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On MRAs and prewavelets based on elliptic splines

Barbara Bacchelli and Milvia Rossini

Abstract We consider shift-invariant multiresolution spaces generated by q-elliptic splines in $\mathbb{R}^d, d \geq 2$, which are tempered distributions characterized by a complex-valued elliptic homogeneous polynomial q of degree $m > d$. To construct Riesz bases of $L^2(\mathbb{R}^d)$, a family of non-separable basic smooth functions are obtained by localizing a fundamental solution of the operator $q(D)$, properly. The construction provides a generalization of some known elliptic scaling functions, the most famous being polyharmonic B-splines.
Here, we prove that real-valued q leads to r-regular multiresolution analysis, with $r = m - d - 1$. In addition, we prove that there exist r-regular non-separable prewavelet systems associated with not necessarily regular multiresolution analyses. These prewavelets have $m - 1$ vanishing moments and the approximation order of the prewavelet decomposition can be established.

Keywords Non-separable wavelets · q-elliptic splines · regular multiresolution analysis · regular prewavelets

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1 Introduction

The class $E_q(\mathbb{R}^d)$ of *q-elliptic splines* can be defined as the subset of tempered distributions f on $\mathbb{R}^d$ such that the differential operator $q(D)$, $D = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$, applied to f is a measure supported on the integer lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$; q is a complex, homogeneous polynomial of degree $m > d$

B. Bacchelli
Università di Milano-Bicocca, Italy
E-mail: barbara.bacchelli@unimib.it

M. Rossini
Università di Milano-Bicocca, Italy
E-mail: milvia.rossini@unimib.it

which is required to be zero in $\mathbb{R}^d$ only in the origin, hence the use of the term *elliptic*.

The first paper introducing multiresolution analysis (MRA) for elliptic spline is [14] where a dyadic prewavelet decomposition of $L^2(\mathbb{R}^d)$ is provided. In that paper, the authors introduced a special family of q-elliptic splines ϕ having Fourier transform of the form

$$\widehat{\phi} = \widehat{T} q^{-1},$$

where $\widehat{T}$ is a trigonometric polynomial which is zero only in the grid $2\pi\mathbb{Z}^d$ and there exists a positive integer s such that $\widehat{T}(\omega) - q(\omega) = O(\|\omega\|_\infty^{m+1+s})$, as $\omega \to 0$. This implies that $\widehat{\phi}(\omega) \to 1$ as $\omega \to 0$. If we specialize that $q(D)$ is the kth-iterate of the Laplace operator ($k = m/2, m$ even) we obtain polyharmonic B-splines, which include the elementary and high level k-harmonic "B-splines" defined by Rabut in his seminal papers [16,17] and all the ones introduced later on (see e.g. [3,18,19,22]). MRAs based on polyharmonic B-splines have been widely studied and explicit constructions of the filters can be found for instance in [1,4,22].

More recently, in the two dimensional setting, the authors of [8] modify the Laplacian of order $\alpha \in \mathbb{R}_0^+$ by a differential operator of Wirtinger type and allow for ϕ less restrictive assumptions than [14] by requiring that $\left|\widehat{\phi}(\omega)\right| \to c > 0$ as $\omega \to 0$. When α is an integer, their special complex q is a homogeneous elliptic polynomial and the introduced scaling functions and prewavelets are in $E_q(\mathbb{R}^2)$ (see also [23]).

In all the above cases, the resulting q-elliptic splines are non-separable and non-compactly supported. Non-compactly supported scaling functions and wavelets may have some drawback when we consider implementation aspects and applications, then r-regularity, in the sense of Meyer [13], is a very desirable property in order to produce reliable numerical results. Furthermore, r-regularity of the MRA has important consequences from a theoretical point of view, as for instance on the convergence properties of the projections onto the multiscale spaces of functions in Sobolev spaces. We refer to the book [13] for a basic treatment of the subject.

In this paper, we deepen the knowledge on elliptic splines with a particular attention to r-regularity. As main results, we provide a class of scaling function with less restrictive assumptions on $\widehat{\phi}$ than those in [14,8] and we prove r-regularity of the MRAs corresponding to real-valued polynomials q, where r is the degree of smoothness of the functions. Moreover, we show that for any q-elliptic spline based MRA, a r-regular complex prewavelet system exists in $E_q(\mathbb{R}^d)$.

More precisely, we introduce a new family $LE_q(\mathbb{R}^d) \subset E_q(\mathbb{R}^d)$ of *localized q-elliptic* splines which turn out to be valid scaling functions for generating stationary MRAs of $L^2(\mathbb{R}^d)$. We provide a simple proof based on a result that can be found in [10]. The novelty consists in requiring that $\widehat{\phi} = \widehat{T} q^{-1}$ is bounded and not null in the d-dimensional torus. Our definition includes those mentioned above and it is mainly motivated when dealing with complex-valued polynomials q. The Fourier transform of ϕ may not be continuous, thus, in that case, $\phi \notin L^1(\mathbb{R}^d)$. Refinability intrinsically binds the dilation matrix A for $\mathbb{Z}^d$ with the generating polynomial q in a relation that, thanks to the homogeneity of q, is always satisfied when A is a dyadic dilation. These scaling functions are non-separable, non-compactly supported with algebraic decay at infinity.

In the special case of real-valued polynomials q, we can introduce the Lagrange q-elliptic function Λ_q just in terms of q, extending the definition given in [11] for the polyharmonic case. We prove that Λ_q is a localized q-elliptic spline, then it is a scaling function, and that Λ_q and any ϕ in $LE_q(\mathbb{R}^d)$ generates the same MRA, associated with an admissible dilation A for $\mathbb{Z}^d$. Moreover, we can prove that Λ_q and its derivatives up to the order $r := m - d - 1$ exponentially decay at infinity. Thus, the Lagrange scaling function Λ_q is r-regular. These results are attractive since we can say that when q is real-valued, the shift-invariant approximation spaces of the multiscale analysis are characterized essentially by q and A and the resulting MRA is r-regular. For example, the well-known MRAs generated by polyharmonic B-splines are r-regular, improving our knowledge on these splines. We conclude the real-valued q analysis by discussing some interpolation properties of Λ_q. In particular, the cardinal interpolation problem for data of polynomial growth has solution in $E_q(\mathbb{R}^d)$ and Λ_q has the polynomial reproducing property up to the degree $m - 1$, thus generalizing the known results for polyharmonic Lagrange splines (see e.g. [11]).

Moving on to the aim of providing r-regular prewavelet basis in $E_q(\mathbb{R}^d)$, we remark that in the mentioned literature the wavelet constructions depend explicitly on a scaling function generating the MRA, so that one may suppose an algebraic decay at infinity. Also, it is well-known that for any r-regular MRA associated with a dilation A such that $|\det A| > (d+1)/2$ there exists a set of wavelets consisting of $|\det A| - 1$ r-regular functions (see e.g. [24]).

In our approach, the MRA may not be r-regular and the scaling function generating the approximation spaces is not explicitly involved in the prewavelet definition that actually depends on the real valued polynomial $|q|^2$. Thanks to some characterizations of a set of wavelets given in Sect. 5.1 and to the regularity properties shown for Lagrange elliptic splines associated with real-valued polynomials, we can construct a family of r-regular complex prewavelets and duals, for any complex-valued q. These non-separable elliptic splines depend only on q and A, have vanishing moments up to the order $m - 1$ and provide approximation order m.

Finally, we provide some particular polynomials q and associated scaling functions in $LE_q(\mathbb{R}^d)$ together with prewavelets.

The paper is organized as follows. In Sect. 2 we give some notations and definitions. In Sect. 3 we define the subspace $LE_q(\mathbb{R}^d)$ of tempered distributions of localized q-elliptic splines and we prove that they are scaling functions generating MRAs of $L^2(\mathbb{R}^d)$. In Sect. 4 we define the Lagrange elliptic spline in $LE_q(\mathbb{R}^d)$ associated with real-valued q and we prove its properties. Sect. 5

deals with the construction of a r-regular prewavelet systems. In Sect. 6 we illustrate connections to prior definitions and some examples.

2 Basic notations and definitions

Let d be the dimension of the space $\mathbb{R}^d$, and $d \geq 2$. We use standard notations for the spaces L^p and ℓ^p with norms $\|\cdot\|_p$, $p \in [1,\infty]$, and we denote by $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^d$. If $z \in \mathbb{C}$, the complex plane, then $|z|$ is the usual modulus and we use the same notation for real z. $\mathbb{N} = \{0,1,2,...\}$. The inner product between functions f,g in $L^2(\mathbb{R}^d)$, or between vectors x,y in $\mathbb{R}^d$ is denoted respectively as

$$(f,g) := \int_{\mathbb{R}^d} f(x)\,\overline{g(x)}\,dx, \qquad < x,y > := \sum_{k=1}^{d} x_k y_k.$$

The space of locally square integrable functions which are $2\pi\mathbb{Z}^d$-periodic is denoted by $L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$, and $\mathbb{T}^d := (-\pi,\pi]^d$ is the d-dimensional torus. We use $\widehat{\ }$ for the Fourier transform, i.e. for any function $f \in L^2(\mathbb{R}^d)$, $\widehat{f}(\omega) := \int_{\mathbb{R}^d} f(x)e^{-i<\omega,x>}dx$. We also use $\widehat{\ }$ for the Fourier transform of distributions. When f is a distribution, and $\zeta \in \mathbb{C}^d$, $\widehat{f}(\zeta)$ is the Fourier-Laplace transform of f. Restricted to $\mathbb{R}^d$, $\widehat{f}$ becomes the Fourier transform of f. All equalities between functions and other related notions are interpreted in the distributional sense whenever possible. If $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d$ is a multi-index, $|\alpha| := \sum_{k=1}^d \alpha_k$, $\partial^\alpha f := \dfrac{\partial^{|\alpha|} f}{(\partial x_1)^{\alpha_1}\ldots(\partial x_d)^{\alpha_d}}$, and $\partial^0 f := f$. $\mathcal{C}^r(\mathbb{R}^d)$ is the set of functions that are r times continuously differentiable on $\mathbb{R}^d$. According to Meyer [13], given $r \in \mathbb{N}$, a function f on $\mathbb{R}^d$ is *r-regular* if $f \in \mathcal{C}^r(\mathbb{R}^d)$ and for each $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq r$

$$|\partial^\alpha f(x)| \leq c_{\alpha,n}(1+\|x\|)^{-n}, \quad \forall\, x \in \mathbb{R}^d, \quad \forall\, n \in \mathbb{N}. \tag{2.1}$$

If f is r-regular then $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.
Given a function ϕ in $L^2(\mathbb{R}^d)$ we denote by $V(\phi)$ the $L^2(\mathbb{R}^d)$-closure of the set generated by all linear combinations of its $\mathbb{Z}^d$-translates

$$V(\phi) := clos_{L^2(\mathbb{R}^d)} \left\{ \sum_{k\in\mathbb{Z}^d} c_k \phi(\cdot - k), c_k \in \ell^2(\mathbb{Z}^d) \right\}.$$

In this paper, a $d \times d$ matrix A denotes a *dilation* for $\mathbb{Z}^d$, that is $A\mathbb{Z}^d \subset \mathbb{Z}^d$, and the eigenvalues of A have modulus greater than one. A dilation matrix enjoys the properties that $|\det A|$ is an integer ≥ 2 and it gives the number of cosets of the quotient group $\mathbb{Z}^d/A\mathbb{Z}^d$. We denote by $\mathcal{F}$ a complete set of representatives of this group, then $|\det A| = \sharp\mathcal{F}$. We assume that $0 \in \mathcal{F}$. We call $\mathcal{F}' = \mathcal{F} \setminus \{0\}$ and $B = A^T$. The identity matrix in $\mathbb{R}^d$ is denoted by I^d.

