UNIQUE CONTINUATION FROM A CRACK'S TIP UNDER NEUMANN BOUNDARY CONDITIONS

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Abstract. We derive local asymptotics of solutions to second order elliptic equations at the edge of a $(N-1)$-dimensional crack, with homogeneous Neumann boundary conditions prescribed on both sides of the crack. A combination of blow-up analysis and monotonicity arguments provides a classification of all possible asymptotic homogeneities of solutions at the crack's tip, together with a strong unique continuation principle.

Keywords. Crack singularities; monotonicity formula; unique continuation; blow-up analysis.

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1. Introduction

In this paper we establish a strong unique continuation principle and analyse the asymptotic behaviour of solutions, from the edge of a flat crack Γ, for the following elliptic problem with homogeneous Neumann boundary conditions on both sides of the crack

$$\begin{cases} -\Delta u = fu, & \text{in } B_R \setminus \Gamma, \\ \dfrac{\partial^+ u}{\partial \nu^+} = \dfrac{\partial^- u}{\partial \nu^-} = 0, & \text{on } \Gamma, \end{cases} \tag{1}$$

where

$$B_R = \{x \in \mathbb{R}^N : |x| < R\} \subset \mathbb{R}^N, \quad N \geq 2,$$

Γ is a closed subset of $\mathbb{R}^{N-1} \times \{0\}$ with $C^{1,1}$-boundary, and the potential f satisfies either assumption (H1) or assumption (H2) below. The boundary operators $\frac{\partial^+}{\partial \nu^+}$ and $\frac{\partial^-}{\partial \nu^-}$ in (1) are defined as

$$\frac{\partial^+ u}{\partial \nu^+} := -\frac{\partial}{\partial x_N}\left(u\big|_{B_R^+}\right) \quad \text{and} \quad \frac{\partial^- u}{\partial \nu^-} := \frac{\partial}{\partial x_N}\left(u\big|_{B_R^-}\right),$$

where we are denoting, for all $r > 0$,

$$B_r^+ := \{(x', x_{N-1}, x_N) \in B_r : x_N > 0\}, \quad B_r^- := \{(x', x_{N-1}, x_N) \in B_r : x_N < 0\},$$

being the total variable $x \in \mathbb{R}^N$ written as $x = (x', x_{N-1}, x_N) \in \mathbb{R}^{N-2} \times \mathbb{R} \times \mathbb{R}$.

The interest in elliptic problems in domains with cracks is motivated by elasticity theory, see e.g. [24, 11]. In particular, in crack problems, the coefficients of the asymptotic expansion of solutions near the crack's tip are related to the so called *stress intensity factor*, see [11]. We refer to [9, 10, 15] and references therein for the study of the behaviour of solutions at the edge of a cut.

We recall that a family of functions $\mathcal{F} = \{f_i\}_{i\in I}$, with $f_i : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^N$, satisfies the *strong unique continuation* property if no function in $\mathcal{F}$, besides possibly the trivial null function, has a zero of infinite order at any point $x_0 \in A$. The first significant contribution to the study of strong unique continuation for second order elliptic equations was given by Carleman in [8] for bounded potentials in dimension 2, by means of weighted a priori inequalities. The so-called *Carleman estimates* are still today one of the main techniques used in this research field. They have been adapted by many authors to generalize Carleman's results and prove unique continuation for more general classes of elliptic equations. Among the numerous contributions in this area we mention [4, 23, 28, 32] and in particular [25], where strong unique continuation is established under sharp

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scale invariant assumptions on the potentials. Garofalo and Lin developed in [21] an alternative approach to the study of unique continuation, based on local doubling inequalities, which are in turn deduced by the monotonicity of an Almgren type frequency function, see [3]. In the present paper we follow this latter approach and study the Almgren frequency function $\mathcal{N}$ around the point 0 lying on the edge of the crack. The frequency $\mathcal{N}$ is defined as the ratio between the local energy function

$$E(r) := \frac{1}{r^N} \int_{B_r \setminus \Gamma} (|\nabla u|^2 - fu^2)\, dx$$

and the local mass or height

$$H(r) := \frac{1}{r^{N-1}} \int_{\partial B_r} u^2 \, d\sigma,$$

i.e.

$$\mathcal{N}(r) := \frac{E(r)}{H(r)}.$$

The boundedness of the frequency function $\mathcal{N}$ will imply a strong unique continuation principle from the edge of Γ. Furthermore, the monotonicity properties of the quotient $\mathcal{N}$ will allow us to obtain energy estimates, which will be combined with a blow-up analysis for scaled solutions. In this way we will be able to prove that any $u \in H^1(B_R \setminus \Gamma)$ weakly solving (1) behaves, asymptotically at the edge of the crack Γ, as a homogeneous function with half-integer degree of homogeneity. We mention that an analogous procedure for classifying all possible asymptotic homogeneity degrees of solutions by monotonicity formula and blow-up analysis was introduced in [18, 19, 20] for equations with singular potentials and adapted to domains with corners in [17].

The derivation of a monotonicity formula around a boundary point presents some additional difficulties with respect to the interior case, due to the role that the regularity and the geometry of the domain may play.

Among papers dealing with unique continuation from the boundary under homogeneous Dirichlet conditions we cite [1, 2, 17, 26]. Instead, for Neumann problems, we refer to [1] and [30] for the homogeneous case and to [14] for unique continuation from the vertex of a cone under non-homogeneous Neumann conditions. We also mention that unique continuation from Dirichlet-Neumann junctions for planar mixed boundary value problems was established in [16].

In order to estimate the derivative of the Almgren frequency function, see Proposition 3.10, a Pohozaev type identity is needed. However, the high non-smoothness of the domain $B_R \setminus \Gamma$ at points on the edge of the crack causes two kinds of difficulties in its proof. A first difficulty is a lack of regularity that can prevent us from integrating Rellich-Nečas identities of type (66). A second issue is related to the interference with the geometry of the crack, which manifests in the form of extra terms, produced by integration by parts, which could be problematic to estimate.

In [12], where homogeneous Dirichlet conditions on the crack are considered, this latter difficulty is overcome by assuming a local star-shapedness condition for the cracked domain. This geometric assumption forces the extra terms, produced by integration by parts, to have a sign favourable to the desired estimates. The problem produced by lack of regularity is instead solved in [12] by approximating $B_r \setminus \Gamma$ with a sequence of smooth domains $\Omega_{n,r} \subset B_r$. The solutions u_n of approximating problems in $\Omega_{n,r}$ converge in $H^1(B_r)$ to the solution of the original cracked problem for $r \in (0, R)$ small enough. Each function u_n is sufficiently regular to satisfy a Pohozaev type identity, in which it is possible to pass to the limit as $n \to \infty$. In this way it is possible to establish the inequality needed to estimate the derivative of the Almgren frequency function.

In the present paper we use a similar approximation technique, which however entails additional difficulties and requires substantial modifications due to the Neumann boundary conditions. In particular, the existence of an extension operator for Sobolev functions on Ω_n, uniform with respect to n, is obvious under Dirichlet boundary conditions but it turns out to be more delicate in the Neumann case, see Proposition 2.11. Furthermore the different boundary conditions produce remainder terms with different signs, requiring a modified profile for the approximating domains, see Section 2.3.

Unlike [12], we do not require any geometric star-shapedness condition on the crack Γ, limiting ourselves to a $C^{1,1}$-regularity assumption, see (4) below. The removal of the star-shapedness condition assumed in [12] requires a more sophisticated monotonicity formula, which is developed for the auxiliary problem (21), obtained after straightening the crack Γ with a diffeomorphism introduced in [1], see Section 2.1. We mention that the same diffeomorphism is used for fractional elliptic equations, with a similar purpose, in [13]. The effect of this transformation straightening the crack is the appearance of a variable coefficient matrix in the divergence-form elliptic operator. As a consequence an adaption of the definition of the energy E and the height H in (57) and (58) is needed.

To state the main results of this paper, we introduce now our assumptions on the crack Γ and the potential f. We suppose that Γ is a closed set of the form

$$\Gamma := \{(x_1, 0) : x_1 \in [0, +\infty)\} \quad \text{if } N = 2 \tag{2}$$

and

$$\Gamma := \{(x', x_{N-1}, 0) \in \mathbb{R}^N : g(x') \le x_{N-1}\} \quad \text{if } N \ge 3, \tag{3}$$

where

$$g : \mathbb{R}^{N-2} \to \mathbb{R}, \quad g \in C^{1,1}(\mathbb{R}^{N-2}), \tag{4}$$

and

$$g(0) = 0, \quad \nabla g(0) = 0. \tag{5}$$

Assumption (5) is not restrictive, being a free consequence of an appropriate choice of the Cartesian coordinate system. We are going to study the behaviour of solutions to (1) near 0, which belongs to the edge of the crack Γ defined in (2)–(3).

Furthermore we assume that $f : B_R \to \mathbb{R}$ is a measurable function for which there exists $\epsilon \in (0,1)$ such that either

$$f \in W^{1,\frac{N}{2}+\epsilon}(B_R \setminus \Gamma), \tag{H1}$$

or

$$N \ge 3 \quad \text{and} \quad |f(x)| \le c|x|^{-2+2\epsilon} \quad \text{for some } c > 0 \text{ and for all } x \in B_R. \tag{H2}$$

For every closed set $K \subseteq \mathbb{R}^{N-1} \times \{0\}$ and $r > 0$, we define the functional space $H^1_{0,\partial B_r}(B_r \setminus K)$ as the closure in $H^1(B_r \setminus K)$ of the set

$$\{v \in H^1(B_r \setminus K) : v = 0 \text{ in a neighbourhood of } \partial B_r\}.$$

A weak solution to (1) is a function $u \in H^1(B_R \setminus \Gamma)$ such that

$$\int_{B_R \setminus \Gamma} (\nabla u \cdot \nabla \phi - f u \phi)\, dy = 0,$$

for all $\phi \in H^1_{0,\partial B_R}(B_R \setminus \Gamma)$.

The following unique continuation principle for solutions to (1) is our main result.

Theorem 1.1. *Let u be a weak solution to* (1) *with Γ as in* (2)–(3) *and f satisfying either* (H1) *or* (H2). *If $u(x) = O(|x|^k)$ as $|x| \to 0^+$ for all $k \in \mathbb{N}$, then $u \equiv 0$ in B_R.*

In Theorem 4.8 we provide a classification of blow-up limits in terms of the eigenvalues of the following problem

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}} \psi = \mu \psi, & \text{on } \mathbb{S}^{N-1} \setminus \Sigma, \\ \dfrac{\partial^+ \psi}{\partial \nu^+} = \dfrac{\partial^- \psi}{\partial \nu^-} = 0, & \text{on } \Sigma, \end{cases} \tag{6}$$

on the unit $(N-1)$-dimensional sphere $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ with a cut on the half-equator

$$\Sigma := \{(x', x_{N-1}, 0) \in \mathbb{S}^{N-1} : x_{N-1} \ge 0\},$$

where, letting $e_N := (0, \dots, 1)$,

$$\mathbb{S}^{N-1}_+ := \left\{(x', x_{N-1}, x_N) \in \mathbb{S}^{N-1} : x_N > 0\right\}, \quad \mathbb{S}^{N-1}_- := \left\{(x', x_{N-1}, x_N) \in \mathbb{S}^{N-1} : x_N < 0\right\},$$

the boundary operators $\frac{\partial^\pm}{\partial \nu^\pm}$ are defined as

$$\frac{\partial^+ \psi}{\partial \nu^+} := -\nabla_{\mathbb{S}^{N-1}_+} \left(\psi\big|_{\mathbb{S}^{N-1}_+}\right) \cdot e_N \quad \text{and} \quad \frac{\partial^- \psi}{\partial \nu^-} := \nabla_{\mathbb{S}^{N-1}_-} \left(\psi\big|_{\mathbb{S}^{N-1}_-}\right) \cdot e_N,$$

see Section 4.1 for the weak formulation of (6). In Section 4.1 we prove that the set of the eigenvalues of (6) is $\{\mu_k : k \in \mathbb{N}\}$ where

$$\mu_k = \frac{k(k + 2N - 4)}{4}, \quad k \in \mathbb{N}.$$

As a consequence of the classification of blow-up limits, we obtain the following unique continuation result from the edge with respect to crack points.

Theorem 1.2. *Let u be a weak solution to* (1) *with Γ as in* (2)–(3) *and f satisfying either* (H1) *or* (H2). *Let us also assume that u vanishes at 0 at any order with respect to crack points, namely that either $\operatorname{Tr}^+_\Gamma u(z) = O(|z|^k)$ as $|z| \to 0^+$, $z \in \Gamma$, for all $k \in \mathbb{N}$ or $\operatorname{Tr}^-_\Gamma u(z) = O(|z|^k)$ as $|z| \to 0^+$, $z \in \Gamma$, for all $k \in \mathbb{N}$, where $\operatorname{Tr}^+_\Gamma u$, respectively $\operatorname{Tr}^-_\Gamma u$, denotes the trace of $u\big|_{B^+_R}$, respectively $u\big|_{B^-_R}$, on Γ. Then $u \equiv 0$ in B_R.*

If $N \geq 3$, we can combine the blow-up analysis with an expansion in Fourier series with respect to a orthonormal basis made of eigenfunctions of (6). This allows us to classify the possible asymptotic homogeneity degrees of solutions at 0.

Theorem 1.3. *Let $N \geq 3$ and let $u \in H^1(B_R \setminus \Gamma)$, $u \not\equiv 0$, be a non-trivial weak solution to* (1), *with Γ defined in* (2)–(3) *and f satisfying either assumption* (H1) *or assumption* (H2). *Then there exist $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem* (6), *associated to the eigenvalue μ_{k_0}, such that, letting*

$$\Phi(x) := |x|^{\frac{k_0}{2}} Y\left(\frac{x}{|x|}\right),$$

we have that

$$\lambda^{-\frac{k_0}{2}} u(\lambda \cdot) \to \Phi \quad \text{and} \quad \lambda^{1-\frac{k_0}{2}} \left(\nabla_{B_R \setminus \Gamma} u\right)(\lambda \cdot) \to \nabla_{\mathbb{R}^N \setminus \tilde{\Gamma}} \Phi \quad \text{in } L^2(B_1)$$

as $\lambda \to 0^+$, where

$$\tilde{\Gamma} := \left\{x = (x', x_{N-1}, 0) \in \mathbb{R}^N : x_{N-1} \geq 0\right\} \tag{7}$$

and $\nabla_{B_R \setminus \Gamma}$ and $\nabla_{\mathbb{R}^N \setminus \tilde{\Gamma}}$ denote the distributional gradients in $B_R \setminus \Gamma$ and $\mathbb{R}^N \setminus \tilde{\Gamma}$ respectively.

A more precise version of Theorem 1.3, relating k_0 to the limit of a frequency function and characterizing the eigenfunction Y, will be proved in Section 5, see Theorem 5.3.

The paper is organized as follows. In Section 2.1 an equivalent problem in a domain with a straightened crack is constructed. Sections 2.2 contains some trace and embedding inequalities for the space $H^1(B_r \setminus \tilde{\Gamma})$. Section 2.3 is devoted to the construction of the approximating problems. In Section 3 we develop the monotonicity argument, which is first used to prove Theorem 1.1 and later, in Section 4.2, to perform a blow-up analysis and prove Theorem 1.2, taking into account the structure of the spherical eigenvalue problem (6) studied in Section 4.1. Finally Theorem 1.3 is proved in Section 5.

2. An equivalent problem with straightened crack and approximation procedure

In this section we first introduce an equivalent problem with a straightened crack; then we develop an approximation procedure regularizing the domain, for which suitable trace and embedding inequalities are needed.

2.1. **An equivalent problem with straightened crack.** In this section we straighten the boundary of the crack in a neighbourhood of 0. If $N \geq 3$ we use the local diffeomorphism F defined in [13, Section 2], see also [1]; for the sake of clarity and completeness we summarize its properties in Propositions 2.1 and 2.2 below, referring to [13, Section 2] for their proofs. If $N = 2$, the crack is a segment and we simply take $F = \mathrm{Id}$, where Id is the identity function on $\mathbb{R}^2$.

Proposition 2.1. [13, Section 2] *Let $N \geq 3$ and Γ be defined in (3) with g satisfying (4) and (5). There exist $F = (F_1, \dots, F_N) \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ and $r_1 > 0$ such that $F\big|_{B_{r_1}} : B_{r_1} \to F(B_{r_1})$ is a diffeomorphism of class $C^{1,1}$,*

$$F(y', 0, 0) = (y', g(y'), 0) \text{ for any } y' \in \mathbb{R}^{N-1}, \quad \text{and} \quad F(\tilde{\Gamma} \cap B_{r_1}) = \Gamma \cap F(B_{r_1}),$$

with $\tilde{\Gamma}$ as in (7). Furthermore, letting $J_F(y)$ be the Jacobian matrix of F at $y = (y', y_{N-1}, y_N) \in B_{r_1}$ and

$$A(y) := |\det J_F(y)| (J_F(y))^{-1} ((J_F(y))^{-1})^T, \tag{8}$$

the following properties hold:

i) *J_F depends only on the variable $y'' = (y', y_{N-1})$ and*

$$J_F(y) = J_F(y'') = \mathrm{Id}_N + O(|y''|) \quad \text{as } |y''| \to 0^+,$$

where Id_N denotes the identity $N \times N$ matrix and $O(|y''|)$ denotes a matrix with all entries being $O(|y''|)$ as $|y''| \to 0^+$;

ii) *$\det J_F(y) = \det J_F(y', y_{N-1}) = 1 + O(|y'|^2) + O(y_{N-1})$ as $|y'| \to 0^+$ and $y_{N-1} \to 0$;*

iii) *$\frac{\partial F_i}{\partial y_N} = \frac{\partial F_N}{\partial y_i} = 0$ for any $i = 1, \dots, N-1$ and $\frac{\partial F_N}{\partial y_N} = 1$;*

iv) *the matrix-valued function A can be written as*

$$A(y) = A(y', y_{N-1}) = \left(\begin{array}{c|c} D(y', y_{N-1}) & 0 \\ \hline 0 & \det J_F(y', y_{N-1}) \end{array} \right), \tag{9}$$

with

$$D(y', y_{N-1}) = \left(\begin{array}{c|c} \mathrm{Id}_{N-2} + O(|y'|^2) + O(y_{N-1}) & O(y_{N-1}) \\ \hline O(y_{N-1}) & 1 + O(|y'|^2) + O(y_{N-1}) \end{array} \right), \tag{10}$$

where Id_{N-2} denotes the identity $(N-2) \times (N-2)$ matrix and $O(y_{N-1})$, respectively $O(|y'|^2)$, denotes blocks of matrices with all entries being $O(y_{N-1})$ as $y_{N-1} \to 0$, respectively $O(|y'|^2)$ as $|y'| \to 0$.

v) *A is symmetric with coefficients of class $C^{0,1}$ and*

$$\frac{1}{2}|z|^2 \leq A(y)z \cdot z \leq 2|z|^2 \quad \text{for all } z \in \mathbb{R}^N \text{ and } y \in B_{r_1}. \tag{11}$$

We note that (11) implies that $\|A(y)\|_{\mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)} \leq 2$ for all $y \in B_{r_1}$. We also observe

$$A = \mathrm{Id}_2 \quad \text{if } N = 2. \tag{12}$$

Moreover (9)– (10) easily imply that

$$A(y) = A(y'') = \mathrm{Id}_N + O(|y''|) \quad \text{as } |y''| \to 0^+. \tag{13}$$

Under the same assuptions and with the same notation of Proposition 2.1, we define

$$\mu(y) := \frac{A(y)y \cdot y}{|y|^2} \quad \text{and} \quad \beta(y) := \frac{A(y)y}{\mu(y)} \quad \text{for any } y \in B_{r_1} \setminus \{0\}. \tag{14}$$

Proposition 2.2. [13, Section 2] *Under the same assumptions as Proposition 2.1, let μ and β be as in* (14). *Then, possibly choosing r_1 smaller from the beginning,*

$$\frac{1}{2} \leq \mu(y) \leq 2 \quad \text{for any } y \in B_{r_1} \setminus \{0\}, \tag{15}$$

$$\mu(y) = 1 + O(|y|) \quad \text{as } |y| \to 0^+, \tag{16}$$

$$\nabla\mu(y) = O(1) \quad \text{as } |y| \to 0^+. \tag{17}$$

Moreover β is well-defined and

$$\beta(y) = y + O(|y|^2) = O(|y|) \quad \text{as } |y| \to 0^+, \tag{18}$$

$$J_\beta(y) = A(y) + O(|y|) = \mathrm{Id}_N + O(|y|) \quad \text{as } |y| \to 0^+,$$

$$\mathrm{div}(\beta)(y) = N + O(|y|) \quad \text{as } |y| \to 0^+. \tag{19}$$

We also define $dA(y)zz$, for every $z = (z_1, \dots, z_N) \in \mathbb{R}^N$ and $y \in B_{r_1}$, as the vector of $\mathbb{R}^N$ with i-th component, for $i = 1, \dots, N$, given by

$$(dA(y)zz)_i = \sum_{h,k=1}^{N} \frac{\partial a_{kh}}{\partial y_i} z_h z_k, \tag{20}$$

where we have defined the matrix $A = (a_{k,h})_{k,h=1,\dots,N}$ in (8).

