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Nonlinear equations with lack of compactness

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Contents

1 Introduction

In the 18th century, with the work of mathematicians such as Euler, d'Alembert, Lagrange, and Laplace, the study of Partial Differential Equations (PDE's) started having a central role. In particular, this tool turned out to be extremely useful for analytically describing a wide range of phenomena arising in physical science. During the mid-19th century, with the work of many mathematicians including Riemann, PDE's also became a tool used to study problems originating from other areas of mathematics. This duality of theoretical aspects of PDE's and real applications was predicted for the first time by H. Poincare in [98] and arrived at the present days. On the one hand, he claimed that many problems arising in different areas (electricity, hydrodynamics, heat, magnetism, optics, elasticity) present very common features, and they can be treated using similar methods. On the other hand, he insisted on the importance of rigorous proofs, even if the models were an approximation of the reality, since he was convinced that the theory that would emerge from this study would have a significant impact on other branches of Mathematics. For instance, we can mention differential geometry, real analysis and functional analysis, topology, probabilistic models, algebraic geometry, chaos theory (the interested reader can consult [29] for a complete survey on the history of PDE's and on the interactions with other research fields).

One of the aspects that has a particular importance in the study of PDE's is the existence of solutions of nonlinear Partial Differential Equations. This research field has been extremely active in the last two centuries. A natural way to approach the problem turned out to be the so-called Variational Methods. The idea behind these techniques is to associate to the equation a functional. Choosing appropriately the functional, it is possible to establish a one-to-one correspondence between the critical points of the functional and the solutions of the PDE's. In this spirit, one of the main results in order to prove the existence of critical points is the Mountain Pass Theorem proved by A. Ambrosetti and P.H. Rabinowitz in their seminal paper [3]. This article and the ideas contained in it lead the way to the development of a sector of mathematics known in literature as Critical Point Theory.

At the end of the previous century, Analysis on non-Euclidean settings started to be an area under great development. This was due to numerous problems arising in Geometry and Physics that lead to the study of some PDE's set in particular on Riemannian Manifolds. As a consequence of that, in order to apply the strategies used in the Euclidean case, it was necessary to build a theory of Sobolev spaces on Riemannian manifolds and in this direction the contributions of T. Aubin and E. Hebey were very relevant. After that, the study of PDE's set in particular on Riemannian manifolds attracted the attention of many researchers, since they are usually quite challenging from a mathematical point of view and existing techniques are inadequate to solve them. On the other hand, if analytical tools are not effective, the geometry of the manifold may help to solve some issues and this make the problems taken under consideration deeply interesting.

Another very active research field in the last decades has been the study of Partial Differential Equations driven by non-local operators. It is well known that the value in a point of a local differential operator, such as the classical Laplacian, depends only on what happens in a neighbourhood of the point, as suggested by the name. Unlike them, the value of a non-local operator is influenced by what happen in the whole space. Because of this feature, non-local operators turned out to be extremely useful to model a wide class of situations in the real world. As a consequence of that, many researchers were attracted by these operators, and they started studying them. Undoubtedly, the most studied non-local operator is the fractional Laplacian and one of the first techniques to obtain existence of solutions and qualitative properties of fractional differential equations was proposed by L. Caffarelli and L. Silvestre in [33]. They showed that it is possible to derive a fractional differential equation from a local equation in higher dimension. More recently, with the publications of [103] and [104], mathematicians started studying these kinds of problem via Variational Methods without the Caffarelli-Silvestre extension.

In all the problems and techniques mentioned above, there is an issue that one usually has to face, and it is the *compactness*. Here, with compactness we mean that the Sobolev space in which we are looking for solutions of a given Partial Differential Equations is compactly embedded into the Lebesgue spaces. When this is not true, the compactness condition introduced by R. Palais and S. Smale, known as the Palais-Smale condition or PS for short, does not hold in general and standard variational methods can not be applied. Hence, one must rely on more sophisticated strategies which are still the subject of great interest and study today.

In this thesis we are going to present some results for some Partial Differential Equations, driven by fractional operators or set on a Riemannian manifold, in which for some reasons we have a loss of compactness, and the problem became demanding. The first problem we are going to analyze is the existence of solutions for the fractional Schrödinger equation with prescribed L^2-mass. Here the loss of compactness is caused by the invariance of $\mathbb{R}^N$ with respect to the non-compact group of translations. To solve the issue, we will use some Concentration-Compactness arguments introduced for the first time by P.L. Lions in [71] and [72]. The second equation we will take under examination is a fractional p-Kirchhoff type equation critical in the sense of Sobolev. The presence of the critical exponent prevents from having a functional associated to the problem that is sequentially weakly lower semicontinuous and that satisfies the Palais-Smale condition. The generalization to the fractional case of the second Concentration-Compactness Principle of P.L. Lions (see [73], [74]) will be crucial to carry out our analysis. After these two problems, we will draw our attention to the Schrödinger equation set on Riemannian manifolds in two particular cases. The first one is on a non-compact Riemannian manifold with very general assumptions on the Ricci tensor, which are usually referred as asymptotically non-negative. In this case, we will deal the non-compactness of the manifold with a coercivity hypothesis on the potential in the differential operator. The second one is on a homogeneous Cartan-Hadamard manifold with a nonlinearity with an oscillating behaviour. Working on a Sobolev space where the functions have some "symmetries" will enable us to recover the compactness and prove the existence of infinitely many solutions. All the problems were studied in collaboration of my Ph.D. advisor Prof. Simone Secchi, Prof. Giovanni Molica Bisci from University of Urbino and Alessio Fiscella from University of Milano-Bicocca. The content of chapters 3, 4, 5 are published respectively in [14], [11], [13]. Chapter 6 was accepted for publication, and a paper version is available on [10].

2 Mathematical background

This chapter is devoted to introducing some mathematical concepts that will be useful throughout the thesis. We will give some information on fractional Sobolev spaces and on the fractional Laplace operator. After that, we will recall some rudiments on Riemannian geometry and on Sobolev spaces on manifolds.

2.1 Fractional Sobolev spaces

In this section we will recall some basic notions on fractional Sobolev spaces. We will present the topics without proofs, and we remind the reader to [41] and the references therein for a more detailed discussion.

We fix $s \in (0,1)$, an integer $N > 2s$ and $p \in [1,+\infty)$. We consider a general open set Ω in $\mathbb{R}^N$ (also non-smooth is allowed). We define the fractional Sobolev spaces $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \mid \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega\times\Omega) \right\}, \tag{2.1}$$

endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy \right)^{\frac{1}{p}}, \tag{2.2}$$

where

$$[u]_{W^{s,p}(\Omega)} := \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy \right)^{\frac{1}{p}} \tag{2.3}$$

is the so-called fractional Gagliardo seminorm of u.

The case $p = 2$ is very relevant and is somehow special, since the fractional Sobolev space $W^{s,2}(\Omega)$ turns out to be a Hilbert space, usually denoted by $H^s(\Omega)$, with scalar product

$$\langle u, v\rangle_{H^s(\Omega)} = \int_\Omega uv \, dx + \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{N+2s}} \, dx\, dy.$$

These spaces, introduced almost simultaneously, are a sort of intermediary spaces between $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

Analogously to the case in which s is an integer, it is possible to define a critical exponent that plays the same role in the embedding theorems. Namely, we define

$$p_s^* := \frac{Np}{N-sp},$$

and we have

Theorem 2.1. *Let* $s \in (0,1)$ *and* $p \in [1,\infty)$ *such that* $sp < N$*. Then there exist a positive constant* $C = C(N,p,s)$ *such that, for any* $u \in W^{s,p}(\mathbb{R}^N)$*, we have*

$$\|u\|_{p_s^*}^p \leq C \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy.$$

Consequently, the space $W^{s,p}(\mathbb{R}^N)$ *is continuously embedded in* $L^q(\mathbb{R}^N)$ *for any* $q \in [p, p_s^*]$*.*

Remark 2.2. The notation $\|\cdot\|_q$ will denote the classic norm on the Lebesgue space $L^q(\mathbb{R}^N)$. In the rest of the thesis we will also study some problems in bounded domains Ω, but since the functions can be extended equal to zero in $\mathbb{R}^N \setminus \Omega$ we will always use the same notation for the norm of $L^q(\Omega)$.

Requiring a hypothesis of regularity on the boundary Ω, it is possible to generalize the celebrated Rellich-Kondrachov Theorem for fractional Sobolev spaces.

Proposition 2.3. *Let* $s \in (0,1)$ *and* $p \in [1,\infty)$ *be such that* $sp < N$*. Let* $q \in [1, p_s^*)$*, let* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain for* $W^{s,p}(\Omega)$ *and let* $\mathfrak{F}$ *be a bounded subset of* $L^p(\Omega)$*. Suppose that*

$$\sup_{u \in \mathfrak{F}} \iint_{\Omega \times \Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dxdy < \infty.$$

Then $\mathfrak{F}$ *is pre-compact in* $L^q(\Omega)$*.*

Remark 2.4. The last Proposition tells us that $W^{s,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ compactly for all $q \in [1, p_s^*)$.

2.2 Fractional Laplacian

Fractional Sobolev spaces are strictly related to the fractional Laplacian operator. Before giving its definition, it is necessary to fix some notation.

We denote with

$$\mathcal{S}(\mathbb{R}^N) := \left\{ u \in C^\infty(\mathbb{R}^N) \mid \sup_{x \in \mathbb{R}^N} |x^\alpha D^\beta u(x)| < \infty \ \forall\ \alpha,\ \beta \in \mathbb{N}^N \right\}$$

where $C^\infty(\mathbb{R}^N)$ is the space of infinitely differentiable functions (functions that admits continuous derivative of any order) and α, β are multi-indexes, i.e.

$$\alpha = (\alpha_1, \ldots, \alpha_N),$$

with $\alpha_i \in \mathbb{N}$ and

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_N}.$$

With the symbol D^β we mean

$$D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_N^{\beta_N}}$$

where

$$|\beta|= \sum_{i=1}^{N} \beta_i.$$

Now, for every $u \in \mathcal{S}(\mathbb{R}^N)$ we can define the fractional Laplacian as

$$\begin{aligned}(-\Delta)^s u(x) &= C(N,s)P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy \\ &= C(N,s) \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy.\end{aligned}$$

Here $P.V.$ stands for "in the principal value sense" (as defined by the previous equation), and $C(N,s)$ is a dimensional constant that depends on N and s, precisely given by

$$C(N,s) = \left(\int \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}}\, d\zeta \right)^{-1}.$$

For our purposes, and since the parameter s is kept fixed in all the problems we are going to study in the next chapters, we will always work with a *rescaled* fractional operator, in such a way that $C(N,s)=1$.

At this point, the relation between the fractional Laplacian and the classical Laplace operator for $s=1$ may be not clear. This connection between the two operators is more clear using an approach via the Fourier transform. Indeed, for any $u \in \mathcal{S}(\mathbb{R}^N)$ an alternative definition for the fractional Laplacian is

$$(-\Delta)^s u = \mathscr{F}^{-1}\left(|\xi|^{2s}\mathscr{F}(u)\right),$$

where

$$\mathscr{F}(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\mathrm{i}\xi\cdot x} u(x)\, dx$$

is the usual Fourier transform, $\mathscr{F}^{-1}$ is its inverse and $\cdot$ is the scalar product in $\mathbb{R}^N$. It is possible to show that these two definitions of fractional Laplacian are equivalent for any $u \in \mathcal{S}(\mathbb{R}^N)$. Furthermore, using the second one and standard properties of the Fourier transform, it is straightforward to verify that when $s=1$ the fractional Laplacian and the Laplacian coincide. With this second definition via the Fourier transform, we also have the following Proposition that relates the fractional Gagliardo semi-norm with the L^2 norm of the operator.

Proposition 2.5. *Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^N)$ then*

$$[u]^2_{H^s(\mathbb{R}^N)} = 2C(N,s)^{-1} \|(-\Delta)^{\frac{s}{2}} u\|_2^2$$

As we did for the fractional Laplacian, if $p \neq 2$ we can also generalize the p-Laplace operator to the fractional case. More precisely, the p-fractional Laplacian can be defined up to a normalization constant as

$$(-\Delta_p)^s u(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy.$$

Unfortunately, if $p \neq 2$, it is not possible to find an equivalent definition utilizing the Fourier transform, so understating the relation of this operator with the classical p-Laplace operator is more intricate. We remind to [25] where the authors proved that

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy = C_1(N,s,p) \int_{\mathbb{R}^N} |\nabla u|^p\, dx$$

where C_1 is a positive constant that depends on N and p. It is also worth mentioning [39] where the reader can find a discussion on three different representations of the fractional p-Laplacian.

2.3 Elements of Riemannian geometry

This section is devoted to recalling some basic concepts of Riemannian geometry and to fix the notation. Throughout the thesis, we assume that the reader is already familiar with the basic definition and results on Riemannian geometry, so we will not go into detail about it. We remind the reader to the classical [42, 52, 53, 54] and [65] for a more in-depth discussion on these topics.

Let $(\mathcal{M}, g)$ be a d-dimensional Riemannian manifold where g is a $(0,2)$ positive definite tensor and g_{ij} are its component. We will denote the tangent space and the cotangent of $\mathcal{M}$ at a point $\sigma \in \mathcal{M}$ with $T_\sigma\mathcal{M}$ and $T^*_\sigma\mathcal{M}$ respectively. We recall that if $f\colon \mathcal{M} \to \mathcal{N}$, where $\mathcal{N}$ is a d'–manifold the differential of $Df_\sigma\colon T_\sigma\mathcal{M} \to T_{f(\sigma)}\mathcal{N}$ is defined as

$$Df_\sigma(v)(h) := v(h \circ f)$$

for all $h \in C^\infty(\mathcal{N})$ and $v \in T_\sigma\mathcal{M}$. From the notion of differential, if A is a covariant k-tensor field of M we can define a covariant k-tensor field f^*A on $\mathcal{M}$ defined as

$$(f^*A)_\sigma(v_1, ..., v_k) = A_{f(\sigma)}(Df_\sigma(v_1), \ldots, Df_\sigma(v_k))$$

for $v_1, ..., v_k \in T_\sigma\mathcal{M}$ called the pullback of A by f. If $\mathcal{N}$ is endowed with a metric $\tilde{g}$ we will say that f is an isometry if $f^*\tilde{g} = g$. It is straightforward to verify that requiring that$f$ is an isometry is equivalent to ask it preserves the scalar product, i.e.

$$\langle Df_\sigma(v_1), Df_\sigma(v_2)\rangle_{f(\sigma)} = \langle v_1, v_2\rangle_\sigma$$

for $v_1, v_2 \in T_\sigma\mathcal{M}$ where $\langle\cdot,\cdot\rangle_\sigma = g_\sigma(\cdot,\cdot)$. In the following, the group of all isometries $\varphi\colon \mathcal{M} \to \mathcal{M}$ will be denoted by $\mathrm{Isom}_g(\mathcal{M})$. If $\mathcal{S} \subset \mathcal{M}$ we can define

$$\mathrm{diam}(\mathcal{S}) := \sup\{d_g(\sigma_1, \sigma_2) \mid \sigma_1, \sigma_2 \in \mathcal{S}\},$$

where $d_g\colon \mathcal{M}\times\mathcal{M}\to[0,+\infty)$ is the geodesic distance associated to the Riemannian metric g. We will denote with ${}^g\nabla$ the Levi-Civita connection associated with the metric g. Fixed a chart, we will denote by ∂_{x_i} and dx^i the orthogonal frame of $T_\sigma\mathcal{M}$ and $T^*_\sigma\mathcal{M}$ respectively. From basic linear algebra, we have that

$$ {}^g\nabla_{\partial_{x_i}}\partial_{x_j} = \Gamma^k_{ij}\partial_{x_k} $$

where Γ^k_{ij} are the so-called Christoffel symbols, and we are assuming the Einstein summation convention. For a general (p,q) tensor T, we will denote with ${}^g\nabla T$ the covariant derivative of T induced by the Levi-Civita connection that is a $(p,q+1)$ tensor field that in local coordinate is

$$ ({}^g\nabla T)^{j_1\cdots j_q}_{i_1\cdots i_{p+1}} = ({}^g\nabla_{\partial_{x_{i_1}}}T)^{j_1\cdots j_q}_{i_2\cdots i_{p+1}} = \frac{\partial T^{j_1\cdots j_q}_{i_2\cdots i_{p+1}}}{\partial x_{i_1}} - \sum_{k=2}^{p+1}\Gamma^\alpha_{i_1 i_k}(T)^{j_1\cdots j_q}_{i_2\cdots i_{k-1}\alpha i_{k+1}\cdots i_{p+1}} + \sum_{k=1}^{q}\Gamma^{j_k}_{i_1\alpha}T^{j_1\cdots j_{k-1}\alpha j_{k+1}\cdots j_q}_{i_2\cdots i_{p+1}}. $$

In particular, given a function $u\in C^\infty(\mathcal{M})$ we denote by ${}^g\nabla^k u$ the k-th covariant derivative and by $|{}^g\nabla^k u|$ the norm that in local coordinates is defined as

$$ |{}^g\nabla^k u|^2 = g^{i_1 j_1}\cdots g^{i_k j_k}({}^g\nabla^k u)_{i_1\cdots i_k}({}^g\nabla^k u)_{j_1\cdots j_k}. $$

Observe that for $k=1$ and with the classical Euclidean metric δ_{ij} we obtain the standard norm of a vector in $\mathbb{R}^N$. When $k=1$ we will drop the dependence of k in the covariant derivative writing simply $|{}^g\nabla u|$.

Given a $(1,1)$ tensor field T that can be written as

$$ T = T^i_j\partial_{x_i}\otimes dx^j $$

we define the contraction as the usual trace

$$ \mathrm{C}(T) = \operatorname{tr} T = T^i_i. $$

and for a vector field X the divergence can be set as

$$ \operatorname{div} X := \mathrm{C}({}^g\nabla X). $$

We point out that, exploiting the isomorphism between the tangent space and the cotangent space induced by the metric at any point, it is possible to transform a $(2,0)$ tensor field in a $(1,1)$ tensor field and compute the contraction.

Let $u\in C^\infty(\mathcal{M})$. The Laplace-Beltrami operator that is defined as

$$ \Delta_g u := \operatorname{tr}({}^g\nabla\,{}^g\nabla u). $$

will be of particular relevance. It is possible to prove that in local coordinates this operator has the expression

$$\Delta_g u := \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial u}{\partial x^j} \right).$$

or

$$\Delta_g u = g^{ij} \left(\partial_{x_i} \partial_{x_j} u - \Gamma^k_{ij} \partial_{x_k} u \right)$$

using the Christoffel symbols. We emphasize that we have defined Δ_g with the "analyst's sign convention", so that $-\Delta_g$ coincides with $-\Delta$ in $\mathbb{R}^d$ with its flat metric. Finally, we recall that in local coordinates the Riemannian volume form can be expressed as

$$dv_g := \sqrt{\det(g)} dx_1 \wedge ... \wedge dx_d.$$

Once one has defined dv_g, it is possible to notice that it induces a measure on $\mathcal{M}$. Namely, if $\mathcal{S} \subset \mathcal{M}$ we have

$$\mathrm{Vol}_g(\mathcal{S}) := \int_{\mathcal{S}} dv_g.$$

2.4 Curvature

Given a Riemannian manifold $(\mathcal{M}, g)$, the Riemann curvature $(1,3)$-tensor field defined by

$$\mathrm{Riem}(X,Y)Z := {}^g\nabla_Y {}^g\nabla_X Z - {}^g\nabla_X {}^g\nabla_Y Z + {}^g\nabla_{[X,Y]} Z$$

where X, Y, Z are vector fields and $[\cdot,\cdot]$ denotes the Lie brackets. Observe that through the identification of the tangent space and the cotangent space, it is possible to see Riem as a $(0,4)$ tensor field. The idea lying behind this definition is to measures the non-commutativity of the covariant derivative, and a as a consequence of that, how far we are from being Euclidean. Despite this, the definition of the Riemann curvature tensor should be considered more or less formal, and for a more precise geometrical interpretation we rely on the notion of sectional curvature that we are going to introduce. Namely, point-wisely the sectional curvature is defined as

$$\mathrm{Sect}_\sigma(v_1, v_2) := \frac{\langle \mathrm{Riem}(v_1, v_2)v_1, v_2 \rangle_\sigma}{\langle v_1, v_1 \rangle_\sigma \langle v_2, v_2 \rangle_\sigma - \langle v_1, v_2 \rangle_\sigma^2}$$

for all $v_1, v_2 \in T_\sigma\mathcal{M}$. Multiple factors contribute to the importance of the sectional curvature. As anticipated, the first is the geometrical interpretation. Indeed, from the definition it is clear it is defined on two-dimensional subspaces of the tangent space, where it corresponds to the notion of Gaussian curvature. Secondly, it characterizes the manifold's curvature completely. In other words, the curvature tensor Riem is determined by the knowledge of Sect for all two-dimensional subspaces of the tangent.

At this point, we are ready to introduce a very important class of manifolds that will play a relevant role in this thesis.

Definition 2.6. A Cartan-Hadamard manifold is a Riemannian Manifold that is complete, simply connected and has everywhere non-positive sectional curvature. We also say that a Riemannian manifold $\mathcal{M}$ is homogeneous if for all $\sigma_1,\ \sigma_2 \in \mathcal{M}$ there is an isometry $\varphi \in \mathrm{Isom}_g(\mathcal{M})$ such that $\varphi(\sigma_1) = \sigma_2$.

These manifolds are very studied in differential geometry because of their remarkable properties. For instance, they are diffeomorphic to $\mathbb{R}^N$ by the Cartan-Hadamard Theorem. In addition to that, from the Hopf-Rinow Theorem it follows that every couple of points in a Cartan–Hadamard manifold could be connected by a unique geodesic line. Sometimes, requiring hypothesis on the Riemann curvature tensor and on the sectional curvature, turned out to be too restrictive. Then, it is necessary to further introduce a notion of curvature that a significant importance in many contexts such as the Sobolev Embedding Theorems. Moreover, some quantities appear with such frequency that they deserve to be named. The Ricci curvature tensor is defined by

$$\begin{aligned}\mathrm{Ric}_\sigma(v_1, v_2) :&= \mathrm{tr}\,(v_3 \mapsto \mathrm{Riem}(v_1, v_3)v_2) \\ &= \sum_{i=1}^{d} \langle \mathrm{Riem}(v_1, e_i)v_2, e_i\rangle_\sigma\end{aligned}$$

where $v_1, v_2, v_3 \in T_\sigma\mathcal{M}$ and $e_1, \ldots, e_d$ is an orthonormal frame for $T_\sigma\mathcal{M}$. The Ricci curvature tensor can be seen as $(0, 2)$ or $(1, 1)$-tensor field.

2.5 Sobolev spaces on Riemannian manifolds

This section is devoted to introducing some basic facts on the theory of Sobolev spaces on Riemannian manifold. Let $(\mathcal{M}, g)$ be a Riemannian manifold. We start defining the space

$$C_g^{k,p}(\mathcal{M}) := \left\{ u \in C^\infty(\mathcal{M}) \mid \int_{\mathcal{M}} |{}^g\nabla^j\, u|^p\, dv_g < \infty \ j = 1, \ldots, k \right\}$$

for $p \geq 1$ and $k \in \mathbb{N}$. On this space, we can define the following norm

$$\|u\|_{k,p} := \sum_{j=1}^{k} \left(\int_{\mathcal{M}} |{}^g\nabla^j\, u|^p\, dv_g \right)^{\frac{1}{p}}.$$

Now, we are ready to define the Sobolev space $H_g^{k,p}(\mathcal{M})$ as the closure of $C_g^{k,p}(\mathcal{M})$ with respect to the norm $\|\cdot\|_{k,p}$. When $\mathcal{M}$ is compact, it is possible to prove that $H_g^{k,p}(\mathcal{M})$ does not depend on the metric g, and all the results valid in the Euclidean case are still true in general. Since in this thesis we are not interested in compact manifolds, we will not go into details. On the other hand, if the manifold $\mathcal{M}$ is non-compact, strange phenomena may appear, and we have to require some assumptions on the curvature tensors.

Since the approach in the problem we are going to study in the next chapters will be variational, we need a Sobolev embedding theorem to have an energy functional well-defined also on the Lebesgue spaces. A result in this direction is the following. We will deal only with the case $k = 1$. Observe that when $p = 2$ we will drop the dependence of p in $H_g^{k,p}(\mathcal{M})$ writing simply $H_g^1(\mathcal{M})$. Next Theorem was proved for the first time by N. Th. Varopoulos in [110].

Theorem 2.7. *Let $(\mathcal{M}, g)$ be a smooth, complete d-Riemannian manifold with Ricci curvature bounded form below and such that*

$$\inf_{\sigma \in \mathcal{M}} \mathrm{Vol}_g \left(B_\sigma(1)\right) > 0$$

where

$$B_\sigma(1) := \{\xi \in \mathcal{M} \mid d_g(\xi, \sigma) < 1\}\,.$$

Then $H_g^{1,p}(\mathcal{M}) \hookrightarrow L^q(\mathcal{M})$ continuously where $1/p = 1/q - 1/d$.

Strengthening a bit the hypothesis on the curvature, it is possible to have a similar statement without requiring the lower bound for the volume of small balls. This result is contained in [55] and is due to D. Hoffman and J. Spruck (see also [54, Lemma 8.1 and Theorem 8.3])

Theorem 2.8. *Let $(\mathcal{M}, g)$ a smooth, complete, simply connected Riemannian manifold of non-positive sectional curvature. Then $H_g^{1,p}(\mathcal{M}) \hookrightarrow L^q(\mathcal{M})$ continuously where $1/p = 1/q - 1/d$.*

3 Normalized solutions for the fractional NLS with mass supercritical nonlinearity

In this chapter we investigate the existence of solutions to the fractional Nonlinear Schrödinger Equation (NLS in the sequel)

$$\mathrm{i}\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi - V(|\psi|)\psi, \tag{3.1}$$

where i denotes the imaginary unit and $\psi = \psi(x,t) \colon \mathbb{R}^N \times (0,\infty) \to \mathbb{C}$ is an unknown function. This type of Schrödinger equation was introduced by Laskin in [64], and the interest in its analysis has grown over the years. An important family of solutions, known under the name of *standing waves*, is characterized by the *ansatz*

$$\psi(x,t) = e^{\mathrm{i}\mu t} u(x) \tag{3.2}$$

for some (unknown) function $u : \mathbb{R}^N \to \mathbb{R}$. These solutions are self-similar and conserve their mass along time, i.e. $\frac{d}{dt}\|\psi(\cdot,t)\|_2 = 0$ at any $t > 0$. Therefore, it is natural and meaningful to seek solutions having a *prescribed* L^2-norm.

Coupling (3.1) with (3.2), we arrive at the problem

$$\begin{cases} (-\Delta)^s u = V(|u|)u - \mu u & \text{in } \mathbb{R}^N, \\ \|u\|_2^2 = m, \end{cases}$$

where $s \in (0,1)$, $N > 2s$, $\mu \in \mathbb{R}$, $m > 0$ is a prescribed parameter, and $(-\Delta)^s$ denotes the usual fractional Laplacian.

In order to ease notation, we will write $f(u) = V(|u|)u$, and study the problem

$$\begin{cases} (-\Delta)^s u = f(u) - \mu u & \text{in } \mathbb{R}^N, \\ \|u\|_2^2 = m. \end{cases} \tag{P_m}$$

The role of the real number μ is twofold: it can either be *prescribed*, or it can arise as a *suitable* parameter in the analysis of (P_m). In the present work, we will choose the second option, and μ will arise as a Lagrange multiplier.

Since we are looking for *bound-state* solutions whose L^2-norm must be finite, it is natural to build a variational setting for (P_m). Since this is by now standard, we will be sketchy. To avoid confusion and ease notation, we stress that in this chapter the norm in $H^s(\mathbb{R}^N)$ will be denoted with

$$\|u\| = \sqrt{\|u\|_2^2 + [u]^2_{H^s(\mathbb{R}^N)}},$$

which naturally arises from an inner product. In the whole chapter we will denote with

$$\langle u, v \rangle_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy$$

and with

$$(u, v)_{L^2(\mathbb{R}^N)} := \int_{\mathbb{R}^N} uv\, dx$$

for all $u, v \in H^s(\mathbb{R}^N)$. We then (formally) introduce the energy functional

$$I(u) = \frac{1}{2}\, [u]^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(u)\, dx$$

where $F(t) = \int_0^t f(\sigma)\, d\sigma$. A standard approach for studying (P_m) consists in looking for critical points of I constrained on the sphere

$$S_m = \left\{ u \in H^s(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |u|^2\, dx = m \right\}.$$

The convenience of this variational approach depends strongly on the behaviour of the nonlinearity f. If $f(t)$ grows slower than $|t|^{1+\frac{4s}{N}}$ as $t \to +\infty$, then I is coercive and bounded from below on S_m: this is the *mass subcritical case*, and the minimization problem

$$\min \{ I(u) \mid u \in S_m \}$$

is the natural approach. On the other hand, if $f(t)$ grows faster than $|t|^{1+\frac{4s}{N}}$ as $t \to +\infty$ then I is unbounded from below on S_m, and we are in the *mass supercritical case*. Since constrained minimizers of I on S_m cannot exist, we have to find critical points at higher levels.

When $s = 1$, i.e. when the fractional Laplace operator $(-\Delta)^s$ reduces to the *local* differential operator $-\Delta$, the literature for (P_m) is huge ([57], [17], [16], [18], [59]). The particular case of a combined nonlinearity of power type, namely $f(t) = t^{p-2} + \mu t^{q-2}$ with $2 < q < p < 2N/(N-2)$ has been widely investigated. The interplay of the parameters p and q add some richness to the structure of the problem.

The situation is different when $0 < s < 1$, and few results are available. Feng *et al.* in [47] deal with particular nonlinearities. Stanislavova *et al.* in [106] add the further complication of a trapping potential. In the recent paper [114] the author proves some existence and asymptotic results for the fractional NLS when a lower order perturbation to a mass supercritical pure power in the nonlinearity is added. It is also worth mentioning [75], where Luo *et al.* studied the problem when the nonlinear term consists in the sum of two pure powers of different order. They provide some existence and non-existence results, analyzing separately what happens in the mass subcritical and supercritical case for both the leading term and the lower order perturbation. The interested reader can also consult [43], [67], [66], [38], [115].

Very recently, Jeanjean *et al.* in [58] provided a thorough treatment of the local case $s = 1$ via a careful analysis based on the Pohozaev identity. In this chapter we propose a partial extension of their results to the non-local case $0 < s < 1$. Since we deal with a fractional operator, our conditions on f must be adapted correspondingly.

We collect here our standing assumptions about the nonlinearity f; we recall that

$$F(t) = \int_0^t f(\sigma)\, d\sigma$$

and define the auxiliary function

$$\tilde{F}(t) = f(t)t - 2F(t).$$

(f_0) $f \colon \mathbb{R} \to \mathbb{R}$ is an odd and locally Lipschitz continuous function;

(f_1) $\displaystyle\lim_{t\to 0} \frac{f(t)}{|t|^{1+4s/N}} = 0;$

(f_2) $\displaystyle\lim_{t\to +\infty} \frac{f(t)}{|t|^{(N+2s)/(N-2s)}} = 0;$

(f_3) $\displaystyle\lim_{t\to +\infty} \frac{F(t)}{|t|^{2+4s/N}} = +\infty;$

(f_4) The function $t \mapsto \dfrac{\tilde{F}(t)}{|t|^{2+4s/N}}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, +\infty)$;

(f_5) $f(t)t < \frac{2N}{N-2s} F(t)$ for all $t \in \mathbb{R} \setminus \{0\}$;

(f_6) $\displaystyle\lim_{t\to 0} \frac{tf(t)}{|t|^{2N/(N-2s)}} = +\infty.$

Remark 3.1. The oddness of f is necessary in order to use the classical genus theory and to get a desired property on the fiber map that we will introduce in detail in the next section (see for instance Lemma 3.11 below). Assumption (f_2) guarantees a Sobolev subcritical growth, whereas (f_3) characterises the problem as mass supercritical. At one point, we will need (f_5) to establish the strict positivity of the Lagrange multiplier μ.

Example 1. As suggested in [58], an explicit example can be constructed as follows. Set $\alpha_{N,s} = \frac{4s^2}{N(N-2s)}$ for simplicity, and define

$$f(t) = \left(\left(2 + \frac{4s}{N} \right) \log\left(1 + |t|^{\alpha_{N,s}}\right) + \frac{\alpha_{N,s} |t|^{\alpha_{N,s}}}{1 + |t|^{\alpha_{N,s}}} \right) |t|^{\frac{4s}{N}} t$$

We briefly outline our results. Firstly, we show that the ground-state level is attained with a strictly positive Lagrange multiplier.

Theorem 3.2. *Assume that f satisfies (f_0)-(f_5). Then (P_m) admits a positive ground-state for any $m > 0$. Moreover, for any ground-state the associated Lagrange multiplier μ is positive.*

Furthermore, we can prove some remarkable properties of the ground-state level energy with respect the variable m and its asymptotic behavior. We refer to (3.16) for the precise definition of the ground-state level E_m.

Theorem 3.3. *Assume that f satisfies (f_0)-(f_6). Then the function $m \mapsto E_m$ is positive, continuous, strictly decreasing. Furthermore, $\lim_{m\to 0^+} E_m = +\infty$ and $\lim_{m\to\infty} E_m = 0$.*

Finally, we have a multiplicity result for the radially symmetric case.

Theorem 3.4. *If (f_0)-(f_5) hold and $N > 2$, then (P_m) admits infinitely many radial solutions $(u_k)_k$ for any $m > 0$. In particular,*

$$I(u_{k+1}) \geq I(u_k)$$

for all $k \in \mathbb{N}$ and $I(u_k) \to +\infty$ as $k \to +\infty$.

The chapter is organised as follows. Section 3.1 contains the proofs of some preliminary Lemmas that will be useful during the whole remaining part of the chapter. Moreover, we introduce a fiber map that will play a crucial role for our purposes. In Section 3.2 we define the ground-state level energy for a fixed mass m and we start analyzing its asymptotic behaviour near zero and infinity. Section 3.3 is devoted to proving our main existence theorem. Using a min-max theorem of linking type and the fiber map cited previously, we construct a Palais-Smale sequence whose value of the Pohozaev functional is zero and we show that a sequence of this kind must be necessarily bounded. Finally, in Section 3.4, for the sake of completeness, we discuss the existence of radial solutions. Here, we use a variant of the min-max theorem already cited in Section 3.3, but this time we are helped by the fact that the space of the radially symmetric functions with finite fractional derivative is compactly embedded in $L^p(\mathbb{R}^N)$ for $p \in (2, 2_s^*)$.

3.1 Preliminary results

We define the *Pohozaev manifold*

$$\mathcal{P}_m = \{u \in S_m \mid P(u) = 0\},$$

where

$$P(u) = [u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2s}\int_{\mathbb{R}^N} \tilde{F}(u)\,dx.$$

Let us collect some technical results that we will frequently used in the chapter. We use the shorthand

$$B_m = \left\{u \in H^s(\mathbb{R}^N) \mid \|u\|_2^2 \leq m\right\}.$$

Lemma 3.5. *Assuming* (f_0), (f_1), (f_2), *the following statements hold*

(i) *for every* $m>0$ *there exists* $\delta>0$ *such that*

$$\frac{1}{4}\,[u]^2_{H^s(\mathbb{R}^N)} \le I(u) \le [u]^2_{H^s(\mathbb{R}^N)}$$

where $u\in B_m$ *and* $[u]_{H^s(\mathbb{R}^N)}\le\delta$.

(ii) *Let* $(u_n)_n$ *be a bounded sequence in* $H^s(\mathbb{R}^N)$. *If* $\lim_{n\to+\infty}\|u_n\|_{2+\frac{4s}{N}}=0$ *we have that*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} F(u_n)\,dx = 0 = \lim_{n\to+\infty}\int_{\mathbb{R}^N}\tilde{F}(u_n)\,dx.$$

(iii) *Let* $(u_n)_n$, $(v_n)_n$ *two bounded sequences in* $H^s(\mathbb{R}^N)$. *If* $\lim_{n\to+\infty}\|v_n\|_{2+\frac{4s}{N}}=0$ *then*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} f(u_n)v_n\,dx = 0.$$

Proof. (i) It suffices to show that there exists $\delta>0$ such that

$$\int_{\mathbb{R}^N}|F(u)|\,dx \le \frac{1}{4}\,[u]^2_{H^s(\mathbb{R}^N)}$$

whenever $u\in B_m$ and $[u]_{H^s(\mathbb{R}^N)}\le\delta$. In order to show that, we start noticing that (f_0), (f_1), and (f_2) imply that for every $\varepsilon>0$ we can find $C_1=C_1(\varepsilon)>0$ such that

$$|F(u)|\le \varepsilon|t|^{2+\frac{4s}{N}}+C_1|t|^{\frac{2N}{N-2s}}. \tag{3.3}$$

Hence, by (3.3), using the interpolation inequality and the fractional Sobolev inequality (see for instance [41, Theorem 6.5]), we get

$$\begin{aligned}
\int_{\mathbb{R}^N}|F(u)|\,dx &\le \varepsilon\int_{\mathbb{R}^N}|u|^{2+\frac{4s}{N}}\,dx + C_1\int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2s}}\,dx\\
&\le \varepsilon m^{\frac{2s}{N}}\|u\|^2_{2^*_s}+C_1\|u\|^{2^*_s}_{2^*_s}\\
&\le \varepsilon m^{\frac{2s}{N}}C_1\,[u]^2_{H^s(\mathbb{R}^N)} + C_2\,[u]^{2^*_s}_{H^s(\mathbb{R}^N)}\\
&= \left[\varepsilon m^{\frac{2s}{N}}C_1 + C_2\,[u]^{2^*_s-2}_{H^s(\mathbb{R}^N)}\right][u]^2_{H^s(\mathbb{R}^N)}\,.
\end{aligned}$$

Choosing

$$\varepsilon=\frac{1}{8m^{\frac{2s}{N}}C_1}\quad\text{and}\quad\delta=\left(\frac{1}{C_2}\right)^{\frac{1}{2^*_s-2}}$$

the assertion is verified.

(ii) Since (f_0), (f_1) and (f_2) hold, for every $\varepsilon > 0$ there exists C_3, $C_4 > 0$ such that

$$|f(t)t|\leq \frac{\varepsilon}{2}|t|^{\frac{2N}{N-2s}}+C_3|t|^{2+\frac{4s}{N}}$$

and

$$|F(t)|\leq \frac{\varepsilon}{4}|t|^{\frac{2N}{N-2s}}+C_4|t|^{2+\frac{4s}{N}},$$

which implies

$$|\tilde{F}(t)|\leq \varepsilon|t|^{\frac{2N}{N-2s}}+(C_3+2C_4)\,|t|^{2+\frac{4s}{N}}. \tag{3.4}$$

By (3.4) we have

$$\begin{aligned}\int_{\mathbb{R}^N}|\tilde{F}(u_n)|\,dx &\leq \varepsilon\int_{\mathbb{R}^N}|u_n|^{\frac{2N}{N-2s}}\,dx+(C_3+2C_4)\int_{\mathbb{R}^N}|u_n|^{2+\frac{4s}{N}}\,dx\\ &\leq \varepsilon C_5\,[u_n]_{H^s(\mathbb{R}^N)}^{\frac{2N}{N-2s}}+(C_3+2C_4]\,\|u_n\|_{2+\frac{4s}{N}}^{2+\frac{4s}{N}}\\ &\leq \varepsilon C_6+(C_3+2C_4)\,\|u_n\|_{2+\frac{4s}{N}}^{2+\frac{4s}{N}}\to 0\end{aligned}$$

as $n\to+\infty$ and $\varepsilon\to 0$. The proof of $\lim_{n\to+\infty}\int_{\mathbb{R}^N}|F(u_n)|\,dx=0$ is similar.