A multiresolution analysis (MRA) of $L^2(\mathbb{R}^d)$ associated with $(\mathbb{Z}^d, A)$ is a family $\mathcal{V} = \{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ with the following properties

- $\mathcal{V}_j \subset \mathcal{V}_{j+1}$, $j \in \mathbb{Z}$,
- $\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j$ is dense in $L^2(\mathbb{R}^d)$,
- for all $f \in L^2(\mathbb{R}^d)$ and all $j \in \mathbb{Z}$, $f(\cdot) \in \mathcal{V}_j \iff f(A^{-1}\cdot) \in \mathcal{V}_{j-1}$,
- there is a function $\phi \in L^2(\mathbb{R}^d)$, which is called scaling function, such that the set $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is a Riesz basis of $\mathcal{V}_0$.

In short, we will say that ϕ generates the MRA with dilation A. Clearly, $\mathcal{V}_0 = V(\phi)$ and if $\phi \in \mathcal{C}^k(\mathbb{R}^d)$, $k \in \mathbb{N}$, then $\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j \subset \mathcal{C}^k(\mathbb{R}^d)$.
A MRA of $L^2(\mathbb{R}^d)$ is r-regular if there exists a r-regular scaling function generating it (see e.g. [13,24]).

Throughout this paper, we denote by q a complex-valued homogeneous polynomial on $\mathbb{R}^d$ of degree $m > d$

$$q(\omega) := \sum_{|\alpha|=m} q_\alpha \omega^\alpha,\ \omega \in \mathbb{R}^d,\ q_\alpha \in \mathbb{C},$$

which is required to be *elliptic*, in other words, $q(\omega) = 0$ for some $\omega \in \mathbb{R}^d$ implies that $\omega = 0$. Consequently, the following bounds hold for some positive constants C_1, C_2

$$C_1 \|\omega\|_\infty^m \leq |q(\omega)| \leq C_2 \|\omega\|_\infty^m,\ w \in \mathbb{R}^d. \tag{2.2}$$

3 Localized q-elliptic scaling functions

Definition 3.1 The set $E_q(\mathbb{R}^d)$ of q-elliptic splines is the subspace of tempered distributions f on $\mathbb{R}^d$ such that the differential operator $q(D)$ applied to f is a measure supported on the integer lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$. Symbolically

$$q(D)f(x) = \sum_{k\in\mathbb{Z}^d} c_k \delta(x-k),\ x \in \mathbb{R}^d, \quad c_k \in \mathbb{C}, \tag{3.1}$$

where $D = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$ and $\delta(x)$ is the unit Dirac measure supported at the origin.

It is known that every linear partial differential operator $q(D)$ with constant coefficients has a fundamental solution (see e.g. [12]). In our case there exists a tempered distribution u such that $q(D)u(x) = \delta(x)$, $x \in \mathbb{R}^d$ whose generalized Fourier transform $\widehat{u}(\omega) = (-i)^m q^{-1}(\omega), \omega \in \mathbb{R}^d$ has a pole of order m at the origin. Due to the fact that Fourier transformation is an isometric isomorphism on $L^2(\mathbb{R}^d)$, $u \notin L^2(\mathbb{R}^d)$ since $\widehat{u} \notin L^2(\mathbb{R}^d)$. Therefore, in order to get a scaling function $\phi \in E_q(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$, we cancel the singularity at the origin by multiplying q^{-1} by an appropriate function. This is a classical procedure to get localized basis functions used not only in wavelet theory but in many contexts of approximation theory (see e.g. [5]).

The following definition introduces a family of tempered distributions ϕ, the *localized q-elliptic splines*, capable to generate MRAs.

Definition 3.2 The set $LE_q(\mathbb{R}^d)$ of localized q-elliptic splines is the subspace of tempered distributions ϕ on $\mathbb{R}^d$ defined by the formula of their Fourier transform

$$\widehat{\phi} := \widehat{T}\ q^{-1} \tag{3.2}$$

where $\widehat{T}$ and q enjoy the following properties:

(A1) $\widehat{T}(\omega)$ is a bounded, $2\pi\mathbb{Z}^d$-periodic function such that $\widehat{T}(\omega) \neq 0$ for all $\omega \in \mathbb{T}^d \setminus \{0\}$;

(A2) there is a positive constant c such that

$$0 < \left|\widehat{\phi}(\omega)\right| < c \quad \text{for } \omega \in \mathbb{T}^d. \tag{3.3}$$

Our definition generalizes and includes the ones of [14,16,17] and [8] by relaxing the requests on the behavior of $\widehat{\phi}(\omega)$ in a neighborhood of the origin. We refer to Sect. 6 for more details.

Proposition 3.1 $LE_q(\mathbb{R}^d) \subset E_q(\mathbb{R}^d)$.

Proof Let $\phi \in LE_q(\mathbb{R}^d)$. The periodicity of the tempered distribution $\widehat{T}$ implies that T satisfies

$$T(x) = \sum_{k\in\mathbb{Z}^d} b_k \delta(x-k),$$

where $b_k = \int_{\mathbb{T}^d} e^{i<k,\omega>}\widehat{T}(\omega)d\omega$. From (3.2), we can write $q\widehat{\phi} = \widehat{T}$ and by inverting the Fourier transform we get

$$q(D)\phi(x) = \sum_{k\in\mathbb{Z}^d} c_k \delta(x-k),$$

where $c_k = i^m b_k, k \in \mathbb{Z}^d$. That is, $\phi \in E_q(\mathbb{R}^d)$. □

We discuss now the properties of localized q-elliptic splines.

Proposition 3.2 *Let ϕ in $LE_q(\mathbb{R}^d)$. Then the following properties hold:*

(i) $\left|\widehat{\phi}(\omega)\right| \leq \dfrac{K}{1+\|\omega\|_\infty^m}$, $\forall \omega \in \mathbb{R}^d$.

(ii) $\widehat{\phi} \in L^2(\mathbb{R}^d)$.

(iii) $\phi \in \mathcal{C}^{m-d-1}(\mathbb{R}^d)$.

(iv) *As* $\|x\| \to \infty$, $\phi(x) = O(\|x\|^{-s})$, $s \in (d/2, d]$.

Proof By (A2) in Definition 3.2 and by (2.2), $\widehat{\phi}$ is bounded on $\mathbb{R}^d$ and $\widehat{\phi} = O(\|\omega\|^{-m})$, as $\|\omega\| \to \infty$. Thus (i) holds and (ii) follows since $m > d$. Moreover, $\omega^\beta\widehat{\phi}$ is in $L^2(\mathbb{R}^d)$ for $|\beta| < m - d/2$. Then ϕ belongs to the Sobolev space $\mathcal{H}^t(\mathbb{R}^d)$ for $t < m - d/2$, and by the embedding theorem $\phi \in \mathcal{C}^{m-d-1}(\mathbb{R}^d)$, and (iii) is proven. To prove (iv), observe that since $\phi \in L^2(\mathbb{R}^d)$, ϕ decays better than $\|x\|^{-d/2}$ as $\|x\| \to \infty$. Also, $\widehat{\phi} \in L^1(\mathbb{R}^d)$, but $\widehat{\phi}$ is not required to be continuous, thus, in general, $\phi \notin L^1(\mathbb{R}^d)$ and ϕ decays not better than $\|x\|^{-d}$ as $\|x\| \to \infty$. □

Note that despite (iv) is true in the general case, a better decay is possible. Given a dilation A for $\mathbb{Z}^d$, ϕ is refinable if

$$\hat{\phi}(\omega) = S(B^{-1}\omega)\hat{\phi}(B^{-1}\omega) \tag{3.4}$$

for some $S(\omega) \in L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$. Replacing (3.2) in (3.4), we get

$$S(\omega) = \frac{\widehat{T}(B\omega)}{\widehat{T}(\omega)} \frac{q(\omega)}{q(B\omega)}.$$

In order to obtain a $2\pi\mathbb{Z}^d$-periodic measurable function $S(\omega)$, $q(w)$ and $q(Bw)$ need to cancel each other up to a scalar factor. Then the matrix $B = A^T$ and q must satisfy

$$\frac{q(\omega)}{q(B\omega)} = \kappa \tag{3.5}$$

for some constant κ. In particular, by virtue of the homogeneity of q, (3.5) holds whenever $A = 2^d I^d$ and $\kappa = 2^{-d}$.

Despite the fact that functions in $LE_q(\mathbb{R}^d)$ may not be in $L^1(\mathbb{R}^d)$, we prove that they are valid scaling functions.

Proposition 3.3 *Any $\phi \in LE_q(\mathbb{R}^d)$ generates a MRA $\mathcal{V} = \{\mathcal{V}_j\}_{j\in\mathbb{Z}^d}$ of $L^2(\mathbb{R}^d)$ associated with $(\mathbb{Z}^d, A)$, whenever A enjoys (3.5).*

Proof If A enjoys (3.5), there exists $S(\omega) = \kappa\widehat{T}(B\omega)/\widehat{T}(\omega)$ in $L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$ such that the refinement equation (3.4) holds. Since $\widehat{\phi} \in L^2(\mathbb{R}^d)$ (see Proposition 3.2), the $2\pi\mathbb{Z}^d$-periodic series

$$G(\omega) := \sum_{k\in\mathbb{Z}^d} \left|\widehat{\phi}(\omega + 2\pi k)\right|^2$$

converges. If $\omega \in \mathbb{T}^d$, $G(\omega) \geq \left|\widehat{\phi}(\omega)\right|^2$ and by (3.3) $\left|\widehat{\phi}(\omega)\right| > 0$. Then $G(\omega) > 0$ in $\mathbb{T}^d$ and by periodicity $G(\omega) > 0$ for $\omega \in \mathbb{R}^d$. Observe that

$$\frac{|\widehat{\phi}(\omega)|^2}{G(\omega)} = \frac{|q(\omega)|^{-2}}{\sum_{k\in\mathbb{Z}^d} |q(\omega + 2\pi k)|^{-2}} = \frac{1}{1 + |q(\omega)|^2 \sum_{k\neq 0} |q(\omega + 2\pi k)|^{-2}}.$$

Thus $|\widehat{\phi}|^2/G$ is a continuous function equal to 1 at the origin. The thesis follows by virtue of [10, Proposition 5]. □

4 q real-valued and Lagrange function Λ_q

In this Section, we assume that the homogeneous and elliptic polynomial q is real-valued. In this case, we can introduce a function Λ_q in $LE_q(\mathbb{R}^d)$ that enjoys some remarkable properties. The definition in the Fourier domain involves only q, and it satisfies the interpolation property that characterizes a Lagrange function: $\Lambda_q(k) = \delta_{0k}$, $k \in \mathbb{Z}^d$. Moreover, any ϕ in $LE_q(\mathbb{R}^d)$ generates the same MRA $\mathcal{V}$ generated by Λ_q, that is, given a dilation A for $\mathbb{Z}^d$ enjoying (3.5), a q-elliptic MRA associated with $(\mathbb{Z}^d, A)$ is essentially determined only by the polynomial q. Then we prove that Λ_q and its derivatives up to the order r, exponentially decay at infinity, where $r = m - d - 1$ is the degree of smoothness of the class $LE_q(\mathbb{R}^d)$. Thus, Λ_q is r-regular. This property will be used in Section 5 to construct r-regular prewavelet systems. Here, we deduce that the MRA $\mathcal{V}$ is r-regular. Of course, the special case of real-valued q includes the well-known polyharmonic scaling functions, where $q = \|\cdot\|^m$, m even, improving our knowledge on the MRAs generated by these splines. Finally, thanks to the properties of Λ_q, we show that the cardinal interpolation problem of data with polynomial growth has solution in $E_q(\mathbb{R}^d)$ and Λ_q has the polynomial reproducing property up to the degree $m-1$.

