Normalized solutions for the fractional NLS with mass supercritical nonlinearity

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Abstract

We investigate the existence of solutions to the fractional nonlinear Schrödinger equation $(-\Delta)^s u = f(u) - \mu u$ with prescribed L^2-norm $\int_{\mathbb{R}^N} |u|^2 \, dx = m$ in the Sobolev space $H^s(\mathbb{R}^N)$. Under fairly general assumptions on the nonlinearity f, we prove the existence of a ground state solution and a multiplicity result in the radially symmetric case.

Keywords: Fractional Schrödinger equation, Supercritical nonlinearity, Prescribed mass

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1. Introduction

In this paper we investigate the existence of solutions to the fractional Nonlinear Schrödinger Equation (NLS in the sequel)

$$\mathrm{i}\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi - V(|\psi|)\psi, \tag{1}$$

where i denotes the imaginary unit and $\psi = \psi(x,t) \colon \mathbb{R}^N \times (0,\infty) \to \mathbb{C}$. This type of Schrödinger equation was introduced by Laskin in [1], and the interest in

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its analysis has grown over the years. An important family of solutions, known under the name of *travelling* or *standing waves*, is characterized by the *ansatz*

$$\psi(x,t) = e^{\mathrm{i}\mu t}u(x) \tag{2}$$

for some (unknown) function $u\colon \mathbb{R}^N \to \mathbb{R}$. These solutions are self-similar and conserve their mass along time, i.e. $\frac{d}{dt}\|\psi(\cdot,t)\|_{L^2(\mathbb{R}^N)} = 0$ at any $t > 0$. It is therefore natural and meaningful to seek solutions having a *prescribed* L^2-norm. Coupling (1) with (2), we arrive at the problem

$$\begin{cases} (-\Delta)^s u = V(|u|)u - \mu u & \text{in } \mathbb{R}^N, \\ \|u\|^2_{L^2(\mathbb{R}^N)} = m, \end{cases}$$

where $s \in (0,1)$, $N > 2s$, $\mu \in \mathbb{R}$, $m > 0$ is a prescribed parameter, and $(-\Delta)^s$ denotes the usual fractional laplacian. We recall that

$$(-\Delta)^s u(x) = C(N,s) \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,$$

where

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{1-\cos\zeta_1}{|\zeta|^{n+2s}}\, d\zeta \right)^{-1}.$$

For further details on the fractional laplacian we refer to [2]. For our purposes, and since the parameter s is kept fixed, we will always work with a *rescaled* fractional operator, in such a way that $C(N,s) = 1$.

In order to ease notation, we will write $f(u) = V(|u|)u$, and study the problem

$$\begin{cases} (-\Delta)^s u = f(u) - \mu u & \text{in } \mathbb{R}^N, \\ \|u\|^2_{L^2(\mathbb{R}^N)} = m. \end{cases} \tag{P_m}$$

The rôle of the real number μ is twofold: it can either be *prescribed*, or it can arise as a *suitable* parameter in the analysis of (P_m). In the present work we will choose the second option, and μ will arise as a Lagrange multiplier.

Since we are looking for *bound-state* solutions whose L^2-norm must be finite, it is natural to build a variational setting for (P_m). Since this is by now standard, we will be sketchy. We introduce the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \mid [u]^2_{H^s(\mathbb{R}^N)} < +\infty \right\},$$

where

$$[u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy$$

is the so-called Gagliardo semi-norm. The norm in $H^s(\mathbb{R}^N)$ is defined by

$$\|u\| = \sqrt{\|u\|^2_{L^2} + [u]^2_{H^s(\mathbb{R}^N)}},$$

which naturally arises from an inner product. We then (formally) introduce the energy functional

$$I(u) = \frac{1}{2} [u]^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(u) \, dx$$

where $F(t) = \int_0^t f(\sigma) \, d\sigma$. A standard approach for studying (P_m) consists in looking for critical points of I constrained on the sphere

$$S_m = \left\{ u \in H^s(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |u|^2 \, dx = m \right\}.$$

The convenience of this variational approach depends strongly on the behavior of the nonlinearity f. If $f(t)$ grows slower than $|t|^{1+\frac{4s}{N}}$ as $t \to +\infty$, then I is coercive and bounded from below on S_m: this is the *mass subcritical case*, and the minimization problem

$$\min \left\{ I(u) \mid u \in S_m \right\}$$

is the natural approach. On the other hand, if $f(t)$ grows faster than $|t|^{1+\frac{4s}{N}}$ as $t \to +\infty$ then I is unbounded from below on S_m, and we are in the *mass supercritical case*. Since constrained minimizers of I on S_m cannot exist, we have to find critical points at higher levels.

When $s = 1$, i.e. when the fractional Laplace operator $(-\Delta)^s$ reduces to the *local* differential operator $-\Delta$, the literature for (P_m) is huge. The particular case of a combined nonlinearity of power type, namely $f(t) = t^{p-2} + \mu t^{q-2}$ with

$2 < q < p < 2N/(N-2)$ has been widely investigated. The interplay of the parameters p and q add some richness to the structure of the problem.

The situation is different when $0 < s < 1$, and few results are available. Feng *et al.* in [3] deal with particular nonlinearities. Stanislavova *et al.* in [4] add the further complication of a trapping potential. In the recent paper [5] the author proves some existence and asymptotic results for the fractional NLS when a lower order perturbation to a mass supercritical pure power in the nonlinearity is added. It is also worth mentioning [6], where Zhang *et al.* studied the problem when the nonlinear term consists in the sum of two pure powers of different order. They provide some existence and non-existence results analysing separately what happens in the mass subcritical and supercritical case for both the leading term and the lower order perturbation.

Very recently, Jeanjean *et al.* in [7] provided a thorough treatment of the local case $s = 1$ via a careful analysis based on the Pohozaev identity. In the present paper we propose a partial extension of their results to the non-local case $0 < s < 1$. Since we deal with a fractional operator, our conditions on f must be adapted correspondingly.

We collect here our standing assumptions about the nonlinearity f; we recall that

$$F(t) = \int_0^t f(\sigma)\, d\sigma$$

and define the auxiliary function

$$\tilde{F}(t) = f(t)t - 2F(t).$$

(f_0) $f\colon \mathbb{R} \to \mathbb{R}$ is an odd and continuous function;

(f_1) $\displaystyle\lim_{t\to 0} \frac{f(t)}{|t|^{1+4s/N}} = 0$;

(f_2) $\displaystyle\lim_{t\to +\infty} \frac{f(t)}{|t|^{(N+2s)/(N-2s)}} = 0$;

(f_3) $\displaystyle\lim_{t\to +\infty} \frac{F(t)}{|t|^{2+4s/N}} = +\infty$;

(f_4) The function $t \mapsto \frac{\tilde{F}(t)}{|t|^{2+4s/N}}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, +\infty)$;

(f_5) $f(t)t < \frac{2N}{N-2s}F(t)$ for all $t \in \mathbb{R} \setminus \{0\}$;

(f_6) $\displaystyle\lim_{t\to 0} \frac{tf(t)}{|t|^{2N/(N-2s)}} = +\infty.$

Remark 1. *The oddness of f is necessary in order to use the classical genus theory and to get a desired property on the fiber map that we will introduce in detail in the next section (see for instance Lemma 5 below). Assumption (f_2) guarantees a Sobolev subcritical growth, whereas (f_3) characterises the problem as mass supercritical. At one point we will need (f_5) to establish the strict positivity of the Lagrange multiplier μ.*

Example 1. *As suggested in [7], an explicit example can be constructed as follows. Set $\alpha_{N,s} = \frac{4s^2}{N(N-2s)}$ for simplicity, and define*

$$f(t) = \left(\left(2 + \frac{4s}{N} \right) \log\left(1 + |t|^{\alpha_{N,s}}\right) + \frac{\alpha_{N,s}|t|^{\alpha_{N,s}}}{1 + |t|^{\alpha_{N,s}}} \right) |t|^{\frac{4s}{N}} t$$

We briefly outline our results. Firstly, we show that the ground state level is attained with a strictly positive Lagrange multiplier.

Theorem 1. *Assume that f satisfies (f_0)-(f_5). Then (P_m) admits a positive ground state for any $m > 0$. Moreover, for any ground state the associated Lagrange multiplier μ is positive.*

Furthermore, we can prove some remarkable properties of the ground state level energy with respect the variable m and its asymptotic behavior. We refer to (11) for the precise definition of the ground-state level E_m.

Theorem 2. *Assume that f satisfies (f_0)-(f_6). Then the function $m \mapsto E_m$ is positive, continuous, strictly decreasing. Furthermore, $\lim_{m\to 0^+} E_m = +\infty$ and $\lim_{m\to\infty} E_m = 0$.*

Finally, we have a multiplicity result for the radially symmetric case.

Theorem 3. *If* (f_0)-(f_5) *hold and* $N > 2$*, then* (P_m) *admits infinitely many radial solutions* $(u_k)_k$ *for any* $m > 0$*. In particular,*

$$I(u_{k+1}) \geq I(u_k)$$

for all $k \in \mathbb{N}$ *and* $I(u_k) \to +\infty$ *as* $k \to +\infty$*.*

Our paper is organised as follows. Section 2 contains the proofs of some preliminary lemmas that will be useful during the whole remaining part of the paper. Moreover, we introduce a fiber map that will play a crucial role for our purposes. In Section 3 we define the ground state level energy for a fixed mass m and we start analysing its asymptotic behaviour near zero and infinity. Section 4 is devoted to prove our main existence theorem. Using a min-max theorem of linking type and the fiber map cited previously, we construct a Palais-Smale sequence whose value on the Pohozaev functional is zero and we show that a sequence of this kind must be necessarily bounded. Finally, in Section 5, for the sake of completeness, we discuss the existence of radial solutions. Here, we use a variant of the min-max theorem already cited in Section 4, but this time we are helped by the fact that the space of the radially symmetric functions with finite fractional derivative is compactly embedded in $L^p(\mathbb{R}^N)$ for $p \in (2, 2_s^*)$.

2. Preliminary results

We define the *Pohozaev manifold*

$$\mathcal{P}_m = \{u \in S_m \mid P(u) = 0\},$$

where

$$P(u) = [u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2s} \int_{\mathbb{R}^N} \tilde{F}(u)\, dx.$$

Let us collect some technical results that we will frequently used in the paper. The first two Lemmas will be proved in the Appendix. We use the shorthand

$$B_m = \left\{ u \in H^s(\mathbb{R}^N) \mid \|u\|^2_{L^2(\mathbb{R}^N)} \leq m \right\}.$$

Lemma 1. *Assuming* (f_0), (f_1), (f_2), *the following statements hold*

(i) for every $m > 0$ *there exists* $\delta > 0$ *such that*

$$\frac{1}{4}\,[u]^2_{H^s(\mathbb{R}^N)} \leq I(u) \leq [u]^2_{H^s(\mathbb{R}^N)}$$

where $u \in B_m$ *and* $[u]_{H^s(\mathbb{R}^N)} \leq \delta$.

(ii) Let $(u_n)_n$ *be a bounded sequence in* $H^s(\mathbb{R}^N)$. *If* $\lim_{n\to+\infty} \|u_n\|_{L^{2+4s/N}(\mathbb{R}^N)} = 0$ *we have that*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} F(u_n)\,dx = 0 = \lim_{n\to+\infty} \int_{\mathbb{R}^N} \tilde{F}(u_n)\,dx.$$

(iii) Let $(u_n)_n$, $(v_n)_n$ *two bounded sequences in* $H^s(\mathbb{R}^N)$. *If* $\lim_{n\to+\infty} \|v_n\|_{L^{2+4s/N}} = 0$ *then*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} f(u_n) v_n\,dx = 0.$$

Remark 2. *An inspection of the proof of this Lemma shows that the inequality*

$$\int_{\mathbb{R}^N} \tilde{F}(u)\,dx \leq \frac{s}{N}\,[u]^2_{H^s(\mathbb{R}^N)}$$

holds true if $u \in B_m$ *and* $[u]_{H^s(\mathbb{R}^N)} \leq \delta$. *It follows that*

$$P(u) \geq \frac{1}{2}\,[u]^2_{H^s(\mathbb{R}^N)}$$

for every $u \in B_m$ *with* $[u]_{H^s(\mathbb{R}^N)} \leq \delta$.

In order to prove the next result we introduce for every $u \in H^s(\mathbb{R}^N)$ and $\rho \in \mathbb{R}$ the scaling map[2]

$$(\rho * u)(x) = e^{\frac{N\rho}{2}} u(e^{\rho} x) \quad x \in \mathbb{R}^N.$$

It easy to verify that $\rho * u \in H^s(\mathbb{R}^N)$ and $\|\rho * u\|_{L^2(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)}$.