Remark 2.3. For any measurable function $f : F(B_{r_1}) \to \mathbb{R}$ we set

$$\tilde{f} : B_{r_1} \to \mathbb{R}, \quad \tilde{f} := |\det J_F| \, (f \circ F).$$

Then, in view of i) and ii) in Proposition 2.1, the function $\tilde{f}$ satisfies assumptions (H1) or (H2) on B_{r_1} if and only if f satisfies assumptions (H1) or (H2) on $F(B_{r_1})$.

It is easy to see that, if u is a solution to (1), then the function $U := u \circ F$ belongs to $H^1(B_{r_1} \setminus \tilde{\Gamma})$ and is a weak solution of the problem

$$\begin{cases} -\mathrm{div}(A\nabla U) = \tilde{f}u, & \text{in } B_{r_1} \setminus \tilde{\Gamma}, \\ A\nabla^+ U \cdot \nu^+ = A\nabla^- U \cdot \nu^- = 0, & \text{on } \tilde{\Gamma}, \end{cases} \tag{21}$$

where

$$\nabla^+ U = \nabla\left(U\big|_{B^+_{r_1}}\right), \quad \nabla^- U = \nabla\left(U\big|_{B^-_{r_1}}\right), \quad \text{and } \nu^- = -\nu^+ = (0, \dots, 1).$$

By saying that U is a weak solution to (21) we mean that $U \in H^1(B_{r_1} \setminus \tilde{\Gamma})$ and

$$\int_{B_{r_1} \setminus \tilde{\Gamma}} (A\nabla U \cdot \nabla\phi - \tilde{f} U \phi) \, dy = 0$$

for all $\phi \in H^1_{0,\partial B_{r_1}}(B_{r_1} \setminus \tilde{\Gamma})$.

2.2. **Traces and embeddings for the space $H^1(B_{r_1} \setminus \tilde{\Gamma})$.** In this section, we present some trace and embedding inequalities for the space $H^1(B_{r_1} \setminus \tilde{\Gamma})$ which will be used throughout the paper.

We define the even reflection operators

$$\mathcal{R}^+(v)(y', y_{N-1}, y_N) = v(y', y_{N-1}, |y_N|),$$
$$\mathcal{R}^-(v)(y', y_{N-1}, y_N) = v(y', x_{N-1}, -|y_N|),$$

and observe that, for all $r > 0$, $\mathcal{R}^+ : H^1(B_r \setminus \tilde{\Gamma}) \to H^1(B_r)$ and $\mathcal{R}^- : H^1(B_r \setminus \tilde{\Gamma}) \to H^1(B_r)$. We have that $\mathcal{R}^+(v), \mathcal{R}^-(v) \in L^p(B_r)$ for some $p \in [1, \infty)$ if and only if $v \in L^p(B_r)$; in such a case we have that

$$\left\|\mathcal{R}^+(v)\right\|^p_{L^p(B_r)} = 2\,\|v\|^p_{L^p(B_r^+)}, \quad \left\|\mathcal{R}^-(v)\right\|^p_{L^p(B_r)} = 2\,\|v\|^p_{L^p(B_r^-)}, \tag{22}$$

and

$$\|v\|^p_{L^p(B_r)} = \frac{1}{2}\left(\left\|\mathcal{R}^+(v)\right\|^p_{L^p(B_r)} + \left\|\mathcal{R}^-(v)\right\|^p_{L^p(B_r)}\right). \tag{23}$$

Furthermore, for every $v \in H^1(B_r \setminus \tilde{\Gamma})$,

$$\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 \, dy = \frac{1}{2} \left(\int_{B_r} |\nabla \mathcal{R}^+(v)|^2 \, dy + \int_{B_r} |\nabla \mathcal{R}^-(v)|^2 \, dy \right). \tag{24}$$

Proposition 2.4. *For any $r > 0$ there exists a linear continuous trace operator*

$$\gamma_r : H^1(B_r \setminus \tilde{\Gamma}) \to L^2(\partial B_r).$$

Furthermore γ_r is compact.

Proof. Since B_r^+ and B_r^- are Lipschitz domains, there exist two linear, continuous and compact trace operators $\gamma_r^+ : H^1(B_r^+) \to L^2(\partial B_r^+ \cap \partial B_r)$ and $\gamma_r^- : H^1(B_r^-) \to L^2(\partial B_r^- \cap \partial B_r)$. By setting

$$\gamma_r(v)(y) := \begin{cases} \gamma_r^+(v)(y), & \text{if } y_N > 0, \\ \gamma_r^-(v)(y), & \text{if } y_N < 0, \end{cases}$$

we complete the proof. □

Letting γ_r be the trace operator introduced in Proposition 2.4, we observe that

$$\int_{\partial B_r} |\gamma_r(v)|^2 \, dS = \frac{1}{2} \left(\int_{\partial B_r} |\gamma_r(\mathcal{R}^+(v))|^2 \, dS + \int_{\partial B_r} |\gamma_r(\mathcal{R}^-(v))|^2 \, dS \right) \tag{25}$$

for every $v \in H^1(B_r \setminus \tilde{\Gamma})$. With a slight abuse of notation we will often write v instead of $\gamma_r(v)$ on ∂B_r.

Proposition 2.5. *If $N \geq 3$ and $r > 0$, then, for any $v \in H^1(B_r \setminus \tilde{\Gamma})$,*

$$\left(\frac{N-2}{2}\right)^2 \int_{B_r} \frac{v^2}{|x|^2} \, dx \leq \int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 \, dx + \frac{N-2}{2r} \int_{\partial B_r} v^2 \, dS. \tag{26}$$

Proof. By scaling, [31, Theorem 1.1] proves the claim for $\mathcal{R}^+(v)$ and $\mathcal{R}^-(v)$. Then we conclude by (23), (24), and (25). □

Proposition 2.6. *Let $N \geq 2$ and $q \geq 1$ be such that $q \leq 2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $q < \infty$ if $N = 2$. Then*

$$H^1(B_r \setminus \tilde{\Gamma}) \subset L^q(B_r) \quad \text{for every } r > 0$$

and there exists $\mathcal{S}_{N,q} > 0$ (depending only on N and q) such that

$$\|v\|^2_{L^q(B_r)} \leq \mathcal{S}_{N,q} \, r^{\frac{N(2-q)+2q}{q}} \left(\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 \, dx + \frac{1}{r} \int_{\partial B_r} v^2 \, dS \right), \tag{27}$$

for all $r > 0$ and $v \in H^1(B_r \setminus \tilde{\Gamma})$.

Proof. Since

$$\left(\int_{B_1} |\nabla v|^2 \, dx + \int_{\partial B_1} v^2 \, dS \right)^{\frac{1}{2}}$$

is an equivalent norm on $H^1(B_1)$, from a scaling argument and Sobolev embedding Theorems it follows that, for all $q \in [1, 2^*]$ if $N \geq 3$ and $q \in [1, \infty)$ if $N = 2$, there exists $\mathcal{S}_{N,q} > 0$ such that, for all $r > 0$ and $v \in H^1(B_r)$,

$$\|v\|^2_{L^q(B_r)} \leq \mathcal{S}_{N,q} r^{\frac{N(2-q)+2q}{q}} \left(\int_{B_r} |\nabla v|^2 \, dx + \frac{1}{r} \int_{\partial B_r} v^2 \, dS \right).$$

Using (22), (23), (24) and (25) we complete the proof. □

Proposition 2.7. *For any $r > 0$, $h \in L^{\frac{N}{2}+\epsilon}(B_r)$ with $\epsilon > 0$, and $v \in H^1(B_r \setminus \tilde{\Gamma})$, there holds*

$$\int_{B_r} |h| v^2 \leq \eta_h(r) \left(\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 \, dx + \frac{1}{r} \int_{\partial B_r} v^2 \, dS \right), \tag{28}$$

where

$$\eta_h(r) = \mathcal{S}_{N,q_\epsilon} \, \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \, r^{\frac{4\epsilon}{N+2\epsilon}} \quad \text{and} \quad q_\epsilon := \frac{2N+4\epsilon}{N-2+2\epsilon}. \tag{29}$$

Proof. For any $v \in H^1(B_r \setminus \tilde{\Gamma})$

$$\int_{B_r} |h| v^2\, dx \le \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \left(\int_{B_r} |v|^{q_\epsilon}\, dx \right)^{2/q_\epsilon}$$
$$\le \mathcal{S}_{N,q_\epsilon} \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} r^{\frac{4\epsilon}{N+2\epsilon}} \left(\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dx + \frac{1}{r} \int_{\partial B_r} v^2 dS \right)$$

thanks to Hölder's inequality and (27). □

Remark 2.8. If f satisfies (H2), then $f \in L^{\frac{N}{2}+\epsilon}(B_R)$, so that Proposition 2.7 applies to potentials satisfying either (H1) or (H2).

Remark 2.9. By (28), (15) and (11), for any $r \in (0, r_1)$, $h \in L^{\frac{N}{2}+\epsilon}(B_r)$, and $v \in H^1(B_r \setminus \tilde{\Gamma})$, we have that

$$\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dy \le 2 \int_{B_r \setminus \tilde{\Gamma}} (A \nabla v \cdot \nabla v - h v^2)\, dy + 2\eta_h(r) \left(\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dy + \frac{2}{r} \int_{\partial B_r} \mu v^2\, dS \right)$$

and therefore, if $\eta_h(r) < \frac{1}{2}$,

$$\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2\, dy \le \frac{2}{1 - 2\eta_h(r)} \int_{B_r \setminus \tilde{\Gamma}} (A \nabla v \cdot \nabla v - h v^2)\, dy + \frac{4\eta_h(r)}{(1 - 2\eta_h(r)) r} \int_{\partial B_r} \mu v^2\, dS. \tag{30}$$

2.3. **Approximating problems.** In this section we construct a sequence of problems in smooth sets approximating the straightened cracked domain. We define, for any $n \in \mathbb{N} \setminus \{0\}$,

$$g_n : \mathbb{R} \to \mathbb{R}, \quad g_n(t) := n t^4$$

and, for any $r \in (0, r_1]$,

$$\Omega_{n,r} := \{(y', y_{N-1}, y_N) \in B_r : y_{N-1} < g_n(y_N)\}$$

and

$$\Gamma_{n,r} := \{(y', y_{N-1}, y_N) \in B_r : y_{N-1} = g_n(y_N)\} = \partial\Omega_{n,r} \cap B_r.$$

The domains $\Omega_{n,r}$ approximate $B_r \setminus \tilde{\Gamma}$ in the following sense: for every $y \in B_r \setminus \tilde{\Gamma}$, there exists $\bar{n} \in \mathbb{N} \setminus \{0\}$ such that $y \in \Omega_{n,r}$ for all $n \ge \bar{n}$. Moreover $\Omega_{n,r} \cap \tilde{\Gamma} = \emptyset$ for any $r \in (0, r_1]$ and $n \in \mathbb{N} \setminus \{0\}$. We also note that $\Omega_{n,r}$ is a Lipschitz domain and $\Gamma_{n,r}$ is a C^2-smooth portion of its boundary.

Proposition 2.10. *Let $\nu(y)$ be the outward normal vector to $\partial\Omega_{n,r_1}$ in y. Then*

$$y \cdot \nu(y) \le 0 \quad \text{for all } y \in \Gamma_{n,r_1}, \tag{31}$$

$$A(y) y \cdot \nu(y) \le 0 \quad \text{for all } y \in \Gamma_{n,r_1}. \tag{32}$$

Proof. As a first step we notice that

$$g_n(t) - \frac{1}{3} t g_n'(t) = n t^4 - \frac{4}{3} n t^4 = -\frac{1}{3} n t^4 \le 0, \quad g_n(t) - t g_n'(t) \le 0 \tag{33}$$

and that

$$\nu(y) = \frac{(0, 1, -g_n'(y_N))}{\sqrt{1 + (g_n'(y_N))^2}} \quad \text{for all } y \in \Gamma_{n,r_1}.$$

Then, for all $y \in \Gamma_{n,r_1}$,

$$\nu(y) \cdot y = \frac{(0, 1, -g_n'(y_N))}{\sqrt{1 + (g_n'(y_N))^2}} \cdot (y', g_n(y_N), y_N) = \frac{g_n(y_N) - y_N g_n'(y_N)}{\sqrt{1 + (g_n'(y_N))^2}} \le 0$$

due to (33). We have then proved (31) (and (32) in the case $N = 2$ in view of (12)).

If $N \geq 3$, possibly choosing r_1 smaller in Proposition 2.1, for all $y \in \Gamma_{n,r_1}$ we have that

$$\sqrt{1+(g_n'(y_N))^2}A(y)y\cdot\nu(y) = g_n(y_N)(1+O(|y'|)+O(y_{N-1})) - \det J_F(y)\, y_N g_n'(y_N)$$
$$\leq \frac{3}{2}g_n(y_N) - \frac{1}{2}y_N g_n'(y_N) = \frac{3}{2}(g_n(y_N) - \frac{1}{3}y_N g_n'(y_N)),$$

thanks to ii) in Proposition 2.1, (9) and (10). Then, by (33) we finally obtain (32) also for $N \geq 3$. □

Let

$$\mathbb{R}^N_+ := \{y = (y', y_{N-1}, y_N) \in \mathbb{R}^N : y_N > 0\} \text{ and } \mathbb{R}^N_- := \{y = (y', y_{N-1}, y_N) \in \mathbb{R}^N : y_N < 0\}.$$

For any $r \in (0, r_1]$ and $n \in \mathbb{N} \setminus \{0\}$ let

$$\Omega^+_{n,r} := \Omega_{n,r} \cap B^+_r, \quad \Omega^-_{n,r} := \Omega_{n,r} \cap B^-_r, \quad S_{n,r} := \partial\Omega_{n,r} \cap \partial B_r. \tag{34}$$

For all $n \in \mathbb{N} \setminus \{0\}$ we also define

$$K^+_{n,r_1} := \{y = (y', y_{N-1}, y_N) \in \mathbb{R}^N_+ : \text{either } y_{N-1} < g_n(y_N) \text{ or } |y| > r_1\},$$
$$K^-_{n,r_1} := \{y = (y', y_{N-1}, y_N) \in \mathbb{R}^N_- : \text{either } y_{N-1} < g_n(y_N) \text{ or } |y| > r_1\}.$$

Since $\Omega_{n,r}$ is a Lipschitz domain, for any $r \in (0, r_1]$ and $n \in \mathbb{N} \setminus \{0\}$ there exists a trace operator

$$\gamma_{n,r} : H^1(\Omega_{n,r}) \to L^2(\partial\Omega_{n,r}).$$

We define

$$H^1_{0,S_{n,r}}(\Omega_{n,r}) := \{u \in H^1(\Omega_{n,r}) : \gamma_{n,r}(u) = 0 \text{ on } S_{n,r}\}.$$

The following proposition provides an extension operator from $H^1_{0,S_{n,r}}(\Omega_{n,r})$ to $H^1(B_{r_1} \setminus \tilde{\Gamma})$ with an operator norm bounded uniformly with respect to n.

Proposition 2.11. *For any $r \in (0, r_1)$ and $n \in \mathbb{N} \setminus \{0\}$ there exists an extension operator*

$$\xi^0_{n,r} : H^1_{0,S_{n,r}}(\Omega_{n,r}) \to H^1(B_{r_1} \setminus \tilde{\Gamma}) \tag{35}$$

such that, for any $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$,

$$\xi^0_{n,r}(\phi)\big|_{\Omega_{n,r}} = \phi, \quad \xi^0_{n,r}(\phi) = 0 \text{ on } \Omega_{n,r_1} \setminus \Omega_{n,r}, \quad \xi^0_{n,r}(\phi) \in H^1_{0,\partial B_{r_1}}(B_{r_1} \setminus \tilde{\Gamma}), \tag{36}$$

and

$$\left\|\xi^0_{n,r}(\phi)\right\|_{H^1(B_{r_1}\setminus\tilde{\Gamma})} \leq c_0 \|\phi\|_{H^1(\Omega_{n,r})} = c_0\left(\int_{\Omega_{n,r}} \left(\phi^2 + |\nabla\phi|^2\right) dy\right)^{1/2}, \tag{37}$$

where $c_0 > 0$ is independent of n, r, and ϕ.

Proof. It is well known that, since K^+_{n,r_1} and K^-_{n,r_1} are uniformly Lipschitz domains, there exist continuous extension operators $\xi^+_n : H^1(K^+_{n,r_1}) \to H^1(\mathbb{R}^N_+)$ and $\xi^-_n : H^1(K^-_{n,r_1}) \to H^1(\mathbb{R}^N_-)$, see [29], [7] and [27]. Furthermore, since the Lipschitz constants of the parameterization of $\partial K^+_{n,r_1}$ and $\partial K^-_{n,r_1}$ are bounded uniformly with respect to n, there exists a constant $C > 0$, which does not depend on n, such that

$$\left\|\xi^+_n(v)\right\|_{H^1(\mathbb{R}^N_+)} \leq C\|v\|_{H^1(K^+_{n,r_1})} \quad \text{and} \quad \left\|\xi^-_n(w)\right\|_{H^1(\mathbb{R}^N_-)} \leq C\|w\|_{H^1(K^-_{n,r_1})} \tag{38}$$

for all $v \in H^1(K^+_{n,r_1})$ and $w \in H^1(K^-_{n,r_1})$.

If $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$ then the trivial extension $\bar{\phi}_+$ of $\phi\big|_{\Omega^+_{n,r}}$ to K^+_{n,r_1} belongs to $H^1(K^+_{n,r_1})$ and the trivial extension $\bar{\phi}_-$ of $\phi\big|_{\Omega^-_{n,r}}$ to K^-_{n,r_1} belongs to $H^1(K^-_{n,r_1})$. Then we define

$$\xi^0_{n,r}(\phi)(y) := \begin{cases} \xi^+_n(\bar{\phi}_+)(y), & \text{if } y \in B^+_{r_1}, \\ \xi^-_n(\bar{\phi}_-)(y), & \text{if } y \in B^-_{r_1}, \end{cases}$$

which belongs to $H^1(B_{r_1} \setminus \tilde{\Gamma})$ and satisfies (37) in view of (38). Furthermore (36) follows directly from the definition of $\xi^0_{n,r}$. □

The following proposition establishes a Poincaré type inequality for $H^1_{0,S_{n,r}}(\Omega_{n,r})$-functions, with a constant independent of n.

