is the only missing case, is exclusively proved for hypersurfaces. We improve this, by showing that the conjecture still holds for weighted complete intersections of general type of codimension up to 3.

Theorem 2. *Let $X \subset \mathbb{P}(a_0, \ldots, a_n)$ be a well-formed quasi-smooth weighted complete intersection and assume that* $\operatorname{codim} X \leq 3$. *Then, for any ample Cartier divisor H on X such that $H - K_X$ is ample, $H^0(X, H) \neq 0$.*

In order to prove the statement, following the ideas of [PST17], we move from the original conjecture on algebraic varieties to a purely numerical problem, with the introduction of h-regular pairs (Definition 1.1.17). A h-regular pair $(d; a)$ is given by a pair of tuples of positive integers $d = (d_1, \ldots, d_c)$ and $a = (a_0, \ldots, a_n)$, which we respectively call degrees and weights of the pair, satisfying properties that generalise the numerical relations between degrees and weights of quasi-smooth weighted complete intersections. Surprisingly, translating the Ambro-Kawamata conjecture to a numerical problem leads to a deep connection to a classical problem in the theory of numerical semigroups, called the Frobenius problem: when $h = 1$, the conjecture on regular pairs is the following.

Conjecture 3. *Let $(d; a) = (d_1, \ldots, d_c; a_0, \ldots, a_n)$ be a regular pair such that $c \leq n$ and $a_i \neq 1$ for any i. Then,*

$$\delta(d; a) \geq F(a_0, \ldots, a_n),$$

where $\delta(d; a) = \sum d_i - \sum a_j$, and $F(a_0, \ldots, a_n)$ is the Frobenius number of $a_0, \ldots, a_n$.

In the language of natural numbers, given a set of coprime natural numbers, the Frobenius number of the set is the largest integer that cannot be written as a nonnegative combination of elements in the set; computing such number is the objective of the Frobenius problem. Despite the apparent simplicity of the problem, there is no close formula to compute Frobenius numbers, and the problem is known to be computationally hard; for these reasons, it is not obvious how to make use of any known property of Frobenius numbers in the present setting. Then, approaching the conjecture on regular pairs requires producing new results aimed at connecting the problem to the known properties of Frobenius numbers. We achieve this by proving a range of new results which allows to only consider h-regular pairs whose prime structure is simpler, together with new recursive bounds on Frobenius numbers.

Since little is known about Conjecture 3 and its generalisation to h-regular pairs, Theorem 2 provides evidence in support of the validity of the conjecture for pairs of any codimension. Lastly, as a corollary to Theorem 2, we prove a new upper bound to the Frobenius number of a set of positive integers, which improves the most known one due to Brauer (Proposition 1.2.12).

The second part of the thesis, which covers Chapter 2, is devoted to foliations in algebraic geometry, and more precisely studying their behaviour in families.

In recent years, foliation theory has received increasing attention in algebraic and birational geometry, thanks to the many applications of foliations to the study of varieties and their maps. After the work of people such as McQuillan, Brunella and Mendes, it has become clear that some of the geometrical properties of a foliation $\mathcal{F}$ are related to the positivity of a rank one sheaf associated to it; studying the sheaf is often done, in the language of divisors, by considering the canonical divisor $K_{\mathcal{F}}$ associated to such sheaf.

In order to understand the ideas behind the main problem of Chapter 2, a digression on the classification of algebraic varieties is needed. First of all, for any variety X there is a canonical divisor K_X associated to it; much of the research of last century has shown a deep relation between the geometry of a variety and the properties of its canonical divisor. For this reason, varieties are very roughly classified on the basis of their Kodaira dimension $\kappa(X)$, which describes how quickly the global sections of mK_X grow with m. On one end of the spectrum there are varieties such that $H^0(mK_X) = 0$ for all $m > 0$: this is the case, for example, of projective spaces of any dimension. On the other, there are varieties such that a multiple mK_X of K_X induces a birational map on the image (that is, K_X is big); such varieties are said to be of general type. Inspired by the techniques that have led to the Enriques-Kodaira classification of surfaces, the modern approach to the classification in higher dimension is, for every mildly singular variety X, to find a variety X' birational to it such that either $K_{X'}$ is nef or there exists a fibration $X' \to Y$ onto a variety of lower dimension; this is achieved by the (still conjectured) Minimal Model Program, or MMP for short. When X is a variety of general type, more can be done: it is possible to find X' so that $K_{X'}$ is not only big and nef, but also ample. Such a variety is called the canonical model of X. The main reason of interest in canonical models comes from the problem of constructing moduli spaces for varieties of general type. In fact, for a fixed dimension n, the existence of a canonical polarisation for all varieties of general type (or rather, of a common multiple M_nK_X which is very ample) is fundamental to study varieties in families. An important consequence is that canonical models of general type with fixed volume K_X^n belong to a bounded family: this means that it is possible to construct a flat and proper morphism between quasi-projective varieties of finite type, whose fibers are such models. This constitutes the first step towards the construction of moduli spaces of varieties of general type.

Recently, an approach to the classification of foliations has been attempted in a similar fashion. This has led, up to dimension 3, to proving the existence of a foliated version of the MMP ([McQ08], [CS20], [CS21]). It is then natural to ask whether foliations of general type (that is, foliations such that $K_{\mathcal{F}}$ is big) can be studied, in analogy to varieties of general type, by finding a birational model $\mathcal{F}'$ such that $K_{\mathcal{F}'}$ is ample. Unfortunately, for foliations this is not always possible (Example 2.2.14): there exist canonical singularities at which the canonical divisor of a foliation might not even be $\mathbb{Q}$-Cartier. Then, in order to study families of foliations of general type, we can investigate in what other way it is possible to replicate the results that hold for canonical models of varieties of general type. One idea is to use a weaker notion of canonical model (Definition 2.2.16), which does not require the ampleness of $K_{\mathcal{F}}$; still, constructing bounded families of foliated surfaces of general type requires additional conditions than are needed for varieties: even when restricting to foliations with only canonical singularities such that $K_{\mathcal{F}}$ is ample, it is possible to construct a family of foliated surfaces with $K_{\mathcal{F}}^2$ fixed, whose underlying surfaces belong to an unbounded family (Example 2.4.2).

Aiming to find sufficient conditions for boundedness of foliated canonical models of surfaces, Hacon-Langer [HL21] and Chen [Che21] prove that some partial results can be achieved by fixing the Hilbert function $P(m) = \chi(mK_{\mathcal{F}})$ of the canonical divisor.

More precisely, Hacon and Langer show that for foliated canonical models of general type with fixed Hilbert function $P(m)$, there exists an integer N_P, only depending on $P(m)$, such that $|NK_{\mathcal{F}}|$ gives a birational map for any $N \geq N_P$. Building on this, Chen proves that some partial resolutions of canonical models with fixed Hilbert function (called minimal partial Du Val resolutions) are bounded.

One issue with the previous results is that they require the Hilbert function to be fixed. As long as $K_{\mathcal{F}}$ is Cartier, Riemann-Roch theorem gives a purely numerical description of $\chi(mK_{\mathcal{F}})$ for any m. In general, when $K_{\mathcal{F}}$ is only a Weil divisor, the formula can be corrected by introducing a constant term depending on the singularities of the foliation, but computing such term is usually unfeasible. Hence, it is interesting to understand whether it is possible to prove the results of Hacon-Langer and Chen without explicitly knowing the correction term appearing in the formula for $\chi(mK_{\mathcal{F}})$. This is precisely what is achieved in Chapter 2, as a consequence of the following theorem.

Theorem 4. *Let $\mathcal{H}_{k_1,k_2,s}$ be the set of Hilbert functions of foliated canonical models $(X, \mathcal{F})$ of general type with $K_{\mathcal{F}}^2 = k_1, K_{\mathcal{F}} \cdot K_X = k_2$ and $i_{\mathbb{Q}}(\mathcal{F}) = s$, where $i_{\mathbb{Q}}(\mathcal{F})$ is the $\mathbb{Q}$-index of $\mathcal{F}$ (Definition 2.2.9). Then, $|\mathcal{H}_{k_1,k_2,s}| < \infty$.*

Note that $K_{\mathcal{F}}^2$, $K_{\mathcal{F}} \cdot K_X$ and $i_{\mathbb{Q}}(\mathcal{F})$ are fixed when the Hilbert function is fixed (Proposition 2.2.22), so the assumptions of Theorem 4 are indeed weaker. Furthermore, given a family of canonical models, in general it is much simpler to check the conditions of Theorem 4 than it is to check that the Hilbert function is fixed in the family: $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$ are numerical conditions, while the index $i_{\mathbb{Q}}$ can be shown to be bounded, for example, when the underlying surfaces are bounded.

Since $\mathcal{H}_{k_1,k_2,s}$ is finite, the result of Hacon and Langer still holds when we only fix $K_{\mathcal{F}}^2, K_{\mathcal{F}} \cdot K_X$ and $i_{\mathbb{Q}}(\mathcal{F})$, by taking the maximum among the integers N_P, for every $P \in \mathcal{H}_{k_1,k_2,s}$; similarly, the boundedness statement of Chen holds for all models with Hilbert function $P \in \mathcal{H}_{k_1,k_2,s}$, hence it is enough to take the union of each family with fixed Hilbert function. In particular, we deduce the following two corollaries.

Corollary 5. *Fix rational numbers k_1, k_2 and a positive integer s, and consider the family of canonical models $(X, \mathcal{F})$ of general type such that $K_{\mathcal{F}}^2 = k_1$, $K_{\mathcal{F}} \cdot K_X = k_2$ and $i(\mathcal{F}) = s$. Then, there exists a constant N_1, only depending on k_1, k_2, s, such that for any $(X, \mathcal{F})$ in the family and $m \geq N_1$, $|mK_{\mathcal{F}}|$ defines a birational map.*

Corollary 6. *Fix rational numbers k_1, k_2 and a positive integer s. The set $\mathcal{S}_{k_1,k_2,s}$ of minimal partial Du Val resolutions of canonical models of general type $(X, \mathcal{F})$ with fixed $K_{\mathcal{F}}^2 = k_1, K_{\mathcal{F}} \cdot K_X = k_2, i(\mathcal{F}) = s$ is bounded.*

The idea behind the proof of Theorem 4 starts from a classical theorem, due to Kollár and Matsusaka, which gives a bound on $h^0(D)$ only depending on D^2 and $D \cdot K_X$, for any big and semiample Cartier divisor D. When $K_{\mathcal{F}}$ is ample, since $K_{\mathcal{F}}^2$

and $K_{\mathcal{F}} \cdot K_X$ are fixed, this can be used to show that the pairs $(X, i_{\mathbb{Q}}(\mathcal{F})K_{\mathcal{F}})$ are bounded as polarised surfaces; this is enough to bound the number of singularities, which in turns restricts the possible values of the constant term in the formula for the Hilbert function of $K_{\mathcal{F}}$. When $K_{\mathcal{F}}$ is only big and nef, it is necessary to pass to a partial resolution $(X', \mathcal{F}')$ and take a perturbation $D_X = K_{\mathcal{F}'} + \epsilon K_{X'}$ of $K_{\mathcal{F}'}$ (for a fixed $\epsilon > 0$) which is ample. With more care the previous argument still holds, after showing that under the assumptions of the theorem D_X^2 and $D_X \cdot K_X$ only assume a finite number of values.

Finally, we conjecture that Theorem 4 does not hold under weaker conditions. First of all, it is natural to expect that $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$ cannot be unbounded, as they are the top two terms in Riemann-Roch theorem; as mentioned before, an example of an unbounded family with $K_{\mathcal{F}}^2$ fixed but $K_{\mathcal{F}} \cdot K_X$ unbounded is already known (Example 2.4.2). The condition on the index is more subtle; while we are not able to prove it is necessary, we show partial results in the opposite direction, as in many natural families of surfaces (such as Fano or Calabi-Yau varieties and minimal models of general type) the index must be bounded nonetheless.

Acknowledgements

I am deeply thankful to my PhD advisor Prof. Luca Tasin, for supervising me throughout the three years of my PhD and supporting me in the world of mathematical research, and to Dr. Calum Spicer, for giving me the opportunity to work under his supervision during my two stays in London and guiding me in a field of research completely new to me.

I would like to express my sincere gratitude to the referees, for their useful comments and suggestions on the structure and content of the thesis.

Finally, I want to thank all the people I have met during my stays in London, in particular all the members of the LSGNT, for making me feel welcome from the first day, for all the productive discussions and talks, and for all the support and encouragement.

Chapter 1

Effective non-vanishing for quasi-smooth WCI

In [PST17], the authors study the Ambro-Kawamata conjecture, which states the non-vanishing of $H^0(X, H)$ for H an ample Cartier divisor with X klt and $H - K_X$ ample, in the case of quasi-smooth weighted complete intersections (WCI). Instead of pursuing a geometrical approach, they first prove necessary and sufficient conditions satisfied by the degrees and weights of quasi-smooth WCIs; this allows the problem to be translated to a purely numerical statement on pairs of tuples of integer numbers, called h-regular pairs. While the new setting is more general, it reduces the problem to showing the existence of a stronger relation between degrees and weights of the WCI. Thanks to this, the authors are able to prove that Conjecture 1.2.1 holds in the Fano or Calabi-Yau case for quasi-smooth WCI. We focus on studying the missing case, that is WCIs of general type. In this setting, the conjecture on h-regular pairs turns out to have an important connection to a problem coming from the theory of numerical semigroups, called the Frobenius problem. We prove the Ambro-Kawamata for quasi-smooth WCI of codimension at most 3, by proving the generalised numerical conjecture for h-regular pairs of codimension at most 3. In order to achieve this, we also prove several intermediate results both on the computation of Frobenius numbers and on conditions under which the problem can be made simpler.

1.1 Preliminaries

In the following, we always work over $\mathbb{C}$. We will often mix sheaf and divisorial notation.

First of all, we review some definitions in order to introduce quasi-smooth weighted complete intersections. An in-depth examination of these objects and their properties can be found in [Ian00], [Dol82].

1.1.1 Weighted projective spaces

Definition 1.1.1. Let $x_0, \dots, x_n$ be affine coordinates on $\mathbb{A}^{n+1}$, $a_0, \dots, a_n$ positive integers. Consider the action of $\mathbb{C}^*$ given by

$$\lambda(x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n).$$

The quotient $\mathbb{P}(a_0, \dots, a_n) \coloneqq (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{C}^*$ is a projective variety called the *weighted projective space* of weights $a_0, \dots, a_n$.

A different way to obtain weighted projective spaces is through a Proj construction of the polynomial ring: in fact, let $S = S(a_0, \dots, a_n)$ be the graded polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ with grading given by $\deg x_i = a_i$. Then,

$$\mathbb{P}(a_0, \dots, a_n) = \text{Proj } S.$$

Remark 1.1.2. For any positive integer $l > 0$,

$$\mathbb{P}(la_0, \dots, la_n) \simeq \mathbb{P}(a_0, \dots, a_n);$$

this follows directly from the fact that the graded rings are isomorphic. As a consequence, it is natural to only consider the case $\gcd(a_0, \dots, a_n) = 1$. Still, we will usually work with weights satisfying a stronger condition:

Definition 1.1.3. A weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is *well-formed* if

$$\gcd(a_0, \dots, \hat{a}_i, \dots, a_n) = 1$$

for any $0 \leq i \leq n$.

The reason to consider well-formed weighted projective spaces is given by the following observation:

Lemma 1.1.4 ([Ian00, Lemma 5.7])**.** *Let $a_0, \dots, a_n$ be positive, coprime integers, and $g = \gcd(a_1, \dots, a_n)$. Then,*

$$\text{Proj } S(a_0, \dots, a_n) \simeq \text{Proj } S(a_0, a_1/g, \dots, a_n/g).$$

This is a consequence of the fact that any weighted ring S is isomorphic to any of its truncations $S^{(k)} = \oplus_{m \geq 0} S_{mk}$, where S_i is the i-th graded part of S, and that under the assumptions of the Lemma,

$$S(a_0, \dots, a_n) \simeq S(ga_0, a_1, \dots, a_n),$$

with the latter being isomorphic to the g-th truncation of $S(a_0, \dots, a_n)$.

Example 1.1.5.

- From both constructions of weighted projective spaces, it follows that $\mathbb{P}^n = \mathbb{P}(1,\dots,1)$. We will refer to this case as the standard projective space.
- $\mathbb{P}(6,10,15)$ is isomorphic to the standard projective plane, despite all the weights being greater than 1. In fact, by repeatedly applying Lemma 1.1.4, we obtain that
$$\mathbb{P}(6,10,15) = \mathbb{P}(3,5,15) = \mathbb{P}(1,5,5) = \mathbb{P}(1,1,1).$$

Remark 1.1.6.

(i) Due to the $\mathbb{C}^*$-action, any well-formed weighted projective space which is not the standard projective space is singular. To see this, for the sake of simplicity suppose that any two weights are coprime, $a_0 > 1$ and consider the open chart $U_0 = \{x_0 \neq 0\}$. For any a_0-root of unity ξ, we get that $[1, p_1, \dots, p_n] = [1, \xi^{a_1} p_1, \dots, \xi^{a_n} p_n]$. Looking at the $x_1, \dots, x_n$-coordinates, this shows that U_0 is a quotient $\mathbb{A}^n/\mathbb{Z}_{a_0}$, and the point $[1,0,\dots,0]$ is a cyclic quotient singularity of type $\frac{1}{a_0}(a_1,\dots,a_n)$. This can be generalised to any set of weights: let $P_j = [0,\dots,1,\dots,0]$ the point with only the j-th coordinate being non-zero. For any subset $I = \{a_{i_1},\dots,a_{i_k}\} \subset \{0,\dots,n\}$ such that $g_I = \gcd_{i\in I} a_i > 1$, any point in the interior of the linear space generated by the P_i, $i \in I$ is a quotient singularity of type
$$\frac{1}{g_I}(a_0,\dots,\hat{a_{i_1}},\dots,\hat{a_{i_k}},\dots,a_n) \times \mathbb{C}^{|I|-1}.$$
Note that all singularities of $\mathbb{P}$ are obtained this way; as a consequence,
$$\operatorname{codim}\operatorname{Sing}(\mathbb{P}) \geq 2.$$

(ii) For any weighted projective space $\mathbb{P}$, its class group $\mathrm{Cl}(\mathbb{P})$ is cyclic; on the other hand, since $\mathbb{P}$ has cyclic quotient singularities, in general it does not coincide with $\mathrm{Pic}(\mathbb{P})$.

(iii) Generalising the standard case, the canonical sheaf is given by [BR86, Corollary 6B.8]
$$K_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-\sum_{i=0}^{n} a_i).$$
Note that $K_{\mathbb{P}}$ is, in general, not Cartier: in fact, it is Cartier if and only if $\operatorname{lcm}(a_0,\dots,a_n)$ divides $a = \sum a_i$ [BR86, Corollary 6B.10].

Example 1.1.7. Consider the space $\mathbb{P} = \mathbb{P}(3,1,1)$. It has an isolated cyclic quotient singularity at $[1,0,0]$, and $K_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-5)$. $K_{\mathbb{P}}$ is not Cartier, because $\operatorname{lcm}(3,1) = 3$ does not divide 5.

1.1.2 Weighted complete intersections

As in the standard projective space, a natural class of varieties is given by complete intersections.

Definition 1.1.8.

- Let X be a variety in $\mathbb{P}(a_0, \dots, a_n)$, and I its homogeneous ideal. Suppose that I is generated by a regular sequence $\{f_i\}$ of homogenous polynomials such that $\deg f_i = d_i$. Then, we say that X is a *weighted complete intersection* (WCI for short) of multidegree $(d_1, \dots, d_c)$. Any WCI of multidegree $d_1, \dots, d_c$ will be denoted by $X_{d_1,\dots,d_c}$.
- A WCI $X_{d_1,\dots,d_c}$ is called a *linear cone* if $a_i = d_j$ for some i and j.

Since weighted projective spaces are usually singular, it should be expected that subvarieties are rarely smooth. This leads to the following, weaker definition.

Definition 1.1.9.