Lemma 2. *Assuming* (f_0), (f_1), (f_2) *and* (f_3), *we have:*

[2]The notation $\rho * u$ is standard in the theory of transformation groups, and is not ambiguous since we never use convolution.

(i) $I(\rho * u) \to 0^+$ *as* $\rho \to -\infty$,

(ii) $I(\rho * u) \to -\infty$ *as* $\rho \to \infty$.

Remark 3. *Assume* $f \in C(\mathbb{R}, \mathbb{R})$, (f_1) *and* (f_4). *Then the function* $g\colon \mathbb{R} \to \mathbb{R}$ *defined as*

$$g(t) = \begin{cases} \frac{f(t)t - 2F(t)}{|t|^{2+\frac{4s}{N}}}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

is continuous, strictly increasing in $(0, \infty)$ *and strictly decreasing in* $(-\infty, 0)$.

Lemma 3. *Assuming* $f \in C(\mathbb{R}, \mathbb{R})$, (f_1), (f_3) *and* (f_4), *we have*

(i) $F(t) > 0$ *if* $t \neq 0$;

(ii) there exists $(\tau_n^+)_n \subset \mathbb{R}^+$ *and* $(\tau_n^-)_n \subset \mathbb{R}^-$, $|\tau_n^\pm| \to 0$ *as* $n \to +\infty$ *such that*

$$f(\tau_n^\pm)\tau_n^\pm > \left(2 + \frac{4s}{N}\right) F(\tau_n^\pm)$$

for any $g \neq 1$;

(iii) there exists $(\sigma_n^+)_n \subset \mathbb{R}^+$ *and* $(\sigma_n^-)_n \subset \mathbb{R}^-$, $|\sigma_n^\pm| \to \infty$ *as* $n \to +\infty$ *such that*

$$f(\sigma_n^\pm)\sigma_n^\pm > \left(2 + \frac{4s}{N}\right) F(\sigma_n^\pm)$$

for any $n \geq 1$.

Proof. *(i)* By contradiction suppose $F(t_0) \leq 0$ for some $t_0 \neq 0$. Because of (f_1) and (f_3) the function $F(t)/|t|^{2+4s/N}$ must attain its global minimum in a point $\tau \neq 0$ such that $F(\tau) \leq 0$. It follows that

$$\frac{d}{dt} \frac{F(t)}{|t|^{2+\frac{4s}{N}}}\bigg|_{t=\tau} = \frac{f(\tau)\tau - \left(2 + \frac{4s}{N}\right) F(\tau)}{|\tau|^{3+\frac{4s}{N}} \operatorname{sgn}(\tau)} = 0. \tag{3}$$

From Remark 3 it follows that $f(t)t > 2F(t)$ if $t \neq 0$. Indeed, were the claim false, there would exists $\bar{t}$ such that $f(\bar{t})\bar{t} \leq 2F(\bar{t})$. Choosing without loss of generality $\bar{t} < 0$, we have that $g(\bar{t}) \leq 0$. This and the fact that $g(0) = 0$ show that g must be strictly increasing on an interval between $\bar{t}$ and 0. Finally, we can have a contradiction observing that

$$0 < f(\tau)\tau - 2F(\tau) = \frac{4s}{N}F(\tau) \leq 0.$$

(ii) We start with the positive case. By contradiction we suppose there is $T_\alpha > 0$ small enough such that

$$f(t)t \leq \left(2 + \frac{4s}{N}\right) F(t)$$

for every $t \in (0, T_\alpha]$. Remembering the expression of (3) computed in the step (i) we have that the derivative of $F(t)/|t|^{2+4s/N}$ is nonpositive on $(0, T_\alpha]$, then

$$\frac{F(t)}{t^{2+\frac{4s}{N}}} \geq \frac{F(T_\alpha)}{T_\alpha^{2+\frac{4s}{N}}} > 0 \quad \text{for every} \quad t \in (0, T_\alpha]\,,$$

that is in contradiction with (f_1). The negative case is similar.

(iii) Being the two cases similar, we will prove only the negative one. Again, by contradiction we suppose there is $T_\gamma > 0$ such that

$$f(t)t \leq \left(2 + \frac{4s}{N}\right) F(t) \quad \text{for every} \quad t \leq -T_\gamma.$$

Since the derivative of $F(t)/|t|^{2+4s/N}$ is nonnegative on $(-\infty, -T_\gamma]$, we can deduce

$$\frac{F(t)}{|t|^{2+\frac{4s}{N}}} \leq \frac{F(-T_\gamma)}{T_\gamma^{2+\frac{4s}{N}}} \quad \text{for every} \quad t \in (-\infty, -T_\gamma]\,,$$

which contradicts (f_3).

Lemma 4. *For any $t > 0$ there results*

$$f(t)t > \left(2 + \frac{4s}{N}\right) F(t).$$

Proof. We start proving that the inequality holds weakly. By contradiction we assume

$$f(t_0)t_0 < \left(2 + \frac{4s}{N}\right) F(t_0)$$

for some $t_0 \neq 0$ and without loss of generality we can suppose $t_0 < 0$. By step (ii) and (iii) of Lemma 3 there are $\tau_{\min}, \tau_{\max} \in \mathbb{R}$, where $\tau_{\min} < t_0 < \tau_{\max} < 0$ such that

$$f(t)t < \left(2 + \frac{4s}{N}\right) F(t) \quad \text{for every} \quad t \in (\tau_{\min}, \tau_{\max}) \tag{4}$$

and

$$f(t)t = \left(2 + \frac{4s}{N}\right) F(t) \quad \text{for every} \quad t \in \{\tau_{\min}, \tau_{\max}\}. \tag{5}$$

By (4) we have

$$\frac{F(\tau_{\min})}{|\tau_{\min}|^{2+\frac{4s}{N}}} < \frac{F(\tau_{\max})}{|\tau_{\max}|^{2+\frac{4s}{N}}}. \tag{6}$$

Besides, by (5) and (f_4) must be

$$\frac{F(\tau_{\min})}{|\tau_{\min}|^{2+\frac{4s}{N}}} = \frac{N}{4s} \frac{\tilde{F}(\tau_{\min})}{|\tau_{\min}|^{2+\frac{4s}{N}}} > \frac{N}{4s} \frac{\tilde{F}(\tau_{\max})}{|\tau_{\max}|^{2+\frac{4s}{N}}} = \frac{F(\tau_{\max})}{|\tau_{\max}|^{2+\frac{4s}{N}}}, \tag{7}$$

and clearly (6) and (7) are in contradiction. From what we have just proved, we have that $F(t)/|t|^{2+4s/N}$ is non-increasing in $(-\infty, 0)$ and non decreasing in $(0, \infty)$. Hence, by virtue of (f_4) the function $f(t)/|t|^{1+4s/N}$ must necessarily be strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$. Then

$$\begin{aligned}\left(2 + \frac{4s}{N}\right) F(t) &= \left(2 + \frac{4s}{N}\right) \int_0^t \frac{f(\kappa)}{|\kappa|^{1+\frac{4s}{N}}} |\kappa|^{1+\frac{4s}{N}}\, d\kappa \\ &< \left(2 + \frac{4s}{N}\right) \frac{f(t)}{|t|^{1+\frac{4s}{N}}} \int_0^t |\kappa|^{1+\frac{4s}{N}}\, d\kappa = f(t)t\end{aligned}$$

completes the proof.

Lemma 5. *Assume* $(f_0) - (f_4)$, $u \in H^s(\mathbb{R}^N) \setminus \{0\}$. *Then the following hold:*

(i) *There is a unique* $\rho(u) \in \mathbb{R}$ *such that* $P(\rho(u) * u) = 0$.

(ii) $I(\rho(u) * u) > I(\rho * u)$ *for any* $\rho \neq \rho(u)$. *Moreover* $I(\rho(u) * u) > 0$.

(iii) *The map* $u \to \rho(u)$ *is continuous for every* $u \in H^s(\mathbb{R}^N)$.

(iv) $\rho(u) = \rho(-u)$ *and* $\rho(u(\cdot + y)) = \rho(u)$ *for ever* $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ *and* $y \in \mathbb{R}^N$.

Proof. (i) Since

$$I(\rho * u) = \frac{1}{2} e^{2\rho s} [u]^2_{H^s(\mathbb{R}^N)} - e^{-N\rho} \int_{\mathbb{R}^N} F(e^{N\rho} u)\, dx$$

it is easy to check that $I(\rho * u)$ is C^1 with respect to ρ. Now, computing

$$\frac{d}{d\rho} I(\rho * u) = \rho e^{2\rho s} [u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2} e^{-N\rho} \int_{\mathbb{R}^N} \tilde{F}\left(e^{\frac{N\rho}{2}} u\right) dx.$$

and observing that

$$P(\rho * u) = e^{2\rho s} [u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2s} e^{-N\rho} \int_{\mathbb{R}^N} \tilde{F}\left(e^{\frac{N\rho}{2}} u\right) dx$$

we deduce

$$\frac{d}{d\rho} I(\rho * u) = s P(\rho * u).$$

Remembering that by lemma 2

$$\lim_{\rho \to -\infty} I(\rho * u) = 0^+ \quad \text{and} \quad \lim_{\rho \to \infty} I(\rho * u) = -\infty$$

we can conclude that $\rho \mapsto I(\rho * u)$ must reach a global maximum at some point $\rho(u)$; since

$$0 = \frac{d}{d\rho} I(\rho(u) * u) = s P(\rho(u) * u),$$

we conclude that $P(\rho(u) * u) = 0$. To check the uniqueness of the point $\rho(u)$, recalling the function g defined in Remark 3, we observe that $\tilde{F}(t) = g(t)|t|^{2+\frac{4s}{N}}$ for every $t \in \mathbb{R}$. Thus we obtain

$$\begin{aligned} P(\rho * u) &= e^{2\rho s} [u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2s} e^{2\rho s} \int_{\mathbb{R}^N} g(e^{\frac{N\rho}{2}} u)|u|^{2+\frac{4s}{N}}\, dx \\ &= e^{2\rho s} \left[[u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2s} \int_{\mathbb{R}^N} g(e^{\frac{N\rho}{2}} u)|u|^{2+\frac{4s}{N}}\, dx \right] = \frac{1}{s} \frac{d}{d\rho} I(\rho * u). \end{aligned}$$

Fixing $t \in \mathbb{R} \setminus \{0\}$, thanks to Remark 3 and (f_4), we notice that the function

$$\rho \mapsto g\left(e^{\frac{N\rho}{2}} t\right)$$

is strictly increasing. Thus, by virtue of the previous computations, it follows that $\rho(u)$ must be unique.

(ii) This follows at once from (i).

(iii) By step (i) the function $u \mapsto \rho(u)$ is well defined. Let $u \in H^s(\mathbb{R}^N)\setminus\{0\}$ and $(u_n)_n \subset H^s(\mathbb{R}^N) \setminus \{0\}$ a sequence such that $u_n \to u$ in $H^s(\mathbb{R}^N)$ as $n \to +\infty$. We set $\rho_n = \rho(u_n)$ for any $n \geq 1$. Let us show that up to a subsequence we have $\rho_n \to \rho(u)$ as $n \to +\infty$.

Claim. The sequence $(\rho_n)_n$ is bounded.