(iii). (f_0), (f_1) and (f_2) imply that for every $\varepsilon>0$ we can find $C_7>0$ such that

$$|f(t)|\leq \varepsilon|t|^{\frac{N+2s}{N-2s}}+C_7|t|^{1+\frac{4s}{N}}. \tag{3.5}$$

Hence, by (3.5), we obtain that

$$\begin{aligned}\int_{\mathbb{R}^N}|f(u_n)||v_n|\,dx &\leq \varepsilon\int_{\mathbb{R}^N}|u_n|^{\frac{N+2s}{N-2s}}|v_n|\,dx+C_7\int_{\mathbb{R}^N}|u_n|^{1+\frac{4s}{N}}|v_n|\,dx\\ &\leq \varepsilon\|u_n\|_{2_s^*}^{\frac{N+2s}{N-2s}}\|v_n\|_{2_s^*}+C_7\|u_n\|_{2+\frac{4s}{N}}^{\frac{N+4s}{N}}\|v_n\|_{2+\frac{4s}{N}}\\ &\leq \varepsilon C_8\|u_n\|_{H^s(\mathbb{R}^N)}^{\frac{N+2s}{N-2s}}\|v_n\|_{H^s(\mathbb{R}^N)}+C_9\|u_n\|_{H^s(\mathbb{R}^N)}^{\frac{N+4s}{N}}\|v_n\|_{2+\frac{4s}{N}}\\ &\leq \varepsilon C_{10}+C_{11}\|v_n\|_{2+\frac{4s}{N}}\to 0\end{aligned}$$

as $n\to+\infty$ and $\varepsilon\to 0$. This completes the proof of the Lemma. □

Remark 3.6. An inspection of the proof of this Lemma shows that the inequality

$$\int_{\mathbb{R}^N}\tilde{F}(u)\,dx\leq \frac{s}{N}\,[u]_{H^s(\mathbb{R}^N)}^2$$

holds true if $u\in B_m$ and $[u]_{H^s(\mathbb{R}^N)}\leq\delta$. It follows that

$$P(u)\geq \frac{1}{2}\,[u]_{H^s(\mathbb{R}^N)}^2$$

for every $u\in B_m$ with $[u]_{H^s(\mathbb{R}^N)}\leq\delta$.

In order to prove the next result, we introduce for every $u \in H^s(\mathbb{R}^N)$ and $\rho \in \mathbb{R}$ the scaling map[1]

$$(\rho * u)(x) = e^{\frac{N\rho}{2}} u(e^{\rho} x) \quad x \in \mathbb{R}^N .$$

It is easy to verify that $\rho * u \in H^s(\mathbb{R}^N)$ and $\|\rho * u\|_2 = \|u\|_2$.

Lemma 3.7. *Assuming* (f_0), (f_1), (f_2) *and* (f_3), *we have:*

(i) $I(\rho * u) \to 0^+$ *as* $\rho \to -\infty$,

(ii) $I(\rho * u) \to -\infty$ *as* $\rho \to \infty$.

Proof. *(i)* Let us fix $m := \|u\|_2^2$. We observe that $\rho * u \in S_m$ and after a change of variables we obtain

$$[\rho * u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^{2N}} \frac{e^{N\rho}(u(e^{\rho}x) - u(e^{\rho}y))^2}{|x-y|^{N+2s}} \, dx \, dy = e^{2\rho s} \, [u]^2_{H^s(\mathbb{R}^N)} .$$

By virtue of the previous computation, choosing $\rho \ll -1$, Lemma 3.5 *(i)* guarantees the existence of a $\delta > 0$ such that if $[\rho * u]_{H^s(\mathbb{R}^N)} \leq \delta$ then

$$\frac{1}{4} e^{2\rho s} \, [u]^2_{H^s(\mathbb{R}^N)} \leq I(\rho * u) \leq e^{2\rho s} \, [u]^2_{H^s(\mathbb{R}^N)} ,$$

thus

$$\lim_{\rho \to -\infty} I(\rho * u) = 0^+.$$

(ii) For every $\lambda \geq 0$ we define the function $h_\lambda \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$h_\lambda(t) = \begin{cases} \frac{F(t)}{|t|^{2+\frac{4s}{N}}} + \lambda & t \neq 0 \\ \lambda & t = 0. \end{cases} \tag{3.6}$$

It is straightforward to verify that $F(t) = h_\lambda(t)|t|^{2+\frac{4s}{N}} - \lambda |t|^{2+\frac{4s}{N}}$. Moreover, from (f_0) and (f_1) it follows that h_λ is continuous, whereas thanks to (f_3) we have

$$h_\lambda(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Putting together the divergence of the limit above at infinity and (f_1), we can find $\lambda > 0$ large enough such that $h_\lambda(t) \geq 0$ for every $t \in \mathbb{R}$. Now, applying the well-known Fatou's Lemma, we obtain

$$\liminf_{\rho \to \infty} \int_{\mathbb{R}^N} h_\lambda(e^{\frac{N\rho}{2}} u)|u|^{2+\frac{4s}{N}} \, dx \geq \int_{\mathbb{R}^N} \lim_{\rho \to \infty} h_\lambda(e^{\frac{N\rho}{2}} u)|u|^{2+\frac{4s}{N}} \, dx = \infty.$$

[1] The notation $\rho * u$ is standard in the theory of transformation groups, and is not ambiguous since we never use convolution.

Then, we observe that

$$I(\rho * u) = \frac{1}{2}\left[\rho * u\right]^2_{H^s(\mathbb{R}^N)} + \lambda \int_{\mathbb{R}^N} |\rho * u|^{2+\frac{4s}{N}}\, dx - \int_{\mathbb{R}^N} h_\lambda(\rho * u)|\rho * u|^{2+\frac{4s}{N}}\, dx \tag{3.7}$$
$$= e^{2\rho s}\left[\frac{1}{2}\left[u\right]^2_{H^s(\mathbb{R}^N)} + \lambda \int_{\mathbb{R}^N} |u|^{2+\frac{4s}{N}}\, dx - \int_{\mathbb{R}^N} h_\lambda(e^{\frac{N\rho}{2}} u)|u|^{2+\frac{4s}{N}}\, dx\right],$$

from which it follows immediately that

$$\lim_{\rho\to\infty} I(\rho * u) = -\infty.$$

□

Remark 3.8. Assume $f \in C(\mathbb{R},\mathbb{R})$, (f_1) and (f_4). Then the function $g\colon \mathbb{R}\to\mathbb{R}$ defined as

$$g(t) = \begin{cases} \frac{f(t)t-2F(t)}{|t|^{2+\frac{4s}{N}}}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

is continuous, strictly increasing in $(0,\infty)$ and strictly decreasing in $(-\infty, 0)$.

Lemma 3.9. *Assuming* $f \in C(\mathbb{R},\mathbb{R})$, (f_1), (f_3) *and* (f_4), *we have*

(i) $F(t) > 0$ *if* $t \neq 0$*;*

(ii) there exist $(\tau_n^+)_n \subset \mathbb{R}^+$ *and* $(\tau_n^-)_n \subset \mathbb{R}^-$, $|\tau_n^\pm| \to 0$ *as* $n \to +\infty$ *such that*

$$f(\tau_n^\pm)\tau_n^\pm > \left(2 + \frac{4s}{N}\right) F(\tau_n^\pm)$$

for any $n \neq 1$*;*

(iii) there exist $(\sigma_n^+)_n \subset \mathbb{R}^+$ *and* $(\sigma_n^-)_n \subset \mathbb{R}^-$, $|\sigma_n^\pm| \to \infty$ *as* $n \to +\infty$ *such that*

$$f(\sigma_n^\pm)\sigma_n^\pm > \left(2 + \frac{4s}{N}\right) F(\sigma_n^\pm)$$

for any $n \geq 1$.

Proof. *(i)* By contradiction suppose $F(t_0) \leq 0$ for some $t_0 \neq 0$. Because of (f_1) and (f_3) the function $F(t)/|t|^{2+4s/N}$ must attain its global minimum in a point $\tau \neq 0$ such that $F(\tau) \leq 0$. It follows that

$$\frac{d}{dt}\frac{F(t)}{|t|^{2+\frac{4s}{N}}}\bigg|_{t=\tau} = \frac{f(\tau)\tau - \left(2+\frac{4s}{N}\right)F(\tau)}{|\tau|^{3+\frac{4s}{N}}\operatorname{sgn}(\tau)} = 0. \tag{3.8}$$

From Remark 3.8 it follows that $f(t)t > 2F(t)$ if $t \neq 0$. Indeed, were the claim false, there would exist $\bar{t}$ such that $f(\bar{t})\bar{t} \leq 2F(\bar{t})$. Choosing without loss of generality $\bar{t} < 0$, we have that $g(\bar{t}) \leq 0$. This and the fact that $g(0) = 0$ show that g must be strictly increasing on an interval between $\bar{t}$ and 0. Finally, we can have a contradiction observing that

$$0 < f(\tau)\tau - 2F(\tau) = \frac{4s}{N}F(\tau) \leq 0.$$

(ii) We start with the positive case. By contradiction, we suppose that there is $T_\alpha > 0$ small enough such that

$$f(t)t \leq \left(2 + \frac{4s}{N}\right) F(t)$$

for every $t \in (0, T_\alpha]$. Recalling the expression of (3.8) computed in the step *(i)* we see that the derivative of $F(t)/|t|^{2+4s/N}$ is non-positive on $(0, T_\alpha]$, then

$$\frac{F(t)}{t^{2+\frac{4s}{N}}} \geq \frac{F(T_\alpha)}{T_\alpha^{2+\frac{4s}{N}}} > 0 \quad \text{for every} \quad t \in (0, T_\alpha]\,,$$

that is in contradiction with (f_1). The negative case is similar.

(iii) Being the two cases similar, we will prove only the negative one. Again, by contradiction we suppose there is $T_\gamma > 0$ such that

$$f(t)t \leq \left(2 + \frac{4s}{N}\right) F(t) \quad \text{for every} \quad t \leq -T_\gamma.$$

Since the derivative of $F(t)/|t|^{2+4s/N}$ is non-negative on $(-\infty, -T_\gamma]$, we can deduce

$$\frac{F(t)}{|t|^{2+\frac{4s}{N}}} \leq \frac{F(-T_\gamma)}{T_\gamma^{2+\frac{4s}{N}}} \quad \text{for every} \quad t \in (-\infty, -T_\gamma]\,,$$

which contradicts (f_3). □

Lemma 3.10. *Assume (f_0), (f_1), (f_3) and (f_4). For any $t > 0$ there results*

$$f(t)t > \left(2 + \frac{4s}{N}\right) F(t).$$

Proof. We start by proving that the inequality holds weakly. By contradiction, we assume

$$f(t_0)t_0 < \left(2 + \frac{4s}{N}\right) F(t_0)$$

for some $t_0 \neq 0$ and without loss of generality, we can suppose $t_0 < 0$. By step *(ii)* and *(iii)* of Lemma 3.9 there are $\tau_{\min}, \tau_{\max} \in \mathbb{R}$, where $\tau_{\min} < t_0 < \tau_{\max} < 0$ such that

$$f(t)t < \left(2 + \frac{4s}{N}\right) F(t) \quad \text{for every} \quad t \in (\tau_{\min}, \tau_{\max}) \tag{3.9}$$

and

$$f(t)t = \left(2 + \frac{4s}{N}\right) F(t) \quad \text{for every} \quad t \in \{\tau_{\min}, \tau_{\max}\}. \tag{3.10}$$

By (3.9) we have

$$\frac{F(\tau_{\min})}{|\tau_{\min}|^{2+\frac{4s}{N}}} < \frac{F(\tau_{\max})}{|\tau_{\max}|^{2+\frac{4s}{N}}}. \tag{3.11}$$

Besides, by (3.10) and (f_4) must be

$$\frac{F(\tau_{\min})}{|\tau_{\min}|^{2+\frac{4s}{N}}} = \frac{N}{4s}\frac{\tilde{F}(\tau_{\min})}{|\tau_{\min}|^{2+\frac{4s}{N}}} > \frac{N}{4s}\frac{\tilde{F}(\tau_{\max})}{|\tau_{\max}|^{2+\frac{4s}{N}}} = \frac{F(\tau_{\max})}{|\tau_{\max}|^{2+\frac{4s}{N}}}, \tag{3.12}$$

and clearly (3.11) and (3.12) are in contradiction. From what we have just proved, we have that $F(t)/|t|^{2+4s/N}$ is non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$. Hence, by virtue of (f_4) the function $f(t)/|t|^{1+4s/N}$ must necessarily be strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$. Then

$$\begin{aligned}\left(2 + \frac{4s}{N}\right) F(t) &= \left(2 + \frac{4s}{N}\right) \int_0^t \frac{f(\kappa)}{|\kappa|^{1+\frac{4s}{N}}} |\kappa|^{1+\frac{4s}{N}}\, d\kappa \\ &< \left(2 + \frac{4s}{N}\right) \frac{f(t)}{|t|^{1+\frac{4s}{N}}} \int_0^t |\kappa|^{1+\frac{4s}{N}}\, d\kappa = f(t)t\end{aligned}$$

completes the proof. □

Lemma 3.11. *Assume* $(f_0) - (f_4)$, $u \in H^s(\mathbb{R}^N) \setminus \{0\}$. *Then the following hold:*

(i) There is a unique $\rho(u) \in \mathbb{R}$ *such that* $P(\rho(u) * u) = 0$.

(ii) $I(\rho(u) * u) > I(\rho * u)$ *for any* $\rho \neq \rho(u)$. *Moreover,* $I(\rho(u) * u) > 0$.

(iii) The map $u \to \rho(u)$ *is continuous on* $H^s(\mathbb{R}^N) \setminus \{0\}$.

(iv) $\rho(u) = \rho(-u)$ *and* $\rho(u(\cdot + y)) = \rho(u)$ *for all* $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ *and* $y \in \mathbb{R}^N$.

Proof. *(i)* Since

$$I(\rho * u) = \frac{1}{2} e^{2\rho s} [u]^2_{H^s(\mathbb{R}^N)} - e^{-N\rho} \int_{\mathbb{R}^N} F(e^{\frac{N}{2}\rho} u)\, dx,$$

it is easy to check that $I(\rho * u)$ is C^1 with respect to ρ. Now, computing

$$\frac{d}{d\rho} I(\rho * u) = s e^{2\rho s} [u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2} e^{-N\rho} \int_{\mathbb{R}^N} \tilde{F}\left(e^{\frac{N\rho}{2}} u\right) dx$$

and observing that

$$P(\rho * u) = e^{2\rho s} [u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2s} e^{-N\rho} \int_{\mathbb{R}^N} \tilde{F}\left(e^{\frac{N\rho}{2}} u\right) dx$$

we deduce

$$\frac{d}{d\rho} I(\rho * u) = sP(\rho * u).$$

Remembering that by Lemma 3.7

$$\lim_{\rho \to -\infty} I(\rho * u) = 0^+ \quad \text{and} \quad \lim_{\rho \to \infty} I(\rho * u) = -\infty$$

we can conclude that $\rho \mapsto I(\rho * u)$ must reach a global maximum at some point $\rho(u)$; since

$$0 = \frac{d}{d\rho} I(\rho(u) * u) = sP(\rho(u) * u),$$

we conclude that $P(\rho(u) * u) = 0$. To check the uniqueness of the point $\rho(u)$, recalling the function g defined in Remark 3.8, we observe that $\tilde{F}(t) = g(t)|t|^{2+\frac{4s}{N}}$ for every $t \in \mathbb{R}$. Thus, we obtain

$$\begin{aligned} P(\rho * u) &= e^{2\rho s} [u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2s} e^{2\rho s} \int_{\mathbb{R}^N} g(e^{\frac{N\rho}{2}} u)|u|^{2+\frac{4s}{N}} \, dx \\ &= e^{2\rho s} \left[[u]^2_{H^s(\mathbb{R}^N)} - \frac{N}{2s} \int_{\mathbb{R}^N} g(e^{\frac{N\rho}{2}} u)|u|^{2+\frac{4s}{N}} \, dx \right] = \frac{1}{s} \frac{d}{d\rho} I(\rho * u). \end{aligned}$$

Fixing $t \in \mathbb{R} \setminus \{0\}$, thanks to Remark 3.8 and (f_4), we notice that the function

$$\rho \mapsto g\left(e^{\frac{N\rho}{2}} t\right)$$

is strictly increasing. Thus, by virtue of the previous computations, it follows that $\rho(u)$ must be unique.

(ii) This follows immediately from (i).

(iii) By step (i) the function $u \mapsto \rho(u)$ is well defined. Let $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ and $(u_n)_n \subset H^s(\mathbb{R}^N) \setminus \{0\}$ a sequence such that $u_n \to u$ in $H^s(\mathbb{R}^N)$ as $n \to +\infty$. We set $\rho_n = \rho(u_n)$ for any $n \geq 1$. Let us show that, up to a subsequence, we have $\rho_n \to \rho(u)$ as $n \to +\infty$.

Claim. The sequence $(\rho_n)_n$ is bounded.

We recall that the function h_λ defined in (3.6) noticing that by Lemma 3.9 (i) $h_0(t) \geq 0$ for every $t \in \mathbb{R}$. We assume by contradiction that up to a subsequence $\rho_n \to +\infty$. By Fatou's Lemma and the fact that $u_n \to u$ a.e. in $\mathbb{R}^N$, we have that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} h_0\left(e^{\frac{N\rho_n}{2}} u_n\right) |u_n|^{2+\frac{4s}{N}} \, dx = \infty.$$

As a consequence of that, by (3.7) with $\lambda = 0$ and step (ii), we obtain

$$0 \leq e^{-2\rho_n s} I(\rho_n * u_n) = \frac{1}{2} [u_n]^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} h_0\left(e^{\frac{N\rho_n}{2}} u_n\right) |u_n|^{2+\frac{4s}{N}} \, dx \to -\infty \tag{3.13}$$

as $n \to +\infty$, which is evidently not possible. Then $(\rho_n)_n$ must be bounded from above. Now we assume, again by contradiction, that $\rho_n \to -\infty$. By step (ii) we observe that

$$I(\rho_n * u_n) \geq I(\rho(u) * u_n),$$

and since $\rho(u) * u_n \to \rho(u) * u$ in $H^s(\mathbb{R}^N)$, it follows that

$$I(\rho(u) * u_n) = I(\rho(u) * u) + o_n(1).$$

We deduce that

$$\liminf_{n \to +\infty} I(\rho_n * u_n) \geq I(\rho(u) * u) > 0. \tag{3.14}$$

Since we have $\rho_n * u_n \subset B_m$ for $m \gg 1$, Lemma 3.5 (i) implies that there exists $\delta > 0$ such that if $[\rho_n * u_n]_{H^s(\mathbb{R}^N)} \leq \delta$, we have

$$\frac{1}{4} [\rho_n * u_n]^2_{H^s(\mathbb{R}^N)} \leq I(\rho_n * u_n) \leq [\rho_n * u_n]^2_{H^s(\mathbb{R}^N)} . \tag{3.15}$$

Since

$$[\rho_n * u_n]_{Hs} = e^{\rho_n s} [u_n]_{H^s(\mathbb{R}^N)} ,$$

(3.15) holds for any n sufficiently large. Therefore, we obtain

$$\liminf_{n \to +\infty} I(\rho_n * u_n) = 0,$$

in contradiction to (3.14). The claim is proved.

The sequence $(\rho_n)_n$ being bounded, we can assume that, up to a subsequence, $\rho_n \to \rho^*$ for some ρ^* in $\mathbb{R}$. Hence, $\rho_n * u_n \to \rho^* * u$ in $H^s(\mathbb{R}^N)$ and since $P(\rho_n * u_n) = 0$ we have

$$P(\rho^* * u) = 0.$$

By the uniqueness proved at step (ii) we obtain $\rho^* = \rho(u)$.

(iv) Since f is odd by (f_0), the fact that

$$P(\rho(u) * (-u)) = P\left(-(\rho(u) * u)\right) = P(\rho(u) * u) = 0$$

implies $\rho(u) = \rho(-u)$. Similarly, changing the variables in the integral, we can verify that ρ is invariant under translation, and it is easy to check that

$$P(\rho(u) * u(\cdot + y)) = P(\rho(u) * u) = 0,$$

thus $\rho(u(\cdot + y)) = \rho(u)$. □

As we are going to see, the functional I constrained on $\mathcal{P}_m$ has some crucial properties.

Lemma 3.12. *Assuming $(f_0) - (f_4)$, the following statements are true:*

(i) $\mathcal{P}_m \neq \emptyset$,

(*ii*) $\inf_{u\in\mathcal{P}_m} [u]_{H^s(\mathbb{R}^N)} > 0$,

(*iii*) $\inf_{u\in\mathcal{P}_m} I(u) > 0$,

(*iv*) *I is coercive on* $\mathcal{P}_m$, *i.e.* $I(u_n) \to \infty$ *if* $(u_n)_n \subset \mathcal{P}_m$ *and* $\|u_n\|_{H^s(\mathbb{R}^N)} \to \infty$ *as* $n \to +\infty$.

Proof. Statement (*i*) follows directly from Lemma 3.11 (*i*).

(*ii*) Were the assertion not true, we would be able to take a sequence $(u_n)_n \subset \mathcal{P}_m$ such that $[u_n]_{H^s(\mathbb{R}^N)} \to 0$, and so, by Lemma 3.5 (*i*) we could also find $\delta > 0$ and $\overline{n}$ so large that $[u_n]_{H^s(\mathbb{R}^N)} \leq \delta$ for every $n \geq \overline{n}$. By Remark 3.6 we would have

$$0 = P(u_n) \geq \frac{1}{2}\, [u_n]^2_{H^s(\mathbb{R}^N)}$$

which is possible only for a constant u_n. But this is not admissible since $u \in S_m$. Hence, the statement must hold.

(*iii*) For every $u \in \mathcal{P}_m$ Lemma 3.11 (*ii*) and (*iii*) implies that

$$I(u) = I(0 * u) \geq I(\rho * u) \quad \text{for every} \quad \rho \in \mathbb{R}\,.$$

Let $\delta > 0$ be the number given by Lemma 3.5 (*i*) and set $\rho := 1/s \log\left(\delta/[u]_{H^s(\mathbb{R}^N)}\right)$. Since $\delta = [\rho * u]_{H^s(\mathbb{R}^N)}$, using again Lemma 3.5 (*i*) we obtain

$$I(u) \geq I(\rho * u) \geq \frac{1}{4}\, [\rho * u]^2_{H^s(\mathbb{R}^N)} = \frac{1}{4}\delta^2$$

proving the statement.

(*iv*) By contradiction we suppose the existence of $(u_n)_n \subset \mathcal{P}_m$ such that $\|u_n\|_{H^s(\mathbb{R}^N)} \to \infty$ with $\sup_{n\geq 1} I(u_n) \leq c$ for some $c \in (0,\infty)$. For any $n \geq 1$ we set

$$\rho_n = \frac{1}{s}\log\left([u_n]_{H^s(\mathbb{R}^N)}\right) \quad \text{and} \quad v_n = (-\rho_n) * u_n.$$

Evidently $\rho_n \to +\infty$, $(v_n)_n \subset S_m$ and $[v_n]_{H^s(\mathbb{R}^N)} = 1$. We denote with

$$\alpha = \limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n|^2\, dx$$

and we distinguish two cases.

Non vanishing: $\alpha > 0$. Up to a subsequence we can assume the existence of a sequence $(y_n)_n \subset \mathbb{R}^N$ and $\omega \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that

$$\omega_n = v_n(\cdot + y_n) \rightharpoonup \omega \text{ in } H^s(\mathbb{R}^N) \quad \text{and} \quad \omega_n \to \omega \text{ a.e. in } \mathbb{R}^N\,.$$

Recalling the definition of the continuous function h_λ with $\lambda = 0$, remembering that $\rho_n \to +\infty$ as $n \to +\infty$ and using the Fatou's Lemma we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} h_0\left(e^{\frac{N\rho_n}{2}}\omega_n\right) |\omega_n|^{2+\frac{4s}{N}}\, dx = \infty.$$

By step (*iii*) and (3.8), after changing the variables in the integral, we obtain

$$
\begin{aligned}
0 \le e^{-2\rho_n s} I(u_n) = e^{-2\rho_n s} I(\rho_n * v_n) &= \frac{1}{2} - \int_{\mathbb{R}^N} h_0\left(e^{\frac{N\rho_n}{2}} v_n\right) |v_n|^{2+\frac{4s}{N}}\, dx \\
&= \frac{1}{2} - \int_{\mathbb{R}^N} h_0\left(e^{\frac{N\rho_n}{2}} \omega\right) |\omega_n|^{2+\frac{4s}{N}}\, dx \to -\infty
\end{aligned}
$$

as $n \to +\infty$.

Vanishing: $\alpha = 0$. By [102, Lemma II.4], we have that $v_n \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$ and by Lemma 3.5 (*ii*) we see that

$$
\lim_{n\to+\infty} e^{N\rho} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho}{2}} v_n\right) = 0 \quad \text{for every} \quad \rho \in \mathbb{R}\,.
$$

Since $P(\rho_n * v_n) = P(u_n) = 0$, by Lemma 3.11 (*ii*) and (*iii*), we obtain

$$
\begin{aligned}
c \ge I(u_n) &= I(\rho_n * v_n) \\
&\ge I(\rho * v_n) = \frac{1}{2} e^{2\rho s} - e^{-N\rho} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho}{2}} v_n\right) dx = \frac{1}{2} e^{2\rho s} - o_n(1).
\end{aligned}
$$

We can conclude choosing $\rho > \log(2c)/2s$ and letting $n \to +\infty$. □

Remark 3.13. Observe that if we assume the validity of $(f_0)-(f_4)$ and we take a sequence $(u_n)_n \subset H^s(\mathbb{R}^N)$ such that

$$
P(u_n) = 0, \quad \sup_{n\ge1} \|u_n\|_2 < +\infty \quad \text{and} \quad \sup_{n\ge1} I(u_n) < +\infty,
$$

then repeating the arguments carried out in the proof of Lemma 3.12 (*iv*) we get that $(u_n)_n$ is bounded in $H^s(\mathbb{R}^N)$.

We conclude with a splitting result *à la* Brezis-Lieb. A proof is included for the reader's convenience.

Lemma 3.14. *Let $f\colon \mathbb{R} \to \mathbb{R}$ continuous, odd and let $(u_n)_n \subset H^s(\mathbb{R}^N)$ a bounded sequence such that $u_n \to u$ pointwise almost everywhere in $\mathbb{R}^N$. If there exists $C > 0$ such that*

$$
|f(t)| \le C\left(|t| + |t|^{2^*_s - 1}\right),
$$

then

$$
\lim_{n\to+\infty} \int_{\mathbb{R}^N} |F(u_n) - F(u_n - u) - F(u)|\, dx = 0
$$

Proof. Let a, $b \in \mathbb{R}$ and $\varepsilon > 0$. We compute

$$\begin{aligned}
|F(a+b) - F(a)| &= \left| \int_0^1 \frac{d}{d\tau} F(a+\tau b)\, d\tau \right| \\
&= \left| \int_0^1 F'(a+\tau b) b \, d\tau \right| \\
&\le C \int_0^1 \left(|a+\tau b| + |a+\tau b|^{2_s^*-1} \right) |b| \, d\tau \\
&\le C \left(|a| + |b| + 2^{2_s^*-1} \left(|a|^{2_s^*-1} + |b|^{2_s^*-1} \right) \right) |b| \\
&\le C \left(|a| + |b| + 2^{2_s^*} \left(|a|^{2_s^*-1} + |b|^{2_s^*-1} \right) \right) |b| \\
&\le C \left(|ab| + b^2 + 2^{2_s^*} \left(|a|^{2_s^*-1} |b| + |b|^{2_s^*} \right) \right).
\end{aligned}$$

We have used that $\tau \le 1$ and the convexity inequality

$$|a+b|^{2_s^*-1} \le 2^{2_s^*-1} \left(|a|^{2_s^*-1} + |b|^{2_s^*-1} \right).$$

Now we use Young's inequality twice:

$$\begin{aligned}
|ab| &\le \varepsilon \frac{a^2}{2} + \frac{1}{2\varepsilon} |b|^2 \\
|a|^{2_s^*-1} |b| &\le \eta^{\frac{2_s^*}{2_s^*-1}} \frac{|a|^{2_s^*}}{\frac{2_s^*}{2_s^*-1}} + \frac{1}{\eta^{2_s^*}} \frac{|b|^{2_s^*}}{2_s^*}.
\end{aligned}$$

Hence, choosing

$$\eta = \varepsilon^{\frac{2_s^*-1}{2_s^*}},$$

we get

$$\begin{aligned}
|ab| + b^2 + 2^{2_s^*} \left(|a|^{2_s^*-1}|b| + |b|^{2_s^*} \right) &\le \varepsilon \frac{a^2}{2} + \frac{1}{2\varepsilon} b^2 + b^2 + 2^{2_s^*} \left(|a|^{2_s^*-1}|b| + |b|^{2_s^*} \right) \\
&\le \varepsilon C \left(a^2 + |2a|^{2_s^*} \right) + C \left[\left(1 + \varepsilon^{-1} \right) b^2 + \left(1 + \varepsilon^{1-2_s^*} \right) |2b|^{2_s^*} \right] \\
&= \varepsilon \varphi(a) + \psi_\varepsilon(b).
\end{aligned}$$

Applying [30, Theorem 2] with $g_n = u_n - u$ and $f = u$ we have the assertion. □

3.2 Properties of the map $m \mapsto E_m$

Under our standing assumptions (f_0)–(f_4), for every $m > 0$ we can define the least level of energy

$$E_m = \inf_{u \in \mathcal{P}_m} I(u). \tag{3.16}$$

This section is devoted to the analysis of the quantity E_m as a *function* of $m > 0$.

Lemma 3.15. *If* (f_0)–(f_4) *hold true, then* $m \mapsto E_m$ *is continuous.*

Proof. Let $m > 0$ and $(m_k)_k \subset \mathbb{R}$ such that $m_k \to m$ in $\mathbb{R}$. We want to show that $E_{m_k} \to E_m$ as $k \to +\infty$. Firstly, we will prove that

$$\limsup_{k\to+\infty} E_{m_k} \leq E_m. \tag{3.17}$$

For any $u \in \mathcal{P}_m$ we define

$$u_k := \sqrt{\frac{m_k}{m}} u \in S_{m_k}, \quad k \in \mathbb{N}\,.$$

It is easy to see that $u_k \to u$ in $H^s(\mathbb{R}^N)$, thus, by Lemma 3.11 (iii) we get $\lim_{k\to+\infty} \rho(u_k) = \rho(u) = 0$. Therefore,

$$\rho(u_k) * u_k \to \rho(u) * u = 0 \quad \text{in } H^s(\mathbb{R}^N)$$

as $k \to +\infty$ and as a consequence

$$\limsup_{k\to+\infty} E_{m_k} \leq \limsup_{k\to+\infty} I(\rho(u_k) * u_k) = I(u).$$

Since this holds for any u, we obtain (3.17). The next step consists in proving

$$\liminf_{k\to+\infty} E_{m_k} \geq E_m. \tag{3.18}$$

From the definition of E_{m_k}, it follows that for every $k \in \mathbb{N}$ there exists $v_k \in \mathcal{P}_{m_k}$ such that

$$I(v_k) \leq E_{m_k} + \frac{1}{k}. \tag{3.19}$$

We set

$$t_k := \left(\frac{m}{m_k}\right)^{\frac{1}{N}} \quad \text{and} \quad \tilde{v}_k := v_k\left(\frac{\cdot}{t_k}\right) \in S_m.$$

By Lemma 3.11 and (3.19) we get

$$\begin{aligned}
E_m &\leq I(\rho(\tilde{v}_k) * \tilde{v}_k) \leq I(\rho(\tilde{v}_k) * v_k) + |I(\rho(\tilde{v}_k) * \tilde{v}_k) - I(\rho(\tilde{v}_k) * v_k)| \\
&\leq I(v_k) + |I(\rho(\tilde{v}_k) * \tilde{v}_k) - I(\rho(\tilde{v}_k) * v_k)| \\
&\leq E_{m_k} + \frac{1}{k} + |I(\rho(\tilde{v}_k) * \tilde{v}_k) - I(\rho(\tilde{v}_k) * v_k)| \\
&=: E_{m_k} + \frac{1}{k} + C(k).
\end{aligned}$$

To prove (3.18) we show that $\lim_{k\to+\infty} C(k) = 0$. Indeed, as a first step, we notice that $\rho * \left(v\left(\frac{\cdot}{t}\right)\right) = (\rho * v)\left(\frac{\cdot}{t}\right)$, and after a change of variable, we get

$$\begin{aligned} C(k) &= \left|\frac{1}{2}\left(t_k^{N-2s}-1\right)[\rho(\tilde{v}_k)*v_k]^2_{H^s(\mathbb{R}^N)} - \left(t_k^N - 1\right)\int_{\mathbb{R}^N} F(\rho(\tilde{v}_k)*v_k)\,dx\right| \\ &\leq \frac{1}{2}\left|t_k^{N-2s}-1\right|[\rho(\tilde{v}_k)*v_k]^2_{H^s(\mathbb{R}^N)} + \left|t_k^N-1\right|\int_{\mathbb{R}^N}|F(\rho(\tilde{v}_k)*v_k)|\,dx \\ &=: \frac{1}{2}\left|t_k^{N-2s}-1\right|A(k) + \left|t_k^N-1\right|B(k). \end{aligned}$$

Since $t_k \to 1$ as $k \to +\infty$, it suffices to prove that

$$\limsup_{k\to+\infty} A(k) < \infty, \quad \limsup_{k\to+\infty} B(k) < \infty. \tag{3.20}$$

We divide the proof of (3.20) into three claims.

Claim 1: $(v_k)_k$ is bounded in $H^s(\mathbb{R}^N)$.
Recalling (3.17) and (3.19) we have that

$$\limsup_{k\to+\infty} I(v_k) \leq E_m.$$

Thus, observing that $v_k \in \mathcal{P}_{m_k}$ and $m_k \to m$, from Remark 3.13 it follows the validity of the claim.

Claim 2: $(\tilde{v}_k)_k$ is bounded in $H^s(\mathbb{R}^N)$, and there are a sequence $(y_k)_k \subset \mathbb{R}$ and $v \in H^s(\mathbb{R}^N)\setminus\{0\}$ such that $\tilde{v}(\cdot + y_k) \to v$ a.e. in $\mathbb{R}^N$ up to a subsequence.
To see the boundedness of $(\tilde{v}_k)_k$ it suffices to notice that $t_k \to 1$ and the statement follows by claim 1. Now, we set

$$\alpha = \limsup_{k\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |\tilde{v}_k|^2\,dx.$$

If $\alpha = 0$, by [102, Lemma II.4] we get $\tilde{v}_k \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$. As a consequence of that, we have

$$\int_{\mathbb{R}^N} |v_k|^{2+\frac{4s}{N}}\,dx = \int_{\mathbb{R}_N} |\tilde{v}_k(t_k\cdot)|^{2+\frac{4s}{N}}\,dx = t_k^{-N}\int_{\mathbb{R}^N}|\tilde{v}_k|^{2+\frac{4s}{N}}\,dx \to 0$$

as $k \to +\infty$, and since $P(v_k) = 0$, by Lemma 3.12 (ii), we deduce that

$$[v_k]^2_{H^s(\mathbb{R}^N)} = \frac{N}{2s}\int_{\mathbb{R}^N} \tilde{F}(v_k)\,dx \to 0.$$

In this case, by virtue of Remark 3.6, we see that

$$0 = P(v_k) \geq \frac{1}{2}[v_k]^2_{H^s(\mathbb{R}^N)},$$

which is admissible only if v_k in constant. But this is in contradiction with the fact that $v_k \in \mathcal{P}_{m_k}$. Hence α must be strictly positive.

Claim 3: $\limsup_{k\to+\infty} \rho(\tilde{v}_k) < \infty$.

By contradiction we assume that up to a subsequence $\rho(\tilde{v}_k) \to \infty$ as $k \to +\infty$. By Claim 2 we can suppose the existence of a sequence $(y_k)_k \subset \mathbb{R}^N$ and $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that

$$\tilde{v}_k(\cdot + y_k) \to v \quad \text{a.e. in } \mathbb{R}^N. \tag{3.21}$$

On the other hand, by Lemma 3.11 we get

$$\rho(\tilde{v}_k(\cdot + y_k)) = \rho(\tilde{v}_k) \to \infty \tag{3.22}$$

and

$$I(\rho(\tilde{v}_k(\cdot + y_k)) * \tilde{v}_k(\cdot + y_k)) \geq 0. \tag{3.23}$$

Now, taking into account (3.21), (3.22), (3.23) and arguing similarly as we have already done to prove (3.13) we have a contradiction. The proof concludes by observing that by Claims 1 and 3

$$\limsup_{k\to+\infty} \|\rho(\tilde{v}_k) * v_k\|_{H^s(\mathbb{R}^N)} < \infty. \tag{3.24}$$

Hence, by virtue of $(f_0) - (f_2)$ and (3.24), (3.20) holds true. □

The next result provides a weak monotonicity property for E_m.

Lemma 3.16. *If $(f_0) - (f_4)$ hold, then $m \mapsto E_m$ is non-increasing in $(0, \infty)$.*

Proof. It suffices to show that for all $\varepsilon > 0$ and m, $m' > 0$ with $m > m'$ we have

$$E_m \leq E_{m'} + \frac{\varepsilon}{2}. \tag{3.25}$$

Now, we take $\chi \in C_c^\infty(\mathbb{R}^N)$ radial such that

$$\chi(x) = \begin{cases} 1 & |x| \leq 1 \\ \in [0,1] & 1 < |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

and $u \in \mathcal{P}_{m'}$. For every $\delta > 0$ we set $u_\delta(x) = u(x)\chi(\delta x)$. By a result of Palatucci *et al.*, see [95, Lemma 5 of Section 6.1], we know that $u_\delta \to u$ as $\delta \to 0^+$, and using Lemma 3.11 *(iii)* we obtain

$$\lim_{\delta\to 0^+} \rho(u_\delta) = \rho(u) = 0.$$

As a consequence of that, we obtain

$$\rho(u_\delta) * u_\delta \to \rho(u) * u \quad \text{in } H^s(\mathbb{R}^N) \tag{3.26}$$

as $\delta \to 0^+$. Now, fixing $\delta > 0$ small enough, by virtue of (3.26) we have

$$I(\rho(u_\delta) * u_\delta) \leq I(u) + \frac{\varepsilon}{4}. \tag{3.27}$$

After that, we choose $v \in C_c^\infty(\mathbb{R}^N)$ with $\mathrm{supp}(v) \subset B\left(0, 1 + \frac{4}{\delta}\right) \setminus B\left(0, \frac{4}{\delta}\right)$ and we set

$$\tilde{v} = \frac{m - \|u_\delta\|_2^2}{\|v\|_2^2} v(x)$$

For every $\lambda \leq 0$ we also define $\omega_\lambda = u_\delta + \lambda * \tilde{v}$. We observe that choosing λ appropriately we have $\mathrm{supp}(u_\delta) \cap \mathrm{supp}(\lambda * \tilde{v}) = \emptyset$, thus $\omega_\lambda \in S_m$.

Claim: $\rho(\omega_\lambda)$ is upper bounded as $\lambda \to -\infty$.

If the claim does not hold, we observe that by Lemma 3.11 (ii) $I(\rho(\omega_\lambda) * \omega_\lambda) \geq 0$ and that $\omega_\lambda \to u_\delta$ a.e. in $\mathbb{R}^N$ as $\lambda \to -\infty$. Hence, arguing as we have already done to obtain (3.13) we reach a contradiction. Then the claim must hold.

By virtue of the claim

$$\rho(\omega_\lambda) + \lambda \to -\infty \quad \text{as } \lambda \to -\infty,$$

thus

$$[(\rho(\omega_\lambda) + \lambda) * \tilde{v}]^2_{H^s(\mathbb{R}^N)} = e^{2s(\rho(\omega_\lambda)+\lambda)} \, [\tilde{v}]^2_{H^s(\mathbb{R}^N)} \to 0$$

implying

$$\|(\rho(\omega_\lambda) + \lambda) * \tilde{v}\|_{2+\frac{4s}{N}} \leq C\|(\rho(\omega_\lambda) + \lambda) * \tilde{v}\|_2^{\frac{2s}{N}} \, [(\rho(\omega_\lambda) + \lambda) * \tilde{v}]_{H^s(\mathbb{R}^N)}^{\frac{N-2s}{N}} \to 0.$$

As a consequence, by Lemma 3.5 (ii), for a suitable λ

$$I((\rho(\omega_\lambda) + \lambda) * \tilde{v}) \leq \frac{\varepsilon}{4}. \tag{3.28}$$

Finally, by Lemma 3.11 and using (3.25), (3.27) and (3.28) it easy to see that

$$\begin{aligned} E_m &\leq I(\rho(\omega_\lambda) * \omega_\lambda) = I(\rho(\omega_\lambda) * u_\delta) + I(\rho(\omega_\lambda) * (\lambda * \tilde{v})) \\ &\leq I(\rho(u_\delta) * u_\delta) + I((\rho(\omega_\lambda) + \lambda) * \tilde{v}) \\ &\leq I(u) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq E_{m'} + \varepsilon \end{aligned}$$

completing the proof. □

The strict monotonicity of E_m holds true only locally, as we now show.