- A subvariety X of $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ is *well-formed* if $\operatorname{codim}(X \cap \operatorname{Sing}(\mathbb{P})) \geq 2$.
- Let $\pi\colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}(a_0, \dots, a_n)$ be the canonical projection. We say that X is *quasi-smooth* if the punctured affine cone $\pi^{-1}(X)$ is smooth.

Note that, if P is a singular point of $\pi^{-1}(X)$, all points in the same fiber of π are singular as well. This means that any singularity on a quasi-smooth variety X only appears due to the $\mathbb{C}^*$-action. As we now show, while these conditions together are weaker than X being smooth, they are enough to prove that X behaves well enough.

Property 1.1.10.

(i) For $X = X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0, \dots, a_n)$, $\dim X = n - c$.

(ii) If X is a well-formed quasi-smooth WCI, then
$$\operatorname{Sing}(X) = X \cap \operatorname{Sing}(\mathbb{P}).$$
Thus, in general a well-formed quasi-smooth WCI is not smooth.

(iii) If $X \subset \mathbb{P}$ is a well-formed quasi-smooth WCI and $\dim X > 2$, then $\operatorname{Cl}(X) \cong \mathbb{Z}$ and is generated by $\mathcal{O}_X(1) \coloneqq \mathcal{O}_{\mathbb{P}}(1)\big|_X$.

(iv) Adjunction holds for a well-formed quasi-smooth WCI X even if it is singular. In particular, if $\dim X > 2$ the canonical sheaf of X is given by
$$K_X = \mathcal{O}_X(\sum_{i=1}^{c} d_i - \sum_{j=0}^{n} a_i).$$
The integer number $\delta = \sum_{i=1}^{c} d_i - \sum_{j=0}^{n} a_i$ is called the *amplitude* of X.

(v) If X is a well-formed quasi-smooth WCI, the space of global sections of $\mathcal{O}_X(k)$ can be computed from the homogenous coordinate ring of X. More precisely, let $A = \mathbb{C}[x_0, \ldots, x_n]/(f_1, \ldots, f_c)$ be the homogenenous coordinate ring of X, A_k its k-graded part. Then,

$$H^0(X, \mathcal{O}_X(k)) \simeq A_k.$$

As a consequence, unlike the standard case, the class of WCI in $\mathbb{P}(a_0, \ldots, a_n)$ of multidegree $(d_1, \ldots, d_c)$ is not necessarily non-trivial. In fact, it is non-trivial only if there are polynomials $f_1, \ldots, f_c$ of weighted degrees $d_1, \ldots, d_c$; in other words, each degree d_i must be a non-negative linear combination of the weights $a_0, \ldots, a_n$.

These properties will be fundamental in translating geometrical statements about well-formed quasi-smooth WCIs into purely numerical problems.

Remark 1.1.11. For the most part, we will suppose that the WCI we work with are *not* linear cones: in fact, a general WCI

$$X = X_{d_1,\ldots,d_c} \subset \mathbb{P}(a_0, \ldots, a_n)$$

with $d_1 = a_0$ is isomorphic to

$$X' = X'_{d_2,\ldots,d_c} \subset \mathbb{P}(a_1, \ldots, a_n).$$

This is a consequence of the fact that, if $x_0, \ldots, x_n$ are the projective coordinates of $\mathbb{P}(a_0, \ldots, a_n)$, the general equation of degree d_1 defining X is of the form $f = x_0 + g$, where g is homogeneous of degree d_1 in the other variables.

In [PST17], it was proved that there are necessary and sufficient numerical conditions for a general WCI X which is not a linear cone to be quasi-smooth. While we will not use the result in its general form, it gives a complete generalisation of previous partial results such as [Ian00, Chapter 8] and [Che15, Proposition 2.3].

Proposition 1.1.12 ([PST17], Proposition 3.1)**.** *Let $X = X_{d_1,\ldots,d_c} \subset \mathbb{P}(a_0, \ldots, a_n)$ be a quasi-smooth WCI which is not a linear cone. For a subset $I = \{i_1, \ldots, i_k\} \subset \{0, \ldots, n\}$ let $\rho_I = \min\{c, k\}$, and for a k-tuple of natural numbers $m = (m_1, \ldots, m_k)$ write $m \cdot a_I = \sum_{j=0}^{k} m_j a_{i_j}$. Then, one of the following conditions holds.*

(Q1) There exist distinct integers $p_1, \ldots, p_{\rho_I} \in \{1, \ldots, c\}$ and k-tuples $M_1, \ldots, M_{\rho_I} \in \mathbb{N}^k$ such that $M_j \cdot a_I = d_{p_j}$ for $j = 1, \ldots, \rho_I$.

(Q2) Up to a permutation of the degrees, there exist:

- *an integer $l < \rho_I$,*

– *integers* $e_{\mu,r} \in \{0,\dots,n\} \setminus I$ *for* $\mu = 1,\dots,k-l$ *and* $r = l+1,\dots,c$,

– *k-tuples* $M_1,\dots,M_l$ *such that* $M_j \cdot a_I = d_j$ *for* $j = 1,\dots,l$,

– *for each* r, *k-tuples* $M_{\mu,r}$, $\mu = 1,\dots,k-l$ *such that* $a_{e_{\mu,r}} + M_{\mu,r} \cdot a_I = d_r$,

satisfying the following property: for any subset $J \subset \{l+1,\dots,c\}$,

$$|\{e_{\mu,r} : r \in J, \mu = 1,\dots,k-l\}| \geq k-l+|J|-1.$$

Conversely, if for all subsets $I \subset \{0,\dots,n\}$ *either (Q1) or (Q2) holds, then a general WCI* $X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0 \dots, a_n)$ *is quasi-smooth.*

A particular case for which Proposition 1.1.12 takes a simpler form is the case of hypersurfaces:

Corollary 1.1.13 ([Ian00, Theorem 8.1]). *Let* $X_d \subset \mathbb{P}(a_0,\dots,a_n)$ *be a general hypersurface of degree* d. *Then,* X_d *is quasi-smooth if and only if, for any subset* $I = \{i_1,\dots,i_k\} \subset \{0,\dots,n\}$, *one of the following holds.*

(Q1) There exist non-negative integers $m_1,\dots,m_k$ *such that* $d = \sum m_j a_{i_j}$.

(Q2) For each $\mu = 1,\dots,k$, *there exist non-negative integers* $m_{1,\mu},\dots,m_{k,\mu}$ *and a weight* a_μ *such that*

$$d = a_\mu + \sum m_{j,\mu} a_{i_j},$$

where the weights a_μ *are all distinct.*

Example 1.1.14.

- The hypersurface $X_{40} \subset \mathbb{P}(5,7,10)$ is quasi-smooth: for the subset $I = \{1\}$ (that is, $a_1 = 7$) (Q1) is not satisfied since 7 does not divide 40, but (Q2) holds because $40 = 5 + 5 \times 7$. For all other subsets, (Q1) holds.

- $X_{20} \subset \mathbb{P}(5,7,10)$ is not quasi-smooth, because for $I = \{1\}$ neither (Q1) nor (Q2) holds.

The main takeaway from Proposition 1.1.12 is that quasi-smoothness of a general WCI X can be checked solely on degrees, weights and their numerical relations. At the same time, quasi-smoothness implies strong constraints on the degrees and weights appearing; the following corollary is a display of this fact, and will be fundamental for our approach to the main problem.

Corollary 1.1.15 ([PST17, Proposition 3.6]). *Let* $X = X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0,\dots,a_n)$ *be a quasi-smooth well-formed WCI which is not a linear cone. Suppose* $\mathrm{Pic}(X)$ *is generated by* $\mathcal{O}(h)$, $h > 0$. *For any subset* $I = \{i_1,\dots,i_k\} \subset \{0,\dots,n\}$ *such that* $a_I = \gcd(a_{i_1},\dots,a_{i_k}) > 1$, *one of the following holds.*

(i) There exist distinct integers $p_1, \ldots, p_k$ such that $a_I \mid d_{p_1}, \ldots, d_{p_k}$.

(ii) $a_I \mid h$.

Remark 1.1.16. In the original proof of Corollary 1.1.15, it is wrongly stated that for a subset $I = \{i_1, \ldots, i_k\} \subset \{0, \ldots, n\}$, Proposition 1.1.12 (Q1) implies (i) even when $|I| > c$, which does not make sense. Still, under this assumption on I we are in the second case: when $k > c$, by a dimension argument X must intersect the linear space generated by $P_{i_1}, \ldots, P_{i_k}$, which is singular of index a_I, hence $a_I \mid h$.

1.1.3 h-regular pairs

Corollary 1.1.15 leads to the following definition:

Definition 1.1.17. Let $c, n \in \mathbb{N}$, $d_1, \ldots, d_c$, $a_0, \ldots, a_n \geq 1$ be natural numbers, and write $(d; a) = (d_1, \ldots, d_c; a_0, \ldots, a_n)$. Let $\bar{c} = \{1, \ldots, c\}$, $\bar{n} = \{0, \ldots, n\}$. We say that $(d; a)$ is *h-regular* for a positive integer h if, for any non-empty subset $I = \{i_1, \ldots, i_k\} \subset \bar{n}$ such that $a_I := \gcd(a_{i_1}, \ldots, a_{i_k}) > 1$, at least one of the following holds.

(i) There exist k distinct integers $p_1, \ldots, p_k \in \bar{c}$ such that $a_I \mid d_{p_1}, \ldots, d_{p_k}$.

(ii) $a_I \mid h$.

If $h = 1$, we say the pair is *regular*. In analogy to the geometrical setting, we call the integers d_i the *degrees* of the pair and c the *codimension*, the integers a_j the *weights* and n the *dimension*, and h the *regularity index*.

Note that the regularity index is not unique, as if a pair $(d; a)$ is h-regular, it is also h'-regular for any $h' > h$, $h \mid h'$. While a minimal index exists and is uniquely determined, in some cases it is useful to allow h to be not minimal (see for example Lemma 1.1.21).

Remark 1.1.18. By Property 1.1.10 and Corollary 1.1.15, if $(d; a)$ are degrees and weights of a well-formed quasi-smooth WCI X which is not a linear cone, and $\dim X > 2$, then $(d; a)$ is a h-regular pair, where $\mathcal{O}(h)$ is a positive generator of $\operatorname{Pic} X$. On the other hand, it is very easy to find regular pairs which do not come from a well-formed quasi-smooth WCI.

Example 1.1.19.

- $(15, 6, 1; 2, 5)$ is regular but does not come from a WCI, as there are more degrees than weights.
- $(60, 2; 4, 5, 6)$ is regular but cannot come from a WCI, as there is a degree smaller than any weight.

- $(30, 14; 6, 7, 10)$ is regular but neither (Q1) nor (Q2) hold: for $I = \{1, 2\}$ (that is, $a_1 = 6, a_2 = 10$), (Q1) does not hold because 14 is not a combination of 6 and 10, while (Q2) is not satisfied because neither 30 nor 14 can be written as $7 + 6m_1 + 10m_2$ for some non-negative integers m_1, m_2.

Notation 1.1.20. Let $(d; a)$ be a h-regular pair.

- We write $|d| = c$ (resp. $|a| = n$) if $d \in \mathbb{N}^c$ (resp. $a \in \mathbb{N}^n$). For integers d_i and a_j, we write $d_i \in d$ (resp. $a_j \in a$) if d_i appears in d (resp. a_j appears in a).
- For a pair $(d; a)$ with $|d| = c$, $|a| = n$, we define

$$\delta(d; a) := \sum_{i=0}^{c} d_i - \sum_{j=0}^{n} a_j,$$

  and call it the amplitude of the pair. If the pair comes from a well-formed quasi-smooth WCI X of dimension > 2, then $K_X \simeq \mathcal{O}(\delta(d; a))$ by Property 1.1.10(iv). When the pair is clear from the context, we will simply write δ for $\delta(d; a)$.

- Let g be a positive integer. Write $I_g = \{i \in \bar{n} \colon g \mid a_i\}$, $J_g = \{j \in \bar{c} \colon g \mid d_j\}$. We can construct two new types of pairs from $(d; a)$:

  - $(d^g; a^g)$, where $d^g = ((d_j/g)_{j \in J_g}, (d_j)_{j \in \bar{c} \setminus J_g})$, $a^g = ((a_i/g)_{i \in I_g}, (a_i)_{i \in \bar{n} \setminus I_g})$, obtained by dividing all divisible degrees and weights by g;
  - $(d(g), a(g))$, where $d(g) = (d_j)_{j \in J_g}$, $a(g) = (a_i)_{i \in I_g}$, obtained by only considering degrees and weights divisible by g.

  When the pair is clear from the context, we will write $\delta(g) = \delta(d(g); a(g))$ and $\delta^g = \delta(d^g; a^g)$.

The following Lemma is useful to use the previous subpairs for inductive purposes:

Lemma 1.1.21 ([PST17, Lemma 4.5 and 4.6])**.**

- *Let $(d; a)$ be a h-regular pair and p a prime not dividing h. Then the pairs $(d(p); a(p))$, $(d(p)/p; a(p)/p)$ and $(d^p; a^p)$ are h-regular;*
- *Let $(d; a)$ be a h-regular pair and p a prime dividing h. Then $(d(p); a(p))$ is h-regular, while $(d(p)/p; a(p)/p)$ and $(d^p; a^p)$ are h/p-regular.*

Corollary 1.1.22. *Let $(d; a)$ be a h-regular pair, and $g = \gcd(a_{i_1}, \ldots, a_{i_m}) > 1$ for some weights $a_{i_1}, \ldots, a_{i_m}$. Then, $(d(g); a(g))$ is h-regular and $(d(g)/g; a(g)/g)$ is h/g-regular.*

Proof. Let $g = \prod p_i^{k_i}$, with p_i prime numbers. We prove the statement on $(d(g); a(g))$ by induction on $k = \sum k_i$; the base case $k = 1$ is Lemma 1.1.21. Now suppose the statement is true up to $\sum k_i = k - 1$. By the same Lemma, the pair $(d'; a') = (d(p_1)/p_1; a(p_1)/p_1)$ is h-regular if $p_1 \nmid h$, or h/p_1-regular if $p_1 \mid h$. Since $p_1 \mid g$, $a_{i_1}/p_1, \dots, a_{i_m}/p_1 \in a'$ and $g' = \gcd(a_{i_1}/p_1, \dots, a_{i_m}/p_1) = g/p_1$. By induction, $(d'(g'); a'(g'))$ is either h-regular or h/p_1 regular as before. Since $(d(g); a(g)) = p_1(d'(g'); a'(g'))$, we get that $(d(g); a(g))$ is h-regular.

The statement on $(d(g)/g; a(g)/g)$, follows directly from the fact that $(d(g); a(g))$ is h-regular, and repeatedly applying Lemma 1.1.21 for every prime p dividing h. $\square$

Remark 1.1.23.

- Even if in Lemma 1.1.21 we take h to be the minimal regularity index of $(d; a)$, it is possible for the index of the subpairs to be smaller than h or h/p. For example, the pair $(d; a) = (140, 63; 4, 7, 10, 21)$ is 2-regular but $(d(7); a(7)) = (140, 63; 7, 21)$ is regular. Similarly, the pair $(d; a) = (14, 50, 60; 6, 7, 10, 12)$ is 6-regular, but $(d^3; a^3) = (14, 50, 20; 2, 7, 10, 4)$ is regular.

- A statement similar to Corollary 1.1.22 is false for pairs of the form $(d^g; a^g)$: $(d; a) = (630, 126, 14; 7, 10, 18, 42)$ is regular, but $(d^6; a^6) = (105, 21, 14; 7, 10, 3, 7)$ is 10-regular.

1.2 The Ambro-Kawamata conjecture and the Frobenius problem

The geometrical conjecture that we want to investigate is the following.

Conjecture 1.2.1 (Ambro-Kawamata; [Kaw00, Conjecture 2.1])**.** *Let (X, Δ) be a klt pair, H an ample Cartier divisor such that $H - K_X - \Delta$ is ample. Then, $H^0(X, H) \neq 0$.*

In the case of X a well-formed quasi-smooth WCI of dimension at least 3, Property 1.1.10 can be used to translate the conjecture (in the case with empty boundary) into a purely numerical problem. In fact, under these assumptions let $\mathcal{O}(h)$ be a generator of $\mathrm{Pic}(X)$, then H is an ample Cartier divisor if $H = \mathcal{O}(k)$ for $k > 0$, $h \mid k$; also, $H - K_X$ is again ample if $k - \delta(d; a) > 0$. Then, the equivalent numerical conjecture is the following.

Conjecture 1.2.2. *Let $(d; a) = (d_1, \dots, d_c; a_0, \dots, a_n)$ be the pair of degrees and weights of a well-formed quasi-smooth WCI X which is not a linear cone with* $\dim X > 2$*, and $\mathcal{O}(h)$ a positive generator of* $\mathrm{Pic}X$*. Let $\delta(d; a)$ be the amplitude of X and k*

a positive integer such that $h \mid k$ and $k > \delta(d;a)$. Then, there exist natural numbers $x_0, \dots, x_n$ such that

$$\sum_{i=0}^{n} x_i a_i = k.$$

In [PST17], the conjecture was proved for Fano and Calabi-Yau well-formed quasi-smooth WCIs which are not linear cones. In these cases, since $H - K_X$ is automatically ample, the conjecture amounts to stating that some weight divides h. This is done in the more general setting of h-regular pairs, for which the following is proved.

Proposition 1.2.3 ([PST17, Proposition 5.12, Corollary 5.13])**.** *Let $(d;a)$ be a h-regular pair. if $a_i \nmid h$ for any $i = 0, \dots, n$, then $\delta(d;a) > 0$. Equivalently, if $\delta(d;a) \leq 0$ then there exists a weight a_i such that $a_i \mid h$.*

The case where X is of general type is more subtle. To fully grasp the content of the conjecture, we first give two definitions.

Definition 1.2.4. Let $m_1, \dots, m_k$ be positive natural numbers with $\gcd(m_1, \dots, m_k) = 1$, and $S = \langle m_1, \dots, m_k \rangle$ the monoid generated by $m_1, \dots, m_k$ by addition. Then, the set $G(S) = \mathbb{N} \setminus S$ is finite, so we can define the *Frobenius number* of $m_1, \dots, m_k$ (or equivalently, of S) as

$$F(S) = F(m_1, \dots, m_k) = \max(\mathbb{Z} \setminus S).$$

In other words, the Frobenius number of $m_1, \dots, m_k$ is the largest integer which cannot be written as a non-negative linear combination of $m_1, \dots, m_k$. When $m_i \neq 1$ for every i, $F(S) \in G(S)$; otherwise, $F(S) = -1$.

Definition 1.2.5. Let $m_1, \dots, m_k$ as in Definition 1.2.4. The $\frac{1}{h}$-*Frobenius number* of $m_1, \dots, m_k$ (or of S) is defined as

$$F^h(S) = F^h(m_1, \dots, m_k) = \max(h\mathbb{Z} \setminus (h\mathbb{Z} \cap S)).$$

Not much differently from before, the $\frac{1}{h}$-Frobenius number is the largest multiple of h which cannot be written as a combination of $m_1, \dots, m_k$, and if $m_1, \dots, m_k \nmid h$ $F^h(S) \in G(S)$, otherwise $F^h(S) = -h$.

By abuse of notation, we still talk about the Frobenius number (or $\frac{1}{h}$-Frobenius number) when $g = \gcd(m_1, \dots, m_k) > 1$, even though it is, in general, not well defined; in that case, we mean the following: if $\gcd(g, h) = 1$, then

$$F^h(m_1, \dots, m_k) = gF^h(m_1/g, \dots, m_k/g).$$

while if $g \mid h$,

$$F^h(m_1, \dots, m_k) = gF^{h/g}(m_1/g, \dots, m_k/g).$$

Thus in general, let $G = \gcd(g, h)$, then

$$F^h(m_1, \dots, m_k) = gF^{h/G}(m_1/g, \dots, m_k/g).$$

Given these definitions, we are ready to generalise Conjecture 1.2.2 to h-regular pairs. In [PST17], only the regular case (h=1) was considered, but we will also study the h-regular case.