We recall that the function h_λ defined in (43) noticing that by lemma 3 (i) $h_0(t) \geq 0$ for every $t \in \mathbb{R}$. We assume by contradiction that up to a subsequence $\rho_n \to +\infty$. By Fatou's lemma and the fact that $u_n \to u$ a.e. in $\mathbb{R}^N$, we have that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} h_0\left(e^{\frac{N\rho_n}{2}} u_n\right) |u_n|^{2+\frac{4s}{N}}\, dx = \infty.$$

As a consequence of that, by (44) with $\lambda = 0$ and step (ii), we obtain

$$0 \leq e^{-2\rho_n s} I(\rho_n * u_n) = \frac{1}{2}\left[u_n\right]^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} h_0\left(e^{\frac{N\rho_n}{2}} u_n\right) |u_n|^{2+\frac{4s}{N}}\, dx \to -\infty \tag{8}$$

as $n \to +\infty$ that is evidently not possible. Then $(\rho_n)_n$ must be bounded from above. Now we assume, again by contradiction, that $\rho_n \to -\infty$. By step (ii) we observe that

$$I(\rho_n * u_n) \geq I(\rho(u) * u_n),$$

and since $\rho(u) * u_n \to \rho(u) * u$ in $H^s(\mathbb{R}^N)$, it follows that

$$I(\rho(u) * u_n) = I(\rho(u) * u) + o_n(1).$$

We deduce that

$$\liminf_{n\to+\infty} I(\rho_n * u_n) \geq I(\rho(u) * u) > 0. \tag{9}$$

Since we have $\rho_n * u_n \subset B_m$ for $m \gg 1$, Lemma 1 (i) implies that there exists $\delta > 0$ such that if $\left[\rho_n * u_n\right]_{H^s(\mathbb{R}^N)} \leq \delta$, we have

$$\frac{1}{4}\left[\rho_n * u_n\right]^2_{H^s(\mathbb{R}^N)} \leq I(\rho_n * u_n) \leq \left[\rho_n * u_n\right]^2_{H^s(\mathbb{R}^N)}. \tag{10}$$

Since

$$[\rho_n * u_n]_{Hs} = e^{\rho_n s} [u_n]_{H^s(\mathbb{R}^N)} ,$$

(10) holds for any n sufficiently large. Therefore we obtain

$$\liminf_{n\to+\infty} I(\rho_n * u_n) = 0,$$

in contradiction to (9). The claim is proved.

The sequence $(\rho_n)_n$ being bounded, we can assume that, up to a subsequence, $\rho_n \to \rho^*$ for some ρ^* in $\mathbb{R}$. Hence, $\rho_n * u_n \to \rho^* * u$ in $H^s(\mathbb{R}^N)$ and since $P(\rho_n * u_n) = 0$ we have

$$P(\rho^* * u) = 0.$$

By the uniqueness proved at step (ii) we obtain $\rho^* = \rho(u)$.

(iv) Since f is odd by (f_0), the fact that

$$P(\rho(u) * (-u)) = P\left(-(\rho(u) * u)\right) = P(\rho(u) * u) = 0$$

imply $\rho(u) = \rho(-u)$. Similarly, changing the variables in the integral, we can verify that ρ is invariant under translation, and it is easy to check that

$$P(\rho(u) * u(\cdot + y)) = P(\rho(u) * u) = 0,$$

thus $\rho(u(\cdot + y)) = \rho(u)$.

As we are going to see, the functional I constrained on $\mathcal{P}_m$ has some crucial properties.

Lemma 6. *Assuming $(f_0) - (f_4)$, the following statements are true:*

(i) $\mathcal{P}_m \neq \emptyset$,

(ii) $\inf_{u\in\mathcal{P}_m} [u]_{H^s(\mathbb{R}^N)} > 0$,

(iii) $\inf_{u\in\mathcal{P}_m} I(u) > 0$,

(iv) I is coercive on $\mathcal{P}_m$, i.e. $I(u_n) \to \infty$ if $(u_n)_n \subset \mathcal{P}_m$ and $\|u_n\|_{H^s(\mathbb{R}^N)} \to \infty$ as $n \to +\infty$.

Proof. Statement (i) follows directly from Lemma 5 (i).

(ii) Were the assertion not true, we would be able to take a sequence $(u_n)_n \subset \mathcal{P}_m$ such that $[u_n]_{H^s(\mathbb{R}^N)} \to 0$, and so, by Lemma 1 (i) we could also find $\delta > 0$ and $\overline{n}$ so large that $[u_n]_{H^s(\mathbb{R}^N)} \leq \delta$ for every $n \geq \overline{n}$. By Remark 2 we would have

$$0 = P(u_n) \geq \frac{1}{2} [u_n]^2_{H^s(\mathbb{R}^N)}$$

which is possible only for a constant u_n. But this is not admissible since $u \in S_m$. Hence the statement must hold.

(iii) For every $u \in \mathcal{P}_m$ Lemma 5 (ii) and (iii) implies that

$$I(u) = I(0 * u) \geq I(\rho * u) \quad \text{for every} \quad \rho \in \mathbb{R}.$$

Let $\delta > 0$ be the number given by Lemma 2 (i) and set $1/\rho := s \log \Big(\delta / [u]_{H^s(\mathbb{R}^N)}\Big)$. Since $\delta = [\rho * u]_{H^s(\mathbb{R}^N)}$, using again Lemma 1 (i) we obtain

$$I(u) \geq I(\rho * u) \geq \frac{1}{4} [\rho * u]^2_{H^s(\mathbb{R}^N)} = \frac{1}{4}\delta^2$$

proving the statement.

(iv) By contradiction we suppose the existence of $(u_n)_n \subset \mathcal{P}_m$ such that $\|u_n\|_{H^s(\mathbb{R}^N)} \to \infty$ with $\sup_{n\geq 1} I(u_n) \leq c$ for some $c \in (0, \infty)$. For any $n \geq 1$ we set

$$\rho_n = \frac{1}{s} \log \Big([u_n]_{H^s(\mathbb{R}^N)}\Big) \quad \text{and} \quad v_n = (-\rho_n) * u_n.$$

Evidently $\rho_n \to +\infty$, $(v_n)_n \subset S_m$ and $[v_n]_{H^s(\mathbb{R}^N)} = 1$. We denote with

$$\alpha = \limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n|^2 \, dx$$

and we distinguish two cases.

Non vanishing: $\alpha > 0$. Up to a subsequence we can assume the existence of a sequence $(y_n)_n \subset \mathbb{R}^N$ and $\omega \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that

$$\omega_n = v_n(\cdot + y_n) \rightharpoonup \omega \text{ in } H^s(\mathbb{R}^N) \quad \text{and} \quad \omega_n \to \omega \text{ a.e. in } \mathbb{R}^N.$$

Recalling the definition of the continuous function h_λ with $\lambda = 0$, remembering that $\rho_n \to +\infty$ as $n \to +\infty$ and using the Fatou's lemma we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} h_0 \left(e^{\frac{N\rho_n}{2}} \omega_n\right) |\omega_n|^{2+\frac{4s}{N}} \, dx = \infty.$$

By step (iii) and (3), after changing the variables in the integral, we obtain

$$0 \leq e^{-2\rho_n s} I(u_n) = e^{-2\rho_n s} I(\rho_n * v_n) = \frac{1}{2} - \int_{\mathbb{R}^N} h_0 \left(e^{\frac{N\rho_n}{2}} v_n \right) |v_n|^{2+\frac{4s}{N}} \, dx$$
$$= \frac{1}{2} - \int_{\mathbb{R}^N} h_0 \left(e^{\frac{N\rho_n}{2}} \omega \right) |\omega_n|^{2+\frac{4s}{N}} \, dx \to -\infty$$

as $n \to +\infty$.

Vanishing: $\alpha = 0$. By [8, Lemma II.4], we have that $v_n \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$ and by Lemma 1 (ii) we see that

$$\lim_{n \to +\infty} e^{N\rho} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho}{2}} v_n \right) = 0 \quad \text{for every} \quad \rho \in \mathbb{R}.$$

Since $P(\rho_n * v_n) = P(u_n) = 0$, by Lemma 5 (ii) and (iii), we obtain

$$\begin{aligned} c &\geq I(u_n) = I(\rho_n * v_n) \\ &\geq P(\rho * v_n) = \frac{1}{2} e^{2\rho s} - e^{-N\rho} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho}{2}} v_n \right) dx = \frac{1}{2} e^{2\rho s} - o_n(1). \end{aligned}$$

We can conclude choosing $\rho > \log(2c)/2s$ and letting $n \to +\infty$.

We conclude with a splitting result *à la* Brezis-Lieb. A proof is included for the reader's convenience.

Lemma 7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ continuous, odd and let $(u_n)_n \subset H^s(\mathbb{R}^N)$ a bounded sequence such that $u_n \to u$ pointwise almost everywhere in $\mathbb{R}^N$. If there exists $C > 0$ such that*

$$|f(t)| \leq C \left(|t| + |t|^{2_s^* - 1} \right),$$

then

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} |F(u_n) - F(u_n - u) - F(u)| \, dx = 0$$

Proof. Let a, $b \in \mathbb{R}$ and $\varepsilon > 0$. We compute

$$
\begin{aligned}
|F(a+b) - F(a)| &= \left| \int_0^1 \frac{d}{d\tau} F(a+\tau b)\, d\tau \right| \\
&= \left| \int_0^1 F'(a+\tau b) b \, d\tau \right| \\
&\le C \int_0^1 \left(|a+\tau b| + |a+\tau b|^{2_s^*-1} \right) |b| \, d\tau \\
&\le C \left(|a| + |b| + 2^{2_s^*-1} \left(|a|^{2_s^*-1} + |b|^{2_s^*-1} \right) \right) |b| \\
&\le C \left(|a| + |b| + 2^{2_s^*} \left(|a|^{2_s^*-1} + |b|^{2_s^*-1} \right) \right) |b| \\
&\le C \left(|ab| + b^2 + 2^{2_s^*} \left(|a|^{2_s^*-1}|b| + |b|^{2_s^*} \right) \right).
\end{aligned}
$$

We have used that $\tau \le 1$ and the convexity inequality

$$
|a+b|^{2_s^*-1} \le 2^{2_s^*-1} \left(|a|^{2_s^*-1} + |b|^{2_s^*-1} \right).
$$

Now we use Young's inequality twice:

$$
\begin{aligned}
|ab| &\le \varepsilon \frac{a^2}{2} + \frac{1}{2\varepsilon} |b|^2 \\
|a|^{2_s^*-1}|b| &\le \eta^{\frac{2_s^*}{2_s^*-1}} \frac{|a|^{2_s^*}}{\frac{2_s^*}{2_s^*-1}} + \frac{1}{\eta^{2_s^*}} \frac{|b|^{2_s^*}}{2_s^*}.
\end{aligned}
$$

Hence, choosing

$$
\eta = \varepsilon^{\frac{2_s^*-1}{2_s^*}},
$$

we get

$$
\begin{aligned}
|ab| + b^2 + 2^{2_s^*} \left(|a|^{2_s^*-1}|b| + |b|^{2_s^*} \right) &\le \varepsilon \frac{a^2}{2} + \frac{1}{2\varepsilon} b^2 + b^2 + 2^{2_s^*} \left(|a|^{2_s^*-1}|b| + |b|^{2_s^*} \right) \\
&\le \varepsilon C \left(a^2 + |2a|^{2_s^*} \right) + C \left[\left(1 + \varepsilon^{-1} \right) b^2 + \left(1 + \varepsilon^{1-2_s^*} \right) |2b|^{2_s^*} \right] \\
&= \varepsilon \varphi(a) + \psi_\varepsilon(b).
\end{aligned}
$$

Applying [9, Theorem 2] with $g_n = u_n - u$ and $f = u$ we have the assertion.

3. Properties of the map $m \mapsto E_m$

Under our standing assumptions (f_0)–(f_4), for every $m > 0$ we can define the least level of energy

$$E_m = \inf_{u \in \mathcal{P}_m} I(u). \tag{11}$$

This section is devoted to the analysis of the quantity E_m as a *function* of $m > 0$.