Let us consider the distribution Λ_q defined by the formula of its Fourier transform

$$\widehat{\Lambda}_q(\omega) := \frac{q^{-1}(\omega)}{\sum_{k\in\mathbb{Z}^d} q^{-1}(\omega - 2\pi k)}. \tag{4.1}$$

We remember that $d \geq 2$, then m is even and q has constant sign. By using (2.2), we get

$$0 \leq \widehat{\Lambda}_q(\omega) \leq \frac{C_2}{C_1} \|\omega\|_\infty^{-m} \left[\sum_{k\in\mathbb{Z}^d} \|\omega - 2\pi k\|_\infty^{-m} \right]^{-1}.$$

Since $m > d$, Λ_q is well defined as an absolutely convergent integral. Moreover, $\widehat{\Lambda}_q$ is a continuous function with $\widehat{\Lambda}_q(0) = 1$ and $\widehat{\Lambda}_q(2\pi k) = 0, k \in \mathbb{Z}^d \setminus \{0\}$. Let us introduce the $2\pi\mathbb{Z}^d$-periodic distribution

$$\widehat{\Phi} := q\widehat{\Lambda}_q. \tag{4.2}$$

Proposition 4.1 *The following properties hold for* Λ_q *:*

(i) $\Lambda_q \in LE_q(\mathbb{R}^d)$.
(ii) $\Lambda_q(k) = \delta_{0k}, \ k \in \mathbb{Z}^d$.
(iii) *For any* $\phi \in LE_q(\mathbb{R}^d)$, $V(\phi)=V(\Lambda_q)$ *and* Λ_q *and* ϕ *generate the same MRA* $\mathcal{V}$ *of* $L^2(\mathbb{R}^d)$ *associated with* $(\mathbb{Z}^d, A)$, *whenever* A *enjoys (3.5).*

Proof Let $\widehat{\Phi}$ be defined as in (4.2). Clearly, $\widehat{\Lambda}_q = \widehat{\Phi}q^{-1}$. $\widehat{\Phi}$ is bounded since continuous. If $\omega \in \mathbb{T}^d$ and $k \neq 0$ then $\omega - 2\pi k \in \mathbb{R}^d \setminus \mathbb{T}^d$ and $q(\omega - 2\pi k) \neq 0$. Hence $\widehat{\Phi}(\omega) \neq 0$ for all $\omega \in \mathbb{T}^d \setminus \{0\}$, namely $\widehat{\Phi}$ enjoys (A1) of Definition 3.2. Since $\widehat{\Lambda}_q(0) = 1$ it follows $\widehat{\Lambda}_q(\omega) \neq 0$ for all $\omega \in \mathbb{T}^d$, and $\widehat{\Lambda}_q$ is bounded on

$\mathbb{T}^d$ because it is continuous; thus $\widehat{\Lambda}_q$ enjoys (A2) of Definition 3.2 and we can conclude that $\Lambda_q \in LE_q(\mathbb{R}^d)$.

To prove (ii), we observe that $\sum_{k\in\mathbb{Z}^d} \widehat{\Lambda}_q(\omega - 2\pi k) = 1$, and for $k \in \mathbb{Z}^d$ we can write

$$\begin{aligned}\Lambda_q(k) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{\Lambda}_q(\omega) e^{i<k,\omega>} d\omega = \sum_{l\in\mathbb{Z}^d} \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \widehat{\Lambda}_q(\omega - 2\pi l) e^{i<k,\omega>} d\omega \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \sum_{l\in\mathbb{Z}^d} \widehat{\Lambda}_q(\omega - 2\pi l) e^{i<k,\omega>} d\omega = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{i<k,\omega>} d\omega = \delta_{0k}.\end{aligned}$$

Let us prove (iii). According to Proposition 3.3, let $\mathcal{V}$ be the MRA generated by Λ_q associated with $(\mathbb{Z}^d, A)$, where A enjoys (3.5). Let ϕ be in $LE_q(\mathbb{R}^d)$ with $\widehat{\phi} = \widehat{T} q^{-1}$, as in Definition 3.2. Consider the periodic distribution

$$H(\omega) := \frac{\widehat{T}(\omega)}{\widehat{\Phi}(\omega)},$$

where $\widehat{\Phi}$ is defined in (4.2). If $\omega \in \mathbb{T}^d$ we can write $H(\omega) = \widehat{\phi}(\omega)/\widehat{\Lambda}_q(\omega)$. Since Λ_q is in $LE_q(\mathbb{R}^d)$, $\left|\widehat{\Lambda}_q\right|$ is positive in $\mathbb{T}^d$ by (3.3). Indeed, by continuity, there exists a positive constant a such that $\left|\widehat{\Lambda}_q(\omega)\right| \geq a$ for all $\omega \in \mathbb{T}^d$. Then

$$0 < |H(\omega)| < c\ a, \quad \omega \in \mathbb{T}^d,$$

where c is the positive constant in (3.3). By periodicity $|H(\omega)| > 0, \omega \in \mathbb{R}^d$. Moreover, $H(\omega) \in L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$. Since

$$\widehat{\phi}(\omega) = H(\omega)\widehat{\Lambda}_q(\omega),$$

by virtue of [10, C6, Lemma 4], $V(\phi)$=$V(\Lambda_q) = \mathcal{V}_0$ and we can conclude that ϕ also generates the MRA $\mathcal{V}$. □

Let $\mathcal{V} = \{\mathcal{V}_j\}_{j\in\mathbb{Z}^d}$ be a MRA of $L^2(\mathbb{R}^d)$ associated with $(\mathbb{Z}^d, A)$ generated by a localized q-elliptic splines with real-valued q. We observe that every element f in $\mathcal{V}_0$ is continuous and can be uniquely expressed as $f(x) = \sum_{k\in\mathbb{Z}^d} f(k)\Lambda_q(x-k)$. Then the MRA $\mathcal{V}$ consists of *generalized splines* in the sense of Meyer (see [13]).

Moreover, since for any ϕ in $LE_q(\mathbb{R}^d)$ $V(\phi)$=$V(\Lambda_q) = \mathcal{V}_0$, we can say that the properties of the multiscale spaces $\mathcal{V}_j$, $j \in \mathbb{Z}^d$ depend only on q and can not be improved by the choice of $\widehat{T}$ in $\widehat{\phi} = \widehat{T} q^{-1}$.

In general, $\widehat{T}$ acts on the shape and decay of ϕ, as it has already been shown for example for polyharmonic B-splines (see e.g. [3,18,19,22]). We note that Λ_q meets the property $\widehat{\Lambda}_q = N q^{-1}$, where N is the Fourier expansion of the continuous and periodic function $\widehat{\Phi}$. Thus, $\widehat{\Lambda}_q$ has quite the same form as $\widehat{\phi}$, the infinite expansion giving the possibility of an exponential decay instead of an algebraic one.

Remember that exponential decay at infinity of Lagrange polyharmonic splines was proven in [11]. Here, we improve that result by extending it to Λ_q and also to all its admissible derivatives, for any real-valued q.

These properties can be derived from the fact that $\widehat{\Lambda}_q$ has a holomorphic extension to a tube about the real axis. We use the following notations. If $\zeta = (\zeta_1, ..., \zeta_d) \in \mathbb{C}^d$ then $\Re\zeta$ and $\Im\zeta$ denote the vector of the real and imaginary parts of the components of ζ, $|\zeta| = \sqrt{|\zeta_1|^2 + ... + |\zeta_d|^2}$, $\|\zeta\|_\infty = \max_{j=1,...,d} |\zeta_j|$ and $\zeta^\alpha = \prod_{k=1}^{d_k} \zeta_k^{\alpha_k}$. Clearly, $|\zeta^\alpha| = \prod_{k=1}^{d_k} |\zeta_k|^{\alpha_k} \leq \|\zeta\|_\infty^{|\alpha|}$. Given a subset $\mathbb{A}^d \subseteqq \mathbb{R}^d$ and $\varepsilon > 0$, we denote by $\mathbb{A}^d_\varepsilon$ the following subset of $\mathbb{C}^d = \mathbb{R}^d + i\mathbb{R}^d$, $i^2 = -1$,

$$\mathbb{A}^d_\varepsilon = \left\{ \xi = \omega + i\nu : \omega \in \mathbb{A}^d \ , \ \|\nu\|_\infty < \varepsilon \right\}. \tag{4.3}$$

Lemma 4.1 *The functions $\widehat{\Lambda}_q$ and $\widehat{\Phi}$, defined as in (4.1) and (4.2) respectively, have holomorphic extensions to a tube $\mathbb{R}^d_\varepsilon$, for some positive ε.*

Proof Let $\omega \in \mathbb{T}^d$ and put

$$F(\omega) := \sum_{k \in \mathbb{Z}^d \setminus \{0\}} q^{-1}(\omega - 2\pi k) \tag{4.4}$$

which is a uniformly convergent series in $\mathbb{T}^d$. Since q is real valued $q(\omega)F(\omega) \geq 0$ and observe that for ω in $\mathbb{T}^d$ $\widehat{\Phi}(\omega) = q(\omega)\left[1 + q(\omega)F(\omega)\right]^{-1}$ and $\widehat{\Lambda}_q(\omega) = \left[1 + q(\omega)F(\omega)\right]^{-1}$. Let $q(\xi)$ be the analytic extension to $\mathbb{C}^d$ of the polynomial q. Since q is homogeneous and elliptic, there is a positive ε such that $q(\xi) \neq 0$, $\xi \in \mathbb{R}^d_\varepsilon \setminus \mathbb{T}^d_\varepsilon$. It follows that if $\xi \in \mathbb{T}^d_\varepsilon$ and $j \neq 0$ then $\xi - 2\pi j \in \mathbb{R}^d_\varepsilon \setminus \mathbb{T}^d_\varepsilon$ and $q(\xi - 2\pi j) \neq 0$. Hence F extends analytically to $\mathbb{T}^d_\varepsilon$. By reducing ε if necessary, we may assume that $1 + q(\xi)F(\xi)$ has no zeros in $\mathbb{T}^d_\varepsilon$. It follows that $\widehat{\Phi}$ extends analytically to $\mathbb{T}^d_\varepsilon$ and hence, by periodicity, it extends analytically to $\mathbb{R}^d_\varepsilon$. Also, it is evident that $\widehat{\Lambda}_q$ has analytic extension in $\mathbb{T}^d_\varepsilon$. Since $q(\xi) \neq 0$ in $\mathbb{R}^d_\varepsilon \setminus \mathbb{T}^d_\varepsilon$, then by (4.2) this extension is analytic in all $\mathbb{R}^d_\varepsilon$. □

Lemma 4.2 *Let q be real valued and let α be a multi-index, with $|\alpha| \leq m - d - 1$. Then*

(i) *the series $\sum_{j \in \mathbb{Z}^d} \widehat{\partial^\alpha \Lambda}_q(\zeta + 2\pi j)$ converges absolutely and uniformly in the cube $\mathbb{T}^d_\varepsilon$, for some $\varepsilon > 0$;*
(ii) *the periodic distribution*

$$\widehat{G}_x(\omega) := \sum_{j \in \mathbb{Z}^d} \widehat{\partial^\alpha \Lambda}_q(\omega + 2\pi j)\ e^{i<x,\ \omega\ +2\pi j>},\ \omega \in \mathbb{R}^d, \tag{4.5}$$

has extension which is analytic in a tube $\mathbb{R}^d_\varepsilon$, for some $\varepsilon > 0$.