Conjecture 1.2.6 ([PST17, Conjecture 4.8])**.** *Let $(d_1, \dots, d_c; a_0, \dots, a_n)$ be a regular pair such that $c \leq n$ and $a_i \neq 1$ for any i. Then,*

$$\delta(d; a) \geq F(a_0, \dots, a_n).$$

Note that by definition, in this case we have that $\gcd(a_0, \dots, a_n) = 1$.

Conjecture 1.2.7. *Let $(d_1, \dots, d_c; a_0, \dots, a_n)$ be a h-regular pair, such that $c \leq n$ and $a_i \nmid h$ for any i. Then,*

$$\delta(d; a) \geq F^h(a_0, \dots, a_n)$$

.

Notation 1.2.8. In analogy to the geometrical problem, when computing $F(m_1, \dots, m_n)$, we will call $m_1, \dots, m_n$ weights rather than generators.

Remark 1.2.9.

- By Proposition 1.2.3, under the hypotheses of Conjectures 1.2.6 and 1.2.7, $\delta(d; a) > 0$ so this is in the general type case.

- For the statement of Conjecture 1.2.6 (and consequently, of Conjecture 1.2.7 as well), we ask the condition $c \leq n$ as in the original statement of [PST17], Conjecture 4.8; note that, since the description of Pic$X$ as a cyclic group in Property 1.1.10 only holds when $\dim X \geq 3$, Conjectures 1.2.6 and 1.2.7 correctly generalise Conjecture 1.2.2 only when $\dim X \leq n - 3$, that is $c \leq n - 3$. Still, we expect that the weaker assumption does not change the validity of the conjecture.

- On a similar note, we usually allow pairs with $d_i = a_j$ for some i, j even if the original statements avoid the linear cone case. If for a pair $(d; a)$ we have $\delta(d; a) \geq F^h(a)$, then adding a pair of identical weights and degrees does not change the inequality: in fact, the amplitude of the pair remains the same and the $\frac{1}{h}$-Frobenius number does not increase.

1.2.1 About the Frobenius problem

Before moving to the main sections, we spend a couple words on the Frobenius problem, that is the following.

Problem. Let $S = \langle a_1, \dots, a_n \rangle$ be a numerical semigroup. Compute $F(S)$, the Frobenius number of S.

Despite the apparent simplicity, it turns out that the computation of $F(S)$ is hard under many points of view. First of all, an explicit answer is known for $n = 2$: this is simply $F(a_1, a_2) = [a_1, a_2] - a_1 - a_2$, where $[a_1, a_2] = \mathrm{lcm}(a_1, a_2)$; for larger n, the natural hope of finding an equally nice formula fails due to the following surprising result.

Theorem 1.2.10 ([Cur90])**.** *Let*

$$A = \{(a_1, a_2, a_3) \in \mathbb{N}^3 \mid a_1 < a_2 < a_3, a_1, a_2 \text{ are prime and } a_1, a_2 \nmid a_3\}.$$

Then, there is no nontrivial polynomial $P \in \mathbb{C}[X_1, X_2, X_3, Y]$ such that

$$P(a_1, a_2, a_3, F(a_1, a_2, a_3)) = 0$$

for all $(a_1, a_2, a_3) \in A$. In other words, it is not possible to find a polynomial relation between a_1, a_2, a_3 and $F(a_1, a_2, a_3)$ holding for all semigroups of given embedding dimension n (that is, the cardinality of the minimal set of generators).

This shows that the case $n = 2$ is the exception, and in general no such formula holding for all numerical semigroups of embedding dimension n can be found. Even from a computational point of view, computing Frobenius numbers is notably hard, since it is known to be a NP-hard problem (see [Alf05] for a state of the art of the known computational aspects and algorithms). Still, it is possible to find formulas when the semigroups have a particular structure, for example if a set of generators forms an arithmetic sequence.

Proposition 1.2.11 ([Rob56])**.** *Let m, k be positive integers, then*

$$F(m, m+k, \dots, m+nk) = \left(\left\lfloor \frac{m-2}{n} \right\rfloor + 1\right) m + (k-1)(m-1) - 1.$$

Another approach is to find upper bounds on Frobenius numbers: this is, for example, the case of the following result, which is one of the best known upper bounds. It also computes exactly the Frobenius number of a semigroup if the generators form a sequence with a particular structure.

Proposition 1.2.12 ([Bra42], [BB54])**.** *Let $(a_1, \dots, a_n)$ be coprime positive numbers, and define $g_i = \gcd(a_1, \dots, a_i)$. Then,*

$$F(a_1, \dots, a_n) \leq \sum_{j=2}^{n} \frac{g_{j-1}}{g_j} a_j - \sum_{i=1}^{n} a_i.$$

Equality holds if and only if $a_1, \ldots, a_n$ *form a telescopic sequence, that is* $a_i/g_i \in S_{i-1}$ *for all* $i = 2, \ldots, n$*, where* S_{i-1} *is the semigroup generated by* $a_1/g_{i-1}, \ldots, a_{i-1}/g_{i-1}$.

This bound has a natural relation with regular pairs: in fact, write $d_j = \frac{g_{j-1}}{g_j} a_j$, then the pair $(d_2, \ldots, d_n; a_1, \ldots, a_n)$, is regular. In general, this pair does not achieve the minimal value of δ for a given set of weights. For example, the pair

$$(d; a) = (6p, 6q, pq; 2p, 3p, 2q, 3q)$$

with p, q primes large enough satisfies

$$\delta(d; a) = F(2p, 3p, 2q, 3q) = pq + p + q,$$

but the sequence $2p, 3p, 2q, 3q$ is not telescopic, hence $\delta(d; a) \neq F(2p, 3p, 2q, 3q)$. This motivates the choice of working with regular pairs, rather than with the geometrical problem, as proving cases of Conjecture 1.2.6 can only improve such bound.

1.3 Preliminary results on h-regular pairs

Here we introduce several results which will be used multiple times in the next section.

We start with a simple observation.

Lemma 1.3.1. *Let* $a_0, \ldots, a_n$ *and* a_0' *be positive integers such that* $a_0 \mid a_0'$*. Suppose that* $g := \gcd(a_0, \ldots, a_n) = \gcd(a_0', \ldots, a_n)$*. Then, for any* $h > 0$*,*

$$F^h(a_0', \ldots, a_n) \geq F^h(a_0, \ldots, a_n).$$

.

Proof. First, consider the case $g = 1$. Since $a_0 \mid a_0'$,

$$\langle a_0', \ldots, a_n \rangle \subset \langle a_0, \ldots, a_n \rangle$$

and the statement follows from the definition of $\frac{1}{h}$-Frobenius number.

For the general case, write $G = \gcd(g, h)$. Since by definition

$$F^h(a_0', \ldots, a_n) = gF^{h/G}(a_0'/g, \ldots, a_n/g)$$

and

$$F^h(a_0, \ldots, a_n) = gF^{h/G}(a_0/g, \ldots, a_n/g),$$

the statement follows like before, as $\gcd(a_0'/g, \ldots, a_n/g) = \gcd(a_0/g, \ldots, a_n/g) = 1$. $\square$

Corollary 1.3.2. *If $a_0, \dots, a_n$ are positive integers, then for any positive integer $m > 0$ such that $\gcd(a_0, \dots, a_k, ma_{k+1}, \dots, ma_n) = \gcd(a_0, \dots, a_k, a_{k+1}, \dots, a_n)$ for some $k \geq 0$, we have that for any $h > 0$,*

$$F^h(a_0, \dots, a_k, ma_{k+1}, \dots, ma_n) \geq F^h(a_0, \dots, a_n).$$

The following is a classical result which, in some cases, allows to compute the Frobenius number of a set of weights from the Frobenius number of a set of smaller weights.

Lemma 1.3.3 ([Alf05, Lemma 3.1.7])**.** *Let $a_0, \dots, a_n$ be positive integers with no common divisor, and $g = \gcd(a_0, \dots, a_{n-1})$. Then,*

$$F(a_0, \dots, a_n) = gF(\frac{a_0}{g}, \dots, \frac{a_{n-1}}{g}, a_n) + (g-1)a_n.$$

We also need a lower bound on $\delta(d; a)$ for regular pairs satisfying the hypotheses of Conjecture 1.2.6.

Lemma 1.3.4 ([PST17, Proposition 5.2])**.** *Let $(d; a) = (d_1, \dots, d_c; a_0, \dots, a_n)$ be a regular pair such that $a_i \neq 1$ for any i. Then, $\delta(d; a) \geq c$.*

Now we can show that in the regular case we can suppose that there is no nontrivial factor dividing all the degrees.

Lemma 1.3.5. *Let $(d; a)$ be a regular pair, $|d| = c \leq |a| - 1 = n$. Let $g \coloneqq \gcd(d_1, \dots, d_c) > 1$ and $p \mid g$ a prime dividing g. Suppose $p \mid a_0, \dots, a_k$, $p \nmid a_{k+1}, \dots, a_n$. If*

$$\delta^p = \delta(d^p; a^p) \geq F(a^p) = F(\frac{a_0}{p}, \dots, \frac{a_k}{p}, a_{k+1}, \dots, a_n),$$

then

$$\delta(d; a) \geq F(a_0, \dots, a_n)$$

also holds.

Proof. Note that by regularity, $k + 1 \leq c \leq n$. Then, the statement follows from Corollary 1.3.2 and Lemma 1.3.3.

$$\begin{aligned}
\delta(d; a) &\geq p\delta^p + (p-1)\sum_{i=k+1}^{n} a_i \geq p\delta^p + (p-1)a_n \\
&\geq pF(\frac{a_0}{p}, \dots, \frac{a_{k-1}}{p}, a_k, \dots, a_n) + (p-1)a_n \\
&= F(a_0, \dots, a_{k-1}, pa_k, \dots, pa_{n-1}, a_n) \geq F(a_0, \dots, a_n).
\end{aligned}$$

□

Corollary 1.3.6. *Suppose that Conjecture 1.2.6 holds for pairs $(d^*; a^*)$ such that $\gcd(d_1^*, \dots, d_c^*) = 1$, then it holds for any pair $(d; a)$ such that $\gcd(d_1, \dots, d_c) > 1$ as well.*

Proof. Let $g = \gcd(d_1, \dots, d_c) > 1$, and write $g = \prod p_i^{k_i}$. We show the statement by induction on $k = \sum k_i$; the base case $k = 0$ is given by hypothesis, so suppose that the statement holds for any pair $(d'; a')$ such that $\gcd(d_1', \dots, d_c') = \prod p_i^{k_i'}$ and $\sum k_i' \leq k - 1$.

Let $p \mid g$ a prime dividing g and consider the pair $(d^p; a^p) = (d_1'', \dots, d_c''; a_0'', \dots, a_n'')$ obtained by dividing all divisible degrees and weights by p; then, $g' = \gcd(d_1'', \dots, d_c'') = g/p$.

- If $a_i'' \neq 1$ for any i, $(d^p; a^p)$ satisfies the hypotheses of Conjecture 1.2.6, hence

$$\delta^p = \delta(d''; a'') \geq F(a_0'', \dots, a_n'') = F(a^p)$$

  by the induction step. Then, $\delta(d; a) \geq F(a_0, \dots, a_n)$ by Lemma 1.3.5.

- If $a_i'' = 1$ for some i, then we may assume that $a_0, \dots, a_m = p$ and $a_{m+1}, \dots, a_n \neq p$, where $m \leq c-1$ by regularity. Now, the subpair $(d''; a''') = (d_1'', \dots, d_c''; a_{m+1}'', \dots, a_n'')$ satisfies the hypotheses of Lemma 1.3.4, therefore $\delta(d''; a''') \geq c$. Since $a_0'' = \dots = a_m'' = 1$ and $m \leq c - 1$, this implies that $\delta(d^p; a^p) = \delta(d'; a') \geq 0$; on the other hand, $F(a_0', \dots, a_n') = -1$ because some weight is equal to 1, hence $\delta(d^p; a^p) > F(a_0', \dots, a_n')$, and the statement follows from Lemma 1.3.5.

□

Applying Corollary 1.3.6 iteratively for any prime p dividing $\gcd(d_1, \dots, d_c)$ shows that Conjecture 1.2.6 only needs to be checked on regular pairs such that $\gcd(d_1, \dots, d_c) = 1$.

Next, we introduce a family of recursive bounds on the Frobenius number, which allows us to use induction on the number of weights.

Lemma 1.3.7. *Let $a_0, \dots, a_n$ be coprime positive integers, $g = \gcd(a_0, \dots, a_k)$, G a positive integer coprime with g. Then,*

$$F(a_0, \dots, a_n) \leq F^g(a_0, \dots, a_k) + F^G(a_{k+1}, \dots, a_n, g) + gG.$$

Proof. First of all, note that under the assumptions, $\gcd(g, a_{k+1}, \dots, a_n) = 1$, so that $F^G(a_{k+1}, \dots, a_n, g)$ is well defined. Let $N > F(a_0, \dots, a_k) + F^G(a_{k+1}, \dots, a_n, g) + gG$, then

$$N - F(a_0, \dots, a_k) - F^G(a_{k+1}, \dots, a_n, g) - g - G > gG - g - G.$$

Since $F(g, G) = gG - g - G$, by definition we get that there exist $y_1, y_2 \geq 0$ such that

$$N - F(a_0 \dots, a_k) - F^G(a_{k+1}, \dots, a_n, g) - g - G = y_1 g + y_2 G,$$

and reordering the terms,

$$N - (F(a_0, \ldots, a_k) + (y_1 + 1)g) = F^G(a_{k+1}, \ldots, a_n, g) + (y_2 + 1)G.$$

Again by definition, since

$$F^G(a_{k+1}, \ldots, a_n, g) + (y_2 + 1)G = \sum_{i=k+1}^{n} x_i a_i + yg$$

with $x_i, y \geq 0$, we get

$$N = F(a_0, \ldots, a_k) + (y + y_1 + 1)g + \sum_{i=k+1}^{n} x_i a_i,$$

hence by definition,

$$N = \sum_{j=0}^{k} x_j a_j + \sum_{i=k+1}^{n} x_i a_i$$

for some $x_j \geq 0$. $\square$

Lemma 1.3.8. *Let $a_0, \ldots, a_n$ be coprime positive integers, $g = \gcd(a_0, \ldots, a_k)$ for $k \geq 0$. Then,*

$$F(a_0, \ldots, a_n) \leq F(a_0, \ldots, a_k) + F(a_{k+1}, \ldots, a_n, g) + g.$$

Proof. This is Lemma 1.3.7 when $G = 1$.

$\square$

Lemma 1.3.9. *Let $a_0, \ldots, a_n$ be coprime positive integers, and consider (not necessarily disjoint) non-empty subsets $I_1, \ldots, I_k \subset \{0, \ldots, n\}$. Let $g_j = \gcd_{i \in I_j}(a_i)$ and write a_{I_j} for the set of weights indexed by I_j. Suppose the g_j are coprime. Then,*

$$F(a_0, \ldots, a_n) \leq F(g_1, \ldots, g_k) + \sum_{j=1}^{k} g_j + \sum_{j=1}^{k} F^{g_j}(a_{I_j}).$$

Proof. Let

$$N > F(g_1, \ldots, g_k) + \sum_{j=1}^{k} g_j + \sum_{j=1}^{k} F^{g_j}(a_{I_j}),$$

then,

$$N - \sum_{j=1}^{k} g_j - \sum_{j=1}^{k} F^{g_j}(a_{I_j}) > F(g_1, \ldots, g_k)$$

and by definition,

$$N - \sum_{j=1}^{k} g_j - \sum_{j=1}^{k} F^{g_j}(a_{I_j}) = \sum_{i=1}^{k} y_i g_i.$$

We can rewrite this as

$$N = \sum_{j=1}^{k} (F^{g_j}(a_{I_j}) + (y_j + 1)g_j).$$

By definition for each Frobenius number, we get

$$N = \sum_{j=1}^{k} \sum_{l \in I_j} x_l a_l,$$

for $x_l \geq 0$. $\square$

1.3.1 Reduction to the regular case

While Conjectures 1.2.6 and 1.2.7 do not seem to be equally strong statements, it turns out that in most cases the h-regular conjecture can be reduced to the regular case.

Lemma 1.3.10. *Suppose the Conjecture 1.2.7 holds for h'-regular pairs, where $h' < h$. Then, the conjecture also holds for a h-regular pair $(d; a)$ satisfying any of the following conditions.*

(i) There is a prime p dividing h such that $\bar{\delta}(p) \leq 0$, where $\bar{\delta}(p) = \delta(d; a) - \delta(p)$.

(ii) There is a prime p dividing h such that $\bar{\delta}(p) \geq 0$ and $|d(p)| < |a(p)|$.

(iii) There is a greatest common divisor $g = \gcd(a_{i_1}, \ldots, a_{i_k}) > 1$ such that $\delta(d; a) \geq \delta(d(g); a(g))$ and $|d(g)| < |a(g)|$.

Proof.

(i) Note that $\delta(d; a) = p\delta^p - (p-1)\bar{\delta}(p)$; since $\bar{\delta}(p) \leq 0$, then $\delta(d; a) \geq p\delta^p$. By Lemma 1.1.21, $(d^p; a^p)$ is h/p-regular; $(d^p; a^p)$ still satisfies the conditions of Conjecture 1.2.7, hence by hypothesis $p\delta^p \geq pF^{h/p}(a'_0, \ldots, a'_n)$, where a'_i are the weights of $(d^p; a^p)$. But $pF^{h/p}(a'_0, \ldots, a'_n) = F^h(pa'_0, \ldots, pa'_n) \geq F^h(a_0, \ldots, a_n)$ by construction, hence

$$\delta(d; a) \geq p\delta^p \geq pF^{h/p}(a'_0, \ldots, a'_n) \geq F^h(a_0, \ldots, a_n).$$

(ii) By Lemma 1.1.21, the pair $(d(p)/p; a(p)/p)$ is h/p-regular, and since $|a(p)| > |d(p)|$ by hypothesis, it satisfies the assumptions of Conjecture 1.2.7. Then, $\delta(p)/p \geq F^{h/p}(a(p)/p)$, which implies that

$$\delta(d; a) \geq \delta(p) \geq pF^{h/p}(a(p)/p) = F^h(a(p)) \geq F^h(a_0, \ldots, a_n).$$

(iii) The proof follows as in the previous case, applying Lemma 1.1.21 and Corollary 1.1.22. It follows that $(d(g); a(g))$ is h-regular and $(d(g)/g; a(g)/g)$ is h/g-regular. Then, as before, $\delta(d; a) \geq \delta(g) \geq F^h(a(g))$.

$\square$

It follows that whenever a h-regular pair $(d; a)$ satisfies any of the conditions of Lemma 1.3.10, Conjecture 1.2.7 for $(d; a)$ follows directly if the conjecture holds for any h'-regular pair with $h' < h$. If any h-regular pair always satisfied one of the conditions of the Lemma, it would mean that Conjectures 1.2.6 and 1.2.7 are equivalent. Thus, if the two conjectures are not equivalent, it must be because there exists a pair $(d; a)$ which does not satisfy any of the conditions of Lemma 1.3.10. More precisely, $(d; a)$ satisfies the following property.

Property $(*)$**.** Let $d(k)$ (resp. $a(k)$) be the subset of degrees (resp. weights) divisible by an integer k. Then both of the following hold.

- For any prime $p \mid h$, $\bar{\delta}(p) > 0$ and $|d(p)| \geq |a(p)|$.
- For any $g = \gcd(a_{i_1}, \dots, a_{i_l}) > 1$ such that $|d(g)| < |a(g)|$, $\delta(d; a) < \delta(d(g); a(g))$.

While it is not clear whether a pair satisfying Property $(*)$ exists, for the purposes of studying the conjecture in low codimension we can show that this case cannot happen.