Proposition 2.12. *For any $r \in (0, r_1]$, $n \in \mathbb{N} \setminus \{0\}$, and $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$*

$$\int_{\Omega_{n,r}} \phi^2 dy \leq \frac{r^2}{N-1} \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy \tag{39}$$

and

$$\|\phi\|_{H^1_{0,S_{n,r}}(\Omega_{n,r})} := \left(\int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy \right)^{\frac{1}{2}} \tag{40}$$

is an equivalent norm on $H^1_{0,S_{n,r}}(\Omega_{n,r})$.

Proof. For any $\phi \in C^\infty(\overline{\Omega}_{n,r})$ such that $\phi = 0$ in a neighbourhood of $S_{n,r}$ we have that

$$\operatorname{div}(\phi^2 y) = 2\phi \nabla \phi \cdot y + N\phi^2$$

so that

$$N \int_{\Omega_{n,r}} \phi^2 \, dy = -2 \int_{\Omega_{n,r}} \phi \nabla \phi \cdot y \, dy + \int_{\Gamma_{n,r}} \phi^2 y \cdot \nu \, dS \leq \int_{\Omega_{n,r}} \phi^2 \, dy + r^2 \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy,$$

since $y \cdot \nu \leq 0$ on $\Gamma_{n,r}$ by(31). Then we may conclude that

$$\int_{\Omega_{n,r}} \phi^2 \, dy \leq \frac{r^2}{N-1} \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy,$$

for all $\phi \in C^\infty(\overline{\Omega}_{n,r})$ such that $\phi = 0$ in a neighbourhood of $S_{n,r}$. Since $\Omega_{n,r}$ is a Lipschitz domain, (39) holds for any $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$ by [6, Theorem 3.1]. The second claim is now obvious. $\square$

From now on we consider on $H^1_{0,S_{n,r}}(\Omega_{n,r})$ the norm $\|\cdot\|_{H^1_{0,S_{n,r}}}$ defined in (40).

Proposition 2.13. *Let $r \in (0, r_1)$, $n \in \mathbb{N} \setminus \{0\}$, $h \in L^{\frac{N}{2}+\epsilon}(B_r)$ with $\epsilon > 0$, and q_ϵ be as in (29). Then, for any $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$,*

$$\int_{\Omega_{n,r}} |h|\phi^2 \, dy \leq c_0^2 \frac{N-1+r_1^2}{N-1} \mathcal{S}_{N,q_\epsilon} r_1^{\frac{4\epsilon}{N+2\epsilon}} \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy. \tag{41}$$

Proof. We have, for every $\phi \in H^1_{0,S_{n,r}}(\Omega_{n,r})$,

$$\begin{aligned}
\int_{\Omega_{n,r}} |h|\phi^2 \, dy &\leq \int_{B_r} |h||\xi^0_{n,r}(\phi)|^2 \, dy \leq \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \left(\int_{B_{r_1}} |\xi^0_{n,r}(\phi)|^{q_\epsilon} \, dy \right)^{\frac{2}{q_\epsilon}} \\
&\leq \mathcal{S}_{N,q_\epsilon} r_1^{\frac{4\epsilon}{N+2\epsilon}} \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \int_{B_{r_1} \setminus \tilde{\Gamma}} |\nabla \xi^0_{n,r}(\phi)|^2 \, dy \\
&\leq c_0^2 \frac{N-1+r^2}{N-1} \mathcal{S}_{N,q_\epsilon} r_1^{\frac{4\epsilon}{N+2\epsilon}} \|h\|_{L^{\frac{N}{2}+\epsilon}(B_r)} \int_{\Omega_{n,r}} |\nabla \phi|^2 \, dy,
\end{aligned}$$

thanks to Hölder's inequality, (27), Proposition 2.11, and Proposition 2.12. $\square$

Hereafter we fix a potential f satisfying either (H1) or (H2) and define $\tilde{f} := |\det J_F| \, (f \circ F)$ as in Remark 2.3. Thanks to Remark 2.3 we have that $\tilde{f}$ satisfies either (H1) or (H2) as well. If f (and consequently $\tilde{f}$) satisfies (H2), we define

$$f_n(y) = \begin{cases} n, & \text{if } \tilde{f}(y) > n, \\ \tilde{f}(y), & \text{if } |\tilde{f}(y)| \leq n, \\ -n, & \text{if } \tilde{f}(y) < -n, \end{cases} \tag{42}$$

so that

$$f_n \in L^\infty(B_{r_1}) \quad \text{and} \quad |f_n| \leq |\tilde{f}| \text{ a.e. in } B_{r_1} \quad \text{for all } n \in \mathbb{N} \setminus \{0\} \tag{43}$$

and

$$f_n \to \tilde{f} \text{ a.e. in } B_{r_1}. \tag{44}$$

If f satisfies (H1), we just let

$$f_n := \tilde{f} \quad \text{for any } n \in \mathbb{N}. \tag{45}$$

We observe that

$$f_n \to \tilde{f} \quad \text{in } L^{\frac{N}{2}+\epsilon}(B_{r_1}) \quad \text{as } n \to \infty \tag{46}$$

as a consequence of (43), (44) and the Dominated Convergence Theorem if assumption (H2) holds and f_n is defined in (42), in view of Remark 2.8; on the other hand (46) is obvious if assumption (H1) holds and f_n is defined in (45).

Since under both assumptions (H1) and (H2) we have that $\tilde{f} \in L^{\frac{N}{2}+\epsilon}(B_{r_1})$ (see Remark 2.8), by the absolute continuity of the Lebesgue integral we can choose $r_0 \in (0, \min\{1, r_1\})$ such that

$$\eta_{\tilde{f}}(r_0) < \frac{1}{2} \quad \text{and} \quad c_0^2 \frac{N-1+r_1^2}{N-1} \mathcal{S}_{N,q_\epsilon} r_1^{\frac{4\epsilon}{N+2\epsilon}} \|\tilde{f}\|_{L^{\frac{N}{2}+\epsilon}(B_{r_0})} < \frac{1}{4}, \tag{47}$$

where q_ϵ and $\eta_{\tilde{f}}$ are defined in (29).

Let $U = u \circ F$, where u is a fixed weak solution to (1) and F is the diffeomorphism introduced in Section 2.1, so that U weakly solves (21). For any $n \in \mathbb{N} \setminus \{0\}$, we consider the following sequence of approximating problems, with potentials f_n defined in (42)–(45):

$$\begin{cases} -\operatorname{div}(A\nabla U_n) = f_n U_n, & \text{in } \Omega_{n,r_0}, \\ A\nabla U_n \cdot \nu = 0, & \text{on } \Gamma_{n,r_0}, \\ \gamma_{n,r_0}(U_n) = \gamma_{n,r_0}(U), & \text{on } S_{n,r_0}, \end{cases} \tag{48}$$

with r_0 as in (47). A weak solution to problem (48) is a function $U_n \in H^1(\Omega_{n,r_0})$ such that $U_n - U \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ and

$$\int_{\Omega_{n,r_0}} (A\nabla U_n \cdot \nabla\phi - f_n U_n \phi)\, dy = 0$$

for all $\phi \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$. If U_n weakly solves (48), then $W_n := U - U_n \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ and

$$\int_{\Omega_{n,r_0}} (A\nabla W_n \cdot \nabla\phi - f_n W_n \phi)\, dy = \int_{\Omega_{n,r_0}} (A\nabla U \cdot \nabla\phi - f_n U \phi)\, dy \tag{49}$$

for any $\phi \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$.

For every $n \in \mathbb{N} \setminus \{0\}$, let us consider the bilinear form

$$B_n : H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}) \times H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}) \to \mathbb{R}, \quad B_n(v,\phi) := \int_{\Omega_{n,r_0}} (A\nabla v \cdot \nabla\phi - f_n v \phi)\, dy, \tag{50}$$

and the functional

$$L_n : H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}) \to \mathbb{R}, \quad L_n(\phi) := \int_{\Omega_{n,r_0}} (A\nabla U \cdot \nabla\phi - f_n U \phi)\, dy. \tag{51}$$

Proposition 2.14. *The bilinear form B_n defined in* (50) *is continuous and coercive; more precisely*

$$B_n(\phi,\phi) \geq \frac{1}{4} \|\phi\|^2_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})} \quad \textit{for all } \phi \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}). \tag{52}$$

Furthermore the functional L_n defined in (51) *belongs to $(H^1_{0,S_{n,r_0}}(\Omega_{n,r_0}))^*$ and there exists a constant $\ell > 0$ independent of n such that*

$$|L_n(\phi)| \leq \ell \|\phi\|_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})} \quad \textit{for all } \phi \in H^1_{0,S_{n,r}}(\Omega_{n,r_0}). \tag{53}$$

Proof. The continuity of B_n and (52) easily follow from (11),(43), (41) and (47). Thanks to Hölder's inequality, (43), (11), (28), (41) and (47)

$$
\begin{aligned}
|L_n(\phi)| &\leq 2\,\|\nabla U\|_{L^2(\Omega_{n,r_0})}\,\|\phi\|_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})} + \left(\int_{B_{r_0}} |\tilde f| U^2\,dx\right)^{\frac12}\left(\int_{\Omega_{n,r_0}} |\tilde f|\phi^2\,dx\right)^{\frac12} \\
&\leq \left(2\,\|\nabla U\|_{L^2(B_{r_0}\setminus\tilde\Gamma)} + \frac12\sqrt{\eta_{\tilde f}(r_0)}\left(\int_{B_{r_0}\setminus\tilde\Gamma}|\nabla U|^2\,dx + \frac{1}{r_0}\int_{\partial B_{r_0}} U^2\,dS\right)^{\frac12}\right)\|\phi\|_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})},
\end{aligned}
$$

thus implying (53). $\square$

Corollary 2.15. *Let u be a weak solution to* (1) *and $U = u\circ F$. Let either* (H1) *hold and $\{f_n\}$ be as in* (45)*, or* (H2) *hold and $\{f_n\}$ be as in* (42)*. Let r_0 be as in* (47) *and ℓ be as in Proposition 2.14. Then, for any $n\in\mathbb{N}\setminus\{0\}$, there exists a solution $W_n\in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ of* (49) *such that*

$$
\|W_n\|_{H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})} \leq 4\ell. \tag{54}
$$

Proof. The existence of a solution W_n of (49) follows from the Lax-Milgram Theorem, taking into account Proposition 2.14. Estimate (54) follows from (52) and (53) with $\phi = W_n$. $\square$

We are now in position to prove the main result of this section.

Theorem 2.16. *Suppose that f satisfies either* (H1) *or* (H2)*, u is a weak solution of* (1)*, and $U = u\circ F$ with F as in Section 2.1. Let $\{f_n\}_{n\in\mathbb{N}}$ satisfies* (45) *under hypothesis* (H1) *or* (42) *under hypothesis* (H2)*. Let $r_0\in(0,r_1)$ be as* (47)*. Then there exists $\{U_n\}_{n\in\mathbb{N}\setminus\{0\}}\subset H^1(B_{r_0}\setminus\tilde\Gamma)$ such that U_n weakly solves* (48) *for any $n\in\mathbb{N}\setminus\{0\}$ and $U_n\to U$ in $H^1(B_{r_0}\setminus\tilde\Gamma)$ as $n\to\infty$. Furthermore $U_n\in H^2(\Omega_{n,r})$ for any $r\in(0,r_0)$ and $n\in\mathbb{N}\setminus\{0\}$.*

Proof. Let $r_0\in(0,r_1)$ be as in (47). For any $n\in\mathbb{N}\setminus\{0\}$, let $W_n\in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ be the solution to (49) given by Corollary 2.15. Then $U - W_n$ weakly solves problem (48) and we define $U_n := U - \xi^0_{n,r_0}(W_n)$, with ξ^0_{n,r_0} being the extension operator introduced in Proposition 2.11. We observe that $U_n\in H^1(B_{r_0}\setminus\tilde\Gamma)$. To prove that U_n converges to U in $H^1(B_{r_0}\setminus\tilde\Gamma)$ as $n\to\infty$, we notice that

$$
\|U-U_n\|^2_{H^1(B_{r_0}\setminus\tilde\Gamma)} \leq c_0^2\,\|W_n\|^2_{H^1(\Omega_{n,r_0})} \leq 4\,c_0^2\,\frac{N-1+r_0^2}{N-1}\int_{\Omega_{n,r_0}}(A\nabla W_n\cdot\nabla W_n - f_nW_n^2)\,dy,
$$

by Proposition 2.11, (39), and (52). Therefore it is enough to prove that

$$
\lim_{n\to\infty}\int_{\Omega_{n,r_0}}(A\nabla W_n\cdot\nabla W_n - f_nW_n^2)\,dy = 0. \tag{55}
$$

Let

$$
O_n := (B_{r_1}\setminus\tilde\Gamma)\setminus\Omega_{n,r_1} \tag{56}
$$

for any $n \in \mathbb{N}\setminus\{0\}$. Since $W_n \in H^1_{0,S_{n,r_0}}(\Omega_{n,r_0})$ solves (49) and U is a solution to (21), by Hölder's inequality, (11) and Proposition 2.11 we have that

$$\begin{aligned}
&\left|\int_{\Omega_{n,r_0}} (A\nabla W_n \cdot \nabla W_n - f_n W_n^2)\,dy\right| = \left|\int_{\Omega_{n,r_1}} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\,\xi^0_{n,r_0}(W_n))\,dy\right| \\
&= \bigg|\int_{B_{r_1}\setminus\tilde\Gamma} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\,\xi^0_{n,r_0}(W_n))\,dy \\
&\qquad - \int_{O_n} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\,\xi^0_{n,r_0}(W_n))\,dy\bigg| \\
&= \bigg|\int_{B_{r_1}\setminus\tilde\Gamma} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - \tilde f U\,\xi^0_{n,r_0}(W_n))\,dy + \int_{B_{r_1}\setminus\tilde\Gamma} (\tilde f - f_n) U\,\xi^0_{n,r_0}(W_n))\,dy \\
&\qquad - \int_{O_n} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\,\xi^0_{n,r_0}(W_n))\,dy\bigg| \\
&\le \left|\int_{O_n} (A\nabla U \cdot \nabla(\xi^0_{n,r_0}(W_n)) - f_n U\,\xi^0_{n,r_0}(W_n))\,dy\right| + \left|\int_{B_{r_1}\setminus\tilde\Gamma} (\tilde f - f_n) U\,\xi^0_{n,r_0}(W_n)\,dy\right| \\
&\le 2\,\|\nabla U\|_{L^2(O_n)} \left\|\nabla \xi^0_{n,r_0}(W_n)\right\|_{L^2(B_{r_1}\setminus\tilde\Gamma)} + \|f_n\|_{L^{\frac N2+\epsilon}(O_n)} \|U\|_{L^{q_\epsilon}(O_n)} \left\|\xi^0_{n,r_0}(W_n)\right\|_{L^{q_\epsilon}(B_{r_1})} \\
&\quad + \|\tilde f - f_n\|_{L^{\frac N2+\epsilon}(B_{r_1})} \|U\|_{L^{q_\epsilon}(B_{r_1})} \left\|\xi^0_{n,r_0}(W_n)\right\|_{L^{q_\epsilon}(B_{r_1})} \\
&\le 4c_0\ell\frac{\sqrt{N-1+r_0^2}}{\sqrt{N-1}}\Big(2\,\|\nabla U\|_{L^2(O_n)} + \sqrt{\mathcal{S}_{N,q_\epsilon}}\,r_1^{\frac{2\epsilon}{N+2\epsilon}}\|\tilde f\|_{L^{\frac N2+\epsilon}(O_n)} \|U\|_{L^{q_\epsilon}(O_n)} \\
&\qquad\qquad + \sqrt{\mathcal{S}_{N,q_\epsilon}}\,r_1^{\frac{2\epsilon}{N+2\epsilon}}\|\tilde f - f_n\|_{L^{\frac N2+\epsilon}(B_{r_1})} \|U\|_{L^{q_\epsilon}(B_{r_1})}\Big),
\end{aligned}$$

where q_ϵ is defined in (29) and we have used (43), (27), (37), (39), and (54) in the last inequality. We observe that

$$\lim_{n\to\infty} |O_n| = 0,$$

where $|O_n|$ is the N-dimensional Lebesgue measure of O_n. Then, since $\nabla U \in L^2(B_{r_1}\setminus\tilde\Gamma)$, $U \in L^{q_\epsilon}(B_{r_1})$ by Proposition 2.6, and $\tilde f \in L^{\frac N2+\epsilon}(B_{r_1})$, (55) follows by the absolute continuity of the integral and convergence (46).

We observe that $f_n U_n \in L^2(\Omega_{n,r_0})$. Indeed, under assumption (H1), by Remark 2.3 we have that $\tilde f \in W^{1,\frac N2+\epsilon}(B_{r_1}\setminus\tilde\Gamma)$ and then, by Sobolev embeddings and Hölder's inequality, we easily obtain that $f_n U_n = \tilde f U_n \in L^2(\Omega_{n,r_0})$. Under assumption (H2), f_n is defined in (42) and $f_n \in L^\infty(B_{r_1})$, hence $f_n U_n \in L^2(\Omega_{n,r_0})$.

Since Γ_{n,r_0} is C^∞-smooth and $f_n U_n \in L^2(\Omega_{n,r_0})$, by classical elliptic regularity theory, see e.g. [22, Theorem 2.2.2.5], we deduce that $U_n \in H^2(\Omega_{n,r})$ for any $r \in (0,r_0)$. The proof is thereby complete. □

3. The Almgren type frequency function

Let $u \in H^1(B_R\setminus\Gamma)$ be a non-trivial weak solution to (1) and $U = u\circ F \in H^1(B_{r_1}\setminus\tilde\Gamma)$ be the corresponding solution to (21). Let $r_0 \in (0,\min\{1,r_1\})$ be as in (47). For any $r \in (0,r_0]$, we define

$$H(r) := \frac{1}{r^{N-1}}\int_{\partial B_r} \mu\,U^2\,dS, \tag{57}$$

where μ is the function introduced in (14), and

$$E(r) := \frac{1}{r^{N-2}}\int_{B_r\setminus\tilde\Gamma} (A\nabla U\cdot\nabla U - \tilde f\,U^2)\,dy. \tag{58}$$

Proposition 3.1. *If $r \in (0,r_0]$ then $H(r) > 0$.*

Proof. We suppose by contradiction that there exists $r \in (0, r_0]$ such that $H(r) = 0$. By (15), it follows that U weakly solves (21) with the extra condition $U = 0$ on ∂B_r. Then by (30) we obtain that $U = 0$ on B_r. By classical unique continuation principles for elliptic equations, see e.g. [21], we conclude that $u = 0$ on B_R, which is a contradiction. □

Proposition 3.2. *We have that $H \in W^{1,1}_{\mathrm{loc}}((0, r_0])$ and*

$$H'(r) = \frac{1}{r^{N-1}} \left(2 \int_{\partial B_r} \mu U \frac{\partial U}{\partial \nu} \, dS + \int_{\partial B_r} U^2 \nabla \mu \cdot \nu \, dS \right) \tag{59}$$
$$= \frac{2}{r^{N-1}} \int_{\partial B_r} \mu U \frac{\partial U}{\partial \nu} \, dS + H(r) O(1) \quad \text{as } r \to 0^+,$$

in a distributional sense and for a.e. $r \in (0, r_0)$.