Proof Let ε as that in Lemma 4.1. The term with $j = 0$, $\widehat{\partial^\alpha \Lambda}_q(\zeta) = (-2\pi i\zeta)^\alpha \widehat{\Lambda}_q(\zeta)$, is bounded in $\mathbb{T}^d_\varepsilon$ because of the analyticity of $\widehat{\Lambda}_q$ in $\mathbb{R}^d_\varepsilon$. Let $j \in \mathbb{Z}^d \setminus \{0\}$ be fixed and assume $\zeta = u + iv$ in $\mathbb{T}^d_\varepsilon$, namely, $u \in [-\pi, \pi]^d$ and $v \in (-\varepsilon, \varepsilon)^d$. We use the notation $\mu_j = \zeta + 2\pi j = (u + 2\pi j) + iv$. When ζ belongs to $\mathbb{T}^d_\varepsilon$ then μ_j belongs to $\mathbb{R}^d_\epsilon \backslash \mathbb{T}^d_\varepsilon$. By virtue of analyticity of $\widehat{\Phi} = q\, \widehat{\Lambda}_q$ and since it is periodic, $\widehat{\Phi}$ is bounded in $\mathbb{R}^d_\epsilon$. The following inequalities hold:

$$\begin{aligned} \|\mu_j\|_\infty &\geq \|\Re \mu_j\|_\infty \geq |\|u\|_\infty - \|2\pi j\|_\infty| \geq \pi \|j\|_\infty, \\ \left|\mu_j^\alpha\right| &\leq \|\mu_j\|_\infty^{|\alpha|} \leq (3\pi + \varepsilon)^\alpha \|j\|_\infty^\alpha. \end{aligned}$$

Then, by using (2.2), it is possible to choose ε small enough so that

$$\left|\widehat{\partial^\alpha \Lambda_q}(\mu_j)\right| = \left|\mu_j^\alpha \widehat{\Lambda}_q(\mu_j)\right| = \left|\widehat{\Phi}(\mu_j) \frac{\mu_j^\alpha}{q(\mu_j)}\right| \leq C \|j\|_\infty^{|\alpha| - m},$$

where C is a positive constant independent of μ_j. Since $m - |\alpha| \geq d + 1$, then (i) is proved. Since $\widehat{\Lambda}_q$ has analytic extension in $\mathbb{R}^d_\epsilon$, for some $\varepsilon > 0$, then each term of the series (4.5) has analytic extension in $\mathbb{R}^d_\epsilon$ too. Let $\zeta \in \mathbb{T}^d_\varepsilon$. Clearly,

$$\left|\widehat{G}_x(\zeta)\right| \leq e^{\|x\|\varepsilon} \sum_{j \in \mathbb{Z}^d} \left|\widehat{\partial^\alpha \Lambda_q}(\zeta + 2\pi j)\right|.$$

Then, for (i), $\widehat{G}_x$ extends analytically to $\mathbb{T}^d_\varepsilon$ and hence, by periodicity, it extends analytically to the whole $\mathbb{R}^d_\varepsilon$. □

Now we are ready to prove the main result on regularity.

Proposition 4.2 *Let q be real valued and let α be a multi-index, with $|\alpha| \leq m - d - 1$. Then there are positive constants C and c, depending on d, m and α but independent of x, such that for all $x \in \mathbb{R}^d$,*

$$|\partial^\alpha \Lambda_q(x)| \leq C\, e^{-\, c\|x\|}. \tag{4.6}$$

Proof Since $\left|\widehat{\Lambda}_q(\omega)\right| \leq C|\omega|^{-m}$, then $\left|\widehat{\partial^\alpha \Lambda_q}(\omega)\right| \leq K\ |\omega|^{|\alpha| - m}, \omega \in \mathbb{R}^d$. According to the hypothesis $|\alpha| \leq m - d - 1$, we can apply the Fourier inversion formula and we can write

$$\begin{aligned} \partial^\alpha \Lambda_q(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{\partial^\alpha \Lambda_q}(\omega)\ e^{i<x,t>} d\omega \\ &= (2\pi)^{-d} \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \widehat{\partial^\alpha \Lambda_q}(\omega + 2\pi j)\ e^{i<x,\ \omega\ + 2\pi j>}\ d\omega \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} \widehat{G}_x(\omega)\ d\omega, \end{aligned}$$

where $\widehat{G}_x$ is the same of (4.5). Now, by virtue of analyticity of $\widehat{G}_x$, the set $\mathbb{T}^d$ in the last integral can be replaced by $S^d = \left\{\zeta \in \mathbb{C}^d : \Re\zeta \in \mathbb{T}^d,\ \text{and } \Im\zeta = \gamma\right\}$, where $\gamma = (\gamma_1, ..., \gamma_d)$, γ_k are constants, $|\gamma| = \varepsilon/2$, $k = 1, ..., d$, (here ε is the same as that in Lemma 4.2) and the sign of γ_k is chosen so that $< x, \gamma > / \|x\| > 0, x \neq 0$; then

$$|\partial^\alpha \Lambda_q(x)| = (2\pi)^{-d} \left| \int_{S^d} \sum_{j\in\mathbb{Z}^d} \widehat{\partial^\alpha \Lambda_q}(\zeta + 2\pi j)\ e^{i<x,\Re\zeta>} e^{-\ c\|x\|} d\zeta \right|$$
$$\leq (2\pi)^{-d} e^{-\ c\|x\|} \int_{S^d} \sum_{j\in\mathbb{Z}^d} \left|\widehat{\partial^\alpha \Lambda_q}(\zeta + 2\pi j)\right|\ d\zeta = C\ e^{-\ c\|x\|},$$

where C and c are positive and independent from x. The convergence of the series in the last integral is established in Lemma 4.2. □

Proposition 4.3 *Let q be real valued of degree $m > d$ and let $r = m - d - 1$. Then*

(i) *Λ_q is r-regular.*
(ii) *MRAs generated by localized q-elliptic splines are r-regular.*
(iii) *For any $x \in \mathbb{R}^d$,*

$$q(D)\Lambda_q(x) = \sum_{k\in\mathbb{Z}^d} a_k \delta(x-k),\ \textit{and}\ |a_k| \leq C \exp(-c\,\|k\|),\ k \in \mathbb{Z}^d, \quad (4.7)$$

where C and c are positive constants independent of k.

(iv) *If A is a dilation matrix enjoying (3.5) then Λ_q meets the refinement equation*

$$\Lambda_q(A^{-1}x) = \sum_{k\in\mathbb{Z}^d} l_k \Lambda_q(x-k),$$

where the so called refinement coefficients $l_k = \Lambda_q(A^{-1}k), k\in \mathbb{Z}^d$, *have exponential decay, as $\|k\| \to \infty$.*

Proof Properties (i) and (ii) are a transparent consequences of Proposition 4.2 and (iii) of Proposition 4.1. By (i) of Proposition 4.1, $\Lambda_q \in LE_q(\mathbb{R}^d)$, and $LE_q(\mathbb{R}^d) \subset E_q(\mathbb{R}^d)$ by Proposition 3.1, then by definition the equality in (4.7) holds for some sequence of constants $\{a_k\}_{k\in\mathbb{Z}^d}$. The proof of the exponential decay of $\{a_k\}_{k\in\mathbb{Z}^d}$ is analogous to the proof of (4.6) in Proposition 4.2. Property (iv) follows in view of the interpolation property (ii) of Proposition 4.1 and because of the exponential decay stated by Proposition 4.2. □

Corollary 4.1 *MRAs generated by polyharmonic spline-based scaling functions are r-regular.* □

We end the Section by generalizing some properties that are well-known in the polyharmonic case. More precisely, we prove that the cardinal interpolating problem for data of polynomial growth has solution in the class $E_q(\mathbb{R}^d)$, for any real-valued q, and Λ_q satisfies the Strang-Fix conditions of order $m-1$ [21], then it reproduces the polynomials up to the degree $m-1$. We remark that any polynomial of degree $s \leq m-1$ is in $E_q(\mathbb{R}^d)$.

Proposition 4.4 *Let q be real valued of degree $m > d$ and let $r = m - d - 1$. Suppose that $v = \{v_k\}_{k\in\mathbb{Z}^d}$ is a sequence of polynomial growth, then*

$$f_v(x) = \sum_{j\in\mathbb{Z}^d} v_j \Lambda_q(x-j), \tag{4.8}$$

is a tempered distribution in the class $\mathcal{C}^r(\mathbb{R}^d)$, f_v is in $E_q(\mathbb{R}^d)$, and $f_v(k) = v_k$ for all k. Moreover, for any polynomial P of degree $s \leq m-1$

$$P(x) = \sum_{k\in\mathbb{Z}^d} P(k)\Lambda_q(x-k),\ x \in \mathbb{R}^d, \tag{4.9}$$

Proof Since the sequence v has a polynomial growth, $\|v_k\|_\infty \leq K\|k\|_\infty^N$ for some $N \in \mathbb{N}$. According to Proposition 4.2, $|\partial^\alpha \Lambda_q(x)|_\infty \leq C\|x\|_\infty^{-(N+d+1)}$ for all α such that $|\alpha| \leq r$, and all the series

$$\sum_{j\in\mathbb{Z}^d} v_j \partial^\alpha \Lambda_q(x-j) = \partial^\alpha \left(\sum_{j\in\mathbb{Z}^d} v_j \Lambda_q(x-j)\right), \quad |\alpha| \leq r,$$

are absolutely and uniformly continuous on any compact subset of $\mathbb{R}^d$. Then f_v is a tempered distribution in the class $\mathcal{C}^r(\mathbb{R}^d)$, with $r = m-d-1$. Clearly, $f_v(k) = v_k$ for all k, by (ii) of Proposition 4.1. Moreover

$$\begin{aligned} q(D)f_v(x) &= \sum_{l\in\mathbb{Z}^d} v_l q(D)\Lambda_q(x-l) = \sum_{l\in\mathbb{Z}^d} v_l \sum_{k\in\mathbb{Z}^d} a_k \delta(x-l-k) \\ &=_{l+k=t} \sum_{l\in\mathbb{Z}^d}\left(\sum_{t\in\mathbb{Z}^d} v_l a_{t-l}\right)\delta(x-t) = \sum_{t\in\mathbb{Z}^d} d_t \delta(x-t), \end{aligned}$$

where the equalities hold because $\{a_k\}_{k\in\mathbb{Z}^d}$ has exponential decay (see (4.7)). Thus f_ν is in $E_q(\mathbb{R}^d)$.

If $\omega \in \mathbb{T}^d$, and we can write $\widehat{\Lambda}_q$ in the form $\widehat{\Lambda}_q(\omega) = [1 + q(\omega)F(\omega)]^{-1}$, where F is defined as in (4.4). $\widehat{\Lambda}_q$ is analytic, then by a Taylor expansion about $\omega = 0$ we get

$$\widehat{\Lambda}_q(\omega) - 1 = O(\|\omega\|_\infty^m), \quad \omega \to 0.$$

Making a Taylor expansion of (4.1) about $\omega = 2k\pi, k \neq 0$ we get

$$\widehat{\Lambda}_q(\omega) = O(\|\omega - 2k\pi\|_\infty^m), \quad \omega \to 2k\pi,\ \text{for all } k \in \mathbb{Z}^d \setminus \{0\}.$$

Then Λ_q satisfies the Strang-Fix conditions of order $m-1$ [21] and we get (4.9). □

5 Prewavelets

5.1 Preliminary results

Let $\mathcal{V} = \{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ be a MRA of $L^2(\mathbb{R}^d)$ associated with $(\mathbb{Z}^d, A)$. For any $j \in \mathbb{Z}$, the wavelet space $\mathcal{W}_j$, is the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$,

$$\mathcal{W}_j := \mathcal{V}_{j+1} \ominus \mathcal{V}_j, \; j \in \mathbb{Z}.$$