Proposition 1.3.11. *Any h-regular pair $(d; a)$ such that $|d| \in \{1, 2, 3\}$ and satisfying the hypotheses of Conjecture 1.2.7 satisfies at least one of the conditions of Lemma 1.3.10. In other word, $(d; a)$ does not satisfy Property $(*)$.*

Proof. When $|d| = 1, 2$ the statement is straightforward. In fact, if $|d| = 1$ condition (ii) of Lemma 1.3.10 is automatically satisfied for any $p \mid h$; when $|d| = 2$, if $g = \gcd(a_0, a_1) > 1$ and $g \mid d_1, g \nmid d_2$, either $p \mid d_1, d_2$ for some $p \mid g$ (case (i)), or $p \mid d_1, p \nmid d_2$, which is either case (i) or (ii). So we consider the case $|d| = 3$.

Without loss of generality, let $g = \gcd(a_0, \dots, a_k) > 1$ be such that $|d(g)| < |a(g)|$. Notice that we can suppose $|d(g)| = 1$ and $|a(g)| = 2$:

- $|d(g)| = 0$: this cannot happen, as no a_i divides h;
- $|d(g)| = 2$: since $|d(g)| < |a(g)|$, then $3 \leq |a(g)| \leq |a(p)|$ for any $p \mid g$. Then, either case (i) or (ii) of Lemma 1.3.10 holds.
- $|d(g)| = 3$: then $|d(p)| = 3$, and we have case (i).
- $|d(g)| = 1, |a(g)| = 3$: $|a(p)| \geq 3$ for all $p \mid g$, hence either case (i) or (ii) holds.

Then suppose $|d(g)| = 1, |a(g)| = 2$; we can also assume that for any $p \mid g$, $a(g) = a(p)$, otherwise $|a(p)| \geq 3 \geq |d(p)|$ and we are again in case (i) or (ii). Without loss of generality, under these assumptions we have $g \mid a_0, a_1$ and

$$\begin{cases} d_1 = g \cdot k \\ d_2 = g_1 \cdot k_1 \\ d_3 = g_2 \cdot k_2 \end{cases}$$

where $g = g_1 g_2$ for $g_1, g_2 > 1$, $\gcd(g_1, g_2) = 1$, and $k, k_1, k_2 \geq 1$. If Property $(*)$ holds, we know that $\delta - \delta(g) < 0$, which means that $d_2 + d_3 - \sum_{i=2}^{n} a_i < 0$; on the other hand, since $\delta - \delta(p) > 0$ for any $p \mid g_1$, we get that $d_3 - \sum_{i=2}^{n} a_i > 0$, which is a contradiction.

Thus, Property $(*)$ cannot hold. □

Corollary 1.3.12. *Conjectures 1.2.6 and 1.2.7 are equivalent for pairs of codimension at most 3.*

Remark 1.3.13. In higher codimensions, it is not clear whether a pair satisfying Property $(*)$ exists; still, we conjecture that it is nonetheless possible to deduce Conjecture 1.2.7 from a reduction to the regular case.

Conjecture 1.3.14. *Suppose that Conjecture 1.2.6 is true. Then, Conjecture 1.2.7 holds as well.*

1.4 Main results

The goal of this section is to prove Conjecture 1.2.2 when $\operatorname{codim}(X) \leq 3$:

Theorem 1.4.1. *Let $X \subset \mathbb{P}$ be a well-formed quasi-smooth WCI which is not a linear cone,* $\operatorname{codim} X \leq 3$ *and H an ample Cartier divisor such that $H - K_X$ is ample. Then, $|H| \neq \emptyset$.*

As with the Fano and Calabi-Yau case of [PST17], this is done by proving the conjecture in the more general setting of h-regular pairs. Thanks to Proposition 1.3.11, we only need to consider the regular case, that is Conjecture 1.2.6.

Proposition 1.4.2 (cf. [PST17, Proposition 6.2])**.** *Let $(d_1; a_0, \dots, a_n)$ be a h-regular pair, $n > 0$, and suppose $a_i \nmid h$ for all i. Then, $\delta(d_1; a_0, \dots, a_n) \geq F^h(a_0, \dots, a_n)$.*

Proof. The hypersurface case of Conjecture 1.2.7 was already proved in [PST17, Proposition 6.2], but we can now notice that it follows directly from Proposition 1.3.11 by reducing to the regular case. Then, for a regular pair $(d; a) = (d_1; a_0, \dots, a_n)$ of codimension 1, it is easy to see that Conjecture 1.2.6 holds, as all weights must be pairwise coprime, hence $d_1 \geq \prod a_i$ and in particular, $\delta(d;a) \geq F(a_i,a_j)$ for all $a_i, a_j \in a$. □

Proposition 1.4.3. *Let $(d;a) = (d_1, d_2; a_0, \dots, a_n)$, $n \geq 2$ be a regular pair such that $a_i \neq 1$ for every i. Then, $\delta(d;a) \geq F(a_0, \dots, a_n)$.*

Proof. By Lemma 1.3.5, we can suppose that $(a_i, a_j) = 1$ for every $0 \leq i, j \leq n$, $i \neq j$. Up to a permutation of the weights, suppose $a_0, \dots, a_k \mid d_1$, $a_{k+1}, \dots, a_n \mid d_2$, with $k > 1$. Both the pairs $(d';a') = (d_1; a_0, \dots, a_k)$ and $(d'';a'') = (d_2; a_{k+1}, \dots, a_n)$ are regular of codimension 1, thus $\delta(d'';a'') > 0$. Then, by Proposition 1.4.2,

$$\delta(d;a) \geq \delta(d';a') \geq F(a_0, \dots, a_k) \geq F(a_0, \dots, a_n).$$

□

From Proposition 1.3.11, we obtain the generalisation to h-regular pairs of codimension 2.

Corollary 1.4.4. *For any h-regular pair $(d;a) = (d_1, d_2; a_0, \dots, a_n)$, $n \geq 2$ such that $a_i \nmid h$ for any i, $\delta(d;a) \geq F^h(a_0, \dots, a_n)$. In particular, Conjecture 1.2.7 holds for $c = 2$.*

Proposition 1.4.5. *Let $(d;a) = (d_1, d_2, d_3; a_0, \dots, a_n)$ be a regular pair such that $n \geq 3$ and $a_i \neq 1$ for all i. Then, $\delta(d;a) \geq F(a_0, \dots, a_n)$.*

Proof. We can suppose that $d_i \neq a_j$ for any i, j (otherwise this reduces to the case of codimension 2) and by Lemma 1.3.5 we can only consider the case $\gcd(a_{i_1}, a_{i_2}, a_{i_3}) = 1$ for all distinct i_1, i_2, i_3. For any degree $d_j \in d$, let $A_j = \{a_i \in a \colon a_i \mid d_j\}$ be the set of weights dividing d_j. Define the pairs $(d';a') = (d_2, d_3; a_l, \dots, a_n)$ where $a_i, \dots, a_{l-1} \in A_1$ and $a_l, \dots, a_n \notin A_1$, and $(d'';a'') = (d_2, d_3; a_2, \dots, a_n, g)$, where $g = \gcd(a_0, a_1)$. By our assumptions, both $(d';a')$ and $(d'';a'')$ are d_1/m-regular, where $m = \operatorname{lcm}\{a_i \in A_1\}$, as a consequence of the fact that any three weights do not share any common factor; we write $\delta' = \delta(d';a')$ and $\delta'' = \delta(d'';a'')$. Also, $\delta(d';a') > 0$ by Proposition 1.2.3 because $a_k, \dots, a_n \nmid d_1$.

For the rest of the proof, we will use the convention that if $a_i \mid a_k$ for some i, k, then a_i and a_k belong to distinct sets A_j. More precisely, even though a_i and a_k must belong to at least one common A_j, since there is at least another $A_{j'}$ such that $a_i \in A_{j'}$, we will say that $a_i \in A_{j'}$ and $a_k \in A_j$, but $a_i \notin A_j$ and $a_k \notin A_{j'}$. Note that the regularity of $(d';a')$ and $(d'';a'')$ is unchanged by this convention.

We prove the statement by induction on $k = \min_j\{|A_j| \colon |A_j| > 1\}$. This is well defined, because the pair is regular and since $|a| > |c|$, there must be two weights dividing the same degree. Without loss of generality, we will always suppose that $k = |A_1|$, and that the weights belonging to A_1 are $a_0, \dots, a_{k-1}$. We prove each part of induction case-by-case; whenever there are two weights satisfying the condition of a case, we can suppose that, up to permutation, they are a_0 and a_1, and that there are not other weights satisfying the assumptions of the previous cases. We first prove the statement under the assumption that $d_1 = [a_0, \dots, a_{k-1}]$, where $[a_0, \dots, a_{k-1}] = \mathrm{lcm}(a_0, \dots, a_{k-1})$, then show how the proof generalises to the (easier) case $d_1 > [a_0, \dots, a_{k-1}]$.

- $\boldsymbol{k = 2}$ **:**

  - $g = \gcd(a_0, a_1) = 1$: then,

    $$\delta(d; a) \geq ([a_0, a_1] - a_0 - a_1) + \delta' > F(a_0, a_1),$$

    and we are done because a_0 and a_1 are coprime.

  - $g > 1$: write $\delta(d; a) = (d_1 - a_0 - a_1) + \delta(d''; a'') + g$. Since $|a''| = n > |d''| = 2$, by Corollary 1.4.4

    $$\delta(d''; a'') \geq F(a_2, \dots, a_n, g),$$

    therefore

    $$\delta(d; a) \geq ([a_0, a_1] - a_0 - a_1) + \delta(d''; a'') + g \geq F(a_0, \dots, a_n)$$

    by Lemma 1.3.8.

- $\boldsymbol{k = 3}$ **:**

  - $[a_0, a_1] = [a_0, a_2] = [a_1, a_2] = [a_0, a_1, a_2] = d_1$: in this case, since

    $$[a_0, a_1, a_2] = \frac{a_0 a_1 a_2}{\gcd(a_0, a_1)\gcd(a_0, a_2)\gcd(a_1, a_2)}$$

    and

    $$[a_i, a_j] = \frac{a_i a_j}{\gcd(a_i, a_j)},$$

    we get

    $$\begin{cases} a_0 = \gcd(a_0, a_1)\gcd(a_0, a_2) \\ a_1 = \gcd(a_0, a_1)\gcd(a_1, a_2) \\ a_2 = \gcd(a_0, a_2)\gcd(a_1, a_2) \end{cases}$$

    Two among $\gcd(a_0, a_1), \gcd(a_0, a_2), \gcd(a_1, a_2)$ (which are $\neq 1$ by the convention on the weights) must divide one of d_2 or d_3, hence also one of a_0, a_1, a_2 divides d_2 or d_3, say a_2. Then, $(d''; a'')$ is regular and the proof follows as in the case $k = 2$.

  - $[a_0, a_1] \neq d_1$:

* $g = \gcd(a_0, a_1) = 1$: then $d_1 \geq [a_0, a_1] + a_2$, hence

$$\delta(d; a) \geq [a_0, a_1] - a_0 - a_1 + \delta(d'; a') > F(a_0, a_1).$$

* $g > 1$, $ga_2 \neq d_1$: in this case, $d_1 \geq [a_0, a_1] + ga_2$, hence

$$\delta(d; a) > ([a_0, a_1] - a_0 - a_1) + (ga_2 - a_2 - g) + g = F(a_0, a_1) + F(a_2, g) + g$$

and we are done by Lemma 1.3.8.

For the cases $g > 1$, $ga_2 = d_1$, we can reduce to a pair with $n = 3$. In fact, when $|a^*| = n - 1 > 2$, by Corollary 1.4.4 $\delta(d^*; a^*) \geq F(a^*)$, where $(d^*; a^*) = (d_2, d_3; a_3, \dots, a_n, g)$ is regular. Since $d_1 \geq [a_0, a_1] + a_2$, we get

$$\delta(d; a) > F(a_0, a_1) + F(a^*) + g \geq F(a_0, \dots, a_n)$$

by Lemma 1.3.8. Then, suppose $n = 3$. We have the following three cases (up to permutations):

* $g, a_3 \mid d_2$: since $d_1 \geq [a_0, a_1] + a_2$,

$$\delta(d; a) > [a_0, a_1] - a_0 - a_1 + (ga_3 - g - a_3) + g = F(a_0, a_1) + F(a_3, g) + g$$

, and we get the result from Lemma 1.3.8.

* $a_3 \mid d_2$, $g \mid d_3$, $g_0 \mid d_3$ where $g_i = a_i/g$: since $a_0 = gg_0$, $a_0 \mid d_3$ and we can conclude by induction by noticing that

$$\delta(d; a) = \delta(d_1; a_1, a_2) + \delta(d_2, d_3; a_0, a_3, g) + g$$

and $(d_2, d_3; a_0, a_3, g)$ is regular, hence we can use Lemma 1.3.8.

* $g_0, g_1, a_3 \mid d_2$, $g = d_3$ (if $g < d_3$, d_3 can be divided and the new pair is still regular): the pair $(d^*; a^*) = (d_1, d_2; a_1, a_2, a_3)$ is still regular, and by induction

$$\delta(d^*; a^*) \geq F(a_1, a_2, a_3).$$

Since $\delta(d; a) = \delta(d^*; a^*) + d_3 - a_0$ and $d_3 = g \mid a_0$, then $\delta(d; a) \geq F(a_1, a_2, a_3)$.

- ***$k > 3$* :**

  - All weights dividing d_1 are pairwise coprime: then, $(d_1; a_0, \dots, a_{k-1})$ is regular and the statement follows directly from Proposition 1.4.2.

  - $\gcd(a_0, a_1) = g > 1$ and $[a_0, a_1] \neq d_1$: then $d_1 \geq [a_0, a_1] + d_1/g$ and the pair $(d^*; a^*) = (d_1/g, d_2, d_3; a_2, \dots, a_n, g)$ is again regular. If $k \geq 5$ (which implies $n \geq 4$), then $|a^*| \geq 3$ and we can use induction to say that

$$\delta(d^*; a^*) \geq F(a_2, \dots, a_n),$$

in which case

$$\delta(d;a) \geq ([a_0,a_1] - a_0 - a_1) + \delta(d^*;a^*) + g \geq F(a_0,\dots,a_n)$$

by Lemma 1.3.8.

Otherwise $k = n+1 = 4$, and let $g' = \gcd(a_2,a_3)$; note that $[a_0,a_1] \leq d_1/g'$ (and $\neq d_1$ by hypothesis), $[a_2,a_3] \leq d_1/g \neq d_1$, and if $g' = 1$ then the statement follows easily, because

$$\delta(d;a) \geq ([a_2,a_3] - a_2 - a_3) + ([a_0,a_1] - a_0 - a_1) + \delta(d';a') > F(a_2,a_3).$$

Thus, we can suppose $g' > 1$, which implies

$$[a_0,a_1] + [a_2,a_3] \leq \frac{d_1(g+g')}{gg'} \leq \frac{5}{6}d_1.$$

If we can show that $gg' \leq \frac{1}{6}d_1$ we are done, because then

$$d_1 \geq [a_0,a_1] + [a_2,a_3] + gg',$$

and

$$\delta(d;a) \geq F(a_0,a_1) + F(a_2,a_3) + gg',$$

hence $\delta(d;a) \geq F(a_0,a_1,a_2,a_3)$ by Lemma 1.3.7. Since neither of a_0 and a_1 divides the other, there must be coprime numbers $q_0,q_1 > 1$ such that $a_0 = gq_0, a_1 = gq_1$, thus

$$gg' \leq \frac{d_1}{q_0q_1} \leq \frac{1}{6}d_1.$$

Hence, $d_1 \geq [a_0,a_1] + [a_2,a_3] + gg'$ and we get the statement.

- $\gcd(a_0,a_1) = g > 1$ and $[a_0,a_1] = d_1$:

We want to show that $d_1 \geq a_0 + a_1 + d_1/g$. Write $a_0 = gg_0$, $a_1 = gg_1$ for $g_0,g_1 > 1$ since any three weights do not have any common factor, g,g_0,g_1 are all distinct), then

$$a_0 + a_1 + \frac{d_1}{g} = \frac{d_1}{g_1} + \frac{d_1}{g_0} + \frac{d_1}{g} = d_1(\frac{1}{g_0} + \frac{1}{g_1} + \frac{1}{g}).$$

While it is possible that g and g_0 or g_1 share a common factor (call it q), in that case $(d_1/qg; a_2,\dots,a_{k-1})$ is regular because q cannot divide any weight among $a_2,\dots,a_{k-1}$. There are very few values of g,g_0,g_1 satisfying the previous assumptions and such that

$$\frac{1}{g_0} + \frac{1}{g_1} + \frac{1}{g} < 1,$$

but they force a_2 and a_3 to be primes dividing a_0 or a_1 (because if at least one of g_0 and g_1 is greater than 5, then $1/g_0 + 1/g_1 + 1/g \geq 1$), against our convention on the weights. Thus, we always have

$$d_1 \geq a_0 + a_1 + \frac{d_1}{g}.$$

Since no two weights among $a_2, \dots, a_k$ have a common factor (because $[a_0, a_1] = d_1$), $(d_1/g; a_2, \dots, a_k)$ is regular, hence

$$\frac{d_1}{g} - a_2 - \dots - a_k \geq F(a_2, \dots, a_k).$$

We can now use the fact that $d_1 \geq a_0 + a_1 + \frac{d_1}{g}$ to obtain the statement.

For $d_1 > [a_0, \dots, a_{k-1}]$, the same proofs still work verbatim except when the regularity of $(d''; a'')$ is used. But if $d_1 > [a_0, \dots, a_{k-1}]$, then $d_1 \geq [a_0, \dots, a_{k-1}] + mg$, where $g = \gcd(a_0, a_1)$ and $m = d_1/[a_0, \dots, a_{k-1}]$. Then, a similar proof holds by Lemma 1.3.7. For the sake of clarity, we give an example by showing how the case $k = 2$ generalises.

Suppose $g = \gcd(a_0, a_1) > 1$ and $d_1 > [a_0, a_1]$. Then $mg < d_1$ ($mg = d_1$ is excluded by our convention on the weights, as it corresponds to $a_0 = a_1 = g$); hence $d_1 \geq [a_0, a_1] + mg$ and

$$\delta(d; a) > ([a_0, a_1] - a_0 - a_1) + F^m(a_2, \dots, a_n, g) + mg.$$

The result now follows from Lemma 1.3.7.

□

Corollary 1.4.6. *For any h-regular pair $(d; a) = (d_1, \dots, d_c; a_0, \dots, a_n)$ such that $c \leq 3$ and $c \leq n$, $\delta(d; a) \geq F(a_0, \dots, a_n)$. In particular, Conjecture 1.2.6 holds for $c \leq 3$.*

In particular, thanks to Proposition 1.3.11, we have proved the general case as well:

Corollary 1.4.7. *For any h-regular pair $(d; a) = (d_1, \dots, d_c; a_0, \dots, a_n)$ such that $c \leq 3$ and $c \leq n$, Conjecture 1.2.7 holds.*

Corollary 1.4.8 (=Theorem 1.4.1)**.** *Let $X \subset \mathbb{P}$ be a well-formed quasi-smooth WCI which is not a linear cone, such that* codim$X \leq 3$. *Let H be an ample Cartier divisor on X such that $H - K_X$ is ample, then $|H| \neq \emptyset$.*

Writing the previous results from a different point of view, we also obtain the following bound on Frobenius numbers:

Corollary 1.4.9. *Let $a_0, \dots, a_n$ be coprime positive integers, then*

$$F(a_0, \dots, a_n) \leq \delta(d; a),$$

where $(d; a) = (d_1, \dots, d_c; a_0, \dots, a_n)$ is any regular pair such that $c \leq 3$ and $n \geq c$.

1.5 Final remarks

We end the chapter with some observations on the minimality of δ for a given set of weights: note that for any set of weights $a = (a_0, \dots, a_n)$, the set

$$\Delta_a = \{\delta(d; a) \in \mathbb{Z} \mid (d; a) \text{ is a regular pair and } |d| < |a|\}$$

admits a minimum, since $\delta(d; a) > 0$ for any regular pair by Lemma 1.3.4; we say that a pair $(d; a)$ with $\delta(d; a) = \min \Delta_a$ is *minimal*. It is then natural to study which properties distinguish the degrees of any minimal pair. A first observation that can be made is that such a pair must be irreducible, in some sense.

