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THEORY

On the Dynamical Behavior of Cellular Automata on Finite Groups

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ABSTRACT During the last few decades, significant efforts have been devoted to analyze the dynamical behavior of cellular automata (CAs) on cyclic groups, their Cartesian power (referred to as linear cellular automata), and on general abelian groups (referred to as additive cellular automata). Many fundamental properties describing the dynamical behavior of a system such as injectivity, surjectivity, sentitivity to the initial conditions, topological transitivity, ergodicity, positive expansivity, denseness of periodic orbits, and chaos have been fully characterized for these classes of cellular automata, i.e., the relation between the cellular automaton (CA) local rule and the CA global behavior was made explicit, being this task a challenging and important problem in CA general theory. A natural step forward leads to investigate the dynamical behavior of group cellular automata, i.e., cellular automata defined on (not necessarily abelian) finite groups. Despite the work recently carried out by some authors, none of the previously mentioned properties has yet been fully characterized in the case of general finite groups. In this paper, we study the dynamical behavior of cellular automata on a number of classes of finite groups such as simple, symmetric, alternating, dihedral, quaternion and decomposable groups and we provide exact characterizations for some of the above mentioned properties. To do this, in each of those classes, we focus our attention to the non-abelian scenarios. Some results are quite surprising because they show that the non-abelianness of the group imposes strong limitations on defining the local rule of the cellular automaton, making the class of group cellular automata very constrained. Finally, we also introduce a graph allowing one to build and study the local rules of any group cellular automaton.

INDEX TERMS Cellular automata, group cellular automata, dynamical behavior.

I. INTRODUCTION

Cellular automata (CAs) are formal models for complex systems that can be described as discrete time dynamical systems consisting of a regular lattice of variables which can take a value from a finite alphabet. The global state of a cellular automaton (CA), specified by the values of all the variables at a given time, evolves in synchronous discrete time steps according to a given local rule which updates the value of each single variable on the basis of those of the variables in its neighoborhood. CAs have been widely studied and find application in a number of disciplines (e.g., computer science, physics, mathematics, biology, chemistry) with different purposes (e.g., simulation of natural phenomena, pseudo-random number generation, image processing, analysis of universal model of computations, cryptography). CAs can display a rich and complex temporal evolution whose exact determination is in general very hard, if not impossible.

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In particular, many properties of the temporal evolution of general CAs are undecidable [6], [7], [21], [22], as, for instance, the non trivial properties of the limit set and the main dynamical properties such as sensitivity to the initial conditions, equicontinuity, topological transitivity, chaos, etc. Luckily, the undecidability issue can be tackled by imposing some constraints on the model. As it often happens - and this is the environment we deal with - the alphabet and the global updating map are constrained to be a group and an additive function, respectively. We stress that such requirements do not prevent such CAs at all from being successfully used for practical purposes. On the contrary, being able to exhibit most of the complex behaviors of general CAs, they are often exploited for designing many applications (see [25], [27], for instance). Furthermore, for significant subclasses of such CAs it was possible to make explicit the relation between the CA global behavior and the local rule, being this task a challenging and important problem in CA general theory (for an introduction to CA theory see [5], [19], [20]). To be clearer, the problem is precisely that of finding out the specific properties of a local rule that determine the main global behaviors. For example, for general CAs, the balance condition of the local rule influences the global surjectivity [20], as well as leftmost/rightmost permutativity gives rise to a global chaotic behavior [3] (for significant subclasses of CAs, the main global behaviors are even characterized in terms of properties of the local rule, as mentioned below).

In this paper, we just focus our attention to the class of group CAs (GCAs) where the alphabet is any finite (possibly non-abelian) group G and a group CA (GCA) is defined by a neighbor vector of length k together a local rule which is a homomorphism from G^k to G (for an introduction to group theory and CAs over algebraic structures, see [2] and [26], respectively). In this way, the GCA global updating rule turns out to be a continuous and shift-invariant endomorphism of $G^{\mathbb{Z}}$.