It is clear that the space $\mathcal{W}_j$ is the A^j-dilate of $\mathcal{W}_0$, $j \in \mathbb{Z}$. The spaces $\mathcal{W}_j$, $j \in \mathbb{Z}$ are mutually orthogonal, and

$$L^2(\mathbb{R}^d) = \oplus \mathcal{W}_j.$$

Given a finite set of functions $\Psi = \{\psi^\nu\}_{\nu\in I}$ from $L^2(\mathbb{R}^d)$, we say that Ψ has *ℓ^2-stable integer translates,* or more simply that Ψ is *ℓ^2-stable,* if there exist constants $0 < M_1 \le M_2 < +\infty$ such that

$$M_1 \|h\|_2^2 \le \left\| \sum_{\nu\in I} \sum_{k\in\mathbb{Z}^d} h_k^\nu \psi^\nu(\cdot - k) \right\|_2^2 \le M_2 \|h\|_2^2, \tag{5.1}$$

for all $h = \left\{h^\nu = \{h_k^\nu\}_{k\in\mathbb{Z}^d}\right\}_{\nu\in I} \in \ell^2(\mathbb{Z}^d)$.
Clearly, $\Psi = \{\psi^\nu\}_{\nu\in I}$ is ℓ^2-stable if and only if the family of functions $\{\psi^\nu(\cdot - k)\}_{k\in\mathbb{Z}^d, \nu\in I}$ is a Riesz basis of the closure in $L^2(\mathbb{R}^d)$ of its linear span S. In this case we say that $\Psi = \{\psi^\nu\}_{\nu\in I}$ *provides* a Riesz basis of S.
We use the following notations for the dilations and shifts of a function f

$$f_{j,k} := p^{j/2} f(A^j \cdot -k), \; k \in \mathbb{Z}^d, \; j \in \mathbb{Z}, \tag{5.2}$$

where here and in the sequel $p := |\det A|$.
A finite set $\Psi = \{\psi^\nu\}_{\nu\in I}$ of functions in $L^2(\mathbb{R}^d)$ is called set of wavelets if

$$\mathcal{R} = \left\{\psi_{j,k}^\nu\right\}_{\nu\in I, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$$

is a Riesz basis of $L^2(\mathbb{R}^d)$ (see e.g. [2]). For every MRA with dilation A there exists an associate wavelet set consisting of $p - 1$ elements [24]. In order to find a set of wavelets, it suffices to find a set of $p - 1$ functions Ψ in $\mathcal{W}_0$ which provides a Riesz basis of $\mathcal{W}_0$.

When the MRA $\mathcal{V} = \{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ is generated by a continuous scaling function then $\cup_{j\in\mathbb{Z}} \mathcal{V}_j \subset \mathcal{C}^0(\mathbb{R}^d)$ and we can give the following characterization of a set of wavelets.

Proposition 5.1 *Let $\mathcal{V} = \{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ be a MRA of $L^2(\mathbb{R}^d)$ with dilation A, generated by a continuous scaling function and let $\Psi = \{\psi^\nu\}_{\nu\in I}$ be a finite set of functions in $\mathcal{W}_0$. Then Ψ provides a Riesz basis of $\mathcal{W}_0$ if and only if $\sharp\Psi = p - 1$, and Ψ has ℓ^2-stable integer translates.*

Proof We premise the following argument. Let $\phi \in \mathcal{V}_0$ be a continuous scaling function generating the MRA $\mathcal{V}$. Since $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is a Riesz basis of $\mathcal{V}_0$, then the family $\left\{\phi_{1,k} = |\det A|^{1/2}\, \phi(A\cdot -k)\right\}_{k\in\mathbb{Z}^d}$ is a Riesz basis of $\mathcal{V}_1$. Since each $k \in \mathbb{Z}^d$ can be expressed in the unique form $k = As+\lambda$, where $s \in \mathbb{Z}^d$ and $\lambda \in \digamma$, then if we define $\phi^\lambda(x) := \phi(Ax+\lambda), \lambda \in \digamma$ it is clear that $\left\{\phi^\lambda\right\}_{\lambda\in\digamma}$ is a set of p functions which provides a Riesz basis of $\mathcal{V}_1$. We consider a set $\Psi = \{\psi^\nu\}_{\nu\in I}$ of functions from $\mathcal{W}_0$, and let $\Phi = \{\phi\} \cup \Psi$. We have $\Phi \subset \mathcal{V}_1$. According to [2, pag.33], since $\Phi \subset \mathcal{C}^0(\mathbb{R}^d)$, Φ provides a Riesz basis of $\mathcal{V}_1$ if and only if $\sharp\Phi = \sharp\digamma = p$ and Φ is ℓ^2-stable.

If Ψ provides a Riesz basis of $\mathcal{W}_0$, then Ψ is ℓ^2-stable and $\Phi = \{\phi\} \cup \Psi$ is ℓ^2-stable too. Since ϕ provides a Riesz basis of $\mathcal{V}_0$, it follows that Φ provides a Riesz basis of $\mathcal{V}_1$. By the previous argument, $\sharp\Phi = p$. But $\phi \notin \Psi$, then $\sharp\Psi = p-1$.

Viceversa, if $\Psi \subset \mathcal{W}_0$, with $\sharp\Psi = p-1$ and Ψ is ℓ^2-stable, then, by definition, Ψ provides a Riesz basis of

$$S = clos_{L^2(\mathbb{R}^d)}\text{span}\{\psi^\nu(\cdot - k), k \in \mathbb{Z}^d, \nu \in I\},$$

Since $\phi \notin \mathcal{W}_0$, then $\Phi = \{\phi\} \cup \Psi$ consists in p functions in $\mathcal{V}_1 \subset \mathcal{C}^0(\mathbb{R}^d)$ which are ℓ^2-stable. By the previous argument, Φ provides a Riesz basis of $\mathcal{V}_1$. But again, $\phi \notin \mathcal{W}_0$. Then $S = \mathcal{W}_0$. □

In view of the previous Proposition, and remembering that $\sharp\digamma' = p-1$, we number the elements of a family of wavelets $\Psi \subset \mathcal{W}_0$ by using the coset representative symbology, that is, $\Psi = \{\psi^\lambda\}_{\lambda\in\digamma'}$.

In the following, we provide a sufficient condition to establish that a family $\Psi = \{\psi^\lambda\}_{\lambda\in\digamma'}$ is ℓ^2-stable whenever the functions $\psi^\lambda, \lambda \in \digamma'$, are *generated* by one single function ψ according to the following definition:

$$\psi^\lambda := \psi(\cdot - A^{-1}\lambda),\ \lambda \in \digamma'. \tag{5.3}$$

We denote by $\mathcal{L}^2(\mathbb{R}^d)$ the linear space of all functions f on $\mathbb{R}^d$ for which

$$\int_{[0,1)^d} \left(\sum_{k\in\mathbb{Z}^d} |f(x-k)| \right)^2 dx < \infty.$$

If $f \in L^2(\mathbb{R}^d)$ is r-regular, $r \in \mathbb{N}$, then $f \in \mathcal{L}^2(\mathbb{R}^d)$.

Proposition 5.2 *Let $\psi \in \mathcal{L}^2(\mathbb{R}^d)$ such that $\widehat{\psi}(\omega) = 0$ if and only if $\omega = 2\pi Bk, k \in \mathbb{Z}^d$. Then*

(i) *ψ has ℓ^2-stable integer translates;*
(ii) *the family $\Psi = \{\psi^\lambda\}_{\lambda\in\digamma'}$ of functions defined as in (5.3) has ℓ^2-stable integer translates.*

Proof It is known that ψ has ℓ^2-stable integer translates if and only if

$$\sup_{l\in\mathbb{R}^d}\left|\widehat{\psi}(\omega+2\pi l)\right|>0 \quad \text{for all} \quad \omega\in\mathbb{R}^d$$

(see e.g. [9, Theorem 3.5]). We have to show that there is no $\omega\in\mathbb{R}^d$ such that

$$\widehat{\psi}(\omega+2\pi l)=0,\ \text{for all } l\in\mathbb{Z}^d. \tag{5.4}$$

By hypothesis, $\widehat{\psi}(\omega)=0$, iff $\omega=2\pi Bk, k\in\mathbb{Z}^d$. Each $l\in\mathbb{Z}^d$ can be expressed in the form $l=Bk+\mu$, $l\in\mathbb{Z}^d, \mu\in\Omega$, where Ω is a complete set of representatives of the distinct cosets of $\mathbb{Z}^d/B\mathbb{Z}^d$, containing 0. Thus (5.4) is equivalent to show that there is no $\omega\in\mathbb{R}^d$ such that $\widehat{\psi}(\omega+2\pi Bk+2\pi\mu)=0$, for all $k\in\mathbb{Z}^d$, and $\mu\in\Omega$. Chose $k=0$. Then it is enough to prove that there is no $\omega\in\mathbb{R}^d$ such that

$$\widehat{\psi}(\omega+2\pi\mu)=0,\ \text{for all } \mu\in\Omega.$$

Observe that there is at most one $\mu\in\Omega$ such that $\widehat{\psi}(\omega+2\pi\mu)=0$. Indeed, if there were two distinct values $\mu',\mu''\in\Omega$, we would have $\mu'-\mu''=Bl$, for some $l\in\mathbb{Z}^d$ which is impossible unless $\mu'=\mu''$. Since $\sharp\Omega=|\det B|\geq 2$, we get a contradiction.

If $\psi\in\mathcal{L}^2(\mathbb{R}^d)$, clearly $\psi^\lambda\in\mathcal{L}^2(\mathbb{R}^d)$, $\lambda\in F'$ as well. Then, in order to prove (ii) we use [9, Theorem 4.1,(i)]. Precisely, we prove that the sequences $(\widehat{\psi^\lambda}(\omega+2\pi l))_{l\in\mathbb{Z}^d}, \lambda\in F'$ are linearly independent. We must show that there is no $\omega\in\mathbb{R}^d$ and $c=\{c_\lambda\}_{\lambda\in F'}\neq 0$ such that

$$\sum_{\lambda\in F'}c_\lambda\widehat{\psi^\lambda}(\omega+2\pi l)=0 \quad \text{for all } l\in\mathbb{Z}^d. \tag{5.5}$$

Since $\widehat{\psi^\lambda}(\omega)=\widehat{\psi}(\omega)\ e^{-i<\omega,A^{-1}\lambda>}$, and every $l\in\mathbb{Z}^d$ can be expressed in the form $l=Bk+\mu$, where $k\in\mathbb{Z}^d$ and $\mu\in\Omega$, it follows that (5.5) is equivalent to

$$\left(\sum_{\lambda\in F'}\left(c_\lambda\ e^{-i<\omega,A^{-1}\lambda>}\right)\left(e^{-i<2\pi\mu,A^{-1}\lambda>}\right)\right)\widehat{\psi}(\omega+2\pi Bk+2\pi\mu)=0, \tag{5.6}$$

for all $k\in\mathbb{Z}^d$ and $\mu\in\Omega$. By hypothesis, $\widehat{\psi}(\omega)=0$ if and only if $\omega=2\pi Bl$, $l\in\mathbb{Z}^d$. We choose $k=0$ into (5.6). As before, we observe that there is at most one $\mu_0\in\Omega$ such that $\widehat{\psi}(\omega+2\pi\mu_0)=0$. Hence

$$\sum_{\lambda\in F'}\left(c_\lambda\ e^{-i<\omega,A^{-1}\lambda>}\right)\left(e^{-i<2\pi\mu,A^{-1}\lambda>}\right)=0, \quad \text{for all } \mu\in\Omega\setminus\{\mu_0\}\,.$$

We finish the proof by proving that the system has only the null solution $c=0$, which is a contradiction. In fact, for any $\mu_0\in\Omega$ the matrix

$$H=\left[e^{-i<2\pi\mu,A^{-1}\lambda>}\right]_{\mu\in\Omega\setminus\{\mu_0\},\lambda\in F'}$$

is non singular. To this end, let us consider the matrix

$$J = \left[e^{-i<2\pi\mu, A^{-1}\lambda>} \right]_{\mu\in\Omega,\lambda\in F} .$$

It is known that $\frac{1}{\sqrt{p}} J$ is a unitary matrix (see for example [6, Lemma 2.3]), and so $J^{-1} = p^{-1} J^T$. Since every element of J is nonzero, then every element of J^{-1} is nonzero too. This is equivalent to say that every minor of J of order $p-1$ is nonzero as well. In particular, the matrix H is nonsingular for any $\mu_0 \in \Omega$. □

We can join Proposition 5.2 with Proposition 5.1 to state the following useful result.