Lemma 8. *If (f_0)–(f_4) hold true, then $m \mapsto E_m$ is continuous.*

Proof. Let $m > 0$ and $(m_k)_k \subset \mathbb{R}$ such that $m_k \to m$ in $\mathbb{R}$. We want to show that $E_{m_k} \to E_m$ as $k \to +\infty$. Firstly, we will prove that

$$\limsup_{k \to +\infty} E_{m_k} \leq E_m. \tag{12}$$

For any $u \in \mathcal{P}_m$ we define

$$u_k := \sqrt{\frac{m_k}{m}} u \in S_{m_k}, \quad k \in \mathbb{N}.$$

It is easy to see that $u_k \to u$ in $H^s(\mathbb{R}^N)$, thus, by Lemma 5 (iii) we get $\lim_{k \to +\infty} \rho(u_k) = \rho(u) = 0$. Therefore

$$\rho(u_k) * u_k \to \rho(u) * u = 0 \quad \text{in } H^s(\mathbb{R}^N)$$

as $k \to +\infty$ and as a consequence

$$\limsup_{k \to +\infty} E_{m_k} \leq \limsup_{k \to +\infty} I(\rho(u_k) * u_k) = I(u).$$

Since this holds for any u, we obtain (12). The next step consists in proving

$$\liminf_{k \to +\infty} E_{m_k} \geq E_m. \tag{13}$$

From the definition of E_{m_k}, it follows that for every $k \in \mathbb{N}$ there exists $v_k \in \mathcal{P}_{m_k}$ such that

$$I(v_k) \leq E_{m_k} + \frac{1}{k}. \tag{14}$$

We set

$$t_k := \left(\frac{m}{m_k}\right)^{\frac{1}{N}} \quad \text{and} \quad \tilde{v}_k := v_k\left(\frac{\cdot}{t_k}\right) \in S_m.$$

By Lemma 5 and (14) we get

$$\begin{aligned}
E_m &\le I(\rho(\tilde{v}_k) * \tilde{v}_k) \le I(\rho(v_k) * \tilde{v}_k) + |I(\rho(\tilde{v}_k) * \tilde{v}_k) - I(\rho(\tilde{v}_k) * v_k)| \\
&\le I(v_k) + |I(\rho(\tilde{v}_k) * \tilde{v}_k) - I(\rho(\tilde{v}_k) * v_k)| \\
&\le E_{m_k} + \frac{1}{k} + |I(\rho(\tilde{v}_k) * \tilde{v}_k) - I(\rho(\tilde{v}_k) * v_k)| \\
&=: E_{m_k} + \frac{1}{k} + C(k).
\end{aligned}$$

In order to prove (13) we show that $\lim_{k\to+\infty} C(k) = 0$. Indeed, as a first step we notice that $\rho * \left(v\left(\frac{\cdot}{t}\right)\right) = (\rho * v)\left(\frac{\cdot}{t}\right)$, and after a change of variable we get

$$\begin{aligned}
C(k) &= \left|\frac{1}{2}\left(t_k^{N-2s} - 1\right)[\rho(\tilde{v}_k) * v_k]^2_{H^s(\mathbb{R}^N)} - \left(t_k^N - 1\right)\int_{R^N} F(\rho(\tilde{v}_k) * v_k)\, dx\right| \\
&\le \frac{1}{2}\left|t_k^{N-2s} - 1\right|[\rho(\tilde{v}_k) * v_k]^2_{H^s(\mathbb{R}^N)} + \left|t_k^N - 1\right|\int_{R^N} |F(\rho(\tilde{v}_k) * v_k)|\, dx \\
&=: \frac{1}{2}\left|t_k^{N-2s} - 1\right| A(k) + \left|t_k^N - 1\right| B(k).
\end{aligned}$$

Since $t_k \to 1$ as $k \to +\infty$, it suffices to prove that

$$\limsup_{k\to+\infty} A(k) < \infty, \quad \limsup_{k\to+\infty} B(k) < \infty. \tag{15}$$

We divide the proof of (15) in three claims.

Claim 1: $(v_k)_k$ is bounded in $H^s(\mathbb{R}^N)$.

Recalling (12) and (14) we have that

$$\limsup_{k\to+\infty} I(v_k) \le E_m.$$

Thus, observing that $v_k \in \mathcal{P}_{m_k}$ and $m_k \to m$ if the claim does not hold, we obtain a contradiction with lemma 6 (iv).

Claim 2: $(\tilde{v}_k)_k$ is bounded in $H^s(\mathbb{R}^N)$, and there are a sequence $(y_k)_k \subset \mathbb{R}$ and $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that $\tilde{v}(\cdot + y_k) \to v$ a.e. in $\mathbb{R}^N$ up to a subsequence .

To see the boundedness of $(\tilde{v}_k)_k$ it suffices to notice that $t_k \to 1$ and the statement follows by claim 1. Now, we set

$$\alpha = \limsup_{k\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |\tilde{v}_k|^2 \, dx.$$

If $\alpha = 0$, by [8, Lemma II.4] we get $\tilde{v}_k \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$. As a consequence we have that

$$\int_{\mathbb{R}^N} |v_k|^{2+\frac{4s}{N}} \, dx = \int_{\mathbb{R}_N} |\tilde{v}_k(t_k\cdot)|^{2+\frac{4s}{N}} \, dx = t_k^{-N} \int_{\mathbb{R}^N} |\tilde{v}_k|^{2+\frac{4s}{N}} \, dx \to 0$$

as $k \to +\infty$, and since $P(v_k) = 0$, by Lemma 2 (i), we deduce that

$$[v_k]^2_{H^s(\mathbb{R}^N)} = \frac{N}{2s} \int_{\mathbb{R}^N} \tilde{F}(v_k) \, dx \to 0.$$

In this case, by virtue of Remark 2, we see that

$$0 = P(v_k) \geq \frac{1}{2} [v_k]^2_{H^s(\mathbb{R}^N)},$$

which is admissible only if v_k in constant. But this is in contradiction with the fact that $v_k \in \mathcal{P}_{m_k}$. Hence α must be strictly positive.

Claim 3: $\limsup_{k\to+\infty} \rho(\tilde{v}_k) < \infty$.

By contradiction we assume that up to a subsequence $\rho(\tilde{v}_k) \to \infty$ as $k \to +\infty$. By Claim 2 we can suppose the existence of a sequence $(y_k)_k \subset \mathbb{R}^N$ and $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that

$$\tilde{v}_k(\cdot + y_k) \to v \quad \text{a.e. in}\, \mathbb{R}^N. \tag{16}$$

Instead, by Lemma 5 we get

$$\rho(\tilde{v}_k(\cdot + y_k)) = \rho(\tilde{v}_k) \to \infty \tag{17}$$

and

$$I(\rho(\tilde{v}_k(\cdot + y_k)) * \tilde{v}_k(\cdot + y_k)) \geq 0. \tag{18}$$

Now, taking into account (16), (17), (18) and arguing similarly as we have already done to prove (8) we have a contradiction. The proof concludes observing that by Claims 1 and 3

$$\limsup_{k\to+\infty} \|\rho(\tilde{v}_k) * v_k\|_{H^s(\mathbb{R}^N)} < \infty. \tag{19}$$

Hence, by virtue of $(f_0) - (f_2)$ and (19), (15) holds true.

The next result provides a weak monotonicity property for E_m.

Lemma 9. *If* $(f_0) - (f_4)$ *hold, then* $m \mapsto E_m$ *is non-increasing in* $(0, \infty)$.

Proof. It suffices to show that for every $\varepsilon > 0$ and m, $m' > 0$ with $m > m'$ we have

$$E_m \leq E_{m'} + \frac{\varepsilon}{2}. \tag{20}$$

Now, we take $\chi \in C_c^\infty(\mathbb{R}^N)$ radial such that

$$\chi(x) = \begin{cases} 1 & |x| \leq 1 \\ [0,1] & 1 < |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

and $u \in \mathcal{P}_{m'}$. For every $\delta > 0$ we set $u_\delta(x) = u(x)\chi(\delta x)$. By a result of Palatucci *et al.*, see [10, Lemma 5 of Section 6.1], we know that $u_\delta \to u$ as $\delta \to 0^+$, and using Lemma 5 (iii) we obtain

$$\lim_{\delta \to 0^+} \rho(u_\delta) = \rho(u) = 0.$$

As a consequence of that, we obtain

$$\rho(u_\delta) * u_\delta \to \rho(u) * u \quad \text{in } H^s(\mathbb{R}^N) \tag{21}$$

as $\delta \to 0^+$. Now, fixing $\delta > 0$ small enough, by virtue of (21) we have

$$I(\rho(u_\delta) * u_\delta) \leq I(u) + \frac{\varepsilon}{4}. \tag{22}$$

After that, we choose $v \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp}(v) \subset B\left(0, 1 + \frac{4}{\delta}\right) \setminus B\left(0, \frac{4}{\delta}\right)$ and we set

$$\tilde{v} = \frac{m - \|u_\delta\|^2_{L^2(\mathbb{R}^N)}}{\|v\|^2_{L^2(\mathbb{R}^N)}}$$

For every $\lambda \leq 0$ we also define $\omega_\lambda = u_\delta + \lambda * \tilde{v}$. We observe that choosing λ appropriately we have $\operatorname{supp}(u_\delta) \cap \operatorname{supp}(\lambda * \tilde{v}) = \emptyset$, thus $\omega_\lambda \in S_m$.

Claim: $\rho(\omega_\lambda)$ is upper bounded as $\lambda \to -\infty$.

If the claim does not hold we observe that by lemma 5 (ii) $I(\rho(\omega_\lambda) * \omega_\lambda) \geq 0$ and that $\omega_\lambda \to u_\delta$ a.e. in $\mathbb{R}^N$ as $\lambda \to -\infty$. Hence, arguing as we have already done to obtain (8) we reach a contradiction. Then the claim must hold.
By virtue of the claim

$$\rho(\omega_\lambda) + \lambda \to -\infty \quad \text{as}\, \lambda \to -\infty,$$

thus

$$[(\rho(\omega_\lambda) + \lambda) * \tilde{v}]^2_{H^s(\mathbb{R}^N)} = e^{2s(\rho(\omega_\lambda)+\lambda)}\, [\tilde{v}]^2_{H^s(\mathbb{R}^N)} \to 0$$

implying

$$\|(\rho(\omega_\lambda) + \lambda) * \tilde{v}\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)} \leq C\|(\rho(\omega_\lambda) + \lambda) * \tilde{v}\|_{L^2(\mathbb{R}^N)}\, [(\rho(\omega_\lambda) + \lambda) * \tilde{v}]_{H^s(\mathbb{R}^N)} \to 0.$$

As a consequence, by Lemma 1 (ii), for a suitable λ

$$I((\rho(\omega_\lambda) + \lambda) * \tilde{v}) \leq \frac{\varepsilon}{4}. \tag{23}$$

Finally, by Lemma 5 and using (20), (22) and (23) it easy to see that

$$\begin{aligned} E_m &\leq I(\rho(\omega_\lambda) * \omega_\lambda) = I(\rho(\omega_\lambda) * u_\delta) + I(\rho(\omega_\lambda) * (\lambda * \tilde{v})) \\ &\leq I(\rho(u_\delta) * u_\delta) + I((\rho(\omega_\lambda) + \lambda) * \tilde{v}) \\ &\leq I(u) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq E_{m'} + \varepsilon \end{aligned}$$

completing the proof.

The strict monotonicity of E_m holds true only locally, as we now show.

Lemma 10. *Assume $(f_0) - (f_4)$ hold true. Moreover, let $u \in S_m$ and $\mu \in \mathbb{R}$ such that*

$$(-\Delta)^s + \mu u = f(u)$$

and $I(u) = E_m$. Then $E_m > E_{m'}$ for every $m' > m$ close enough if $\mu > 0$ and for any $m' < m$ close enough if $\mu < 0$.

Proof. Let $t > 0$ and $\rho \in \mathbb{R}$. Defining $u_{t,\rho} := u(\rho * (tu)) \in S_{mt^2}$ and

$$\alpha(t,\rho) := I(u_{t,\rho}) = \frac{1}{2} t^2 e^{2\rho s} \, [u]^2_{H^s(\mathbb{R}^N)} - e^{-N\rho} \int_{\mathbb{R}^N} F(te^{\frac{N\rho}{2}} u) \, dx$$

it is straightforward to verify that

$$\begin{aligned} \frac{\partial}{\partial t} \alpha(t,\rho) &= t e^{2\rho s} \, [u]^2_{H^s(\mathbb{R}^N)} - e^{-N\rho} \int_{\mathbb{R}^N} f\left(te^{\frac{N\rho}{2}} u\right) e^{\frac{N\rho}{2}} u \, dx \\ &= t^{-1} I'(u_{t,\rho}) \, [u_{t,\rho}] \, . \end{aligned}$$

In the case $\mu > 0$, we observe that $u_{t,\rho} \to u$ in $H^s(\mathbb{R}^N)$ as $(t,\rho) \to (1,0)$. Moreover, we notice that

$$I'(u) \, [u] = -\mu \|u\|^2_{L^2(\mathbb{R}^N)} = -\mu m < 0$$

and so, choosing $\delta > 0$ small enough we have

$$\frac{\partial \alpha}{\partial t}(t,\rho) < 0 \quad \text{for any} \, (t,\rho) \in (1, 1+\delta) \times [-\delta, \delta] \, .$$

Using the Mean Value Theorem, there exists $\xi \in (1,t)$ such that

$$\frac{\partial \alpha}{\partial t}(\xi,\rho) = \frac{\alpha(t,\rho) - \alpha(1,\rho)}{t-1}$$

whenever $(t,\rho) \in (1, 1+\delta) \times [-\delta, \delta]$, hence

$$\alpha(t,\rho) = \alpha(1,\rho) + (t-1) \frac{\partial}{\partial t} \alpha(\xi,\rho) < \alpha(1,\rho). \tag{24}$$

Since by Lemma 5 (*iii*) $\rho(tu) \to \rho(u) = 0$ as $t \to 1^+$, setting for any $m' > m$ close enough to m

$$t := \sqrt{\frac{m'}{m}} \in (1, 1+\delta) \quad \text{and} \quad \rho := \rho(tu) \in [-\delta, \delta] \, ,$$

and using (24) together with Lemma 5 (*ii*) we obtain that

$$E_m \leq \alpha(t, \rho(tu)) < \alpha(1, \rho(tu)) = I(\rho(tu) * u) \leq I(u) = E_m.$$

The proof for $\mu < 0$ is similar, and we omit it.