Many papers have addressed the case of GCAs on cyclic groups (isomorphic to $\mathbb{Z}/m\mathbb{Z}$), also known as linear CAs (LCAs) over $\mathbb{Z}/m\mathbb{Z}$ (see, for example, [4], [8], [16], [18], [23], [24]). Local rules of linear CAs on $\mathbb{Z}/m\mathbb{Z}$ take the form $f(a_1, \dots, a_k) = \lambda_1 a_1 + \dots + \lambda_k a_k$, where $\lambda_i, a_i \in \mathbb{Z}/m\mathbb{Z}$ and the operations are meant modulo m. For LCAs over $\mathbb{Z}/m\mathbb{Z}$, exact and efficiently computable characterizations of several global properties have been carried out in terms of conditions on the local rules, hence involving their coefficients and m. These properties are the fundamental ones describing the dynamical behavior of a system and they include surjectivity, injectivity, sensitivity to initial conditions, equicontinuity, topological transitivity, strong transitivity, ergodicity, Lyapunov exponents, topological entropy, positive expansivity, denseness of periodic orbits, and chaos.

Linear CAs over $(\mathbb{Z}/m\mathbb{Z})^n$ are a generalization of the above mentioned LCAs. Their local rules take the form $f(a_1, \dots, a_k) = M_1 a_1 + \dots + M_k a_k$ where M_i are matrices and a_i are now vectors of $(\mathbb{Z}/m\mathbb{Z})^n$. The investigation of LCAs over $(\mathbb{Z}/m\mathbb{Z})^n$ turns out to be much more challenging than the case $n = 1$. Indeed, one of the main obstacles in the analysis of such GCAs originates from the non-commutative nature of matrix multiplication. In spite of that, exact and efficiently computable characterizations of several among the previously mentioned properties have been carried out also for LCAs over $(\mathbb{Z}/m\mathbb{Z})^n$ (see, for example, [9], [10], [12], [13], [14], [15]). In [11], it has been proved how such characterizations can be exploited to decide the dynamical behavior of a class of GCAs that are an extension of LCAs over $(\mathbb{Z}/m\mathbb{Z})^n$, namely, Additive CA, i.e., GCAs over a finite abelian group. These results are non immediate consequences of the fact that any finite abelian group can be represented as a direct product of appropriate cyclic groups.

The aim of this paper is to extend the works conducted on LCAs and Additive CAs, i.e., GCAs on a finite abelian group, to the more general case of GCAs. Therefore, the focus is on GCAs on a non-abelian finite group. Clearly, the fundamental theorem of finite abelian group and the linearity of the local rule can be no longer exploited for providing results in the same direction as in the above mentioned previous works. So, although the goal is the same, i.e, trying to characterize the dynamical behavior, this can not be achieved in the same manner. Indeed, for GCAs there are very few results in the literature (see [1]) and none of the above mentioned properties have been characterized or efficiently computed.

The main results of this paper can be summarized as follows.

- We analyze the structure of GCA local rules providing an exact characterization (Theorem 1) and a number of additional properties they must satisfy (Corollary 1 and Theorems 2, 3, and 4).
- For the class of GCAs on non-abelian simple groups we prove that all of them are bijective, no of them can be strongly transitive or positively expansive, and we characterize topological transitivity (Theorem 5).
- We consider GCAs over non-abelian alternating groups, especially the alternating group A_4 (all the other alternating groups are simple) and we provide for GCAs over A_4 an exact characterization of surjectivity, injectivity, and topological transitivity, beside proving that no of them is strongly transitive or positively expansive (Theorem 7).
- For the class of GCAs on non-abelian symmetric groups we get the same scenario than GCAs over A_4 (Theorem 8).
- For the class of GCAs on the Quaternion group we show an exact characterization of surjectivity, injectivity, and topological transitivity, beside proving that no of them is strongly transitive or positively expansive (Theorem 11).
- We provide some preliminary results for the class of GCAs on Dihedral group (Theorem 9).

Finally, we introduce an important formal tool, namely, a graph called images graph, allowing one to visualize and understand the possible local rules of a GCA on a certain group G that can be built on the basis of all the endomorphisms of G itself. In particular, the vertexes of the images graph are all the possible distinct images of the non-trivial endomorphisms of G. We characterize the set of the endomorphisms defining a GCA local rule by a condition on the subgraph induced by their images (Proposition 1) and we provide a necessary condition expressed in terms of the images graph to build local rules giving rise to surjective GCAs (Theorem 12).