Definition 1.5.1. A regular pair $(d; a)$ is *reducible* if there exists a degree d_i and a prime $p \mid d_i$ such that the pair

$$(d'; a') = (d_1, \dots, d_i/p, \dots, d_c; a_0, \dots, a_n)$$

is still regular.

In fact, if a regular pair $(d; a)$ is reducible, there is another pair $(d'; a)$ such that $\delta(d'; a) \leq \delta(d; a)$, obtained by replacing a reducible degree d_i with d_i/p.

A harder problem is understanding if there is any constraint on the codimension of any minimal pair. Since the degrees can be, on average, smaller the more degrees a pair has, a naive guess is that a minimal pair must have maximal codimension, that is $|d| = |a| - 1$. While it is hard to give a complete answer, we can notice two facts that support this idea.

- Suppose that for a set of weights $a = (a_0, \dots, a_n)$, the minimal pair has $(d; a) =$ has codimension $c = |d| < n$. Then, there is a pair $(d'; a)$ of maximal codimension which is, in a sense, almost minimal: in fact, consider the pair

  $$(d'; a) = (d_1, \dots, d_c, 1, \dots, 1; a_0, \dots, a_n),$$

  where d' has $n - c$ degrees equal to 1. Then, $\delta(d'; a) = \delta(d; a) + n - c$. Hence, even if $(d'; a)$ is not a minimal pair, it is close to being one.

- Suppose again that $c < n$. If there is a prime p such that $(d(p); a(p))$ is reducible, then there is a regular pair $(d'; a)$ such that $\delta(d'; a) \leq \delta(d; a)$ and $|d'| = c+1$. In fact, suppose we can reduce the degree d_1 in $d(p)$ by some prime q. Then, the pair $(d'; a) = (d_1/p, d_1/q, d_2, \dots, d_c; a_0, \dots, a_n)$ is regular and $\delta(d'; a) \leq \delta(d; a)$. To see that $(d'; a)$ is still regular, let $g = \gcd(a_{i_1}, \dots, a_{i_k}) > 1$ for some weights $a_{i_1}, \dots, a_{i_k}$. We only need to check the cases of p or q dividing g. First, suppose $p \mid g$. Then, g divides at least k degrees because it does in the reduced pair

$(d'(p); a(p))$ by assumption. If $q \mid g$, $p \nmid g$, g still divides d_1/p, hence divides at least k degrees. Therefore, $(d'; a)$ is indeed regular. This means that for $c < n$, if there exists some prime p such that the pair $(d(p); a(p))$ is reducible, then $\delta(d; a)$ is not minimal.

Based on the previous observations, we end with the following questions:

Question 1.5.2.

- For any regular pair $(d; a)$ with $|d| < |a| - 1$, is there a prime p such that the pair $(d(p); a(p))$ is reducible?
- If the answer to the previous question is false, for a given set of weights $a = (a_0, \ldots, a_n)$ is there a minimal pair $(d; a)$ such that $|d| = |a| - 1$?

Chapter 2

Boundedness of foliated surfaces

In this chapter, we study under which conditions canonical models of foliated surfaces of general type are bounded in some way. By previous works of [HL21] and [Che21], it is known that a first condition towards the boundedness of a family of canonical models is that the Hilbert function $\chi(mK_{\mathcal{F}})$ is fixed. The goal of the chapter is to improve the main results on boundedness of [HL21] and [Che21], by showing that they still hold under weaker assumptions, namely if only $K_{\mathcal{F}}^2$, $K_{\mathcal{F}} \cdot K_X$ and $i_{\mathbb{Q}}(\mathcal{F})$ are fixed. We do this by using a classical result due to Kollár and Matsusaka, which gives a bound on $h^0(D)$, depending only on D^2 and $D \cdot K_X$, for any big and semiample Cartier divisor D. While $K_{\mathcal{F}}$ is not necessarily $\mathbb{Q}$-Cartier, we can still make use of the theorem by passing to a partial resolution and taking a perturbation of the canonical divisor of the pullback foliation. Finally, we give partial results on the sharpness of these conditions. We first give an example showing that it is necessary to fix $K_{\mathcal{F}} \cdot K_X$, then we present several results related to the condition on the index $i_{\mathbb{Q}}(\mathcal{F})$, showing that for many natural families of surfaces the index of foliated canonical models of general type with fixed $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$ must be bounded.

2.1 Preliminaries

In the following, we always work over $\mathbb{C}$. With variety we mean a reduced and irreducible complex algebraic space. A surface is a 2-dimensional variety.

2.1.1 Intersection theory on normal surfaces

On normal surfaces, it is possible to define an intersection pairing on Weil divisors (due to Mumford) which generalises the intersection of Cartier divisors (for a reference, see [Sak84]).

Let X be a complete normal surface, and let $\mathrm{Div}(X, \mathbb{Q}) = \mathrm{Div}(X) \otimes \mathbb{Q}$ be the group of Weil $\mathbb{Q}$-divisors on X. Define the intersection pairing

$$\mathrm{Div}(X,\mathbb{Q}) \times \mathrm{Div}(X,\mathbb{Q}) \to \mathbb{Q}$$

in the following way. First, let $D \in \mathrm{Div}(X,\mathbb{Q})$, $f\colon Y \to X$ a proper birational morphism from a smooth surface Y. Let $E = \sum E_i$ be the exceptional divisor of f. Then, since the matrix $(E_i \cdot E_j)$ is negative definite, there exist unique real numbers x_i such that, for any exceptional curve E_j, $(f_*^{-1}D + \sum x_i E_i) \cdot E_j = 0$; define f^*D to be $f_*^{-1}D + \sum x_i E_i$. Then, given two divisors $D_1, D_2 \in \mathrm{Div}(X,\mathbb{Q})$, we define their intersection to be

$$D_1 \cdot D_2 = f^*D_1 \cdot f^*D_2,$$

where $f^*D_1 \cdot f^*D_2$ is defined in the usual sense as f^*D_1 and f^*D_2 are $\mathbb{Q}$-divisors on a smooth surface. Furthermore, the intersection pairing on $\mathrm{Div}(X,\mathbb{Q})$ coincides with the usual one when restricted to $\mathbb{Q}$-Cartier divisors. As with $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, a Weil $\mathbb{Q}$-divisor D is nef if $D \cdot C \geq 0$ for any irreducible curve C.

Remark 2.1.1. The definition of f^*D for a Weil $\mathbb{Q}$-divisor D can be generalised to the case of f being any birational morphism between normal surfaces. More precisely, let $f\colon Y \to X$ be a birational morphism of complete normal surfaces, and let $E = \sum E_i$ be the exceptional divisor of f. Again, the matrix $(E_i \cdot E_j)$ is negative definite, and for any Weil $\mathbb{Q}$-divisor D on X we can define the pullback divisor

$$f^*D = f_*^{-1}D + \sum x_i E_i,$$

where x_i are uniquely defined by the identities $(f_*^{-1}D + \sum x_i E_i) \cdot E_j = 0$. Note that the definition is consistent with the previous one in the case of Y smooth, and with the intersection pairing on Weil $\mathbb{Q}$-divisors.

2.1.2 Riemann-Roch theorem for normal surfaces

For normal surfaces, the Riemann-Roch theorem for smooth surfaces can be generalised to singular surfaces (and any Weil divisor) as well:

Theorem 2.1.2 ([Rei87], [Lan00])**.** *Let X be a complete normal surface, D a Weil divisor on X. Then,*

$$\chi(X,D) = \frac{1}{2}(D^2 - K_X \cdot D) + \chi(X) + \sum_{x \in \mathrm{Sing}\,X} a(x,D),$$

where $a(x,D)$ depends only on the local isomorphism class of the reflexive sheaf $\mathcal{O}_X(D)$ at x.

The integer $a(x, D)$ is computed in the following way (cf. [Lan00, Definition 2.7]; [HL21, Section 1.1.3]). Let (X, x) be a surface singularity, $(Y, E = \sum E_i) \to (X, x)$ any resolution of the singularity and $\tilde{D}$ any divisor such that $f_*\tilde{D} = D$. There is a unique $\mathbb{Q}$-divisor $c_1(x, \tilde{D})$, supported on E, such that $c_1(x, \tilde{D}) \cdot E_i = \deg \mathcal{O}_{E_i}(\tilde{D})$ for all exceptional curves E_i. Set

$$\chi(x, \mathcal{O}_Y(\tilde{D})) = \dim(\mathcal{O}_X(D)/f_*\mathcal{O}_Y(\tilde{D}))_x + \dim(R^1 f_*\mathcal{O}_Y(\tilde{D}))_x.$$

Then, $a(x, D)$ is given as

$$a(x, D) = \frac{1}{2}c_1(x, \tilde{D})(c_1(x, \tilde{D})) - c_1(x, K_Y)) + \chi(x, \mathcal{O}_Y(\tilde{D}) - \dim(R^1 f_*\mathcal{O}_Y)_x.$$

In particular, if D is Cartier at x, then $a(x, D) = 0$; thus, if D is Cartier, we recover the classical Riemann-Roch theorem.

In general, computing $a(x, D)$ is not simple. Still, for the scope of this work, where D will be the canonical divisor of a foliation (Definition 2.2.2), a complete description has been done by Hacon and Langer [HL21, Section 2], and will be reviewed in Section 2.2.1.

2.1.3 The Kollár-Matsusaka Theorem

When studying the Hilbert polynomial of a Cartier divisor D on a projective variety X, it is desirable to have some kind of bound on the dimension of its cohomology groups. For example, if D is ample, finding a bound to its Hilbert polynomial amounts to finding a bound on $h^0(\mathcal{O}_X(D))$. It turns out that in some cases, $h^0(\mathcal{O}_X(D))$ can be estimated using only the two top coefficients of the Hilbert polynomial. This is the content of the Kollár-Matsusaka theorem, which we state in two versions; still, we will only use the original statement (that is, Theorem 2.1.3) despite the stronger conditions on D, as in general the projective varieties we consider will be singular. When working with a divisor which is not semiample, we will still be able to use the theorem by considering a perturbation of the divisor which will be semiample.

Theorem 2.1.3 (Kollár-Matsusaka Theorem; [KM83, Theorem 2])**.** *Let X be a normal projective variety of dimension n, D a big and semiample Cartier divisor. Then there is a polynomial $Q(m)$ of degree $n-1$, uniquely determined by D^n and $K_X \cdot D^{n-1}$, such that*

$$|h^0(X, mD) - \frac{D^n m^n}{n!}| \leq Q(m).$$

Theorem 2.1.4 ([Luo89, Theorem 3.2])**.** *Let X be a nonsingular projective variety of dimension n, and D a nef and big divisor on X. Then for every $m \in \mathbb{N}$,*

$$|h^0(X, mD) - \frac{D^n m^n}{n!}| \leq Q(m),$$

where $Q(m)$ is a polynomial of degree at most $n-1$ whose coefficients are uniquely determined by D^n and $K_X \cdot D^{n-1}$.

2.2 Foliations

We now give some definitions and properties of foliations and their singularities.

Definition 2.2.1. A *foliation* $\mathcal{F}$ of rank r on a normal variety X is a rank r coherent subsheaf $T_{\mathcal{F}}$ of T_X which is saturated (that is, $T_X/T_{\mathcal{F}}$ is torsion-free) and closed under Lie bracket. The pair $(X, \mathcal{F})$ is called a *foliated variety.*

Note that when $\mathrm{rank}(\mathcal{F}) = 0$, $\mathcal{F}$ is the foliation by points on X, and if $\mathrm{rank}(\mathcal{F}) = \dim X$ it is the trivial foliation. In the following, we always consider *proper foliations*, that is foliations $\mathcal{F}$ such that $0 < \mathrm{rank}(\mathcal{F}) < n$, unless otherwise stated.

Definition 2.2.2.

- Let $(X, \mathcal{F})$ be a foliated variety of rank r. For any positive integer d, let $\Omega_X^{[d]} := (\bigwedge^d \Omega_X^1)^{**}$. The inclusion $T_{\mathcal{F}} \to T_X$ induces a map $\Omega_X^{[1]} \to T_{\mathcal{F}}^*$ by taking the dual, and a map
$$\phi\colon \Omega_X^{[r]} \to \mathcal{O}_X(K_{\mathcal{F}})$$
by taking the r-wedge product, for some divisor $K_{\mathcal{F}}$ such that $\mathcal{O}(-K_{\mathcal{F}}) \simeq \det T_{\mathcal{F}}$. $K_{\mathcal{F}}$ is called the *canonical divisor* of $\mathcal{F}$, and the cosupport of the image of the map
$$\phi'\colon (\Omega_X^{[r]} \otimes \mathcal{O}_X(-K_{\mathcal{F}}))^{**} \to \mathcal{O}_X$$
is called the *singular locus* of $\mathcal{F}$.

- The *Kodaira dimension* of $\mathcal{F}$ is given by
$$\kappa(\mathcal{F}) := \kappa(K_{\mathcal{F}}) = \max\{\dim \phi_{mK_{\mathcal{F}}}(X) \mid m \in \mathbb{N}\},$$
where $\phi_{mK_{\mathcal{F}}}$ is the m-th pluricanonical map induced by $mK_{\mathcal{F}}$. If $h^0(mK_{\mathcal{F}}) = 0$ for any m, we set $\kappa(\mathcal{F}) = -\infty$.

- A *leaf* L of $\mathcal{F}$ is given by a maximal connected and immersed holomorphic submanifold in the smooth locus $U = X \setminus (\mathrm{Sing}X \cup \mathrm{Sing}\mathcal{F})$, such that $T_L = \mathcal{F}\big|_L$.

- A subvariety W of X is *tangent* to the foliation $\mathcal{F}$ on X if, on the open set $U = X \setminus (\mathrm{Sing}(X) \cup \mathrm{Sing}(W) \cup \mathrm{Sing}(\mathcal{F}))$, the inclusion $T_W\big|_U \to T_X\big|_U$ factors through $\mathcal{F}\big|_U$. Otherwise, W is said to be *transverse* to $\mathcal{F}$.

- W is *invariant* if the inclusion $\mathcal{F}\big|_U \to T_X\big|_U$ factors through $T_W\big|_U$.

Remark 2.2.3.

- If X is a surface, $K_{\mathcal{F}}$ is simply given by $K_{\mathcal{F}} = T_{\mathcal{F}}^*$.

- A foliation $\mathcal{F}$ can be seen as a way to partition the smooth locus of a variety X and $\mathcal{F}$ in equidimensional submanifolds: in fact, they are disjoint, have dimension equal to the rank of $\mathcal{F}$, and cover X. This is a consequence of Frobenius' theorem: away from the singular points of X and $\mathcal{F}$, $T_{\mathcal{F}}$ is a subbundle of T_X, hence it gives a distribution which is involutive by definition; then, at every point p there is only one maximal submanifold tangent to $T_{\mathcal{F}}$. Each of these submanifolds is a leaf of the restriction of $\mathcal{F}$ to the smooth locus.

Definition 2.2.4.

- Given a dominant rational map $f\colon Y \dashrightarrow X$ and a foliation $\mathcal{F}$ of rank r on X, it is possible to define a *pullback foliation* on Y, as in [Dru21, Section 3.2]. When f is a morphism, $f^*\mathcal{F}$ is given by the kernel of the differential $df\colon T_Y \to f^*(T_X/T_{\mathcal{F}})$. If f is birational, it can be described as follows: let U an open subset of X such that $f\big|_U\colon V = f^{-1}(U) \to U$ is an isomorphism (in particular, $T_U \cong T_V$). By [Har77, Exercise II.5.15], there is a coherent subsheaf $\mathcal{G} \subset T_Y$ such that $\mathcal{G}\big|_V = \mathcal{F}\big|_U$. The pullback foliation of $\mathcal{F}$ on Y is defined as the saturation of $\mathcal{G}$.

- For a birational map $g\colon X \to Y$, the *pushforward* foliation $g_*(\mathcal{F})$ of a foliation $\mathcal{F}$ on X is given by $g_*(\mathcal{F}) = (g^{-1})^*\mathcal{F}$.

- Given a dominant rational map $f\colon Y \to X$, the pullback foliation of the foliation by points on X, that is $T_{\mathcal{F}} = 0$, is called the *induced foliation* of f. A foliation which is induced by some dominant rational map is said to be *algebraically integrable.*

Remark 2.2.5. Let $f\colon X \to Y$ be a morphism of normal projective varieties, and $\mathcal{F}$ the induced foliation on X. If f is equidimensional, the canonical divisor of $\mathcal{F}$ is tightly related to the relative canonical divisor of f. In fact, define

$$R(f) = \sum_D (f^*D - f^{-1}D),$$

where the sum runs through all prime divisors of Y. Then, $K_{\mathcal{F}}$ is given by

$$K_{\mathcal{F}} = K_{X/Y} - R(f).$$

In particular, if the fibers of f are reduced, then $K_{\mathcal{F}} = K_{X/Y}$.

Remark 2.2.6. For the most part, we will work with foliations on surfaces. In this case, we will always suppose the foliation has *reduced* singularities: let p be a singular point of a foliation $\mathcal{F}$ given by a vector field v around p. The linear part $(Dv)(p)$ has eigenvalues defined up to multiplication by a non-zero constant; p is a reduced singularity of $\mathcal{F}$ if at least one of the eigenvalues is non-zero and their quotient is not a positive rational number. Any foliation on a surface can be reduced to one with only reduced singularities by Seidenberg's Theorem [Bru15, p.5].

Remark 2.2.7. Let X be a normal surface. On the smooth locus of X, a foliation on X with isolated singularities can be described by a family of local vector fields in the following way: let $\{U_i\}$ be a finite open cover of X, and for every i, let v_i be a holomorphic vector field defined on U_i with isolated zeroes. Suppose that for any i, j, $v_i = g_{ij} v_j$ on $U_i \cap U_j$, where $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$. Then, the local integral curves agree on the intersection and give global leaves of a foliation $\mathcal{F}$; the singular locus of the foliation is the set of points where the local vector fields vanish. Furthermore, the functions $\{g_{ij}\}$ form a cocycle that corresponds to $\mathcal{O}_X(K_{\mathcal{F}})$ on the smooth locus of X.

Definition 2.2.8. A *bounded family* of foliated surfaces is given by a foliated variety $(\mathcal{X}, \mathcal{F})$ where $\mathcal{F}$ has rank one and both K_X and together with a proper morphism $f \colon \mathcal{X} \to \mathcal{T}$ with $\mathcal{T}$ of finite type, such that:

- for any fiber F of f, $\operatorname{codim}(\operatorname{Sing}(\mathcal{X}) \cap F) \geq 2$;
- $\mathcal{F} \subset T_{\mathcal{X}/\mathcal{T}}$;
- for any $t \in \mathcal{T}$, the pair $(X_t, \mathcal{F}_t)$ is a foliated surface, where $\mathcal{F}_t \coloneqq (\mathcal{F}\big|_{X_t})^{**}$.

We now introduce some standard definitions for singularities of foliations; these are given in analogy to the singularities of MMP, the main difference arising for log terminal and log canonical singularities. We will be mostly interested in terminal and canonical singularities, which are the singularities appearing on canonical models, the main object we will study. We will give the definitions on algebraic spaces, rather than solely on algebraic varieties; as we will see, this is required to work with canonical models, even if we want to consider canonical models of foliated projective surfaces.

Definition 2.2.9.

- Let $(X, \mathcal{F})$ be a foliated normal variety such that $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier, and $p \colon X \to Y$ a proper birational morphism. Write
$$K_{p^*\mathcal{F}} = p^* K_{\mathcal{F}} + \sum_E a(E, X, \mathcal{F}) E,$$
where E are the prime divisors contracted by p. We call $a(E, X, \mathcal{F})$ the *discrepancy* of $\mathcal{F}$ along E, and if E is contracted to a point x we say E is a divisor *over* x.
- A point $x \in X$ is a *terminal* (resp. *canonical*) singularity of $\mathcal{F}$ if $a(E, X, \mathcal{F}) > 0$ (resp. ≥ 0) for any exceptional divisor over x.
- Define
$$\epsilon(E) \coloneqq \begin{cases} 1 & \text{if } E \text{ is invariant by } \mathcal{F} \\ 0 & \text{if } E \text{ is not invariant by } \mathcal{F} \end{cases}$$

Then a point x is a *log terminal* (resp. *log canonical*) singularity if for any divisor E over x, $a(E, X, \mathcal{F}) > -\epsilon(E)$ (resp. $\geq -\epsilon(E)$).