As a direct consequence of the previous two lemmas we have the following result.

Lemma 11. *Assume* $(f_0) - (f_4)$ *hold true. In addition let* $u \in S_m$ *and* $\mu \in \mathbb{R}$ *such that* $(-\Delta)^s u + \mu u = f(u)$ *with* $I(u) = E_m$*. Then* $\mu \geq 0$*, and if* $\mu > 0$ *it is* $E_m > E_{m'}$ *for any* $m' > m > 0$*.*

To make a step ahead, we describe the asymptotic behaviour of E_m as $m \to 0^+$ and $m \to +\infty$.

Lemma 12. *Assume* $(f_0) - (f_4)$ *hold true, then* $E_m \to +\infty$ *as* $m \to 0^+$*.*

Proof. In order to prove the Lemma, we will show that for every sequence $(u_n)_n \subset H^s(\mathbb{R}^N) \setminus \{0\}$ such that

$$P(u_n) = 0 \quad \text{and} \quad \lim_{n \to +\infty} \|u_n\|_{L^2(\mathbb{R}^N)} = 0$$

it must be $I(u_n) \to +\infty$. We set

$$\rho_n := \frac{1}{s} \log \left([u_n]_{H^s(\mathbb{R}^N)} \right) \quad \text{and} \quad v_n := (-\rho_n) * u_n$$

Trivially $[v_n]_{H^s(\mathbb{R}^N)} = 1$ and $\|v_n\|_{L^2(\mathbb{R}^N)} \to 0$. Moreover, thanks to these two facts we also have by interpolation that $v_n \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$, thus, by Lemma 1 (ii) we have

$$\lim_{n \to +\infty} e^{-N\rho} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho}{2}} v_n\right) dx = 0.$$

Since $P(\rho_n * v_n) = P(u_n) = 0$, using Lemma 5 (i) and (ii) we obtain that

$$\begin{aligned} I(u_n) = I(\rho_n * v_n) \geq I(\rho * v_n) &= \frac{1}{2} e^{2\rho s} - e^{N\rho} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho}{2}} v_n\right) dx \\ &= \frac{1}{2} e^{2\rho s} + o_n(1). \end{aligned}$$

Since ρ is arbitrary, we get the statement as $\rho \to +\infty$.

Lemma 13. *Assume* $(f_0) - (f_4)$ *and* (f_6)*. Then* $E_m \to 0$ *as* $m \to +\infty$*.*

Proof. We fix $u \in L^\infty(\mathbb{R}^N) \cap S_1$ and we set $u_m = \sqrt{m} u \in S_m$. By Lemma 5 (ii) we can find a unique $\rho(m) \in \mathbb{R}$ such that $\rho(m) * u_m \in \mathcal{P}_m$. Since by Lemma 3 (i) F is non negative, we get

$$0 < E_m \leq I(\rho(m) * u_m) \leq \frac{1}{2} e^{2\rho(m)s} [u]^2_{H^s(\mathbb{R}^N)} . \tag{25}$$

Thus, by (25) it suffices to show that

$$\lim_{m\to\infty} \sqrt{m}\, e^{\rho(m)s} = 0. \tag{26}$$

Using the function g defined in Remark 3, and recalling that $P(\rho(m) * u_m) = 0$ we get

$$[u]^2_{H^s(\mathbb{R}^N)} = \frac{N}{2s} m^{\frac{2s}{N}} \int_{\mathbb{R}^N} g\left(\sqrt{m} e^{\frac{N\rho(m)}{2}} u\right) |u|^{2+\frac{4s}{N}}\, dx,$$

which implies

$$\lim_{m\to\infty} \sqrt{m}\, e^{\frac{N\rho(m)}{2}} = 0. \tag{27}$$

Now, using (f_6) and Lemma 4, for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\tilde{F}(t) \geq \frac{4s}{N} F(t) \geq \frac{1}{\varepsilon} |t|^{\frac{2N}{N-2s}}$$

if $|t| \leq \delta$. Hence, taking into account the fact that $P(\rho(m) * u_m) = 0$ and (27), we get

$$\begin{aligned}
[u]^2_{H^s(\mathbb{R}^N)} &= \frac{N}{2s} \frac{1}{m} e^{-(N+2s)\rho(m)} \int_{\mathbb{R}^N} \tilde{F}\left(\sqrt{m} e^{\frac{N\rho(m)}{2}} u\right) dx \\
&\geq \frac{N}{2s} \frac{1}{\varepsilon} \left(\sqrt{m} e^{\rho(m)s}\right)^{\frac{4s}{N-2s}} \int_{\mathbb{R}^N} \tilde{F}\left(\sqrt{m} e^{\frac{N\rho(m)}{2}} u\right) dx
\end{aligned}$$

for m large enough. Then (26) holds, and the proof is complete.

4. Ground states

We introduce the functional

$$\Psi(u) = I(\rho(u) * u) = \frac{1}{2} e^{2\rho(u)s} [u]^2_{H^s(\mathbb{R}^N)} - e^{-N\rho(u)} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho(u)}{2}} u\right) dx.$$

Lemma 14. *The functional $\Psi\colon H^s(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}$ is of class C^1, and*

$$d\Psi(u)\,[\varphi] = dI(\rho(u) * u)\,[\rho(u) * \varphi]$$

for every $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ and $\varphi \in H^s(\mathbb{R}^N)$.

Proof. A proof appears in [7] for the case $s = 1$. Only minor adjustments are needed in the fractional case, so we omit the details.

For $m > 0$, we consider the constrained functional $J \colon S_m \to \mathbb{R}$ defined by $J = \Psi_{|S_m}$. Lemma 14 yields the following statement.

Lemma 15. *The functional $J \colon S_m \to \mathbb{R}$ is C^1 and*

$$dJ(u)\,[\varphi] = d\Psi(u)\,[\varphi] = dI(\rho(u) * u)\,[\rho(u) * \varphi]$$

for any $u \in S_m$ and $\varphi \in T_u S_m$, where $T_u S_m$ is the tangent space at u to the manifold S_m.

We recall from [11, Definition 3.1] a definition that will be useful to construct a min-max principle.

Definition 1. Let B be a closed subset of a metric space X. We say that a class $\mathcal{G}$ of compact subsets of X is a homotopy stable family with closed boundary B provided

(i) every set in $\mathcal{G}$ contains B,

(ii) for any set A in $\mathcal{G}$ and any homotopy $\eta \in C\left([0,1] \times X, X\right)$ that satisfies $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times X) \cup ([0,1] \times B)$, one has $\eta\left(\{1\} \times A\right) \in \mathcal{G}$.

We remark that $B = \emptyset$ is admissible.

Lemma 16. *Let $\mathcal{G}$ be a homotopy stable family of compact subset with (with $B = \emptyset$). We set*

$$E_{m,\mathcal{G}} = \inf_{A \in \mathcal{G}} \max_{u \in A} J(u).$$

If $E_{m,\mathcal{G}} > 0$, then there exists a Palais-Smale sequence $(u_n)_n \in \mathcal{P}_m$ for the constrained functional $I_{|S_m}$ at level $E_{m,\mathcal{G}}$. In particular, if $\mathcal{G}$ is the class of all singletons in S_m, one has that $\|u_n^-\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to +\infty$.

Proof. Let $(A_n)_n \subset \mathcal{G}$ be a minimizing sequence of $E_{m,\mathcal{G}}$. We define the map

$$\eta \colon\; [0,1] \times S_m \to S_m$$

where $\eta(t,u) = (t\rho(u)) * u$ is continuous and well defined by lemma 5 (ii) and (iii). Noticing $\eta(t,u) = u$ for every $(t,u) \in \{0\} \times S_m$ we obtain that

$$D_n := \eta(1, A_n) = \{\rho(u) * u \mid u \in A_n\} \in \mathcal{G}.$$

In particular we can see that $D_n \subset \mathcal{P}_m$ for any $m > 0$, with $m > 0$. Since $J(\rho(u) * u) = J(u)$ for every $\rho \in \mathbb{R}$ and $u \in S_m$, we can observe that

$$\max_{u \in D_n} J(u) = \max_{u \in A_n} J(u) \to E_{m,\mathcal{G}}$$

thus, $(D_n)_n$ is another minimizing sequence for $E_{m,\mathcal{G}}$. Now, using [11, Theorem 3.2] we get a Palais-Smale sequence $(v_n)_n \subset S_m$ for J at level $E_{m,\mathcal{G}}$ such that $\text{dist}_{H^s(\mathbb{R}^N)}(v_n, D_n) \to 0$ as $n \to +\infty$. We will denote

$$\rho_n := \rho(v_n) \quad \text{and} \quad u_n := \rho_n * v_n.$$

Claim: There exists $C > 0$ such that $e^{-2\rho_n s} \leq C$ for any $n \in \mathbb{N}$.
We start pointing out that

$$e^{-2\rho_n s} = \frac{[v_n]^2_{H^s(\mathbb{R}^N)}}{[u_n]^2_{H^s(\mathbb{R}^N)}}.$$

By virtue of the fact that $(u_n)_n \subset \mathcal{P}_m$, using lemma 6 (ii) we obtain that $\left\{[u_n]_{H^s(\mathbb{R}^N)}\right\}_n$ is bounded from below. Moreover, since $D_n \subset \mathcal{P}_m$ and the fact that

$$\max_{u \in D_n} I = \max_{u \in D_n} J \to E_{m,\mathcal{G}},$$

Lemma 6 (iv) implies that D_n is uniformly bounded in $H^s(\mathbb{R}^N)$. Finally, from $\text{dist}(v_n, D_n) \to 0$ we can deduce that $\sup_{n\in\mathbb{N}} [v_n]_{H^s(\mathbb{R}^N)} < \infty$. Thus the claim holds.
Now, from $(u_n) \subset \mathcal{P}_m$ we get

$$I(u_n) = J(u_n) = J(v_n) \to E_{m,\mathcal{G}}.$$

Instead, for any $\psi \in T_{u_n} S_m$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} v_n\left[(-\rho_n) * \psi\right] dx &= \int_{\mathbb{R}^N} v_n e^{-\frac{N\rho_n}{2}} \psi\left(e^{-\rho_n} x\right) dx = \int_{\mathbb{R}^N} e^{\frac{N\rho_n}{2}} v_n\left(e^{\rho_n} x\right) \psi \, dx \\ &= \int_{\mathbb{R}^N} (\rho_n * v_n) \psi \, dx = \int_{\mathbb{R}^N} u_n \psi \, dx = 0\end{aligned}$$

implying $(-\rho_n * \psi) \in T_{v_n} S_m$. Besides, by the claim

$$\|(-\rho_n) * v_n\|_{H^s(\mathbb{R}^N)} \leq \max\{C, 1\} \|\psi\|_{H^s(\mathbb{R}^N)}.$$

Denoting with $\|\cdot\|_{u,*}$ the dual norm of the space $(T_u S_m)^*$ and using Lemma 10 we get

$$\begin{aligned}
\|dI(u_n)\|_{u_n,*} &= \sup_{\substack{\psi \in T_{u_n} S_m \\ \|\psi\|_{H^s(\mathbb{R}^N)} \leq 1}} |dI(u_n)\,[\psi]\,| = \sup_{\substack{\psi \in T_{u_n} S_m \\ \|\psi\|_{H^s(\mathbb{R}^N)} \leq 1}} |dI(\rho_n * v_n)\,[\rho_n * ((-\rho_n) * \psi)]\,| \\
&= \sup_{\substack{\psi \in T_{u_n} S_m \\ \|\psi\|_{H^s(\mathbb{R}^N)} \leq 1}} |dJ(v_n)\,[(-\rho_n) * \psi]\,| \\
&\leq \|dJ(v_n)\|_{v_n,*} \sup_{\substack{\psi \in T_{u_n} S_m \\ \|\psi\|_{H^s(\mathbb{R}^N)} \leq 1}} \|(-\rho_n) * \psi\|_{H^s(\mathbb{R}^N)} \\
&\leq \max\{C, 1\}\ \|dJ(v_n)\|_{v_n,*} \to 0
\end{aligned}$$

as $n \to +\infty$ remembering that $(v_n)_n$ is a Palais-Smale sequence for the functional J. We have just proved $(u_n)_n$ is a Palais-Smale sequence for the functional $I_{|S_m}$ at level $E_{m,\mathcal{G}}$ with the additional property that $(u_n)_n \subset \mathcal{P}_m$. Finally, noticing that the family of singleton of S_m is a particular homotopy stable family of compact subsets of S_m, and doing this particular choice as $\mathcal{G}$, arguing similarly as we have just done, we can obtain a minimizing sequence $(D_n)_n$ with the additional property that its elements are non negative: we only need to replace the functions with their absolute value. Moreover, $(A_n)_n$ will inherit this property, and recalling that $\operatorname{dist}(v_n, D_n) \to 0$ as $n \to +\infty$ we have

$$\|u_n^-\|_{L^2(\mathbb{R}^N)} = \|\rho_n * v_n^-\|_{L^2(\mathbb{R}^N)} = \|v_n^-\|_{L^2(\mathbb{R}^N)} \to 0.$$

This concludes the proof of the lemma.