The rest of this paper is organized as follows. Section II contains basic definitions and notations. In Section III we analyze the general structure of GCA local rules. Section IV is devoted to the study of GCAs on simple groups. In Section V we show how to investigate the dynamical behavior of decomposable GCAs passing through the analysis of its direct indecomposable factors. Sections VI, VII, VIII, and IX deal with the study of GCAs on alternating, symmetric, dihedral and quaternion groups, respectively. In Section X, we introduce the images graph and we investigate the set of the endomorphisms defining a GCA local rule. Section XI contains some concluding remarks and a list of open questions.

II. BASIC DEFINITIONS

A. ABOUT GROUPS

A *finite group* G is a mathematical structure consisting of a finite set of elements along with an operation (we will use multiplication) satisfying the following conditions: the operation is associative and has an identity element $e \in G$, and every element $g \in G$ has an inverse element $g^{-1} \in G$.

Let G be a finite group. The *order* $ord(g)$ of an element $g \in G$ is the smallest natural $n > 0$ such that $g^n = e$, while the order of G is just its cardinality $|G|$. A set $H \subseteq G$ is a subgroup of G (denoted by $H \leq G$) if H forms a group with the same operation of G. The set $\{e\}$ is called the trivial group and is a subgroup of any group. A set $N \subseteq G$ is a *normal subgroup* of G (denoted by $H \lhd G$) if for all $g \in G$ and for all $n \in N$ it holds that $gng^{-1} \in N$. A non-trivial group G is simple if it has only two normal subgroups: the trivial group and G itself. According to this definition the trivial group is not *simple*. For any $g \in G$, the set $C_G(g) = \{x \in G : xg = gx\}$ is called the centralizer of g and it is nothing but the set of all elements of G that commute with g. The centralizer of a subset $S \subseteq G$ is the set $C_G(S) = \{x \in G : \forall g \in S, xg = gx\}$. The *center* of G is the set $Z(G) = \{z \in G : \forall g \in G, zg = gz\}$, i.e., the set of elements commuting with every element of G. Clearly, $C_G(G) = Z(G)$ and it is well-known that $Z(G)$ is an abelian and normal subgroup of G.

Given two finite groups G and H, a *group homomorphism* from G to H is a function $h : G \to H$ such that for all $g_1, g_2 \in G$ it holds that $h(g_1g_2) = h(g_1)h(g_2)$ where the group operations on the left side and on the right side of the equality are that of G and of H, respectively. If $G = H$ the homomorphism h is called group *endomorphism*. The image and the *kernel* of a homomorphism $h : G \to H$ are the sets $Img(h) = \{h(g) : g \in G\}$ and $Ker(h) = \{g \in G : h(g) = e\}$, respectively. It is well-known that the kernel of any group homomorphism is a normal subgroup of G and any endomorphism h of a finite group G is injective if and only if it is surjective if and only if $Ker(h) = \{e\}$. Any bijective endomorphism is called *automorphism*.

A subset $S \subseteq G$ is called generating set of G if every element of G can be written as a product of elements in S and their inverses. If G has generating set with only one element g then G is said to be *cyclic* (denoted by C_n, where $n = ord(G)$) and g is called *generator* of G.

A *permutation group* is a group G whose elements are permutations of $\{1, \dots, n\}$ and whose group operation is the composition of permutations in G. By Cayley's theorem, every group is isomorphic to some permutation group.

Let A, B be two subgroups of a group G. The product of A and B is $AB = \{ab : a \in A, b \in B\}$. A group G is said to be the *internal direct product* of two subgroups N_1 and N_2 if all the following conditions are satisfied: N_1 and N_2 are normal subgroups of G, $N_1 \cap N_2 = \{e\}$, and $N_1N_2 = G$. In that case, N_1 and N_2 are called direct factors of G. It is well-known that if G is the internal direct product of two its normal subgroups N_1, N_2 then G is isomorphic to the *external direct product* $N_1 \times N_2$. Moreover, if G is isomorphic to the external direct product $N_1 \times N_2$ then there exist two its normal subgroups N_1', N_2' such that G is the internal direct product of N_1' and N_2'. A non-trivial group is said to be directly indecomposable if it can not be expressed as an internal direct product of non-trivial subgroups, or, equivalently, it is not isomorphic to the external direct product of any two non-trivial groups. The Krull-Remak-Schmidt Theorem restricted to finite groups ensures that any finite group can be expressed as an internal direct product of finitely many directly indecomposable groups.