Remark 3.3. To explain in what sense the term $\frac{\partial U}{\partial \nu}$ in (59) is meant, we observe that, if ∇U is the distributional gradient of U in $B_{r_1} \setminus \tilde{\Gamma}$, then $\nabla U \in L^2(B_{r_1}, \mathbb{R}^N)$ and $\frac{\partial U}{\partial \nu} := \nabla U \cdot \frac{y}{|y|} \in L^2(B_{r_1})$. By the Coarea Formula it follows that $\nabla U \in L^2(\partial B_r, \mathbb{R}^N)$ and $\frac{\partial U}{\partial \nu} \in L^2(\partial B_r)$ for a.e. $r \in (0, r_1)$.

Proof. For any $\phi \in C_0^\infty(0, r_0)$ we define $v(y) := \phi(|y|)$. Then we have

$$\begin{aligned}
\int_0^{r_0} H(r)\phi'(r) \, dy &= \int_0^{r_0} \frac{1}{r^{N-1}} \left(\int_{\partial B_r} \mu U^2 \, dS \right) \phi'(r) \, dr \\
&= \int_{B^+_{r_0}} \frac{1}{|y|^N} \mu(y) U^2(y) \nabla v(y) \cdot y \, dy + \int_{B^-_{r_0}} \frac{1}{|y|^N} \mu(y) U^2(y) \nabla v(y) \cdot y \, dy \\
&= -\int_{B_{r_0} \setminus \tilde{\Gamma}} \frac{1}{|y|^N} (2\mu(y) v(y) U(y) \nabla U(y) \cdot y + v(y) U^2(y) \nabla \mu(y) \cdot y) \, dy \\
&= -\int_0^{r_0} \frac{2}{r^{N-1}} \left(\int_{\partial B_r} \mu U \frac{\partial U}{\partial \nu} \, dS \right) \phi(r) \, dr - \int_0^{r_0} \frac{1}{r^{N-1}} \left(\int_{\partial B_r} U^2 \nabla \mu \cdot \nu \, dS \right) \phi(r) \, dr,
\end{aligned}$$

which proves (59) thanks to (17). Since r^{-N+1} is bounded in any compact subset of $(0, r_0]$, then, by (15), (17) and the Coarea Formula, H and H' are locally integrable so that $H \in W^{1,1}_{\mathrm{loc}}((0, r_0])$. □

Now we turn our attention to E. Henceforth we let $\{f_n\}$ be as in (45), if f satisfies (H1), or as in (42), if f satisfies (H2), and we consider the sequence $\{U_n\}$ converging to U in $H^1(B_{r_0} \setminus \tilde{\Gamma})$ provided by Theorem 2.16.

Remark 3.4. By Proposition 2.6 and (29), $U_n \to U$ in $L^{q_\epsilon}(B_{r_0})$. Then, since $f_n \to \tilde{f}$ in $L^{\frac{N}{2}+\epsilon}(B_{r_0})$ by (46), from Hölder's inequality it easily follows that

$$\lim_{n \to \infty} \int_{B_{r_0}} |\tilde{f} \, U^2 - f_n \, U_n^2| \, dy = 0. \tag{60}$$

Moreover, if f satisfies (H1), $\nabla \tilde{f} \in L^{\frac{N}{2}+\epsilon}(B_{r_0}, \mathbb{R}^N)$ and hence

$$\lim_{n \to \infty} \int_{B_{r_0} \setminus \Gamma} |(\nabla \tilde{f} \cdot \beta) \, (U^2 - U_n^2)| \, dx = 0, \tag{61}$$

since the vector field β defined in (14) is bounded in view of (18).

Lemma 3.5. *If $F_n \to F$ in $L^1(B_{r_0})$, then there exists a subsequence $\{F_{n_k}\}_{k \in \mathbb{N}}$ such that, for a.e. $r \in (0, r_0)$,*

$$\lim_{k \to \infty} \int_{\partial B_r} |F - F_{n_k}| \, dS = 0 \quad \text{and} \quad \lim_{k \to \infty} \int_{S_{n_k, r}} F_{n_k} \, dS = \int_{\partial B_r} F \, dS,$$

where the notation $S_{n,r}$ has been introduced in (34).

Proof. Let $h_n(r) := \int_{\partial B_r} |F_n - F| \, dS$. Since, by assumption and the Coarea Formula,

$$\lim_{n \to \infty} \int_{B_{r_0}} |F - F_n| \, dy = \lim_{n \to \infty} \int_0^{r_0} h_n(r) dr = 0,$$

we have that $h_n \to 0$ in $L^1(0,r_0)$. Hence there exists a subsequence $\{h_{n_k}\}_{k\in\mathbb{N}}$ converging to 0 a.e. in $(0,r_0)$. Therefore $F_{n_k} \to F$ in $L^1(\partial B_r)$ for a.e. $r \in (0,r_0)$. It follows that, for a.e. $r \in (0,r_0)$,

$$\int_{S_{n_k,r}} F_{n_k}\,dS - \int_{\partial B_r} F\,dS = \int_{\partial B_r} \chi_{S_{n_k,r}}(F_{n_k} - F)\,dS + \int_{\partial B_r} (\chi_{S_{n_k}} - 1)F\,dS \to 0$$

as $k \to \infty$, thus yielding the conclusion. □

Proposition 3.6. *We have that $E \in W^{1,1}_{\rm loc}((0,r_0])$,*

$$E(r) = \frac{1}{r^{N-2}} \int_{\partial B_r} UA\nabla U \cdot \nu\,dS = \frac{r}{2}H'(r) + rH(r)O(1) \quad \text{as } r \to 0^+ \tag{62}$$

and

$$E'(r) = (2-N)\frac{1}{r^{N-1}} \int_{B_r\setminus\tilde\Gamma} (A\nabla U\cdot\nabla U - \tilde f U^2)\,dy + \frac{1}{r^{N-2}}\int_{\partial B_r} (A\nabla U\cdot\nabla U - \tilde f U^2)\,dS \tag{63}$$

in the sense of distributions and for a.e. $r \in (0,r_0)$.

Proof. The fact that $E \in W^{1,1}_{\rm loc}((0,r_0])$ and (63) follow from the Coarea Formula and (28). To prove (62) we consider the sequence $\{U_n\}$ introduced in Theorem 2.16. For every $r \in (0,r_0)$ and $n \in \mathbb{N}\setminus\{0\}$,

$$\frac{1}{r^{N-2}}\int_{\Omega_{n,r}} (A\nabla U_n\cdot\nabla U_n - f_n U_n^2)\,dy = \frac{1}{r^{N-2}}\int_{S_{n,r}} U_n A\nabla U_n\cdot\nu\,dS$$

since U_n solve (48) and $U_n \in H^2(\Omega_{n,r})$ by Theorem 2.16. Thanks to Remark 3.4,the Dominated Convergence Theorem, and Lemma 3.5, we can pass to the limit, up to a subsequence, as $n\to\infty$ in the above identity for a.e. $r \in (0,r_0)$, thus proving the first equality in (62). To prove the second equality in (62) we define

$$\zeta(y) := \frac{\mu(y)(\beta(y) - y)}{|y|} = \frac{A(y)y}{|y|} - \frac{A(y)y\cdot y}{|y|^3}y.$$

Then, since $\zeta(y)\cdot y = 0$ and $\zeta\cdot(0,\dots,0,1) = 0$ on $\tilde\Gamma$, we have that

$$\begin{aligned}\int_{\partial B_r} UA\nabla U\cdot\nu\,dS - \int_{\partial B_r}\mu U\frac{\partial U}{\partial\nu}\,dS &= \frac12\int_{\partial B_r}\zeta\cdot\nabla(U^2)\,dS\\ &= -\frac12\int_{\partial B_r}\operatorname{div}(\zeta)U^2\,dS = r^{N-1}H(r)O(1)\end{aligned}$$

as $r\to 0$, where we have used in the last equality the estimate

$$\operatorname{div}(\zeta)(y) = \left(\frac{\nabla\mu(y)}{|y|} - \frac{\mu(y)y}{|y|^3}\right)(\beta(y)-y) + \frac{\mu(y)}{|y|}\big(\operatorname{div}(\beta)(y) - N\big) = O(1)$$

which follows from Proposition 2.2. Then we conclude by (59). □

The approximation procedure developed above also allows us to derive the following integration by parts formula.

Proposition 3.7. *There exists a set $\mathcal{M} \subset [0,r_0]$ having null 1-dimensional Lebesgue measure such that, for all $r \in (0,r_0]\setminus\mathcal{M}$, $A\nabla U\cdot\nu \in L^2(\partial B_r)$ and*

$$\int_{B_r\setminus\tilde\Gamma} A\nabla U\cdot\nabla\phi\,dx = \int_{B_r}\tilde f U\phi\,dx + \int_{\partial B_r}(A\nabla U\cdot\nu)\phi\,dS$$

for every $\phi \in H^1(B_{r_0}\setminus\tilde\Gamma)$, where $A\nabla U\cdot\nu$ on ∂B_r is meant in the sense of Remark 3.3.

Proof. Since $U_n \to U$ in $H^1(B_{r_0}\setminus\tilde\Gamma)$ in view of Theorem 2.16, by Lemma 3.5 there exist a subsequence $\{U_{n_k}\}$ and a set $\mathcal{M}\subset[0,r_0]$ having null 1-dimensional Lebesgue measure such that

$A\nabla U\cdot\nu\in L^2(\partial B_r)$ and $A\nabla U_{n_k}\cdot\nu\to A\nabla U\cdot\nu$ in $L^2(\partial B_r)$ for all $r\in(0,r_0]\setminus\mathcal{M}$. Since $U_n\in H^2(\Omega_{n,r})$ for any $r\in(0,r_0)$ and $n\in\mathbb{N}\setminus\{0\}$ by Theorem 2.16, from (48) it follows that

$$\int_{\Omega_{n,r}}(A\nabla U_n\cdot\nabla\phi-f_nU_n\phi)\,dy=\int_{S_{n,r}}\phi A\nabla U_n\cdot\nu\,dS.$$

Arguing as in the proof of Proposition 3.6, we can pass to the limit along $n=n_k$ as $k\to\infty$ in the above identity for all $r\in(0,r_0]\setminus\mathcal{M}$, thus obtaining the conclusion. □

Theorem 3.8. *(Pohozaev type inequality) Under either assumption* (H1) *or assumption* (H2)*, for any $r\in(0,r_0]$ we have that*

$$r\int_{\partial B_r}A\nabla U\cdot\nabla U\,dS\geq 2r\int_{\partial B_r}\frac{|A\nabla U\cdot\nu|^2}{\mu}\,dS+\int_{B_r\setminus\tilde\Gamma}(A\nabla U\cdot\nabla U)\operatorname{div}(\beta)\,dy$$
$$+2\int_{B_r\setminus\tilde\Gamma}\frac{A\nabla U\cdot y}{\mu}\tilde f\,U\,dy+\int_{B_r\setminus\tilde\Gamma}(dA\nabla U\nabla U)\cdot\beta\,dy-2\int_{B_r\setminus\tilde\Gamma}J_\beta(A\nabla U)\cdot\nabla U\,dy,\tag{64}$$

which can be rewritten as

$$r\int_{\partial B_r}(A\nabla U\cdot\nabla U-\tilde f\,U^2)\,dS\geq 2r\int_{\partial B_r}\frac{|A\nabla U\cdot\nu|^2}{\mu}\,dS$$
$$+\int_{B_r\setminus\tilde\Gamma}(A\nabla U\cdot\nabla U)\operatorname{div}(\beta)\,dy+\int_{B_r\setminus\tilde\Gamma}(\tilde f\operatorname{div}(\beta)+\nabla\tilde f\cdot\beta)\,U^2\,dy$$
$$+\int_{B_r\setminus\tilde\Gamma}(dA\nabla U\nabla U)\cdot\beta\,dy-2\int_{B_r\setminus\tilde\Gamma}J_\beta(A\nabla U)\cdot\nabla U\,dy\tag{65}$$

if f satisfies (H1).

Proof. By Theorem 2.16 we have that $U_n\in H^2(\Omega_{n,r})$ for any $r\in(0,r_0)$ and $n\in\mathbb{N}\setminus\{0\}$. Then, since A is symmetric by Proposition 2.1, we may write the following Rellich-Nečas identity in a distributional sense in $\Omega_{n,r}$:

$$\operatorname{div}((A\nabla U_n\cdot\nabla U_n)\beta-2(\beta\cdot\nabla U_n)A\nabla U_n)=(A\nabla U_n\cdot\nabla U_n)\operatorname{div}(\beta)$$
$$-2(\beta\cdot\nabla U_n)\operatorname{div}(A\nabla U_n)+(dA\nabla U_n\nabla U_n)\cdot\beta-2J_\beta(A\nabla U_n)\cdot\nabla U_n.\tag{66}$$

Since $U_n\in H^2(\Omega_{n,r})$ and the components of A and β are Lipschitz continuous by Propositions 2.1 and 2.2, then $(A\nabla U_n\nabla U_n)\beta-2(\beta\cdot\nabla U_n)A\nabla U_n)\in W^{1,1}(\Omega_{n,r})$. Therefore we can integrate both sides of (66) on the Lipschitz domain $\Omega_{n,r}$ and apply the Divergence Theorem to obtain, in view of (14) and (48),

$$r\int_{S_{n,r}}\left(A\nabla U_n\cdot\nabla U_n-2\frac{|A\nabla U_n\cdot\nu|^2}{\mu}\right)dS+\int_{\Gamma_{n,r}}(A\nabla U_n\cdot\nabla U_n)\frac{Ay\cdot\nu}{\mu}\,dS$$
$$=\int_{\Omega_{n,r}}(A\nabla U_n\cdot\nabla U_n)\operatorname{div}(\beta)\,dy+2\int_{\Omega_{n,r}}\frac{A\nabla U_n\cdot y}{\mu}f_nU_n\,dy$$
$$+\int_{\Omega_{n,r}}(dA\nabla U_n\nabla U_n)\cdot\beta\,dy-2\int_{\Omega_{n,r}}J_\beta(A\nabla U_n)\cdot\nabla U_n\,dy.\tag{67}$$

From Proposition 2.10, (11), and (15) it follows that, for all $n\in\mathbb{N}\setminus\{0\}$ and $r\in(0,r_0)$,

$$\int_{\Gamma_{n,r}}(A\nabla U_n\cdot\nabla U_n)\frac{Ay\cdot\nu}{\mu}\,dS\leq 0.\tag{68}$$

From Theorem 2.16, we recall that $U_n\to U$ strongly in $H^1(B_{r_0}\setminus\tilde\Gamma)$, while Propositions 2.1 and 2.2 imply that

$$\mu\in L^\infty(B_{r_0},\mathbb{R}),\quad \beta\in L^\infty(B_{r_0},\mathbb{R}^N),\quad \operatorname{div}\beta\in L^\infty(B_{r_0},\mathbb{R}),$$
$$A\in L^\infty(B_{r_0},\mathbb{R}^{N^2}),\quad \left\{\frac{\partial a_{i,j}}{\partial y_h}\right\}_{i,j,h=1,\dots,N}\in L^\infty(B_{r_0},\mathbb{R}^{N^3}).\tag{69}$$

Furthermore, under assumption (H1), we have that, by Sobolev embeddings (see Proposition 2.6), if $N \geq 3$, then $f_n = \tilde{f} \in L^N(B_{r_0})$ and $U_n \to U$ strongly in $L^{2^*}(B_{r_0})$, whereas, if $N = 2$, then $f_n = \tilde{f} \in L^{2(1+\epsilon)/(1-\epsilon)}(B_{r_0})$ and $U_n \to U$ strongly in $L^{(1+\epsilon)/\epsilon}(B_{r_0})$; then, since $\nabla U_n \to \nabla U$ in $L^2(B_{r_0})$, Hölder's inequality ensures that

$$f_n U_n A\nabla U_n \cdot y \to \tilde{f} U A\nabla U \cdot y \quad \text{in } L^1(B_{r_0}). \tag{70}$$

Under assumption (H2), we have that Hardy's inequality (see Proposition 2.5), Proposition 2.4 and (43) yield that

$$\int_{B_{r_0}} |f_n y(U_n - U)|^2 \, dy \leq \text{const}\, r_0^{4\epsilon} \int_{B_{r_0}} |y|^{-2}|U_n - U|^2 \, dy \to 0 \quad \text{as } n \to \infty$$

which, thanks to Proposition 2.5 again and the Dominated Convergence Theorem, easily implies that

$$f_n y U_n \to \tilde{f} y U \quad \text{in } L^2(B_{r_0}),$$

thus proving (70) also under assumption (H2).

Then, thanks to the Dominated Convergence Theorem, (20), (70) and Lemma 3.5, we can pass to the limit in (67) as $n \to \infty$, up to a subsequence, and, taking into account (68), we obtain inequality (64).