Lemma 17. *We assume $(f_0) - (f_4)$ hold. Then there exists a Palais-Smale sequence $(u_n)_n \subset \mathcal{P}_m$ for the constrained functional $I_{|S_m}$ at level E_m such that $\|u_n^-\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to +\infty$.*

Proof. We apply lemma 16 with $\mathcal{G}$ the class of all singletons in S_m. Lemma 6 imply that $E_m > 0$, thus the only thing that remains to prove is $E_m = E_{m,\mathcal{G}}$.

In order to do that, as a first step we notice that

$$E_{m,\mathcal{G}} = \inf_{A\in\mathcal{G}} \max_{u\in A} J(u) = \inf_{u\in S_m} I(\rho(u)*u).$$

Since for every $u \in S_m$ we have that $\rho(u)*u \in \mathcal{P}_m$ it must be $I(\rho(u)*u) \geq E_m$, thus $E_{m,\mathcal{G}} \geq E_m$. On the other hand, if $u \in \mathcal{P}_m$ we have $\rho(u) = 0$ and $I(u) \geq E_{m,\mathcal{G}}$, that implies $E_m \geq E_{m,\mathcal{G}}$.

Lemma 18. *Let $(u_n)_n \subset S_m$ be a bounded Palais-Smale sequence for the constrained functional $I_{|S_m}$ at level $E_m > 0$ such that $P(u_n) \to 0$ as $n \to +\infty$. Then we have the existence of $u \in S_m$ and $\mu > 0$ such that, up to a subsequence and translations in $\mathbb{R}^N$, $u_n \to u$ strongly in $H^s(\mathbb{R}^N)$ and*

$$(-\Delta)^s u + \mu u = f(u).$$

Proof. It is clear that $(u_n)_n \subset S_m$ is bounded in $H^s(\mathbb{R}^N)$ and is a Palais-Smale sequence. Together, these two facts enable us to assume without loss of generality that $\lim_{n\to+\infty} [u_n]_{H^s(\mathbb{R}^N)}$, $\lim_{n\to+\infty} \int_{\mathbb{R}^N} F(u_n)\,dx$, and $\lim_{n\to+\infty} \int_{\mathbb{R}^N} f(u_n)u_n\,dx$ exist. Besides, [12, Lemma 3] implies

$$(-\Delta)^s u_n + \mu_n u_n - f(u_n) \to 0 \quad \text{in } H^s(\mathbb{R}^N)^*$$

where we denote

$$\mu_n = \frac{1}{m}\left(\int_{\mathbb{R}^N} f(u_n)u_n\,dx - [u_n]^2_{H^s(\mathbb{R}^N)}\right).$$

By the assumptions done above we can see that $\mu_n \to \mu$ for some $\mu \in \mathbb{R}$ and we also have that for any $(y_n)_n \subset \mathbb{R}^N$

$$(-\Delta)^s u_n(\cdot + y_n) + \mu u_n(\cdot + y_n) - f(u_n(\cdot + y_n)) \to 0 \quad \text{in } H^s(\mathbb{R}^N)^*. \qquad (28)$$

Claim: $(u_n)_n$ is non vanishing.
Otherwise by [8, Lemma II.4] we would get $u_n \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$. Taking into account that $P(u_n) \to 0$ and using lemma 1 (ii) we get

$$[u_n]^2_{H^s(\mathbb{R}^N)} = P(u_n) + \frac{N}{2s}\int_{\mathbb{R}^n} \tilde{F}(u_n)\,dx \to 0$$

and as a consequence of that,

$$E_m = \lim_{n\to+\infty} I(u_n) = \frac{1}{2}\lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} - \lim_{n\to+\infty}\int_{\mathbb{R}^N} F(u_n)\,dx$$

contradicting $E_m > 0$. Then the claim must hold.

Since $(u_n)_n$ in non vanishing we can find $(y_n^1)_n \subset \mathbb{R}^N$ and $\omega_1 \in B_m \setminus \{0\}$ such that $u_n(\cdot + y_n^1) \rightharpoonup \omega_1$ in $H^s(\mathbb{R}^N)$, $u_n(\cdot + y_n^1) \to \omega_1$ in $L^p_{\rm loc}(\mathbb{R}^N)$ for $p \in [1, 2_s^*]$ and $u_n(\cdot + y_n^1) \to \omega$ a.e. in $\mathbb{R}^N$. Now, we want to apply [13, Lemma A.1] with $P(t) = f(t)$ and $Q(t) = |t|^{(N+2s)/(N-2s)}$ and we notice that

$$\begin{aligned}\lim_{n\to+\infty}&\int_{\mathbb{R}^N} \left|\left[f(u_n(\cdot + y_n^1) - f(\omega_1)\right]\varphi\right|\,dx\\ &\leq \|\varphi\|_{L^\infty(\mathbb{R}^N)}\lim_{n\to+\infty}\int_{\mathrm{supp}(\varphi)}\left|f(u_n(\cdot + y_n^1) - f(\omega_1)\right|\,dx \qquad (29)\end{aligned}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Hence, by (28) and (29) we get

$$(-\Delta)^s\omega_1 + \mu\omega_1 = f(\omega_1) \qquad (30)$$

and through the Pohozaev Identity (see for instance [14, Proposition 4.1]) associated to (30) we also have $P(\omega_1) = 0$. Now, we set $v_n^1 := u_n - \omega_1(\cdot - y_n^1)$ for every $n \in \mathbb{N}$. Clearly $v_n^1(\cdot + y_n^1) = u_n(\cdot + y_n^1) - \omega_1 \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, thus

$$m = \lim_{n\to+\infty}\|u_n(\cdot + y_n^1)\|_{L^2(\mathbb{R}^N)} = \lim_{n\to+\infty}\|v_n^1\|^2_{L^2(\mathbb{R}^N)} + \|\omega_1\|^2_{L^2(\mathbb{R}^N)}. \qquad (31)$$

By lemma 7 we also have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} F(u_n(\cdot + y_n^1))\,dx = \int_{\mathbb{R}^N} F(\omega_1)\,dx + \lim_{n\to+\infty}\int_{\mathbb{R}^N} F(v_n^1(\cdot + y_n^1))\,dx$$

hence

$$\begin{aligned}E_m = \lim_{n\to+\infty} I(u_n) &= \lim_{n\to+\infty} I(u_n(\cdot + y_n^1)) = \lim_{n\to+\infty} I(v_n^1(\cdot + y_n^1)) + I(\omega_1) \qquad (32)\\ &= \lim_{n\to+\infty} I(v_n^1) + I(\omega_1).\end{aligned}$$

Claim: $\lim_{n\to+\infty} I(v_n^1) \geq 0$.

If the claim does not hold, i.e $\lim_{n\to+\infty} I(v_n^1) < 0$, $(v_n^1)_n$ is non vanishing, then there exists $(y_n^2)_n \subset \mathbb{R}^N$ such that

$$\lim_{n\to+\infty}\int_{B(y_n^2,1)} |v_n^1|^2 > 0.$$

Since $v_n^1(\cdot + y_n^1) \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, it must be $|y_n^2 - y_n^1| \to \infty$, and up to a subsequence $v_n^1(\cdot + y_n^2) \to \omega_2$ in $H^s(\mathbb{R}^N)$ for some $\omega_2 \in B_m \setminus \{0\}$. We notice

$$u_n(\cdot + y_n^2) = v_n^1(\cdot + y_n^2) + \omega_1(\cdot - y_n^1 + y_n^2) \rightharpoonup \omega_2$$

thus, arguing as before, we get $P(\omega_2) = 0$ and $I(\omega_2) > 0$. We set

$$v_n^2 = v_n^1 - \omega^2(\cdot - y_n^2) = u_n - \sum_{\ell=1}^{2} \omega_\ell(\cdot - y_n^\ell)$$

and we observe that

$$\begin{aligned}
\lim_{n\to+\infty} \left[v_n^2\right]^2_{H^s(\mathbb{R}^N)} &= \lim_{n\to+\infty} \left[v_n^1\right]^2_{H^s(\mathbb{R}^N)} + [\omega_2]^2_{H^s(\mathbb{R}^N)} - 2 \lim_{n\to+\infty} \langle v_n^1, \omega_2(\cdot - y_n^2)\rangle_{H^s(\mathbb{R}^N)} \\
&= \lim_{n\to+\infty} \left[v_n^1\right]^2_{H^s(\mathbb{R}^N)} + [\omega_2]^2_{H^s(\mathbb{R}^N)} - 2 \lim_{n\to+\infty} \langle v_n^1(\cdot + y_n^2), \omega_2\rangle_{H^s(\mathbb{R}^N)} \\
&= \lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} + [\omega_1]^2_{H^s(\mathbb{R}^N)} - [\omega_2]^2_{H^s(\mathbb{R}^N)} \\
&\quad - 2 \lim_{n\to+\infty} \langle u_n(\cdot + y_n^1), \omega_1\rangle_{H^s(\mathbb{R}^N)} \\
&= \lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} - \sum_{\ell=1}^{2} [\omega_\ell]^2_{H^s(\mathbb{R}^N)}
\end{aligned}$$

and

$$0 > \lim_{n\to+\infty} I(v_n^1) = I(\omega_2) + \lim_{n\to+\infty} I(v_n^2) > \lim_{n\to+\infty} I(v_n^2).$$

Iterating, we can build an infinite sequence $(\omega_k) \subset B_m \setminus \{0\}$ such that $P(\omega_k) = 0$ and

$$\sum_{\ell=1}^{k} [\omega_k]^2_{H^s(\mathbb{R}^N)} \leq [u_n]^2_{H^s(\mathbb{R}^N)} < \infty$$

for every $k \in \mathbb{N}$. Though, this is a contradiction. Indeed, recalling remark 2, for any $\omega \in B_m \setminus \{0\}$ such that $P(\omega) = 0$, we can find $\delta > 0$ such that $[\omega]^2_{H^s(\mathbb{R}^N)} \geq \delta$. Hence, the claim must hold and $\lim_{n\to+\infty} I(v_n^1) \geq 0$.

Now, we denote with $h := \|\omega_1\|^2_{L^2(\mathbb{R}^N)} \in (0, m]$. By virtue of the claim, (32) and the fact that $\omega_1 \in \mathcal{P}_h$, we get

$$E_m = I(\omega_1) + \lim_{n\to+\infty} I(v_n^1) \geq I(\omega^1) \geq E_h$$

but, recalling that E_m in non-increasing by lemma 9, we obtain

$$I(\omega_1) = E_m = E_h \tag{33}$$

and

$$\lim_{n\to+\infty} I(v_n^1) = 0. \tag{34}$$

To prove that $\mu \geq 0$ it suffices to put together (30), (33) and Lemma 11. Instead, to see that μ is strictly positive, using (f_5), lemma 2 and the Pohozaev Identity corresponding to (30), we get

$$\mu = \frac{1}{ms}\int_{\mathbb{R}^N}\left(NF(\omega_1) - \frac{N-2s}{2}f(\omega_1)\omega_1\right)\,dx > 0. \tag{35}$$

At this point, we suppose by contradiction that $h < m$, but taking into account (30), (35) and Lemma (11) we would have

$$I(\omega_1) = E_h > E_m$$

which is not compatible with (34). Thus $h = m$. Moreover, by (31) $v_n^1 \to 0$ in $L^2(\mathbb{R}^N)$. It remains only to prove the strong convergence of $(v_n^1)_n$ in $H^s(\mathbb{R}^N)$. To do that, it is sufficient to notice that by lemma 1 (ii) we have $\lim_{n\to+\infty}\int_{\mathbb{R}^N} F(v_n^1)\,dx$, and so we obtain the assertion thanks to (34).