B. ABOUT CELLULAR AUTOMATA

Let G be a finite set. A CA *configuration* is any function from $\mathbb{Z}$ to G. Given a configuration $c \in G^{\mathbb{Z}}$ and any integer $i \in \mathbb{Z}$, the value of c in position i is denoted by $c(i)$. The set $G^{\mathbb{Z}}$ is as usual equipped with the standard Tychonoff distance d defined as

$$\forall c, c' \in G^{\mathbb{Z}},\ d(c, c') = \begin{cases} 0, & \text{if} \quad c = c', \\ 2^{-\min\{|j| \,:\, j \in \mathbb{Z},\, c(j) \neq c'(j)\}}, & \text{otherwise} \end{cases} .$$

A *CA* on G is any continuous (with respect to d) and shift commuting map $F : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ of $G^{\mathbb{Z}}$, where the shift map $\sigma : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ is defined as follows

$$\forall c \in G^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad \sigma(c)(i) = c(i-1) \ .$$

Any CA can be equivalently defined by means of a *local rule* $f : G^k \to G$ (see [20]) that is paired with an ordered integer vector $v \in \mathbb{Z}^k$ called the *neighbor vector*. Namely, a CA F based on (f, v) is defined as follows:

$$\forall c \in G^{\mathbb{Z}},\ \forall i \in \mathbb{Z} :\ F(c)(i) = f(c(i + v_1), \dots, c(i + v_k)) \ .$$

When G is a finite group, $G^{\mathbb{Z}}$ is a group too (along with the operation defined componentwise by the group operation of G) and a CA F is said to be a GCA if F is also an endomorphism of $G^{\mathbb{Z}}$. In that case, the local rule of F is a homomorphism $f : G^k \to G$ (see [2] for a proof as far as an arbitrary algebraic structure is concerned). We denote by $e^{\mathbb{Z}}$ the configuration having e as a value in every integer positions, i.e., $e^{\mathbb{Z}}$ is the identity element of the group $G^{\mathbb{Z}}$. The *kernel of a GCA* F is $Ker(F) = \{c \in G^{\mathbb{Z}} : F(c) = e^{\mathbb{Z}}\}$. A configuration $c \in G^{\mathbb{Z}}$ is said to be *finite* if the number of positions $i \in \mathbb{Z}$ such that $c(i) \neq e$ is finite.

A CA F is said to be injective (resp., surjective) simply if the map F is injective (resp., surjective). We recall that injective CAs are surjective and a CA is surjective iff every configuration has a finite and uniformly bounded number of pre-images [20].

A CA F is *topologically transitive*, or, simply *transitive* if for any pair of nonempty open subsets $U, V \subseteq G^{\mathbb{Z}}$ there exists a natural $t > 0$ such that $F^t(U) \cap V \neq \emptyset$, while it is said to be *strongly transitive* (a stronger condition) if for any nonempty open subset $U \subseteq G^{\mathbb{Z}}$ it holds that $\bigcup_{t\in\mathbb{N}} F^t(U) = X$. Transitive CAs are surjective as well as strongly transitive CAs but the latter are never injective.

A CA F is *sensitive to the initial conditions* if there exists $\epsilon > 0$ such that for any $\delta > 0$ and $c \in G^{\mathbb{Z}}$ there is a configuration $c' \in G^{\mathbb{Z}}$ with $0 < d(c', c) < \delta$ such that $d(F^t(c'), F^t(c)) \geq \epsilon$ for some natural t. Sensitivity is the well-known basic component and essence of the chaotic behavior of discrete time dynamical systems. Indeed, sensitivity, topological transitivity and dense periodic orbits are the features that together define the popular notion of *chaos* according to the Devaney definition (see [17]).

We recall that a CA F is *positively expansive* if for some constant $\varepsilon > 0$ it holds that for any pair of distinct configurations $c, c' \in G^{\mathbb{Z}}$ there exists a natural number t such that $d(F^t(c), F^t(c')) \geq \varepsilon$. We stress that CA positive expansivity is a condition of strong chaos. Indeed, on a hand, CA positive expansivity is a stronger condition than CA sensitivity. On the other hand, any positively expansive CA is also topologically transitive (even strongly transitive) and, at the same time, it has dense periodic orbits. Therefore, any positively expansive CA is chaotic according to the Devaney definition of chaos. Clearly, if a CA F is positively expansive then it is surjective but not injective.