Lemma 3.17. *Assume $(f_0) - (f_4)$ hold true. Moreover, let $u \in S_m$ and $\mu \in \mathbb{R}$ such that*

$$(-\Delta)^s \, u + \mu u = f(u)$$

and $I(u) = E_m$. Then $E_m > E_{m'}$ for every $m' > m$ close enough if $\mu > 0$ and for any $m' < m$ close enough if $\mu < 0$.

Proof. Let $t > 0$ and $\rho \in \mathbb{R}$. Defining $u_{t,\rho} := u(\rho * (tu)) \in S_{mt^2}$ and

$$\alpha(t,\rho) := I(u_{t,\rho}) = \frac{1}{2}t^2 e^{2\rho s}\,[u]^2_{H^s(\mathbb{R}^N)} - e^{-N\rho}\int_{\mathbb{R}^N} F(te^{\frac{N\rho}{2}}u)\,dx,$$

it is straightforward to verify that

$$\begin{aligned}\frac{\partial}{\partial t}\alpha(t,\rho) &= te^{2\rho s}\,[u]^2_{H^s(\mathbb{R}^N)} - e^{-N\rho}\int_{\mathbb{R}^N} f\left(te^{\frac{N\rho}{2}}u\right)e^{\frac{N\rho}{2}}u\,dx \\ &= t^{-1}I'(u_{t,\rho})\,[u_{t,\rho}]\,.\end{aligned}$$

In the case $\mu > 0$, we observe that $u_{t,\rho} \to u$ in $H^s(\mathbb{R}^N)$ as $(t,\rho) \to (1,0)$. Moreover, we notice that

$$I'(u)\,[u] = -\mu\|u\|_2^2 = -\mu m < 0$$

and so, choosing $\delta > 0$ small enough, we have

$$\frac{\partial \alpha}{\partial t}(t,\rho) < 0 \quad \text{for any } (t,\rho) \in (1, 1+\delta) \times [-\delta,\delta]\,.$$

Using the Mean Value Theorem, there exists $\xi \in (1,t)$ such that

$$\frac{\partial \alpha}{\partial t}(\xi,\rho) = \frac{\alpha(t,\rho) - \alpha(1,\rho)}{t-1}$$

whenever $(t,\rho) \in (1,1+\delta) \times [-\delta,\delta]$, hence

$$\alpha(t,\rho) = \alpha(1,\rho) + (t-1)\frac{\partial}{\partial t}\alpha(\xi,\rho) < \alpha(1,\rho). \tag{3.29}$$

Since by Lemma 3.11 *(iii)* $\rho(tu) \to \rho(u) = 0$ as $t \to 1^+$, setting for any $m' > m$ close enough to m

$$t := \sqrt{\frac{m'}{m}} \in (1, 1+\delta) \quad \text{and} \quad \rho := \rho(tu) \in [-\delta,\delta]\,,$$

and using (3.29) together with Lemma 3.11 *(ii)* we obtain that

$$E_{m'} \le \alpha(t,\rho(tu)) < \alpha(1,\rho(tu)) = I(\rho(tu) * u) \le I(u) = E_m.$$

The proof for $\mu < 0$ is similar, and we omit it. □

As a direct consequence of the previous two Lemmas, we have the following result.

Lemma 3.18. *Assume $(f_0) - (f_4)$ hold true. In addition, let $u \in S_m$ and $\mu \in \mathbb{R}$ such that $(-\Delta)^s u + \mu u = f(u)$ with $I(u) = E_m$. Then $\mu \ge 0$, and if $\mu > 0$ it is $E_m > E_{m'}$ for any $m' > m > 0$.*

To make a step ahead, we describe the asymptotic behaviour of E_m as $m \to 0^+$ and $m \to +\infty$.

Lemma 3.19. *Assume $(f_0) - (f_4)$ hold true, then $E_m \to +\infty$ as $m \to 0^+$.*

Proof. In order to prove the Lemma, we will show that for every sequence $(u_n)_n \subset H^s(\mathbb{R}^N) \setminus \{0\}$ such that

$$P(u_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|u_n\|_2 = 0$$

it must be $I(u_n) \to +\infty$. We set

$$\rho_n := \frac{1}{s} \log\left([u_n]_{H^s(\mathbb{R}^N)}\right) \quad \text{and} \quad v_n := (-\rho_n) * u_n$$

Trivially $[v_n]_{H^s(\mathbb{R}^N)} = 1$ and $\|v_n\|_2 \to 0$. Moreover, thanks to these two facts we also have by interpolation that $v_n \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$, thus, by Lemma 3.5 (ii) we have

$$\lim_{n\to+\infty} e^{-N\rho} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho}{2}} v_n\right) dx = 0.$$

Since $P(\rho_n * v_n) = P(u_n) = 0$, using Lemma 3.11 (i) and (ii) we obtain that

$$\begin{aligned} I(u_n) = I(\rho_n * v_n) \geq I(\rho * v_n) &= \frac{1}{2} e^{2\rho s} - e^{N\rho} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho}{2}} v_n\right) dx \\ &= \frac{1}{2} e^{2\rho s} + o_n(1). \end{aligned}$$

Since ρ is arbitrary, we get the statement as $\rho \to +\infty$. □

Lemma 3.20. *Assume $(f_0) - (f_4)$ and (f_6). Then $E_m \to 0$ as $m \to +\infty$.*

Proof. We fix $u \in L^\infty(\mathbb{R}^N) \cap S_1$ and we set $u_m = \sqrt{m} u \in S_m$. By Lemma 3.11 (ii) we can find a unique $\rho(m) \in \mathbb{R}$ such that $\rho(m) * u_m \in \mathcal{P}_m$. Since by Lemma 3.9 (i) F is non-negative, we get

$$0 < E_m \leq I(\rho(m) * u_m) \leq \frac{1}{2} e^{2\rho(m)s} [u]^2_{H^s(\mathbb{R}^N)}. \tag{3.30}$$

Thus, by (3.30) it suffices to show that

$$\lim_{m\to\infty} \sqrt{m}\, e^{\rho(m)s} = 0. \tag{3.31}$$

Using the function g defined in Remark 3.8, and recalling that $P(\rho(m) * u_m) = 0$ we get

$$[u]^2_{H^s(\mathbb{R}^N)} = \frac{N}{2s} m^{\frac{2s}{N}} \int_{\mathbb{R}^N} g\left(\sqrt{m} e^{\frac{N\rho(m)}{2}} u\right) |u|^{2+\frac{4s}{N}} dx,$$

which implies

$$\lim_{m\to\infty} \sqrt{m}\, e^{\frac{N\rho(m)}{2}} = 0. \tag{3.32}$$

Now, using (f_6) and Lemma 3.10, for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\tilde{F}(t) \geq \frac{4s}{N} F(t) \geq \frac{1}{\varepsilon} |t|^{\frac{2N}{N-2s}}$$

if $|t| \leq \delta$. Hence, taking into account the fact that $P(\rho(m) * u_m) = 0$ and (3.32), we get

$$\begin{aligned}[u]^2_{H^s(\mathbb{R}^N)} &= \frac{N}{2s} \frac{1}{m} e^{-(N+2s)\rho(m)} \int_{\mathbb{R}^N} \tilde{F}\left(\sqrt{m} e^{\frac{N\rho(m)}{2}} u\right) dx \\ &\geq \frac{N}{2s} \frac{1}{\varepsilon} \left(\sqrt{m} e^{\rho(m)s}\right)^{\frac{4s}{N-2s}} \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}} dx\end{aligned}$$

for m large enough. Then (3.31) holds and the proof is complete. □

3.3 Ground-states

We introduce the functional

$$\Psi(u) = I(\rho(u) * u) = \frac{1}{2} e^{2\rho(u)s} [u]^2_{H^s(\mathbb{R}^N)} - e^{-N\rho(u)} \int_{\mathbb{R}^N} F\left(e^{\frac{N\rho(u)}{2}} u\right) dx.$$

Throughout this section we will assume that f satisfies $(f_0) - (f_5)$.

Lemma 3.21. *The functional $\Psi \colon H^s(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}$ is of class C^1, and*

$$d\Psi(u)[\varphi] = dI(\rho(u) * u)[\rho(u) * \varphi]$$

for every $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ and $\varphi \in H^s(\mathbb{R}^N)$.

Proof. A proof appears in [58] for the case $s = 1$. Only minor adjustments are needed in the fractional case, so we omit the details. □

For $m > 0$, we consider the constrained functional $J \colon S_m \to \mathbb{R}$ defined by $J = \Psi_{|S_m}$. Lemma 3.21 yields the following statement.

Lemma 3.22. *The functional $J \colon S_m \to \mathbb{R}$ is C^1 and*

$$dJ(u)[\varphi] = d\Psi(u)[\varphi] = dI(\rho(u) * u)[\rho(u) * \varphi]$$

for any $u \in S_m$ and $\varphi \in T_u S_m$, where $T_u S_m$ is the tangent space at u to the manifold S_m.

We recall from [50, Definition 3.1] a definition that will be useful to construct a min-max principle.

Definition 3.23. Let B be a closed subset of a metric space X. We say that a class $\mathcal{G}$ of compact subsets of X is a homotopy stable family with closed boundary B provided that

(i) every set in $\mathcal{G}$ contains B,

(ii) for any set A in $\mathcal{G}$ and any homotopy $\eta \in C([0,1] \times X, X)$ that satisfies $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times X) \cup ([0,1] \times B)$, one has $\eta(\{1\} \times A) \in \mathcal{G}$.

We remark that $B = \emptyset$ is admissible.

Lemma 3.24. *Let $\mathcal{G}$ be a homotopy stable family of compact subsets (with $B = \emptyset$). We set*

$$E_{m,\mathcal{G}} = \inf_{A \in \mathcal{G}} \max_{u \in A} J(u).$$

If $E_{m,\mathcal{G}} > 0$, then there exists a Palais-Smale sequence $(u_n)_n \in \mathcal{P}_m$ for the constrained functional $I_{|S_m}$ at level $E_{m,\mathcal{G}}$. In particular, if $\mathcal{G}$ is the class of all singletons in S_m, one has that $\|u_n^-\|_2 \to 0$ as $n \to +\infty$.

Proof. Let $(A_n)_n \subset \mathcal{G}$ be a minimizing sequence of $E_{m,\mathcal{G}}$. We define the map

$$\eta \colon [0,1] \times S_m \to S_m$$

where $\eta(t,u) = (t\rho(u)) * u$ is continuous and well-defined by Lemma 3.11 (i) and (iii). Noticing $\eta(t,u) = u$ for every $(t,u) \in \{0\} \times S_m$ we obtain that

$$D_n := \eta(1, A_n) = \{\rho(u) * u \mid u \in A_n\} \in \mathcal{G}.$$

In particular we can see that $D_n \subset \mathcal{P}_m$ for any $n \geq 1$, with $m > 0$. Since $J(\rho(u) * u) = J(u)$ for every $\rho \in \mathbb{R}$ and $u \in S_m$, we can observe that

$$\max_{u \in D_n} J(u) = \max_{u \in A_n} J(u) \to E_{m,\mathcal{G}}$$

thus, $(D_n)_n$ is another minimizing sequence for $E_{m,\mathcal{G}}$. Now, using [50, Theorem 3.2] we get a Palais-Smale sequence $(v_n)_n \subset S_m$ for J at level $E_{m,\mathcal{G}}$ such that $\mathrm{dist}_{H^s(\mathbb{R}^N)}(v_n, D_n) \to 0$ as $n \to +\infty$. We will denote

$$\rho_n := \rho(v_n) \quad \text{and} \quad u_n := \rho_n * v_n.$$

Claim: There exists $C > 0$ such that $e^{-2\rho_n s} \leq C$ for any $n \in \mathbb{N}$.
We start pointing out that

$$e^{-2\rho_n s} = \frac{[v_n]^2_{H^s(\mathbb{R}^N)}}{[u_n]^2_{H^s(\mathbb{R}^N)}}.$$

By virtue of the fact that $(u_n)_n \subset \mathcal{P}_m$, using Lemma 3.12 (ii), we obtain that $\left\{[u_n]_{H^s(\mathbb{R}^N)}\right\}_n$ is bounded from below. Moreover, since $D_n \subset \mathcal{P}_m$ and that

$$\max_{u \in D_n} I = \max_{u \in D_n} J \to E_{m,\mathcal{G}},$$

Lemma 3.12 (iv) implies that D_n is uniformly bounded in $H^s(\mathbb{R}^N)$. Finally, from $\text{dist}_{H^s(\mathbb{R}^N)}(v_n, D_n) \to 0$ we can deduce that $\sup_{n\in\mathbb{N}} [v_n]_{H^s(\mathbb{R}^N)} < \infty$. Thus, the claim holds.

Now, from $(u_n) \subset \mathcal{P}_m$ we get

$$I(u_n) = J(u_n) = J(v_n) \to E_{m,\mathcal{G}}.$$

On the other hand, for any $\psi \in T_{u_n} S_m$ we have

$$\begin{aligned} \int_{\mathbb{R}^N} v_n\,[(-\rho_n) * \psi]\,dx &= \int_{\mathbb{R}^N} v_n e^{-\frac{N\rho_n}{2}} \psi\left(e^{-\rho_n} x\right) dx = \int_{\mathbb{R}^N} e^{\frac{N\rho_n}{2}} v_n\left(e^{\rho_n} x\right) \psi\,dx \\ &= \int_{\mathbb{R}^N} (\rho_n * v_n)\psi\,dx = \int_{\mathbb{R}^N} u_n \psi\,dx = 0 \end{aligned}$$

implying $(-\rho_n * \psi) \in T_{v_n} S_m$. Besides, by the claim

$$\|(-\rho_n) * v_n\|_{H^s(\mathbb{R}^N)} \leq \max\{C, 1\} \|\psi\|_{H^s(\mathbb{R}^N)}.$$

Denoting by $\|\cdot\|_{u,*}$ the dual norm of the space $(T_u S_m)^*$ and using Lemma 3.22 we get

$$\begin{aligned} \|dI(u_n)\|_{u_n,*} &= \sup_{\substack{\psi\in T_{u_n} S_m \\ \|\psi\|_{H^s(\mathbb{R}^N)}\leq 1}} |dI(u_n)\,[\psi]\,| = \sup_{\substack{\psi\in T_{u_n} S_m \\ \|\psi\|_{H^s(\mathbb{R}^N)}\leq 1}} |dI(\rho_n * v_n)\,[\rho_n * ((-\rho_n) * \psi)]\,| \\ &= \sup_{\substack{\psi\in T_{u_n} S_m \\ \|\psi\|_{H^s(\mathbb{R}^N)}\leq 1}} |dJ(v_n)\,[(-\rho_n) * \psi]\,| \\ &\leq \|dJ(v_n)\|_{v_n,*} \sup_{\substack{\psi\in T_{u_n} S_m \\ \|\psi\|_{H^s(\mathbb{R}^N)}\leq 1}} \|(-\rho_n) * \psi\|_{H^s(\mathbb{R}^N)} \\ &\leq \max\{C, 1\}\ \|dJ(v_n)\|_{v_n,*} \to 0 \end{aligned}$$

as $n \to +\infty$ remembering that $(v_n)_n$ is a Palais-Smale sequence for the functional J. We have just proved $(u_n)_n$ is a Palais-Smale sequence for the functional $I_{|S_m}$ at level $E_{m,\mathcal{G}}$ with the additional property that $(u_n)_n \subset \mathcal{P}_m$. Finally, noticing that the family of singleton of S_m is a particular homotopy stable family of compact subsets of S_m, and doing this particular choice as $\mathcal{G}$, arguing similarly as we have just done, we can obtain a minimizing sequence $(D_n)_n$ with the additional property that its elements are non-negative: we only need to replace the functions with their absolute value. Moreover, $(A_n)_n$ will inherit this property, and recalling that $\text{dist}_{H^s(\mathbb{R}^N)}(v_n, D_n) \to 0$ as $n \to +\infty$ we have

$$\|u_n^-\|_2 = \|\rho_n * v_n^-\|_2 = \|v_n^-\|_2 \to 0.$$

This concludes the proof of the Lemma. □

Lemma 3.25. *We assume* $(f_0) - (f_4)$ *hold. Then there exists a Palais-Smale sequence* $(u_n)_n \subset \mathcal{P}_m$ *for the constrained functional* $I_{|S_m}$ *at level* E_m *such that* $\|u_n^-\|_2 \to 0$ *as* $n \to +\infty$.

Proof. We apply Lemma 3.24 with $\mathcal{G}$ the class of all singletons in S_m. Lemma 3.12 imply that $E_m > 0$, thus the only thing that remains to prove is $E_m = E_{m,\mathcal{G}}$. In order to do that, as a first step, we notice that

$$E_{m,\mathcal{G}} = \inf_{A\in\mathcal{G}} \max_{u\in A} J(u) = \inf_{u\in S_m} I(\rho(u) * u).$$

Since for every $u \in S_m$ we have that $\rho(u) * u \in \mathcal{P}_m$ it must be $I(\rho(u) * u) \geq E_m$, thus $E_{m,\mathcal{G}} \geq E_m$. On the other hand, if $u \in \mathcal{P}_m$ we have $\rho(u) = 0$ and $I(u) \geq E_{m,\mathcal{G}}$, that implies $E_m \geq E_{m,\mathcal{G}}$. □

Lemma 3.26. *Let $(u_n)_n \subset S_m$ be a bounded Palais-Smale sequence for the constrained functional $I_{|S_m}$ at level $E_m > 0$ such that $P(u_n) \to 0$ as $n \to +\infty$. Then we have the existence of $u \in S_m$ and $\mu > 0$ such that, up to a subsequence and translations in $\mathbb{R}^N$, $u_n \to u$ strongly in $H^s(\mathbb{R}^N)$ and*

$$(-\Delta)^s u + \mu u = f(u).$$

Proof. It is clear that $(u_n)_n \subset S_m$ is bounded in $H^s(\mathbb{R}^N)$ and is a Palais-Smale sequence. Together, these two facts enable us to assume without loss of generality that $\lim_{n\to+\infty} [u_n]_{H^s(\mathbb{R}^N)}$, $\lim_{n\to+\infty} \int_{\mathbb{R}^N} F(u_n)\,dx$, and $\lim_{n\to+\infty} \int_{\mathbb{R}^N} f(u_n)u_n\,dx$ exist. Besides, [21, Lemma 3] implies

$$(-\Delta)^s u_n + \mu_n u_n - f(u_n) \to 0 \quad \text{in } H^s(\mathbb{R}^N)^*$$

where we denote

$$\mu_n = \frac{1}{m}\left(\int_{\mathbb{R}^N} f(u_n)u_n\,dx - [u_n]^2_{H^s(\mathbb{R}^N)}\right).$$

By the assumptions made above, we can see that $\mu_n \to \mu$ for some $\mu \in \mathbb{R}$ and we also have that for any $(y_n)_n \subset \mathbb{R}^N$

$$(-\Delta)^s u_n(\cdot + y_n) + \mu u_n(\cdot + y_n) - f(u_n(\cdot + y_n)) \to 0 \quad \text{in } H^s(\mathbb{R}^N)^*. \tag{3.33}$$

Claim: $(u_n)_n$ is non vanishing.

Otherwise, by [102, Lemma II.4] we would get $u_n \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$. Taking into account that $P(u_n) \to 0$ and using Lemma 3.5 *(ii)* we get

$$[u_n]^2_{H^s(\mathbb{R}^N)} = P(u_n) + \frac{N}{2s}\int_{\mathbb{R}^n} \tilde{F}(u_n)\,dx \to 0$$

and as a consequence of that,

$$E_m = \lim_{n\to+\infty} I(u_n) = \frac{1}{2}\lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} - \lim_{n\to+\infty}\int_{\mathbb{R}^N} F(u_n)\,dx = 0$$

contradicting $E_m > 0$. Then the claim must hold.

Since $(u_n)_n$ in non-vanishing we can find $(y_n^1)_n \subset \mathbb{R}^N$ and $\omega_1 \in B_m \setminus \{0\}$ such that $u_n(\cdot+y_n^1) \rightharpoonup \omega_1$ in $H^s(\mathbb{R}^N)$, $u_n(\cdot+y_n^1) \to \omega_1$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for $p \in [1, 2_s^*)$ and $u_n(\cdot+y_n^1) \to \omega$ a.e. in $\mathbb{R}^N$. Now, we want to apply [20, Theorem A.1] with $P(t) = f(t)$ and $Q(t) = |t|^{(N+2s)/(N-2s)}$ and notice that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \left|\left[f(u_n(\cdot + y_n^1) - f(\omega_1)\right] \varphi\right| \, dx \\ \leq \|\varphi\|_{L^\infty(\mathbb{R}^N)} \lim_{n\to+\infty} \int_{\mathrm{supp}(\varphi)} \left|f(u_n(\cdot + y_n^1) - f(\omega_1)\right| \, dx \quad (3.34)$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Hence, by (3.33) and (3.34) we get

$$(-\Delta)^s \omega_1 + \mu\omega_1 = f(\omega_1) \quad (3.35)$$

and through the Pohozaev Identity (see for instance [36, Proposition 4.1]) associated to (3.35) we also have $P(\omega_1) = 0$. Now, we set $v_n^1 := u_n - \omega_1(\cdot - y_n^1)$ for every $n \in \mathbb{N}$. Clearly $v_n^1(\cdot + y_n^1) = u_n(\cdot + y_n^1) - \omega_1 \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, thus

$$m = \lim_{n\to+\infty} \|u_n(\cdot + y_n^1)\|_2 = \lim_{n\to+\infty} \|v_n^1\|_2^2 + \|\omega_1\|_2^2. \quad (3.36)$$

By Lemma 3.14 we also have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} F(u_n(\cdot + y_n^1)) \, dx = \int_{\mathbb{R}^N} F(\omega_1) \, dx + \lim_{n\to+\infty} \int_{\mathbb{R}^N} F(v_n^1(\cdot + y_n^1)) \, dx$$

hence

$$E_m = \lim_{n\to+\infty} I(u_n) = \lim_{n\to+\infty} I(u_n(\cdot + y_n^1)) = \lim_{n\to+\infty} I(v_n^1(\cdot + y_n^1)) + I(\omega_1) \quad (3.37) \\ = \lim_{n\to+\infty} I(v_n^1) + I(\omega_1).$$

Claim: $\lim_{n\to+\infty} I(v_n^1) \geq 0$.

If the claim does not hold, i.e. $\lim_{n\to+\infty} I(v_n^1) < 0$, $(v_n^1)_n$ is non vanishing, then there exists $(y_n^2)_n \subset \mathbb{R}^N$ such that

$$\lim_{n\to+\infty} \int_{B(y_n^2,1)} |v_n^1|^2 \, dx > 0.$$

Since $v_n^1(\cdot + y_n^1) \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, it must be $|y_n^2 - y_n^1| \to \infty$, and up to a subsequence $v_n^1(\cdot + y_n^2) \to \omega_2$ in $H^s(\mathbb{R}^N)$ for some $\omega_2 \in B_m \setminus \{0\}$. We notice

$$u_n(\cdot + y_n^2) = v_n^1(\cdot + y_n^2) + \omega_1(\cdot - y_n^1 + y_n^2) \rightharpoonup \omega_2$$

thus, arguing as before, we get $P(\omega_2) = 0$ and $I(\omega_2) > 0$. We set

$$v_n^2 = v_n^1 - \omega_2(\cdot - y_n^2) = u_n - \sum_{\ell=1}^{2} \omega_\ell(\cdot - y_n^\ell)$$

and we observe that

$$\begin{aligned}\lim_{n\to+\infty}\left[v_n^2\right]^2_{H^s(\mathbb{R}^N)} &= \lim_{n\to+\infty}\left[v_n^1\right]^2_{H^s(\mathbb{R}^N)} + [\omega_2]^2_{H^s(\mathbb{R}^N)} - 2\lim_{n\to+\infty}\langle v_n^1, \omega_2(\cdot - y_n^2)\rangle_{H^s(\mathbb{R}^N)} \\ &= \lim_{n\to+\infty}\left[v_n^1\right]^2_{H^s(\mathbb{R}^N)} + [\omega_2]^2_{H^s(\mathbb{R}^N)} - 2\lim_{n\to+\infty}\langle v_n^1(\cdot + y_n^2), \omega_2\rangle_{H^s(\mathbb{R}^N)} \\ &= \lim_{n\to+\infty}[u_n]^2_{H^s(\mathbb{R}^N)} + [\omega_1]^2_{H^s(\mathbb{R}^N)} - [\omega_2]^2_{H^s(\mathbb{R}^N)} \\ &\quad - 2\lim_{n\to+\infty}\langle u_n(\cdot + y_n^1), \omega_1\rangle_{H^s(\mathbb{R}^N)} \\ &= \lim_{n\to+\infty}[u_n]^2_{H^s(\mathbb{R}^N)} - \sum_{\ell=1}^{2}[\omega_\ell]^2_{H^s(\mathbb{R}^N)}\end{aligned}$$

and

$$0 > \lim_{n\to+\infty} I(v_n^1) = I(\omega_2) + \lim_{n\to+\infty} I(v_n^2) > \lim_{n\to+\infty} I(v_n^2).$$

Iterating, we can build an infinite sequence $(\omega_k) \subset B_m \setminus \{0\}$ such that $P(\omega_k) = 0$ and

$$\sum_{\ell=1}^{k}[\omega_k]^2_{H^s(\mathbb{R}^N)} \le [u_n]^2_{H^s(\mathbb{R}^N)} < \infty$$

for every $k \in \mathbb{N}$. However, this is a contradiction. Indeed, recalling Remark 3.6, for any $\omega \in B_m \setminus \{0\}$ such that $P(\omega) = 0$, we can find $\delta > 0$ such that $[\omega]^2_{H^s(\mathbb{R}^N)} \ge \delta$. Therefore, the claim must be valid and $\lim_{n\to+\infty} I(v_n^1) \ge 0$.

Now, we denote by $h := \|\omega_1\|_2^2 \in (0, m]$. By virtue of the claim, (3.37) and the fact that $\omega_1 \in \mathcal{P}_h$, we get

$$E_m = I(\omega_1) + \lim_{n\to+\infty} I(v_n^1) \ge I(\omega^1) \ge E_h$$

but, recalling that E_m in non-increasing by Lemma 3.16, we obtain

$$I(\omega_1) = E_m = E_h \tag{3.38}$$

and

$$\lim_{n\to+\infty} I(v_n^1) = 0. \tag{3.39}$$

To prove that $\mu \ge 0$ it suffices to put together (3.35), (3.38) and Lemma 3.18. Instead, to see that μ is strictly positive, using (f_5), Lemma 3.7 and the Pohozaev identity corresponding to (3.35), we get

$$\mu = \frac{1}{ms}\int_{\mathbb{R}^N}\left(NF(\omega_1) - \frac{N-2s}{2}f(\omega_1)\omega_1\right)dx > 0. \tag{3.40}$$

At this point, we suppose by contradiction that $h < m$, but taking into account (3.35), (3.40) and Lemma 3.18 we would have

$$I(\omega_1) = E_h > E_m$$

which is not compatible with (3.38). Thus $h = m$. Moreover, by (3.36) $v_n^1 \to 0$ in $L^2(\mathbb{R}^N)$. It remains only to prove the strong convergence of $(v_n^1)_n$ in $H^s(\mathbb{R}^N)$. To do that, it is sufficient to notice that by Lemma 3.5 (ii) we have $\lim_{n\to+\infty} \int_{\mathbb{R}^N} F(v_n^1)\, dx = 0$, and so we obtain the assertion thanks to (3.39). □

Proof of theorem 3.2. Applying Lemma 3.25 we obtain a Palais-Smale sequence $(u_n)_n \subset \mathcal{P}_m$ at level $E_m > 0$ for the constrained functional $I_{|S_m}$. This sequence is bounded in $H^s(\mathbb{R}^N)$ by Lemma 3.12 and through Lemma 3.26 we get a critical point $u \in S_m$ at the level $E_m > 0$ that results to be a ground-state energy. Finally, since $\|u_n^-\|_2 \to 0$ we deduce that $u \geq 0$ and after applying the strong maximum principle we obtain $u > 0$. □

Proof of theorem 3.3. The proof is a direct consequence of Theorem 3.2 and Lemmas 3.12, 3.15, 3.16, 3.19, 3.20. □

3.4 Existence of radial solutions

This section is devoted to proving the existence of infinitely many radial solutions to problem (P_m). Before doing this, we recall some basic definitions and provide some notation.

Denote by $\sigma\colon H^s(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$ the transformation $\sigma(u) = -u$ and let $X \subset H^s(\mathbb{R}^N)$. A set $A \subset X$ is called σ-invariant if $\sigma(A) = A$. A homotopy $\eta\colon [0,1] \times X \to X$ is σ-equivariant if $\eta(t, \sigma(u)) = \sigma(\eta(t,u))$ for all $(t,u) \in [0,1] \times X$. Next definition is in [50, Definition 7.1].

Definition 3.27. Let B be a closed σ-invariant subset of $X \subset H^s(\mathbb{R}^N)$. We say that a class $\mathcal{G}$ of compact subsets of X is a σ-homotopy stable family with closed boundary B provided

(i) every set in $\mathcal{G}$ is σ-invariant.

(ii) every set in $\mathcal{G}$ contains B,

(iii) for any set A in $\mathcal{G}$ and any σ-equivariant homotopy $\eta \in C([0,1] \times X, X)$ that satisfies $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times X) \cup ([0,1] \times B)$, one has $\eta(\{1\} \times A) \in \mathcal{G}$.

We denote with $H_r^s(\mathbb{R}^N)$ the space of radially symmetric functions in $H^s(\mathbb{R}^N)$ and recall that $H_r^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R})$ compactly for all $p \in (2, 2_s^*)$ (see [70, Proposition I.1]).

In order to prove the main result of this section, we need to build a sequence of σ-homotopy stable families of compact subsets of $S_m \cap H_r^s(\mathbb{R}^N)$. We point out that in the above definition, the case in which $B = \emptyset$ is not excluded. The idea is borrowed from [58]. Let $(V_k)_k$ be a sequence of finite dimensional linear subspaces of $H_r^s(\mathbb{R}^N)$ such that $V_k \subset V_{k+1}$, $\dim V_k = k$ and $\bigcup_{k\geq 1} V_k$ is dense in $H_r^s(\mathbb{R}^N)$. Denote by π_k the orthogonal projection from $H_r^s(\mathbb{R}^N)$ onto V_k. We recall to the reader the definition of the genus of σ-invariant sets introduced by M. A. Krasnoselskii and we refer to [100, Section 7] or [2, chapter 10] for its basic properties.

Definition 3.28. Let A be a non-empty compact σ-invariant subset of $H_r^s(\mathbb{R}^N)$. The genus $\gamma(A)$ of A is the least integer k such that there exists $\phi \in C(H_r^s(\mathbb{R}^N), \mathbb{R}^k)$ such that ϕ is odd and $\phi(x) \neq 0$ for all $x \in A$. We set $\gamma(A) = \infty$ if there are no integers with the above property and $\gamma(\emptyset) = 0$.

Let $\mathcal{A}$ be the family of closed σ-invariant subsets of $S_m \cap H_r^s(\mathbb{R}^N)$. For each $k \in \mathbb{N}$, set

$$\mathcal{G}_k := \{A \in \mathcal{A} \mid \gamma(A) \geq k\}$$

and

$$E_{m,k} = \inf_{A \in \mathcal{G}_k} \max_{u \in A} J(u).$$

Next, we give a result about the weak convergence of the nonlinearity f.

Lemma 3.29. *Assume $(f_0)-(f_2)$ hold true. Let $(u_n)_n \subset H_r^s(\mathbb{R}^N)$. If $u_n \rightharpoonup u$ in $H_r^s(\mathbb{R}^N)$ for some $u \in H_r^s(\mathbb{R}^N)$, then $f(u_n) \rightharpoonup f(u)$ in $L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$.*

Proof. We borrow some ideas from [88, Theorem 2.6]. We start exploiting the compact embedding $H_r^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for any $p \in (2, 2_s^*)$. Hence, up to a subsequence, $u_n \to u$ in $L^p(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. From equation (3.5), we get

$$|f(u_n)|^{\frac{2N}{N+2s}} \leq C_\varepsilon |u_n|^{\frac{2N}{N-2s}} + C|u_n|^{2\frac{N+4s}{N+2s}}$$

for some C_ε, $C > 0$. As a consequence of that, recalling the fractional Sobolev inequality and observing that $2\frac{N+4s}{N+2s} \in (2, 2_s^*)$, we obtain that $(f(u_n))_n$ is bounded in $L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$. Thus, there exists $y \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ such that $f(u_n) \rightharpoonup y$. At this point, we fix a cover $(\Omega_j)_j$ of $\mathbb{R}^N$ made of subsets with finite measure. For any $\upsilon > 0$, Severini-Egorov's Theorem yields the existence of $B_\upsilon^j \subset \Omega_j$, with measure $\left|B_\upsilon^j\right| < \upsilon$, such that $u_n \to u$ uniformly in $\Omega_j \setminus B_\upsilon^j$. Clearly, $y = f(u)$ in $\Omega_j \setminus B_\upsilon^j$. Now, we set

$$\mathcal{Q} := \left\{x \in \mathbb{R}^N \mid y \neq f(u)\right\} \quad \text{and} \quad Q_j := \{x \in \Omega_j \mid y \neq f(u)\}.$$

Since υ is arbitrary and $Q_j \subset B_\upsilon^j$, we have that Q_j is a set of measure zero. Furthermore, it is easy to see that $\mathcal{Q} = \bigcup_{j=1}^\infty Q_j$, therefore Q has measure zero and the proof is complete. □

From now on, we will always assume $(f_0) - (f_5)$ hold until the end of the section.

Lemma 3.30. *Let $\mathcal{G}$ be a σ-homotopy stable family of compact subsets of $S_m \cap H_r^s(\mathbb{R}^N)$ (with $B = \emptyset$) and set*

$$E_{m,\mathcal{G}} := \inf_{A \in \mathcal{G}} \max_{u \in A} J(u).$$

If $E_{m,\mathcal{G}} > 0$ then there exists a Palais-Smale sequence $(u_n)_n$ in $\mathcal{P}_m \cap H_r^s(\mathbb{R}^N)$ for $I_{|S_m \cap H_r^s(\mathbb{R}^N)}$ at level $E_{m,\mathcal{G}}$.

Proof. It suffices to replace Theorem 3.2 with 7.2 of [50] in the proof of Lemma 3.24. □

Lemma 3.31. *For any $k \in \mathbb{N}$ we have*

(i) $\mathcal{G}_k \neq \emptyset$ and $\mathcal{G}_k$ is a σ-homotopy stable family of compact subsets of $S_m \cap H^s_r(\mathbb{R}^N)$ (with $B = \emptyset$),

(ii) $E_{m,k+1} \geq E_{m,k} > 0$.

Proof. (*i*) It suffices to notice that for any $k \in \mathbb{N}$ one has $S_m \cap V_k \in \mathcal{A}$ and that by [2, Theorem 10.5]

$$\gamma(S_m \cap V_k) = k.$$

Thus $\mathcal{G}_k \neq \emptyset$. The conclusion is a direct consequence of the definition of $\mathcal{A}$.

(*ii*) By the previous step $E_{m,k}$ is well defined. Furthermore, recalling that $\rho(u)*u \in \mathcal{P}_m$ for all $u \in A$, where A is chosen arbitrarily in $\mathcal{G}$, we have

$$\max_{u\in A} J(u) = \max_{u\in A} I(\rho(u) * u) \geq \inf_{v\in\mathcal{P}_m} I(v),$$

hence $E_{m,k} > 0$. The other part of the statement follows easily from $\mathcal{G}_{k+1} \subset \mathcal{G}_k$. □

Lemma 3.32. *Let $(u_n)_n \subset S_m \cap H^s_r(\mathbb{R}^N)$ be a bounded Palais-smale sequence for $I_{|S_m}$ at an arbitrary level $c > 0$ satisfying $P(u_n) \to 0$. Then there exists $u \in S_m \cap H^s_r(\mathbb{R}^N)$ and $\mu > 0$ such that, up to a subsequence, $u_n \to u$ strongly in $H^s_r(\mathbb{R}^N)$ and*

$$(-\Delta)^s u + \mu u = f(u).$$

Proof. By the boundedness of the Palais-Smale sequence we may assume $u_n \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$, $u_n \to u$ in $L^p(\mathbb{R}^N)$ for any $p \in (2, 2^*_s)$ and a.e. in $\mathbb{R}^N$. Besides, as already seen in the previous section, using [21, Lemma 3] we get

$$(-\Delta)^s u_n + \mu_n u_n - f(u_n) \to 0 \quad \text{in } (H^s_r(\mathbb{R}^N))^* \tag{3.41}$$

where

$$\mu_n := \frac{1}{m}\left(\int_{\mathbb{R}^N} f(u_n)u_n\,dx - [u_n]^2_{H^s(\mathbb{R}^N)}\right).$$

Again, similarly to the proof of Lemma 3.26, we can assume the existence of $\mu \in \mathbb{R}$ such that $\mu_n \to \mu$, from which we derive

$$(-\Delta)^s u + \mu u = f(u). \tag{3.42}$$

Claim: $u \neq 0$.

If $u = 0$, then by the compact embedding $u_n \to 0$ in $L^{2+\frac{4s}{N}}(\mathbb{R}^N)$. Hence, using Lemma 3.5 (*ii*) and the fact that $P(u_n) \to 0$, we have $\int_{\mathbb{R}^N} F(u_n)\,dx \to 0$ and

$$[u_n]^2_{H^s(\mathbb{R}^N)} = P(u_n) + \frac{N}{2s}\int_{\mathbb{R}^N} \tilde{F}(u_n)\,dx \to 0,$$

from which

$$c = \lim_{n\to+\infty} I(u_n) = \frac{1}{2}\lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} - \lim_{n\to+\infty}\int_{\mathbb{R}^N} F(u_n)\,dx = 0,$$

that contradicts the hypothesis of $c > 0$. Now, since $u \neq 0$, as we obtained ((3.40)), we get

$$\mu := \frac{1}{ms}\int_{\mathbb{R}^N}\left(NF(u) - \frac{N-2s}{2}f(u)u\right)\,dx > 0.$$

Since $u_n \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$, by Lemma 3.29

$$\int_{\mathbb{R}^N} [f(u_n) - f(u)]\,u\,dx \to 0.$$

Indeed, the fractional Sobolev inequality implies that $u \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$, and the multiplication by u turns out to be a continuous linear operator from $L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$. Now, observing that $\int_{\mathbb{R}^N} f(u_n)(u_n - u)\,dx \to 0$ by Lemma 3.5 *(iii)* we get

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} f(u_n)u_n\,dx = \int_{\mathbb{R}^N} f(u)u\,dx.$$

Finally, from (3.41) and (3.42) one has

$$\begin{aligned}[u]^2_{H^s(\mathbb{R}^N)} + \mu\int_{\mathbb{R}^N} u^2\,dx &= \int_{\mathbb{R}^N} f(u)u\,dx \\ &= \lim_{n\to+\infty}\int_{\mathbb{R}^N} f(u_n)u_n\,dx = \lim_{n\to+\infty}[u_n]^2_{H^s(\mathbb{R}^N)} + \mu m,\end{aligned}$$

and since $\mu > 0$,

$$\lim_{n\to+\infty}[u_n]^2_{H^s(\mathbb{R}^N)} = [u]^2_{H^s(\mathbb{R}^N)}, \quad \lim_{n\to+\infty}\int_{\mathbb{R}^N} u_n^2\,dx = m = \int_{\mathbb{R}^N} u^2\,dx.$$

Thus $u_n \to u$ in $H^s_r(\mathbb{R}^N)$. □

Lemma 3.33. *For any $c > 0$, there exist $\beta = \beta(c) > 0$ and $k(c) \in \mathbb{N}$ such that for any $k \geq k(c)$ and any $u \in \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$*

$$\|\pi_k u\|_{H^s(\mathbb{R}^N)} \leq \beta \quad \textit{implies} \quad I(u) \geq c.$$

Proof. By contradiction, we assume that there exists c_0 such that for any $\beta > 0$ and any $k \in \mathbb{N}$ it is possible to find $\ell \geq k$ and $u \in \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ such that

$$I(u) < c_0 \quad \text{with } \|\pi_\ell u\|_{H^s(\mathbb{R}^N)} \leq \beta.$$

In view of that, one can find a sequence $(k_j)_j \subset \mathbb{N}$, with $k_j \to \infty$ as $j \to \infty$, and a sequence $(u_j)_j \subset \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ such that

$$\|\pi_{k_j} u_j\|_{H^s(\mathbb{R}^N)} \leq \frac{1}{j} \quad \text{and} \quad I(u_j) < c_0 \tag{3.43}$$

for any $j \in \mathbb{N}$. Noticing that by Lemma 3.12 (iv) $(u_j)_j$ is bounded, up to a subsequence we have $u_j \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$.