If assumption (H1) holds then by (14), (45) and Proposition 2.2 we have that

$$\begin{aligned} 2\int_{\Omega_{n,r}} \frac{A\nabla U_n \cdot y}{\mu} f_n U_n \, dy &= 2\int_{\Omega_{n,r}} (\beta \cdot \nabla U_n)\tilde{f} U_n \, dy \\ &= -\int_{\Omega_{n,r}} (\tilde{f}\,\mathrm{div}(\beta) + \nabla\tilde{f} \cdot \beta)\, U_n^2 \, dy + r\int_{S_{n,r}} \tilde{f}\, U_n^2 \, dS + \int_{\Gamma_{n,r}} \tilde{f}\, U_n^2 \, \beta \cdot \nu \, dS. \end{aligned} \tag{71}$$

We define

$$\begin{aligned} O^+_{n,r} &:= O_n \cap B^+_r, \quad O^-_{n,r} := O_n \cap B^-_r, \\ \Gamma^+_{n,r} &:= \Gamma_{n,r} \cap B^+_r, \quad \Gamma^-_{n,r} := \Gamma_{n,r} \cap B^-_r, \end{aligned}$$

where O_n is defined in (56). Taking into account that $\beta \cdot \nu = \frac{Ay}{\mu} \cdot \nu = 0$ on $\partial O^+_{n,r} \cap \partial \mathbb{R}^N_+$ since $\nu = -(0, \ldots, 1)$ and (9) holds, the Divergence Theorem yields that

$$\begin{aligned} \int_{\Gamma^+_{n,r}} \tilde{f} U_n^2 \, \beta \cdot \nu \, dS = -r\int_{\partial O^+_{n,r} \cap \partial B_r} &\tilde{f} U_n^2 \, \beta \cdot \nu \, dS \\ &+ \int_{O^+_{n,r}} \Big(\tilde{f} U_n^2 \,\mathrm{div}\beta + U_n^2 \, \nabla\tilde{f} \cdot \beta \, + 2\tilde{f} U_n \nabla U_n \cdot \beta\Big) \, dy. \end{aligned} \tag{72}$$

By (60), (69), and Lemma 3.5 there exists a subsequence $\{\tilde{f}\, U^2_{n_k} \beta \cdot \nu\}_{k\in\mathbb{N}}$ converging in $L^1(\partial B_r)$ and hence equi-integrable in ∂B_r for a.e. $r \in (0, r_0)$, hence

$$\lim_{k\to\infty} \int_{\partial O^+_{n_k,r} \cap \partial B_r} \tilde{f}\, U^2_{n_k} \beta \cdot \nu \, dS = 0 \quad \text{for a.e. } r \in (0, r_0).$$

Since $\nabla U_n \to \nabla U$ in $L^2(B^+_{r_0}, \mathbb{R}^N)$, $U_n \to U$ in $L^{q_\epsilon}(B^+_{r_0})$ and $\tilde{f} \in L^{N+2\epsilon}(B^+_{r_0})$ by (H1) and classical Sobolev embeddings, from (69) and Hölder's inequality we deduce that

$$\tilde{f} U_n \nabla U_n \cdot \beta \to \tilde{f} U \nabla U \cdot \beta \quad \text{in } L^1(B^+_{r_0}),$$

so that $\{\tilde{f} U_n \nabla U_n \cdot \beta\}_{n\in\mathbb{N}}$ is equi-integrable in $B^+_{r_0}$. Therefore

$$\lim_{n\to\infty} \int_{O^+_{n,r}} \tilde{f} U_n \nabla U_n \cdot \beta \, dy = 0 \quad \text{for all } r \in (0, r_0).$$

Moreover, also $\{\mathrm{div}\beta\, \tilde{f}\, U_n^2 + U_n^2\, \nabla\tilde{f} \cdot \beta\}_{n\in\mathbb{N}}$ is equi-integrable thanks to (60) and (61). It follows that

$$\lim_{n\to\infty} \int_{O^+_{n,r}} (\mathrm{div}\beta\, \tilde{f} U_n^2 + \nabla\tilde{f} \cdot \beta\, U_n^2) \, dy = 0 \quad \text{for all } r \in (0, r_0).$$

Then from (72) we conclude that

$$\lim_{k\to\infty}\int_{\Gamma^+_{n_k,r}} \tilde f U_{n_k}^2 \beta\cdot\nu\, dS = 0.$$

In a similar way we obtain that $\lim_{k\to\infty}\int_{\Gamma^-_{n_k,r}} \tilde f U_{n_k}^2 \beta\cdot\nu\, dS = 0$ so that

$$\lim_{k\to\infty}\int_{\Gamma_{n_k,r}} \tilde f U_{n_k}^2 \beta\cdot\nu\, dS = 0.$$

Therefore (65) follows by passing to the limit in (67) and (71) as $n\to\infty$ along a subsequence, taking into account Proposition 2.10, the Dominated Convergence Theorem, (20), Remark 3.4 and Lemma 3.5. □

Proposition 3.9. *For a.e.* $r\in(0,r_0)$

$$\begin{aligned}E'(r) \geq{}& 2r^{2-N}\int_{\partial B_r}\frac{|A\nabla U\cdot\nu|^2}{\mu}\,dS + r^{1-N}\int_{B_r\setminus\tilde\Gamma}(\mathrm{div}(\beta)+2-N)A\nabla U\cdot\nabla U\,dy\\ &+r^{1-N}\int_{B_r\setminus\tilde\Gamma}\left(\tilde f(\mathrm{div}(\beta)+N-2)+\nabla\tilde f\cdot\beta\right)U^2\,dy\\ &+r^{1-N}\int_{B_r\setminus\tilde\Gamma}(dA\nabla U\nabla U)\cdot\beta\,dy - 2r^{1-N}\int_{B_r\setminus\tilde\Gamma}J_\beta(A\nabla U)\cdot\nabla U\,dy,\end{aligned}\tag{73}$$

if (H1) *holds, and*

$$\begin{aligned}E'(r) \geq{}& 2r^{2-N}\int_{\partial B_r}\frac{|A\nabla U\cdot\nu|^2}{\mu}\,dS - r^{2-N}\int_{\partial B_r}\tilde f U^2\,dS + (N-2)r^{1-N}\int_{B_r}\tilde f U^2\,dy\\ &+r^{1-N}\int_{B_r\setminus\tilde\Gamma}(A\nabla\bar u\cdot\nabla U)(\mathrm{div}(\beta)+2-N)\,dy \;+2r^{1-N}\int_{B_r\setminus\tilde\Gamma}\frac{A\nabla U\cdot y}{\mu}\tilde f U\,dy\\ &+r^{1-N}\int_{B_r\setminus\tilde\Gamma}(dA\nabla U\nabla U)\cdot\beta\,dy - 2r^{1-N}\int_{B_r\setminus\tilde\Gamma}J_\beta(A\nabla U)\cdot\nabla U\,dy\end{aligned}\tag{74}$$

if (H2) *holds.*

Proof. Estimates (73)–(74) are direct consequences of (63), (64), and (65). □

We now introduce the Almgren frequency function, defined as

$$\mathcal N:(0,r_0]\to\mathbb R,\quad \mathcal N(r):=\frac{E(r)}{H(r)}.\tag{75}$$

The above definition of $\mathcal N$ is well posed thanks to Proposition 3.1.

Proposition 3.10. *If either assumption* (H1) *or assumption* (H2) *hold, then* $\mathcal N\in W^{1,1}_{\mathrm{loc}}((0,r_0])$ *and, for any* $r\in(0,r_0]$,

$$\mathcal N(r)\geq -2\eta_{\tilde f}(r).\tag{76}$$

Furthermore, for a.e. $r\in(0,r_0)$,

$$\mathcal N'(r)\geq \mathcal V(r)+\mathcal W(r)\tag{77}$$

where

$$\mathcal V(r)=\frac{2r\left(\left(\int_{\partial B_r}\frac{|A\nabla U\cdot\nu|^2}{\mu}\,dS\right)\left(\int_{\partial B_r}\mu U^2 dS\right)-\left(\int_{\partial B_r}UA\nabla U\cdot\nu\,dS\right)^2\right)}{\left(\int_{\partial B_r}\mu U^2\,dS\right)^2}\geq 0\tag{78}$$

and

$$\mathcal W(r)=O\left(r^{-1+\frac{4\epsilon}{N+2\epsilon}}\right)(1+\mathcal N(r))\quad \text{as } r\to 0^+.\tag{79}$$

Proof. Since $1/H, E \in W^{1,1}_{\rm loc}((0,r_0])$, then $\mathcal{N} \in W^{1,1}_{\rm loc}((0,r_0])$. Furthermore (30) directly implies (76).

By(62), for a.e. $r \in (0,r_0)$

$$\mathcal{N}'(r) = \frac{E'(r)H(r) - E(r)H'(r)}{H^2(r)} = \frac{E'(r)H(r) - \frac{2}{r}E^2(r)}{H^2(r)} + \frac{E(r)O(1)}{H(r)}$$
$$= \frac{E'(r)H(r) - \frac{2}{r}r^{4-2N}\big(\int_{\partial B_r} UA\nabla U\cdot\nu\, dS\big)^2}{H^2(r)} + O(1)\mathcal{N}(r)$$

as $r \to 0^+$. By Proposition 2.1, Proposition 2.2, (29) and (30)

$$\left|\int_{B_r\setminus\tilde\Gamma}\Big((A\nabla U\cdot\nabla U)(\operatorname{div}(\beta)+2-N) - 2J_\beta(A\nabla U)\cdot\nabla U + (dA\nabla U\nabla U)\cdot\beta\Big)\,dy\right|$$
$$\le O(r)\int_{B_r\setminus\tilde\Gamma}|\nabla U|^2\,dy$$
$$\le O(r)\int_{B_r\setminus\tilde\Gamma}(A\nabla U\cdot\nabla U - \tilde f U^2)\,dy + O\left(r^{\frac{4\epsilon}{N+2\epsilon}}\right)\int_{\partial B_r}\mu U^2\,dS \quad \text{as } r\to 0^+.$$

By (28), (30), and (15)

$$\int_{B_r}\tilde f U^2\,dy \le O\left(r^{\frac{4\epsilon}{N+2\epsilon}}\right)\int_{B_r\setminus\tilde\Gamma}|\nabla U|^2\,dy + O\left(r^{\frac{2\epsilon-N}{N+2\epsilon}}\right)\int_{\partial B_r}U^2\,dS$$
$$\le O\left(r^{\frac{4\epsilon}{N+2\epsilon}}\right)\int_{B_r\setminus\tilde\Gamma}(A\nabla U\cdot\nabla U\,dy - \tilde f U^2) + O\left(r^{\frac{2\epsilon-N}{N+2\epsilon}}\right)\int_{\partial B_r}\mu U^2\,dS$$

as $r\to 0^+$ and, by (19), the same holds for $\int_{B_r}(\operatorname{div}\beta - N + 2)\tilde f U^2\,dy$. In the same way from (18) it follows that, if (H1) holds,

$$\int_{B_r}\nabla\tilde f\cdot\beta U^2\,dy \le O\left(r^{\frac{4\epsilon}{N+2\epsilon}}\right)\int_{B_r\setminus\tilde\Gamma}(A\nabla U\cdot\nabla U\,dy - \tilde f U^2) + O\left(r^{\frac{2\epsilon-N}{N+2\epsilon}}\right)\int_{\partial B_r}\mu U^2\,dS$$

as $r\to 0^+$.

Under assumption (H2), by Remark 2.3, (16), (11), (29) (28), (30) and Hölder's inequality,

$$\int_{B_r\setminus\tilde\Gamma}\frac{A\nabla U\cdot y}{\mu}\tilde f U\,dy = O(r)\int_{B_r\setminus\tilde\Gamma}|\nabla U||\tilde f|U\,dy$$
$$\le O(r^\epsilon)\,\|\nabla U\|_{L^2(B_r\setminus\tilde\Gamma)}\left(\int_{B_r}|\tilde f|U^2\,dx\right)^{\frac12}$$
$$\le O\left(r^{\epsilon+\frac{2\epsilon}{N+2\epsilon}}\right)\left(\int_{B_r\setminus\tilde\Gamma}(A\nabla U\cdot\nabla U - \tilde f U^2)\,dy + \frac{2}{\eta_f(r)}r\int_{\partial B_r}\mu U^2\,dS\right)^{\frac12}\times$$
$$\times\left(\int_{B_r\setminus\tilde\Gamma}(A\nabla U\cdot\nabla U - \tilde f U^2)\,dy + \frac{2}{r}\int_{\partial B_r}\mu U^2\,dS\right)^{\frac12}$$
$$\le O\left(r^{\epsilon+\frac{2\epsilon}{N+2\epsilon}}\right)\int_{B_r\setminus\tilde\Gamma}(A\nabla U\cdot\nabla U - \tilde f U^2)\,dy + O\left(r^{-1+\epsilon+\frac{2\epsilon}{N+2\epsilon}}\right)\int_{\partial B_r}\mu U^2\,dS.$$

Under assumptions (H2), thanks to Remark 2.3 and (15),

$$\int_{\partial B_r}\tilde f U^2\,dS = O\left(r^{2\epsilon-2}\right)\int_{\partial B_r}\mu U^2\,dS.$$

Collecting the above estimates, we conclude that (77), (78) and (79) follow from (73)or (74) under hypotheses (H1) or (H2) respectively. From the Cauchy–Schwarz inequality we also deduce that $\mathcal{V}\ge 0$ a.e. in $(0,r_0)$. $\square$

We now prove that $\mathcal{N}$ is bounded.

Proposition 3.11. *There exists a constant $C > 0$ such that, for every $r \in (0, r_0]$,*

$$\mathcal{N}(r) \le C. \tag{80}$$

Proof. By Proposition 3.10 there exists a constant $\kappa > 0$ such that, for a.e. $r \in (0, r_0)$,

$$(\mathcal{N} + 1)'(r) \ge \mathcal{W}(r) \ge -\kappa\, r^{-1+\frac{4\epsilon}{N+2\epsilon}}(\mathcal{N}(r) + 1).$$

Since $\mathcal{N} + 1 > 0$ by (76) and the choice of r_0 in (47), it follows that

$$(\log(\mathcal{N} + 1))' \ge -\kappa r^{-1+\frac{4\epsilon}{N+2\epsilon}}.$$

An integration over (r, r_0) yields

$$\mathcal{N}(r) \le -1 + \exp\left(\kappa \frac{N + 2\epsilon}{4\epsilon} r_0^{\frac{4\epsilon}{2\epsilon+N}}\right)(\mathcal{N}(r_0) + 1)$$

and the proof is thereby complete. □

Proposition 3.12. *There exists the limit*

$$\gamma := \lim_{r\to 0^+} \mathcal{N}(r). \tag{81}$$

Furthermore γ is finite and $\gamma \ge 0$.

Proof. From Proposition 3.10 and (80) there exists a constant $\kappa > 0$ such that

$$\mathcal{N}'(r) \ge \mathcal{W}(r) \ge -\kappa\, r^{-1+\frac{4\epsilon}{N+2\epsilon}}(\mathcal{N}(r) + 1) \ge -\kappa(C + 1)r^{-1+\frac{4\epsilon}{N+2\epsilon}} \quad \text{for a.e. } r \in (0, r_0).$$

Then

$$\frac{d}{dr}\left(\mathcal{N}(r) + \frac{\kappa(C + 1)(N + 2\epsilon)}{4\epsilon} r^{\frac{4\epsilon}{N+2\epsilon}}\right) \ge 0$$

for a.e. $r \in (0, r_0)$. We conclude that $\lim_{r\to 0^+} \mathcal{N}(r)$ exists; moreover such a limit is finite thanks to (80) and (76). Furthermore from (29) and (76) we deduce that $\gamma \ge 0$. □

Proposition 3.13. *There exists a constant $\alpha > 0$ such that, for every $r \in (0, r_0]$,*

$$H(r) \le \alpha\, r^{2\gamma}. \tag{82}$$

Furthermore for every $\sigma > 0$ there exist $\alpha_\sigma > 0$ and $r_\sigma \in (0, r_0)$ such that, for every $r \in (0, r_\sigma]$,

$$H(r) \ge \alpha_\sigma r^{2\gamma+\sigma}. \tag{83}$$

Proof. For the proof in a similar situation we refer to [18, Lemma 5.6]. □

Proposition 3.14. *The limit $\lim_{r\to 0^+} r^{-2\gamma} H(r)$ exists and is finite.*

Proof. For the proof in a similar situation we refer to [18, Lemma 6.4]. □

From the properties of the height function H derived above, in particular from estimate (83), we deduce the unique continuation property stated in Theorem 1.1.

Proof of Theorem 1.1. Let u be a weak solution to (1) such that $u(x) = O(|x|^k)$ as $|x| \to 0^+$ for all $k \in \mathbb{N}$. To prove that $u \equiv 0$ in B_R, we argue by contradiction and assume that $u \not\equiv 0$. Then we can define a frequency function for $U = u \circ F$ as in (57), (58) and (75). Choosing $k \in \mathbb{N}$ such that $k > \gamma + \frac{\sigma}{2}$, we would obtain that $H(r) = O(r^{2k}) = o(r^{2\gamma+\sigma})$ as $r \to 0$, contradicting estimate (83). □

4. The blow-up analysis

In this section we perform a blow-up analysis for scaled solutions to (21). To this aim we first study the spectrum of (6), which plays a crucial role in the classification of blow-up profiles.

4.1. **Neumann eigenvalues on** $\mathbb{S}^{N-1} \setminus \Sigma$**.** In this section we study the spectrum of (6). We recall that $\mu \in \mathbb{R}$ is an eigenvalue of (6) if there exists $\psi \in H^1(\mathbb{S}^{N-1} \setminus \Sigma) \setminus \{0\}$ such that

$$\int_{\mathbb{S}^{N-1}\setminus\Sigma} \nabla_{\mathbb{S}^{N-1}\setminus\Sigma}\psi \cdot \nabla_{\mathbb{S}^{N-1}\setminus\Sigma}\phi \, dS = \mu \int_{\mathbb{S}^{N-1}\setminus\Sigma} \psi\phi \, dS \quad \text{for any } \phi \in H^1(\mathbb{S}^{N-1} \setminus \Sigma). \tag{84}$$

A Rellich-Kondrakov type theorem is needed to apply the classical Spectral Theorem to problem (6).

Proposition 4.1. *The embedding* $H^1(\mathbb{S}^{N-1} \setminus \Sigma) \hookrightarrow L^2(\mathbb{S}^{N-1})$ *is compact.*

Proof. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{S}^{N-1}\setminus\Sigma)$. We observe that $\mathbb{S}^{N-1}_+$ and $\mathbb{S}^{N-1}_-$ are smooth compact manifolds with boundary and that the sequences of restrictions $\big\{\phi_n\big|_{\mathbb{S}^{N-1}_+}\big\}_{n\in\mathbb{N}}$ and $\big\{\phi_n\big|_{\mathbb{S}^{N-1}_-}\big\}_{n\in\mathbb{N}}$ are bounded in $H^1(\mathbb{S}^{N-1}_+)$ and $H^1(\mathbb{S}^{N-1}_-)$ respectively. Then we can extract a subsequence $\{\phi_{n_k}\}_{k\in\mathbb{N}}$ such that $\big\{\phi_n\big|_{\mathbb{S}^{N-1}_+}\big\}_{n\in\mathbb{N}}$ converges in $L^2(\mathbb{S}^{N-1}_+)$ by the classical Rellich-Kondrakov Theorem on compact manifolds with boundary, see [5]. Proceeding in the same way for $\big\{\phi_{n_k}\big|_{\mathbb{S}^{N-1}_-}\big\}_{n\in\mathbb{N}}$ in $H^1(\mathbb{S}^{N-1}_-)$, we conclude that there exists a subsequence $\{\phi_{n_{k_h}}\}_{h\in\mathbb{N}}$ which converges both in $L^2(\mathbb{S}^{N-1}_-)$ and in $L^2(\mathbb{S}^{N-1}_+)$, hence in $L^2(\mathbb{S}^{N-1})$. □

Proposition 4.2.

(i) The point spectrum of (6) *is a diverging and increasing sequence of non-negative eigenvalues* $\{\mu_k\}_{k\in\mathbb{N}}$ *of finite multiplicity and the eigenvalue* $\mu_0 = 0$ *is simple. Letting* N_k *be the multiplicity of* μ_k *and* V_k *be the eigenspace associated to* μ_k*, there exists an orthonormal basis of* $L^2(\mathbb{S}^{N-1})$ *consisting of eigenfunctions* $\{Y_{k,i}\}_{k\in\mathbb{N},i=1,\dots,N_k}$ *such that* $\{Y_{k,i}\}_{i=1,\dots N_k}$ *is a basis of* V_k *for any* $k \in \mathbb{N}$.

(ii) For any $k \in \mathbb{N}$

$$\mu_k = \frac{k(k + 2N - 4)}{4}. \tag{85}$$

Moreover any eigenfunction of (6) *belongs to* $L^\infty(\mathbb{S}^{N-1})$.

Proof. The proof of (i) follows from the classical Spectral Theorem for compact self-adjoint operators, taking into account Proposition 4.1. We prove now (ii). If μ is an eigenvalue of (6) and Ψ an associated eigenfunction, let $\sigma := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu}$ and

$$W(r\theta) := r^\sigma \Psi(\theta), \quad \text{for any } r \in [0, \infty),\, \theta \in \mathbb{S}^{N-1} \setminus \Sigma.$$

Since Ψ is an eigenfunction of (6) then W is harmonic on $B_1 \setminus \tilde{\Gamma}$ and $\frac{\partial^+ W}{\partial\nu^+} = \frac{\partial^- W}{\partial\nu^-} = 0$ on $\tilde{\Gamma}$. Therefore we deduce from [10] that there exists $k \in \mathbb{N}$ such that $\sigma = \frac{k}{2}$ and so $\mu = \frac{k(k+2N-4)}{4}$. Moreover from [10] it also follows that $W \in L^\infty(B_1)$ hence $\Psi \in L^\infty(\mathbb{S}^{N-1})$.