Proposition 5.3 *Let $\psi \in \mathcal{L}^2(\mathbb{R}^d)$ such that $\widehat{\psi}(\omega) = 0$ if and only if $\omega = 2\pi Bk, k \in \mathbb{Z}^d$. Then the family $\Psi = \{\psi^\lambda\}_{\lambda\in F'}$ of functions defined as $\psi^\lambda := \psi(\cdot - A^{-1}\lambda)$, $\lambda \in F'$ provides a Riesz basis of W_0 if and only if $\Psi \subset W_0$. In this case Ψ is a set of wavelets.* □

It is well known that corresponding to a Riesz basis $\mathcal{R}$, there is a unique dual Riesz basis $\widetilde{\mathcal{R}}$ relative to $\mathcal{R}$. If $\widetilde{\mathcal{R}}$ is of the form $\widetilde{\mathcal{R}} = \left\{ \widetilde{\psi}^\lambda_{j,k} \right\}_{\lambda\in F', j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ we call $\widetilde{\Psi} = \left\{ \widetilde{\psi}^\lambda \right\}_{\lambda\in F'}$ the dual set of Ψ and the pair $(\Psi, \widetilde{\Psi})$ is called *wavelet system*, in the sense that every function $f \in L^2(\mathbb{R}^d)$ can be obtained equivalently from $\mathcal{R}$ or $\widetilde{\mathcal{R}}$:

$$\begin{aligned} f(x) &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{\lambda\in F'} \left(f, \widetilde{\psi}^\lambda_{j,k} \right) \psi^\lambda_{j,k}(x) \\ &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{\lambda\in F'} \left(f, \psi^\lambda_{j,k} \right) \widetilde{\psi}^\lambda_{j,k}(x). \end{aligned} \tag{5.7}$$

Finally, we remark that any two functions in $\mathcal{R}$, as well as two functions in $\widetilde{\mathcal{R}}$, are orthogonal across different levels $j, l \in \mathbb{Z}$. However, two functions in the same level may be not orthogonal. In this case the functions in $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are called *prewavelets.*

5.2 q-elliptic r-regular prewavelets and duals

As usual, let q be an elliptic and homogeneous polynomial of degree $m > d$ and A be a dilation for $\mathbb{Z}^d$ which enjoys (3.5). Let $\mathcal{V} = \{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ be a MRA of $L^2(\mathbb{R}^d)$ associated with $(\mathbb{Z}^d, A)$ generated by a scaling function $\phi \in LE_q(\mathbb{R}^d)$. We underline that q is not required to be real-valued and thus the MRA $\mathcal{V}$ may not be r-regular. Nonetheless, we can generate a complex non-separable prewavelet system $(\Psi, \widetilde{\Psi})$ enjoying the very desirable properties of being r-regular and of having a certain of number vanishing moments.

Let us consider the polynomial

$$|q|^2 = q\overline{q},$$

which is real-valued, homogeneous and elliptic, of degree $2m$. We make use of the Lagrange function $\Lambda_{|q|^2}$ in $LE_{|q|^2}(\mathbb{R}^d)$ to define a finite set $\Psi = \{\psi^\lambda\}_{\lambda\in\digamma'}$ of r-regular functions which provides a Riesz basis of the space $\mathcal{W}_0$. Thanks to the structure of the wavelet spaces $\mathcal{W}_j$, $\mathcal{R} = \{\psi^\lambda_{j,k}\}_{\lambda\in\digamma',k\in\mathbb{Z}^d,j\in\mathbb{Z}}$ is a Riesz basis of $L^2(\mathbb{R}^d)$.

Proposition 5.4 *Let $\mathcal{V} = \{\mathcal{V}_j\}_{j\in\mathbb{Z}^d}$ be a MRA of $L^2(\mathbb{R}^d)$ associated with $(\mathbb{Z}^d, A)$, and generated by $\phi \in LE_q(\mathbb{R}^d)$. Let ψ and ψ^λ be defined as follows:*

$$\psi := \overline{q}(D)\ \Lambda_{|q|^2}(A\cdot), \tag{5.8}$$

$$\psi^\lambda := \psi(\cdot - A^{-1}\lambda), \quad \lambda \in \digamma'. \tag{5.9}$$

Then ψ is r-regular and the set $\Psi := \{\psi^\lambda\}_{\lambda\in\digamma'}$ provides an r-regular Riesz basis of $\mathcal{W}_0$, with $r = m - d - 1$.

Proof The polynomial $|q|^2$ is real valued, then by Proposition 4.3 $\Lambda_{|q|^2}$ is in the class $\mathcal{C}^{2m-d-1}(\mathbb{R}^d)$ and it is $(2m-d-1)$-regular. Since $q(D)$ is a differential operator of order m, it follows that ψ and consequently $\psi^\lambda, \lambda \in \digamma'$, are $(m-d-1)$-regular. In particular, ψ is in $\mathcal{L}^2(\mathbb{R}^d)$. The Fourier transform of ψ is given by

$$\widehat{\psi}(\omega) = i^m p^{-1}\overline{q}(B^{-1}\omega)\widehat{\Lambda}_{|q|^2}(B^{-1}\omega). \tag{5.10}$$

Observe that $\widehat{\psi}(\xi) = 0$, iff $\xi = 2\pi Bk, k \in \mathbb{Z}^d$. This follows from (5.10), since $q(B^{-1}\xi) = 0$ iff $\xi = 0$, and $\widehat{\Lambda}_{|q|^2}(B^{-1}\xi) = 0$ iff $\xi = 2\pi Bk, k \in \mathbb{Z}^d \setminus \{0\}$. Then ψ satisfies the hypotheses of Proposition 5.3 and, in order to prove that Ψ provides a Riesz basis of $\mathcal{W}_0$, we have only to show that $\Psi \subset \mathcal{W}_0$. First, let us show that ψ is in $\mathcal{V}_1$, and consequently $\psi^\lambda \in \mathcal{V}_1, \lambda \in \digamma'$. Let $\widehat{\phi} = \widehat{T}q^{-1}$. According to (4.1) we can write

$$\begin{aligned}\widehat{\psi}(\omega) &= i^m p^{-1}\overline{q}(B^{-1}\omega)\frac{|q|^{-2}(B^{-1}\omega)}{\sum_{k\in\mathbb{Z}^d}|q|^{-2}(B^{-1}\omega+2k\pi)} \\ &= i^m p^{-1}\frac{\widehat{T}(B^{-1}\omega)\ q^{-1}(B^{-1}\omega)}{\widehat{T}(B^{-1}\omega)\ \sum_{k\in\mathbb{Z}^d}|q|^{-2}(B^{-1}\omega+2k\pi)}, \quad \omega \in \mathbb{R}^d.\end{aligned}$$

That is,

$$\widehat{\psi}(\omega) = i^m p^{-1}S(B^{-1}\omega)\widehat{\phi}(B^{-1}\omega), \quad \omega \in \mathbb{R}^d, \tag{5.11}$$

with

$$S(\omega) = \frac{1}{\sum_{k\in\mathbb{Z}^d}\widehat{T}(\omega+2k\pi)\,|q|^{-2}(\omega+2k\pi)}, \quad \omega \in \mathbb{R}^d.$$

It is now sufficient to prove that $S(\omega)$ is in $L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$. We observe that there is no $\omega \in \mathbb{R}^d$ such that the equality $\widehat{T}(\omega+2k\pi)\,|q|^{-2}(\omega+2k\pi) = 0$ is satisfied for all $k \in \mathbb{Z}^d$. Since $\widehat{T}\,|q|^{-2} = \widehat{\phi}\overline{q}^{-1}$, this is equivalent to say that there is no $\omega \in \mathbb{R}^d$ such that

$$\widehat{\phi}(\omega + 2\pi k) = 0 \quad \text{for all } k \in \mathbb{Z}^d. \tag{5.12}$$

To the contrary, suppose that there exists such an $\omega \in \mathbb{R}^d$, but then $\widehat{T}(\omega + 2\pi k) = 0$ for all $k \in \mathbb{Z}^d$. Let $l \in \mathbb{Z}^d$ be such that $\|\omega + 2\pi l\|_\infty < \pi$. By (A1) of Definition 3.2 this implies $\omega + 2\pi l = 0$, i.e. $\omega = -2\pi l$ for such a l. Choosing $k = l$ in (5.12), we get $\widehat{\phi}(0) = 0$, which is a contradiction. Thus, $S(\omega)$ is in $L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$.

The following equalities show that ψ^λ is orthogonal to any elliptic spline f in $\mathcal{V}_0$:

$$\begin{aligned}(f, \psi^\lambda) &= (f, \overline{q}(D)\Lambda_{|q|^2}(A\cdot - \lambda)) = (q(D)f, \Lambda_{|q|^2}(A\cdot - \lambda)) \\ &= (\sum_{k\in\mathbb{Z}^d} c_k\delta(\cdot - k), \Lambda_{|q|^2}(A\cdot - \lambda)) = \sum_{k\in\mathbb{Z}^d} c_k\Lambda_{|q|^2}(Ak - \lambda) = 0,\end{aligned}$$

where the last equality holds since $\lambda \neq Ak$, for all $k \in \mathbb{Z}^d$ and $\lambda \in \Gamma'$. Recalling that $\mathcal{W}_0 = \mathcal{V}_1 \ominus \mathcal{V}_0$, we have proved that $\Psi \subset \mathcal{W}_0$. □

We note that the family Ψ introduced in Proposition 5.4 is a generalization of the one given in [14] with $A = 2I$. However, in that paper the Lagrange function is defined in dependence on the scaling function, that is not our case. Moreover, the regularity results here stated are new to our knowledge.

In the following Proposition the dual set $\widetilde{\Psi}$ of Ψ is defined. Note that this space is generated by the shifts of one function $\widetilde{\psi}$ whose definition in the Fourier domain depends only on the Fourier transform of the generator ψ.

Proposition 5.5 *Let ψ and $\psi^\lambda, \lambda \in \Gamma'$ be defined as in Proposition 5.4. Let $\widetilde{\psi}$ be defined in the Fourier domain as follows:*

$$\widehat{\widetilde{\psi}}(\omega) := \frac{\widehat{\psi}(\omega)}{\sum_{k\in\mathbb{Z}^d} \left|\widehat{\psi}(\omega + 2k\pi)\right|^2}, \tag{5.13}$$

and let

$$\widetilde{\psi}^\lambda := \widetilde{\psi}(\cdot - A^{-1}\lambda), \;\; \lambda \in \Gamma'. \tag{5.14}$$

Then $\widetilde{\psi}$ is r-regular and the set $\widetilde{\Psi} := \left\{\widetilde{\psi}^\lambda\right\}_{\lambda\in\Gamma'}$ provides a r-regular Riesz basis of $\mathcal{W}_0$ with $r = m - d - 1$. Moreover,

$$\left(\widetilde{\psi}^\lambda, \psi^\mu(\cdot - l)\right) = \delta_{\lambda,\mu}\delta_{l,0}, \tag{5.15}$$

for all $\lambda, \mu \in \Gamma'$ and $l \in \mathbb{Z}^d$.