Claim: $u = 0$.

Since $k_j \to \infty$, it follows that $\pi_{k_j} u \to u$ in $L^2(\mathbb{R}^N)$, hence

$$(\pi_{k_j} u_j, u)_{L^2(\mathbb{R}^N)} = (u_j, \pi_{k_j} u)_{L^2(\mathbb{R}^N)} \to (u, u)_{L^2(\mathbb{R}^N)}$$

as $j \to \infty$.

On the other hand, using (3.43) we get $\pi_{k_j} u_j \to 0$ in $L^2(\mathbb{R}^N)$, thus the claim must hold. Now, since $\|u_j\|_{2+\frac{4s}{N}} \to 0$ by the compact embedding, $(u_j)_j \subset \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$, and Lemma 3.5 (ii), we obtain

$$[u_j]^2_{H^s(\mathbb{R}^N)} = \frac{N}{2s} \int_{\mathbb{R}^N} \tilde{F}(u_j)\, dx \to 0$$

as $j \to \infty$, which contradicts Lemma 3.12 (ii). □

Lemma 3.34. *$E_{m,k} \to \infty$ as $k \to +\infty$.*

Proof. We assume by contradiction that there exists $c > 0$ such that

$$\liminf_{k \to +\infty} E_{m,k} < c.$$

Denote with $\beta(c)$ and $k(c)$ the numbers given in Lemma 3.33. Up to choosing a bigger c, we can find $k > k(c)$ such that $E_{m,k} < c$. Moreover, by definition of $E_{m,k}$ there must be $A \in \mathcal{G}_k$ such that

$$\max_{u \in A} I(\rho(u) * u) = \max_{u \in A} J(u) < c.$$

Now, recalling Lemma 3.11 (iii) and (iv) we get that the map $\varphi : A \to \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ defined by $\varphi(u) = \rho(u) * u$ is odd and continuous. Thus, setting $\overline{A} := \varphi(A) \subset \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ we have

$$\max_{v \in \overline{A}} I(v) < c$$

and

$$\gamma(\overline{A}) \geq \gamma(A) \geq k > k(c) \tag{3.44}$$

by the properties of the genus. On the other hand, Lemma 3.33 implies that

$$\inf_{v \in \overline{A}} \|\pi_{k(c)} v\|_{H^s(\mathbb{R}^N)} \geq \beta(c) > 0,$$

and after setting

$$\phi(v) := \frac{\pi_{k(c)} v}{\|\pi_{k(c)} v\|_{H^s(\mathbb{R}^N)}} \quad \text{for any}\, v \in \overline{A}$$

we get

$$\gamma(\overline{A}) \leq \gamma(\phi(\overline{A})) \leq k(c)$$

noticing that ϕ is odd, continuous and that $\phi(\overline{A}) \subset V_{k(c)}$. That is against (3.44). Therefore $E_{m,k} \to \infty$ as $k \to +\infty$. □

Proof of Theorem 3.4. For each $k \in \mathbb{N}$, by Lemmas 3.30 and 3.31 one can find a Palais-Smale sequence $(u_n)_n \subset \mathcal{P}_m \cap H^s_r(\mathbb{R}^N)$ of the constrained functional $I_{|S_m \cap H^s_r(\mathbb{R}^N)}$ at level $E_{m,k} > 0$. By Lemma 3.12 $(u_n)_n$ is bounded and by virtue of Lemma 3.32 we deduce that (P_m) has a radial solution u_k such that $I(u_k) = E_{m,k}$. Moreover, using Lemma 3.31 *(ii)* and Lemma 3.34, we get

$$I(u_{k+1}) \geq I(u_k) > 0 \quad \text{for any } k \geq 1$$

and $I(u_k) \to \infty$. □

4 A perturbed fractional p-Kirchhoff problem with critical nonlinearity

In this chapter we are concerned in the study of the problem

$$\begin{cases} \left(a + b \displaystyle\int_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy\right) (-\Delta_p)^s u = |u|^{p_s^* - 2} u + \lambda g(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{$P_{a,b}^{\lambda}$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\mathcal{Q} = \mathbb{R}^{2N} \setminus \mathcal{O}$ and $\mathcal{O} = \Omega^c \times \Omega^c$, a and b are strictly positive real numbers, $s \in (0, 1)$. If $1 < p < 2$ we require $N > 2ps$ while if $p > 2$ we suppose $N > p^2 s$. Here, $p_s^* := Np/(N - ps)$ denotes the critical exponent for the Sobolev embedding of $W^{s,p}(\mathbb{R}^N)$ into Lebesgue spaces.

Problem $(P_{a,b}^{\lambda})$ can be seen as a non-local stationary generalized version of the classical Kirchhoff equation

$$\rho h\, \partial_{tt}^2 u - \left(p_0 + \frac{\mathcal{E} h}{2L} \int_0^L |\partial_x u|^2\, dx\right) \partial_{xx}^2 u + \delta\, \partial_t u + g(x, u) = 0 \tag{4.1}$$

for $t \geq 0$ and $0 < x < L$, where $u = u(t, x)$ is the lateral displacement at time t and position x, $\mathcal{E}$ is the Young modulus, ρ is the mass density, h is the cross section area, L the length of the string, p_0 is the initial stress tension, δ the resistance modulus and g the external force. As pointed out by Murthy in [111], Kirchhoff in [60], attempting to generalize the well known d'Alembert equation of the vibrating string, introduced this model taking into account not only the transversal displacement. At a later time, the Kirchhoff equation found application in various fields. Indeed, Alves et al. in [1] emphasized that the solutions u of the Kirchhoff equation can also describe a process which depends on the average of itself such as the population density. Moreover, operators such as the one introduced by Kirchhoff also arise in phase transition phenomena, continuum mechanics, population dynamics, game theory, nonlinear optic, and minimal surfaces. The interested reader can consult [9], [32], [33], [34], [79] and the references therein. The interest in generalizing this kind of problems to the fractional case is not only for mathematical purposes. In fact, Fiscella and Valdinoci in [48] constructed a model for the vibrating string in which the tension of the string is related to non-local measurements of the displacement of the string from its rest position. In recent years, the fractional quasilinear Kirchhoff case has attracted the attention of many researchers. For instance, Franzina and Palatucci in [49] and Lindgren and Lindqvist in [69] studied some properties of the eigenvalues of $(-\Delta_p)^s$. Furthermore, Brasco and Lindgren in [26], Di Castro et al. in [40] and Iannizzoto et al. in [56] obtained some results regarding the regularity of solutions involving the fractional p-Laplace operator. Also the attention to the fractional quasilinear case has grown considerably in the last years. We refer to [5, 6] for results on existence, multiplicity and concentration of positive solutions for a singularly perturbed fractional p-Schrödinger equation by means of variational methods and the Lyusternik-Shnirel'man theory. Pucci et al. in [99] obtained a multiplicity result for the so-called Kirchhoff-Schrödinger equation in $\mathbb{R}^N$ where a potential was added to the Kirchhoff operator. Xiang et al. in [112] proved the existence of a non-trivial weak solution to a problem driven by a non-local operator with a more general kernel than the one taken into consideration here. Moreover Xiang et al. in [113] proved the existence of a non-trivial solution for a problem with the fractional p-Laplace operator and a critical exponent. It is also worth mentioning [81] where the authors obtained the existence of a sequence of non-trivial solutions by using the symmetric mountain pass theorem under the assumption that the nonlinear term f satisfies a superlinear growth condition. We finally cite [7], where the authors investigate fractional p-Kirchhoff type problems in $\mathbb{R}^N$ with subcritical, critical and supercritical growth.

The aim of the present chapter is to generalize to the fractional quasilinear case some results obtained by Appolloni et. al in [12] following the approach proposed in [45]. We point out that to the best of our knowledge these results we are going to prove are new even for the local case $s = 1$. The main mathematical difficulty we have to face in order to study existence of solutions for problem $(P^{\lambda}_{a,b})$ is the presence of the term $|u|^{p^*_s-2}u$. Due to the lack of compactness of the embedding $W^{s,p}(\Omega) \hookrightarrow L^{p^*_s}(\Omega)$, the energy functional associated to problem $(P^{\lambda}_{a,b})$ is not even weakly sequentially lower semicontinuous. Moreover, the validity of the Palais-Smale condition is not ensured. In order to overcome these difficulties, we will invoke the concentration-compactness principle developed by Lions in [73] and [74] and generalized to the p-fractional case by Mosconi and Squassina in [90]. Helped by this result and choosing the quantity $a^{(N-2ps)/ps}b$ adequately, we will show that the functional associated to the problem is weakly sequentially lower semicontinuous and satisfies the Palais-Smale condition at any level. In addition to that, while in the semilinear case $p = 2$ the minimizers for the best Sobolev constant are completely characterized, if $p \neq 2$ we can only rely on some asymptotic estimates at infinity. As regards studying the different levels of energy on which the solutions are, we will use a fiber type approach. Defining appropriately a map depending on a parameter, we will identify a parameter $\bar{\lambda}^s_0$ that will play a crucial role in establishing whether the ground state is attained at a negative level. Since we will assume that the function g has a subcritical growth, for the sake of simplicity at the beginning we will focus our attention on the auxiliary problem

$$\begin{cases} \left(a + b \displaystyle\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy\right) (-\Delta_p)^s u = |u|^{p^*_s-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \qquad (P_{a,b})$$

Throughout the chapter we will denote with

$$\|u\|^p = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.$$

Since the solutions of $(P_{a,b}^{\lambda})$ must satisfy some kind of boundary condition, we introduce the space

$$X_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}(\mathbb{R}^N) \mid u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

Remark 4.1. The norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$ in $X_0^{s,p}(\Omega)$.

We define the functional $\mathcal{I}\colon X_0^{s,p}(\Omega) \to \mathbb{R}$ by

$$\mathcal{I}(u) := \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*},$$

whose critical points are weak solutions to $(P_{a,b})$. To see a complete summary of the notation used we refer the reader to the next section and Chapter 2. The chapter is structured as follows. Section 4.1 is devoted to introducing the notation and to collect some preliminary lemmas. In Section 4.2 we give the proof of the main results for the auxiliary problem $P_{a,b}$. Finally, in Section 4.3 we investigate the existence of solutions for problem $(P_{a,b}^{\lambda})$. To conclude the section, we collect here the main results we are going to prove along the chapter.

Theorem 4.2. *Let*

$$L(N,p,s) := 2ps \frac{(N-2ps)^{\frac{N-2ps}{ps}}}{N^{\frac{N-ps}{ps}} S_{s,p}^{\frac{N}{ps}}},$$

where $S_{s,p}$ is the best Sobolev constant defined in (4.2) *below. The functional $\mathcal{I}$ is sequentially weakly lower semicontinuous on $X_0^{s,p}(\Omega)$ if and only if $a^{\frac{N-2ps}{ps}} b \geq L(N,p,s)$.*

Theorem 4.3. *Define*

$$\mathsf{PS}(N,p,s) := ps \frac{(N-2ps)^{\frac{N-2ps}{ps}}}{(N-ps)^{\frac{N-ps}{ps}} S_{s,p}^{\frac{N}{ps}}}.$$

If $a^{(N-2ps)/ps} b > \mathsf{PS}(N,p,s)$, the functional $\mathcal{I}$ satisfies the compactness Palais–Smale condition at any level $c \in \mathbb{R}$.

Remark 4.4. We point out that $\mathsf{PS}(N,p,s) \geq L(N,p,s)$ in our setting. Indeed, this inequality is equivalent to

$$\left(\frac{N}{N-ps}\right)^{\frac{N-ps}{ps}} \geq 2,$$

or

$$\left(1 + \frac{ps}{N-ps}\right)^{\frac{N-ps}{ps}} \geq 2.$$

The generalized Bernouilli inequality

$$(1+x)^r \geq 1+rx, \quad r\geq 1,\ x\geq -1$$

and the assumption that $N>2ps$ yield

$$\left(1+\frac{ps}{N-ps}\right)^{\frac{N-ps}{ps}} \geq 1+\frac{N-ps}{ps}\cdot\frac{ps}{N-ps}=2.$$

We next prove an existence result for ground states of problem $(P^{\lambda}_{a,b})$.

Theorem 4.5. *Let a, $b\in\mathbb{R}^+$ such that $a^{(N-2ps)/ps}b\geq L(N,p,s)$, and set*

$$\iota^s_\lambda := \inf\left\{\mathcal{I}^\lambda(u)\mid\ u\in X^{s,p}_0(\Omega)\setminus\{0\}\right\}\quad \textit{for any } \lambda>0.$$

There exists $\overline{\lambda}^s_0\geq 0$ such that for any $\lambda>\overline{\lambda}^s_0$ there exists $u^s_\lambda\in X^{s,p}_0(\Omega)\setminus\{0\}$ satisfying $\mathcal{I}^\lambda(u^s_\lambda)=\iota^s_\lambda<0$.

Theorem 4.6. *Let $\lambda=\overline{\lambda}^s_0$. The following statements hold:*

(i) if $a^{(N-2ps)/ps}b>L(N,p,s)$ then there exists $u^s_\lambda\in X^{s,p}_0(\Omega)\setminus\{0\}$ such that $\iota^s_{\overline{\lambda}^s_0}=\mathcal{I}^{\overline{\lambda}^s_0}=0$;

(ii) if $a^{(N-2ps)/ps}b=L(N,p,s)$, then $u=0$ in the only minimizer for $\iota^s_{\overline{\lambda}^s_0}$.

The following Theorem states a sort of stability when the quantity $a^{(N-2ps)/ps}b$ converges to $L(N,p,s)$.

Theorem 4.7. *Let $(a_k)_k$, $(b_k)_k$ be sequences of real positive numbers such that $a_k\to a$, $b_k\to b$ and $a_k^{(N-2ps)/ps}b_k\searrow L(N,p,s)$. Setting $\lambda_k:=\overline{\lambda}^s_0(a_k,b_k)$ we have that $\lambda_k\to 0$ as $k\to\infty$. Furthermore, if $(u_k)_k\subset X^{s,p}_0(\Omega)\setminus\{0\}$ satisfies $\lambda_k=\lambda^s_0(u_k)$ then $u_k/\|u_k\|\rightharpoonup 0$ and*

$$\frac{\|u_k\|^p_{p^*_s}}{\|u_k\|^p}\to S_{s,p}.$$

Next statement describes the situations for mountains pass solutions.

Theorem 4.8. *If $\lambda\geq\overline{\lambda}^s_0$ and $a^{(N-2ps)/ps}b>\mathsf{PS}(N,p,s)$, then there exists a $v^s_\lambda\in X^{s,p}_0(\Omega)\setminus\{0\}$ such that $\mathcal{I}^\lambda(v^s_\lambda)=c^s_\lambda$ and $\left(\mathcal{I}^\lambda\right)'(v^s_\lambda)=0$ where*

$$c^s_\lambda := \inf_{h\in\Gamma^s_\lambda}\max_{\zeta\in[0,1]}\mathcal{I}^\lambda(h(\zeta))$$

and

$$\Gamma^s_\lambda := \left\{h\in C\left([0,1],X^{s,p}_0(\Omega)\right)\mid h(0)=0,\ h(1)=u^s_{\overline{\lambda}^s_0}\right\}.$$

The last two Theorems analyze what happens to the set of solutions of $(P^\lambda_{a,b})$ when $\lambda<\bar{\lambda}^s_0$.

Theorem 4.9. *Assume* $a^{(N-2ps)/ps}b > \mathsf{PS}(N,p,s)$. *There exist* $\delta > 0$, $r > 0$ *such that for any* $\lambda \in (\overline{\lambda}_0^s - \delta, \overline{\lambda}_0^s)$ *the value*

$$\hat{\iota}_\lambda^s := \inf \left\{ \mathcal{I}^\lambda(u) \mid u \in X_0^{s,p}(\Omega),\ \|u\| \geq r \right\}$$

is attained at a function $w_\lambda^s \in X_0^{s,p}(\Omega)$ *satisfying* $\|w_\lambda^s\| > r$.

Theorem 4.10. *Suppose* $a^{(N-2ps)/ps}b > \mathsf{PS}(N,p,s)$. *For any* $\lambda \in (\overline{\lambda}_0^s - \delta, \overline{\lambda}_0^s)$ *there is* $v_\lambda^s \in X_0^{s,p}(\Omega) \setminus \{0\}$ *such that* $\mathcal{I}^\lambda(v_\lambda^s) = c_\lambda^s$ *and* $\left(\mathcal{I}^\lambda\right)'(v_\lambda^s) = 0$, *where*

$$c_\lambda^s := \inf_{h \in \Gamma_\lambda^s} \max_{\zeta \in [0,1]} \mathcal{I}^\lambda(h(\zeta))$$

and

$$\Gamma_\lambda^s := \{ h \in C([0,1], X_0^{s,p}(\Omega)) \mid h(0) = 0,\ h(1) = w_\lambda^s \}.$$

4.1 Abstract framework and preliminary results

We consider the potential operator A_p associated to the functional $u \mapsto \|u\|^p/p$ on $X_0^{s,p}(\Omega)$, i.e. the operator $A_p \colon X_0^{s,p}(\Omega) \to (X_0^{s,p}(\Omega))^*$ such that

$$\langle A_p(u), v \rangle = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+sp}} dx\, dy$$

for every u, $v \in X_0^{s,p}(\Omega)$. Trivially,

$$\langle A_p(u), u \rangle = \|u\|^p, \quad |\langle A_p(u), v \rangle| \leq \|u\|^{p-1} \|v\|.$$

Lemma 4.11. *If a sequence* $(u_n)_n$ *converges weakly to* u *in* $X_0^{s,p}(\Omega)$ *and*

$$\langle A_p(u_n), u_n - u \rangle \to 0,$$

then $\|u_n - u\| \to 0$.

Proof. We refer to [4] for a proof. □

The following Lemma will be particularly useful in the proof of the Palais-Smale condition.

Lemma 4.12. *Let* $q \in \mathbb{R}^N$, $\varepsilon \in (0,1)$ *and* $u \in L^{p_s^*}(\mathbb{R}^N)$. *Suppose that either* $U = B(q,\varepsilon)$ *and* $V = \mathbb{R}^N$, *or* $U = \mathbb{R}^N$ *and* $V = B(q,\varepsilon)$. *Then,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_U \int_{V \cap \{|x-y| \leq \varepsilon\}} \frac{|u(y)|^p}{|x-y|^{N+ps-p}} dx\, dy = 0$$

and

$$\lim_{\varepsilon \to 0} \int_U \int_{V \cap \{|x-y| > \varepsilon\}} \frac{|u(y)|^p}{|x-y|^{N+ps}} dx\, dy = 0$$

Proof. The verification of the two limits is similar to [48, Proposition 7]. We omit the details. □

Proposition 4.13. *Let $(u_n)_n \subset X_0^{s,p}(\Omega)$ be a bounded sequence. Suppose that $\vartheta \in C^\infty(\mathbb{R}^N)$ is such that $0 \leq \vartheta \leq 1$, $\vartheta = 1$ in $B(0,1)$ and $\vartheta = 0$ in $\mathbb{R}^N \setminus B(0,2)$. For $q \in \mathbb{R}^N$, let $\vartheta_\varepsilon(x) = \vartheta\left(\frac{x-q}{\varepsilon}\right)$. Then*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^{2N}}|u_n(y)|^p\frac{|\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = 0.$$

Proof. The verification of the limit is similar to [48, Theorem 2]. We omit the details. □

We conclude this section recalling that the best Sobolev constant is defined as

$$S_{s,p} := \inf_{u\in X_0^{s,p}(\Omega)\setminus\{0\}}\frac{\|u\|^p}{\|u\|_{p_s^*}^p} \tag{4.2}$$

A natural conjecture is that all the minimizers for $S_{s,p}$ are of the form $V(|\cdot - x_0|/\varepsilon)$, where

$$V(x) = \frac{1}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N-ps}{p}}}$$

in analogy to the case $p = 2$ and $s = 1$ (see for instance [68]). Unfortunately, this problem is still open and we can only rely on some asymptotic estimates at infinity proved by Mosconi et al. in [89].

4.2 Weakly sequentially lower semicontinuity and validity of the Palais-Smale

Proof of Theorem 4.2. We start assuming

$$a^{\frac{N-2ps}{ps}}b \geq L(N,p,s).$$

Take a sequence $(u_n)_n \subset X_0^{s,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $X_0^{s,p}(\Omega)$. Recalling that the embedding $X_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $q \in [1, p_s^*)$ by Proposition 2.3, we deduce $u_n \to u$ in $L^q(\Omega)$ for all $q \in [1, p_s^*)$ and in particular $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. At this point we use [96, Lemma 3.2] getting

$$\|u_n - u\|^p = \|u_n\|^p - \|u\|^p + o(1) \quad \text{as } n\to\infty. \tag{4.3}$$

Furthermore, we observe

$$\begin{aligned}\|u_n\|^{2p} - \|u\|^{2p} &= \left(\|u_n\|^p - \|u\|^p\right)\left(\|u_n\|^p + \|u\|^p\right)\\ &= \left(\|u_n - u\|^p + o(1)\right)\left(\|u_n - u\|^p + 2\|u\|^p + o(1)\right)\end{aligned} \tag{4.4}$$

where we used (4.3). We also apply the classical Brezis-Lieb Lemma (see [30, Theorem 1]) to get

$$\|u_n - u\|_{p_s^*}^{p_s^*} = \|u_n\|_{p_s^*}^{p_s^*} - \|u\|_{p_s^*}^{p_s^*} + o(1). \tag{4.5}$$

Now, we assemble (4.3), (4.4), (4.5) and we compute

$$\begin{aligned}
\mathcal{I}(u_n) - \mathcal{I}(u) &= \frac{a}{p}\left(\|u_n\|^p - \|u\|^p\right) + \frac{b}{2p}\left(\|u_n\|^{2p} - \|u\|^{2p}\right) - \frac{1}{p_s^*}\left(\|u_n\|_{p_s^*}^{p_s^*} - \|u\|_{p_s^*}^{p_s^*}\right) \\
&= \frac{a}{p}\|u_n - u\|^p + \frac{b}{2p}\left(\|u_n - u\|^{2p} + 2\|u\|^p\|u_n - u\|^p\right) \\
&- \frac{1}{p_s^*}\|u_n - u\|_{p_s^*}^{p_s^*} + o(1) \\
&\geq \frac{a}{p}\|u_n - u\|^p + \frac{b}{2p}\|u_n - u\|^{2p} - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\|u_n - u\|^{p_s^*} + o(1) \\
&= \|u_n - u\|^p \left[\frac{a}{p} + \frac{b}{2p}\|u_n - u\|^p - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\|u_n - u\|^{p_s^* - p}\right] + o(1)
\end{aligned} \tag{4.6}$$

as $n \to \infty$, where we also used the Sobolev inequality given in (4.2). We introduce the auxiliary function

$$f_{s,p}(\zeta) = \frac{a}{p} + \frac{b}{2p}\zeta^p - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\zeta^{p_s^* - p}, \quad \zeta \geq 0$$

and we notice that $f_{s,p}$ attains its global minimum at the point

$$m_{s,p} = \left(\frac{b}{2}\frac{p_s^*}{p_s^* - p}S_{s,p}^{\frac{p_s^*}{p}}\right)^{\frac{1}{p_s^* - 2p}}.$$

Besides, one easily verifies that

$$a^{\frac{N-2ps}{ps}} b \geq L(N,p,s) \Leftrightarrow f_{s,p}(m_{s,p}) = \frac{1}{p}\left(a - b^{-\frac{ps}{N-2ps}} L(N,p,s)^{\frac{ps}{N-2ps}}\right) \geq 0. \tag{4.7}$$

From (4.6) and (4.7) it follows that

$$\liminf_{n\to\infty}\left(\mathcal{I}(u_n) - \mathcal{I}(u)\right) \geq \liminf_{n\to\infty}\|u_n - u\|^p f_{s,p}(\|u_n - u\|) \geq 0,$$

proving the sufficiency implication. In order to prove the other part of the theorem. we argue by contradiction. Under the assumption that $\mathcal{I}$ is sequentially weakly lower semicontinuous we suppose that

$$a^{\frac{N-2ps}{ps}} b < L(N,p,s). \tag{4.8}$$

Consider a minimizing sequence $(u_n)_n \subset X_0^{s,p}(\Omega)$ for (4.2). Since problem (4.2) is homogeneous, we can assume that the sequence $(u_n)_n$ is bounded on $X_0^{s,p}(\Omega)$. As a consequence, up to a subsequence, we have $u_n \rightharpoonup u$ in $X_0^{s,p}(\Omega)$ for some $u \in X_0^{s,p}(\Omega) \setminus \{0\}$. Set

$L := \liminf_{n\to\infty}\|u_n\|$ and observe that exploiting the sequentially weakly semicontinuity of the norm, we get $0 < \|u\| \leq L$. Now, there exists a subsequence $(u_{n_k})_k$ such that $\lim_{k\to\infty}\|u_{n_k}\| = L$. We have already seen that the function $f_{s,p}$ has a minimum in $m_{s,p}$ which is global since $p > p_s^* - p > 0$ implies $\lim_{\zeta\to+\infty} f_{s,p}(\zeta) = +\infty$. At this point, we set $c = m_{s,p}/L$. On the one hand, also $(cu_{n_j})_j$ is a minimizing sequence for $S_{s,p}$, so

$$\begin{aligned}\liminf_{n\to\infty}\mathcal{I}(cu_n) &\leq \liminf_{k\to\infty}\mathcal{I}(cu_{n_k})\\ &= \liminf_{k\to\infty}\|cu_{n_k}\|^p f_{s,p}(\|cu_{n_k}\|) = (cL)^p f_{s,p}(cL)\\ &= (cL)^p f_{s,p}(m_{s,p}) \leq \|cu\|^p f_{s,p}(m_{s,p}) \leq \|cu\|^p f_{s,p}(\|cu\|)\end{aligned} \tag{4.9}$$

where in the second to last inequality we used the inequality $f_{s,p}(m_{s,p}) < 0$, since $a^{\frac{N-2ps}{ps}} b < L(N,p,s)$. On the other hand, from the Sobolev inequality it follows

$$\begin{aligned}\|cu\|^p f_{s,p}(\|cu\|) &= \frac{a}{p}\|cu\|^p + \frac{b}{2p}\|cu\|^{2p} - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\|cu\|^{p_s^*}\\ &\leq \frac{a}{p}\|cu\|^p + \frac{b}{2p}\|cu\|^{2p} - \frac{1}{p_s^*}\int_\Omega |cu|^{p_s^*}\,dx = \mathcal{I}(cu).\end{aligned} \tag{4.10}$$

Coupling (4.9) and (4.10) we get

$$\liminf_{n\to\infty}\mathcal{I}(cu_n) \leq \mathcal{I}(cu). \tag{4.11}$$

The strict inequality in (4.11) would contradict the weakly sequentially lower sequentially of the functional $\mathcal{I}$, so in (4.11) the equality must hold. However, this means that cu would be a minimazer for (4.2), but recalling that $u = 0$ in $\mathbb{R}^N \setminus \Omega$, we have a contradiction with [27, Theorem 1.1] since $0 < \|u\| \leq L$. □

Remark 4.14. In the second part of the previous proof, we assert that the weak limit u of a minimizing sequence $(u_n)_n$ for $S_{s,p}$ is different from 0. Since the embedding $X_0^{s,p}(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$ is not compact, in general it is not true that $u \neq 0$, but it is always possible to modify $(u_n)_n$, making it still remain a minimizing sequence for $S_{s,p}$, in order to have the desired property. Since these arguments are standard, we prefer to omit the details to make the proof more concise.

Proof of Theorem 4.3. Let $(u_n)_n \subset X_0^{s,p}(\Omega)$ be a $(PS)_c$ sequence, i.e. $\mathcal{I}(u_n) \to c$, and $\mathcal{I}'(u_n) \to 0$ as $n \to \infty$. From (4.2) it follows that

$$\mathcal{I}(u) = \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{p_s^*}\int_\Omega |u|^{p_s^*}\,dx \geq \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\|u\|^{p_s^*}.$$

Recalling $2p > p_s^*$, we can deduce that the functional $\mathcal{I}$ is bounded from below. As a consequence of that, we have that the sequence $(u_n)_n$ is bounded since $\mathcal{I}(u_n) \to c$ as

$n \to \infty$. Thus, we are allowed to suppose

$$\begin{cases} u_n \rightharpoonup u & \text{in } X_0^{s,p}(\Omega) \\ u_n \to u & \text{in } L^q(\Omega) \text{ for all } q \in [1, p_s^*) \\ u_n \to u & \text{a.e in } \mathbb{R}^N. \end{cases}$$

Exploiting the Hölder inequality, we can deduce the boundedness of the sequence $(u_n)_n$ also in the space of measures $\mathcal{M}(\Omega)$. At this point, invoking [90, Theorem 2.5] there exist two Borel regular measures μ and ν such that

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}}\, dy \rightharpoonup^* \mu \quad \text{and} \quad |u_n|^{p_s^*} \rightharpoonup^* \nu \quad \text{in } \mathcal{M}(\Omega)$$

where

$$\nu = |u|^{p_s^*} + \sum_{j \in J} \nu_j \delta_{x_j} \tag{4.12}$$

and

$$\mu \geq \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dy + \sum_{j \in J} \mu_j \delta_{x_j} \tag{4.13}$$

with

$$\nu_j = \nu(\{x_j\}) \qquad \mu_j = \mu(\{x_j\})$$

and the set J is at most countable. We also have

$$\mu_j \geq S_{s,p} \nu_j^{\frac{p}{p_s^*}}. \tag{4.14}$$

We claim that the set J is empty. If the claim were false, there would exist at least an index $j_0 \in J$ and a point x_{j_0} with $\nu_{j_0} \neq 0$ associated to it. Pick $\varepsilon > 0$ and consider a cut-off function such that

$$\begin{cases} 0 \leq \vartheta_\varepsilon \leq 1 & \text{in } \Omega \\ \vartheta_\varepsilon = 1 & \text{in } B(x_{j_0}, \varepsilon) \\ \vartheta_\varepsilon = 0 & \text{in } \Omega \setminus B(x_{j_0}, 2\varepsilon). \end{cases}$$

We also notice that the sequence $(u_n \vartheta_\varepsilon)_n$ is bounded in $X_0^{s,p}(\Omega)$, hence

$$\lim_{n \to \infty} \mathcal{I}'(u_n)\,[u_n \vartheta_\varepsilon] = 0.$$

As a consequence of that

$$\begin{aligned}
o(1) &= \mathcal{I}'(u_n)\,[u_n\vartheta_\varepsilon] \\
&= (a+b\|u_n\|^p)\times \\
&\times \int_{\mathcal{Q}} \frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))\,(u_n(x)\vartheta_\varepsilon(x)-u_n(y)\vartheta_\varepsilon(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&- \int_\Omega |u_n|^{p_s^*}\vartheta_\varepsilon\,dx \\
&= (a+b\|u_n\|^p)\left[\int_{\mathcal{Q}} u_n(y)\frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))(\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y))}{|x-y|^{N+ps}}\,dx\,dy\right. \\
&+ \left.\int_{\mathcal{Q}} \vartheta_\varepsilon(x)\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right] - \int_\Omega |u_n|^{p_s^*}\vartheta_\varepsilon\,dx. \qquad (4.15)
\end{aligned}$$

We estimate the first term of $\mathcal{I}'(u_n)\,[u_n\vartheta_\varepsilon]$ with the Hölder inequality, obtaining

$$\begin{aligned}
&(a+b\|u_n\|^p)\int_{\mathcal{Q}} u_n(y)\frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))(\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&\le C\left(\int_{\mathcal{Q}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{p-1}{p}}\left(\int_{\mathcal{Q}} |u_n(y)|^p\frac{|\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}} \\
&\le \tilde{C}\left(\int_{\mathcal{Q}} |u_n(y)|^p\frac{|\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}
\end{aligned}$$

for some constants $C>0$, $\tilde{C}>0$. Now, Proposition 4.13 yields

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^{2N}} |u_n(y)|^p\frac{|\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = 0,$$

so that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}(a+b\|u_n\|^p)\times \\
&\qquad\times\int_{\mathcal{Q}} u_n(y)\frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))(\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y))}{|x-y|^{N+ps}}\,dx\,dy = 0. \qquad (4.16)
\end{aligned}$$

Now, exploiting (4.13), we have

$$
\begin{aligned}
\lim_{n\to\infty}(a+b\|u_n\|^p)&\int_{\mathcal{Q}}\vartheta_\varepsilon(x)\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\
&\geq \lim_{n\to\infty}\left[a\int_{\mathbb{R}^{2N}\setminus B(x_{j_0},2\varepsilon)^c\times\Omega^c}\vartheta_\varepsilon(x)\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right.\\
&\qquad\left.+b\left(\int_{\mathcal{Q}}\vartheta_\varepsilon(x)\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^2\right]\\
&\geq a\int_{\mathbb{R}^{2N}\setminus B(x_{j_0},2\varepsilon)^c\times\Omega^c}\vartheta_\varepsilon(x)\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy+a\mu_{j_0}\\
&\qquad+b\left(\int_{\mathcal{Q}}\vartheta_\varepsilon(x)\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^2+b\mu_{j_0}^2.
\end{aligned}
$$

Thus

$$
\lim_{\varepsilon\to0}\lim_{n\to\infty}(a+b\|u_n\|^p)\int_{\mathcal{Q}}\vartheta_\varepsilon(x)\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\geq a\mu_{j_0}+b\mu_{j_0}^2. \tag{4.17}
$$

Furthermore, it follows from (4.12) that

$$
\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\Omega}|u_n|^{p_s^*}\vartheta_\varepsilon\,dx=\lim_{\varepsilon\to0}\int_{\Omega}|u|^{p_s^*}\vartheta_\varepsilon\,dx+\nu_{j_0}=\nu_{j_0}. \tag{4.18}
$$

At this point, from (4.15), taking into account (4.16), (4.17), (4.18) and using (4.14) we deduce

$$
0\geq a\mu_{j_0}+b\mu_{j_0}^2-\nu_{j_0}\geq a\mu_{j_0}+b\mu_{j_0}^2-S_{s,p}^{-\frac{p_s^*}{p}}\mu_{j_0}^{\frac{p_s^*}{p}}=\mu_{j_0}\left(a+b\mu_{j_0}-S_{s,p}^{-\frac{p_s^*}{p}}\mu_{j_0}^{\frac{p_s^*}{p}-1}\right). \tag{4.19}
$$

We define

$$
\tilde{f}_{s,p}(\zeta)=a+b\zeta-S_{s,p}^{-\frac{p_s^*}{p}}\zeta^{\frac{p_s^*}{p}-1}\quad\text{for }\zeta\geq0.
$$

We observe that the function $\tilde{f}_{s,p}$ has a global minimum in

$$
\tilde{m}_{s,p}:=\left(bS_{s,p}^{\frac{p_s^*}{p}}\frac{p}{p_s^*-p}\right)^{\frac{p}{p_s^*-2p}}
$$

and that

$$
a^{\frac{N-2ps}{ps}}b>\mathsf{PS}(N,p,s)\Leftrightarrow\tilde{f}_{s,p}(\tilde{m}_{s,p})=a-b^{-\frac{ps}{N-2ps}}\mathsf{PS}(N,p,s)^{\frac{ps}{N-2ps}}>0.
$$

Hence

$$
a+b\mu_{j_0}-S_{N,s}^{-\frac{p_s^*}{2}}\mu_{j_0}^{\frac{p_s^*}{2}-1}>0.
$$

and the only admissible case in (4.19) is $\mu_{j_0} = 0$. From this, recalling (4.14), we also have $\nu_{j_0} = 0$ that is absurd. Hence $J = \emptyset$, which means

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p_s^*}\, dx = \int_\Omega |u|^{p_s^*}\, dx.$$

This coupled with (4.5) implies

$$u_n \to u \quad \text{in } L^{p_s^*}(\Omega).$$

From this and the Hölder inequality it follows that

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p_s^*-2} u_n(u-u_n)\, dx = 0. \tag{4.20}$$

Computing the derivative of $\mathcal{I}(u_n)$ along $u_n - u$, we get

$$\begin{aligned}
0 &= \lim_{n\to\infty} \mathcal{I}'(u_n)\,[u_n-u] \\
&= \lim_{n\to\infty} \Big[\big(a+b\|u_n\|^2\big)\times \\
&\quad \times \int_\mathcal{Q} \frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))\,((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{N+ps}}\, dx\, dy \\
&\quad - \int_\Omega |u_n|^{2_s^*-2} u_n(u_n-u)\, dx\Big] \\
&= \lim_{n\to\infty} \big(a+b\|u_n\|^2\big)\times \\
&\quad \times \int_\mathcal{Q} \frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))\,((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{N+ps}}\, dx\, dy
\end{aligned}$$

which implies

$$\int_\mathcal{Q} \frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))\,((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{N+ps}}\, dx\, dy \to 0 \tag{4.21}$$

since $(u_n)_n$ is bounded in $X_0^{s,p}(\Omega)$. Along a subsequence, u_n converges weakly to u in $X_0^{s,p}(\Omega)$, and Lemma 4.11 implies that $u_n \to u$ in $X_0^{s,p}(\Omega)$ as $n\to\infty$. □

4.3 The perturbed problem

In this section, applying the result obtained in Theorem 4.2, we investigate the existence of solutions of different kind of the perturbed problem

$$\begin{cases} \left(a + b\displaystyle\int_\mathcal{Q} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy\right)(-\Delta_p)^s u = |u|^{p_s^*-2}u + \lambda g(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{$P_{a,b}^\lambda$}$$

where as before a, b are real positive parameter, Ω is a bounded domain and $\lambda > 0$. As for g, we make the same assumptions present in [12], but adapted to the case of the fractional p-Laplacian. Namely, we make the following assumptions:

(H_1) $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(x,0)=0$ a.e. in Ω;

(H_2) $g(x,t)>0$ for every $t>0$ and $g(x,t)<0$ for every $t<0$ a.e. in Ω. In addition, we require that there is a $\mu>0$ such that $g(x,t)\geq \mu>0$ a.e in Ω and for every $t\in I$, where I is some open interval of $(0,\infty)$;

(H_3) there is a constant $c>0$ and $q\in(p,p_s^*)$ such that $|g(x,t)|\leq c(1+|t|^{q-1})$ a.e. in Ω;

(H_4) $\lim_{t\to 0} g(x,t)/|t|^{p-1}=0$ uniformly with respect to $x\in\Omega$.

Using a variational approach, we investigate the existence of critical points of the functional defined on the space $X_0^s(\Omega)$

$$\mathcal{I}^\lambda(u) := \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*} - \lambda\int_\Omega G(x,u)\,dx$$

where we denote with $G(x,t)=\displaystyle\int_0^t g(x,\tau)d\tau$. Before starting the analysis of our problem we need to prove some technical results that will be useful up to the end of the section.

Remark 4.15. For the reader's convenience, we remember the definitions of the functions

$$f_{s,p}(\zeta) := \frac{a}{p} + \frac{b}{2p}\zeta^p - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\zeta^{p_s^*-p}$$

and

$$\tilde{f}_{s,p}(\zeta) = a + b\zeta - S_{s,p}^{-\frac{p_s^*}{p}}\zeta^{\frac{p_s^*}{p}-1}$$

defined in the previous section. We also recall that these functions have a unique local minimum attained respectively at

$$m_{s,p} = \left[\frac{b}{2}\frac{p_s^*}{p_s^*-p}S_{s,p}^{\frac{p_s^*}{p}}\right]^{\frac{1}{p_s^*-2p}},$$

and

$$\tilde{m}_{s,p} = \left[b\frac{p}{p_s^*-p}S_{s,p}^{\frac{p_s^*}{p}}\right]^{\frac{1}{p_s^*-2p}}.$$

Besides, $f_{s,p}(m_{s,p})>0$ if and only if $a^{\frac{N-2ps}{ps}}b > L(N,p,s)$ and $f_{s,p}(m_{s,p})=0$ when $a^{\frac{N-2ps}{ps}}b = L(N,p,s)$. Similarly $\tilde{f}_{s,p}(\tilde{m}_{s,p})>0$ if and only if $a^{(N-2ps)/ps}b > \mathsf{PS}(N,p,s)$ and $\tilde{f}_{s,p}(\tilde{m}_{s,p})=0$ when $a^{(N-2ps)/ps}b=\mathsf{PS}(N,p,s)$.