Viceversa, if we let $k \in \mathbb{N}$ and define W in cylindrical coordinates as

$$W(x', r\cos(t), r\sin(t)) := r^{\frac{k}{2}} \cos\left(\frac{kt}{2}\right) \quad \text{for any } x' \in \mathbb{R}^{N-2},\ r \in [0, \infty),\ \text{and } t \in [0, 2\pi],$$

then W is harmonic on $B_1 \setminus \tilde{\Gamma}$ and $\frac{\partial^+ W}{\partial\nu^+} = \frac{\partial^- W}{\partial\nu^-} = 0$ on $\tilde{\Gamma}$. Since W is homogeneous of degree $k/2$, then

$$W(r\theta) = r^{\frac{k}{2}} \Psi(\theta), \quad \text{for any } r \in [0, \infty),\ \text{and } \theta \in \mathbb{S}^{N-1} \setminus \Sigma,$$

where $\Psi = W_{|_{\mathbb{S}^{N-1}}}$. Then from

$$r^{\frac{k-4}{2}} \left(\frac{k(k-2)}{4} \Psi(\theta) + \frac{k(N-1)}{2} \Psi(\theta) + \Delta_{\mathbb{S}^{N-1}} \Psi(\theta) \right) = 0, \quad r \in [0, \infty),\ \theta \in \mathbb{S}^{N-1} \setminus \Sigma,$$

we deduce that Ψ solves (6) with $\mu = \frac{k(k+2N-4)}{4}$. □

Remark 4.3. The traces of eigenfunctions of problem (6) on both sides of Σ (i.e. the traces of restrictions to $\mathbb{S}^{N-1}_+$ and $\mathbb{S}^{N-1}_+$) cannot vanish identically.

Indeed, if an eigenfunction Ψ associated to the eigenvalue μ_k is such that the trace of $\Psi\big|_{\mathbb{S}^{N-1}_+}$ on Σ vanishes, then the function $W(x) := |x|^{k/2}\Psi(x/|x|)$ would be a harmonic function in $\mathbb{R}^N \setminus \tilde{\Gamma}$ satisfying both Dirichlet and Neumann homogeneous boundary conditions on the upper side of the crack, thus violating classic unique continuation principles.

4.2. **The blow-up analysis.** Throughout this section we let $u \in H^1(B_R \setminus \Gamma)$ be a non-trivial weak solution to (1) with f satisfying either (H1) or (H2), $U = u \circ F \in H^1(B_{r_1} \setminus \tilde{\Gamma})$ be the corresponding solution to (21), r_0 be as in (47) and r_1 be as in Proposition 2.1. For all $\lambda \in (0, r_0)$, let

$$W^\lambda(y) := \frac{U(\lambda y)}{\sqrt{H(\lambda)}} \quad \text{for any } y \in B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma}. \tag{86}$$

For any $\lambda \in (0, r_0)$ it is easy to verify that $W^\lambda \in H^1(B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma})$ and W^λ satisfies

$$\int_{B_{\lambda^{-1}r_1}\setminus\tilde{\Gamma}} A(\lambda y)\nabla W^\lambda(y) \cdot \nabla\phi(y)\, dy - \lambda^2 \int_{B_{\lambda^{-1}r_1}} \tilde{f}(\lambda y) W^\lambda(y)\phi(y)\, dy = 0$$

for any $\phi \in H^1_{0,\partial B_{\lambda^{-1}r_1}}(B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma})$. In other words W^λ is a weak solution of

$$\begin{cases} -\operatorname{div}(A(\lambda\cdot)\nabla W^\lambda) = \lambda^2 \tilde{f}(\lambda\cdot)W^\lambda, & \text{in } B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma}, \\ A(\lambda\cdot)\nabla^+ W^\lambda \cdot \nu^+ = A(\lambda\cdot)\nabla^- W^\lambda \cdot \nu^- = 0, & \text{on } \tilde{\Gamma}, \end{cases}$$

for any $\lambda \in (0, r_0)$. Since $B_1 \subset B_{\lambda^{-1}r_1}$ for all $\lambda \in (0, r_0)$, it follows that, for any $\lambda \in (0, r_0)$,

$$\int_{B_1\setminus\tilde{\Gamma}} A(\lambda y)\nabla W^\lambda(y) \cdot \nabla\phi(y)\, dy - \lambda^2 \int_{B_1} \tilde{f}(\lambda y) W^\lambda(y)\phi(y)\, dy = 0, \tag{87}$$

for any $\phi \in H^1_{0,\partial B_1}(B_1 \setminus \tilde{\Gamma})$. Furthermore by a change of variables, (86) and (57),

$$\int_{\mathbb{S}^{N-1}} \mu(\lambda\theta)|W^\lambda(\theta)|^2 dS = 1 \quad \text{for every } \lambda \in (0, r_0). \tag{88}$$

Proposition 4.4. *Let W^λ be as in* (86)*. Then $\{W^\lambda\}_{\lambda\in(0,r_0)}$ is bounded in $H^1(B_1 \setminus \tilde{\Gamma})$.*

Proof. We have

$$\int_{B_1\setminus\tilde{\Gamma}} |\nabla W^\lambda|^2\, dy = \frac{\lambda^{2-N}}{H(\lambda)} \int_{B_\lambda\setminus\tilde{\Gamma}} |\nabla U(y)|^2\, dy \le \frac{2}{1 - 2\eta_{\tilde{f}}(\lambda)}\mathcal{N}(\lambda) + \frac{4\eta_{\tilde{f}}(\lambda)}{1 - 2\eta_{\tilde{f}}(\lambda)}.$$

by (30). Then thanks to (80), (29), (47), (27), (15), and (88) we conclude. □

The following proposition is a doubling type result.

Proposition 4.5. *There exists a constant $C_1 > 0$ such that for any $\lambda \in (0, \frac{r_0}{2})$ and $T \in [1, 2]$*

$$\frac{1}{C_1}H(T\lambda) \le H(\lambda) \le C_1 H(T\lambda), \tag{89}$$

$$\int_{B_T} |W^\lambda(y)|^2 dy \le 2^N C_1 \int_{B_1} |W^{T\lambda}(y)|^2\, dy, \tag{90}$$

and

$$\int_{B_T\setminus\tilde{\Gamma}} |\nabla W^\lambda(y)|^2 dy \le 2^{N-2} C_1 \int_{B_1\setminus\tilde{\Gamma}} |\nabla W^{T\lambda}(y)|^2\, dy. \tag{91}$$

Proof. From (80), (76), (62), and (47) we deduce that there exist two constants $\kappa_1 > 0$ and $\kappa_2 > 0$ such that, for any $r \in (0, r_0)$,

$$-\frac{2}{r} \le -\frac{2\eta_f(r)}{r} \le \frac{H'(r)}{H(r)} \le \frac{2\mathcal{N}(r) + \kappa_1}{r} \le \frac{\kappa_2}{r}.$$

Then (89) follows from an integration in $(\lambda, T\lambda)$ of the above inequality. Furthermore from (89) we obtain that, for any $\lambda \in (0, \frac{r_0}{2})$ and $T \in [1, 2]$,

$$\int_{B_T} |W^\lambda(y)|^2 \, dy = \frac{\lambda^{-N}}{H(\lambda)} \int_{B_{\lambda T}} |U(y)|^2 \, dy \le \frac{C_1 2^N}{(\lambda T)^N H(T\lambda)} \int_{B_{\lambda T}} |U(y)|^2 \, dy$$
$$= C_1 2^N \int_{B_1} |W^{T\lambda}(y)|^2 \, dy.$$

In the same way (91) follows from (89). □

Proposition 4.6. *Let $\mathcal{M}$ be as in Proposition 3.7 and W^λ be defined in* (86)*. Then there exist $M > 0$ and $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ there exists $T_\lambda \in [1, 2]$ such that $\lambda T_\lambda \notin \mathcal{M}$ and*

$$\int_{\partial B_{T_\lambda}} |\nabla W^\lambda|^2 \, dS \le M \int_{B_{T_\lambda} \setminus \tilde\Gamma} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy. \tag{92}$$

Proof. Since $\{W^\lambda\}_{\lambda \in (0, r_0/2)}$ is bounded in $H^1(B_2 \setminus \tilde\Gamma)$ by Proposition 4.4, (90) and (91), then

$$\limsup_{\lambda \to 0^+} \int_{B_2 \setminus \tilde\Gamma} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy < +\infty. \tag{93}$$

By the Coarea formula, for any $\lambda \in (0, \frac{r_0}{2})$ the function

$$g_\lambda(r) := \int_{B_r \setminus \tilde\Gamma} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy$$

is absolutely continuous in $[1, 2]$ with weak derivative

$$g_\lambda'(r) = \int_{\partial B_r} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dS \quad \text{for a.e. } r \in [1, 2],$$

where the integral $\int_{\partial B_r} |\nabla W^\lambda|^2 dS$ is meant in the sense of Remark 3.3. To prove the statement we argue by contradiction. If the conclusion does not hold, for any $M > 0$ there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, r_0/2)$ such that $\lim_{n \to \infty} \lambda_n = 0$ and

$$\int_{\partial B_r} (|\nabla W^{\lambda_n}|^2 + |W^{\lambda_n}|^2) \, dS > M \int_{B_r \setminus \tilde\Gamma} (|\nabla W^{\lambda_n}|^2 + |W^{\lambda_n}|^2) \, dy$$

for any $n \in \mathbb{N}$ and $r \in [1, 2] \setminus \frac{1}{\lambda_n}\mathcal{M}$, and hence for a.e. $r \in [1, 2]$. Hence

$$g_{\lambda_n}'(r) > M g_{\lambda_n}(r) \quad \text{for any } n \in \mathbb{N} \text{ and a.e. } r \in [1, 2].$$

An integration in $[1, 2]$ yields

$$\limsup_{n \to \infty} g_{\lambda_n}(1) \le e^{-M} \limsup_{n \to \infty} g_{\lambda_n}(2)$$

hence

$$\liminf_{\lambda \to 0^+} g_\lambda(1) \le e^{-M} \limsup_{\lambda \to 0^+} g_\lambda(2).$$

In view of (93), letting $M \to \infty$ we conclude that

$$\liminf_{\lambda \to 0^+} \int_{B_1 \setminus \tilde\Gamma} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy = 0.$$

Then there exists a sequence $\{\rho_n\}_{k \in \mathbb{N}}$ such that $W^{\rho_n} \to 0$ strongly in $H^1(B_1 \setminus \tilde\Gamma)$ as $n \to \infty$. Due to the continuity of the trace operator γ_1 defined in Proposition 2.4 and (16), this is in contradiction with (88). □

Proposition 4.7. *There exists $\overline{M} > 0$ such that*

$$\int_{\mathbb{S}^{N-1}} |\nabla W^{\lambda T_\lambda}|^2 \, dS \le \overline{M} \quad \text{for all } \lambda \in \left(0, \min\left\{\frac{r_0}{2}, \lambda_0\right\}\right).$$

Proof. Since

$$\int_{\mathbb{S}^{N-1}} |\nabla W^{\lambda T_\lambda}|^2 \, dS = \frac{\lambda^2 T_\lambda^{3-N}}{H(\lambda T_\lambda)} \int_{\partial B_{T_\lambda}} |\nabla U(\lambda y)|^2 \, dS = T_\lambda^{3-N} \frac{H(\lambda)}{H(\lambda T_\lambda)} \int_{\partial B_{T_\lambda}} |\nabla W^\lambda|^2 \, dS,$$

then, by (89), (90), (91), (92), and the fact that $1 \le T_\lambda \le 2$, for any $\lambda \in \left(0, \min\left\{\frac{r_0}{2}, \lambda_0\right\}\right)$ we have that

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} |\nabla W^{\lambda T_\lambda}|^2 \, dS &\le 2C_1 M \int_{B_{T_\lambda} \setminus \tilde{\Gamma}} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy \\ &\le 2^{N+1} C_1^2 M \int_{B_1 \setminus \tilde{\Gamma}} (|\nabla W^{T_\lambda \lambda}|^2 + |W^{T_\lambda \lambda}|^2) \, dy.\end{aligned}$$

Therefore we conclude thanks to Proposition 4.4. □

Thanks to the estimates established above, we can now prove a first blow-up result.

Proposition 4.8. *Let $u \in H^1(B_R \setminus \Gamma)$, $u \not\equiv 0$, be a non-trivial weak solution to* (1)*, with Γ defined in* (2)–(3) *and f satisfying either* (H1) *or* (H2)*, and let $U = u \circ F$ be the corresponding solution to* (21)*. Let γ be as in* (81)*. Then*

$$\text{there exists } k_0 \in \mathbb{N} \text{ such that } \gamma = \frac{k_0}{2}. \tag{94}$$

For any sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \lambda_n = 0$ there exists a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and an eigenfunction Ψ of problem (6) *associated to the eigenvalue μ_{k_0} such that $\|\Psi\|_{L^2(\mathbb{S}^{N-1})} = 1$ and*

$$\frac{U(\lambda_{n_k} y)}{\sqrt{H(\lambda_{n_k})}} \to |y|^\gamma \Psi\left(\frac{y}{|y|}\right) \quad \text{strongly in } H^1(B_1 \setminus \tilde{\Gamma}).$$

Proof. Let W^λ be as in (86) for any $\lambda \in \left(0, \min\left\{\frac{r_0}{2}, \lambda_0\right\}\right)$ and let us consider a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = 0$. From Proposition 4.4 $\{W^{\lambda T_\lambda} : \lambda \in \left(0, \min\left\{\frac{r_0}{2}, \lambda_0\right\}\right)\}$ is bounded in $H^1(B_1 \setminus \tilde{\Gamma})$. Therefore there exists a subsequence $\{W^{\lambda_{n_k} T_{\lambda_{n_k}}}\}_{k\in\mathbb{N}} \subset H^1(B_1 \setminus \tilde{\Gamma})$ and a function $W \in H^1(B_1 \setminus \tilde{\Gamma})$ such that $W^{\lambda_{n_k} T_{\lambda_{n_k}}} \rightharpoonup W$ weakly in $H^1(B_1 \setminus \tilde{\Gamma})$. By compactness of the trace operator γ_1 (see Proposition 2.4), (16), and (88), it follows that

$$\int_{\partial B_1} W^2 \, dS = 1 \tag{95}$$

and so $W \not\equiv 0$ on $B_1 \setminus \tilde{\Gamma}$.

By Hölder's inequality and (28) we have that, for every $\phi \in H^1(B_1 \setminus \tilde{\Gamma})$,

$$\begin{aligned}&\left|\lambda^2 \int_{B_1} \tilde{f}(\lambda y) W^\lambda(y) \phi(y) \, dy\right| \\ &\le \lambda^2 \eta_{\tilde{f}(\lambda\cdot)}(1) \left(\int_{B_1 \setminus \tilde{\Gamma}} |\nabla W^\lambda|^2 \, dy + \int_{\partial B_1} |W^\lambda|^2 \, dS\right)^{\frac{1}{2}} \left(\int_{B_1 \setminus \tilde{\Gamma}} |\nabla \phi|^2 \, dy + \int_{\partial B_1} \phi^2 \, dS\right)^{\frac{1}{2}}.\end{aligned} \tag{96}$$

By (29) and a change of variables we have that

$$\begin{aligned}\lambda^2 \eta_{\tilde{f}(\lambda\cdot)}(1) &= S_{N,q_\epsilon} \lambda^2 \left(\int_{B_1} |\tilde{f}(\lambda y)|^{\frac{N}{2}+\epsilon} \, dy\right)^{\frac{2}{N+2\epsilon}} \\ &= S_{N,q_\epsilon} \lambda^{\frac{4\epsilon}{N+2\epsilon}} \|\tilde{f}\|_{L^{\frac{N}{2}+\epsilon}(B_\lambda)} \to 0 \quad \text{as } \lambda \to 0^+.\end{aligned} \tag{97}$$

From (96), (97), the boundedness of $\{W^\lambda\}$ in $H^1(B_1 \setminus \tilde\Gamma)$ (established in Proposition 4.4) and of the traces (following from Proposition 2.4), we deduce that

$$\lim_{k\to\infty} \lambda^2_{\lambda_{n_k}T_{\lambda_{n_k}}} \int_{B_1} \tilde f(\lambda_{n_k}T_{\lambda_{n_k}}y)W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\phi(y)\,dy = 0, \tag{98}$$

for every $\phi \in H^1(B_1 \setminus \tilde\Gamma)$.

Let $\phi \in H^1_{0,\partial B_1}(B_1 \setminus \tilde\Gamma)$. We can test (87) with ϕ to obtain

$$\begin{aligned}\int_{B_1\setminus\tilde\Gamma} A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\cdot\nabla\phi(y)\,dy \\ = (\lambda_{n_k}T_{\lambda_{n_k}})^2 \int_{B_1} \tilde f(\lambda_{n_k}T_{\lambda_{n_k}}y)W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\phi(y)\,dy,\end{aligned} \tag{99}$$

for any $k\in\mathbb{N}$. Since $W^{\lambda_{n_k}T_{\lambda_{n_k}}} \rightharpoonup W$ weakly in $H^1(B_1\setminus\tilde\Gamma)$, by (13) we have that

$$\lim_{k\to\infty}\int_{B_1\setminus\tilde\Gamma} A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\cdot\nabla\phi(y)\,dy = \int_{B_1\setminus\tilde\Gamma}\nabla W\cdot\nabla\phi\,dy. \tag{100}$$

Therefore, for any $\phi \in H^1_{0,\partial B_1}(B_1 \setminus \tilde\Gamma)$ we can pass to the limit as $k\to\infty$ in (99) thus obtaining, in view of (100) and (98),

$$\int_{B_1\setminus\tilde\Gamma}\nabla W\cdot\nabla\phi\,dy = 0,$$

i.e. W is a weak solution of

$$\begin{cases} -\Delta W = 0, & \text{on } B_1\setminus\tilde\Gamma, \\ \dfrac{\partial^+ W}{\partial\nu^+} = \dfrac{\partial^- W}{\partial\nu^-} = 0, & \text{on } \tilde\Gamma. \end{cases} \tag{101}$$

We note that, by classical elliptic regularity theory, W is smooth in $B_1\setminus\tilde\Gamma$.