A map $f : G^k \to G$ is said to *permutative in the variable* $i \in \{1, \dots, k\}$ iff for every $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k) \in G^{k-1}$ and every $b \in G$ there exists a unique $a \in G$ such that $f(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_k) = b$. CAs defined by permutative local rule have been studied in depth. In particular, CAs defined by a rightmost (resp., leftmost permutative) local rule and by a neighbor vector v such that $v_k > 0$ (resp., $v_k < 0$) turn out to be chaotic.

III. GCA LOCAL RULES

We now characterize the local rules of GCAs. In particular, we prove that for any GCA local rule, i.e., for any group homomorphism $f : G^k \to G$, it holds that f can be written as the multiplication (group operation) of k endomorphisms $h_1, \dots, h_k$ of G and the images of these endomorphisms must commute. More precisely, the following theorem holds.

Theorem 1: Consider any function $f : G^k \to G$. It holds that f is a group homomorphism if and only if there exist k endomorphisms $h_1, \dots, h_k$ of G such that both the following properties hold:

$$\forall (g_1, \dots, g_k) \in G^k,\ f(g_1, \dots, g_k) = h_1(g_1) \cdots h_k(g_k) \quad (1)$$

and

$$\forall i, j \in \{1, \dots, k\} \text{ with } i \neq j,\ Img(h_i) \subseteq C_G(Img(h_j)) \quad (2)$$

Proof: Assume that both properties (1) and (2) hold. Then, for every pair $(a_1, \dots, a_k), (b_1, \dots, b_k) \in G^k$ it holds that

$$\begin{aligned} &f((a_1, \dots, a_k)(b_1, \dots, b_k)) \\ &\quad = f(a_1b_1, \dots, a_kb_k) = \\ &\quad \overset{\text{by (1)}}{=} h_1(a_1b_1) \cdots h_k(a_kb_k) \\ &\quad = h_1(a_1)h_1(b_1) \cdots h_k(a_k)h_k(b_k) \\ &\quad \overset{\text{by (2)}}{=} h_1(a_1) \cdots h_k(a_k)h_1(b_1) \cdots h_k(b_k) \\ &\quad = f(a_1, \dots, a_k)f(b_1, \dots, b_k) \end{aligned}$$

Therefore, f is a group homomorphism from G^k to G.

For a sake of simplicity, we are going to prove the converse implication for $k = 3$. It is not hard to extend the proof to a generic k. Assume now that $f : G^3 \to G$ is a homomorphism. Then, for every $(a, b, c) \in G^3$ it holds that

$$\begin{aligned} f(a, b, c) &= f(aee, e\,be, eec) = f(a, e, e)f(e, b, e)f(e, e, c) \\ f(a, b, c) &= f(aee, e\,eb, ece) = f(a, e, e)f(e, e, c)f(e, b, e) \\ f(a, b, c) &= f(eae, bee, eec) = f(e, b, e)f(a, e, e)f(e, e, c) \\ f(a, b, c) &= f(eea, bee, ece) = f(e, b, e)f(e, e, c)f(a, e, e) \\ f(a, b, c) &= f(eae, eeb, cee) = f(e, e, c)f(a, e, e)f(e, b, e) \\ f(a, b, c) &= f(eea, e\,be, cee) = f(e, e, c)f(e, b, e)f(a, e, e) \end{aligned}$$

We now define the functions $h_1, h_2, h_3 : G \to G$ as follows: for any $\in G$,

$$\begin{aligned} h_1(g) &= f(g, e, e) \\ h_2(g) &= f(e, g, e) \\ h_3(g) &= f(e, e, g) \end{aligned}$$

Since f is a homomorphism, it follows that h_1, h_2, h_3 are endomorphisms of G. Combining all the previous equations, we get that for every $(a, b, c) \in G^3$ the following equalities are true:

$$\begin{aligned} f(a, b, c) &= h_1(a)h_2(b)h_3(c) \\ f(a, b, c) &= h_1(a)h_3(c)h_2(b) \\ f(a, b, c) &= h_2(b)h_1(a)h_3(c) \\ f(a, b, c) &= h_2(b)h_3(c)h_1(a) \\ f(a, b, c) &= h_3(c)h_1(a)h_2(b) \\ f(a, b, c) &= h_3(c)h_2(b)h_1(a) \end{aligned}$$

and, as a consequence, it also holds that

$$h_1(a)h_2(b)h_3(c) = h_1(a)h_3(c)h_2(b) = h_2(b)h_1(a)h_3(c) =$$
$$h_2(b)h_3(c)h_1(a) = h_3(c)h_1(a)h_2(b) = h_3(c)h_2(b)h_1(a)$$

that is, both the properties (1) and (2) are satisfied. □

Corollary 1: Let $f = (h_1, \ldots, h_k)$ be the local rule of any GCA on a finite non-abelian group G. It holds that the number of surjective h_i is at most one. In other terms, any GCA on a finite non-abelian group can be permutative in at most one variable.