Proposition 4.16. *Let $u\in X_0^{s,p}(\Omega)\setminus\{0\}$. We have that:*

(i) for every $\zeta > 0$ *it holds*

$$\frac{a}{p}\|u\|^p+\frac{b}{2p}\zeta^p\|u\|^{2p}-\frac{1}{p_s^*}\zeta^{p_s^*-p}\|u\|_{p_s^*}^{p_s^*}> f_{s,p}(\zeta\|u\|)\|u\|^p;$$

(ii) for every $\zeta > 0$ *it holds*

$$a\|u\|^p+b\zeta^p\|u\|^{2p}-\|u\|_{p_s^*}^{p_s^*}\zeta^{p_s^*-p} > \tilde{f}_{s,p}(\zeta\|u\|)\|u\|^p.$$

Proof. We only prove (i), since (ii) follows in a similar way. Considering $u = 0$ in $\mathbb{R}^N \setminus \Omega$, taking into account [27, Theorem 1.1] and the Sobolev inequality, we have

$$\begin{aligned}
&\zeta^p\left[\frac{a}{p}\|u\|^p+\frac{b}{2p}\zeta^p\|u\|^{2p}-\frac{1}{p_s^*}\zeta^{p_s^*-p}\|u\|_{p_s^*}^{p_s^*}\right]\\
&=\frac{a}{p}\left(\zeta\|u\|\right)^p+\frac{b}{2p}\left(\zeta\|u\|\right)^{2p}-\frac{\|u\|_{p_s^*}^{p_s^*}}{\|u\|^{p_s^*}}\frac{\left(\zeta\|u\|\right)^{p_s^*}}{p_s^*}\\
&>\frac{a}{p}\left(\zeta\|u\|\right)^p+\frac{b}{2p}\left(\zeta\|u\|\right)^{2p}-S_{s,p}^{-\frac{p_s^*}{p}}\frac{\left(\zeta\|u\|\right)^{p_s^*}}{p_s^*}.
\end{aligned}\tag{4.22}$$

□

We are now going to prove that the functional $\mathcal{I}^\lambda$ is sequentially lower semi continuous and satisfies the Palais-Smale condition for a and b large enough.

Lemma 4.17. *Let* $a, b \in \mathbb{R}^+$, $(u_k)_k \subset X_0^{s,p}(\Omega)$ *and* $\lambda_k \to \lambda \geq 0$ *as* $k \to \infty$*:*

(1) *if* $a^{(N-2ps)/ps}b \geq L(N,p,s)$ *and* $u_k \rightharpoonup u$ *in* $X_0^{s,p}(\Omega)$ *then*

$$\mathcal{I}^\lambda(u) \leq \liminf_{k\to\infty} \mathcal{I}^{\lambda_k}(u_k);$$

(2) *if* $a^{(N-2ps)/2s}b > \mathsf{PS}(N,p,s)$, $\mathcal{I}^\lambda(u_k) \to c$ *and* $\left(\mathcal{I}^\lambda\right)'(u_k) \to 0$ *then* $(u_k)_k$ *is convergent to some* u *in* $X_0^{s,p}(\Omega)$ *up to subsequence.*

Proof. Since the proof is essentially the same of Theorems 4.2 and 4.3 we omit it. □

At this point fix $\lambda \geq 0$ and $u \in X_0^s(\Omega)$. For all $\zeta > 0$ we define the fiber map

$$\mathcal{J}^{\lambda,u}(\zeta) := \mathcal{I}^\lambda(\zeta u) = \frac{a}{p}\zeta^p\|u\|^p+\frac{b}{2p}\zeta^{2p}\|u\|^{2p}-\frac{\zeta^{p_s^*}}{p_s^*}\|u\|_{p_s^*}^{p_s^*}-\lambda\int_\Omega G(x,\zeta u)\,dx.$$

Proposition 4.18. *Let* $\lambda \in \mathbb{R}$ *be non-negative and* $u \in X_0^{s,p}(\Omega) \setminus \{0\}$*. Then it is possible to find a neighbourhood* V_λ *of* 0 *such that* $\mathcal{J}^{\lambda,u}(\zeta) > 0$ *for every* $\zeta \in V_\lambda \cap (0,\infty)$*. Furthermore* $\mathcal{J}^{\lambda,u}(\zeta) \to \infty$ *as* $\zeta \to \infty$*. In particular, the map* $\zeta \mapsto \mathcal{J}^{\lambda,u}(\zeta)$ *is bounded from below.*

Proof. Fix $\varepsilon > 0$ small and $L > 0$ arbitrary large. Observe that (H_3) implies

$$\int_{\Omega\cap\{|u|\leq L\}} \frac{G(x,\zeta u)}{\zeta^p}\,dx \leq \varepsilon \int_\Omega |u|^p\,dx \tag{4.23}$$

for ζ small enough. On the other hand, (H_3) and (H_4) implies

$$|G(x,t)| < C(|t|^p + |t|^q)$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. Thus,

$$\int_{\Omega\cap\{|u|>L\}} \frac{G(x,\zeta u)}{\zeta^p}\,dx \leq C\left(\int_{\Omega\cap\{|u|>L\}} |u|^p\,dx + \zeta^{q-p}\int_\Omega |u|^q\,\mathrm{d}x\right) \leq \varepsilon \tag{4.24}$$

for L large enough and ζ small enough. Coupling (4.23) and (4.24), keeping in mind that $\Omega = \{|u| \leq L\} \cup \{|u| > L\}$, we get

$$\begin{aligned}
\mathcal{J}^{\lambda,u}(\zeta) &= \zeta^p\left[\frac{a}{p}\|u\|^p + \frac{b}{2p}\zeta^p\|u\|^{2p} - \frac{\zeta^{p_s^*-p}}{p_s^*}\|u\|_{p_s^*}^{p_s^*} - \lambda\int_\Omega \frac{G(x,\zeta u)}{\zeta^p}\,dx\right] \\
&\geq \zeta^p\left[\frac{a}{p}\|u\|^p + \frac{b}{2p}\zeta^p\|u\|^{2p} - \frac{\zeta^{p_s^*-p}}{p_s^*}\|u\|_{p_s^*}^{p_s^*} - \lambda\varepsilon\left(\|u\|_p^p + 1\right)\right].
\end{aligned}$$

Applying the Sobolev inequality, selecting ε adequately and taking ζ even smaller if needed we get the first part of the assertion. To conclude, it is sufficient to notice that G has subcritical growth and that $p < q < p_s^* < 2p$. □

Now we fix $u \in X_0^{s,p}(\Omega)$ and we consider the system

$$\begin{cases}
\mathcal{J}^{\lambda,u}(\zeta) = 0 \\
(\mathcal{J}^{\lambda,u})'(\zeta) = 0 \\
\mathcal{J}^{\lambda,u}(\zeta) = \inf_{\varrho>0} \mathcal{J}^{\lambda,u}(\varrho)
\end{cases} \tag{4.25}$$

in the unknowns λ and ζ.

Proposition 4.19. *Let $a, b \in \mathbb{R}^+$ such that $a^{(N-2ps)/ps}b \geq L(N,p,s)$. For any $u \in X_0^{s,p}(\Omega) \setminus \{0\}$ there is a unique $\lambda = \lambda_0^s(u)$ that solves (4.25).*

Proof. The statement follows as in [12, Proposition 4]. □

Corollary 4.20. *Let $u \in X_0^{s,p}(\Omega) \setminus \{0\}$. The number $\lambda_0^s(u)$ is the only parameter such that*

$$\inf_{\zeta\in(0,\infty)} \mathcal{J}^{\lambda_0^s(u),u}(\zeta) = 0.$$

In addition,

$$\inf_{\zeta\in(0,\infty)} \mathcal{J}^{\lambda,u}(\zeta) \begin{cases} < 0 & \text{if } \lambda > \lambda_0^s(u) \\ = 0 & \text{if } 0 \leq \lambda \leq \lambda_0^s(u). \end{cases}$$

Proof. The assertion comes as an immediate consequence of the proof of Proposition 4.19. □

Now we define

$$\overline{\lambda}_0^s := \inf_{u\in X_0^{s,p}(\Omega)\setminus\{0\}} \lambda_0^s(u).$$

We emphasize that $\overline{\lambda}_0^s$ is independent from u. In addition, as we are going to see, $\overline{\lambda}_0^s$ has a key importance in determining at what level of energy the minimum is attained. The next Proposition exhibits the relation between $\overline{\lambda}_0^s$ and the parameters a and b.

Proposition 4.21. *The following statements hold:*

(i) if $a^{(N-2ps)/ps}b > L(N,p,s)$ then $\overline{\lambda}_0^s > 0$;

(ii) if $a^{(N-2ps)/ps}b = L(N,p,s)$ then $\overline{\lambda}_0^s = 0$. Furthermore, if $(u_k)_k \subset X_0^{s,p}(\Omega)\setminus\{0\}$ is a sequence such that $\lambda_0^s(u_k)\to 0$ as $k\to\infty$, we have that $u_k/\|u_k\| \rightharpoonup 0$ and

$$\frac{\|u_k\|_p^p}{\|u_k\|_{p_s^*}^p} \to S_{s,p}.$$

Before giving the proof we need some estimates on the minimizers of (4.2). Consider the function $u_{\varepsilon,\delta}(r)$ defined on [89, Lemma 2.7]. In particular, the support of $u_{\varepsilon,\delta}(r)$ is compact and there exists $\tilde{R}>0$ such that

$$u_{\varepsilon,\delta}(r) = U_\varepsilon(r),$$

for $r\le \tilde{R}$, where

$$U_\varepsilon(r) = U_\varepsilon(x) = \frac{1}{\varepsilon^{\frac{N-ps}{p}}} U\left(\frac{|x|}{\varepsilon}\right)$$

and U is a minimizer for (4.2) whose existence is guaranteed by [89, Proposition 2.1].

Now, take the rescaled function

$$w_\varepsilon(x) := \left(\varepsilon^{\frac{1}{p}}\right)^{-\frac{N-ps}{p(p-1)}} u_{\sqrt[p]{\varepsilon},\delta}(x).$$

We point out that we omitted the dependence of δ since is not relevant for our purposes and can be fixed arbitrarily. Taking under consideration this rescaling, from [89, Lemma 2.7] it follows

$$\|w_\varepsilon\|^p \le S_{s,p}^{\frac{N}{ps}} \varepsilon^{-\frac{N-ps}{p(p-1)}} + O(1), \quad \|w_\varepsilon\|_{p_s^*}^{p_s^*} \ge S_{s,p}^{\frac{N}{ps}} \varepsilon^{-\frac{N}{p(p-1)}} - O(1).$$

From this, denoting with $v_\varepsilon := w_\varepsilon/\|w_\varepsilon\|$ the normalized function we get

$$\|v_\varepsilon\| = 1, \quad \|v_\varepsilon\|_{p_s^*}^{p_s^*} \ge S_{s,p}^{-\frac{p_s^*}{p}} + O\left(\varepsilon^{\frac{N-ps}{p(p-1)}}\right), \quad \|w_\varepsilon\| \le S_{s,p}^{\frac{1}{p}} \varepsilon^{-\frac{N-ps}{p^2(p-1)}} + O\left(\varepsilon^{\frac{N-ps}{p^2}}\right) \tag{4.26}$$

as $\varepsilon\to 0$.

Proof of Proposition 4.21. (i) We start noticing that the function $u \to \lambda_0^s(u)$ is well defined and homogeneous of degree zero. Indeed, considering a solution $(\zeta, \lambda_0^s(u))$ of (4.25) and $\alpha > 0$, observing that $\mathcal{J}^{\lambda,\alpha u}(\zeta) = \mathcal{J}^{\lambda,u}(\alpha\zeta)$ and $\left(\mathcal{J}^{\lambda,\alpha u}\right)'(\zeta) = \left(\mathcal{J}^{\lambda,u}\right)'(\alpha\zeta)$ we get that also $(\frac{\zeta}{\alpha}, \lambda_0^s(u))$ solves (4.25) with αu. From the uniqueness of the parameter $\lambda_0^s(\alpha u)$ it follows that $\lambda_0^s(\alpha u) = \lambda_0^s(u)$. To see the positivity of λ_0^s, we argue by contradiction supposing $\overline{\lambda}_0^s = 0$. If so, there would be a sequence $(u_k)_k \subset X_0^{s,p}(\Omega) \setminus \{0\}$ such that $\lambda_k := \lambda_0^s(u_k) \to 0$. Exploiting the homogeneity of the map $u \to \lambda_0^s(u)$, it is not restrictive to assume $\|u_k\|= 1$. Now, Proposition 4.19 implies the existence of $\zeta_k > 0$ such that $\mathcal{J}^{\lambda_k,u_k}(\zeta_k) = 0$, that is

$$\frac{a}{p} + \frac{b}{2p}\zeta_k^p - \frac{1}{p_s^*}\|u_k\|_{p_s^*}^{p_s^*}\zeta_k^{p_s^*-p} - \lambda_k \int_\Omega \frac{G(x, \zeta_k u)}{\zeta_k^p}\, dx = 0.$$

Applying Proposition 4.16, we obtain

$$f_{s,p}(\zeta_k) < \frac{a}{p} + \frac{b}{2p}\zeta_k^p - \frac{1}{p_s^*}\|u_k\|_{p_s^*}^{p_s^*}\zeta_k^{p_s^*-p} = \lambda_k \int_\Omega \frac{G(x, \zeta_k u)}{\zeta_k^p}\, dx. \tag{4.27}$$

From hypotheses (H_3) and (H_4) it follows that for any $\varepsilon > 0$ there is a positive constant $c > 0$ such that $|G(x,t)|< \frac{\varepsilon}{p}|t|^p+\frac{c}{q}|t|^q$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. As a consequence, the sequence $(\zeta_k)_k$ must be bounded, and up to subsequence it converges to some $\overline{\zeta} > 0$. Finally, letting $k \to \infty$ and taking into account Remark 4.15, from (4.27) we obtain

$$0 < f_{s,p}(\overline{\zeta}) = \lim_{k\to\infty} \lambda_k \int_\Omega \frac{G(x, \zeta_k u_k)}{\zeta_k^p}\, dx = 0$$

which is impossible.

(ii) Up to a translation, we can suppose that $0 \in \Omega$. In virtue of the estimates in (4.26), we have

$$\begin{aligned}\mathcal{J}^{\lambda,v_\varepsilon}(\zeta) &= \frac{a}{p}\zeta^p + \frac{b}{2p}\zeta^{2p} - \frac{\zeta^{p_s^*}}{p_s^*}\|v_\varepsilon\|_{2_s^*}^{2_s^*} - \lambda \int_\Omega G(x, \zeta v_\varepsilon(|x|))\, dx \\ &\le \zeta^p f_{s,p}(\zeta) - \frac{\zeta^{p_s^*}}{p_s^*} O\left(\varepsilon^{\frac{N-ps}{p(p-1)}}\right) - \lambda \int_\Omega G(x, \zeta v_\varepsilon(|x|))\, dx.\end{aligned}$$

Selecting as $\zeta = m_{s,p}$ we get

$$\mathcal{J}^{\lambda,v_\varepsilon}(m_{s,p}) \le -\frac{m_{s,p}^{p_s^*}}{p_s^*} O\left(\varepsilon^{\frac{N-ps}{p(p-1)}}\right) - \lambda \int_\Omega G(x, m_{s,p} v_\varepsilon(|x|))\, dx. \tag{4.28}$$

Claim: There exists a constant $C_1 > 0$ such that $\int_\Omega G(x, m_{N,s} u_\varepsilon)\, dx \ge C_1 \varepsilon^{\frac{N}{p^2}}$ as $\varepsilon \to 0$.

Indeed, hypothesis (H_2) asserts the existence of $\mu > 0$ such that $g(x,t) \ge \chi_I$ where I is an open interval of $(0, \infty)$ and χ_I is its characteristic function. Hence we can find a

$\beta > 0$ such that $G(x,t) \geq \tilde{G}(t) := \mu \int_0^t \chi_I(\tau) d\tau \geq \beta$ for any $t \geq \alpha$ where $\alpha := \inf I > 0$. At this point, we have

$$\begin{aligned} \int_\Omega G(x, m_{s,p} v_\varepsilon(|x|))\, dx &\geq \int_{|x|\leq R} G(x, m_{s,p} v_\varepsilon(|x|))\, dx = \int_{|x|\leq R} G\left(x, \frac{m_{s,p}}{\|w_\varepsilon\|} w_\varepsilon(|x|)\right) dx \\ &\geq \int_{|x|\leq R} \tilde{G}\left(\frac{m_{s,p}}{\|w_\varepsilon\|} w_\varepsilon(|x|)\right) dx \\ &\geq \int_{|x|\leq R} \tilde{G}\left(\frac{m_{s,p}}{\|w_\varepsilon\|} \varepsilon^{-\frac{N-ps}{p(p-1)}} U\left(\left|\frac{x}{\sqrt[p]{\varepsilon}}\right|\right)\right) dx \\ &= \omega_N \int_0^R \tilde{G}\left(m_{s,p} \frac{w_\varepsilon(w)}{\|w_\varepsilon\|}\right) w^{N-1}\, dw \\ &\geq \omega_N \int_0^{\sqrt[p]{\varepsilon} R} \tilde{G}\left(m_{s,p} \frac{w_\varepsilon(w}{\|w_\varepsilon\|}\right) w^{N-1}\, dw. \end{aligned} \tag{4.29}$$

With the change of variable $x = \sqrt[p]{\varepsilon} y$ (4.29) becomes

$$\begin{aligned} \int_\Omega G(x, m_{s,p} v_\varepsilon(|x|))\, dx &\geq \varepsilon^{\frac{N}{p}} \int_{|x|\leq R\varepsilon^{-\frac{1}{p}}} \tilde{G}\left(\frac{m_{s,p}}{\|w_\varepsilon\|} \varepsilon^{-\frac{N-ps}{p(p-1)}} U\left(|y|\right)\right) dy \\ &= \omega_N \varepsilon^{\frac{N}{p}} \int_0^{R\varepsilon^{-\frac{1}{p}}} \tilde{G}\left(\frac{m_{s,p}}{\|w_\varepsilon\|} \varepsilon^{-\frac{N-ps}{p(p-1)}} U\left(w\right)\right) w^{N-1} dw \\ &\geq \omega_N \varepsilon^{\frac{N}{p}} \int_0^{R\varepsilon^{-\frac{p-1}{p^2}}} \tilde{G}\left(\frac{m_{s,p}}{\|w_\varepsilon\|} \varepsilon^{-\frac{N-ps}{p(p-1)}} U\left(w\right)\right) w^{N-1} dw \end{aligned}$$

We point out that if

$$\frac{m_{s,p}}{\|w_\varepsilon\|} \varepsilon^{-\frac{N-ps}{p(p-1)}} U\left(w\right) \geq \alpha \quad \text{for } w \in \left[0, \varepsilon^{-\frac{p-1}{p^2}} R\right],$$

then

$$\int_0^{\varepsilon^{-\frac{p-1}{p^2}} R} \tilde{G}\left(m_{s,p} \frac{w_\varepsilon(w)}{\|w_\varepsilon\|}\right) w^{N-1}\, dw \geq \beta \int_0^{\varepsilon^{-\frac{p-1}{p^2}} R} w^{N-1}\, dw = \frac{C_1}{\omega_N} \varepsilon^{-N\frac{p-1}{p^2}}.$$

Since $w \in \left[0, \varepsilon^{-\frac{p-1}{p^2}} R\right]$, and recalling that w_ε is monotone decreasing by [89, Proposition 2.1] and [27, Theorem 1.1], we have

$$\frac{m_{s,p}}{\|w_\varepsilon\|} \varepsilon^{-\frac{N-ps}{p(p-1)}} U\left(w\right) \geq \frac{m_{s,p}}{\|w_\varepsilon\|} \varepsilon^{-\frac{N-ps}{p(p-1)}} U\left(\varepsilon^{-\frac{p-1}{p^2}} R\right). \tag{4.30}$$

Now, applying again [27, Theorem 1.1], for ε small enough we have

$$\frac{1}{2} U_\infty \left(\varepsilon^{-\frac{p-1}{p^2}} R\right)^{-\frac{N-ps}{p-1}} \leq U\left(\varepsilon^{-\frac{p-1}{p^2}} R\right) \leq \frac{3}{2} U_\infty \left(\varepsilon^{-\frac{p-1}{p^2}} R\right)^{-\frac{N-ps}{p-1}}$$

where U_∞ is a constant that can be supposed positive without restrictions. From this, (4.26) and (4.30) we get

$$\begin{aligned}\frac{m_{s,p}}{\|w_\varepsilon\|}\varepsilon^{-\frac{N-ps}{p(p-1)}}U(w) &\geq \frac{U_\infty}{2}\frac{m_{s,p}}{\|w_\varepsilon\|}\varepsilon^{-\frac{N-ps}{p^2(p-1)}}R^{-\frac{N-ps}{p-1}}\\ &\geq m_{s,p}\frac{U_\infty}{2}\frac{R^{-\frac{N-ps}{p-1}}}{S_{s,p}^{\frac{1}{p}}+O\left(\varepsilon^{\frac{N-ps}{p(p-1)}}\right)}\geq \alpha\end{aligned}$$

restricting eventually R, and the claim is proved.

At this point, exploiting the claim, from (4.28) it follows

$$\mathcal{J}^{\lambda,v_\varepsilon}(m_{s,p})\leq \varepsilon^{\frac{N}{p^2}}\left(-\frac{m_{s,p}^{p_s^*}}{p_s^*}O\left(\varepsilon^{\frac{N-p^2s}{p^2(p-1)}}\right)-\lambda C_1\right)<0$$

for ε sufficiently small. As a consequence, $\lambda_0^s(u_\varepsilon)<\lambda$. Since all arguments above are independent of the choice of λ, we may let $\lambda\to 0$ and obtain $\overline{\lambda}_0^s=0$. To prove the remaining part of the Proposition, take a sequence $(u_k)_k\subset X_0^{s,p}(\Omega)\setminus\{0\}$ such that $\lambda_k:=\lambda_0^s(u_k)\to\overline{\lambda}_0^s=0$. Analogously to part (i), it is not restrictive to assume $\|u_k\|=1$, $u_k\rightharpoonup u$ and that there exists $\zeta_k>0$ such that

$$\frac{a}{p}+\frac{b}{2p}\zeta_k^p-\frac{\zeta_k^{p_s^*-p}}{p_s^*}\|u_k\|_{p_s^*}^{p_s^*}-\lambda_k\int_\Omega\frac{G(x,\zeta_k u_k)}{\zeta_k^p}\,dx=0. \tag{4.31}$$

Putting together assumptions (H_3), (H_4) and (4.31), we can see that, up to subsequence, $\zeta_k\to\overline{\zeta}$ and $\|u_k\|_{p_s^*}^{p_s^*}\to\gamma$ as $k\to\infty$. Letting $k\to\infty$ in (4.31), we get

$$\frac{a}{p}+\frac{b}{2p}\overline{\zeta}^p-\frac{\overline{\zeta}^{p_s^*-p}}{p_s^*}\gamma=0.$$

Since $a^{(N-2ps)/ps}b=L(N,p,s)$ it is easy to see that $\gamma=S_{s,p}^{-\frac{p_s^*}{p}}$, implying that $(u_k)_k$ is a minimizing sequence for $S_{s,p}$. Finally, suppose by contradiction $u\neq 0$. By the lower semicontinuity of the norm we have $\|u\|\leq 1$. From this, taking under consideration Remark 4.15 and Theorem 4.2, we obtain

$$\begin{aligned}0\leq\frac{a}{p}+\frac{b}{2p}\overline{\zeta}^p-\frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\overline{\zeta}^{p_s^*-p}\|u\|^{p_s^*}&\leq\frac{a}{p}+\frac{b}{2p}\overline{\zeta}^p-\frac{\overline{\zeta}^{p_s^*-p}}{p_s^*}\|u\|_{p_s^*}^{p_s^*}\\ &\leq\limsup_{k\to\infty}\left(\frac{a}{p}+\frac{b}{2p}\zeta_k^p-\frac{\zeta_k^{p_s^*}}{p_s^*}\|u_k\|_{p_s^*}^{p_s^*}-\lambda_k\int_\Omega\frac{G(x,\zeta_k u_k)}{\zeta_k^p}\,dx\right)=0,\end{aligned}$$

which would imply that u is a minimizer for (4.2). However, this is not possible if we compare [27, Theorem 1.1] and the fact that $u=0$ in $\mathbb{R}^N\setminus\Omega$. □

Proposition 4.22. *If* $\lambda \leq \overline{\lambda}_0^s$ *then* $\inf_{\zeta>0} \mathcal{J}^{\lambda,u}(\zeta) = 0$ *for any* $u \in X_0^{s,p}(\Omega) \setminus \{0\}$. *On the other hand, if* $\lambda > \overline{\lambda}_0^s$ *there exists* $u \in X_0^{s,p}(\Omega) \setminus \{0\}$ *such that* $\inf_{\zeta>0} \mathcal{J}^{\lambda,u}(\zeta) < 0$.

Proof. The proof follows closely the line of [12, Proposition 6] □

We are now ready to prove Theorem 4.5 and Theorem 4.6.

Proof of Theorem 4.5. The thesis comes as in [12, Theorem 2]. □

Proof of Theorem 4.6. (i) Consider a sequence $(\lambda_k)_k \subset \mathbb{R}^+$ such that $\lambda_k \searrow \overline{\lambda}_0^s$. In virtue of Theorem 4.5 we can find a sequence $(u_k)_k \subset X_0^{s,p}(\Omega)\setminus\{0\}$ such that $\iota^s_{\lambda_k} = \mathcal{I}^{\lambda_k}(u_k) < 0$. Similarly to what we have done in Proposition 4.22, after choosing $\varepsilon > 0$ we have

$$|G(x,t)|\leq \frac{\varepsilon}{p}|t|^p+\frac{c}{q}|t|^q \tag{4.32}$$

for all $t \in \mathbb{R}$ and a.e. in Ω. As a consequence of that

$$\begin{aligned}\frac{a}{p}\|u_k\|^p+\frac{b}{2p}\|u_k\|^{2p}-\frac{1}{p_s^*}\|u_k\|_{p_s^*}^{p_s^*}&< \lambda_k \int_\Omega G(x,u_k)\,dx\\ \leq \lambda_k\left(\frac{\varepsilon}{p}\|u_k\|_p^p+\frac{c}{q}\|u_k\|_q^q\right) &\leq \tilde{C}\left(\|u_k\|^p+\|u_k\|^q\right)\end{aligned} \tag{4.33}$$

for some $\widetilde{C} > 0$ since $X_0^{s,p}(\Omega) \hookrightarrow L^{\upsilon}(\Omega)$ continuously for any $\upsilon \in [1, p_s^*]$. Since $2p > p_s^*$ the sequence $(\|u_k\|)_k$ needs to be bounded and we are allowed to suppose $u_k \rightharpoonup u$ in $X_0^{s,p}(\Omega)$. Now, on one hand we use Lemma 4.17[(1)] and we get

$$\mathcal{I}^{\overline{\lambda}_0^s}(u) \leq \liminf_{k\to\infty} \mathcal{I}^{\overline{\lambda_k}}(u_k) \leq 0.$$

On the other hand, Proposition 4.22 implies that $\mathcal{I}^{\overline{\lambda}_0^s}(v) \geq 0$ for any $v \in X_0^{s,p}(\Omega)$. Hence, the only admissible scenario is

$$\iota^s_{\overline{\lambda}_0^s} = \mathcal{I}^{\overline{\lambda}_0^s}_{a,b}(u) = 0. \tag{4.34}$$

In order to show that u is a non-trivial minimizer, we start noticing that

$$\begin{aligned}&\frac{a}{p}\|u_k\|^p+\frac{b}{2p}\|u_k\|^{2p}-\frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\|u_k\|^{p_s^*}\\ &\qquad\leq \frac{a}{p}\|u_k\|^p+\frac{b}{2p}\|u_k\|^{2p}-\frac{1}{p_s^*}\|u_k\|_{p_s^*}^{p_s^*}< \lambda_k \int_\Omega G(x,u_k)\,dx\end{aligned}$$

where we used the fractional Sobolev inequality. Dividing by $\|u_k\|^p$ and exploiting (4.32), we obtain

$$f_{s,p}(\|u_k\|) \leq \lambda_k \tilde{C}\left(\frac{\varepsilon}{p}+\frac{c}{q}\|u_k\|^{q-p}\right).$$

for some $\tilde{C}$. If $u = 0$, recalling that $X_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we would obtain $f_{s,p}(\|u_k\|) \to 0$ as $k \to \infty$ since $\varepsilon > 0$ is arbitrary. However, Remark 4.15 yields

$$f_{s,p}(\|u_k\|) \geq f_{s,p}(m_{s,p}) > 0$$

since $a^{(N-2ps)/ps}b > L(N,p,s)$. This contradiction shows that $u \neq 0$.
(ii) Proposition 4.21[(ii)] implies $\overline{\lambda}_0^s = 0$, so

$$\mathcal{I}^{\overline{\lambda}_0^s}(u) = \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*}.$$

At this point, keeping in mind Remark 4.15 and Proposition 4.16, we have

$$\mathcal{I}^{\overline{\lambda}_0^s}(u) > \|u\|^p f_{s,p}(\|u\|) \geq 0$$

for all $u \in X_0^{s,p}(\Omega) \setminus \{0\}$. In virtue of the inequality above, recalling that (4.34) still holds, it is evident that the infimum can be achieved only if $u = 0$. □

Corollary 4.23. *If $a^{(N-2ps)/ps}b > L(N,p,s)$ and $u \in X_0^{s,p}(\Omega) \setminus \{0\}$ is such that $\iota_{\overline{\lambda}_0^s} = \mathcal{I}^{\overline{\lambda}_0^s}(u)$ then $\overline{\lambda}_0^s = \lambda_0^s(u)$.*

Proof. Observe that $(\overline{\lambda}_0^s, u)$ solves (4.25) and conclude by recalling the uniqueness. □

We are now in position to give the proof of Theorem 4.7. We point out that in the next proof we will highlight the dependence on $\mathcal{I}^\lambda$ and $\mathcal{J}^{\lambda,u}$ from a and b by writing respectively $\mathcal{I}_{a,b}^\lambda$ and $\mathcal{J}_{a,b}^{\lambda,u}$.

Proof of Theorem 4.7. Up to translations it is not restrictive to assume $0 \in \Omega$. Recall the function v_ε considered after the statement of Proposition 4.21 and select $\zeta > 0$. We have

$$\begin{aligned}\mathcal{J}_{a_k,b_k}^{\lambda,v_\varepsilon}(\zeta) &= \frac{a_k}{p}\zeta^p + \frac{b_k}{2p}\zeta^{2p} - \frac{\zeta^{p_s^*}}{p_s^*}\|v_\varepsilon\|_{p_s^*}^{p_s^*} - \lambda\int_\Omega G(x, \zeta v_\varepsilon(|x|))\,dx \\ &\leq \zeta^p f_{s,p}^k(\zeta) - \frac{\zeta^{p_s^*}}{p_s^*} O\left(\varepsilon^{\frac{N-ps}{p(p-1)}}\right) - \lambda\int_\Omega G(x, \zeta v_\varepsilon(|x|))\,dx\end{aligned}$$

where we denoted with $f_{s,p}^k$ the map $f_{s,p}$ emphasizing the dependence on the parameters a_k, b_k. We choose $\zeta = m_{s,p}^k$ where we called $m_{s,p}^k$ the point in which $f_{s,p}^k$ attains its minimum, and since $m_{s,p}^k \to m_{s,p}$ as $k \to \infty$, we get

$$\lim_{k\to\infty} \mathcal{J}_{a_k,b_k}^{\lambda,v_\varepsilon}(m_{s,p}^k) \leq -\frac{m_{s,p}^{p_s^*}}{p_s^*} O\left(\varepsilon^{\frac{N-ps}{p(p-1)}}\right) - \lambda\int_\Omega G(x, m_{s,p}v_\varepsilon(|x|))\,dx. \tag{4.35}$$

At this point, as we did in Proposition 4.21, we estimate

$$\int_\Omega G(x, m_{s,p}v_\varepsilon(|x|))\,dx \geq C_1\varepsilon^{\frac{N}{p^2}},$$

and from (4.35) we get

$$\lim_{k\to\infty} \mathcal{J}^{\lambda,v_\varepsilon}_{a_k,b_k}(m^k_{s,p}) \leq \mathcal{J}^{\lambda,v_\varepsilon}(m_{s,p}) \leq \varepsilon^{\frac{N}{p^2}}\left(-\frac{m^{p^*_s}_{s,p}}{p^*_s}O\left(\varepsilon^{\frac{N-p^2s}{p^2(p-1)}}\right)-\lambda C_1\right)<0.$$

Thus, choosing k big enough and a small ε

$$\mathcal{J}^{\lambda,v_\varepsilon}_{a_k,b_k}(m^k_{s,p})<0.$$

Hence, from Corollary 4.20 $\lambda_k \leq \lambda^s_0(v_\varepsilon)(a_k,b_k)<\lambda$. Now, we point out that no restrictions were made on λ so we are free to let $\lambda\to 0$ and deduce that $\lambda_k\to 0$ as $k\to\infty$.

In order to prove the remaining part of the statement, we recall that in Proposition 4.21 we proved that the map $u\to\lambda^s_0(u)$ is homogeneous degree zero. As a consequence of that, it is not restrictive to suppose $\|u_k\|=1$ and $u_k\rightharpoonup u$. Arguing as for (4.31), it is possible to find $\zeta_k>0$ such that

$$\frac{a_k}{p}+\frac{b_k}{2p}\zeta^p_k-\frac{\zeta_k^{p^*_s-p}}{p^*_s}\|u_k\|^{p^*_s}_{p^*_s}-\lambda_k\int_\Omega \frac{G(x,\zeta_k u_k)}{\zeta^p_k}\,dx=0 \tag{4.36}$$

Furthermore, combining and (4.32) and (4.36),we can deduce the boundedness of $(\zeta_k)_k$ and suppose up to a subsequence that $\zeta_k\to\overline{\zeta}>0$ and that $\|u_k\|^{p^*_s}_{p^*_s}\to\gamma$ as $k\to\infty$. Hence, passing to the limit in (4.36) we obtain

$$\frac{a}{p}+\frac{b}{2p}\overline{\zeta}^p-\frac{1}{p^*_s}\gamma\overline{\zeta}^{p^*_s-p}=0.$$

From $a^{(N-2ps)/ps}b=L(N,p,s)$ it follows $\gamma=S_{p,s}^{-\frac{p^*_s}{p}}$ implying that $(u_k)_k$ is a minimizing sequence for the optimal Sobolev constant. Finally we can see $u=0$. In fact, if we assume $u\neq 0$ we have that $\|u\|\leq 1$ exploiting the sequentially lower semicontinuity of the norm. From this, Lemma 4.17[(1)] and Remark 4.15, we obtain

$$\begin{aligned}0&\leq \frac{a}{p}+\frac{b}{2p}\overline{\zeta}^p-\frac{S_{p,s}^{-\frac{p^*_s}{p}}}{p^*_s}\overline{\zeta}^{p^*_s-p}\|u\|^{p^*_s}\leq \frac{a}{p}+\frac{b}{2p}\overline{\zeta}^p-\frac{\overline{\zeta}^{p^*_s-p}}{p^*_s}\|u\|^{p^*_s}_{p^*_s}\\&\leq \liminf_{k\to\infty}\left(\frac{a_k}{p}+\frac{b_k}{2p}\zeta^p_k-\frac{\zeta_k^{p^*_s-p}}{p^*_s}\|u_k\|^{p^*_s}_{p^*_s}-\lambda_k\int_\Omega\frac{G(x,\zeta_k u_k)}{\zeta^p_k}\,dx\right)=0\end{aligned}$$

So, u is a minimizer for $S_{s,p}$ but this is in not admissible since $u=0$ in $\mathbb{R}^N\setminus\Omega$ as shown in [27, Theorem 1.1]. □

At this point, we start giving the proofs regarding the existence of mountain pass solutions. Namely we are going to prove Theorem 4.8.

Proof of Theorem 4.8. Fix $\varepsilon > 0$. Recalling (4.32) and that $X_0^{s,p}(\Omega) \hookrightarrow L^v(\Omega)$ continuously for any $v \in [1, p_s^*]$ we obtain

$$\mathcal{I}^\lambda(u) \geq \left(\frac{a}{p} - \lambda C\varepsilon\right) \|u\|^p + \frac{b}{2p}\|u\|^{2p} - C\|u\|^{p_s^*} - \lambda C\|u\|^q \tag{4.37}$$

selecting $C > 0$ appropriately. At this point, to conclude the proof, it suffices to argue as in [12, Theorem 5] □

After analyzing the situation to the case $\lambda \geq \overline{\lambda}_0^s$ we focus to the case $\lambda \leq \overline{\lambda}_0^s$. In particular we are interested in looking for local minimizer or mountain pass critical point of $\mathcal{I}^\lambda$.

Proposition 4.24. *If $\lambda \leq \overline{\lambda}_0^s$ then it is possible to find $r = r(s), M = M(s) > 0$ such that*

$$\inf\left\{\mathcal{I}^\lambda(u) \mid u \in X_0^{s,p}(\Omega),\ \|u\| = r\right\} \geq M. \tag{4.38}$$

Proof. Fix $\varepsilon > 0$. From (4.37) and $\lambda \leq \overline{\lambda}_0^s$ it follows

$$\mathcal{I}^\lambda(u) \geq \left(\frac{a}{p} - \overline{\lambda}_0^s C\varepsilon\right) \|u\|^p + \frac{b}{2p}\|u\|^{2p} - C\|u\|^{p_s^*} - \overline{\lambda}_0^s C\|u\|^q$$

for any $u \in X_0^{s,p}(\Omega)$. Choosing ε such that $a/p - \overline{\lambda}_0^s C\varepsilon > 0$ we obtain the assertion. □

After showed the validity of the previous proposition we can finally prove the remaining two theorems.

Proof of Theorem 4.9. Consider the number r given by Proposition 4.24, and argue as in [12, Theorem 6]. □

Remark 4.25. It is immediate to see that $\hat{\iota}_0^s \to 0$ as $\lambda \to \overline{\lambda}_0^s$. In fact, take a function $u \in X_0^{s,p}(\Omega)$ such that $\overline{\lambda}_0^s = \lambda_0^s(u)$ whose existence was shown in Theorem 4.6) and notice that

$$0 \leq \hat{\iota}_\lambda^s \leq \mathcal{I}^\lambda(u) \to 0 \quad \text{as} \quad \lambda \to \overline{\lambda}_0^s.$$

Remark 4.26. The function w_λ^s obtained in the previous theorem is a critical point for the functional $\mathcal{I}^\lambda$, and more precisely it is a local minimizer.

Proof of Theorem 4.10. Observe that $\max\{\mathcal{I}^\lambda(0), \mathcal{I}^\lambda(w_\lambda^s)\} < M$, recall $\|w_\lambda^s\| > M$ and (4.38). Hence, we have a mountain pass geometry. Furthermore, the Palais-Smale condition holds as showed in Lemma 4.17. At this point, in order to conclude, it suffices to apply the Mountain Pass Theorem (see [3]). □

5 Schrödinger equation on Cartan-Hadamard manifolds with oscillating nonlinearities

Let $(\mathcal{M}, g)$ be a d-dimensional homogeneous Cartan-Hadamard Manifold with $d \geq 3$. The aim of this Chapter is to study

$$\begin{cases} -\Delta_g w + w = \lambda \alpha(\sigma) f(w) & \text{in } \mathcal{M} \\ w \in H^1_g(\mathcal{M}) \end{cases} \tag{P_λ}$$

where $-\Delta_g$ denotes the Laplace-Beltrami operator, $\alpha \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M}) \setminus \{0\}$ is a.e positive, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $\lambda > 0$ a real parameter.

The stationary nonlinear Schrödinger equation is undoubtedly one of the most attractive topics in nonlinear analysis. In the last years many researchers studied this equations under various hypothesis on the nonlinear term and in different setting. Among them, the study of the nonlinear Schrödinger equation on Riemannian manifold has received a particular attention recently. Faraci and Farkas in [44] using variational methods proved a characterization result for existence of solutions for the Schrödinger equation with a divergent potential in a non-compact Riemannian manifold with asymptotically non-negative Ricci curvature. In the same setting of this Chapter, Molica Bisci and Secchi in [86] proved some existence and non-existence results for a similar problem, while Appolloni et al. in [10] showed the existence of three critical points for the energy functional associated to a perturbed problem. Kristály in [61] proved a multiplicity result for the equation without a potential and with $\mathcal{M} = \mathbb{S}^d$. We also quote [87] where Molica Bisci and Vilasi obtained an existence result regarding positive solutions which are invariant under the action of a specific family of isometries and [24] where Molica Bisci and Repovš showed the existence of positive solutions when the nonlinear term is critical in the sense of Sobolev. It is also worth mentioning [35] where Cencelj et al. by applying the Palais principle of symmetric criticality and suitable group theoretical arguments are able to prove the existence of non-trivial weak solutions.