In view of (86) and Propositions 4.6 and 3.7, by scaling we have that, for every $\phi\in H^1(B_1\setminus\tilde\Gamma)$,

$$\begin{aligned}\int_{B_1\setminus\tilde\Gamma} A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\cdot\nabla\phi(y)\,dy \\ - (\lambda_{n_k}T_{\lambda_{n_k}})^2 \int_{B_1} \tilde f(\lambda_{n_k}T_{\lambda_{n_k}}y)W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\phi(y)\,dy \\ = \int_{\partial B_1}\left(A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\cdot\nu\right)\phi(y)\,dS.\end{aligned} \tag{102}$$

Thanks to Proposition 4.7 and (11) there exists a function $h\in L^2(\partial B_1)$ such that

$$\left(A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\cdot\nu\right) \rightharpoonup h \quad \text{weakly in } L^2(\partial B_1), \tag{103}$$

up to a subsequence. By the weak convergence $W^{\lambda_{n_k}T_{\lambda_{n_k}}} \rightharpoonup W$ in $H^1(B_1\setminus\tilde\Gamma)$, (13), (98), and (103), passing to the limit as $k\to\infty$ in (102), we obtain that

$$\int_{B_1\setminus\tilde\Gamma}\nabla W\cdot\nabla\phi\,dy = \int_{\partial B_1} h\phi\,dS \tag{104}$$

for any $\phi\in H^1(B_1\setminus\tilde\Gamma)$. From the compactness of the trace operator γ_1 (see Proposition 2.4) and (103) it follows that

$$\lim_{k\to\infty}\int_{\partial B_1}\left(A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\cdot\nu\right)W^{\lambda_{n_k}T_{\lambda_{n_k}}}(y)\,dS = \int_{\partial B_1} hW\,dS.$$

Therefore, recalling estimates (96), (97), and the boundedness of $\{W^\lambda\}$ in $H^1(B_1\setminus\tilde\Gamma)$, choosing $\phi = W^{\lambda_{n_k}T_{\lambda_{n_k}}}$ in (102) and passing to the limit as $k\to\infty$, we obtain that

$$\lim_{k\to\infty}\int_{B_1\setminus\tilde\Gamma} A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\cdot\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\,dy = \int_{\partial B_1} hW\,dS. \tag{105}$$

From (104) and (105) it follows that

$$\lim_{k\to\infty}\int_{B_1\setminus\tilde\Gamma} A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\cdot\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\,dy=\int_{B_1\setminus\tilde\Gamma}|\nabla W|^2\,dy$$

and so, thanks to (13),

$$W^{\lambda_{n_k}T_{\lambda_{n_k}}}\to W\quad\text{strongly in } H^1(B_1\setminus\tilde\Gamma). \tag{106}$$

For any $k\in\mathbb{N}$ and $r\in(0,1)$ let us define

$$\begin{aligned}E_k(r)&:=r^{2-N}\int_{B_r\setminus\tilde\Gamma}(A(\lambda_{n_k}T_{\lambda_{n_k}}y)\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\cdot\nabla W^{\lambda_{n_k}T_{\lambda_{n_k}}}\\&\qquad-(\lambda_{n_k}T_{\lambda_{n_k}})^2\tilde f(\lambda_{n_k}T_{\lambda_{n_k}}y)|W^{\lambda_{n_k}T_{\lambda_{n_k}}}|^2)\,dy,\\H_k(r)&:=r^{1-N}\int_{\partial B_r}\mu(\lambda_{n_k}T_{\lambda_{n_k}}y)|W^{\lambda_{n_k}T_{\lambda_{n_k}}}|^2\,dS,\quad\text{and}\quad\mathcal{N}_k(r):=\frac{E_k(r)}{H_k(r)}.\end{aligned}$$

By a change of variables it is easy to verify that, for any $r\in(0,1)$,

$$\mathcal{N}_k(r)=\frac{E_k(r)}{H_k(r)}=\frac{E(\lambda_{n_k}T_{\lambda_{n_k}}r)}{H(\lambda_{n_k}T_{\lambda_{n_k}}r)}=\mathcal{N}(\lambda_{n_k}T_{\lambda_{n_k}}r). \tag{107}$$

For any $r\in(0,1)$, we also define

$$H_W(r):=r^{1-N}\int_{\partial B_r}|W|^2\,dS,\quad E_W(r):=r^{2-N}\int_{B_r\setminus\tilde\Gamma}|\nabla W|^2\,dy\quad\text{and}\quad\mathcal{N}_W(r):=\frac{E_W(r)}{H_W(r)}.$$

The definition of $\mathcal{N}_W$ is well posed. Indeed, if $H_W(r)=0$ for some $r\in(0,1)$, then we may test the equation (101) on B_r with W and conclude that $W=0$ in B_r. Thanks to classical unique continuation principles for harmonic functions, this would imply that $W=0$ in B_1, thus contradicting (95).

Thanks to (106), (96)-(97) together with the boundedness of $\{W^\lambda\}$ in $H^1(B_1\setminus\tilde\Gamma)$, (13), (16), and Proposition 3.12, passing to the limit as $k\to\infty$ in (107) we obtain that

$$\mathcal{N}_W(r)=\lim_{k\to\infty}\mathcal{N}_k(r)=\lim_{k\to\infty}\mathcal{N}(\lambda_{n_k}T_{\lambda_{n_k}}r)=\gamma\quad\text{for any } r\in(0,1). \tag{108}$$

Then $\mathcal{N}_W$ is constant in $(0,1)$. Following the proof of Proposition 3.10 in the case $f\equiv0$ and $g\equiv0$ (where g is the function defined in (4)–(5)), so that $A=\mathrm{Id}_N$ and $\mu=1$, we obtain that

$$0=\mathcal{N}_W'(r)\ge\frac{2r\left(\left(\int_{\partial B_r}\left|\frac{\partial W}{\partial\nu}\right|^2dS\right)\left(\int_{\partial B_r}W^2dS\right)-\left(\int_{\partial B_r}W\frac{\partial W}{\partial\nu}\,dS\right)^2\right)}{\left(\int_{\partial B_r}W^2\,dS\right)^2}\ge0$$

for a.e. $r\in(0,1)$. It follows that $\left(\int_{\partial B_r}\left|\frac{\partial W}{\partial\nu}\right|^2dS\right)\left(\int_{\partial B_r}W^2dS\right)=\left(\int_{\partial B_r}W\frac{\partial w}{\partial\nu}\,dS\right)^2$ for a.e. $r\in(0,1)$, i.e. equality holds in the Cauchy-Schwartz inequality for the vectors W and $\frac{\partial W}{\partial\nu}$ in $L^2(\partial B_r)$ for a.e. $r\in(0,1)$. It follows that there exists a function $\zeta(r)$ such that

$$\frac{\partial W}{\partial\nu}(r\theta)=\zeta(r)W(r\theta)\quad\text{for any }\theta\in\mathbb{S}^{N-1}\setminus\Sigma\text{ and a.e. } r\in(0,1]. \tag{109}$$

Multiplying by $W(r\theta)$ and integrating on $\mathbb{S}^{N-1}$ we obtain

$$\int_{\mathbb{S}^{N-1}}\frac{\partial W}{\partial\nu}(\theta r)W(r\theta)\,dS=\zeta(r)\int_{\mathbb{S}^{N-1}}W^2(\theta r)\,dS,$$

so that $\zeta(r)=\frac{H_W'(r)}{2H_W(r)}=\frac{\gamma}{r}$ by Proposition 3.2 and (108). Integrating (109) between $r\in(0,1)$ and 1 we obtain that

$$W(r\theta)=r^\gamma W(1\theta)=r^\gamma\Psi(\theta)\qquad\text{for any }\theta\in\mathbb{S}^{N-1}\setminus\Sigma\text{ and any } r\in(0,1],$$

where $\Psi=W|_{\mathbb{S}^{N-1}\setminus\Sigma}$. Then $\Psi\in H^1(\mathbb{S}^{N-1}\setminus\Sigma)$; furthermore, substituting $W(r\theta)=r^\gamma\Psi(\theta)$ in (101) we find out that Ψ is an eigenfunction of (6) with $(\gamma+N-2)\gamma$ as an associated eigenvalue.

Hence by Proposition 4.2 there exists $k_0 \in \mathbb{N}$ such that $(\gamma + N - 2)\gamma = \frac{k_0(k_0+2N-4)}{4}$. Recalling from Proposition 3.12 that $\gamma \geq 0$, we then obtain (94).

To conclude the proof it is enough to show that $W^{\lambda_{n_k}} \to W$ strongly in $H^1(B_1 \setminus \tilde{\Gamma})$ (possibly along a subsequence). Since $\{W^{\lambda_{n_k}}\}_{k\in\mathbb{N}}$ is bounded in $H^1(B_1 \setminus \tilde{\Gamma})$ by Proposition 4.4, there exists a function $\tilde{W} \in H^1(B_1 \setminus \tilde{\Gamma})$ and $T \in [1, 2]$ such that $W^{\lambda_{n_k}} \rightharpoonup \tilde{W}$ weakly in $H^1(B_1 \setminus \tilde{\Gamma})$ and $T_{\lambda_k} \to T$, up to a subsequence.

Moreover, since $\{W^{\lambda_{n_k} T_{\lambda_{n_k}}}\}_{k\in\mathbb{N}}$ and $\{|\nabla W^{\lambda_{n_k} T_{\lambda_{n_k}}}|\}_{k\in\mathbb{N}}$ converge strongly in $L^2(B_1)$ by (106), they are dominated by a measurable $L^2(B_1)$-function, up to a subsequence. Similarly, thanks to (89), we can suppose that, up to a subsequence, the limit

$$\zeta := \lim_{k\to\infty} \frac{H(\lambda_{n_k} T_{\lambda_{n_k}})}{H(\lambda_{n_k})}$$

exists and it is finite and strictly positive. Then for any $\phi \in C_c^\infty(B_1)$ we have that

$$\begin{aligned}
&\lim_{k\to\infty} \int_{B_1} W^{\lambda_{n_k}}(y)\phi(y)\,dy = \lim_{k\to\infty} T^N_{\lambda_{n_k}} \int_{B_{T^{-1}_{\lambda_{n_k}}}} W^{\lambda_{n_k}}(T_{\lambda_{n_k}} y)\phi(T_{\lambda_{n_k}} y)\,dy \\
&= \lim_{k\to\infty} T^N_{\lambda_{n_k}} \sqrt{\frac{H(\lambda_{n_k} T_{\lambda_{n_k}})}{H(\lambda_{n_k})}} \int_{B_{T^{-1}_{\lambda_{n_k}}}} W^{T_{\lambda_{n_k}}\lambda_{n_k}}(y)\phi(T_{\lambda_{n_k}} y)\,dy \\
&= T^N \sqrt{\zeta} \int_{B_{T^{-1}}} W(y)\phi(Ty)\,dy = \sqrt{\zeta} \int_{B_1} W(y/T)\phi(y)\,dy,
\end{aligned}$$

thanks to the Dominated Convergence Theorem. By density the same holds for any $\phi \in L^2(B_1)$. It follows that $W^{\lambda_{n_k}} \rightharpoonup \sqrt{\zeta} W(\cdot/T)$ weakly in $L^2(B_1)$. Hence, by uniqueness of the weak limit, we have that $\tilde{W}(\cdot) = \sqrt{\zeta} W(\cdot/T)$ and $W^{\lambda_{n_k}} \rightharpoonup \sqrt{\zeta} W(\cdot/T)$ weakly in $H^1(B_1 \setminus \tilde{\Gamma})$. Furthermore

$$\begin{aligned}
&\lim_{k\to\infty} \int_{B_1\setminus\tilde{\Gamma}} |\nabla W^{\lambda_{n_k}}(y)|^2\,dy = \lim_{k\to\infty} T^N_{\lambda_{n_k}} \int_{B_{T^{-1}_{\lambda_{n_k}}}\setminus\tilde{\Gamma}} |\nabla W^{\lambda_{n_k}}(T_{\lambda_{n_k}} y)|^2\,dy \\
&= \lim_{k\to\infty} T^{N-2}_{\lambda_{n_k}} \frac{H(\lambda_{n_k} T_{\lambda_{n_k}})}{H(\lambda_{n_k})} \int_{B_{T^{-1}_{\lambda_{n_k}}}\setminus\tilde{\Gamma}} |\nabla W^{T_{\lambda_{n_k}}\lambda_{n_k}}(y)|^2 dy \\
&= T^{N-2}\zeta \int_{B_{T^{-1}}\setminus\tilde{\Gamma}} |\nabla W(y)|^2 dy = \int_{B_1\setminus\tilde{\Gamma}} |\sqrt{\zeta}\nabla(W(\cdot/T))|^2\,dy.
\end{aligned}$$

Then we can conclude that $W^{\lambda_{n_k}} \to \tilde{W} = \sqrt{\zeta} W(\cdot/T)$ strongly in $H^1(B_1 \setminus \tilde{\Gamma})$. Moreover, by compactness of the trace operator γ_1 (see Proposition 2.4), (16), and (88), we deduce that $\int_{\partial B_1} \tilde{W}^2\,dS = 1$. Then, since $W(r\theta) = r^{\frac{k_0}{2}}\Psi(\theta)$, we deduce that

$$\tilde{W}(r\theta) = \sqrt{\zeta} W\left(\frac{r}{T}\theta\right) = \left(\frac{\zeta}{T^{k_0}}\right)^{\frac{1}{2}} r^{\frac{k_0}{2}}\Psi(\theta) = \left(\frac{\zeta}{T^{k_0}}\right)^{\frac{1}{2}} W(r\theta)$$

and

$$1 = \int_{\partial B_1} \tilde{W}^2\,dS = \frac{\zeta}{T^{k_0}} \int_{\partial B_1} W^2\,dS = \frac{\zeta}{T^{k_0}},$$

thanks to (95). Therefore $W = \tilde{W}$ and the proof is complete. □

We are now in position of prove Theorem 1.2.

Proof of Theorem 1.2. Let us assume that $\mathrm{Tr}^+_\Gamma u(z) = O(|z|^k)$ as $|z| \to 0^+$, $z \in \Gamma$, for all $k \in \mathbb{N}$ (a similar argument works under the assumption $\mathrm{Tr}^-_\Gamma u(z) = O(|z|^k)$). Letting $U = u \circ F$, by the properties of the diffeomorphism F described in Proposition 2.1, we have that $\mathrm{Tr}^+_{\tilde{\Gamma}} U(z) = O(|z|^k)$ as $|z| \to 0^+$, so that, for all $k \in \mathbb{N}$,

$$\|\lambda^{-k}\,\mathrm{Tr}^+_{\tilde{\Gamma}} U(\lambda\cdot)\|_{L^2(B_1\cap\tilde{\Gamma})} \to 0 \quad \text{as } \lambda \to 0^+. \tag{110}$$

On the other hand, if, by contradiction, $u \not\equiv 0$, by Proposition 4.8 and classical trace theorems there exist $k_0 \in \mathbb{N}$, a sequence $\lambda_n \to 0^+$, and an eigenfunction Ψ of problem (6) such that

$$\lim_{n\to\infty} \frac{\|\operatorname{Tr}^+_{\tilde\Gamma} U(\lambda_n \cdot)\|_{L^2(B_1\cap\tilde\Gamma)}}{\sqrt{H(\lambda_n)}} = \left\|\operatorname{Tr}^+_{\tilde\Gamma}\left(|y|^\gamma \Psi\left(\tfrac{y}{|y|}\right)\right)\right\|_{L^2(B_1\cap\tilde\Gamma)} \neq 0, \tag{111}$$

where the above limit is nonzero thanks to Remark 4.3. Combining (110) and (111) we obtain that

$$\lim_{n\to\infty} \frac{\sqrt{H(\lambda_n)}}{\lambda_n^k} = 0 \quad \text{for all } k \in \mathbb{N},$$

thus contradicting estimate (83). □

5. Asymptotics of the height function $H(\lambda)$ as $\lambda \to 0^+$, when $N \geq 3$

In dimension $N \geq 3$, we can further specify the behaviour of $U(\lambda\cdot)$ as $\lambda \to 0^+$, deriving the asymptotics of the function $H(\lambda)$ appearing as a normalization factor in the blowed-up family (86). Let $\{Y_{k,i}\}_{k\in\mathbb{N},i=1,\dots,N_k}$ be the basis of $L^2(\mathbb{S}^{N-1})$ given by Proposition 4.2. Let $N \geq 3$, $u \in H^1(B_R \setminus \Gamma)$ be a weak solution to (1), with Γ defined in (2)–(3) and f satisfying either (H1) or (H2), and let $U = u \circ F$ be the corresponding solution to (21). For any $\lambda \in (0, r_0)$, $k \in \mathbb{N}$ and $i = 1, \dots, N_k$ we define

$$\varphi_{k,i}(\lambda) := \int_{\mathbb{S}^{N-1}} U(\lambda\theta) Y_{k,i}(\theta)\, dS \tag{112}$$

and

$$\begin{aligned}\Upsilon_{k,i}(\lambda) := &-\int_{B_\lambda\setminus\tilde\Gamma} (A - \mathrm{Id}_N)\nabla U \cdot \frac{\nabla_{\mathbb{S}^{N-1}} Y_{k,i}(y/|y|)}{|y|}\, dy \\ &+ \int_{B_\lambda} \tilde f(y) U(y) Y_{k,i}(y/|y|)\, dy + \int_{\partial B_\lambda} (A - \mathrm{Id}_N)\nabla U \cdot \frac{y}{|y|} Y_{k,i}(y/|y|)\, dS.\end{aligned} \tag{113}$$

Proposition 5.1. *Let k_0 be as in Proposition 4.8. Then, for any $i = 1, \dots, N_{k_0}$ and $r \in (0, r_0]$,*

$$\begin{aligned}\varphi_{k_0,i}(\lambda) = \lambda^{\frac{k_0}{2}} \Big(r^{-\frac{k_0}{2}} \varphi_{k_0,i}(r) &+ \frac{2N + k_0 - 4}{2(N + k_0 - 2)} \int_\lambda^r s^{-N+1-\frac{k_0}{2}} \Upsilon_{k_0,i}(s)\, ds \\ &+ \frac{k_0 r^{-N+2-k_0}}{2(N + k_0 - 2)} \int_0^r s^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(s)\, ds \Big) + o(\lambda^{\frac{k_0}{2}}) \quad \text{as } \lambda \to 0^+.\end{aligned} \tag{114}$$

Proof. For any $k \in \mathbb{N}$ and any $i = 1, \dots, N_k$ we consider the distribution $\zeta_{k,i}$ on $(0, r_0)$ defined as

$$\begin{aligned}{}_{\mathcal{D}'(0,r_0)}\langle \zeta_{k,i}, \omega\rangle_{\mathcal{D}(0,r_0)} := &\int_0^{r_0} \omega(\lambda) \left(\int_{\mathbb{S}^{N-1}} \tilde f(\lambda\theta) U(\lambda\theta) Y_{m,k}(\theta)\, dS_\theta \right) d\lambda \\ &- \int_{B_{r_0}\setminus\tilde\Gamma} (A - \mathrm{Id}_N)\nabla U \cdot \nabla\Big(|y|^{1-N} \omega(|y|) Y_{m,k}(y/|y|) \Big)\, dy,\end{aligned}$$

for any $\omega \in \mathcal{D}(0, r_0)$.