Proof (of theorem 1). Applying lemma 17 we obtain a Palais-Smale sequence $(u_n)_n \subset \mathcal{P}_m$ at level $E_m > 0$ for the constrained functional $I_{|S_m}$. This sequence is bounded in $H^s(\mathbb{R}^N)$ by Lemma 6 and through Lemma 18 we get a critical point $u \in S_m$ at the level $E_m > 0$ that results to be a ground state energy. Finally, since $\|u_n^-\|_{L^2(\mathbb{R}^N)} \to 0$ we deduce that $u \geq 0$ and after applying the strong maximum principle we obtain $u > 0$.

Proof (of theorem 2). The proof is a direct consequence of Theorem 1 and Lemmas 6, 8, 9, 12, 13.

5. Existence of radial solutions

This section is devoted to prove the existence of infinitely many radial solutions to problem (P_m). Before doing this, we recall some basic definitions and we provide some notation.

Denote by $\sigma\colon H^s(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$ the transformation $\sigma(u) = -u$ and let $X \subset H^s(\mathbb{R}^N)$. A set $A \subset X$ is called σ-invariant if $\sigma(A) = A$. A homotopy $\eta\colon [0,1]\times X \to X$ is σ-equivariant if $\eta(t,\sigma(u)) = \sigma(\eta(t,u))$ for all $(t,u) \in [0,1]\times X$. Next definition is in [11, Definition 7.1].

Definition 2. Let B be a closed σ-invariant subset of $X \subset H^s(\mathbb{R}^N)$. We say that a class $\mathcal{G}$ of compact subsets of X is a σ-homotopy stable family with closed boundary B provided

(i) every set in $\mathcal{G}$ is σ-invariant.

(ii) every set in $\mathcal{G}$ contains B,

(iii) for any set A in $\mathcal{G}$ and any σ-equivariant homotopy $\eta \in C\left([0,1]\times X, X\right)$ that satisfies $\eta(t,u) = u$ for all $(t,u) \in (\{0\}\times X) \cup ([0,1]\times B)$, one has $\eta\left(\{1\}\times A\right) \in \mathcal{G}$.

We denote with $H^s_r(\mathbb{R}^N)$ the space of radially symmetric functions in $H^s(\mathbb{R}^N)$ and recall that $H^s_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R})$ compactly for all $p \in (2, 2^*_s)$ (see [15, Proposition I.1]).

In order to prove the main result of this section, we need to build a sequence of σ-homotopy stable families of compact subsets of $S_m \cap H^s_r(\mathbb{R}^N)$. We point out that in the definition above, the case in which $B = \emptyset$ is not excluded. The idea is borrowed from [7]. Let $(V_k)_k$ be a sequence of finite dimensional linear subspaces of $H^s_r(\mathbb{R}^N)$ such that $V_k \subset V_{k+1}$, $\dim V_k = k$ and $\bigcup_{k\geq 1} V_k$ is dense in $H^s_r(\mathbb{R}^N)$. Denote by π_k the orthogonal projection from $H^s_r(\mathbb{R}^N)$ onto V_k. We recall to the reader the definition of the genus of σ-invariant sets introduced by M. A. Krasnoselskii and we refer to [16, Section 7] or [17, chapter 10] for its basic properties.

Definition 3. Let A be a nonempty compact σ-invariant subset of $H^s_r(\mathbb{R}^N)$. The genus $\gamma(A)$ of A is the least integer k such that there exists $\phi \in C(H^s_r(\mathbb{R}^N), \mathbb{R}^k)$ such that ϕ is odd and $\phi(x) \neq 0$ for all $x \in A$. We set $\gamma(A) = \infty$ if there are no integers with the above property and $\gamma(\emptyset) = 0$.

Let $\mathcal{A}$ be the family of closed σ-invariant subset of $S_m \cap H^s_r(\mathbb{R}^N)$. For each $k \in \mathbb{N}$, set

$$\mathcal{G}_k := \{A \in \mathcal{A} \mid \gamma(A) \geq k\}$$

and

$$E_{m,k} = \inf_{A \in \mathcal{A}} \max_{u \in A} J(u).$$

Next, we give a result about the weak convergence of the nonlinearity f.

Lemma 19. *Assume $(f_0) - (f_2)$ hold true. Let $(u_n)_n \subset H^s_r(\mathbb{R}^N)$. If $u_n \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$ for some $u \in H^s_r(\mathbb{R}^N)$, then $f(u_n) \rightharpoonup f(u)$ in $L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$.*

Proof. We borrow some ideas from [18, Theorem 2.6]. We start exploiting the compact embeding $H^s_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for any $p \in (2, 2^*_s)$. Hence, up to a subsequence, $u_n \to u$ in $L^p(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. From equation (42), we get

$$|f(u_n)|^{\frac{2N}{N+2s}} \leq C_\varepsilon |u_n|^{\frac{2N}{N-2s}} + C|u_n|^{2\frac{N+4s}{N+2s}}$$

for some C_ε, $C > 0$. As a consequence of that, recalling the fractional Sobolev inequality and observing that $2\frac{N+4s}{N+2s} \in (2, 2^*_s)$, we obtain that $(f(u_n))_n$ is bounded in $L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$. Thus, there exists $y \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ such that $f(u_n) \rightharpoonup y$. At this point, we fix a cover $(\Omega_j)_j$ of $\mathbb{R}^N$ made of subsets with finite measure. For any $\upsilon > 0$, Severini-Egorov's Theorem yields the existence of $B^j_\upsilon \subset \Omega_j$, with measure $\left|B^j_\upsilon\right| < \upsilon$, such that $u_n \to u$ uniformly in $\Omega_j \setminus B^j_\upsilon$. Clearly $y = f(u)$ in $\Omega_j \setminus B^j_\upsilon$. Now, we set

$$\mathcal{Q} := \left\{x \in \mathbb{R}^N \mid y \neq f(u)\right\} \quad \text{and} \quad Q_j := \{x \in \Omega_j \mid y \neq f(u)\}.$$

Since υ is arbitrary and $Q_j \subset B^j_\upsilon$, we have that Q_j is a set of measure zero. Furthermore, it is easy to see that $\mathcal{Q} = \bigcup_{j=1}^\infty Q_j$, thus Q has measure zero and the proof is complete.

From now on, we will always assume $(f_0) - (f_5)$ hold until the end of the section.

Lemma 20. *Let $\mathcal{G}$ be a σ-homotopy stable family of compact subset of $S_m \cap H^s_r(\mathbb{R}^N)$ (with $B = \emptyset$) and set*

$$E_{m,\mathcal{G}} := \inf_{A \in \mathcal{G}} \max_{u \in A} J(u).$$

If $E_{m,\mathcal{G}} > 0$ then there exists a Palais-Smale sequence $(u_n)_n$ in $\mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ for $I_{|S_m \cap H^s_r(\mathbb{R}^N)}$ at level $E_{m,\mathcal{G}}$.

Proof. It suffices to replace Theorem 3.2 with 7.2 of [11] in the proof of Lemma 16.

Lemma 21. *For any $k \in \mathbb{N}$ we have,*

(i) $\mathcal{G}_k \neq \emptyset$ and $\mathcal{G}_k$ is a σ-homotopy stable family of compact subsets of $S_m \cap H^s_r(\mathbb{R}^N)$ (with $B = \emptyset$),

(ii) $E_{m,k+1} \geq E_{m,k} > 0$.

Proof. (i) It suffices to notice that for any $k \in \mathbb{N}$ one has $S_m \cap V_k \in \mathcal{A}$ and that by [17, Theorem 10.5]

$$\gamma(S_m \cap V_k) = k.$$

Thus $\mathcal{G}_k \neq \emptyset$. The conclusion is a direct consequence of the definition of $\mathcal{A}$.
(ii) By the previous step $E_{m,k}$ is well defined. Furthermore, recalling that $\rho(u) * u \in \mathcal{P}_m$ for all $u \in A$, where A is chosen arbitrarily in $\mathcal{G}$, we have

$$\max_{u \in A} J(u) = \max I(\rho(u) * u) = \inf_{v \in \mathcal{P}_m} I(v),$$

hence $E_{m,k} > 0$. The other part of the statement follows easily from $\mathcal{G}_{k+1} \subset \mathcal{G}_k$.

Lemma 22. *Let $(u_n)_n \subset S_m \cap H^s_r(\mathbb{R}^N)$ be a bounded Palais-smale sequence for $I_{|S_m}$ at an arbitrary level $c > 0$ satisfying $P(u_n) \to 0$. Then there exists $u \in S_m \cap H^s_r(\mathbb{R}^N)$ and $\mu > 0$ such that, up to a subsequence, $u_n \to u$ strongly in $H^s_r(\mathbb{R}^N)$ and*

$$(-\Delta)^s + \mu u = f(u).$$

Proof. By the boundedness of the Palais-Smale sequence we may assume $u_n \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$, $u_n \to u$ in $L^p(\mathbb{R}^N)$ for any $p \in (2, 2^*_s)$ and a.e. in $\mathbb{R}^N$. Besides, as already seen in the previous section, using [12, Lemma 3] we get

$$(-\Delta)^s u_n + \mu_n u_n - f(u_n) \to 0 \quad \text{in}\, (H^s_r(\mathbb{R}^N))^* \tag{36}$$

where

$$\mu_n := \frac{1}{m}\left(\int_{\mathbb{R}^N} f(u_n)u_n\, dx - [u_n]^2_{H^s(\mathbb{R}^N)}\right).$$

Again, similarly to the proof of Lemma 18, we can assume the existence of $\mu \in \mathbb{R}$ such that $\mu_n \to \mu$, from which we derive

$$(-\Delta)^s + \mu u = f(u). \tag{37}$$

Claim: $u \neq 0$.

If $u = 0$, then by the compact embedding $u_n \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$. Hence, using Lemma 1 (ii) and the fact that $P(u_n) \to 0$, we have $\int_{\mathbb{R}^N} F(u_n)\, dx \to 0$ and

$$[u_n]^2_{H^s(\mathbb{R}^N)} = P(u_n) + \frac{N}{2s}\int_{\mathbb{R}^N} \tilde{F}(u_n)\, dx \to 0,$$

from which

$$c = \lim_{n\to+\infty} I(u_n) = \frac{1}{2}\lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} - \lim_{n\to+\infty} F(u_n)\, dx = 0,$$

that contradicts the hypothesis of $c > 0$. Now, since $u \neq 0$, as we obtained (35), we get

$$\mu := \frac{1}{ms}\int_{\mathbb{R}^N}\left(NF(u) - \frac{N-2s}{2} f(u)u\right)\, dx > 0.$$

Since $u_n \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$, by Lemma 19

$$\int_{\mathbb{R}^N} [f(u_n) - f(u)]\, u\, dx \to 0.$$

Indeed, the fractional Sobolev inequality implies that $u \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$, and the multiplication by u turns out to be a continuous linear operator from $L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$

into $L^1(\mathbb{R}^N)$. Now, observing that $\int_{\mathbb{R}^N} f(u_n)(u_n - u)\,dx \to 0$ by Lemma 1 (iii) we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} f(u_n)u_n\,dx = \int_{\mathbb{R}^N} f(u)u\,dx.$$

Finally, from (36) and (37) one has

$$\begin{aligned}[u]^2_{H^s(\mathbb{R}^N)} + \mu \int_{\mathbb{R}^N} u^2\,dx &= \int_{\mathbb{R}^N} f(u)u\,dx \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^N} f(u_n)u_n\,dx = \lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} + \mu m,\end{aligned}$$

and since $\mu > 0$,

$$\lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} = [u]^2_{H^s(\mathbb{R}^N)}, \quad \lim_{n\to+\infty} \int_{\mathbb{R}^N} u_n^2\,dx = m = \int_{\mathbb{R}^N} u^2\,dx.$$

Thus $u_n \to u$ in $H^s_r(\mathbb{R}^N)$.