Motived by the great interest in this field, in this Chapter we are going to study the Schrödinger equation on a non-compact homogeneous Cartan-Hadamard manifold with a nonlinear term f that oscillates near zero or at infinity. As regards oscillating nonlinearities there is a wide literature dealing with this kind of problems with numerous differential operator. To the best of our knowledge, one of the first contribution in this direction was given in [51] by Habets et al. where the authors exhibit that the problem they are considering admits an unbounded sequence of solutions with $d = 1$

with a technique based on phase-plane analysis and time-mapping estimates. At a later time, Omari and Zanolin in [93] were able to show the existence of infinitely many solutions for a problem with a general operator in divergence form building a sequence of arbitrarily large negative lower solutions and a sequence of arbitrarily large positive upper solutions. More recently Anello and Cordaro in [8] proved the existence of a sequence of critical points converging to zero with respect to the L^∞ norm for a problem with a nonlinear oscillating term at zero. In the same spirit of the previous one, Molica Bisci and Pizzimenti obtained in [23] similar results for the p-Kirchhoff problem analyzing also what happens in presence of oscillations at infinity. Finally, Molica Bisci and Rădulescu in [85] showed the existence of a sequence of invariant solutions tending to zero both in the Sobolev norm and in the L^∞ norm on the Poincaré ball model.

One of the main task we have to face in order to study Problem (P_λ) is the loss of compactness of the embedding $H^1_g(\mathcal{M}) \hookrightarrow L^q(\mathcal{M})$ due to the non-compactness of the manifold $\mathcal{M}$. In order to overcome this difficulty, we will use an embedding result for a Sobolev space which is invariant under the action of a certain group proved by Skrzypczak and Tintarev in [105] generalizing the well known fact that the embedding $H^1_r(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ is compact for all $q \in (2, 2^*)$ for functions invariant under the group of the rotations. Coupling this fact with the principle of symmetric criticality proved by Palais in [94] and the continuity of the superposition operator, whose validity is established in [76] for the Euclidean case and generalized to manifold in [54, Proposition 2.5], we will consider an auxiliary problem with a truncated nonlinearity and we will show the existence of infinitely many local minima. We emphasize that in dealing with the case of oscillations near zero we will assume no growth condition on the nonlinear term f. The Chapter is organised as follows. At the end of this Section we collect our main results. In Section 5.1 present the abstract framework. In Section 5.2 we prove Theorem 5.1 showing the existence of infinitely many critical points for the energy functional associated to (P_λ) and with both L^∞ and Sobolev norm going to zero. In Section 5.3 we address the problem of oscillations at infinity proving Theorem 5.2. More precisely, given a group G that acts on $\mathcal{M}$ we will denote with

$$\mathrm{Fix}_{\mathcal{M}}(G) := \{\sigma \in \mathcal{M} \mid \varphi(\sigma) = \sigma \text{ for all } \varphi \in G\}.$$

the fixed points of G. The following hypothesis will be crucial in the sequel:

$(\mathcal{H}^{\sigma_0}_G)$ G is a compact, connected subgroup of the isometries $\mathrm{Isom}_g(\mathcal{M})$ of $(\mathcal{M}, g)$ such that

$$\mathrm{Fix}_{\mathcal{M}}(G) = \{\sigma_0\}$$

for some point $\sigma_0 \in \mathcal{M}$.

To ease notation, since now to the end of the Chapter we will denote by

$$\|w\| := \left(\int_{\mathcal{M}} |^g\nabla w(\sigma)|^2 \, dv_g + \int_{\mathcal{M}} |w(\sigma)|^2 \, dv_g \right)^{\frac{1}{2}}.$$

The main results we are going to prove during the rest of the Chapter are the following.

Theorem 5.1. *Assume that* $(\mathcal{H}_G^{\sigma_0})$ *holds and let* $\alpha \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M}) \setminus \{0\}$ *be a a.e. positive map such that* $\alpha(\sigma) = \alpha(d_g(\sigma_0, \sigma))$. *Moreover, let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function for which*

(f_0) *there exist two sequences* $(t_j)_j$ *and* $(t'_j)_j$ *with* $\lim_{j\to+\infty} t'_j = 0$ *and* $0 \le t_j < t'_j$ *such that*

$$F(t_j) = \sup_{t\in[t_j, t'_j]} F(t),$$

where $F(t) := \int_0^t f(\tau)\, d\tau$;

(f_1) *there exist a constant* $K_1 > 0$ *and a sequence* $(\xi_j)_j \subset (0, +\infty)$ *with* $\lim_{j\to+\infty} \xi_j = 0$ *such that*

$$\lim_{j\to+\infty} \frac{F(\xi_j)}{\xi_j^2} = +\infty,$$

and

$$\inf_{t\in[0,\xi_j]} F(t) \ge -K_1 F(\xi_j).$$

Then for every $\lambda > 0$ *it is possible to find a sequence* $(w_j)_j \subset H_G^1(\mathcal{M})$ *of non-negative and not identically zero solutions of* (P_λ) *such that*

$$\lim_{j\to+\infty} \|w_j\| = \lim_{j\to+\infty} \|w_j\|_{L^\infty(\mathcal{M})} = 0.$$

Theorem 5.2. *Assume that* $(\mathcal{H}_G^{\sigma_0})$ *holds and let* $\alpha \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M}) \setminus \{0\}$ *be a a.e. positive map such that* $\alpha(\sigma) = \alpha(d_g(\sigma_0, \sigma))$. *Moreover, let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $f(0) \ge 0$ *for which*

(f'_0) *there are a constant* $K_2 > 0$ *and* $q \in (2, 2^* - 1)$ *such that*

$$|f(t)| \le K_2(1 + |t|^q);$$

(f'_1) *there are two sequences* $(t_j)_j$ *and* $(t'_j)_j$ *with* $\lim_{j\to+\infty} t_j = +\infty$ *and* $0 \le t_j < t'_j$ *such that*

$$F(t_j) = \sup_{t\in[t_j, t'_j]} F(t);$$

(f'_2) *there is a constant* $K_3 > 0$ *and a sequence* $(\xi_j)_j \subset (0, +\infty)$ *with* $\lim_{j\to+\infty} \xi_j = \infty$ *such that*

$$\lim_{j\to+\infty} \frac{F(\xi_j)}{\xi_j^2} = +\infty,$$

and

$$\inf_{t\in[0,\xi_j]} F(t) \ge -K_3 F(\xi_j).$$

Then for every $\lambda > 0$ *it is possible to find a sequence* $(w_j)_j \subset H_G^1(\mathcal{M})$ *of non-negative and not identically zero weak solutions of* (P_λ).

5.1 Abstract framework

We begin introducing the notion of coerciveness for a group acting continuosly on the manifold.

Definition 5.3. A group G acting continuously on $\mathcal{M}$ is said to be coercive if for every $t > 0$ the set

$$\{\sigma \in \mathcal{M} \mid \operatorname{diam} G\sigma \leq t\}$$

is bounded, where

$$G\sigma := \{\varphi \cdot \sigma \mid \varphi \in G\}.$$

As we will see later being coercive will play a determining role to have compact embedding for Sobolev spaces invariant under the action of a group G. Despite the coerciveness of a group G has a clear geometrical meaning, it is a property that in most cases turns out to be difficult to verify. In order to overcome this problem, we introduce a condition that is equivalent in a Cartan-Hadamard manifold.

As pointed out in [105, Proposition 3.1] in a simply-connected Riemannian manifold with non-positive Sectional curvature, a subgroup G of $\mathrm{Isom}_g(\mathcal{M})$ satisfies $(\mathcal{H}_G^{\sigma_0})$ if and only if it is coercive. For the sake of completeness we write down here the Proposition omitting the proof.

Proposition 5.4. *Let $\mathcal{M}$ be a simply connected complete Riemannian manifold, and assume that the Sectional curvature is non-positive. Let G be a compact, connected subgroup of $\mathrm{Isom}_g(\mathcal{M})$ that fixes some point $\sigma_0 \in \mathcal{M}$. Then G is coercive if and only if G has no other fixed point but σ_0.*

There are several examples present in literature of homogeneous Cartan-Hadamard manifold with a group acting transitively on it, fixing only one point. For instance, $\mathbb{R}^d$ equipped with the Euclidean metric and the special orthogonal group $SO(d)$ or $SO(d_1) \times \ldots \times SO(d_h)$ where $\sum_{i=1}^{d_h} d_i = d$ with $d_i > 1$. Another common example is the Poincaré model $\mathbb{H}^d := \left\{x \in \mathbb{R}^d \ : \ |x| < 1\right\}$ endowed with the metric

$$g_{ij}(x) := \frac{4}{(1 - |x|^2)^2}\delta_{ij}$$

with the same choices as above for the group. In addition to that, we can also consider the set $P(d, \mathbb{R})$ of the symmetric positive definite matrices with determinant equal to one. It turns out that it has a structure of homogeneous Cartan-Hadamard manifold and that the special orthogonal group $O(d)$ acts transitively on it, fixing the identity matrix I_d. For further detail we suggest the reader to consult [31, Chapter II.10], [46], [62] and [63, Chapter XII].

Now we fix a point $\sigma_0 \in \mathcal{M}$ and a group G satisfying $(\mathcal{H}_G^{\sigma_0})$. We consider the Sobolev space

$$H^1_G(\mathcal{M}) = \left\{w \in H^1_g(\mathcal{M}) \mid \varphi \circledast w = w \text{ for all } \varphi \in G\right\}$$

where

$$\varphi \circledast w := w(\varphi^{-1} \cdot \sigma) \quad \text{for a.e. } \sigma \in \mathcal{M}.$$

In virtue of the previous Remark, we are able to state the following compactness result.

Lemma 5.5. *If G satisfies $(\mathcal{H}_G^{\sigma_0})$, then the embedding*

$$H_G^1(\mathcal{M}) \hookrightarrow L^\nu(\mathcal{M})$$

is compact for all $\nu \in (2, 2^*)$ where $2^* := 2d/(d-2)$.

Proof. According to [54, Lemma 8.1 and Theorem 8.3] or [55] the embedding $H_G^1(\mathcal{M}) \hookrightarrow L^\nu(\mathcal{M})$ is continuous for all $\nu \in [2, 2^*]$ and co-compact for [109, Chapter 9]. At this point, taking into account Proposition 5.4 we can apply [105, Theorem 1.3] to complete the proof. □

5.2 Oscillation at the origin

In this Section, we investigate the existence of solutions for problem (P_λ)

$$\begin{cases} -\Delta_g w + w = \lambda\alpha(\sigma)f(w) & \text{in } \mathcal{M} \\ w \in H_g^1(\mathcal{M}) \end{cases}$$

where f represents a continuous function that oscillates near 0. More precisely, since now till the end of the Section the function f satisfies hypothesis (f_0) and (f_1) of Theorem 5.1. As an immediate consequence of these hypothesis we have the following Lemma.

Lemma 5.6. *If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (f_0) and (f_1), then $f(0) = 0$.*

Proof. We first notice that

$$f(t_j) = \lim_{h\to 0^+} \frac{\displaystyle\int_{t_j}^{t_j+h} f(\tau)\, d\tau}{h} = \lim_{h\to 0^+} \frac{F(t_j+h) - F(t_j)}{h} \le 0$$

by (f_0). Thus, exploiting the continuity of f we have

$$f(0) = \lim_{j\to+\infty} f(t_j) \le 0.$$

On the other hand, suppose by contradiction that $f(0) < 0$. Then, again the continuity of f implies that $f(t) < 0$ for all $t \in [0, \delta)$ for some $\delta > 0$. Then, we would have

$$\lim_{j\to+\infty} \frac{F(\xi_j)}{\xi_j^2} \le 0,$$

in contradiction with (f_1). □

The relation $\alpha(\sigma) = \alpha(d_g(\sigma_0, \sigma))$ is a symmetry condition which replaces the radial symmetry of α is $\mathbb{R}^d$.

Proof of Theorem 5.1. Let $\lambda > 0$. Since $t_j \to 0$ and $\xi_j \to 0$ as $j \to +\infty$, we may assume that $0 \le t_j \le t_0$ and $0 \le \xi_j \le t_0$ for some $t_0 > 0$ and for every j. Let $\kappa = \max\{|f(t)|\,|\, t \in [0, t_0]\}$. In view of Lemma 5.6, we define the continuous truncated function

$$h(t) := \begin{cases} f(t_0) & \text{if } t > t_0 \\ f(t) & \text{if } 0 \le t \le t_0 \\ 0 & \text{if } t < 0 \end{cases}$$

and we consider the auxiliary problem

$$\begin{cases} -\Delta_g w + w = \lambda\alpha(\sigma)h(w) & \text{in } \mathcal{M} \\ w \in H^1_G(\mathcal{M}). \end{cases} \tag{P_0}$$

We also set the energy functional associated to Problem (P_0)

$$J_{G,\lambda}(w) := \frac{1}{2}\|w\|^2 - \lambda \int_{\mathcal{M}} \alpha(\sigma) \left(\int_0^{w(\sigma)} h(\tau)\, d\tau \right) dv_g$$

and we emphasize that $J_{G,\lambda} \in C^1(H^1_G(\mathcal{M}), \mathbb{R})$ thanks to Lemma 5.5 and that is is sequentially lower semicontinuous. Now, for all $j \in \mathbb{N}$ we define the set

$$\mathbb{E}^G_j := \left\{ w \in H^1_G(\mathcal{M}) \mid 0 \le w(\sigma) \le t'_j \text{ a.e in } \mathcal{M} \right\}.$$

We divide the remaining part of the proof in 6 steps.

Step 1: the functional $J_{G,\lambda}$ in bounded from below on $\mathbb{E}^G_j$ and attains its infimum on $\mathbb{E}^G_j$ at a function $u^G_j \in \mathbb{E}^G_j$. Clearly for all $w \in \mathbb{E}^G_j$

$$\begin{aligned} \int_{\mathcal{M}} \alpha(\sigma) \left(\int_0^{w(\sigma)} h(\tau)\, d\tau \right) dv_g &\le \int_{\mathcal{M}} \alpha(\sigma) \left| \int_0^{w(\sigma)} h(\tau)\, d\tau \right| dv_g \\ &\le \kappa \int_{\mathcal{M}} \alpha(\sigma) w(\sigma)\, dv_g \le \kappa \|\alpha\|_{L^1(\mathcal{M})} t'_j \end{aligned}$$

and so

$$J_{G,\lambda}(w) \ge -\kappa \|\alpha\|_{L^1(\mathcal{M})} t'_j. \tag{5.1}$$

At this point set

$$\iota^G_j := \inf_{w \in \mathbb{E}^G_j} J_{G,\lambda}(w).$$

From the definition of infimum, for all $k \in \mathbb{N}$ we can find $w_k \in \mathbb{E}^G_j$ such that

$$\iota^G_j \le J_{G,\lambda}(w_k) \le \iota^G_j + \frac{1}{k}.$$

From this it follows

$$\begin{aligned} \|w_k\|^2 &= J_{G,\lambda}(w_k) + \lambda \int_{\mathcal{M}} \alpha(\sigma) \left(\int_0^{w_k(\sigma)} h(\tau)\, d\tau \right) dv_g \\ &\le \kappa \|\alpha\|_{L^1(\mathcal{M})} t'_j + \iota^G_j + 1 \end{aligned}$$

which implies that $(w_k)_k$ must be bounded in $H^1_G(\mathcal{M})$. Then, up to a subsequence, we can assume $w_k \rightharpoonup u^G_j$ for some $u^G_j \in H^1_G(\mathcal{M})$. In order to prove that $u^G_j \in \mathbb{E}^G_j$ it sufficient to notice that the set $\mathbb{E}^G_j$ is closed and convex, thus weakly closed. Now, exploiting the sequentially lower semicontinuity of $J_{G,\lambda}$ we get

$$\iota^G_j \le J_{G,\lambda}(u^G_j) \le \liminf_{k\to\infty} J_{G,\lambda}(w_k) \le \iota^G_j$$

hence

$$\iota^G_j = J_{G,\lambda}(u^G_j).$$

Step 2: for all $j \in \mathbb{N}$ one has that $0 \le u^G_j(\sigma) \le t_j$ a.e. in $\mathcal{M}$.

In order to show that, we set the Lipschitz continuous function $\varrho_j\colon \mathbb{R} \to \mathbb{R}$

$$\varrho_j(t) := \begin{cases} t_j & \text{if } t > t_j \\ t & \text{if } 0 \le t \le t_j \\ 0 & \text{if } t < 0 \end{cases}$$

we can consider the superposition operator $T_j\colon H^1_g(\mathcal{M}) \to H^1_g(\mathcal{M})$ defined as

$$T_j w(\sigma) := \varrho_j(w(\sigma)) \quad \text{a.e. in } \mathcal{M}.$$

From [54, Proposition 2.5] it follows that T_j is a continuous operator. Furthermore, if we restrict T_j to the G-invariant functions we have $T_j\colon H^1_G(\mathcal{M}) \to H^1_G(\mathcal{M})$. In fact, one can readily see that

$$\begin{aligned} \varphi \circledast T_j w(\sigma) &= T_j w(\varphi^{-1}\cdot\sigma) = (\varrho_j \circ w)(\varphi^{-1}\cdot\sigma) \\ &= \varrho_j(w(\varphi^{-1}\cdot\sigma)) = \varrho_j(w(\sigma)) = (\varrho_j \circ w)(\sigma) \\ &= T_j w(\sigma) \quad \text{a.e. in } \mathcal{M} \end{aligned}$$

for all $w \in H^1_G(\mathcal{M})$ and $\varphi \in G$. In addition, from its definition, it is clear that $T_j w \in \mathbb{E}^G_j$ for all $j \in \mathbb{N}$. At this point we set $v^\star_{G,j} := T_j u^G_j$ and

$$X^G_j := \left\{\sigma \in \mathcal{M} \mid t_j < u^G_j(\sigma) \le t'_j \right\}.$$

Observe that for all $\sigma \in X^G_j$ one has

$$v^\star_{G,j}(\sigma) = T_j u^G_j(\sigma) = t_j.$$

Now, exploiting (f_0) we get

$$\int_0^{u^G_j(\sigma)} h(\tau)\,d\tau \le \sup_{t\in[t_j,t'_j]} \int_0^t h(\tau)\,d\tau = \int_0^{t_j} h(\tau)\,d\tau = \int_0^{v^\star_{G,j}(\sigma)} h(\tau)\,d\tau,$$

thus

$$\int_{u^G_j(\sigma)}^{v^\star_{G,j}(\sigma)} h(\tau)\,d\tau \ge 0 \tag{5.2}$$

for all $\sigma \in X_j^G$. Moreover, taking into account the fact that $|{}^g\nabla v_{G,j}^\star(\sigma)|= 0$ a.e. in X_j^G, we obtain

$$\begin{aligned}
\|v_{G,j}^\star\|^2-\|u_j^G\|^2 &= \int_{\mathcal{M}} \left(|{}^g\nabla v_{G,j}^\star(\sigma)|^2-|{}^g\nabla u_j^G(\sigma)|^2\right) dv_g \\
&\quad + \int_{\mathcal{M}} \left(|v_{G,j}^\star(\sigma)|^2-|u_j^G(\sigma)|^2\right) dv_g \\
&= -\int_{X_j^G} |{}^g\nabla u_j^G(\sigma)|^2 \, dv_g + \int_{X_j^G} \left(t_j^2 - |u_j^G(\sigma)|^2\right) dv_g \qquad (5.3)\\
&\le -\int_{X_j^G} |{}^g\nabla v_{G,j}^\star(\sigma) - {}^g\nabla u_j^G(\sigma)|^2 \, dv_g - \int_{X_j^G} \left|u_j^G(\sigma) - t_j\right|^2 dv_g \\
&= -\int_{\mathcal{M}} |{}^g\nabla v_{G,j}^\star(\sigma) - {}^g\nabla u_j^G(\sigma)|^2 \, dv_g - \int_{\mathcal{M}} \left|u_j^G(\sigma) - v_{G,j}^\star(\sigma)\right|^2 dv_g \\
&= -\|v_{G,j}^\star - u_j^G\|^2.
\end{aligned}$$

At this point, in virtue of (5.2) and (5.3), recalling $v_{G,j}^\star \in \mathbb{E}_j^G$ we have

$$\begin{aligned}
0 \le J_{G,\lambda}(v_{G,j}^\star) - J_{G,\lambda}(u_j^G) &= \frac{\|v_{G,j}^\star\|^2-\|u_j^G\|^2}{2} - \lambda \int_{\mathcal{M}} \alpha(\sigma) \left(\int_{u_j^G(\sigma)}^{v_{G,j}^\star(\sigma)} h(\tau)\, d\tau\right) dv_g \\
\le -\frac{1}{2}\|v_{G,j}^\star - u_j^G\|^2 - \lambda \int_{X_j^G} \alpha(\sigma) \left(\int_{u_j^G(\sigma)}^{v_{G,j}^\star(\sigma)} h(\tau)\, d\tau\right) dv_g \\
\le -\frac{1}{2}\|v_{G,j}^\star - u_j^G\|^2.
\end{aligned}$$

From this, we can deduce

$$\|v_{G,j}^\star - u_j^G\|^2 = 0.$$

Since $v_{G,j}^\star \neq u_j^G$ except on X_j^G, we deduce that $\mathrm{Vol}_g(X_j^G) = 0$ as desired.

Step 3: the function u_j^G is a local minimum for $J_{G,\lambda}$ in the Sobolev space $H_G^1(\mathcal{M})$ for all $j \in \mathbb{N}$.

In order to do that, we select $w \in H_G^1(\mathcal{M})$ and we set

$$Z_j^G := \{\sigma \in \mathcal{M} \mid w(\sigma) \notin [0, t_j]\}$$

for every $j \in \mathbb{N}$. Recalling the superposition operator defined in step 2 we set

$$v_j^\star(\sigma) := T_j w(\sigma) = \begin{cases} t_j & \text{if } w(\sigma) > t_j \\ w(\sigma) & \text{if } 0 \le w(\sigma) \le t_j \\ 0 & \text{if } w(\sigma) < 0. \end{cases}$$

Now, on the one hand one can easily see that

$$\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\, d\tau = 0$$

for every $\sigma \in \mathcal{M} \setminus Z_j^G$. On the other hand, if $\sigma \in Z_j^G$ only three alternatives can occur

1. If $w(\sigma) \leq 0$ it is immediate to see
$$\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\,d\tau = \int_0^{w(\sigma)} h(\tau)\,d\tau = 0.$$

2. If $t_j < w(\sigma) \leq t'_j$ we have
$$\begin{aligned}\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\,d\tau &= \int_0^{w(\sigma)} h(\tau)\,d\tau - \int_0^{v_j^\star(\sigma)} h(\tau)\,d\tau \\ &= \int_0^{w(\sigma)} h(\tau)\,d\tau - \int_0^{t_j} h(\tau)\,d\tau \\ &\leq \int_0^{w(\sigma)} h(\tau)\,d\tau - \sup_{t\in[t_j,t'_j]} \int_0^{t} h(\tau)\,d\tau \leq 0.\end{aligned}$$

3. If $w(\sigma) > t'_j$ we obtain
$$\int_{v_j^\star(\sigma)}^{w(\sigma)} |h(\tau)|\,d\tau = \int_{t_j}^{w(\sigma)} |h(\tau)|\,d\tau \leq \kappa(w(\sigma) - t_j). \tag{5.4}$$

   At this point set
$$C := \kappa\|\alpha\|_{L^\infty(\mathcal{M})} \sup_{t\geq t'_j} \frac{t-t_j}{(t-t_j)^\nu}$$

   where $\nu \in (2, 2^*)$. From this and from (5.4) we have
$$\begin{aligned}\int_{\mathcal{M}} \alpha(\sigma)\left(\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\,d\tau\right) dv_g &\leq \int_{\{w>t'_j\}} \alpha(\sigma)\left(\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\,d\tau\right) dv_g \\ &\leq \int_{\{w>t'_j\}} \alpha(\sigma)\left(\int_{v_j^\star(\sigma)}^{w(\sigma)} |h(\tau)|\,d\tau\right) dv_g \\ &\leq \|\alpha\|_{L^\infty(\mathcal{M})} \int_{\{w>t'_j\}} \left(\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\,d\tau\right) dv_g \qquad (5.5) \\ &\leq C\int_{\mathcal{M}} (w(\sigma) - t_j)^\nu\,dv_g \\ &\leq C\int_{\mathcal{M}} |w(\sigma) - t_j|^\nu\,dv_g.\end{aligned}$$

   Denote with
$$\gamma := \sup_{w\in H^1_G(\mathcal{M})\setminus\{0\}} \frac{\|w\|_{L^\nu(\mathcal{M})}}{\|w\|}$$

   and observe that is finite by Lemma 5.5. From (5.5) we deduce
$$\int_{\mathcal{M}} \alpha(\sigma)\left(\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\,d\tau\right) dv_g \leq C\gamma^\nu \|w - v_j^\star\|^\nu. \tag{5.6}$$

Now, we compute

$$\begin{aligned}
\|w\|^2-\|v_j^\star\|^2 &= \int_{\mathcal{M}} \left(|^g\nabla w(\sigma)|^2-|^g\nabla v_j^\star(\sigma)|^2\right) dv_g + \int_{\mathcal{M}} \left(|w(\sigma)|^2-|v_j^\star|^2\right) dv_g \\
&\geq \int_{Z_j^G} |^g\nabla w(\sigma)|^2 \, dv_g + \int_{Z_j^{G,-}} |w(\sigma)|^2 \, dv_g + \int_{Z_j^{G,+}} |w(\sigma) - t_j|^2 \, dv_g \\
&= \int_{Z_j^G} |^g\nabla w(\sigma) - {}^g\nabla v_j^\star(\sigma)|^2 \, dv_g + \int_{Z_j^{G,-}} |w(\sigma) - v_j^\star(\sigma)|^2 \, dv_g \qquad (5.7) \\
&\quad + \int_{Z_j^{G,+}} |w(\sigma) - t_j|^2 \, dv_g \\
&= \|w - v_j^\star\|^2
\end{aligned}$$

where

$$Z_j^{G,+} := \left\{\sigma \in Z_j^G \mid w(\sigma) > 0\right\} \quad \text{and} \quad Z_j^{G,-} := \left\{\sigma \in Z_j^G \mid w(\sigma) < 0\right\}.$$

Coupling (5.6) and (5.7) we get

$$\begin{aligned}
J_{G,\lambda}(w) - J_{G,\lambda}(v_j^\star) &= \frac{\|w\|^2-\|v_j^\star\|^2}{2} - \lambda \int_{\mathcal{M}} \alpha(\sigma) \left(\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\, d\tau \right) dv_g \\
&\geq \frac{1}{2}\|w - v_j^\star\|^2-\lambda C\gamma^\nu \|w - v_j^\star\|^\nu.
\end{aligned}$$

In view of that, recalling $J_{G,\lambda}(v_j^\star) \geq J_{G,\lambda}(u_j^G)$ since $v_j^\star \in \mathbb{E}_j^G$, we obtain

$$J_{G,\lambda}(w) \geq J_{G,\lambda}(u_j^G) + \|w - v_j^\star\|^2 \left(\frac{1}{2} - \lambda C\gamma^\nu \|w - v_j^\star\|^{\nu-2} \right) \qquad (5.8)$$

At this point, we notice that

$$\|w - v_j^\star\| \leq \|w - u_j^G\| + \|u_j^G - v_j^\star\| = \|w - u_j^G\| + \|T_j u_j^G - v_j^\star\|$$

thus, exploiting the continuity of the superposition operator, it is possible to find a $\delta > 0$ such that

$$\|w - v_j^\star\|^{\nu-2} \leq \frac{1}{4\lambda C\gamma^\nu}$$

if $\|w - u_j^G\| \leq \delta$. Hence, from (5.8) we get

$$J_{G,\lambda}(w) \geq J_{G,\lambda}(u_j^G)$$

that means u_j^G is a local minimizer.

Step 4: If

$$\iota_j^G := \inf_{w\in\mathbb{E}_j^G} J_{G,\lambda}(w).$$

then

$$\lim_{j\to\infty} \iota_j^G = \lim_{j\to\infty} \|u_j^G\| = 0.$$

Recalling that $u_j^G \in \mathbb{E}_j^G$ and that $\iota_j^G = J_{G,\lambda}(u_j^G)$ we have

$$\begin{aligned}\int_{\mathcal{M}} |{}^g\nabla u_j^G(\sigma)|^2\, dv_g + \int_{\mathcal{M}} |u_j^G(\sigma)|^2\, dv_g &= J_{G,\lambda}(u_j^G) + \lambda \int_{\mathcal{M}} \alpha(\sigma) \left(\int_0^{u_j^G(\sigma)} h(\tau)\, d\tau \right) dv_g \\ &= \iota_j^G + \lambda \int_{\mathcal{M}} \alpha(\sigma) \left(\int_0^{u_j^G(\sigma)} h(\tau)\, d\tau \right) dv_g \qquad (5.9) \\ &\le \iota_j^G + \lambda\kappa \|\alpha\|_{L^1(\mathcal{M})} t'_j\end{aligned}$$

At this point, we notice that the function $w_0 = 0$ belongs to $\mathbb{E}_j^G$ and so

$$\iota_j^G = \inf_{w\in\mathbb{E}_j^G} J_{G,\lambda}(w) \le 0.$$

From this and (5.9) we can deduce

$$\lim_{j\to\infty} \|u_j^G\| = 0$$

since $t'_j \to 0$ as $j \to \infty$. Furthermore, recalling (5.1) we obtain

$$-\kappa \|\alpha\|_{L^1(\mathcal{M})} t'_j \le \iota_j^G \le 0$$

which implies

$$\lim_{j\to\infty} \iota_j^G = 0.$$

Step 5: for all $j \in \mathbb{N}$ we have

$$\iota_j^G < 0.$$

In order to do that, we select $j \in \mathbb{N}$ and $0 < a < b$ such that

$$\operatorname*{ess\,inf}_{\sigma\in A_a^b} \alpha(\sigma) \ge \alpha_0 > 0 \qquad (5.10)$$

where

$$A_a^b = B_{\sigma_0}(a+b) \setminus B_{\sigma_0}(b-a)$$

and, after fixing $\varepsilon \in (0,1)$ we define the function

$$\vartheta_{a,b}^{\varepsilon}(\sigma) := \begin{cases} 0 & \text{if } \sigma \in \mathcal{M} \setminus A_a^b \\ 1 & \text{if } \sigma \in A_{\varepsilon a}^b \\ \dfrac{a - |d_g(\sigma_0,\sigma) - b|}{(1-\varepsilon)a} & \text{if } \sigma \in A_a^b \setminus A_{\varepsilon a}^b. \end{cases}$$

It is straightforward to verify that $\vartheta^{\varepsilon}_{a,b} \in H^1_G(\mathcal{M})$ since at each point its value depends only on the distance from σ_0. Moreover, one can easily verify that $\mathrm{supp}(\vartheta^{\varepsilon}_{a,b}) \subset A^b_a$ and $\|\vartheta^{\varepsilon}_{a,b}\|_{L^\infty(\mathcal{M})} \leq 1$. At this point we define the map $\mu_g\colon (0,1) \to \mathbb{R}$ where

$$\mu_g(\varepsilon) = \frac{\displaystyle\int_{A^b_{\varepsilon a}} \alpha(\sigma)\, dv_g}{\displaystyle\int_{A^b_a \setminus A^b_{\varepsilon a}} \alpha(\sigma)\, dv_g}$$

and we notice that

$$\lim_{\varepsilon \to 0^+} \mu_g(\varepsilon) = 0, \qquad \lim_{\varepsilon \to 1^-} \mu_g(\varepsilon) = +\infty.$$

In view of that, it is possible to find $\varepsilon_0 \in (0,1)$ such that

$$\frac{\displaystyle\int_{A^b_{\varepsilon a}} \alpha(\sigma)\, dv_g}{\displaystyle\int_{A^b_a \setminus A^b_{\varepsilon a}} \alpha(\sigma)\, dv_g} = K_1 + 1$$

where $K_1 > 0$ is the constant given in hypothesis (f_1). From (f_1) we also have the existence of an index k_0, with $\xi_{k_0} \leq t'_j$ such that for every $k \geq k_0$

$$\frac{\displaystyle\int_0^{\xi_k} h(\tau)\, d\tau}{\xi_k^2} > \frac{1}{2\lambda} \left(\frac{\displaystyle\int_{A^b_a \setminus A^b_{\varepsilon_0 a}} \alpha(\sigma)\, dv_g}{\|\vartheta^{\varepsilon_0}_{a,b}\|^2} \right)^{-1}.$$

From this, (f_1) and (5.10) it follows

$$\begin{aligned}
&\frac{\displaystyle\int_{A^b_a} \alpha(\sigma) \left(\int_0^{\xi_k \vartheta^{\varepsilon_0}_{a,b}(\sigma)} h(\tau)\, d\tau \right) dv_g}{\|\xi_k \vartheta^{\varepsilon_0}_{a,b}\|^2} = \\
&= \frac{\displaystyle\int_{A^b_{\varepsilon_0 a}} \alpha(\sigma) \left(\int_0^{\xi_k} h(\tau)\, d\tau \right) dv_g}{\xi_k^2 \|\vartheta^{\varepsilon_0}_{a,b}\|^2} + \frac{\displaystyle\int_{A^b_a \setminus A^b_{\varepsilon_0 a}} \alpha(\sigma) \left(\int_0^{\xi_k \vartheta^{\varepsilon_0}_{a,b}(\sigma)} h(\tau)\, d\tau \right) dv_g}{\xi_k^2 \|\vartheta^{\varepsilon_0}_{a,b}\|^2} \\
&\geq \frac{\displaystyle\int_{A^b_{\varepsilon_0 a}} \alpha(\sigma) \left(\int_0^{\xi_k} h(\tau)\, d\tau \right) dv_g}{\xi_k^2 \|\vartheta^{\varepsilon_0}_{a,b}\|^2} + \frac{\displaystyle\int_{A^b_a \setminus A^b_{\varepsilon_0 a}} \alpha(\sigma) \left(\inf_{t \in [0,\xi_k]} \int_0^{t} h(\tau)\, d\tau \right) dv_g}{\xi_k^2 \|\vartheta^{\varepsilon_0}_{a,b}\|^2} \\
&\geq \frac{\displaystyle\int_{A^b_{\varepsilon_0 a}} \alpha(\sigma) \left(\int_0^{\xi_k} h(\tau)\, d\tau \right) dv_g}{\xi_k^2 \|\vartheta^{\varepsilon_0}_{a,b}\|^2} - K_1 \frac{\displaystyle\int_{A^b_a \setminus A^b_{\varepsilon_0 a}} \alpha(\sigma) \left(\int_0^{\xi_k} h(\tau)\, d\tau \right) dv_g}{\xi_k^2 \|\vartheta^{\varepsilon_0}_{a,b}\|^2}
\end{aligned}$$

$$= \frac{\displaystyle\int_{A_a^b \setminus A_{\varepsilon_0 a}^b} \alpha(\sigma)\, dv_g}{\|\vartheta_{a,b}^{\varepsilon_0}\|^2} \frac{\displaystyle\int_0^{\xi_k} h(\tau)\, d\tau}{\xi_k^2} > \frac{1}{2\lambda}$$

for all $k \geq k_0$. Now, from the definition of $\xi_k \vartheta_{a,b}^{\varepsilon}$ it is clear that $\xi_k \vartheta_{a,b}^{\varepsilon_0} \in \mathbb{E}_j^G$. Hence $J_{G,\lambda}(\xi_k \vartheta_{a,b}^{\varepsilon_0}) < 0$ and as a consequence of that $\iota_j^G < 0$ as desired.

Step 6: the function u_j^G is a local minimum for the functional $J_{G,\lambda}$ in the Sobolev space $H_g^1(\mathcal{M})$ for all $j \in \mathbb{N}$.

Since $\|u_j^G\|_{L^\infty(\mathcal{M})} \to 0$ as $j \to \infty$, up to relabel the indexes, we can assume the existence of a sequence $(u_j^G)_j \subset H_g^1(\mathcal{M})$ such that

$$\|u_j^G\|_{L^\infty(\mathcal{M})} \leq t_0. \tag{5.11}$$

At this point, in virtue of the Principle of Symmetric Criticality of Palais (see [94] for details), to conclude the proof, it is sufficient to show that $J_{G,\lambda}$ is invariant under the action of G. Consider first $\|\cdot\|$. For all $\varphi \in G$ and $w \in H_g^1(\mathcal{M})$ we have

$$\begin{aligned}
\|\varphi \circledast w\|^2 &= \int_{\mathcal{M}} |{}^g\nabla(\varphi \circledast w)(\sigma)|^2\, dv_g + \int_{\mathcal{M}} |(\varphi \circledast w)(\sigma)|^2\, dv_g \\
&= \int_{\mathcal{M}} |{}^g\nabla(w(\varphi^{-1}\cdot\sigma))|^2\, dv_g + \int_{\mathcal{M}} |w(\varphi^{-1}\cdot\sigma)|^2\, dv_g \\
&= \int_{\mathcal{M}} \langle D\varphi_{\varphi^{-1}\cdot\sigma}\, {}^g\nabla w(\varphi^{-1}\cdot\sigma), D\varphi_{\varphi^{-1}\cdot\sigma}\, {}^g\nabla w(\varphi^{-1}\cdot\sigma)\rangle_\sigma\, dv_g \\
&\quad + \int_{\mathcal{M}} |w(\varphi^{-1}\cdot\sigma)|^2\, dv_g \\
&= \int_{\mathcal{M}} \langle {}^g\nabla w(\varphi^{-1}\cdot\sigma), {}^g\nabla w(\varphi^{-1}\cdot\sigma)\rangle_{\varphi^{-1}\cdot\sigma}\, dv_g + \int_{\mathcal{M}} |w(\varphi^{-1}\cdot\sigma)|^2\, dv_g \qquad (5.12) \\
&= \int_{\mathcal{M}} \langle {}^g\nabla w(\tilde\sigma), {}^g\nabla w(\tilde\sigma)\rangle_{\tilde\sigma}\, dv_{(\varphi^{-1})^* g} + \int_{\mathcal{M}} |w(\tilde\sigma)|^2\, dv_{(\varphi^{-1})^* g} \\
&= \int_{\mathcal{M}} \langle {}^g\nabla w(\tilde\sigma), {}^g\nabla w(\tilde\sigma)\rangle_{\tilde\sigma}\, dv_g + \int_{\mathcal{M}} |w(\tilde\sigma)|^2\, dv_g = \|w\|^2
\end{aligned}$$

since φ is an isometry and preserves scalar products. Furthermore

$$\alpha(\varphi^{-1}\cdot\sigma) = \alpha(d_g(\sigma_0, \varphi^{-1}\sigma)) = \alpha(d_g(\varphi^{-1}\cdot\sigma_0, \varphi^{-1}\sigma)) = \alpha(d_g(\sigma_0, \sigma)) = \alpha(\sigma)$$

which implies

$$\begin{aligned}
\int_{\mathcal{M}} \alpha(\sigma) \left(\int_0^{w(\varphi^{-1}\cdot\sigma)} h(\tau)\, d\tau \right) dv_g &= \int_{\mathcal{M}} \alpha(\varphi^{-1}\cdot\sigma) \left(\int_0^{w(\varphi^{-1}\cdot\sigma)} h(\tau)\, d\tau \right) dv_g \\
&= \int_{\mathcal{M}} \alpha(\tilde\sigma) \left(\int_0^{w(\tilde\sigma)} h(\tau)\, d\tau \right) dv_{(\varphi^{-1})^* g} \qquad (5.13)
\end{aligned}$$

$$= \int_{\mathcal{M}} \alpha(\tilde{\sigma}) \left(\int_0^{w(\tilde{\sigma})} h(\tau)\, d\tau \right) dv_g.$$

Putting together (5.12) and (5.13) we obtain

$$J_{G,\lambda}(\varphi \circledast w) = J_{G,\lambda}(w)$$

hence, applying the Principle of Symmetric Criticality of Palais, we have that each element of the sequence u_j^G is a critical point of the functional $J_{G,\lambda}$ and a weak solution of (P_0). Furthermore, recalling Step 2 and (5.11) we also have that u_j^G is a solution of our original problem (P_λ). □

Example 5.7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(t) := \begin{cases} 9\sqrt{t} \sin\left(\frac{1}{\sqrt[3]{t}}\right) - 2\sqrt[6]{t} \cos\left(\frac{1}{\sqrt[3]{t}}\right) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases}$$

whose primitive is

$$F(t) = \int_0^t f(s)\, ds = \begin{cases} 6t^{3/2} \sin\left(\frac{1}{\sqrt[3]{t}}\right) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

As in [8] one can check that conditions (f_0) and (f_1) are satisfied.