- If $K_{\mathcal{F}}$ is Cartier (resp. $\mathbb{Q}$-Cartier) at a point x, we say $\mathcal{F}$ is *Gorenstein* (resp. $\mathbb{Q}$-Gorenstein) at x, or equivalently that x is a Gorenstein (resp. $\mathbb{Q}$-Gorenstein) point of $(X, \mathcal{F})$.

- The *index* $i(\mathcal{F})$ of a foliation $\mathcal{F}$ is the smallest positive integer m such that $mK_{\mathcal{F}}$ is Cartier (we set $i(\mathcal{F}) = \infty$ if $\mathcal{F}$ is not $\mathbb{Q}$-Gorenstein at some point). The $\mathbb{Q}$-*index* $i_{\mathbb{Q}}(\mathcal{F})$ is the smallest positive integer m such that $mK_{\mathcal{F}}$ is Cartier at the $\mathbb{Q}$-Gorenstein points. When $K_{\mathcal{F}}$ is $\mathbb{Q}$-Gorenstein at every point, the two definitions coincide.

When working with foliated surfaces,the previous definitions can be extended to include the case of non-$\mathbb{Q}$-Gorenstein foliations by using the notions introduced in Section 2.1.1. In fact, $p^*K_{\mathcal{F}}$ is well defined even when $K_{\mathcal{F}}$ is not $\mathbb{Q}$-Cartier, hence we can still talk about terminal, canonical, log terminal and log canonical singularities of a foliated surface $(X, \mathcal{F})$ even though $K_{\mathcal{F}}$ is only a Weil divisor.

2.2.1 Canonical singularities and their contribution to the Riemann-Roch theorem

For the purpose of the main results of the chapter, it is necessary to better understand how the Hilbert function of foliated surfaces can be computed. In particular, it is fundamental to get an explicit description of the terms $a(x, K_{\mathcal{F}})$ appearing in Theorem 2.1.2 for any terminal or canonical singularity of $\mathcal{F}$; this is the content of [HL21, Section 2]. The computation of such terms relies on the following formal description of terminal and canonical singularities [McQ08, Corollary I.2.2 and Fact I.2.4].

Proposition 2.2.10. *Let $(X, \mathcal{F})$ be a normal foliated surface, and $p \in X$ a terminal or canonical singularity of $\mathcal{F}$. Then, locally around p, $\mathcal{F}$ is formally given by a quotient of a (possibly singular) foliation around a smooth point of a surface Y, namely:*

- Terminal singularities: *A quotient of a smooth foliation by a $\mathbb{Z}/n\mathbb{Z}$-action, preserving both the point and the foliation.*

- Canonical singularities:

  (1) A quotient by a $\mathbb{Z}/n\mathbb{Z}$-action of

$$\partial = x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$$

  for $\lambda \notin \mathbb{Q}$.

(2) *A quotient of*

$$\partial = x\frac{\partial}{\partial x} + (\lambda y + x^\lambda)\frac{\partial}{\partial y}$$

by a $\mathbb{Z}/n\mathbb{Z}$*-action given by*

$$\begin{aligned} \sigma\colon x &\mapsto \chi_1(\sigma)x \\ y &\mapsto \chi_2(\sigma)y \end{aligned}$$

for faithful characters χ_1, χ_2 *of* $\mathbb{Z}/n$ *such that* $\chi_1^\lambda = \chi_2$.

(3) *The quotient of*

$$\partial = x\frac{\partial}{\partial x} + (\frac{y^{p+1}}{1+\nu y^p})\frac{\partial}{\partial y}$$

by a $\mathbb{Z}/n\mathbb{Z}$*-action such that* $\chi_2^p = 1$.

(4) *A quotient of*

$$\partial = px(1 + a((x^q y^p)^d))\frac{\partial}{\partial x} - qy(1 + b((x^q y^p)^d))\frac{\partial}{\partial y}$$

with $p, q \in \mathbb{N}$ *coprime and* a, b *formal functions, by a* $\mathbb{Z}/n\mathbb{Z}$*-action such that* d *is the smallest integer satisfying* $(\chi_1^q \chi_2^p)^d = 1$.

(5) *A quotient of*

$$\partial = x(1 + a((xy)^l))\frac{\partial}{\partial x} - y(1 + a(-(xy)^l)\frac{\partial}{\partial y}$$

for l *an odd integer and* a *a formal function vanishing at the origin, by some action of a dihedral type group* G*. More precisely:* G *is an extension of* $\mathbb{Z}/2\mathbb{Z}$ *by* $\mathbb{Z}/2n\mathbb{Z}$ *such that* $\mathbb{Z}/2\mathbb{Z}$ *gives an action on* $\mathbb{Z}/2n\mathbb{Z}$ *as multiplication by some element* p *such that* $p^2 \equiv 1 \mod 2n$*; write* $2n = 2^k lm$ *where* l, m *are odd and coprime,* $p \equiv -1 \mod 2k$*,* $p \equiv 1 \mod l$*,* $p \equiv -1 \mod m$*, and let* ζ *be a* $4n$*-root of unity;* G *has a representation in* $\mathrm{GL}(2, \mathbb{C})$ *generated by*

$$g_1 = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^{2p} \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and the action on the foliation is described by such representation.

(6) *A quotient of*

$$\partial = x(1 + a((xy)^{2^{k-1}l}))\frac{\partial}{\partial x} - y(1 + a(-(xy)^{2^{k-1}l})\frac{\partial}{\partial y}$$

(with notation as above), by an action of a dihedral type group G*. In this case,* $p \equiv 1 \mod 2^k$ *and* G *has a representation by*

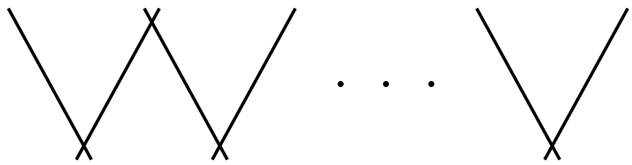
$$g_1 = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^{2p} \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & \zeta^{ml} \\ \zeta^{ml} & 0 \end{pmatrix}$$

giving the action on the foliation.

Definition 2.2.11. Let C a curve with normal crossing such that its irreducible components $C_1, \ldots, C_r$ are projective lines with $C_i^2 = p_i$, where $p_i \leq -2$ for all i, and $C_i \cdot C_j = 1$ if $|i - j| = 1$, 0 otherwise. Then, C is called a *Hirzbruch-Jung string.*

Proposition 2.2.12 ([McQ08, Theorem III.3.2])**.** *The minimal resolution of a foliated terminal or canonical singularity x of $(X, \mathcal{F})$ is given by one of the following (all the curves appearing have self-intersection ≤ -2).*

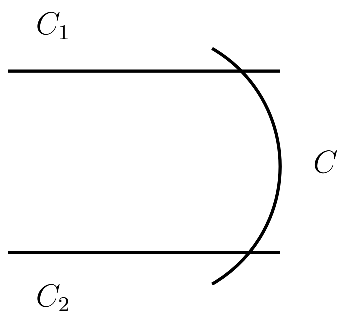
- Terminal singularities*: a* $\mathcal{F}$-chain*, that is a Hirzebruch-Jung string $C = \bigcup_{i=1}^{k} C_i$ such that $K_{\mathcal{F}} \cdot C_1 = -1$ and $K_{\mathcal{F}} \cdot C_i = 0$ for $i > 1$:*

  At x, X has a cyclic quotient singularity and the index of $\mathcal{F}$ is n.

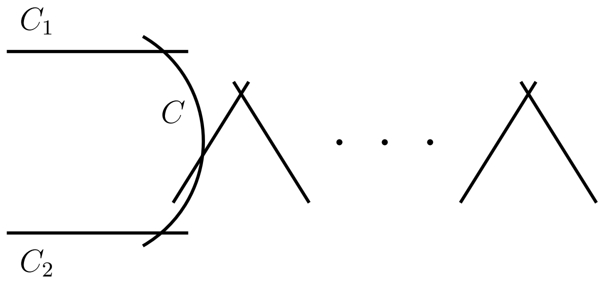
- Canonical singularities*:*

  (a) Either a chain of smooth rational curves T_i such that $K_{\mathcal{F}} \cdot T_i = 0$, or two smooth rational curves C_1, C_2 with $K_{\mathcal{F}} \cdot C_i = -1$, joined by another smooth rational curve C with $K_{\mathcal{F}} \cdot C = 0$ (in the notation of [McQ08]*, C is called* a bad tail*). The latter case can be represented as follows:*

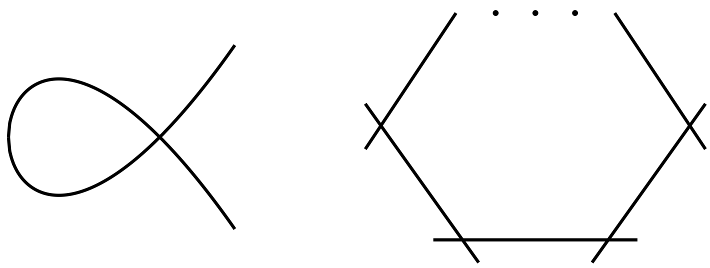
  These correspond to cases (1)-(4) of Proposition 2.2.10; X has a cyclic quotient singularity at x, at which $\mathcal{F}$ is Gorenstein.

  (b) two smooth rational curves C_1, C_2 with $K_{\mathcal{F}} \cdot C_i = -1$ joined by a bad tail, which is connected to a chain of smooth rational curves T_i with $K_{\mathcal{F}} \cdot T_i = 0$:



This is case (5); x is a dihedral quotient singularity of X, and $\mathcal{F}$ is 2-Gorenstein at x.

(c) Elliptic Gorenstein leaves (e.g.l.)*: these are given either by a smooth rational curve with one node, or by a cycle of smooth rational curves T_i with $K_{\mathcal{F}} \cdot T_i = 0$:*



This corresponds to case (6); X has a cusp singularity at x and $\mathcal{F}$ is not $\mathbb{Q}$-Gorenstein there.

In the classification of singularities, particular attention must be given to case (c) (elliptic Gorenstein leaves): as already mentioned, such singularities are not $\mathbb{Q}$-Gorenstein. This is a consequence of the following result.

Proposition 2.2.13 ([McQ08, Fact III.0.4])**.** *Let $\pi\colon (Y, \mathcal{G}) \to (X, \mathcal{F})$ be the contraction of an elliptic Gorenstein leaf Z to an elliptic Gorenstein singularity p of X. Then, $(X, \mathcal{F})$ is $\mathbb{Q}$-Gorenstein at p if and only if $K_{\mathcal{G}}\big|_Z$ is torsion.*

By [McQ08, Theorem IV.2.2], if $(Y, \mathcal{G})$ is a foliated surface with at worst canonical singularities and Z is an elliptic Gorenstein leaf on $(Y, \mathcal{G})$, then $K_{\mathcal{G}}\big|_Z$ is never torsion. It follows that $(X, \mathcal{F})$ is not $\mathbb{Q}$-Gorenstein at such singularities. In particular, abundance fails whenever $(X, \mathcal{F})$ has singularities resolving to elliptic Gorenstein leaves. Later, we will see that one consequence of such pathological behaviour arises when working with big $K_{\mathcal{F}}$, as in that case we will be forced to work with big and nef non-$\mathbb{Q}$-Cartier divisors (and in particular, not ample).

Given the special behaviour of such singularities, it is worth giving an example in which they naturally appear.

Example 2.2.14. Consider the space $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane. Let $F = \mathbb{Q}(\sqrt{d})$ be an algebraic number field, where d is a squarefree positive integer, and $\mathcal{O}_F$ its ring of integers. There is an action of $\Gamma = \mathrm{PSL}_2(\mathcal{O}_F)$ on $\mathbb{H} \times \mathbb{H}$ given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} z_1 & z_2 \end{pmatrix} = \begin{pmatrix} \dfrac{az_1+b}{cz_1+d} & \dfrac{a'z_2+b'}{c'z_2+d'} \end{pmatrix},$$

where a', b', c', d' are the Galois conjugates in F of a, b, c, d respectively. The quotient $\mathbb{H} \times \mathbb{H}/\Gamma$ gives a singular quasi-projective variety X_F, which is not compact; the singularities appearing due to the action of Γ are quotient singularities. In order to obtain a variety, it is enough to take the Baily-Borel compactification X_0 of X_F, which introduces cusp singularities whose resolution is a cycle of smooth rational curves, or a rational curve with a node. Let $f\colon X \to X_0$ be the resolution of X_0; X is called a *Hilbert modular surface*.

There are two natural foliations on $\mathbb{H} \times \mathbb{H}$ coming from the projection on each factor of the product. Let $\mathcal{F}_{\mathbb{H}}$ be one of them; it descends to a foliation on X_F which can be extended to a foliation $\mathcal{F}_0$ on X_0, and the resolution of each cusp of X_0 is an e.g.l. of $\mathcal{F} = f^*\mathcal{F}_0$.

$(X, \mathcal{F})$ is a very special type of foliation, as foliations on Hilbert modular surfaces obtained this way are the only known type of foliations on smooth surfaces with Kodaira dimension $\kappa(\mathcal{F}) = -\infty$ which are not rational foliations. Furthermore, since $K_{\mathcal{F}}$ is nef, this gives an example of the failure of abundance for foliations. As we will focus on the case of foliated surfaces with big $K_{\mathcal{F}}$, it is also possible to construct such an example with $K_{\mathcal{F}}$ big, starting from the e.g.l. appearing on Hilbert modular surfaces (cf. [McQ08, Corollary IV.2.3]). A more concrete example is given by ramified covers of a Hilbert modular surface: in fact, for a sufficiently ample divisor A on X_0 we can construct a smooth cover Y_0 ramified along a smooth, irreducible curve linearly equivalent to A which does not pass through the singular points; let $f\colon Y_0 \to X_0$ be such a covering. By [Bru15, Chapter 2.3(4)], $K_{f^*\mathcal{F}_0} = f^*(K_{\mathcal{F}_0}) + (k-1)\tilde{C}$, where $\tilde{C}$ is the preimage of the ramification locus C on X and k is the ramification index of the map; for a suitable A, $K_{f^*\mathcal{F}_0}$ is big and nef. $\mathcal{F}_0$ only has singular points over each cusp of X_0, and over the tangent points of C and $\mathcal{F}_0$, which are smooth for the surface and reduced, hence canonical. Then, $(Y_0, f^*\mathcal{F}_0)$ is a foliated surface of general type with cusp singularities whose resolution are e.g.l.

We can now give the explicit description of the terms $a(x, K_{\mathcal{F}})$ appearing in Theorem 2.1.2 when $D = K_{\mathcal{F}}$ and $(X, \mathcal{F})$ is a foliated canonical model of general type. For each type of canonical singularity, we refer to the respective case of Proposition 2.2.12. Note that, as mentioned in Section 2.1.2, $a(x, K_{\mathcal{F}}) = 0$ at $\mathcal{F}$-Gorenstein points.

Proposition 2.2.15 ([HL21, Section 2])**.** *Let x be a terminal or canonical foliation singularity of a foliated surface $(X, \mathcal{F})$. Then:*

- *If x is a terminal singularity, then $a(x, K_{\mathcal{F}}) = -\frac{n-1}{2n}$, where n is the index of x.*
- *If x is a canonical non-terminal $\mathbb{Q}$-Gorenstein singularity, then either $a(x, mK_{\mathcal{F}}) = 0$ for any m (case (a) of Proposition 2.2.12), or $a(x, mK_{\mathcal{F}}) = -\frac{1}{2}$ for odd m, and 0 otherwise (case (b)).*

- *If x is canonical such that $\mathcal{F}$ is non-$\mathbb{Q}$-Gorenstein at x (case (c)), then $a(x, mK_{\mathcal{F}}) = -1$ for $m > 0$ and 0 for $m = 0$.*

2.2.2 Canonical models of foliated surfaces

In the following, we want to focus on foliated surfaces of general type (that is, $\kappa(\mathcal{F}) = 2$). It is known by the work of [McQ08] that any foliated surface $(X, \mathcal{F})$ such that $\kappa(K_{\mathcal{F}}) \geq 0$, X is smooth and $\mathcal{F}$ only has canonical singularities admits a minimal model, that is a foliated surface $(Y, \mathcal{G})$ with a birational morphism $f\colon (X, \mathcal{F}) \to (Y, \mathcal{G})$ such that Y is projective, $K_{\mathcal{G}}$ is nef and $\mathcal{F} = f^*\mathcal{G}$. The existence of minimal models of foliated surfaces is akin to the existence of minimal models for varieties, and as in the case of varieties, $(Y, \mathcal{G})$ is constructed by contracting curves, tangent to $\mathcal{F}$, with negative intersection with $K_{\mathcal{F}}$. In a similar fashion, following known results about projective varieties of general type, for a projective foliated surface $(X, \mathcal{F})$ of general type with $K_{\mathcal{F}}$ nef, we would like to show the existence of a birational morphism $f\colon (X, \mathcal{F}) \to (X_c, \mathcal{F}_c)$ such that $(X_c, \mathcal{F}_c)$ is projective, has canonical singularities and $K_{\mathcal{F}_c}$ is ample. As a consequence of Nakai-Moishezon ampleness criterion, a natural way to construct $(X_c, \mathcal{F}_c)$ is to take f to be the contraction of all curves C on X such that $K_{\mathcal{F}} \cdot C = 0$, so that any curve on X_c has positive intersection with $K_{\mathcal{F}}$. Unfortunately, as mentioned in the previous section, it is possible that the foliation $\mathcal{F}_c$ (such that $\mathcal{F} = f^*\mathcal{F}_c$) is not $\mathbb{Q}$-Gorenstein. As a consequence, $K_{\mathcal{F}_c}$ cannot be ample: otherwise, a multiple of $K_{\mathcal{F}_c}$ would be the pullback of a hyperplane section of a projective space, hence Cartier. For this reason, canonical models are required to satisfy weaker properties.

Definition 2.2.16. A foliated surface $(X, \mathcal{F})$ is called a *canonical model* if X is normal, $\mathcal{F}$ only has canonical singularities, $K_{\mathcal{F}}$ is nef, and for all irreducible curves C, $K_{\mathcal{F}} \cdot C = 0$ implies $C^2 \geq 0$.

When $(X, \mathcal{F})$ is a canonical model of general type, $K_{\mathcal{F}}$ satisfies the following weaker condition of ampleness.

Lemma 2.2.17. *If $(X, \mathcal{F})$ is a canonical model of general type, then $K_{\mathcal{F}}^2 > 0$ and $K_{\mathcal{F}} \cdot C > 0$ for every irreducible curve C on X.*

Proof. We prove the statement by contradiction. Suppose there exist a curve C such that $K_{\mathcal{F}} \cdot C = 0$, then by the Hodge index theorem

$$K_{\mathcal{F}}^2 C^2 \leq (K_{\mathcal{F}} \cdot C)^2 = 0.$$

Since $K_{\mathcal{F}}$ is big and nef, then $K_{\mathcal{F}}^2 > 0$ and $C^2 \leq 0$, which means that $C^2 = 0$ by the definition of canonical model. Again by the Hodge index theorem, the class of C must be proportional to the class of $K_{\mathcal{F}}$, so the only possibility is that C is numerically trivial. Let $f\colon (X_m, \mathcal{F}_m) \to (X, \mathcal{F})$ be the minimal resolution of the non-$\mathbb{Q}$-Gorenstein singularities of $(X, \mathcal{F})$. We have that $K_{\mathcal{F}_m} = f^*K_{\mathcal{F}}$, and for any curve $C' \subset X_m$, $K_{\mathcal{F}_m} \cdot C' = 0$ if and only if C' is contracted by f. Therefore,

$$K_{\mathcal{F}} \cdot C = f^*K_{\mathcal{F}} \cdot f^*C = \mathcal{F}_m \cdot (f_*^{-1}C + \sum x_i E_i)$$

for some real numbers x_i, where E_i are the irreducible curves contracted by f. Since every E_i is $K_{\mathcal{F}_m}$-trivial, we get that

$$K_{\mathcal{F}} \cdot C = K_{\mathcal{F}_m} \cdot f_*^{-1}C = 0,$$

hence $f_*^{-1}C$ is contracted by f, which gives the desired contradiction. □

Remark 2.2.18. Cusp singularities give the main obstruction to working with canonical models in the projective category. This is a consequence of the following result:

Theorem 2.2.19 (cf. [Art62, Theorem 2.3]; [HL21, Theorem 1.1])**.**
Let X be a normal complete surface with at most rational singularities, then X is projective.