Lemma 23. *For any $c > 0$, there exists $\beta = \beta(c) > 0$ and $k(c) \in \mathbb{N}$ such that for any $k \geq k(c)$ and any $u \in \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$*

$$\|\pi_k u\|_{H^s(\mathbb{R}^N)} \leq \beta \quad \textit{implies} \quad I(u) \geq c.$$

Proof. By contradiction, we assume that there exists c_0 such that for any $\beta > 0$ and any $k \in \mathbb{N}$ it is possible to find $\ell \geq k$ and $u \in \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ such that

$$I(u) < c_0 \quad \text{with } \|\pi_k u\|_{H^s(\mathbb{R}^N)} \leq \beta.$$

In view of that, one can find a sequence $(k_j)_j \subset \mathbb{N}$, with $k_j \to \infty$ as $j \to \infty$, and a sequence $(u_j)_j \subset \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ such that

$$\|\pi_{k_j} u_j\|_{H^s(\mathbb{R}^N)} \leq \frac{1}{j} \quad \text{and} \quad I(u_j) < c_0 \tag{38}$$

for any $j \in \mathbb{N}$. Noticing that by Lemma 6 (iv) $(u_j)_j$ is bounded, up to a subsequence we have $u_j \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$.

Claim: $u = 0$.

Since $k_j \to \infty$, it follows that $\pi_{k_j} u \to u$ in $L^2(\mathbb{R}^N)$, hence

$$(\pi_{k_j} u_j, u)_{L^2(\mathbb{R}^N)} = (u_j, \pi_{k_j} u)_{L^2(\mathbb{R}^N)} \to (u, u)_{L^2(\mathbb{R}^N)}$$

as $j \to \infty$.

On the other hand, using (38) we get $\pi_{k_j} u_j \to 0$ in $L^2(\mathbb{R}^N)$, thus the claim must hold. Now, since $\|u_j\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)} \to 0$ by the compact embedding, $(u_j)_j \subset \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$, and Lemma 1 (ii), we obtain

$$[u_j]^2_{H^s(\mathbb{R}^N)} = \frac{N}{2s} \int_{\mathbb{R}^N} \tilde{F}(u_j)\, dx \to 0$$

as $j \to \infty$, which contradicts Lemma 6 (ii).

Lemma 24. *$E_{m,k} \to \infty$ as $k \to +\infty$.*

Proof. We assume by contradiction that there exists $c > 0$ such that

$$\liminf_{k \to +\infty} E_{m,k} < c.$$

Denote with $\beta(c)$ and $k(c)$ the numbers given in Lemma 23. Up to choose a bigger c, we can find $k > k(c)$ such that $E_{m,k} < c$. Moreover, by definition of $E_{m,k}$ there must be $A \in \mathcal{G}_k$ such that

$$\max_{u \in A} I(\rho(u) * u) = \max_{u \in A} J(u) < c.$$

Now, recalling Lemma 5 (iii) and (iv) we get that the map $\varphi : A \to \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ defined by $\varphi(u) = \rho(u) * u$ is odd and continuous. Thus, setting $\overline{A} := \varphi(A) \subset \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ we have

$$\max_{v \in \overline{A}} I(v) < c$$

and

$$\gamma(\overline{A}) \geq \gamma(A) \geq k > k(c) \tag{39}$$

by the properties of the genus. On the other hand, Lemma 23 implies that

$$\inf_{v \in \overline{A}} \|\pi_{k(c)} v\|_{H^s(\mathbb{R}^N)} \geq \beta(c) > 0,$$

and after setting

$$\phi(v) := \frac{\pi_{k(c)} v}{\|\pi_{k(c)} v\|_{H^s(\mathbb{R}^N)}} \quad \text{for any } v \in \overline{A}$$

we get

$$\gamma(\overline{A}) \leq \gamma(\phi(\overline{A})) \leq k(c)$$

noticing that ϕ is odd, continuous and that $\phi(\overline{A}) \subset V_{k(c)}$. That is against (39). Therefore $E_{m,k} \to \infty$ as $k \to +\infty$.

Proof (Proof of Theorem 3). For each $k \in \mathbb{N}$, by Lemmas 20 and 21 one can find a Palais-Smale sequence $(u_n)_n \subset \mathcal{P}_m \cap H_r^s(\mathbb{R}^N)$ of the constrained functional $I_{|S_m \cap H_r^s(\mathbb{R}^N)}$ at level $E_{m,k} > 0$. By Lemma 6 $(u_n)_n$ is bounded and by virtue of Lemma 22 we deduce that (P_m) has a radial solution u_k such that $I(u_k) = E_{m,k}$. Moreover, using Lemma 21 (ii) and Lemma 24, we get

$$I(u_{k+1}) \geq I(u_k) > 0 \quad \text{for any } k \geq 1$$

and $I(u_k) \to \infty$.

6. Appendix

Proof (Proof of Lemma 1). (i) It suffices to show that there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^N} |F(u)|\, dx \leq \frac{1}{4} [u]^2_{H^s(\mathbb{R}^N)}$$

whenever $u \in B_m$ and $[u]_{H^s(\mathbb{R}^N)} \leq \delta$. In order to show that, we start noticing that (f_0), (f_1), and (f_2) imply that for every $\varepsilon > 0$ we can find $C_1 = C_1(\varepsilon) > 0$ such that

$$|F(u)| \leq \varepsilon |t|^{2+\frac{4s}{N}} + C_1 |t|^{\frac{2N}{N-2s}}. \tag{40}$$

Hence, by (40), using the interpolation inequality and the fractional Sobolev inequality (see for instance [2, Theorem 6.5]), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |F(u)|\, dx &\leq \varepsilon \int_{\mathbb{R}^N} |u|^{2+\frac{4s}{N}}\, dx + C_1 \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}}\, dx \\
&\leq \varepsilon m^{\frac{2s}{N}} \|u\|^2_{L^{2_s^*}(\mathbb{R}^N)} + C_1 \|u\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} \\
&\leq \varepsilon m^{\frac{2s}{N}} C_1 [u]^2_{H^s(\mathbb{R}^N)} + C_2 [u]^{2_s^*}_{H^s(\mathbb{R}^N)} \\
&= \left[\varepsilon m^{\frac{2s}{N}} C_1 + C_2 [u]^{2_s^*-2}_{H^s(\mathbb{R}^N)}\right] [u]^2_{H^s(\mathbb{R}^N)}.
\end{aligned}$$

Choosing

$$\varepsilon = \frac{1}{8m^{\frac{2s}{N}} C_1} \quad \text{and} \quad \delta = \left(\frac{1}{C_2}\right)^{\frac{1}{2_s^*-2}}$$

the assertion is verified.

(ii) Since (f_0), (f_1) and (f_2) hold, for every $\varepsilon > 0$ there exists C_3, $C_4 > 0$ such that

$$|f(t)t| \leq \frac{\varepsilon}{2}|t|^{\frac{2N}{N-2s}} + C_3|t|^{2+\frac{4s}{N}}$$

and

$$|F(t)| \leq \frac{\varepsilon}{2}|t|^{\frac{2N}{N-2s}} + C_4|t|^{2+\frac{4s}{N}},$$

which implies

$$|\tilde{F}(t)| \leq \varepsilon|t|^{\frac{2N}{N-2s}} + (C_3 + C_4)\,|t|^{2+\frac{4s}{N}}. \tag{41}$$

By (41) we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\tilde{F}(u_n)|\,dx &\leq \varepsilon \int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2s}}\,dx + \int_{\mathbb{R}^N} |u_n|^{2+\frac{4s}{N}}\,dx \\
&\leq \varepsilon C_5\,[u_n]_{H^s(\mathbb{R}^N)}^{\frac{2N}{N-2s}} + (C_3 + C_4]\,\|u_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)}^{2+\frac{4s}{N}} \\
&\leq \varepsilon C_6 + (C_3 + C_4)\,\|u_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)}^{2+\frac{4s}{N}} \to 0
\end{aligned}$$

as $n \to +\infty$ and $\varepsilon \to 0$. The proof of $\lim_{n\to+\infty} \int_{\mathbb{R}^N} |F(u_n)|\,dx = 0$ is similar.

(iii). (f_0), (f_1) and (f_2) imply that for every $\varepsilon > 0$ we can find $C_7 > 0$ such that

$$|f(t)| \leq \varepsilon|t|^{\frac{N+2s}{N-2s}} + C_7|t|^{1+\frac{4s}{N}}. \tag{42}$$

Hence, by (42), we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(u_n)||v_n|\,dx &\leq \varepsilon \int_{\mathbb{R}^N} |u_n|^{\frac{N+2s}{N-2s}}|v_n|\,dx + C_7 \int_{\mathbb{R}^N} |u_n|^{1+\frac{4s}{N}}|v_n|\,dx \\
&\leq \varepsilon\|u_n\|_{L^{2_s^*}(\mathbb{R}^N)}^{\frac{N+2s}{2N}}\|v_n\|_{L^{2_s^*}(\mathbb{R}^N)}^{\frac{N-2s}{2N}} + C_7\|u_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)}^{\frac{N+4s}{2(N+2s)}}\|v_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)}^{\frac{N}{2(N+2s)}} \\
&\leq \varepsilon C_8\|u_n\|_{H^s(\mathbb{R}^N)}^{\frac{N+2s}{2N}}\|v_n\|_{H^s(\mathbb{R}^N)}^{\frac{N-2s}{2N}} + C_9\|u_n\|_{H^s(\mathbb{R}^N)}^{\frac{N+4s}{2(N+2s)}}\|v_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)}^{\frac{N}{2(N+2s)}} \\
&\leq \varepsilon C_{10} + C_{11}\|v_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)}^{\frac{N}{2(N+2s)}} \to 0
\end{aligned}$$

as $n \to +\infty$ and $\varepsilon \to 0$. This completes the proof of the Lemma.

PROOF (OF LEMMA 2). (i) Let us fix $m := \|u\|^2_{L^2(\mathbb{R}^N)}$. We observe that $\rho * u \in S_m$ and after a change of variables we obtain

$$[\rho * u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^{2N}} \frac{e^{N\rho}(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy = e^{2\rho s}\,[u]^2_{H^s(\mathbb{R}^N)}\,.$$

By virtue of the previous computation, choosing $\rho \ll -1$, Lemma 1 (i) guarantees the existence of a $\delta > 0$ such that if $[\rho * u]_{H^s(\mathbb{R}^N)} \le \delta$ then

$$\frac{1}{4}e^{2\rho s}\,[u]^2_{H^s(\mathbb{R}^N)} \le I(\rho * u) \le e^{2\rho s}\,[u]^2_{H^s(\mathbb{R}^N)}\,,$$

thus

$$\lim_{\rho\to-\infty} I(\rho * u) = 0^+.$$

(ii) For every $\lambda \ge 0$ we define the function $h_\lambda\colon \mathbb{R} \to \mathbb{R}$ as follows

$$h_\lambda(t) = \begin{cases} \frac{F(t)}{|t|^{2+\frac{4s}{N}}} + \lambda & t \ne 0 \\ \lambda & t = 0. \end{cases} \tag{43}$$

It is straightforward to verify that $F(t) = h_\lambda(t)|t|^{2+\frac{4s}{N}} - \lambda|t|^{2+\frac{4s}{N}}$. Moreover, from (f_0) and (f_1) it follows that h_λ is continuous, whereas thanks to (f_3) we have

$$h_\lambda(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Putting together the divergence of the limit above at infinity and (f_1), we can find $\lambda > 0$ large enough such that $h_\lambda(t) \ge 0$ for every $t \in \mathbb{R}$. Now, applying the well known Fatou's Lemma, we obtain

$$\liminf_{\rho\to\infty} \int_{\mathbb{R}^N} h_\lambda(e^{\frac{N\rho}{2}}u)|u|^{2+\frac{4s}{N}}\,dx \ge \int_{\mathbb{R}^N} \lim_{\rho\to\infty} h_\lambda(e^{\frac{N\rho}{2}}u)|u|^{2+\frac{4s}{N}}\,dx = \infty.$$

Then, we observe that

$$\begin{aligned} I(\rho * u) &= \frac{1}{2}\,[\rho * u]^2_{H^s(\mathbb{R}^N)} + \lambda \int_{\mathbb{R}^N} |\rho * u|^{2+\frac{4s}{N}}\,dx - \int_{\mathbb{R}^N} h_\lambda(\rho * u)|\rho * u|^{2+\frac{4s}{N}}\,dx \\ &= e^{2\rho s}\left[\frac{1}{2}\,[u]^2_{H^s(\mathbb{R}^N)} + \lambda \int_{\mathbb{R}^N} |u|^{2+\frac{4s}{N}}\,dx - \int_{\mathbb{R}^N} h_\lambda(e^{\frac{N\rho}{2}}u)|u|^{2+\frac{4s}{N}}\,dx\right], \end{aligned} \tag{44}$$

from which it follows immediately that

$$\lim_{\rho\to\infty} I(\rho * u) = -\infty.$$

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