Since $\Upsilon_{k,i} \in L^1_{\rm loc}(0, r_0)$ by (113), we may consider its derivative in the sense of distributions. A direct calculation shows that

$$\Upsilon'_{k,i}(\lambda) = \lambda^{N-1} \zeta_{k,i}(\lambda) \tag{115}$$

in the sense of distributions on $(0, r_0)$. From the definition of $\zeta_{k,i}$, (21), and the fact that $Y_{k,i}$ is a solution of (84) we deduce that

$$-\varphi''_{k,i}(\lambda) - \frac{N-1}{\lambda} \varphi'_{k,i}(\lambda) + \frac{\mu_k}{\lambda^2} \varphi_{k,i}(\lambda) = \zeta_{k,i}(\lambda)$$

in the sense of distribution in $(0, r_0)$; the above equation can be rewritten as

$$-(\lambda^{N-1+k}(\lambda^{-\frac{k}{2}} \varphi_{k,i}(\lambda))')' = \lambda^{N-1+\frac{k}{2}} \zeta_{k,i}(\lambda),$$

thanks to (85). Integrating the right-hand side of the equation above by parts, since (115) holds, we obtain that, for every $r \in (0, r_0)$, $k \in \mathbb{N}$ and $i = 1, \dots, N_k$ there exists a constant $c_{k,i}(r)$ such that

$$(\lambda^{-\frac{k}{2}}\varphi_{k,i}(\lambda))' = -\lambda^{-N+1-\frac{k}{2}}\Upsilon_{k,i}(\lambda) - \frac{k}{2}\lambda^{-N+1-k}\left(c_{k,i}(r) + \int_\lambda^r s^{\frac{k}{2}-1}\Upsilon_{k,i}(s)\,ds\right)$$

in the sense of distribution on $(0, r_0)$. Then $\varphi_{k,i}(\lambda) \in W^{1,1}_{\rm loc}(0, r_0)$ and a further integration yields

$$\begin{aligned}\varphi_{k,i}(\lambda) &= \lambda^{\frac{k}{2}}\left(r^{-\frac{k}{2}}\varphi_{k,i}(r) + \int_\lambda^r s^{-N+1-\frac{k}{2}}\Upsilon_{k,i}(s)\,ds\right)\\ &\quad + \frac{k}{2}\lambda^{\frac{k}{2}}\left(\int_\lambda^r s^{-N+1-k}\left(c_{k,i}(r) + \int_s^r t^{\frac{k}{2}-1}\Upsilon_{k,i}(t)\,dt\right)ds\right.\\ &= \lambda^{\frac{k}{2}}\left(r^{-\frac{k}{2}}\varphi_{k,i}(r) + \frac{2N+k-4}{2(N+k-2)}\int_\lambda^r s^{-N+1-\frac{k}{2}}\Upsilon_{k,i}(s)\,ds\right)\\ &\quad - \lambda^{\frac{k}{2}}\frac{kc_{k,i}(r)r^{-N+2-k}}{2(N+k-2)} + \frac{k\lambda^{-N+2-\frac{k}{2}}}{2(N+k-2)}\left(c_{k,i}(r) + \int_\lambda^r t^{\frac{k}{2}-1}\Upsilon_{k,i}(t)\,dt\right).\end{aligned} \tag{116}$$

Now we claim that, if k_0 is as in Proposition 4.8, then

$$\text{the function } s \to s^{-N+1-\frac{k_0}{2}}\Upsilon_{k_0,i}(s) \text{ belongs to } L^1(0, r_0). \tag{117}$$

To this end we will estimate each terms in (113). Thanks to (13), Hölder's inequality, a change of variables and Proposition 4.4, we have that

$$\begin{aligned}&\left|\int_{B_s\setminus\tilde\Gamma}(A - \mathrm{Id}_N)\nabla U \cdot \frac{\nabla_{\mathbb{S}^{N-1}}Y_{k_0,i}(y/|y|)}{|y|}dy\right| \leq \mathrm{const}\int_{B_s\setminus\tilde\Gamma}|y||\nabla U|\frac{|\nabla_{\mathbb{S}^{N-1}}Y_{k_0,i}(y/|y|)|}{|y|}\,dy\\ &\leq \mathrm{const}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla U|^2 dy\right)^{\frac12}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla_{\mathbb{S}^{N-1}}Y_{k_0,i}(y/|y|)|^2 dy\right)^{\frac12}\\ &\leq \mathrm{const}\, s^{\frac{N-2}{2}}s^{\frac{N}{2}}\sqrt{H(s)}\left(\int_{B_1\setminus\tilde\Gamma}|\nabla W^s(y)|^2\,dy\right)^{\frac12} \leq \mathrm{const}\, s^{N-1}\sqrt{H(s)}.\end{aligned}$$

From Hölder's inequality, (28), (15), and Proposition 4.4 it follows that

$$\begin{aligned}&\left|\int_{B_s}\tilde f(y)U(y)Y_{k_0,i}(y/|y|)\,dy\right| \leq \left(\int_{B_s}|\tilde f(y)|U^2(y)\,dy\right)^{\frac12}\left(\int_{B_s}|\tilde f(y)|Y^2_{k_0,i}(y/|y|)\,dy\right)^{\frac12}\\ &\leq \mathrm{const}\, s^{\frac{4\epsilon}{N+2\epsilon}}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla U|^2\,dy + s^{N-2}H(s)\right)^{\frac12}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla Y_{k_0,i}(y/|y|)|^2\,dy + s^{N-2}\right)^{\frac12}\\ &\leq \mathrm{const}\, s^{(N-2)+\frac{4\epsilon}{N+2\epsilon}}\sqrt{H(s)}.\end{aligned}$$

Furthermore, in view of (13), for a.e. $s \in (0, r_0)$ we have that

$$\left|\int_{\partial B_s}(A - \mathrm{Id}_N)\nabla U \cdot \frac{y}{|y|}Y_{k_0,i}(y/|y|)\,dS\right| \leq \mathrm{const}\, s\int_{\partial B_s}|\nabla U||Y_{k_0,i}(y/|y|)|\,dS$$

and an integration by parts and Hölder's inequality yield, for any $r \in (0, r_0]$,

$$\begin{aligned}&\int_0^r s^{-N+2-\frac{k_0}{2}}\left(\int_{\partial B_s}|\nabla U||Y_{k_0,i}(y/|y|)|\,dS\right)ds = r^{-N+2-\frac{k_0}{2}}\int_{B_r\setminus\tilde\Gamma}|\nabla U||Y_{k_0,i}(y/|y|)|\\ &+ \left(N - 2 + \frac{k_0}{2}\right)\int_0^r s^{-N+1-\frac{k_0}{2}}\left(\int_{B_s\setminus\tilde\Gamma}|\nabla U||Y_{k_0,i}(y/|y|)|\,dS\right)ds\\ &\leq \mathrm{const}\left(r^{1-\frac{k_0}{2}}\sqrt{H(r)} + \int_0^r s^{-\frac{k_0}{2}}\sqrt{H(s)}\,ds\right),\end{aligned}$$

reasoning as above. In conclusion, combining the above estimates with (94) and (82), we obtain that, for any $r \in (0, r_0]$,

$$\int_0^r s^{-N+1-\frac{k_0}{2}} |\Upsilon_{k_0,i}(s)|\, ds \leq \text{const} \left(r^{1-\frac{k_0}{2}} \sqrt{H(r)} + \int_0^r s^{-\frac{k_0}{2}-1+\frac{4\epsilon}{N+2\epsilon}} \sqrt{H(s)}\, ds \right) \tag{118}$$
$$\leq \text{const} \left(r + \int_0^r s^{\frac{2\epsilon-N}{N+2\epsilon}}\, ds \right) \leq \text{const}\, r^{\frac{4\epsilon}{N+2\epsilon}}$$

which in particular implies (117). By (117), it follows that, for every $r \in (0, r_0]$,

$$\lambda^{\frac{k_0}{2}} \left(r^{-\frac{k_0}{2}} \varphi_{k_0,i}(r) + \frac{2N+k_0-4}{2(N+k_0-2)} \int_\lambda^r s^{-N+1-\frac{k_0}{2}} \Upsilon_{k_0,i}(s)\, ds - \frac{k_0 c_{k_0,i}(r) r^{-N+2-k_0}}{2(N+k_0-2)} \right) \tag{119}$$
$$= O\left(\lambda^{\frac{k_0}{2}}\right) = o\left(\lambda^{-N+2-\frac{k_0}{2}}\right) \quad \text{as } \lambda \to 0^+$$

and $s \to s^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(s)$ belongs to $L^1(0, r_0)$.

Next we show that for every $r \in (0, r_0)$

$$c_{k_0,i}(r) + \int_0^r t^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(t)\, dt = 0. \tag{120}$$

We argue by contradiction assuming that there exists $r \in (0, r_0)$ such that (120) does not hold. Then by (116) and (119)

$$\varphi_{k_0,i}(\lambda) \sim \frac{k_0 \lambda^{-N+2-\frac{k_0}{2}}}{2(N+k_0-2)} \left(c_{k_0,i}(r) + \int_\lambda^r t^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(t)\, dt \right) \quad \text{as } \lambda \to 0^+. \tag{121}$$

From Hölder's inequality, a change of variables, and (26)

$$\int_0^{r_0} \lambda^{N-3} |\varphi_{k_0,i}(\lambda)|^2\, d\lambda \leq \int_0^{r_0} \lambda^{N-3} \left(\int_{\mathbb{S}^{N-1}} |U(\lambda\theta)|^2 dS \right) d\lambda = \int_{B_{r_0}} \frac{|U|^2}{|y|^2} dy < +\infty$$

thus contradicting (121). Hence (120) is proved.

Furthermore from (118) and (120)

$$\left| \lambda^{-N+2-\frac{k_0}{2}} \left(c_{k_0,i}(r) + \int_\lambda^r t^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(t)\, dt \right) \right| = \lambda^{-N+2-\frac{k_0}{2}} \left| \int_0^\lambda t^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(t)\, dt \right| \tag{122}$$
$$\leq \lambda^{-N+2-\frac{k_0}{2}} \int_0^\lambda t^{N-2+k_0} \left| t^{-N+1-\frac{k_0}{2}} \Upsilon_{k_0,i}(t) \right| dt$$
$$\leq \lambda^{\frac{k_0}{2}} \int_0^\lambda \left| t^{-N+1-\frac{k_0}{2}} \Upsilon_{k_0,i}(t) \right| dt = O\left(\lambda^{\frac{4\epsilon}{N+2\epsilon}+\frac{k_0}{2}}\right) \quad \text{as } \lambda \to 0^+.$$

Then the conclusion follows form (116), (120), and (122). □

Proposition 5.2. *Let γ be as in* (81)*. Then*

$$\lim_{r\to 0^+} r^{-2\gamma} H(r) > 0.$$

Proof. For any $\lambda \in (0, r_0)$ the function $U(\lambda\cdot)$ belongs to $L^2(\mathbb{S}^{N-1})$. Then we can expand it in Fourier series respect to the basis $\{Y_{k,i}\}_{k\in\mathbb{N}, i=1,\dots,N_k}$ introduced in Proposition 4.2:

$$U(\lambda\,\cdot) = \sum_{k=0}^\infty \sum_{i=1}^{N_k} \varphi_{k,i}(\lambda) Y_{k,i} \quad \text{in } L^2(\mathbb{S}^{N-1}),$$

where we have defined $\varphi_{k,i}(\lambda)$ in (112) for any $k \in \mathbb{N}$ and any $i = 1, \dots, N_k$. From (16), a change of variables and the Parseval identity

$$H(\lambda) = (1 + O(\lambda)) \int_{\mathbb{S}^{N-1}} U^2(\lambda\theta)\, dS = (1 + O(\lambda)) \sum_{k=0}^\infty \sum_{i=1}^{N_k} |\varphi_{k,i}(\lambda)|^2. \tag{123}$$

We argue by contradiction assuming that $\lim_{r\to 0^+} r^{2\gamma}H(r)=0$. Then by (123), letting k_0 be as in (94),

$$\lim_{\lambda\to 0^+}\lambda^{-\frac{k_0}{2}}\varphi_{k_0,i}(\lambda)=0 \quad \text{for any } i=1,\dots,N_{k_0}.$$

From (114) it follows that

$$r^{-\frac{k_0}{2}}\varphi_{k_0,i}(r)+\frac{2N+k_0-4}{2(N+k_0-2)}\int_0^r s^{-N+1-\frac{k_0}{2}}\Upsilon_{k_0,i}(s)\,ds + \frac{k_0 r^{-N+2-k_0}}{2(N+k_0-2)}\int_0^r s^{\frac{k_0}{2}-1}\Upsilon_{k_0,i}(s)\,ds=0 \tag{124}$$

for any $r\in(0,r_0)$ and any $i=1,\dots,N_{k_0}$.

In view of (86), (112), (118), and (122), (124) implies that

$$\sqrt{H(\lambda)}\int_{\mathbb{S}^{N-1}} W^\lambda Y_{k_0,i}\,dS=\varphi_{k_0,i}(\lambda)=O\Big(\lambda^{\frac{4\epsilon}{N+2\epsilon}+\frac{k_0}{2}}\Big) \quad \text{as } \lambda\to 0^+ \tag{125}$$

for all $i=1,\dots,N_{k_0}$. From (83) with $\sigma=\frac{4\epsilon}{N+2\epsilon}$ we have that $\sqrt{H(\lambda)}\geq\sqrt{\alpha_{\frac{4\epsilon}{N+2\epsilon}}}\lambda^{\frac{k_0}{2}+\frac{2\epsilon}{N+2\epsilon}}$ in a neighbourhood of 0, so that (125) implies that

$$\int_{\mathbb{S}^{N-1}} W^\lambda Y_{k_0,i}\,dS=O\Big(\lambda^{\frac{2\epsilon}{N+2\epsilon}}\Big)=o(1) \quad \text{as } \lambda\to 0^+ \tag{126}$$

for all $i=1,\dots,N_{k_0}$.

On the other hand, by Proposition 4.8 and continuity of the trace map γ_1 (see Proposition 2.4), for every sequence $\lambda_n\to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}$ and $\Psi\in\operatorname{span}\{Y_{k_0,i}: m=i,\dots,N_{k_0}\}$ such that

$$\|\Psi\|_{L^2(\mathbb{S}^{N-1})}=1 \quad \text{and} \quad W^{\lambda_{n_k}}\to\Psi \quad \text{in } L^2(\mathbb{S}^{N-1}). \tag{127}$$

From (126) and (127) it follows that

$$0=\lim_{k\to\infty}\int_{\mathbb{S}^{N-1}} W^{\lambda_{n_k}}\Psi\,dS=\|\Psi\|^2_{L^2(\mathbb{S}^{N-1})}=1,$$

thus reaching a contradiction. □

We are now ready to prove he following result, which is a more complete version of Theorem 1.3.

Theorem 5.3. *Let $N\geq 3$ and let $u\in H^1(B_R\setminus\Gamma)$ be a non-trivial weak solution to* (1), *with Γ defined in* (2)–(3) *and f satisfying either assumption* (H1) *or assumption* (H2). *Then there exists $k_0\in\mathbb{N}$ such that, letting $\mathcal{N}$ be as in Section 3,*

$$\lim_{r\to 0^+}\mathcal{N}(r)=\frac{k_0}{2}. \tag{128}$$

Moreover if N_{k_0} is the multiplicity of the eigenvalue μ_{k_0} of problem (6) *and $\{Y_{k_0,i}\}_{i=1,\dots,N_{k_0}}$ is a $L^2(\mathbb{S}^{N-1})$-orthonormal basis of the eigenspace associated to μ_{k_0}, then*

$$\lambda^{-\frac{k_0}{2}}u(\lambda\cdot)\to\Phi \quad \textit{and} \quad \lambda^{1-\frac{k_0}{2}}\left(\nabla_{B_R\setminus\Gamma}u\right)(\lambda\cdot)\to\nabla_{\mathbb{R}^N\setminus\tilde\Gamma}\Phi \quad \textit{in } L^2(B_1) \quad \textit{as } \lambda\to 0^+, \tag{129}$$

where

$$\Phi=\sum_{i=1}^{N_{k_0}}\alpha_i Y_{k_0,i}\left(\frac{y}{|y|}\right)$$

$(\alpha_1,\dots,\alpha_{N_{k_0}})\neq(0,\dots,0)$ *and, for all $i\in\{1,2,\dots,N_{k_0}\}$,*

$$\alpha_i=r^{-k_0/2}\int_{\mathbb{S}^{N-1}} u(F(r\theta))Y_{k_0,i}(\theta)\,dS + \frac{1}{2-N-k_0}\int_0^r\left(\frac{2-N-\frac{k_0}{2}}{s^{N+\frac{k_0}{2}-1}}-\frac{k_0 s^{\frac{k_0}{2}-1}}{2r^{N-2+k_0}}\right)\Upsilon_{k_0,i}(s)\,ds \tag{130}$$

for any $r \in (0, r_0)$ for some $r_0 > 0$, where we have defined $\Upsilon_{k_0,i}$ in (113) *and F is the diffeomorphism introduced in Proposition 2.1.*

Proof. (128) directly comes from (94). Let $U = u \circ F$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} \lambda_n = 0^+$. By Proposition 4.8 and Proposition 5.2 there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and constants $\alpha_1, \dots, \alpha_{N_{k_0}}$ such that $(\alpha_1, \dots, \alpha_{N_{k_0}}) \neq (0, \dots, 0)$ and

$$\lambda_{n_k}^{-\frac{k_0}{2}} U(\lambda_{n_k} y) \to |y|^{\frac{k_0}{2}} \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i}\left(\frac{y}{|y|}\right) \quad \text{in } H^1(B_1 \setminus \tilde{\Gamma}) \quad \text{as } k \to \infty.$$

Now we show that the coefficients $\alpha_1, \dots, \alpha_{N_{k_0}}$ do not depend on $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Thanks to the continuity of the trace operator γ_1 introduced in Proposition 2.4

$$\lambda_{n_k}^{-\frac{k_0}{2}} U(\lambda_{n_k} \cdot) \to \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i} \quad \text{in } L^2(\mathbb{S}^{N-1}) \quad \text{as } k \to \infty$$

and therefore, letting $\varphi_{k_0,i}$ be as in (112) for any $i = 1, \dots, N_{k_0}$,

$$\lim_{k\to\infty} \lambda^{-\frac{k_0}{2}} \varphi_{k_0,i}(\lambda_{n_k}) = \lim_{k\to\infty} \int_{\mathbb{S}^{N-1}} \lambda_{n_k}^{-k_0/2} U(\lambda_{n_k}\theta) Y_{k_0,i}(\theta)\, dS = \sum_{j=1}^{N_{k_0}} \alpha_j \int_{\mathbb{S}^{N-1}} Y_{k_0,j} Y_{k_0,i}\, dS = \alpha_i.$$

On the other hand by (114)

$$\lim_{k\to\infty} \lambda^{-\frac{k_0}{2}} \varphi_{k_0,i}(\lambda_{n_k}) = r^{-\frac{k_0}{2}} \varphi_{k_0,i}(r) + \frac{2N + k_0 - 4}{2(N + k_0 - 2)} \int_0^r s^{-N+1-\frac{k_0}{2}} \Upsilon_{k_0,i}(s)\, ds$$
$$+ \frac{k_0 r^{-N+2-k_0}}{2(N + k_0 - 2)} \int_0^r s^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(s)\, ds,$$

for all $i = 1, \dots, N_{k_0}$ and $r \in (0, r_0]$, where we have defined $\Upsilon_{k_0,i}$ in (113). We deduce that

$$\alpha_i = r^{-\frac{k_0}{2}} \varphi_{k_0,i}(r) + \frac{2N + k_0 - 4}{2(N + k_0 - 2)} \int_0^r s^{-N+1-\frac{k_0}{2}} \Upsilon_{k_0,i}(s)\, ds + \frac{k_0 r^{-N+2-k_0}}{2(N + k_0 - 2)} \int_0^r s^{\frac{k_0}{2}-1} \Upsilon_{k_0,i}(s)\, ds \tag{131}$$

and so α_i does not depend on $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ thus implying that

$$\lambda^{-\frac{k_0}{2}} U(\lambda y) \to |y|^{\frac{k_0}{2}} \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i}\left(\frac{y}{|y|}\right) \quad \text{in } H^1(B_1 \setminus \tilde{\Gamma}) \quad \text{as } \lambda \to 0^+. \tag{132}$$

To prove (129) we note that

$$\lambda^{-\frac{k_0}{2}} u(\lambda x) = \lambda^{-\frac{k_0}{2}} U(\lambda G_\lambda(x)), \quad \nabla\left(\lambda^{-\frac{k_0}{2}} u(\lambda x)\right) = \nabla\left(\lambda^{-\frac{k_0}{2}} U(\lambda x)\right)(G_\lambda(x)) J_{G_\lambda}(x),$$

where $G_\lambda(x) = \frac{1}{\lambda} F^{-1}(\lambda x)$ and F is the diffeomorphism introduced in Proposition 2.1. We also have by Proposition 2.1 that

$$G_\lambda(x) = x + O(\lambda) \quad \text{and} \quad J_G(x) = \mathrm{Id}_N + O(\lambda)$$

as $\lambda \to 0^+$ uniformly respect to $x \in B_1$. Then from (132) we deduce (129) and (130) follows from (131) and (112). □

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