Since all terminal singularities, and the canonical singularities of cases (a)-(b) of Proposition 2.2.12 are rational surface singularities, a canonical model with no cusp singularities is projective. for a canonical model $(X, \mathcal{F})$ with $K_{\mathcal{F}}$ big and nef, another way to show this is by noticing that e.g.l. are the only non-$\mathbb{Q}$-Gorenstein singularities of $\mathcal{F}$. Since the Nakai-Moishezon criterion holds for line bundles on algebraic spaces as well, $i(\mathcal{F})K_{\mathcal{F}}$ is an ample Cartier divisor by Lemma 2.2.17, thus by definition there is some multiple of $K_{\mathcal{F}}$ giving an embedding into a projective space. In particular, we recover the ampleness of the canonical divisor, which characterises canonical models in the classical setting of varieties of general type.

2.2.3 Minimal partial Du Val resolutions

Since canonical models are not necessarily $\mathbb{Q}$-Gorenstein, in some cases it can be useful to pass to some partial resolution such that the canonical divisor of the pullback foliation is $\mathbb{Q}$-Cartier. This is the case, for example, of [Che21], where the author introduces the concept of *minimal partial Du Val resolution* of a canonical model:

Definition 2.2.20. Let $(X_c, \mathcal{F}_c)$ be a canonical model of general type, and let $(X^m, \mathcal{F}^m)$ be the minimal resolution of the canonical non-terminal singularities of $(X_c, \mathcal{F}_c)$ together with its pullback foliation; let $g\colon (X^m, \mathcal{F}^m) \to (X_c, \mathcal{F}_c)$ be the associated morphism. By running a classical MMP, let $h\colon X^m \to X$ be the relative canonical model over X_c, which is obtained by contracting smooth rational curves C with $C^2 = -2$ in the fibers of g (in particular, K_X is ample over X_c), and let $\mathcal{F}$ be the pushforward foliation on X. $(X, \mathcal{F})$ is called the *minimal partial Du Val resolution* (MPDVR) of $(X_c, \mathcal{F}_c)$.

The construction is described by the following diagram:

$$\begin{tikzcd} X^m \arrow[r, "g"] \arrow[dr, "h"'] & X \arrow[d, "f"] \\ & X_c \end{tikzcd}$$

Remark 2.2.21. A canonical model is uniquely determined by its minimal partial Du Val resolution. In fact, suppose that $(Y, \mathcal{G})$ is the minimal partial Du Val resolution of two canonical models $f_1\colon (Y, \mathcal{G}) \to (X_1, \mathcal{F}_1)$ and $f_2\colon (Y, \mathcal{G}) \to (X_2, \mathcal{F}_2)$. Then, $f_1^* K_{\mathcal{F}_1} = f_2^* K_{\mathcal{F}_2} = K_{\mathcal{G}}$; it follows that, for any curve C contracted by f_1, we have that $f_1^* K_{\mathcal{F}_1} \cdot C = f_2^* K_{\mathcal{F}_2} \cdot C = 0$, that is f_1 and f_2 contract the same curves.

2.2.4 Previous results

In [HL21] and [Che21], the authors study families of canonical models of general type with fixed Hilbert function $\chi(\mathcal{F})$. It turns out that fixing the Hilbert function gives useful information on the foliated surface: besides the obvious data on $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$ which follows straight from Theorem 2.1.2, something can be said about the singularities appearing in the canonical models as well:

Proposition 2.2.22 ([HL21, Proposition 4.1])**.** *Let $P\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$. For any canonical model of general type $(X, \mathcal{F})$ with Hilbert function $\chi(mK_{\mathcal{F}}) = P(m)$ (and independently from the choice of such model), there exist:*

- *rational numbers k_1, k_2 such that $K_{\mathcal{F}}^2 = k_1$, $K_{\mathcal{F}} \cdot K_X = k_2$;*
- *integer numbers C, C_1 such that $\chi(\mathcal{O}_X) = C$ and the number of cusps of X is C_1;*
- *integer numbers C_2, s such that the number of terminal and dihedral singularities of $(X, \mathcal{F})$ is at most C_2 and the index of the terminal singularities is at most s.*

The bound on the $\mathbb{Q}$-index allows to prove the following:

Theorem 2.2.23 ([HL21, Theorem 4.3])**.** *Let $P\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ and consider the family of canonical models $(X, \mathcal{F})$ of general type such that $\chi(mK_{\mathcal{F}}) = P(m)$. Then, there exists a constant positive integer N_P, only depending on $P(m)$, such that for any $(X, \mathcal{F})$ in the family and $M \geq N_P$, $|MK_{\mathcal{F}}|$ defines a birational map.*

While this proves effective birationality for the family of canonical models (which does not hold under weaker assumptions, such as only fixing $K_{\mathcal{F}}^2$, see Example 2.4.4), it does not give any direct information on the existence of a bounded family of such models. Still, using Theorem 2.2.23, in [Che21] a first result in this direction was proved:

Theorem 2.2.24 ([Che21, Theorem 3.4]). *Let S_P be the set of minimal partial Du Val resolutions of canonical models $(X, \mathcal{F})$ of general type with fixed Hilbert Function $P(m) = \chi(mK_{\mathcal{F}})$. Then, S_P is bounded.*

As a partial improvement to Proposition 2.2.22, the following holds as well:

Theorem 2.2.25 ([Che21, Theorem 4.3]). *Let $P \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$. There exists an integer N_P, depending only on $P(m)$, such that for any canonical model $(X, \mathcal{F})$ of general type such that $\chi(mK_{\mathcal{F}}) = P(m)$, then for any $M > 0$ divisible by M_P, $|MK_{\mathcal{F}}|$ defines a birational map which is an isomorphism on the complement of the cusp singularities.*

It is worth noting that these boundedness statements resemble, in a way, classical results on polarised varieties, that is pairs (X, D) where D is an ample Cartier divisor on X. This is the case, for example, of the boundedness of polarised varieties (X, D) with fixed Hilbert polynomial $\chi(mD)$ [Kol85, Theorem 2.1.2]. While $K_{\mathcal{F}}$ is, in general, not Cartier, the previous results show that canonical divisors of foliations have additional properties which allow, under suitable assumptions, to deduce more information than it would normally be possible with a general divisor. Following the same argument, it is natural to investigate whether weaker assumptions are enough to obtain similar statements: in fact, computing the Hilbert function of a non-Cartier divisor is not always feasible, hence hypotheses which are easier to check would greatly improve the significance of the work of [HL21] and [Che21]. This motivates the main result, stated in the next section.

2.3 Main result

Theorem 2.3.1. *Let k_1, k_2 be rational numbers, s a positive integer. Let $\mathcal{H}_{k_1,k_2,s}$ be the set of Hilbert functions $P(m) = \chi(X, mK_{\mathcal{F}})$ of canonical models $(X, \mathcal{F})$ of general type such that $K_{\mathcal{F}}^2 = k_1$, $K_{\mathcal{F}} \cdot K_X = k_2$ and $i_{\mathbb{Q}}(\mathcal{F}) = s$. Then $\mathcal{H}_{k_1,k_2,s}$ is finite.*

Proof. We first prove the statement under the assumption that the foliation is $\mathbb{Q}$-Gorenstein then generalise to the non-$\mathbb{Q}$-Gorenstein case as well. Let $(X, \mathcal{F})$ be a canonical model of general type with $K_{\mathcal{F}}^2 = k_1$, $K_{\mathcal{F}} \cdot K_X = k_2$ and $i_{\mathbb{Q}}(\mathcal{F}) = s$. Since $(X, \mathcal{F})$ is $\mathbb{Q}$-Gorenstein, $i_{\mathbb{Q}}(\mathcal{F}) = i(\mathcal{F})$ and $sK_{\mathcal{F}}$ is an ample Cartier divisor. For $m \gg 0$ we have that $h^0(X, msK_{\mathcal{F}}) = \chi(X, msK_{\mathcal{F}}) = P(ms)$. From Theorem 2.1.3 it follows that for $m \gg 0$,

$$|P(ms) - \frac{m^2 s^2 K_{\mathcal{F}}^2}{2}| \leq Q(m),$$

where $Q(m)$ is a degree 1 polynomial only depending on $s^2K_{\mathcal{F}}^2$ and $s(K_X \cdot K_{\mathcal{F}})$; it follows that $Q(m)$ is independent on the choice of the canonical model, as long as its Hilbert function is in $\mathcal{H}_{k_1,k_2,s}$. Then, $P(ms) - \frac{m^2s^2K_{\mathcal{F}}^2}{2}$ is bounded by $Q(m)$ for infinite values of m so that, for $m > 0$, there is a finite set of degree 2 polynomials in the variable m, to which $P(ms)$ can belong: these only differ for the constant term, as $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$ are fixed. In particular, since $msK_{\mathcal{F}}$ is Cartier, the constant term $\chi(\mathcal{O}_X) = P(0)$ can only achieve a finite number of values.

From Theorem 2.1.2, if we fix $\chi(\mathcal{O}_X)$ as well, $P(m)$ is determined up to the term $\sum a(x, mK_{\mathcal{F}})$; each $a(x, mK_{\mathcal{F}})$ can only assume a finite number of values by Proposition 2.2.15 as the index is bounded, so we only need to show that the number of singularities is bounded.

We have shown that $\chi(X, msK_{\mathcal{F}})$ is a polynomial in m belonging to a finite family. Then, from [Kol85, Theorem 2.1.2], we deduce that the family of polarised surfaces $(X, sK_{\mathcal{F}})$ is bounded; in particular, the surfaces X belong to a bounded family $f\colon \mathcal{X} \to \mathcal{T}$. What is left to do is to show that the number of singularities appearing on each surface is bounded. Since normality is an open condition, we can restrict the family and suppose that every fiber of f is normal. By generic smoothness, f is smooth outside a closed set $K \subset \mathcal{X}$, where $K = \bigcup K_i$ and each K_i is irreducible; consider the restriction $f\big|_{K_i}\colon K_i \to \mathcal{T}$. Since the fibers of f are normal, every fiber of $f\big|_{K_i}$ is a finite set and $f\big|_{K_i}$ is quasi-finite. Furthermore, since f is proper $f\big|_{K_i}$ is proper as well; then, f is finite the cardinality of each fiber is bounded by the degree of $f\big|_{K_i}$. This implies that the number of singularities on the fibers is bounded by $\sum \deg(f\big|_{K_i})$.

Now consider the general case of a canonical model $(X_c, \mathcal{F}_c)$ which is not necessarily $\mathbb{Q}$-Gorenstein. Let $f\colon (X, \mathcal{F}) \to (X_c, \mathcal{F}_c)$ be the MPDVR of the canonical model $(X_c, \mathcal{F}_c)$. Note that by [HL21, Theorem 5], $R^1f_*\mathcal{O}_X(mK_{\mathcal{F}}) = 0$, hence $H^i(mK_{\mathcal{F}}) = H^i(mK_{\mathcal{F}_C})$ for all $m \geq 0$, and in particular $\chi(mK_{\mathcal{F}}) = \chi(mK_{\mathcal{F}_c})$. As a consequence, we also have $K_{\mathcal{F}}^2 = K_{\mathcal{F}_c}^2$, $K_{\mathcal{F}} \cdot K_X = K_{\mathcal{F}_c} \cdot K_{X_c}$ and $i(\mathcal{F}) = i_{\mathbb{Q}}(\mathcal{F}) = i_{\mathbb{Q}}(\mathcal{F}_c)$. Therefore, in order to show that $\mathcal{H}_{k_1,k_2,s}$ is a finite set, it is equivalent to show that the set of Hilbert functions of MPDVRs of canonical models of general type with fixed $K_{\mathcal{F}}^2 = k_1$, $K_{\mathcal{F}} \cdot K_X = k_2$ and $i(\mathcal{F}) = s$ is finite.

Let $E = \sum E_i$ be the exceptional divisor of f, and let $D_X = 4i(\mathcal{F})K_{\mathcal{F}} + K_X$, then D_X is ample. To see this, by Nakai-Moishezon criterion it is enough to check that the intersection with every curve is positive. We consider three cases:

- $C = E_i$: in this case,

$$D_X \cdot C = (4i(\mathcal{F})K_{\mathcal{F}} + K_X) \cdot C = K_X \cdot C > 0$$

  as by construction, K_X is ample over X_c.

- $K_X \cdot C \geq 0$: then,

$$D_X \cdot C \geq 4i(\mathcal{F})K_{\mathcal{F}} \cdot C = 4i(\mathcal{F})K_{\mathcal{F}_c} \cdot f_*C > 0,$$

because $K_{\mathcal{F}} = f^*K_{\mathcal{F}_c}$ and $K_{\mathcal{F}_c}$ is numerically ample.

- $K_X \cdot C < 0$: by [Fuj12, Theorem 3.8], every K_X-negative extremal ray is spanned by a rational curve with $-3 \leq K_X \cdot C < 0$, so

$$D_X \cdot C \geq 4i(\mathcal{F})K_{\mathcal{F}} \cdot C - 3 \geq 1.$$

Thus, D_X is ample. Since $i(\mathcal{F}) = s$ is bounded and $i(K_X) \mid i(\mathcal{F})$, sD_X is an ample Cartier divisor and for $m \gg 0$, $\chi(X, msD_X) = h^0(X, msD_X)$. So we can apply Proposition 2.1.3 to say that for $m \gg 0$,

$$|P(msD_X) - \frac{m^2s^2D_X^2}{2}| \leq Q(m),$$

where $Q(m)$ only depends on $s^2D_X^2$ and $s(D_X \cdot K_X)$.

Next, we show that D_X^2 and $D_X \cdot K_X$ can only assume a finite number of values. In particular, since $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$ are fixed, we need to prove that K_X^2 has only a finite number of values.

From the Hodge index theorem,

$$K_{\mathcal{F}}^2 K_X^2 \leq (K_{\mathcal{F}} \cdot K_X)^2,$$

and since $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$ are fixed, K_X^2 is bounded from above. On the other hand, since D_X is ample, $D_X^2 > 0$. But

$$D_X^2 = 16i(\mathcal{F})K_{\mathcal{F}}^2 + 8i(\mathcal{F})K_{\mathcal{F}} \cdot K_X + K_X^2 > 0,$$

which implies that K_X^2 is bounded from below. Since $i(\mathcal{F})K_X$ is Cartier, $i(\mathcal{F})^2K_X^2$ is an integer, thus $K_X^2 = m/i(\mathcal{F})^2$ for some $m \in \mathbb{Z}$; in particular, as K_X^2 is bounded from above and below, it can only assume a finite number of values. Then, we can suppose K_X^2 is fixed, so that both D_X^2 and $D_X \cdot K_X$ are fixed. Since $Q(m)$ only depends on D_X^2 and $D_X \cdot K_X$, in particular we can suppose that $Q(m)$ is independent from the choice of canonical model or minimal partial resolution. Then, arguing as in the $\mathbb{Q}$-Gorenstein case, the number of possible values of $\chi(\mathcal{O}_X)$ is finite. The rest of the proof follows as before: under these assumptions, the family of polarised pairs (X, D_X) is bounded, which implies that the number of singularities is bounded. We conclude that there are only a finite number of possible values for the term $\sum a(x, K_{\mathcal{F}})$, hence $P(m) = \chi(mK_{\mathcal{F}})$ belongs to a finite set.

□

Thanks to Theorem 2.3.1, all the boundedness results of [HL21] and [Che21] holding under the assumption of the Hilbert function being fixed still hold under the weaker hypotheses of Theorem 2.3.1. In particular, from Theorem 2.2.23 we obtain the following.

Corollary 2.3.2. *Fix rational numbers k_1, k_2 and a positive integer s, and consider the family of canonical models $(X, \mathcal{F})$ of general type such that $K_{\mathcal{F}}^2 = k_1$, $K_{\mathcal{F}} \cdot K_X = k_2$ and $i(\mathcal{F}) = s$. Then, there exists a constant N_1, only depending on k_1, k_2, s, such that for any $(X, \mathcal{F})$ in the family and $m \geq N_1$, $|mK_{\mathcal{F}}|$ defines a birational map.*

Proof. Take
$$N_1 = \max_{P \in \mathcal{H}_{k_1,k_2,s}} \{N_P\},$$
where N_P is as in Theorem 2.2.23. Let $(X, \mathcal{F})$ be a canonical model of general type such that $P(m) = \chi(mK_{\mathcal{F}}) \in \mathcal{H}_{k_1,k_2,s}$. Then, for every $M \geq N_1$, $M \geq N_P$, hence by Theorem 2.2.23 $|MK_{\mathcal{F}}|$ defines a birational map. □

From Theorem 2.2.25, we get the following partial improvement.

Corollary 2.3.3. *Fix rational numbers k_1, k_2 and a positive integer s, and consider the family of canonical models $(X, \mathcal{F})$ of general type such that $K_{\mathcal{F}}^2 = k_1$, $K_{\mathcal{F}} \cdot K_X = k_2$ and $i(\mathcal{F}) = s$. Then, there exists an integer M_1, depending only on k_1, k_2, s, such that for any canonical model $(X, \mathcal{F})$ in the family, for any $M > 0$ divisible by M_1, $|MK_{\mathcal{F}}|$ defines a birational map which is an isomorphism on the complement of the cusp singularities.*

Proof. It follows as before, by taking
$$M_1 = \operatorname{lcm}_{P \in \mathcal{H}_{k_1,k_2,s}} \{M_P\},$$
with M_P as in Theorem 2.2.25. □

While from Theorem 2.2.24, by taking the union of the bounded families with fixed Hilbert function, we deduce the following.

Corollary 2.3.4. *Fix rational numbers k_1, k_2 and a positive integer s. The set $\mathcal{S}_{k_1,k_2,s}$ of minimal partial Du Val resolutions of canonical models of general type $(X_c, \mathcal{F}_c)$ with fixed $K_{\mathcal{F}}^2 = k_1, K_{\mathcal{F}} \cdot K_X = k_2, i(\mathcal{F}) = s$ is bounded.*

2.4 Examples and other remarks

This section focuses on giving some insight on the relation between fibrations and foliations, and shedding light on the necessity of the assumptions of Theorem 2.3.1. For the latter problem, while we are not able to give a complete answer, we can show that some of the hypotheses cannot be removed (namely, $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$ being fixed); at the same time, in order to study the necessity of the condition on $i(\mathcal{F})$, we give some examples and remarks that show that, if the underlying surfaces belong to some common families of varieties, the assumption on $i(\mathcal{F})$ is redundant. These allow to put strong constraints on the construction (if it exists) of a family of canonical models of general type with fixed $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$ but unbounded $i(\mathcal{F})$.

In the following, we will consider varieties in the classical sense, that is integral schemes of finite type. Therefore, we will always suppose that canonical models are projective.

2.4.1 Unbounded fibrations which are bounded as foliations

Definition 2.4.1. A *fibration* is a surjective morphism with connected fibers.

As with many algebraic objects, a natural problem in studying fibrations is understanding their behaviour in families.

Definition 2.4.2.

- A *family of fibrations* is given by a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"] \arrow[d, "f_1"'] & \mathcal{Y} \arrow[dl, "f_2"] \\ \mathcal{T} & \end{tikzcd}$$

  where $\mathcal{X}$, $\mathcal{Y}$ are normal schemes, f_1, f_2 are flat morphisms and the map $\pi_t \colon \mathcal{X}_t \to \mathcal{Y}_t$ over $t \in \mathcal{T}$ is a fibration for all $t \in \mathcal{T}$.