5.3 Oscillations at infinity

In this Section we investigate the solutions of problem (P_λ)

$$\begin{cases} -\Delta_g w + w = \lambda \alpha(\sigma) f(w) & \text{in } \mathcal{M} \\ w \in H_g^1(\mathcal{M}) \end{cases}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function that oscillates at infinity. Preferring a variational approach, we define the energy functional $J_\lambda\colon H_g^1(\mathcal{M}) \to \mathbb{R}$ associated to problem (P_λ) where

$$J_\lambda(w) := \frac{1}{2}\|w\|^2 - \lambda \int_{\mathcal{M}} \alpha(\sigma) F(w(\sigma))\, dv_g,$$

and $F(t) := \int_0^t f(\tau)\, d\tau$. As regard the right hand side of (P_λ), we make on the nonlinear term f the hypothesis (f_0')-(f_2') of Theorem 5.2 till the end of the Section. As we already did in the previous Section we will fist look for solutions for a truncated problem and then we will show that they also solves (P_λ). In order to do that, we start defining the function

$$h(t) := \begin{cases} f(t) & \text{if } t \geq 0 \\ f(0) & \text{if } t < 0 \end{cases}$$

and considering the auxiliary problem

$$\begin{cases} -\Delta_g w + w = \lambda\alpha(\sigma)h(w) & \text{in } \mathcal{M} \\ w \in H^1_G(\mathcal{M}). \end{cases} \tag{P_∞}$$

We associate to problem (P_∞) the functional

$$J_{G,\lambda}(w) := \frac{1}{2}\|w\|^2 - \lambda \int_{\mathcal{M}} \alpha(\sigma) \left(\int_0^{w(\sigma)} h(\tau)\, d\tau \right) dv_g$$

and we point out that $J_{G,\lambda} \in C^1(H^1_G(\mathcal{M}), \mathbb{R})$ and again thanks to Lemma 5.5 that is sequentially lower semicontinuous. We emphasize that non-negative critical points of $J_{G,\lambda}(w)$ are also critical point for the functional J_λ.

Proof of Theorem 5.2. Since some arguments of the proof are very similar to the ones described in Theorem 5.1 we will omit them. Fix $\lambda > 0$. We start for every $j \in \mathbb{N}$ setting

$$\mathbb{E}^G_j := \left\{ w \in H^1_G(\mathcal{M}) \mid 0 \le w(\sigma) \le t'_j \text{ a.e in } \mathcal{M} \right\}.$$

Step 1: the functional $J_{G,\lambda}$ in bounded from below on $\mathbb{E}^G_j$ and attains its infimum on $\mathbb{E}^G_j$ at a function $w^G_j \in \mathbb{E}^G_j$.

From hypothesis (f'_0) we obtain

$$\int_0^{w(\sigma)} h(\tau)\, d\tau \le K_2 \left(t'_j + \frac{\left(t'_j\right)^{q+1}}{q+1} \right).$$

As a consequence of that

$$J_{G,\lambda}(w) \ge -\lambda K_2 \|\alpha\|_{L^1(\mathcal{M})} \left(t'_j + \frac{\left(t'_j\right)^{q+1}}{q+1} \right)$$

which implies that $J_{G,\lambda}$ is bounded from below on $\mathbb{E}^G_j$ for every $j \in \mathbb{N}$. At this point, by following the line of Step 1 in Theorem 5.1 we can find u^G_j such that

$$\iota^G_j := \inf_{w \in \mathbb{E}^G_j} J_{G,\lambda}(w) = J_{G,\lambda}(u^G_j).$$

Step 2: for all $j \in \mathbb{N}$ one has that $0 \le u^G_j(\sigma) \le t_j$ a.e. in $\mathcal{M}$.

The statement follows following closely the line of the proof of Step 2 on Theorem 5.1.

Step 3: the function u^G_j is a local minimum for $J_{G,\lambda}$ in the Sobolev space $H^1_G(\mathcal{M})$ for all $j \in \mathbb{N}$

To show this, we choose $w \in H^1_G(\mathcal{M})$ and we set

$$Z^G_j := \{\sigma \in \mathcal{M} \mid w(\sigma) \notin [0, t_j]\}$$

for every $j \in \mathbb{N}$. Recalling the superposition operator defined in step 2 of Theorem 5.1 we set

$$v_j^\star(\sigma) := T_j w(\sigma) = \begin{cases} t_j & \text{if } w(\sigma) > t_j \\ w(\sigma) & \text{if } 0 \le w(\sigma) \le t_j \\ 0 & \text{if } w(\sigma) < 0. \end{cases}$$

Now, on one hand one can easily see that

$$\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\, d\tau = 0$$

for every $\sigma \in \mathcal{M} \setminus Z_j^G$. On the other hand, if $\sigma \in Z_j^G$ we analyze the situation according to the three different possible alternatives.

1. If $w(\sigma) \le 0$ it is immediate to see

$$\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\, d\tau = \int_0^{w(\sigma)} f(0)\, d\tau \le 0.$$

2. If $t_j < w(\sigma) \le t_j'$ we can show

$$\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\, d\tau \le 0.$$

arguing similarly to Step 3 in Theorem 5.1.

3. If $w(\sigma) > t_j'$ we obtain

$$\begin{aligned}\int_{v_j^\star(\sigma)}^{w(\sigma)} |h(\tau)|\, d\tau &= \int_{t_j}^{w(\sigma)} |h(\tau)|\, d\tau \\ &\le \int_{t_j}^{w(\sigma)} h(\tau)\, d\tau \le K_2 \left[(w(\sigma) - t_j) + \frac{1}{q+1}(w(\sigma)^{q+1} - t_j^{q+1}) \right]\end{aligned} \tag{5.14}$$

At this point set

$$\tilde{C} := \frac{K_2 \|\alpha\|_{L^\infty(\mathcal{M})}}{q+1} \sup_{t \ge t_j'} \frac{(q+1)(t - t_j) + (t^{q+1} - t_j^{q+1})}{(t - t_j)^{q+1}}.$$

From this and (5.14) we have

$$\begin{aligned}\int_{\mathcal{M}} \alpha(\sigma) \left(\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\, d\tau \right) dv_g &\le \|\alpha\|_{L^\infty(\mathcal{M})} \int_{\mathcal{M}} \left(\int_{v_j^\star(\sigma)}^{w(\sigma)} |h(\tau)|\, d\tau \right) dv_g \\ &\le \tilde{C} \int_{\mathcal{M}} (w(\sigma) - v_j^\star)^{q+1}\, dv_g.\end{aligned} \tag{5.15}$$

Denote

$$\tilde{\gamma} := \sup_{w \in H^1_G(\mathcal{M}) \setminus \{0\}} \frac{\|w\|_{L^{q+1}(\mathcal{M})}}{\|w\|}$$

and observe that is finite by Lemma 5.5. From (5.15) we deduce

$$\int_{\mathcal{M}} \alpha(\sigma) \left(\int_{v_j^\star(\sigma)}^{w(\sigma)} h(\tau)\, d\tau \right) dv_g \leq \tilde{C} \tilde{\gamma}^{q+1} \|w - v_j^\star\|^{q+1}. \tag{5.16}$$

At this point, the conclusion is achieved as in Step 3 of Theorem 5.1.

Step 4 We have that

$$\liminf_{j \to \infty} \iota_j^G = -\infty.$$

Replacing (f_1) with (f_2') and repeating the calculations done in Step 5 of Theorem 5.1 we can find a constant $\tilde{\kappa} > 0$ and a divergent sequence $(\xi_k)_k$ such that

$$J_{G,\lambda}(\xi_k \vartheta^{\varepsilon_0}_{a,b}) < -\tilde{\kappa} \|\xi_k \vartheta^{\varepsilon_0}_{a,b}\|^2$$

for $k \geq k_0$ (see the proof of Theorem 5.1 for the definition of $\vartheta^{\varepsilon}_{a,b}$). At this point, we notice that we can find a subsequence $(t_{j_k'})_k$ so that $t_{j_k'} \geq \xi_k$ and $\xi_k \vartheta^{\varepsilon_0}_{a,b} \in \mathbb{E}^G_{j_k}$. Then

$$\lim_{k \to \infty} \iota^G_{j_k} \leq \lim_{k \to \infty} J_{G,\lambda}(\xi_k \vartheta^{\varepsilon_0}_{a,b}) < - \lim_{k \to \infty} \tilde{\kappa} \|\xi_k \vartheta^{\varepsilon_0}_{a,b}\|^2 = -\infty.$$

From this, we can conclude using the definition of inferior limit getting

$$\liminf_{j \to \infty} \iota_j^G = -\infty.$$

To conclude the proof, it is sufficient to argue as in Step 6 of Theorem 5.1 proving that $J_{G,\lambda}$ is invariant under the action of the group G and applying the Principle of Symmetric Criticality of Palais. □

To conclude we exhibit an example of a nonlinearity that satisfies hypothesis (f_0')-(f_2').

Example 5.8. *Consider the function*

$$f(t) := \begin{cases} \frac{2(d-1)}{d-2} t^{\frac{d}{d-2}} \sin\left(\sqrt[3]{t}\right) + \frac{1}{3} t^{\frac{2(2d-1)}{3(d-2)}} \cos\left(\sqrt[3]{t}\right) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

whose primitive is

$$F(t) = \begin{cases} t^{2\frac{d-1}{d-2}} \sin\left(\sqrt[3]{t}\right) & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Hypothesis (f_0') is trivially satisfied since the trigonometric functions are bounded and

$$\frac{d}{d-2} < 2^* - 1 \qquad \text{and} \qquad \frac{2(2d-1)}{3(d-2)} < 2^* - 1.$$

In order to see the validity of (f_1') *one can choose for instance*

$$t_j := \left[\frac{\pi}{2}(1+4j)\right]^3 \qquad \text{and} \qquad t_j' := \left[\frac{\pi}{2}(3+4j)\right]^3.$$

It is easy to check that F *is decreasing in the interval* $[t_j, t_j']$, *hence*

$$F(t_j) = \sup_{t\in[t_j,t_j']} F(t).$$

To prove that f *satisfies* (f_2'), *we choose* $\xi_j = t_j \to +\infty$, *so that*

$$\lim_{j\to+\infty} \frac{F(\xi_j)}{\xi_j^2} = \lim_{j\to+\infty} \frac{\xi_j^{2\frac{d-1}{d-2}}}{\xi_j^2} = \lim_{j\to+\infty} \xi_j^{\frac{2}{d-2}} = +\infty.$$

Moreover,

$$\inf_{t\in[0,\xi_j]} F(t) = F\left(t_{j-1}'\right) = -\left(t_{j-1}'\right)^{2\frac{d-1}{d-2}} \geq -\left(\xi_j\right)^{2\frac{d-1}{d-2}} = -F(\xi_j),$$

which shows that (f_2') *is verified with* $K_3 = 1$.

6 Multiple solutions for Schrödinger equations on Riemannian manifolds via ∇-theorems

The study of existence and multiplicity of solutions to semilinear partial differential equations of Schrödinger type is by far one of the richest fields in Nonlinear Analysis, where Variational Methods and Critical Point Theory provide a powerful setting for existence results. The occurrence of more than one solution to such equations is guaranteed, at a basic level, by some symmetry condition together with the use of topological indices such as the genus or the relative category as we already saw in Chapter 3 and 5. We refer to the classical monograph [107] for a survey.

Semilinear elliptic equations of Schrödinger type are typically set in the whole Euclidean space $\mathbb{R}^d$, $d \geq 3$, which has a rather poor geometric structure. Multiplicity results may then appear as a consequence of the presence of potential functions with suitable properties. The situation is much different if $\mathbb{R}^d$ is replaced by a more general Riemannian manifold $\mathcal{M}$, since the geometry of $\mathcal{M}$ may influence the existence of one or more solutions to the equation. Analysis on Manifolds and Geometric Analysis become the necessary language to work with these problems: we refer to [15, 53, 54, 65, 83] and to the references therein for an introduction. For the sake of brevity, we will assume that the reader is familiar with the basic definitions of Riemannian Geometry, and we refer to Chapter 2.

In this chapter, we will consider a d-dimensional smooth complete non-compact Riemannian manifold $(\mathcal{M}, g)$ with $d \geq 3$. The aim of this chapter is to study the existence of solutions for problem

$$\begin{cases} -\Delta_g w + V(\sigma)w = \alpha(\sigma)f(w) + \lambda w & \text{in } \mathcal{M} \\ w(\sigma) \to 0 & \text{as } d_g(\sigma_0, \sigma) \to \infty, \end{cases} \tag{P_λ}$$

where $\alpha \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M})$, $\alpha > 0$ a.e. in $\mathcal{M}$, $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda \in \mathbb{R}$ is a real parameter. We assume that $V\colon \mathcal{M} \to \mathbb{R}$ is a continuous function such that

(V_1) $\upsilon_0 := \inf_{\sigma \in \mathcal{M}} V(\sigma) > 0$;

(V_2) there exists $\sigma_0 \in \mathcal{M}$ such that

$$\lim_{d_g(\sigma_0,\sigma)\to\infty} V(\sigma) = +\infty.$$

The nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies

(f_1)
$$\lim_{t\to 0}\frac{f(t)}{|t|}=0;$$

(f_2) there results
$$\lim_{t\to+\infty}\frac{f(t)}{|t|^{r-1}}<\infty$$
where $r\in\left(2,\dfrac{2d}{d-2}\right)$;

(f_3) $0<rF(t)<f(t)t$ for all $t\in\mathbb{R}\setminus\{0\}$ where $F(t):=\displaystyle\int_0^t f(\tau)\,d\tau$.

To introduce the main assumptions on the manifold $(\mathcal{M},d)$, we suppose that there exists a function $H\colon[0,\infty)\to\mathbb{R}$ of class C^1 such that
$$\int_0^\infty tH(t)\,dt<\infty$$
and

(Ric) for some $\bar\sigma_0\in\mathcal{M}$ there results
$$\mathrm{Ric}_{(\mathcal{M},g)}(\sigma)\geq(1-d)H(d_g(\bar\sigma_0,\sigma)).$$

Moreover, we will assume throughout the chapter that
$$\inf_{\sigma\in\mathcal{M}}\mathrm{Vol}_g\left(B_\sigma(1)\right)>0$$
where
$$B_\sigma(1):=\left\{\xi\in\mathcal{M}\mid \mathrm{dist}(\xi,\sigma)<1\right\}.$$
Since we want to prove a multiplicity result for (P_λ), a natural approach could be based on Morse Theory, see [37, 78]. Unfortunately, Morse Theory requires in general more regularity of the Euler functional associated to the variational problem, and this would require a more regular nonlinearity f in (P_λ).

We propose here a different approach via ∇-Theorems, a family of variational tools which were introduced by Marino and Saccon in [77] to study the multiplicity of solutions of some asymptotically non-symmetric semilinear elliptic problems with jumping nonlinearities. More precisely, we will make use of the sphere-torus linking Theorem with mixed type assumptions (see [77, Theorem 2.10]). The main condition of this theorem can be roughly summarized in these terms: the Euler functional constrained on a closed subspace must not have critical values in a certain prescribed range with "some uniformity". A rigorous definition is as follows.

Definition 6.1. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{I}: \mathcal{H} \to \mathbb{R}$ a C^1 functional. Let also $\mathcal{X}$ be a closed subspace of $\mathcal{H}$, a, $b \in \mathbb{R} \cup \{-\infty, \infty\}$; we say that $\mathcal{I}$ satisfies the condition $(\nabla)(\mathcal{I}, \mathcal{X}, a, b)$ if there exists $\gamma > 0$ such that

$$\inf \{ \| P_{\mathcal{X}} \nabla \mathcal{I}(w) \| \mid a \leq \mathcal{I}(w) \leq b,\ \operatorname{dist}(w, \mathcal{X}) \leq \gamma \} > 0$$

where $P_{\mathcal{X}}: \mathcal{H} \to \mathcal{X}$ denotes the standard orthogonal projection. In the following, we will refer to it as (∇)-condition for short.

In order to make the chapter self-contained, we also write the statement of the ∇-theorem.

Theorem 6.2. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{X}_i$, $i = 1, 2, 3$ three subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ and $\dim \mathcal{X}_i < \infty$ for $i = 1, 2$. Denote with $P_{\mathcal{X}_i}: \mathcal{H} \to \mathcal{X}_i$ the standard orthogonal projection. Let $\mathcal{I}: \mathcal{H} \to \mathbb{R}$ a $C^{1,1}$ functional. Let $\rho, \rho', \rho'', \rho_1$ be such that $\rho_1 > 0$, $0 \leq \rho' < \rho < \rho''$ and define*

$$\Delta = \left\{ w \in \mathcal{X}_1 \oplus \mathcal{X}_2 \mid \rho' \leq \| P_{\mathcal{X}_2} w \| \leq \rho'',\ \| P_{\mathcal{X}_1} w \| \leq \rho_1 \right\} \quad \text{and} \quad T = \partial_{\mathcal{X}_1 \oplus \mathcal{X}_2} \Delta,$$

$$S_{23} = \{ w \in \mathcal{X}_2 \oplus \mathcal{X}_3 \mid \| w \| = \rho \} \quad \text{and} \quad B_{23} = \{ w \in \mathcal{X}_2 \oplus \mathcal{X}_3 \mid \| w \| \leq \rho \}.$$

Assume that

$$a' = \sup \mathcal{I}(T) < \inf \mathcal{I}(S_{23}) = a''.$$

Let a and b such that $a' < a < a''$ and $b > \sup \mathcal{I}(\Delta)$. Assume $(\nabla)(\mathcal{I}, \mathcal{X}_1 \oplus \mathcal{X}_3, a, b)$ holds and that $(PS)_c$ is verified for all $c \in [a, b]$. Then $\mathcal{I}$ has at least two critical points in $\mathcal{I}^{-1}([a, b])$. Moreover, if $a_1 < \inf \mathcal{I}(B_{23}) > -\infty$ and $(PS)_c$ holds for all $c \in [a_1, b]$, then $\mathcal{I}$ has another critical level in $[a_1, a']$.

We define the Sobolev space

$$H^1_V(\mathcal{M}) := \{ w \in H^1_g(\mathcal{M}) \mid \| w \|^2 < \infty \}$$

where throughout the chapter we denote by

$$\| w \| := \left(\int_{\mathcal{M}} |{}^g\nabla w(\sigma)|^2 \, dv_g + \int_{\mathcal{M}} V(\sigma) \, |w(\sigma)|^2 \, dv_g \right)^{1/2}$$

the norm induced by the scalar product

$$\langle w_1, w_2 \rangle := \int_{\mathcal{M}} \langle {}^g\nabla w_1(\sigma), {}^g\nabla w_2(\sigma) \rangle_g \, dv_g + \int_{\mathcal{M}} V(\sigma) w_1(\sigma) w_2(\sigma) \, dv_g.$$

We recall that under the assumptions we made on the potential and the manifold, the embedding $H^1_V(\mathcal{M}) \hookrightarrow L^q(\mathcal{M})$ is continuous for any $q \in [2, 2^*]$. Furthermore, as a result of the Hypothesis (V_1) and (V_2) we also have the following Lemma, whose proof can be found in [44, Lemma 2.1].

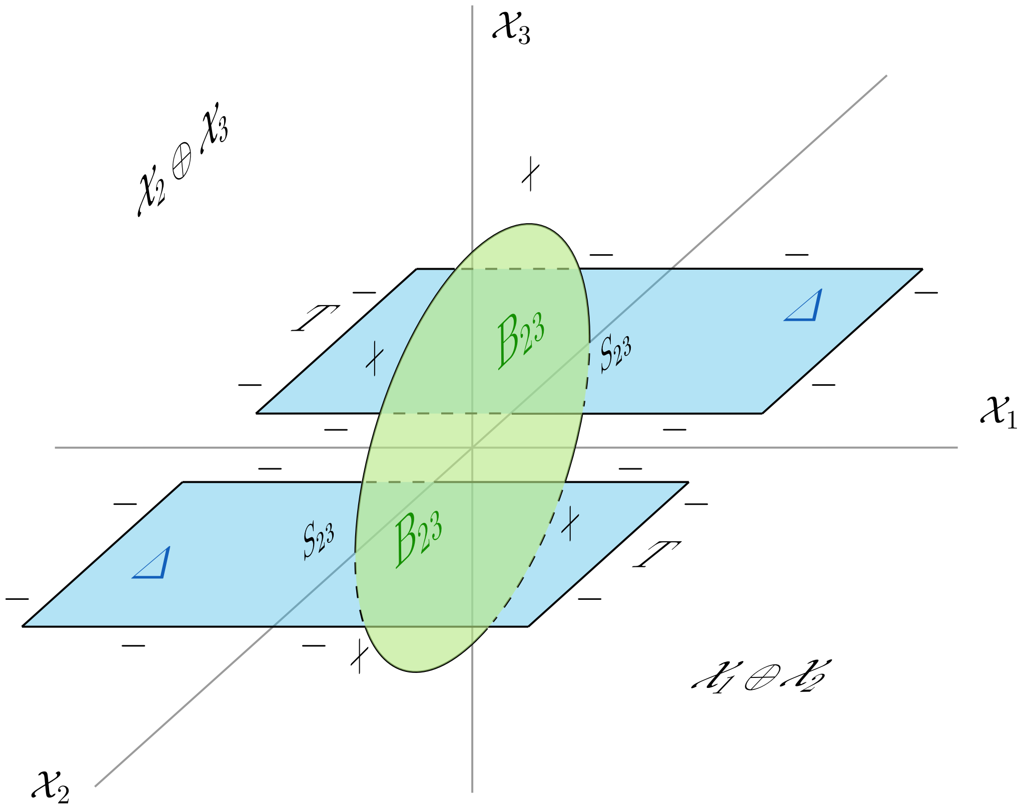

Figure 6.1: The topological situation described in Theorem 6.2

Lemma 6.3. *Let $\mathcal{M}$ be a complete, non-compact d-dimensional Riemannian manifold satisfying the curvature condition* (Ric) *and* $\inf_{\sigma\in\mathcal{M}} \mathrm{Vol}_g(B_\sigma(1)) > 0$*. If V satisfies* (V_1) *and* (V_2) *the embedding* $H^1_V(\mathcal{M}) \hookrightarrow L^q(\mathcal{M})$ *is compact for all* $q \in [2, 2^*)$*.*

∇-Theorems turned out to be a powerful tool when one is interested in studying the multiplicity of solutions for nonlinear equations. In particular, in [97] Pistoia proved the existence of four solutions for a superlinear elliptic problem on a bounded domain of $\mathbb{R}^d$. At a later time, in the same spirit of the paper of Pistoia, Mugnai proved in [91] the existence of three solutions for a superlinear boundary problem with a more general nonlinearity. ∇-Theorems are useful also when one deal with problems with higher order operators, as showed in [80] by Micheletti, Pistoia and Saccon. It is also worth mentioning [82] where Molica Bisci, Mugnai and Servadei showed the existence of three solutions for an equation driven by the fractional Laplacian on a bounded domain of $\mathbb{R}^d$ with Dirichlet condition and a general nonlinearity. When one draws his attentions to problems settled in unbounded domains, the situation is completely different. Indeed, in order to apply the sphere-torus linking Theorem it is necessary to split the space on which is defined the functional in three linear subspaces, two of them finite dimensional, while the third infinite dimensional. When Ω is a bounded domain of $\mathbb{R}^d$ it is well known that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. As a consequence of that, the resolvent of the Schrödinger operator or the Laplacian is compact and with standard arguments it is possible to prove that the spectrum of these operators is discrete and that the eigenfunctions are dense in the space under considerations. So, a common approach to select the three subspaces is to consider the whole space as a direct sum of eigenspaces. Unfortunately, this strategy fails in the case of unbounded domains, since the spectrum of the Schrödinger operator or the Laplacian is not even discrete in general. A contribution in this direction was given by Tehrani in [108] where the existence of two solution for the Nonlinear Schrödinger equation in $\mathbb{R}^d$. Following the characterization of the essential spectrum of a Schrödinger operator present in [22], they are able to decompose the space and apply the theorem. The drawback of their approach is that they don't give sufficient conditions on the potential to ensure the existence of eigenvalues subsequent to the first one. A recent result was also obtained by Mugnai in [92] proving the existence of at least two solutions for an equation in which the nonlinearity is allowed to have an exponential growth in $\mathbb{R}^2$.

In the present chapter, we want to extend the results quoted previously in two directions. The first one is to give sufficient condition that will enable us to completely characterize the spectrum of the operator taken into account. Secondly, the problem we want to investigate is settled in a non compact Riemannian manifold and, as far as we know, results as the one we are going to prove are not present in literature. One of the first contribution for the Nonlinear Schrödinger equation on Riemannian manifolds was given in [44], where Faraci and Farkas established a necessary and sufficient condition for the existence of non-trivial solutions with hypothesis on the manifold equal to the ones we will assume. More recently, Molica Bisci and Secchi in [86] showed the existence of at least two solutions for (P_λ) requiring λ large enough under our assumptions on f.

The main result of the chapter is a multiplicity result for problem (P_λ) whenever λ is sufficiently close to an eigenvalue of $-\Delta_g$.

Theorem 6.4. *Assume $f:\mathbb{R}\to\mathbb{R}$ and $V:\mathcal{M}\to\mathbb{R}$ are continuous functions that verify respectively (f_1) – (f_3) and (V_1) – (V_2). For every eigenvalue λ_k of $-\Delta_g$, there exists $\mu>0$ such that if $\lambda_k-\mu<\lambda<\lambda_k$, then problem (P_λ) admits at least three non-trivial and sign-changing weak solutions w_1, w_2 and w_3. Furthermore, these solutions belong to $L^\infty(\mathcal{M})$ and for each $i\in\{1,2,3\}$ there results*

$$\lim_{d_g(\sigma,\sigma_0)\to+\infty} w_i(\sigma)=0. \tag{6.1}$$

The proof of the previous Theorem is based on a precise description of the spectral properties of the operator $-\Delta_g+V$ which governs (P_λ). In Section 6.2 we list in detail these properties, since they seem to be new in the setting of a non-compact manifold $\mathcal{M}$.

Remark 6.5. The boundedness of our solutions and their decay at infinity (6.1) follow from [44, Theorem 3.1]. This Remark applies to the eigenfunctions considered in Theorem 6.11 as well.

To the best of our knowledge, our results are new even in the Euclidean case $\mathcal{M}=\mathbb{R}^d$, $d\geq 3$. In this case, our assumptions on V can be relaxed, and we can rely on some conditions introduced in [19] which ensure both the discreteness of the spectrum of the operator $-\Delta+V$ and the necessary compact embedding of the Sobolev space $H^1_V(\mathbb{R}^d)$. In our setting, the compactness of the embedding of $H^1_V(\mathcal{M})$ into $L^p(\mathcal{M})$ for all $p\in[2,2^*)$ follows from [44, Lemma 2.1]. As a concrete example, we propose the following result.

Theorem 6.6. *Assume $V:\mathbb{R}^d\to\mathbb{R}$ is a function in $L^\infty_{\text{loc}}(\mathbb{R}^d)$ which verifies $V(x)\geq V_0>0$ for almost every $x\in\mathbb{R}^d$ and*

$$\lim_{|x|\to+\infty}\int_{B_1(x)}\frac{dy}{V(y)}=0.$$

Then the same conclusions as in Theorem 6.4 hold for

$$\begin{cases} -\Delta w+V(x)w=\dfrac{1}{(1+|x|^d)^2}|w|^{r-2}w+\lambda w & \text{in } \mathbb{R}^d\\ w(x)\to 0 & \text{as } |x|\to\infty,\end{cases}$$

where $r\in\left(2,\dfrac{2d}{d-2}\right)$.

6.1 A setting for (P_λ)

Let us consider

$$\begin{cases} -\Delta_g w+V(\sigma)w=\alpha(\sigma)f(w)+\lambda w & \text{in } \mathcal{M}\\ w(\sigma)\to 0 & \text{as } d_g(\sigma_0,\sigma)\to\infty,\end{cases}$$

where $\alpha \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M}) \setminus \{0\}$ is a non-negative function and f satisfies assumptions (f_1) – (f_3).

In order to find solutions for problem (P_λ) we introduce the energy functional associated to the problem. Namely, let $J_\lambda \colon H^1_V(\mathcal{M}) \to \mathbb{R}$ be such that

$$J_\lambda(w) = \frac{1}{2}\|w\|^2 - \frac{\lambda}{2}\|w\|^2_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma)F(w(\sigma))\, dv_g.$$

By virtue of the embedding results presented in the previous sections, this functional is well-defined, and it is standard to prove that it is of class C^1. Moreover, as is well known, critical points of J_λ correspond to weak solutions of problem (P_λ), i.e.

$$\langle w, \varphi \rangle = \lambda \langle w, \varphi \rangle_{L^2(\mathcal{M})} + \int_{\mathcal{M}} \alpha(\sigma) f(w(\sigma))\varphi(\sigma)\, dv_g$$

for any $\varphi \in H^1_V(\mathcal{M})$. More in general, one can show that the derivative of the functional J_λ along a function $v \in H^1_V(\mathcal{M})$ is

$$J'_\lambda(w)\,[w] = \langle w, v \rangle - \lambda \langle w, v \rangle_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma) f(w(\sigma)) v(\sigma)\, dv_g. \tag{6.2}$$

Now, take $s \in [2, 2^*)$ and consider its conjugate exponent s' such that $1/s + 1/s' = 1$. We select a function $h \in L^{s'}(\mathcal{M})$ and we focus on the equation

$$S_V w = h \quad \sigma \in \mathcal{M}. \tag{6.3}$$

where $S_V := -\Delta_g + V$.

By applying the classical Riesz or Lax-Milgram Theorem, one can easily show that the problem above has a unique weak solution. In virtue of that, we are able to define

$$\begin{aligned} S_V^{-1} \colon L^{s'}(\mathcal{M}) &\to H^1_V(\mathcal{M}) \\ h &\mapsto w = S_V^{-1} h \end{aligned}$$

where $\Delta_g^{-1} h$ is the only weak solution of (6.3), which means

$$\langle S_V^{-1} h, \varphi \rangle = \langle h, \varphi \rangle_{L^2(\mathcal{M})}. \tag{6.4}$$

Remark 6.7. We emphasize that the operator S_V^{-1} is compact. Indeed, it is possible to write it by the composition of two maps

$$L^{s'}(\mathcal{M}) \xrightarrow{\ j\ } \left(H^1_V(\mathcal{M})\right)^* \xrightarrow{\ S_V^{-1}\ } H^1_V(\mathcal{M})$$

where the first is compact, recalling that $H^1_V(\mathcal{M}) \hookrightarrow L^s(\mathcal{M})$ is compact and applying [28, Theorem 6.4]. Since $H^1_V(\mathcal{M})$ is a Hilbert space, there is a unique element called the gradient of J_λ and denoted ∇J_λ such that

$$\langle \nabla J_\lambda(w), v \rangle = J'_\lambda(u)\,[v]\,. \tag{6.5}$$

It is also possible to verify that the gradient of J_λ can be written as

$$\nabla J_\lambda(w) = w - S_V^{-1}\left(\lambda w + \alpha f(w)\right). \tag{6.6}$$

We begin our analysis by proving a technical lemma that will provide some useful estimates we will use throughout the chapter.

Lemma 6.8. *If $f:\mathbb{R} \to \mathbb{R}$ is a function that satisfies (f_1) – (f_3), then we have the following estimates:*

(i) for any $\varepsilon > 0$ there exists a constant $A_1^\varepsilon > 0$ such that

$$|f(t)| \le 2\varepsilon\,|t| + rA_1^\varepsilon\,|t|^{r-1} \tag{6.7}$$

and

$$F(t) \le \varepsilon t^2 + A_1^\varepsilon\,|t|^r \tag{6.8}$$

for every $t \in \mathbb{R}$;

(ii) for any $\varepsilon > 0$ there exist $A_2, A_2^{\tilde{\varepsilon}} > 0$ such that

$$|f(t)| \le A_2 + A_2^\varepsilon\,|t|^{r-1} \tag{6.9}$$

for every $t \in \mathbb{R}$;

(iii) there exists $A_3, A_4 > 0$ such that

$$F(t) \ge A_3\,|t|^r - A_4 \tag{6.10}$$

for every $t \in \mathbb{R}$.

Proof. The verification of the three inequalities is standard, and we omit the details. □

We end this section by proving that the functional J_λ satisfies a good compactness condition in Critical Point Theory.

Definition 6.9. We say that a sequence $(w_j)_j \subset H_V^1(\mathcal{M})$ is a Palais-Smale sequence at level $c \in \mathbb{R}$, $(PS)_c$ sequence for short, if $J_\lambda(w_j) \to c$ in $\mathbb{R}$ and $J_\lambda'(w_j) \to 0$ in $\left(H_V^1(\mathcal{M})\right)^*$ as $j \to \infty$. Furthermore, the functional J_λ is said to satisfy the $(PS)_c$ condition if every $(PS)_c$ sequence for J_λ admits a strongly convergent subsequence in $H_V^1(\mathcal{M})$.

Proposition 6.10. *Let f be a map that satisfies (f_1)–(f_3) and $\lambda > 0$ a real parameter. Then, $(PS)_c$ condition holds for every $c \in \mathbb{R}$ for functional J_λ.*

Proof. Let $(w_j)_j \subset H_V^1(\mathcal{M})$ a $(PS)_c$ sequence for functional J_λ, i.e.

$$J_\lambda(w_j) \to c \quad \text{in } \mathbb{R} \tag{6.11}$$

and

$$J'_\lambda(w_j) \to 0 \quad \text{in } H^1_V(\mathcal{M}) \tag{6.12}$$

as $j \to \infty$. We first prove that $(w_j)_j$ is bounded in $H^1_V(\mathcal{M})$, adapting the ideas of [116, Proof of Theorem 6.1]. We proceed by contradiction, assuming without loss of generality that $\rho_j = \|w_j\| \to +\infty$ as $j \to +\infty$. Let us set $v_j = w_j/\rho_j$, so that we may assume that $v_j \rightharpoonup v$ in $H^1_V(\mathcal{M})$ and $v_j \to v$ strongly in $L^2(\mathcal{M})$.

Now,

$$c + o(1) = J_\lambda(w_j) = \frac{1}{2}\|w_j\|^2 - \frac{\lambda}{2}\|w_j\|_2^2 - \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g,$$

hence

$$o(1) = \frac{1}{2} - \frac{\lambda}{2}\|v_j\|_2^2 - \int_{\mathcal{M}} \alpha(\sigma) \frac{F(w_j(\sigma))}{\rho_j^2}\, dv_g,$$

and

$$\lim_{j\to+\infty} \int_{\mathcal{M}} \alpha(\sigma) \frac{F(w_j(\sigma))}{\rho_j^2}\, dv_g = \frac{1}{2} - \frac{\lambda}{2}\|v\|_2^2. \tag{6.13}$$

We consider

$$\mathcal{M}_0 = \{\sigma \in \mathcal{M} \mid v(\sigma) \neq 0\},$$

and we notice that $w_j(\sigma) \to +\infty$ when $\sigma \in \mathcal{M}_0$. From Lemma 6.8 (iii) it is straightforward to verify

$$\lim_{t\to\infty} \frac{F(t)}{t^2} = \infty$$

thus, applying the Fatou's Lemma, we get

$$\lim_{j\to\infty} \int_{\mathcal{M}_0} \alpha(\sigma) \frac{F(w_j(\sigma))}{\|w_j\|^2}\, dv_g = \infty.$$

This obviously implies that

$$\lim_{j\to+\infty} \int_{\mathcal{M}} \alpha(\sigma) \frac{F(w_j(\sigma))}{\rho_j^2}\, dv_g = +\infty. \tag{6.14}$$

Comparing (6.13) and (6.14) we must conclude that $\mathrm{Vol}_g(\mathcal{M}_0) = 0$, which means that $v = 0$ a.e. on $\mathcal{M}$ and in particular $v_j \to 0$ strongly in $L^2(\mathcal{M})$. From

$$C\|w_j\| \geq \langle \nabla J_\lambda(w_j), w_j \rangle = \|w_j\|^2 - \lambda\|w_j\|_2^2 - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g$$

we see that

$$\lim_{j\to+\infty} \int_{\mathcal{M}} \alpha(\sigma) \frac{f(w_j(\sigma)) w_j(\sigma)}{\rho_j^2}\, dv_g = 1 - \lambda\|v\|_2^2 = 1.$$

Therefore

$$\lim_{j\to+\infty} \int_{\mathcal{M}} \alpha(\sigma) \frac{rF(w_j(\sigma)) - f(w_j(\sigma)) w_j(\sigma)}{\rho_j^2} dv_g = \frac{r}{2} - \frac{\lambda r}{2}\|v\|_2^2 - 1 + \lambda\|v\|_2^2 = \frac{r}{2} - 1.$$

Coupling this with assumption (f_3), we conclude that $\frac{r}{2} \leq 1$, against the assumption that $r > 2$. This contradiction implies that $(w_j)_j$ is a bounded sequence in $H^1_V(\mathcal{M})$.

We can now use (6.6) and Remark 6.7 (see also [107, Proposition 2.2] for a general approach) to conclude the proof. □

6.2 Geometry of the ∇-Theorem

As mentioned at the beginning of the chapter, our aim is to prove an existence result through the so-called ∇-Theorem. In order to apply this tool, it is necessary to split the space in three closed subspaces, two of finite dimension and one of infinite dimension. Furthermore, the functional is required to have a precise geometrical structure. A standard decomposition of $H^1_V(\mathcal{M})$ into three subspaces can be made through an adequate selection of some eigenspaces associated to the operator S_V. The following theorem characterizes completely the spectrum of the resolvent of the Schrödinger operator under the assumptions that guarantees the compact embedding $H^1_V(\mathcal{M}) \hookrightarrow L^s(\mathcal{M})$ for $s \in [2, 2^*)$.

Theorem 6.11. *The following statements hold true:*

(*a*) *the smallest eigenvalue of problem* (6.19) *is positive, and it can be characterized as*

$$\lambda_1 := \min_{\substack{w \in H^1_V(\mathcal{M}) \\ \|w\|_{L^2(\mathcal{M})} = 1}} \|w\|^2 \tag{6.15}$$

or analogously

$$\lambda_1 := \min_{w \in H^1_V(\mathcal{M}) \setminus \{0\}} \frac{\|w\|^2}{\|w\|^2_{L^2(\mathcal{M})}};$$

(*b*) *there is a non-negative eigenfunction $e_1 \in H^1_V(\mathcal{M})$ that is an associated eigenfunction to λ_1 where the minimum in* (6.15) *is attained. Moreover, $\|e_1\|_{L^2(\mathcal{M})} = 1$ and $\lambda_1 = \|e_1\|^2$;*

(*c*) *the eigenvalue λ_1 is simple, i.e. if $w \in H^1_V(\mathcal{M})$ is such that*

$$\int_{\mathcal{M}} \langle {}^g\nabla w(\sigma), {}^g\nabla \varphi(\sigma) \rangle_g \, dv_g + \int_{\mathcal{M}} V(\sigma) w(\sigma) \varphi(\sigma) \, dv_g = \lambda_1 \int_{\mathcal{M}} w(\sigma) \varphi(\sigma) \, dv_g$$

for any $\varphi \in H^1_V(\mathcal{M})$ then there exists $\xi \in \mathbb{R}$ such that $w = \xi e_1$;

(*d*) *the set of eigenvalues of problem* (6.19) *can be arranged into a sequence $(\lambda_k)_k$ such that*

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots$$

where $\lim_{k \to \infty} \lambda_k = +\infty$. Moreover, every eigenvalue can be characterized as

$$\lambda_{k+1} := \min_{\substack{w \in E_k^{\perp} \\ \|w\|_{L^2(\mathcal{M})} = 1}} \|w\| \tag{6.16}$$

or equivalently

$$\lambda_{k+1} := \min_{w \in E_k^\perp} \frac{\|w\|^2}{\|w\|^2_{L^2(\mathcal{M})}}$$

where

$$E_k := \operatorname{span}\{e_1, \ldots, e_k\};$$

(*e*) *for any* $k \in \mathbb{N}$ *there is an eigenfunction* $e_k \in E_{k-1}^\perp$ *associated to the eigenvalue* λ_k *such that the minimum in* (6.16) *is attained, i.e.* $\|e_k\|_{L^2(\mathcal{M})} = 1$ *and*

$$\lambda_k = \|e_k\|^2; \tag{6.17}$$

(*f*) *the eigenfunctions* $(e_k)_k$ *are an orthonormal basis for* $L^2(\mathcal{M})$ *and an orthogonal basis for* $H^1_V(\mathcal{M})$;

(*g*) *each eigenvalue has finite multiplicity. Namely, if* λ_k *is such that*

$$\lambda_{k-1} < \lambda_k = \ldots = \lambda_{k+h} < \lambda_{k+h+1} \tag{6.18}$$

for some $h \in \mathbb{N}_0$, *then* $\operatorname{span}\{e_k, \ldots, e_{k+h}\}$ *is the eigenspace associated to* λ_k.