Proof Let $r = m - d - 1$. The r-regularity of $\widetilde{\psi}^\lambda, \lambda \in \digamma'$ follows from the r-regularity of the generator $\widetilde{\psi}$. Let us show that $\widetilde{\psi}$ is r-regular. It is known that, if f is r-regular and g is defined by the condition $\widehat{g}(\omega) = \eta(\omega)\widehat{f}(\omega)$ for a $2\pi\mathbb{Z}^d$-periodic C^∞ function $\eta(\omega)$, then $g(x)$ is r-regular ([24], Lemma 5.13). Let us consider (5.8) defining ψ. By Proposition 5.4, ψ is r-regular. If we look at (5.13) which defines $\widehat{\widetilde{\psi}}$, it is clear that it suffices to show that the function

$$\eta(\omega) := \left[\sum_{k\in\mathbb{Z}^d} \left|\widehat{\psi}(\omega + 2k\pi)\right|^2\right]^{-1} \tag{5.16}$$

is C^∞. Since $\psi \in L^1(\mathbb{R}^d)$, then $t(\omega) := \sum_{k\in\mathbb{Z}^d} \left|\widehat{\psi}(\omega + 2k\pi)\right|^2$ is C^∞ ([24], Corollary 5.14). Moreover, ψ satisfy the hypotheses of Proposition 5.2, then ψ has ℓ^2-stable integer translates, and it follows $t(\omega) > 0$ and $\eta(\omega) \in C^\infty(\mathbb{R}^d)$. So, $\widetilde{\psi}_q$ is r-regular.

Let us show that $\widetilde{\Psi}$ provides a Riesz basis of $\mathcal{W}_0$. We can apply Proposition 5.3. Since $\widetilde{\psi}$ is r-regular, then $\widetilde{\psi}$ is in $\mathcal{L}^2(\mathbb{R}^d)$. We observe that $\widehat{\widetilde{\psi}}(\xi) = 0$ iff $\widehat{\psi}(\xi) = 0$, namely iff $\xi = 2\pi Bk, k \in \mathbb{Z}^d$ (see the proof of Proposition 5.4). Let us show that $\widetilde{\Psi} \subset \mathcal{W}_0$. Definition (5.13) implies that $\widetilde{\psi} = \sum_{k\in\mathbb{Z}^d} b_k\psi(\cdot - k)$, $\{b_k\}_{k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. Since ψ is in $\mathcal{V}_1$, $\widetilde{\psi}$ is in $\mathcal{V}_1$, as well as $\widetilde{\psi}^\lambda$, $\lambda \in \digamma'$. Moreover, by using (5.8), (5.14), $\widetilde{\psi}^\lambda = \sum_{k\in\mathbb{Z}^d} b_k\overline{q}(D)\Lambda_{|q|^2}(A\cdot - \lambda - Ak)$, and by using an argument similar to the one used for ψ^λ in Proposition 5.4, we can conclude that $\widetilde{\psi}^\lambda$ is orthogonal to any f in $\mathcal{V}_0$. Then $\widetilde{\Psi} \subset \mathcal{W}_0$.

Finally, we show the duality relation (5.15). By the Parseval identity, and writing $\mathbb{R}^d$ as $\bigcup_{k\in\mathbb{Z}^d}\left(\mathbb{T}^d + 2\pi k\right)$, we get

$$\begin{aligned}\left(\widetilde{\psi}^\lambda, \psi^\mu(\cdot - l)\right) &= (2\pi)^{-d}\int_{\mathbb{R}^d} \widehat{\widetilde{\psi}^\lambda}(\omega)\overline{\widehat{\psi^\mu}(\omega)}e^{i<l,\omega>}d\omega \\ &= (2\pi)^{-d}\int_{\mathbb{T}^d} e^{i<l,\omega>}\sum_{\beta\in\mathbb{Z}^d}\widehat{\widetilde{\psi}^\lambda}(\omega + 2\pi\beta)\overline{\widehat{\psi^\mu}(\omega + 2\pi\beta)}d\omega.\end{aligned}$$

By a straightforward calculation,

$$\sum_{\beta\in\mathbb{Z}^d}\widehat{\widetilde{\psi}^\lambda}(\omega + 2\pi\beta)\overline{\widehat{\psi^\mu}(\omega + 2\pi\beta)} = e^{-i<A^{-1}(\lambda-\mu),\omega>}.$$

If $\lambda = \mu$,

$$\left(\widetilde{\psi}^\lambda, \psi^\mu(\cdot - l)\right) = (2\pi)^{-d}\int_{\mathbb{T}^d} e^{i<l,\omega>}e^{-i<0,\omega>}d\omega = \delta_{l,0}.$$

If $\lambda \neq \mu$, since for all l, $Al + \mu$ is an integer different from λ,

$$\left(\widetilde{\psi}^\lambda, \psi^\mu(\cdot - l)\right) = (2\pi)^{-d}\int_{\mathbb{T}^d} e^{i<l,\omega>}e^{-i<A^{-1}(\lambda-\mu),\omega>}d\omega,$$

and by the change of variable $\xi = A^T\omega$, we get

$$\left(\widetilde{\psi}^\lambda, \psi^\mu(\cdot - l)\right) = (2\pi)^{-d} p^{-1} \int_{\mathbb{T}^d} e^{i<Al+\mu,\xi>} e^{-i<\lambda,\xi>} d\xi = 0.$$

□

According to the above construction, $\widetilde{\mathcal{R}} = \{\widetilde{\psi}_{j,k}^\lambda\}_{\lambda\in F',k\in\mathbb{Z}^d,j\in\mathbb{Z}}$ is the dual basis of $\mathcal{R} = \{\psi_{j,k}^\lambda\}_{\lambda\in F',k\in\mathbb{Z}^d,j\in\mathbb{Z}}$, and $(\Psi, \widetilde{\Psi})$ is a system of prewavelets consisting of r-regular functions, with $r = m - d - 1$.

5.3 Vanishing moments and approximation order

We conclude this Section by discussing the number of vanishing moments of the prewavelets and their duals. Vanishing moments is a very desirable property since it guarantees that the prewavelet representation (5.7) of piecewise smooth function is sparse. Moreover, it provides the approximation order.

The prewavelets and their duals are in the form $\psi^\lambda = \psi(\cdot - A^{-1}\lambda)$ and $\widetilde{\psi}^\lambda = \widetilde{\psi}(\cdot - A^{-1}\lambda)$, $\lambda \in F'$, then it suffices to consider the generators ψ and $\widetilde{\psi}$. Since ψ has ℓ^2-stable integer translates, $\widehat{\widetilde{\psi}}$, $\widehat{\psi}$, as defined in (5.10), (5.13) have the same behavior around $\omega = 0$ and the number of their vanishing moments is the same. Due to the r-regularity of ψ and $\widetilde{\psi}$, for each multi-integer α, $|\alpha| \geq 0$, their moments of order α are well defined. The function $\widehat{\psi}$ is analytic and $\eta(\omega)$, as defined in (5.16), belongs to $C^\infty(\mathbb{R}^d)$, then $\widehat{\widetilde{\psi}}$ is analytic too. Recalling that $\widehat{\Lambda}_{|q|^2}(0) = 1$ and using the bounds (2.2) for $\overline{q}$, we get

$$\widehat{\psi}(\omega),\ \widehat{\widetilde{\psi}}(\omega) = O(\|\omega\|^m), \quad \omega \to 0,$$

and

$$\int_{\mathbb{R}^d} x^\alpha \psi(x)\, dx = \partial^\alpha \widehat{\psi}(0) = 0, \quad \int_{\mathbb{R}^d} x^\alpha \widetilde{\psi}(x)\, dx = \partial^\alpha \widehat{\widetilde{\psi}}(0) = 0, \quad |\alpha| \leq m - 1.$$

Therefore, the system of prewavelets $(\Psi, \widetilde{\Psi})$ has $m - 1$ vanishing moments.

This prewavelet system satisfies the decay hypotheses in [20, Theorem 4], so that the vanishing moment property for $\widetilde{\psi}$ provides the corresponding approximation order for the prewavelet decomposition (12).

Proposition 5.6 *Let the prewavelet system $(\Psi, \widetilde{\Psi})$ be defined as in Proposition 5.4, and Proposition 5.5. Then for any function f in the Sobolev space $\mathcal{H}^m$, the decomposition (12) has approximation order m :*

$$\left\| f - \sum_{l<j, l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{\lambda\in F'} (f, \widetilde{\psi}_{l,k}^\lambda) \psi_{l,k}^\lambda \right\|_2 \leqslant C(\varepsilon) \|f\|_{\mathcal{H}^m} \left(\frac{1}{|\mu| - \varepsilon} \right)^{jm},$$

where μ is a minimal (in module) eigenvalue of A, $\varepsilon > 0$, $|\mu| - \varepsilon > 1$.

6 Prior definitions and examples

In order to highlight the results studied in this paper, we discuss some examples both related to prior definitions and to the more general Definition 3.2.

Elliptic splines were initially studied in some details in [14]. In this seminal paper, the authors provide dyadic MRAs and prewavelet decomposition of $L^2(\mathbb{R}^d)$ based on a class of functions ϕ with $\widehat{\phi} = \widehat{T}q^{-1}$ where the homogeneous and elliptic polynomial q of degree $m > d$ and $\widehat{T}$ satisfy the following conditions:

(B1) $\widehat{T}$ is a trigonometric polynomial

$$\widehat{T}(\omega) = \sum_{k \in \mathbb{Z}^d} c_k e^{-i<k,\omega>}, \quad \omega \in \mathbb{R}^d$$

such that $\widehat{T}(\omega) \neq 0$ for all $\omega \in \mathbb{T}^d \setminus \{0\}$;

(B2) $\widehat{T}(\omega) - q(\omega) = O(||\omega||_\infty^{m+1+s})$ for $\omega \to 0$ and some positive integer s.

Condition (B2) implies that $\widehat{\phi}(\omega) \to 1$ for $\omega \to 0$; moreover, it is proved that ϕ algebraically decays at infinity at least of order $d + 1 + s$, thus $\phi \in LE_q(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

We note that for complex q and real $\widehat{T}$ (B2) is never met. While, for example, if m is even and q is an elliptic polynomial of the form $q(\omega) = \sum_{|j|=m/2} a_j \omega^{2j}, a_j \in \mathbb{C}$, then a possible $\widehat{T}$ is $\sum_{|j|=m/2} a_j \sin^{2j}(\omega/2)$.

If $q(\omega) = \|\omega\|^m, m$ even, we get the polyharmonic B-splines. These functions are characterized by special choices of $\widehat{T}$ which lead to different orders of decay at infinity (see e.g. [3,16–19,22]). Actually, they are refinable functions with respect to any dilation matrix which is a similarity, that is, $A = \varrho A_0$ where A_0 is an orthogonal matrix and ϱ is a real number such that $|\det A|$ is an integer ≥ 2 (see e.g. [10]).

In [14], following the ideas given in [7] and [15], a prewavelet generator ψ_0 is defined in terms of a Lagrange function Λ explicitly involving the scaling function ϕ, namely

$$\widehat{\Lambda} = \frac{\left|\widehat{\phi}\right|^2}{\sum_{k \in \mathbb{Z}^d} \left|\widehat{\phi}(\cdot + 2\pi k)\right|^2} \quad \text{and} \quad \widehat{\psi}_0 = \frac{i^m}{|\det A|} \overline{q}\ \widehat{\Lambda}(B^{-1} \cdot). \tag{6.1}$$

Comparing (6.1) with (4.1) and (5.8), it is clear that $\widehat{\Lambda} \equiv \widehat{\Lambda}_{|q|^2}$ and $\widehat{\psi}_0 \equiv \widehat{\psi}$. Actually, (4.1) and (5.8) are a different formulation of (6.1). The advantage is that the prewavelet generator ψ just depends on q, and it is evident that its smoothness and decay are independent of $\widehat{T}$. Indeed, we have proven that this construction provides r-regular prewavelet system, with $r = m - d - 1$.

Formulas based on (6.1) have been used to generate MRAs and prewavelet decomposition of $L^2(\mathbb{R}^d)$ based on polyharmonic B-splines (see e.g. [1,4,18]). However, in these cases, q is real-valued and according to Proposition 4.3 we don't even need to define the periodic function $\widehat{T}$ to design both the MRA and the prewavelet decomposition. In fact, when q is real-valued, any localized q-elliptic spline generates the same MRA generated by the Lagrange q-elliptic spline (4.1). Moreover, the MRA and the prewavelets are r-regular. We illustrate the real-valued case in two dimensions by showing the Lagrange scaling function Λ_q and the prewavelet generator ψ.