- Given a set of fibrations $\{f_\lambda \colon X_\lambda \ \to Y_\lambda\}$, we will say that it is a *bounded* if there exists a family of fibrations such that, using the notation above, $\mathcal{X} \to \mathcal{T}$, $\mathcal{Y} \to \mathcal{T}$ are bounded families of varieties (that is, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{T}$ are quasi-projective varieties of finite type, and f_1, f_2 are flat and proper) and for every fibration f_λ there exist isomorphisms $g \colon X_\lambda \xrightarrow{\cong} \mathcal{X}_t$, $h \colon Y_\lambda \xrightarrow{\cong} \mathcal{Y}_t$ for some $t \in \mathcal{T}$, which are compatible with π; in other words, for every $y \in Y$ the following diagram commutes:

$$\begin{tikzcd} (X_\lambda)_y \subset X_\lambda \arrow[r, "g"] \arrow[d, "f_\lambda"] & (\mathcal{X}_t)_{h(y)} \subset \mathcal{X}_t \arrow[d, "\pi_t"] \\ y \in Y_\lambda \arrow[r, "h"] & h(y) \in \mathcal{Y}_t \end{tikzcd}$$

Since fibrations give algebraically integrable foliations, for a set of fibrations it is possible for them to belong to a bounded family of foliations (by looking at the induced foliations) but not to a bounded family of fibrations. The following is such an example:

Example 2.4.3. Let E be an elliptic curve, $P \coloneqq E \times E$. We can consider two different morphisms of P onto E: besides the coordinate projections (we call π_x and π_y the projections onto the first and second coordinate, respectively), the n-multiplication map on E, $[n]\colon x \mapsto n \cdot x$, allows us to view its graph $\Gamma_n = \{(x, n \cdot x) \mid x \in E\}$ as a subvariety of P, isomorphic to E. Then, all the translations $(0, y_0) + \Gamma_n$ of Γ_n form a family of disjoint curves, isomorphic to E and covering P. Thus, we get another projection $P \to E$, defined by sending a point (x, y) to $y - n \cdot x$; we call this projection π_n.

For a suitable divisor D of degree $d > 1$ on E, let $A \sim \pi_x^*(2D) + \pi_y^*(2D)$ be a very ample divisor on P, which we can choose smooth, reduced and irreducible by Bertini's theorem. Let S be the double cover of P ramified along A, $\sigma\colon S \to P$ the covering map, and f_n the composition $\pi_n \circ \sigma$. We have that $K_S \sim_{\mathbb{Q}} \sigma_n^*(K_P + \frac{1}{2}A)$; in this case, since both K_E and K_P are trivial, we get that $K_{S/E} = K_S = \sigma^*(\frac{1}{2}A)$. We can consider the foliation $\mathcal{F}_n$ whose leaves are the fibers of f_n. The fibers are reduced, so that $K_{\mathcal{F}_n} = K_{S/E}$; we get that $K_{\mathcal{F}_n}^2 = K_S \cdot K_{\mathcal{F}_n} = (\sigma^*(\frac{1}{2}A))^2 = \frac{1}{2}A^2$. Note that the genus of the fibers of f_n depends on n: in fact, if F is a fiber of π_n, we have that $A \cdot F = 2d(n^2+1)$. Then, the Riemann-Hurwitz formula implies that the genus of a fiber F_n of f_n is equal to $d(n^2+1)+1$.

Despite the genus of the fibers being arbitrary, the family of foliations is bounded by Corollary 2.3.4, as $(S, \mathcal{F}_n)$ are canonical models of general type, $K_{\mathcal{F}_n}^2$ and $K_{\mathcal{F}_n} \cdot K_S$ are constant, and $i(\mathcal{F}_n) = 1$ for all n because the surfaces are smooth. On the other hand, the fibrations f_n cannot belong to a bounded family. In fact, suppose the fibrations f_n are bounded, that is, there exists a diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"] \arrow[d, "g_1"'] & \mathcal{Y} \arrow[dl, "g_2"] \\ \mathcal{T} & \end{tikzcd}$$

that is, each f_n is the restriction of π over some point t of the base space $\mathcal{T}$. Since the fibers of π are connected, the general fiber of f_n is a fiber of π as well. Then, by generic smoothness the general fiber of f is smooth; on the other hand, by possibly restricting $\mathcal{Y}$ to its smooth locus, we can suppose it is nonsingular. After stratifying the base, f is generically flat and over each component of the base the general fiber of f must have fixed genus, which contradicts the construction of the fibrations f_n.

2.4.2 Unbounded family of canonical models with fixed $K_{\mathcal{F}}^2$

We give an example of an unbounded family of canonical models of general type with fixed volume $K_{\mathcal{F}}^2$ [Xia87, Example 2].

Example 2.4.4. Let C be a smooth curve, k an even integer, $g \geq 2$ an integer. Let D a divisor on C of degree k such that $|(2g+1)D|$ is basepoint free, consider the ruled surface $P = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(D))$ over C, and let π be the projection on C. By the properties of ruled surfaces (for a reference, see [Har77, Chapter V.2]), we can find two disjoint global sections C_0, C_1 of π such that $C_0^2 = -k$, $C_1^2 = k$, $C_0 \cdot C_1 = 0$; more precisely, C_1 is a section such that $C_1 \sim C_0 + kF_P$ where F_P is a fiber of π.

Let $\Lambda = |(2g+1)C_1|$: the system is basepoint free, as it contains the divisor $(2g+1)C_1$ and the system $(2g+1)C_0+\pi^*|(2g+1)D|$, which have no fixed points in common. Therefore, by Bertini's theorem, Λ contains a divisor B which is irreducible, reduced and also smooth. As a consequence, since $C_0 \cdot C_1 = 0$, B and C_0 are disjoint, hence the divisor $R = B + C_0$ is smooth and reduced, and in $\mathrm{Pic}(P)$ $R = (2g+1)kF_P + (2g+2)C_0$ is divisible by 2; we can then consider the double cover $\sigma : S \to P$, ramified along R. By composition, we get a new fibration $f : S \to C$. First, note that the fibers have genus g: the restriction of σ to a fiber F of f is a double cover $f\big|_F : F \to \mathbb{P}^1$ ramified in $2g+2$ points; by the Riemann-Hurwitz formula, we get the result. Next, f induces a foliation on S, whose leaves are the fibers of f; as the fibers are reduced, $K_F = K_{S/C}$. If we let R' be a divisor such that in $\mathrm{Pic}(P)$ $2R' = R$, then $K_{S/C} = \sigma^*(K_{P/C}+R')$, and for the ruled surface P $K_P \equiv_{\text{num}} -2C_0 + (2g(C) - 2 - k)F_P$. Then we can compute the volume of $K_{\mathcal{F}}^2$:

$$
\begin{aligned}
K_{\mathcal{F}}^2 &= (\sigma^*(K_{P/C} + R'))^2 = 2(K_{P/C} + R')^2 \\
&= 2(-2C_0 + (2g(C) - 2 - k)F_P - (2g(C) - 2)F_P + \frac{k}{2}(2g+1)F_P + (g+1)C_0)^2 \\
&= 2((g-1)C_0 + \frac{k}{2}(2g-1)F_P)^2 = 2(-k(g-1)^2 + k(g-1)(2g-1)) = 2kg(g-1).
\end{aligned}
$$

In particular, the volume does not depend on the genus of the base curve C. Furthermore, $(S, K_{\mathcal{F}})$ is a canonical model, as $K_{\mathcal{F}}$ is ample and both the surface and the foliation are nonsingular. If we repeat the same construction by taking C of arbitrarily large genus, we obtain a family of foliations of fixed volume which is unbounded: this follows from the fact that the family is unbounded as a family of surfaces, as K_S^2 is unbounded.

Remark 2.4.5.

- Since the surfaces of the example are smooth, this also means that $i(\mathcal{F}) = 1$. Therefore, we have also constructed an unbounded family of foliated canonical models of general type with fixed $K_{\mathcal{F}}^2$ and $i(\mathcal{F})$.

- Note that in the example, boundedness fails because the underlying surfaces are not bounded. Since the Hilbert function of the foliations has to be constant in the family, it is expected that it should be possible to find an example of canonical models $(X, \mathcal{F})$ with fixed $K_{\mathcal{F}}^2$ and $i(\mathcal{F})$ such that the underlying surfaces belong to a bounded family but the foliations are unbounded.

2.4.3 Unbounded families of Del Pezzo surfaces with fixed volume

When trying to construct an unbounded family of canonical models with $K_{\mathcal{F}}^2, K_{\mathcal{F}} \cdot K_X$ fixed and unbounded $i(\mathcal{F})$, one of the most natural families to consider are quasi-smooth weighted complete intersections (in short, WCI) of dimension 2 (for all the definitions, notations and properties, we refer to Section 1.1). Since a bounded family of klt surfaces has bounded Cartier index, the failure of such a family of canonical models to be bounded must be due to the family of the underlying surfaces being unbounded. Note that by the following result, such families cannot be comprised of Calabi-Yau surfaces:

Theorem 2.4.6 ([Che15, Theorem 1.1]). *For any positive integer m, there are only finitely many families of Calabi-Yau quasi-smooth weighted complete intersections of dimension m.*

We can rule out weighted surfaces of general type as well, if we fix K_X^2: suppose that $(d_1, \ldots, d_c; a_0, \ldots, a_n)$ is the pair of degrees and weights associated to X with $\delta(d; a) > 0$, then $\mathcal{O}_{\mathbb{P}}(\delta)$ is ample, hence $K_X = \mathcal{O}_{\mathbb{P}}(\delta)\big|_X$ is ample as well. Thus, such surfaces are bounded by [HMX18, Theorem 1.1]. Therefore, we are naturally led to considering del Pezzo quasi-smooth WCI, that is dimension 2 quasi-smooth WCI $X_{d_1,\ldots,d_{n-2}} \subset \mathbb{P}(a_0, \ldots, a_n)$ such that $\delta = \sum d_i - \sum a_j < 0$. It is known that for fixed $\epsilon > 0$, ϵ-lc Fano varieties with fixed volume K_X^m are bounded [Bir21, Theorem 1.1], hence an unbounded family of del Pezzo surfaces must have unbounded Cartier index. In [JK01, Theorem 8] (for $\delta = -1$), [CS08, Corollary 1.13] (for $\delta = -2$) and [Pae18, Theorem 1.7] (for the general case), the authors give a complete description of all quasi-smooth del Pezzo weighted hypersurfaces in 3-dimensional weighted projective spaces, which belong to infinite families. Using this classification, we can construct an unbounded family of del Pezzo quasi-smooth WCI with fixed volume K_X^2. Using similar ideas, we also construct a family of weighted projective spaces of dimension 2 with $K_{\mathbb{P}}^2$ fixed and unbounded Cartier index.

Example 2.4.7. Consider the family of del Pezzo weighted hypersurfaces X of degree $a+b$ in $\mathbb{P} = \mathbb{P}(1, k-1, a, b)$ for $a, b, k > 0$; these are quasi-smooth by [Pae18, Theorem 1.7, Class 1]. We want to show that it is possible to choose an infinite number of values of a, b, k so that the corresponding surfaces have the same volume K_X^2. Note that even though $\mathrm{Cl}(X)$ might not be cyclic, intersections can be computed easily using the fact that $\mathrm{Cl}(\mathbb{P})$ is cyclic.

We use the following facts:

Property 2.4.8.

- If $\mathcal{O}_{\mathbb{P}}(1)$ is a positive generator of $\mathrm{Pic}(\mathbb{P})$, then we can compute the self-intersection of $\mathcal{O}(1)$ as
$$\mathcal{O}_{\mathbb{P}}(1)^3 = \frac{1}{(k-1)ab}.$$
- $K_{\mathbb{P}} = -\sum a_i = -(k+a+b)$; it follows, by adjunction:
$$K_X = (K_{\mathbb{P}} + X)\big|_X = \mathcal{O}(k)\big|_X.$$

Then,

$$K_X^2 = \mathcal{O}_{\mathbb{P}}(a+b) \cdot \mathcal{O}_{\mathbb{P}}(k)^2 = \frac{k^2(a+b)}{(k-1)ab}.$$

Thus, we only need to construct an infinite series of values of a, b, k such that the fraction is constant. To do this, suppose that

$$\begin{cases} ab = 6k^2 \\ a+b = 6(k-1) \end{cases} \tag{1}$$

which means that, if solutions to the system exist, then $K_X^2 = 1$.

By substituting b, we obtain

$$a^2 - 6(k-1)a + 6k^2 = 0. \tag{2}$$

A solution is given by $k = 6, a = 12, b = 6(k-1) - a = 18$. An infinite number of solutions can then be obtained recursively by

$$\begin{cases} k_0 = 6, a_0 = 12 \\ k_{m+1} = 5k_m - a_m - 6 \\ a_{m+1} = 6k_m - a_m - 6 \\ b_{m+1} = 6(k_{m+1} - 1) - a_{m+1} = 24k_m - 5a_m - 36 \end{cases}$$

Note that a, b and k must have the same sign: the conditions of (1) are symmetric in a and b, which means that if (a, k) is a solution of (2), then (b, k) is a solution as well. If we fix k and see (2) as an equation in a, it has two solutions with same sign, which shows that a and b have same sign. Then, the middle term in (2) is always negative, which means that the two solutions must be positive; (1) thus implies that k must be positive as well. Therefore, the recursion gives admissible solutions to our problem: the weighted hypersurfaces $X_m \subset \mathbb{P}(1, k_m - 1, a_m, b_m)$ of degree $a_m + b_m$ give an unbounded family of del Pezzo surfaces with fixed volume $K_{X^m}^2 = 1$.

In a similar fashion, we can construct an unbounded family of weighted projective spaces of dimension 2:

Example 2.4.9. Let $\mathbb{P}(1, a, b)$ be a weighted projective space of dimension 2. We want to show that it is possible to choose a series of values a_m, b_m for a and b such that the surfaces $\mathbb{P}(1, a_m, b_m)$ have fixed volume $K_{\mathbb{P}}^2$. From Property 2.4.8,

$$K_{\mathbb{P}}^2 = \frac{(a+b+1)^2}{ab}.$$

Put $K_{\mathbb{P}}^2 = 8$, then we need to find solutions to the equation

$$a^2 + b^2 - 6ab + 2a + 2b + 1 = 0.$$

A solution is given by $a = 2, b = 1$, and recursive solutions are given by

$$\begin{cases} a_0 = 2, b_0 = 1 \\ a_{m+1} = 6a_m - b_m - 2 \\ b_{m+1} = a_m \end{cases}$$

As before, a and b are positive and define weighted projective surfaces $\mathbb{P}_m = \mathbb{P}(1, a_m, b_m)$ with fixed volume $K_{\mathbb{P}_m}^2 = 8$. The family is unbounded since the Gorenstein index of the surfaces $\mathbb{P}(1, a_m, b_m)$ is equal to $\operatorname{lcm}(a, b)$ and grows to infinity.

2.4.4 Remarks on the case of unbounded $i(\mathcal{F})$

Following the original problem of the previous section, we want to study foliated canonical models of general type with fixed $K_{\mathcal{F}}^2$, $K_{\mathcal{F}} \cdot K_X$ but unbounded $i(\mathcal{F})$. We note that, as mentioned before, such an example cannot come from a bounded family of surfaces. In that case, the index of the singularities is bounded, hence $i(\mathcal{F})$ is bounded as well. This allows us to rule out two cases that would be naturally considered.

- Minimal surfaces of general type: $K_X^2 > 0$ and K_X^2 is bounded from above by the Hodge index theorem, as

  $$K_{\mathcal{F}}^2 K_X^2 \leq (K_{\mathcal{F}} \cdot K_X)^2.$$

  Thus, the family of surfaces is bounded by [Ale94, Theorem 7.7];

- Calabi-Yau surfaces: since $K_{\mathcal{F}}$ is nef and big, $K_{\mathcal{F}}^2$ is fixed and X is klt, by [Bir23, Corollary 1.6], the family is bounded.

While in general Fano surfaces satisfying the previous conditions form unbounded families, they can be ruled out as well:

Proposition 2.4.10. *Let $\{(X,\mathcal{F})\}$ be the collection of canonical models of general type such that X is Fano and $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}}\cdot K_X$ are fixed. Then, the family is bounded.*

Proof. Since $-K_X$ is ample, by the Kawamata-Viehweg theorem

$$\chi(K_{\mathcal{F}}) = \chi(K_X + (K_{\mathcal{F}} - K_X)) = h^0(K_{\mathcal{F}}),$$

and

$$\chi(\mathcal{O}_X) = \chi(K_X + (-K_X)) = 1.$$

Hence, from Theorem 2.1.2,

$$-\sum a(x, K_{\mathcal{F}}) = \frac{1}{2}K_{\mathcal{F}}(K_{\mathcal{F}} - K_X) - h^0(K_{\mathcal{F}}) + 1.$$

Since $h^0(K_{\mathcal{F}}) \geq 0$ and is an integer, $-\sum a(x, K_{\mathcal{F}})$ is bounded from above and it can assume only a finite number of values. Then, we can suppose that $-\sum a(x, K_{\mathcal{F}})$ is fixed, so that we can argue as in [HL21, Proposition 4.1] to show that the number of non-Gorenstein singularities of $\mathcal{F}$ is bounded: let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ be the set of singular points of $\mathcal{F}$, where Σ_1 are terminal singularities, Σ_2 are the dihedral quotient singularities and Σ_3 the cusps. Then,

$$-\sum_{x\in\Sigma} a(x, K_{\mathcal{F}}) = \sum_{x\in\Sigma_1} \frac{n_x - 1}{2n_x} + \sum_{x\in\Sigma_2} \frac{1}{2} + \sum_{x\in\Sigma_3} 1 \geq \frac{1}{4}|\Sigma|,$$

where n_x is the index of the cyclic quotient singularity at x. This shows that $|\Sigma|$ is bounded, which implies that

$$\sum_{x\in\Sigma_1} \frac{1}{n} = |\Sigma| + |\Sigma_3| + 2\sum_{x\in\Sigma} a(x, K_{\mathcal{F}})$$

can assume only finitely many values. By [HL21, Lemma 3.4], then n_x must be bounded, and we can use Theorem 2.3.4 to say that the family is bounded.

$\square$

We conclude by studying the case of algebraically integrable foliations. Let $f\colon X \to C$ be a fibration with reduced fibers, $\mathcal{F}$ the induced foliation, hence $K_{\mathcal{F}} = K_{X/C} = K_X - f^*K_C$. We consider the case of $K_{\mathcal{F}}$ ample and $(X,\mathcal{F})$ with only canonical singularities. Let F be a general fiber of f, then

$$K_{\mathcal{F}}^2 = K_{X/C}^2 = K_X^2 - 8(g(F) - 1)(g(C) - 1),$$

and

$$K_{\mathcal{F}} \cdot K_X = K_X^2 - 4(g(F) - 1)(g(C) - 1).$$

We notice that for fixed $K_{\mathcal{F}}^2$ and $K_{\mathcal{F}} \cdot K_X$, K_X^2 is fixed as well, since

$$K_X^2 = 2(K_{\mathcal{F}} \cdot K_X) - K_{\mathcal{F}}^2.$$

For $g(C) \geq 1$, K_X is ample as well; since K_X^2 is fixed, we get that the family of surfaces is bounded. Thus, the foliated surfaces $(X, \mathcal{F})$ with only foliated canonical singularities are bounded as well. For $g(C) = 0$, consider the linear system $|-f^*K_C|$, which is basepoint free. Let $D_1, D_2 \in |-f^*K_C|$, $D_1 = F_1 + F_2$, $D_2 = F_3 + F_4$ be two general members with $F_1, \dots, F_4$ distinct fibers of f, so that $-f^*K_C \sim_{\mathbb{Q}} \frac{1}{2}(D_1 + D_2)$. Then by [KM98, Lemma 5.17], (X, D) is again klt with $(K_X + D)^2 = K_{\mathcal{F}}^2$ fixed. By [HMX18, Theorem 1.1], we deduce that the pairs (X, D) belong to a bounded family.

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