Proof. All these results are a byproduct of the classical theorems of functional analysis on the basic properties of compact self-adjoint operators defined on Hilbert spaces. As a consequence of that, we will omit the proof, and we remind the interested reader to [84] where an elementary proof is presented that can be easily adapted to our new setting. □

We point out that the previous Theorem completely describes the set of solutions of the eigenvalues problem

$$\begin{cases} -\Delta_g w + V(\sigma)w = \lambda w & \text{in } \mathcal{M} \\ w(\sigma) \to 0 & \text{as } d_g(\sigma_0, \sigma) \to \infty. \end{cases} \tag{6.19}$$

The condition $w(\sigma) \to 0$ as $d_g(\sigma, \sigma_0) \to +\infty$ follows from Remark 6.5.

In this section, we are going to show that the functional J_λ associated to problem (P_λ) possesses the geometrical structure required by (∇)-Theorem under the assumption we made on the nonlinearity f and the potential V. Before doing that, for the sake of simplicity, we fix some notation. Henceforth, k positive and h non-negative will be integers such that

$$\lambda_{k-1} < \lambda_k = \ldots = \lambda_{k+h} < \lambda_{k+h+1}.$$

We define

$$X_1 := E_{k-1}, \quad X_2 := \operatorname{span}\{e_k, \ldots e_{k+h}\}, \quad X_3 := E_{k+h}^\perp.$$

We point out that the existence of such integers h and k is guaranteed by Theorem 6.11.

Next Lemma generalize the Poincaré inequality to the case in which the functions belong to eigenspaces or its orthogonal.

Lemma 6.12. *Let $k \in \mathbb{N}$. The following inequalities hold:*

(*a*) *if $w \in E_k^{\perp}$ then*
$$\|w\|^2 \geq \lambda_{k+1}\|w\|^2_{L^2(\mathcal{M})}; \tag{6.20}$$

(*b*) *if $w \in E_k$ then*
$$\|w\|^2 \leq \lambda_k\|w\|^2_{L^2(\mathcal{M})}. \tag{6.21}$$

Proof. We start with the case (a). Since $w \in E_k^{\perp}$ we can write

$$w = \sum_{j=k+1}^{\infty} \alpha_j e_j$$

for some coefficients $\alpha_j \in \mathbb{R}$. Thus, we compute

$$\|w\|^2 = \langle w, w\rangle = \sum_{j=k+1}^{\infty} \alpha_j^2\lambda_j \geq \lambda_{k+1}\|w\|^2_{L^2(\mathcal{M})}$$

where we used Theorem 6.11 (f), (6.17) and the Bessel-Parseval's identity (see for instance [28, Theorem 5.9]). On the other hand, when $w \in E_k$ we have

$$w = \sum_{j=1}^{k} \alpha_j e_j.$$

As a consequence, similarly as we did above we get

$$\|w\|^2 = \sum_{j=1}^{k} \alpha_j^2\lambda_j \leq \lambda_k\|w\|^2_{L^2(\mathcal{M})}.$$

□

Next Proposition will show the functional J_λ verifies the desired geometrical property we need to apply the ∇-Theorem.

Proposition 6.13. *If assumptions (f_1) – (f_3) hold and $\lambda \in (\lambda_{k-1}, \lambda_k)$, then there are $\rho, R, R' \in \mathbb{R}$, with $R' > R > \rho > 0$ such that*

$$\sup_{\{w\in X_1 \ | \ \|w\|\leq R\}\cup\{w\in X_1\oplus X_2 \ | \ \|w\|=\varsigma\}} J_\lambda < \inf_{\{w\in X_2\oplus X_3 \ | \ \|w\|=\rho\}} J_\lambda$$

for all $\varsigma \in [R, R']$

Proof. We start showing
$$\inf_{\{w\in X_2\oplus X_3 \ | \ \|w\|=\rho\}} J_\lambda > 0$$

choosing ρ adequately and observing that $X_2 \oplus X_3 = E_{k-1}^{\perp}$. Applying twice the Hölder inequality, we get

$$\int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^2 \, dv_g \leq \|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})} \|w\|^2_{L^{2^*}(\mathcal{M})} \tag{6.22}$$

and

$$\int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^r \, dv_g \leq \|\alpha\|_{L^{\frac{2^*}{2^*-r}}(\mathcal{M})} \|w\|^r_{L^{2^*}(\mathcal{M})}. \tag{6.23}$$

From Lemma 6.8 (i), (6.22) and (6.23) we obtain

$$\begin{aligned} J_\lambda(w) &\geq \frac{1}{2}\|w\|^2 - \frac{\lambda}{2}\|w\|^2_{L^2(\mathcal{M})} - \varepsilon \int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^2 \, dv_g - A_1^\varepsilon \int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^r \, dv_g \\ &\geq \frac{1}{2}\|w\|^2 - \frac{\lambda}{2}\|w\|^2_{L^2(\mathcal{M})} - \varepsilon\|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})} \|w\|^2_{L^{2^*}(\mathcal{M})} - A_1^\varepsilon \|\alpha\|_{L^{\frac{2^*}{2^*-r}}(\mathcal{M})} \|w\|^r_{L^{2^*}(\mathcal{M})}. \end{aligned}$$

Now, recalling $H_V^1(\mathcal{M}) \hookrightarrow L^s(\mathcal{M})$ for every $s \in [2, 2^*]$ continuously, it is possible to find $C > 0$ such that

$$J_\lambda(w) \geq \frac{1}{2}\|w\|^2 - \frac{\lambda}{2}\|w\|^2_{L^2(\mathcal{M})} - \varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})} \|w\|^2 - A_1^\varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-r}}(\mathcal{M})} \|w\|^r.$$

Finally, Lemma 6.12 yields

$$J_\lambda(w) \geq \left[\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right) - \varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})}\right] \|w\|^2 - A_1^\varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-r}}(\mathcal{M})} \|w\|^r.$$

At this point, choosing $\varepsilon > 0$ such that

$$\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right) - \varepsilon C\|\alpha\|_{L^{\frac{2^*}{2^*-2}}(\mathcal{M})} > 0$$

and ρ sufficiently small, the desired assertion is proved. On the other hand, it is possible to prove

$$\sup_{\{w \in X_1 \,|\, \|w\| \leq R\} \cup \{w \in X_1 \oplus X_2 \,|\, \|w\| = R\}} J_\lambda \leq 0.$$

Indeed, in the case $w \in X_1$, from Lemma 6.12 and (f_3), recalling $\alpha \geq 0$ for a.e. $\sigma \in \mathcal{M}$, it follows that

$$J_\lambda(w) \leq \frac{\lambda_{k-1} - \lambda}{2}\|w\|^2_{L^2(\mathcal{M})} \leq 0.$$

Instead, when $w \in X_1 \oplus X_2$ it suffices to use Lemma 6.8 (iii) to obtain

$$J_\lambda(w) \leq \frac{1}{2}\|w\|^2 - A_3 \int_{\mathcal{M}} \alpha(\sigma)|w(\sigma)|^r \, dv_g + A_4\|\alpha\|_{L^1(\mathcal{M})}.$$

Since $X_1 \oplus X_2$ has finite dimension all norms are equivalent, then choosing $R > 0$ big enough it is straightforward to see that $r > 2$ implies $J_\lambda(w) \leq 0$. □

6.3 Validity of the (∇)-condition

This section is devoted to showing the validity of the (∇)-condition introduced in Definition 6.1. Before proving the main result of this section, we need two preliminary Lemmas.

Proposition 6.14. *Assume Hypotheses* (f_1) – (f_3) *hold. Then for every* $\varrho > 0$ *there exists* $\delta_\varrho > 0$ *such that for each* $\lambda \in [\lambda_{k-1} + \varrho, \lambda_{k+h+1} - \varrho]$ *the only critical point* u *of* J_λ *constrained on* $X_1 \oplus X_3$ *with* $J_\lambda(u) \in [-\delta_\varrho, \delta_\varrho]$ *is the trivial one.*

Proof. By contradiction, we suppose the statement false. So, we assume the existence of $\tilde{\varrho} > 0$, a sequence $\mu_j \subset [\lambda_{k-1} + \tilde{\varrho}, \lambda_{k+h+1} - \tilde{\varrho}]$ and a sequence $(w_j)_j \subset X_1 \oplus X_3$ of non-trivial critical points, i.e.

$$\langle \nabla J_{\mu_j}(w_j), \varphi \rangle = 0 \quad \text{for any } \varphi \in X_1 \oplus X_3 \tag{6.24}$$

such that

$$\lim_{j \to +\infty} J_{\mu_j}(w_j) = 0. \tag{6.25}$$

Since $(w_j)_j \subset X_1 \oplus X_3$, we can choose $\varphi = w_j$ in (6.24). As a consequence we have

$$0 = \|w_j\|^2 - \mu_j \|w_j\|^2_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g. \tag{6.26}$$

Then, we notice that (6.26) can be rewritten as

$$0 = 2J_{\mu_j}(w_j) + 2\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g.$$

Exploiting (f_3) in (6.26) we obtain

$$0 \le 2J_{\mu_j}(w_j) + (2 - r)\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g. \tag{6.27}$$

Reordering the terms in (6.27) we get

$$0 \le (r - 2)\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g \le 2J_{\mu_j}(w_j). \tag{6.28}$$

Putting together (6.25) and (6.28) we obtain

$$\lim_{j \to \infty} \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g = 0. \tag{6.29}$$

Now, recalling $w_j \in X_1 \oplus X_3$ for all $j \in \mathbb{N}$, we are able to find $w_{1,j} \in X_1$ and $w_{3,j} \in X_3$ such that $w_j = w_{1,j} + w_{3,j}$. At this point, on the one hand, we test (6.24) with $\varphi = w_{1,j} - w_{3,j}$ and exploiting the properties of orthogonality of $w_{1,j}$ and $w_{3,j}$ we have

$$\begin{aligned} 0 &= \langle \nabla J_{\mu_j}(w_j), w_{1,j} - w_{3,j} \rangle \\ &= \|w_{1,j}\|^2 - \|w_{3,j}\|^2 - \mu_j \|w_{1,j}\|^2_{L^2(\mathcal{M})} + \mu_j \|w_{3,j}\|^2_{L^2(\mathcal{M})} \\ &\quad - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma))(w_{1,j}(\sigma) - w_{3,j}(\sigma))\, dv_g. \end{aligned} \tag{6.30}$$

Rearranging (6.30) and applying Lemma 6.12 we get

$$\begin{aligned}\int_{\mathcal{M}} \alpha(\sigma)f(w_j(\sigma))(w_{1,j}(\sigma)-w_{3,j}(\sigma))\,dv_g &= \|w_{1,j}\|^2-\|w_{3,j}\|^2-\mu_j\|w_{1,j}\|^2_{L^2(\mathcal{M})}\\ &\quad+\mu_j\|w_{3,j}\|^2_{L^2(\mathcal{M})}\\ &\le \|w_{1,j}\|^2-\|w_{3,j}\|^2-\frac{\mu_j}{\lambda_{k-1}}\|w_{1,j}\|^2\\ &\quad+\frac{\mu_j}{\lambda_{k+h+1}}\|w_{3,j}\|^2 \qquad (6.31)\\ &= \frac{\lambda_{k-1}-\mu_j}{\lambda_{k-1}}\|w_{1,j}\|^2+\frac{\mu_j-\lambda_{k+h+1}}{\lambda_{k+h+1}}\|w_{3,j}\|^3\\ &< -\frac{\tilde{\varrho}}{\lambda_{k-1}}\|w_{1,j}\|^2-\frac{\tilde{\varrho}}{\lambda_{k+h+1}}\|w_{3,j}\|^2\\ &< -\frac{2\tilde{\varrho}}{\lambda_{k+h+1}}\|w_j\|^2. \qquad (6.32)\end{aligned}$$

On the other hand, thanks to Hölder and the continuous embedding $H^1_V(\mathcal{M}) \hookrightarrow L^r(\mathcal{M})$, we have

$$\begin{aligned}\left|\int_{\mathcal{M}} \alpha(\sigma)f(w_j(\sigma))(w_{1,j}(\sigma)-w_{3,j}(\sigma))\,dv_g\right| &\le \|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})}\|w_{1,j}-w_{3,j}\|_{L^r(\mathcal{M})}\\ &\le C\|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})}\|w_j\| \qquad (6.33)\end{aligned}$$

for some $C>0$, where we used

$$\langle w_{1,j}-w_{3,j}, w_{1,j}-w_{3,j}\rangle = \|w_{1,j}\|^2-\|w_{3,j}\|^2=\|w_j\|^2.$$

Coupling (6.31) and (6.33) we have

$$-C\|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})}\|w_j\|\le -\frac{2\tilde{\varrho}}{\lambda_{k+h+1}}\|w_j\|^2$$

from which it follows that

$$\frac{2\tilde{\varrho}}{\lambda_{k+h+1}}\|w_j\|\le C\|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})}. \qquad (6.34)$$

Then, we use Lemma 6.8 (ii) and we obtain

$$\int_{\mathcal{M}} |\alpha(\sigma)f(w_j(\sigma))|^{r'}\,dv_g \le \int_{\mathcal{M}} \left[\alpha(\sigma)\left(A_2+A_2^{\varepsilon}|w_j|^{r-1}\right)\right]^{\frac{r}{r-1}}. \qquad (6.35)$$

Recalling that for any $a,b\ge 0$ we have

$$(a+b)^{r'} \le 2^{r'}(a^{r'}+b^{r'}),$$

it follows from (6.35) that

$$\int_{\mathcal{M}} |\alpha(\sigma)f(w_j(\sigma))|^{r'}\,dv_g \le \left(2A_2\|\alpha\|_{L^{r'}(\mathcal{M})}\right)^{r'}+(2A_2^{\varepsilon})^{r'}\int_{\mathcal{M}}(\alpha(\sigma))^{r'}|w_j|^r\,dv_g. \qquad (6.36)$$

Finally, we exploit Lemma 6.8 in (6.36) and we obtain

$$\int_{\mathcal{M}} |\alpha(\sigma)f(w_j(\sigma))|^{r'}\, dv_g \leq \left(2A_2\|\alpha\|_{L^{r'}(\mathcal{M})}\right)^{r'} + \frac{A_4}{A_3}\left(2A_2^{\varepsilon}\right)^{r'}\|\alpha\|^{r'}_{L^{r'}(\mathcal{M})} + \left(2A_2^{\varepsilon}\right)^{r'}\frac{A_4}{A_3}\|\alpha\|^{r'-1}_{L^{\infty}(\mathcal{M})}\int_{\mathcal{M}}\alpha(\sigma)F(w_j(\sigma))\, dv_g. \tag{6.37}$$

From (6.29), (6.34) and (6.37), we can deduce that $(w_j)_j$ is bounded in $H^1_V(\mathcal{M})$. Hence, up to a subsequence

$$w_j \rightharpoonup w_\infty \quad \text{in } H^1_V(\mathcal{M}).$$

Furthermore, recalling that $H^1_V(\mathcal{M}) \hookrightarrow L^r(\mathcal{M})$ is compact, we have

$$\begin{aligned} &w_j \to w_\infty \quad \text{in } L^r(\mathcal{M}),\\ &w_j(\sigma) \to w_\infty(\sigma) \quad \text{for a.e. } \sigma \in \mathcal{M} \end{aligned}$$

as $j \to \infty$. Now, from (6.34), Lemma 6.8 (i) and the Minkowski inequality it follows

$$\begin{aligned} 0 < \frac{2\tilde{\varrho}}{C\lambda_{k+h+1}} &\leq \frac{\|\alpha f(w_j)\|_{L^{r'}(\mathcal{M})}}{\|w_j\|}\\ &\leq \frac{\left(\int_{\mathcal{M}}\left[\alpha(\sigma)\left(2\varepsilon|w_j|+rA_1^{\varepsilon}|w_j|^{r-1}\right)\right]^{\frac{r}{r-1}}\, dv_g\right)^{\frac{r-1}{r}}}{\|w_j\|}\\ &\leq \frac{4\varepsilon\left(\int_{\mathcal{M}}\alpha(\sigma)^{\frac{r}{r-1}}|w_j|^{\frac{r}{r-1}}\, dv_g\right)^{\frac{r-1}{r}} + 2rA_1^{\varepsilon}\left(\int_{\mathcal{M}}\alpha(\sigma)^{\frac{r}{r-1}}|w_j|^{r}\, dv_g\right)^{\frac{r-1}{r}}}{\|w_j\|}. \end{aligned} \tag{6.38}$$

Recalling that the embedding $H^1_V(\mathcal{M}) \hookrightarrow L^s(\mathcal{M})$ is continuous for every $s \in [2, 2^*]$ we deduce from (6.38) that

$$0 < \frac{2\tilde{\varrho}}{C\lambda_{k+h+1}} \leq \tilde{C}\left(2\varepsilon + rA_1^{\varepsilon}\|w_j\|^{r-2}\right) \tag{6.39}$$

for some optimal $\tilde{C} > 0$. With similar estimates, it is straightforward to check that

$$|\alpha(\sigma)f(w_j(\sigma))|^{\frac{r}{r-1}} \leq C_1^{\varepsilon}|\alpha(\sigma)|^{\frac{r}{r-1}} + C_2^{\varepsilon}|w_j(\sigma)|^r$$

and

$$|\alpha(\sigma)F(w_j(\sigma))| \leq C_3^{\varepsilon}|w_j(\sigma)|^2 + C_4^{\varepsilon}|w_j(\sigma)|^r$$

choosing adequately $C_1^{\varepsilon}, C_2^{\varepsilon}, C_3^{\varepsilon}, C_4^{\varepsilon} > 0$. Hence, the general Lebesgue dominated convergence Theorem [101, Section 4.4, Theorem 19] implies

$$\lim_{j\to\infty}\int_{\mathcal{M}}\alpha(\sigma)F(w_j(\sigma))\, dv_g = \int_{\mathcal{M}}\alpha(\sigma)F(w_\infty(\sigma))\, dv_g \tag{6.40}$$

and

$$\lim_{j\to\infty}\int_{\mathcal{M}}|\alpha(\sigma)f(w_j(\sigma))|^{\frac{r}{r-1}}\,dv_g=\int_{\mathcal{M}}|\alpha(\sigma)f(w_\infty(\sigma))|^{\frac{r}{r-1}}\,dv_g. \tag{6.41}$$

Coupling (6.29) and (6.40), keeping into account (f_3), we see that $w_\infty=0$ is the only admissible case. At this point, only two possible scenarios are possible. The first one is that $w_j\to 0$ in $H^1_V(\mathcal{M})$, but if that were true, letting $j\to\infty$, utilizing (6.39), then we would have

$$0<\frac{2\tilde{\varrho}}{C\lambda_{k+h+1}}\le 2\varepsilon\tilde{C}$$

which is impossible since $\varepsilon>0$ is arbitrary. The second one is that there exist $\eta>0$ such that $\|w_j\|\ge\eta$ for each $j\in\mathbb{N}$. In this case, firstly we notice that from $w_\infty=0$ and $f(0)=0$ it follows

$$\lim_{j\to\infty}\int_{\mathcal{M}}|\alpha(\sigma)f(w_j(\sigma))|^{\frac{r}{r-1}}\,dv_g=0. \tag{6.42}$$

Then, thanks to (6.42), (6.34) becomes

$$0<\frac{2\tilde{\varrho}\eta}{\lambda_{k+h+1}}\le 0,$$

which is clearly a contradiction. □

In the sequel, given a closed subspace Y of $H^1_V(\mathcal{M})$ we will denote with $P_Y\colon H^1_V(\mathcal{M})\to Y$ the usual orthogonal projection.

Proposition 6.15. *Suppose f satisfies (f_1) – (f_3), $\lambda\in\mathbb{R}$ and let $(w_j)_j\subset H^1_V(\mathcal{M})$ be a sequence such that*

$$\left(J_\lambda(w_j)\right)_j\quad\text{is bounded} \tag{6.43}$$

$$P_{X_2}w_j\to 0\quad\text{in } H^1_V(\mathcal{M}) \tag{6.44}$$

$$P_{X_1\oplus X_3}\nabla J_\lambda(w_j)\to 0\quad\text{in } H^1_V(\mathcal{M}). \tag{6.45}$$

Then $(w_j)_j$ is bounded in $H^1_V(\mathcal{M})$.

Proof. We argue by contradiction, and we suppose that

$$\|w_j\|\to\infty \tag{6.46}$$

as $j\to\infty$. Normalizing we assume up to a subsequence

$$\frac{w_j}{\|w_j\|}\rightharpoonup w_\infty\quad\text{in } H^1_V(\mathcal{M})$$

and

$$\frac{w_j}{\|w_j\|}\to w_\infty\quad\text{in } L^s(\mathcal{M}) \tag{6.47}$$

as $j\to\infty$ for all $s\in[2,2^*)$.

Clearly, we can write

$$w_j = P_{X_2} w_j + P_{X_1 \oplus X_3} w_j \tag{6.48}$$

with $P_{X_2} w_j \to 0$. Recalling (6.5), (6.6) and (6.48) we have

$$\begin{aligned} \langle P_{X_1 \oplus X_3} \nabla J_\lambda(w_j), w_j \rangle &= \langle \nabla J_\lambda(w_j), w_j \rangle - \langle P_{X_2} \nabla J_\lambda(w_j), w_j \rangle \\ &= \|w_j\|^2 - \lambda \|w_j\|^2_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g \\ &- \langle P_{X_2} \left(w_j - S_V^{-1} \left(\lambda w_j + \alpha f(w_j) \right) \right), w_j \rangle \end{aligned} \tag{6.49}$$

By orthogonality we get

$$\langle P_{X_2} w, v \rangle = \langle P_{X_2} w, P_{X_1 \oplus X_3} v + P_{X_2} v \rangle = \langle P_{X_2} w, P_{X_2} v \rangle$$

and

$$\langle w, P_{X_2} v \rangle = \langle P_{X_1 \oplus X_3} w + P_{X_2} w, P_{X_2} v \rangle = \langle P_{X_2} w, P_{X_2} v \rangle$$

for every $w, v \in H^1_V(\mathcal{M})$, which means that P_{X_2} is a symmetric operator. In virtue of that, we have

$$\begin{aligned} \langle P_{X_2} \left(w_j - S_V^{-1} \left(\lambda w_j + \alpha f(w_j) \right) \right), w_j \rangle &= \|P_{X_2} w_j\|^2 - \lambda \langle S_V^{-1} w_j, P_{X_2} w_j \rangle \\ &- \langle S_V^{-1} \left(\alpha f(w_j) \right), P_{X_2} w_j \rangle. \end{aligned} \tag{6.50}$$

Recalling (6.4) we get

$$\begin{aligned} &\lambda \langle P_{X_2} w_j, S_V^{-1} w_j \rangle + \langle P_{X_2} w_j, S_V^{-1} \left(\alpha f(w_j) \right) \rangle \\ &\qquad = \lambda \|P_{X_2} w_j\|^2_{L^2(\mathcal{M})} + \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j(\sigma)\, dv_g \end{aligned} \tag{6.51}$$

Inserting (6.50) and (6.51) in (6.49) we obtain

$$\begin{aligned} \langle P_{X_1 \oplus X_3} \nabla J_\lambda(w_j), w_j \rangle &= 2 J_\lambda(w_j) + 2 \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g \\ &- \|P_{X_2} w_j\|^2 + \lambda \|P_{X_2} w_j\|^2_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g \\ &+ \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j(\sigma)\, dv_g. \end{aligned} \tag{6.52}$$

Reordering the terms in (6.52) and using (6.43), (6.44), (6.45) and (6.46) we get

$$\begin{aligned} &\frac{1}{\|w_j\|^r} \left(2 \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma))\, dv_g - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma)\, dv_g \right. \\ &\qquad \left. + \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j\, dv_g \right) \to 0 \end{aligned} \tag{6.53}$$

as $j \to \infty$.

Claim: $w_\infty = 0$
We first need to show

$$\frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j \, dv_g}{\|w_j\|^r} \to 0 \tag{6.54}$$

as $j \to \infty$. As a first step, observe that all eigenfunctions are bounded by [44, Theorem 3.1]. Moreover, having X_2 finite dimension, all norms are equivalent. Therefore, from (6.44) it follows that

$$\|P_{X_2} w_j\|_{L^\infty(\mathcal{M})} \to 0$$

as $j \to \infty$. Then, from Lemma 6.8 (i)

$$\left| \frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j(\sigma) \, dv_g}{\|w_j\|^r} \right| \le \frac{\displaystyle 2\varepsilon \int_{\mathcal{M}} \alpha(\sigma) w_j(\sigma) \, dv_g + r A_1^\varepsilon \|P_{X_2} w_j\|_{L^\infty(\mathcal{M})} \int_{\mathcal{M}} \alpha(\sigma) |w_j(\sigma)|^{r-1} \, dv_g}{\|w_j\|^r}.$$

Applying the Hölder inequality twice and recalling $H^1_V(\mathcal{M}) \hookrightarrow L^2(\mathcal{M})$ it follows

$$\left| \frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) P_{X_2} w_j(\sigma) \, dv_g}{\|w_j\|^r} \right| \le \frac{2\varepsilon C \|\alpha\|_{L^2(\mathcal{M})}}{\|w_j\|^{r-2}} + \frac{r A_1^\varepsilon \|P_{X_2} w_j\|_{L^\infty(\mathcal{M})} \|\alpha\|^r_{L^r(\mathcal{M})} \left\| \frac{w_j}{\|w_j\|} \right\|^{r-1}_{L^r(\mathcal{M})}}{\|w_j\|}$$

for some $C > 0$. Now, the validity of (6.54) follows from the boundedness of the sequence $w_j/\|w_j\|$ in $L^r(\mathcal{M})$. In virtue of (6.54), combining (6.43) with (f_3), we obtain

$$o(1) = \frac{\displaystyle 2\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma)) \, dv_g - \int_{\mathcal{M}} \alpha(\sigma) f(w_j(\sigma)) w_j(\sigma) \, dv_g}{\|w_j\|^r} \le \frac{\displaystyle (2-r) \int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma)) \, dv_g}{\|w_j\|^r} \le 0 \tag{6.55}$$

from which we deduce

$$\lim_{j \to \infty} \frac{\displaystyle\int_{\mathcal{M}} \alpha(\sigma) F(w_j(\sigma)) \, dv_g}{\|w_j\|^r} = 0. \tag{6.56}$$

At this point, Lemma 6.8 *(iii)* implies

$$\frac{\int_{\mathcal{M}} \alpha(\sigma)|w_j|^r \, dv_g}{\|w_j\|^r} \le \frac{A_4\|\alpha\|_{L^1(\mathcal{M})}}{A_3\|w_j\|^r} + \frac{1}{A_3\|w_j\|^r}\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\, dv_g.$$

Combining this with (6.56) we get that $\alpha(\sigma)|w_j(\sigma)|^r \to 0$ a.e. in $\mathcal{M}$ as $j \to \infty$, but then the claim follows because of the positivity a.e of α. Now, we observe that

$$0 \leftarrow \frac{J_\lambda(w_j)}{\|w_j\|^2} = \frac{1}{2} - \frac{\lambda}{2}\left\|\frac{w_j}{\|w_j\|}\right\|^2_{L^2(\mathcal{M})} - \frac{1}{\|w_j\|^2}\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\, dv_g.$$

Recalling $w_j/\|w_j\| \to 0$ in $L^2(\mathcal{M})$ we obtain

$$\frac{1}{\|w_j\|^2}\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\, dv_g \to \frac{1}{2} \tag{6.57}$$

as $j \to \infty$. Furthermore, from Lemma 6.8 *(iii)* it follows

$$\frac{1}{\|w_j\|^2}\int_{\mathcal{M}} \alpha(\sigma)|w_j(\sigma)|^r \, dv_g \le \frac{A_4\|\alpha\|_{L^1(\mathcal{M})}}{A_3\|w_j\|^2} + \frac{1}{A_3\|w_j\|^2}\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\, dv_g. \tag{6.58}$$

Because of (6.57), the second member of (6.58) is bounded and so there exist a $\tilde{C} > 0$ such that

$$\int_{\mathcal{M}} \alpha(\sigma)|w_j(\sigma)|^r \, dv_g \le \tilde{C}\|w_j\|^2. \tag{6.59}$$

At this point, applying Lemma 6.8 *(ii)*, the Hölder inequality and (6.59), we notice

$$\begin{aligned}
&\frac{\int_{\mathcal{M}} |\alpha(\sigma)f(w_j(\sigma))P_{X_2}w_j(\sigma)| \, dv_g}{\|w_j\|^2} \\
&\le \frac{\|P_{X_2}w_j\|_{L^\infty(\mathcal{M})}}{\|w_j\|^2}\left(A_2\|\alpha\|_{L^1(\mathcal{M})} + A_2^\varepsilon \int_{\mathcal{M}} |\alpha(\sigma)|^{\frac{1}{r}}|\alpha(\sigma)|^{\frac{r-1}{r}}|w_j(\sigma)|^{r-1}\right) \\
&\le \|P_{X_2}w_j\|_{L^\infty}\left[\frac{A_2\|\alpha\|_{L^1(\mathcal{M})}}{\|w_j\|^2} + \frac{A_2^\varepsilon\|\alpha\|^{\frac{1}{r}}_{L^1(\mathcal{M})}}{\|w_j\|^{\frac{2}{r}}}\left(\frac{\int_{\mathcal{M}} \alpha(\sigma)|w_j(\sigma)|^r \, dv_g}{\|w_j\|^2}\right)^{\frac{r-1}{r}}\right] \\
&\le \|P_{X_2}w_j\|_{L^\infty}\left[\frac{A_2\|\alpha\|_{L^1(\mathcal{M})}}{\|w_j\|^2} + \frac{A_2^\varepsilon\tilde{C}^{1-\frac{1}{r}}\|\alpha\|^{\frac{1}{r}}_{L^1(\mathcal{M})}}{\|w_j\|^{\frac{2}{r}}}\right],
\end{aligned}$$

which implies

$$\lim_{j\to\infty}\frac{\int_{\mathcal{M}} |\alpha(\sigma)f(w_j(\sigma))P_{X_2}w_j(\sigma)| \, dv_g}{\|w_j\|^2} = 0. \tag{6.60}$$

Dividing (6.52) by $\|w_j\|^2$ and using (6.43), (6.44), (6.45) and (6.60) we get

$$\frac{1}{\|w_j\|^2}\left(\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\,dv_g - \int_{\mathcal{M}} \alpha(\sigma)f(w_j(\sigma))w_j(\sigma)\,dv_g\right) \to 0$$

as $j \to \infty$. To conclude the proof, we argue as did in (6.55) to obtain

$$\lim_{j\to\infty} \frac{1}{2}\int_{\mathcal{M}} \alpha(\sigma)F(w_j(\sigma))\,dv_g = 0. \tag{6.61}$$

Clearly, (6.57) and (6.61) are not compatible. □

Proposition 6.16. *Assume f satisfies (f_1) – (f_3). For any $\varrho > 0$ there exists $\eta_\varrho > 0$ such that for any $\eta', \eta'' \in (0, \eta_\varrho)$, with $\eta' < \eta''$ we have that $\nabla(J_\lambda, X_1 \oplus X_3, \eta', \eta'')$ is verified for all $\lambda \in (\lambda_{k-1} + \varrho, \lambda_{k+h+1} - \varrho)$.*

Proof. By contradiction, we suppose that there is $\tilde{\varrho} > 0$ such that for any $\eta_{\tilde{\varrho}} > 0$ we can find $\tilde{\lambda} \in [\lambda_{k-1} + \tilde{\varrho}, \lambda_{k+h+1} - \tilde{\varrho})$ and $\eta' < \eta''$ such that

$$(\nabla)\left(J_{\tilde{\lambda}}, X_1 \oplus X_3, \eta', \eta''\right)$$

does not hold. If so, it is possible to find a sequence $(w_j)_j \subset H^1_V(\mathcal{M})$ such that

$$J_{\tilde{\lambda}}(w_j) \in \left[\eta', \eta''\right]$$

$$\mathrm{dist}(w_j, X_1 \oplus X_3) \to 0 \quad \text{as } j \to \infty \tag{6.62}$$

$$P_{X_1\oplus X_3}\nabla J_{\tilde{\lambda}}(w_j) \to 0 \quad \text{as } j \to \infty. \tag{6.63}$$

Because of that, Proposition 6.15 can be applied, thus $(w_j)_j$ is bounded in $H^1_V(\mathcal{M})$. Hence, up to a subsequence,

$$w_j \rightharpoonup w_\infty \quad \text{in } H^1_V(\mathcal{M}) \tag{6.64}$$

$$w_j \to w_\infty \quad \text{in } L^s(\mathcal{M}) \quad \text{for all } s \in [2, 2^*) \tag{6.65}$$

$$w_j(\sigma) \to w_\infty(\sigma) \quad \text{a.e in } \mathcal{M}$$

as $j \to \infty$. Now, arguing as we did to obtain (6.36), we can find $\tilde{A}_1^\varepsilon, \tilde{A}_2^\varepsilon > 0$ such that

$$\int_{\mathcal{M}} |\alpha(\sigma)f(w_j(\sigma))|^{\frac{r}{r-1}}\,dv_g \le \tilde{A}_1^\varepsilon + \tilde{A}_2^\varepsilon \int_{\mathcal{M}} |w_j(\sigma)|^r\,dv_g.$$

Since $w_j \to w_\infty$ in $L^r(\mathcal{M})$ there is $\tilde{C} > 0$ such that

$$\int_{\mathcal{M}} |\alpha(\sigma)f(w_j(\sigma))|^{\frac{r}{r-1}}\,dv_g \le \tilde{C}.$$

Then, recalling that S_V^{-1} is a compact operator,

$$P_{X_1\oplus X_3}S_V^{-1}\left(\tilde{\lambda}w_j + \alpha f(w_j)\right) \to P_{X_1\oplus X_3}S_V^{-1}\left(\tilde{\lambda}w_\infty + \alpha f(w_\infty)\right). \tag{6.66}$$

Recalling (6.6), we have

$$P_{X_1\oplus X_3}\nabla J_\lambda(w_j) = w_j - P_{X_2}w_j - P_{X_1\oplus X_3}S_V^{-1}\left(\tilde\lambda w_j + \alpha f(w_j)\right).$$

Since that, (6.66), (6.62) and (6.63) we deduce

$$w_j \to P_{X_1\oplus X_3}S_V^{-1}\left(\tilde\lambda w_\infty + \alpha f(w_\infty)\right)$$

in $H_V^1(\mathcal{M})$ as $j\to\infty$. Now, on the one hand, from (6.5) and (6.63) it follows

$$\langle \nabla J_{\tilde\lambda}(w_j),\varphi\rangle = \langle w_j,\varphi\rangle - \tilde\lambda\langle w_j,\varphi\rangle_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma)f(w_j(\sigma))\varphi(\sigma)\,dv_g \to 0 \tag{6.67}$$

for any $\varphi\in X_1\oplus X_3$ as $j\to\infty$. □

On the other hand, from (6.64) and (6.65) we also have

$$\langle \nabla J_{\tilde\lambda}(w_j),\varphi\rangle \to \langle w_\infty,\varphi\rangle - \tilde\lambda\langle w_\infty,\varphi\rangle_{L^2(\mathcal{M})} - \int_{\mathcal{M}} \alpha(\sigma)f(w_j(\sigma))\varphi(\sigma)\,dv_g \tag{6.68}$$

for any $\varphi\in X_1\oplus X_3$. Coupling (6.67) and (6.68) we get that w_∞ is a critical point for $J_{\tilde\lambda}$ constrained on $X_1\oplus X_3$. Then, we can apply Proposition 6.14 to obtain $w_\infty=0$. But, since $J_{\tilde\lambda}(w_j)\geq\eta'$, $w_j\to w_\infty$ in $H_V^1(\mathcal{M})$, exploiting the continuity of $J_{\tilde\lambda}$ we obtain $J_{\tilde\lambda}(w_\infty)>0$. This is a contradiction, as $J_{\tilde\lambda}(0)=0$.

6.4 Proof of Theorem 6.4

We begin with a technical result.

Lemma 6.17. *If f verifies (f_1)–(f_3) then*

$$\lim_{\lambda\to\lambda_k}\sup_{w\in E_{k+h}} J_\lambda(w) = 0$$

Proof. We start noticing that from Lemma 6.8 *(iii)* it follows

$$\lim_{\xi\to\pm\infty} J_\lambda(\xi w) = -\infty$$

for all $w\in E_{k+h}$, thus

$$\sup_{w\in E_{k+h}} J_\lambda(w) \quad \text{is achieved.}$$

Now, by contradiction we suppose there is a sequence $\tau_j\to\lambda_k$ as $j\to\infty$ and a sequence $(w_j)_j\subset E_{k+h}$ such that

$$J_{\tau_j}(w_j) = \sup_{w\in E_{k+h}} J_\lambda(w) > \gamma \tag{6.69}$$

for some $\gamma>0$. We split the proof analyzing separately the case $(w_j)_j$ bounded and unbounded. In the first one, since the weak and the strong topology coincide, we can suppose $w_j \to w_\infty$ in E_{k+h}. In order to reach a contradiction, keeping into account (6.69) and letting $j \to \infty$, it suffices to apply Lemma 6.12 to obtain

$$\gamma \leq J_{\lambda_k}(w_\infty) = (\lambda_{k+h} - \lambda_k) - \int_{\mathcal{M}} \alpha(\sigma) F(w_\infty(\sigma))\, dv_g \leq 0.$$

Instead, if $(w_j)_j$ is unbounded, we can assume $\|w_j\| \to \infty$ as $j \to \infty$. From Lemma 6.8 (iii) it follows

$$0 < \gamma \leq J_{\tau_j}(w_j) \leq \frac{1}{2}\|w_j\|^2 - \frac{\tau_j}{2}\|w_j\|^2_{L^2(\mathcal{M})} - A_3\|w_j\|^r_{L^r(\mathcal{M})} + A_4\|\alpha\|_{L^1(\mathcal{M})}.$$

Exploiting again the fact that on the finite-dimensional subspace E_{h+k} all norms are equivalent, the right-hand side of the above inequality goes to $-\infty$ concluding the proof. □

Proof of Theorem 6.4. We want to apply [77, Theorem 2.10]. We start choosing $\varrho > 0$. In correspondence of that, thanks to Proposition 6.16 there are $\eta_\varrho, \eta', \eta'' > 0$, with $\eta' < \eta'' < \eta_\varrho$ such that $\nabla(J_\lambda, X_1 \oplus X_3, \eta', \eta'')$ is verified for all $\lambda \in (\lambda_{k-1} + \varrho, \lambda_{k+h+1} - \varrho)$. Exploiting Lemma 6.17 we also have the existence of $\overline{\varrho} > 0$, with $\overline{\varrho} \leq \varrho$ such that

$$\sup_{w \in E_{k+h}} J_\lambda(w) \leq \eta'$$

for $\lambda \in (\lambda_k - \overline{\varrho}, \lambda_k)$. At this point, recalling Propositions 6.10 and 6.13, all hypothesis of Theorem 2.10 in [77] are satisfied, and we have the existence of two non-trivial critical points w_1 and w_2 such that

$$J_\lambda(w_i) \in [\eta', \eta''] \quad (i = 1, 2).$$

The third critical point w_3 is a consequence of the classical Linking Theorem. Furthermore, from Lemma 6.17, choosing λ sufficiently close to λ_k, we can see that

$$J_\lambda(w_i) < \sup_{w \in E_{k+h}} J_\lambda(w) \leq J_\lambda(w_3), \quad (i = 1, 2)$$

proving that w_1, w_2, w_3 are distinct. □

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