In Fig. 6.1 we consider the polyharmonic case $q(\omega) = \|\omega\|^4$ and the prewavelet generator ψ associated with the quincunx dilation matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Since $|\det A| = 2$, only one prewavelet spans the spaces $\mathcal{W}_j$ and both Λ_q and ψ are 1-regular.

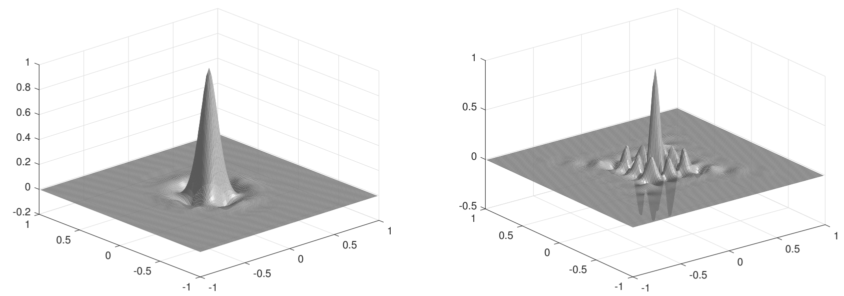

Fig. 6.1 $q(\omega) = \|\omega\|^4$. Left: Λ_q. Right: ψ.

In Fig. 6.2 we consider the elliptic polynomial $q(\omega_1, \omega_2) = 10\ \omega_1^6 + \omega_2^6$ and the prewavelet generator ψ associated with the dyadic dilation matrix. In this case, both Λ_q and the generator ψ are 3-regular.

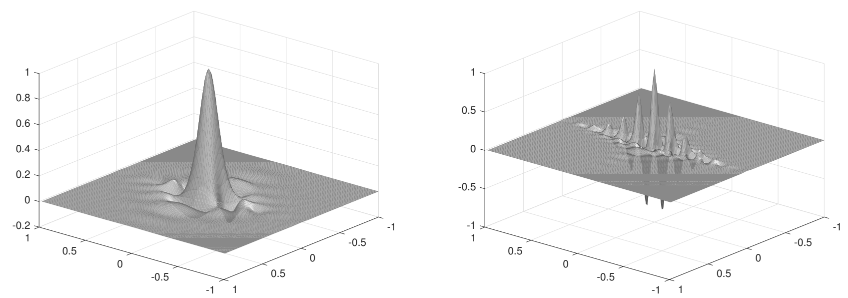

Fig. 6.2 $q(\omega_1, \omega_2) = 10\,\omega_1^6 + \omega_2^6$. Left: Λ_q. Right: ψ.

A different class of two dimensional q-elliptic splines scaling functions can be found, as a particular case, in [23] where according to (6.1), the authors provide a prewavelet generator ψ associated with complex isotropic polyharmonic B-splines in $\mathbb{R}^2$. These functions have Fourier transform $\widehat{\phi} = \widehat{T}\rho^{-1}$ where

$$\rho(\omega_1, \omega_2) = (\omega_1^2 + \omega_2^2)^{\alpha}(\omega_1 - i\omega_2)^N, \ \alpha \in \mathbb{R}_0^+, \ \ N \in \mathbb{N}, \ \ 2\alpha + N > 2, \quad (6.2)$$

$\widehat{T}$ is a $2\pi\mathbb{Z}^2$-periodic function such that $\widehat{T}(\omega) \neq 0$ for all $\omega \in \mathbb{T}^d \setminus \{0\}$ that enjoys

$$\left|\widehat{T}\right| = \nu^{\alpha+N/2}, \quad \text{with} \quad \nu(\omega) = \|\omega\|^2 + O(\|\omega\|^4), \quad \omega \to 0, \qquad (6.3)$$

where ν is a suitable trigonometric polynomial.

Clearly, $\left|\widehat{\phi}(\omega)\right| \to 1$ as $\omega \to 0$, and it is proven in [8] that these ϕ are scaling functions generating MRAs of $L^2(\mathbb{R}^2)$ associated with scaled rotation dilation matrices, i.e. similarities of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

When α is an integer, then ρ is a homogeneous elliptic polynomial of degree $2\alpha + N > 2$ and it is evident that ϕ is in $LE_\rho(\mathbb{R}^2)$. As it is observed in that paper, the prewavelet generator ψ is independent of any particular scaling function in $LE_\rho(\mathbb{R}^2)$. Here, according to Proposition 5.4, we can say that ψ is $(2\alpha + N - 3)$-regular. In Fig. 6.3 the real and imaginary part of ψ associated with $A = 2I$ are depicted for $\alpha = N = 1$.

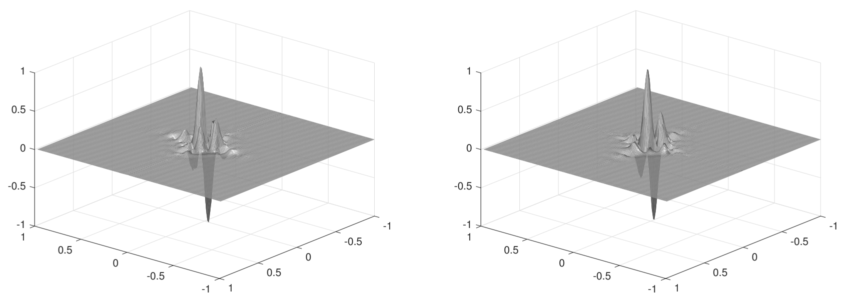

Fig. 6.3 $\alpha = n = N = 1$. From left to right, real and imaginary part of ψ.

We have to mention that the function (6.2) was initially proposed in [8] where several rotation covariant scaling functions are obtained with real valued $\widehat{T}$ enjoying (6.3). In that paper it is required that $\widehat{T}$ is a bounded, $2\pi\mathbb{Z}^2$-periodic function such that $\widehat{T}(\omega) \neq 0$ for all $\omega \in \mathbb{T}^2 \setminus \{0\}$, and that $\left|\widehat{\phi}(\omega)\right| \to c > 0$ as $\omega \to 0$. But the construction of the prewavelets follows a different approach from (6.1) and the resulting functions are bounded, uniformly continuous and slowly decay at infinity like $\|x\|^{-\gamma}$, where γ depends on $\widehat{T}$.

The class $LE_q(\mathbb{R}^d)$ especially motivates for complex polynomials q when the conditions given in the above mentioned literature are disregarded. In order to provide an appropriate example, we generalize (6.2), (6.3) for $\alpha \in \mathbb{N}_0$ integer. More precisely, we consider the two dimensional homogeneous and elliptic polynomials of degree $2\alpha + nN > 2$

$$q(\omega_1, \omega_2) := (\omega_1^2 + \omega_2^2)^\alpha \ (\omega_1^n - i\omega_2^n)^N, \quad \alpha \in \mathbb{N}_0, \ n, N \in \mathbb{N}, \tag{6.4}$$

and let $\widehat{T}$ be a $2\pi\mathbb{Z}^2$-periodic function such that $\widehat{T}(\omega) \neq 0$ for all $\omega \in \mathbb{T}^d \setminus \{0\}$ that enjoys

$$\left|\widehat{T}\right| = \nu^{\alpha + nN/2}, \quad \nu(\omega) = \|\omega\|^2 + O(\|\omega\|^4), \text{ as } \omega \to 0, \tag{6.5}$$

where ν is a trigonometric polynomial.

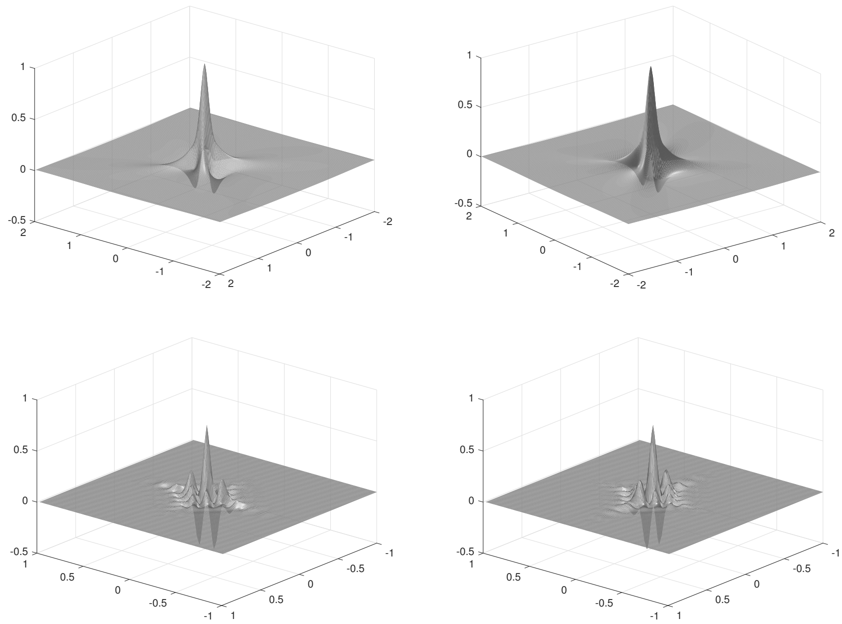

Fig. 6.4 $\alpha = N = 1, n = 2$. From left to right, top to bottom: real and imaginary part of ϕ, real and imaginary part of ψ.

Polyharmonic B-splines are obtained when $nN = 0$. If $n = 1$ we get the class introduced in [8,23] corresponding to $\alpha \in \mathbb{N}_0$.

We note that $\left|\widehat{\phi}\right|$ is positive apart from the grid $2\pi\mathbb{Z}^2$ and

$$\left|\widehat{\phi}(\omega)\right| = \frac{(\omega_1^2 + \omega_2^2)^{nN/2}}{(\omega_1^{2n} + \omega_2^{2n})^{N/2}} + \mathcal{O}(\|\omega\|^2), \text{ as } \omega \to 0.$$

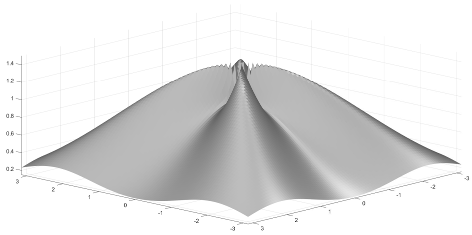

Fig. 6.5 $\alpha = N = 1, n = 2$. $|\widehat{\phi}(\omega)|$, $\omega \in \mathbb{T}^d$.

The case $n \geq 2$ is new and $\left|\widehat{\phi}(\omega)\right|$ has no limit as $\omega \to 0$, but condition (A2) of Definition 3.2 is satisfied, thus ϕ belong to $LE_q(\mathbb{R}^2)$.

In order to specify a particular scaling function in the class $LE_q(\mathbb{R}^2)$, we chose $\widehat{T}$ in the form (6.5) as follows,

$$\widehat{T}(\omega_1, \omega_2) = \left(\sin^2 \frac{\omega_1}{2} + \sin^2 \frac{\omega_2}{2}\right)^{\alpha + nN/2}.$$

If nN is even $\widehat{T}$ is a trigonometric polynomial. According to Proposition 5.4, when we define the prewavelet generator ψ by (5.8), ψ is r-regular with $r := 2\alpha + nN - 3$, and the set $\Psi := \left\{\psi^\lambda\right\}_{\lambda \in F'}$ provides r-regular Riesz basis of $\mathcal{W}_0$. In Fig. 6.4, we show for $\alpha = N = 1$ and $n = 2$ the real and imaginary part of the scaling function and of the prewavelet generator associated with $A = 2I$, which is 1-regular. The oscillating behavior in a neighborhood of the origin of $\left|\widehat{\phi}\right|$ is highlighted in Fig. 6.5.

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