Proof: For a sake of argument, assume that there exists two distinct surjective endomorphisms h_i and h_j defining f. Then, by property (2) applied to h_i and h_j, we get that $G = Z(G)$, contradicting that G is non abelian. □

From now on, any GCA local rule f will be identified as a k-tuple $f = (h_1, \ldots, h_k)$ of endomorphisms of G satisfying properties (1) and (2). Moreover, we will assume that for all $i \in \{1, \ldots k\}$, $Img(h_i) \neq \{e\}$.

We now define two classes of GCAs that will play a crucial role throughout this paper, namely shift-like and identity-like GCAs.

Definition 1 (Shift-like GCA): A GCA with local rule $f = (h_1, \ldots, h_k)$ is a shift-like GCA if and only if $k = 1$ and the neighbor vector is $\langle d \rangle \in \mathbb{Z}$, for some $d \neq 0$.

Definition 2 (Identity-like GCA): A GCA with local rule $f = (h_1, \ldots, h_k)$ is an identity-like GCA if and only if $k = 1$ and the neighbor vector is $\langle 0 \rangle$.

We stress that any shift-like or identity-like GCA is surjective if and only if its local rule is surjective if and only if its local rule is injective. Moreover, any surjective (injective) shift-like GCA is topologically transitive.

Theorem 2: Let $f = (h_1, \ldots, h_k)$ be the local rule of any GCA on a finite non-abelian group G. If there exists $I \subseteq \{1, \ldots, k\}$ such that $\Pi_{i \in I} Img(h_i) = G$ then for every $j \in \{1, \ldots, k\} \setminus I$ it holds that $Img(h_j) \subseteq Z(G)$.

Proof: Assume by contradiction that there exists $j \notin I$ such that $Img(h_j) \nsubseteq Z(G)$ and let $a \in Img(h_j)$ be an element such that $a \notin Z(G)$. So, there exists $b \in G$ such that $ab \neq ba$. Since $\Pi_{i \in I} Img(h_i) = G$, it holds that $b \in Img(h_i)$ for some $i \in I$ and, hence, by property (2) we get $ab = ba$, i.e., a contradiction. □

Theorem 3: Let F be any GCA on a finite group G and let $f = (h_1, \ldots, h_k)$ be its local rule. If

$$|Ker(h_1) \cap \cdots \cap Ker(h_k)| > 1$$

then F is not surjective.

Proof: Set $A = Ker(h_1) \cap \cdots \cap Ker(h_k)$. For any configuration $c \in A^{\mathbb{Z}}$ it holds that $F(c) = e^{\mathbb{Z}}$. Since $|A| > 1$, the configuration $e^{\mathbb{Z}}$ has an infinite number of pre-images and, hence, F is not surjective. □

Lemma 1: For any any group automorphism h of a finite group G both the following equalities hold: $h(Z(G)) = Z(G)$ and $h(G \setminus Z(G)) = G \setminus Z(G)$.

Proof: For every $a \in h(Z(G))$ and every $g \in G$ it holds that $ga = h(g')h(a') = h(g'a') = h(a'g') = h(a')h(g') = ag$, where $a' \in Z(G)$ and $g' \in G$ are the pre-images of a and g. Hence, $h(Z(G)) \subseteq Z(G)$. Since h is injective, it follows that $h(Z(G)) = Z(G)$ and $h(G \setminus Z(G)) = G \setminus Z(G)$. □

Lemma 2: Let F be any GCA on a finite non-abelian group G and let $f = (h_1, \ldots, h_k)$ be its local rule. If $Img(h_i) = G$ for some $i \in \{1, \ldots, k\}$ then both the following conditions hold: $F(Z(G)^{\mathbb{Z}}) \subseteq Z(G)^{\mathbb{Z}}$ and $F(G^{\mathbb{Z}} \setminus Z(G)^{\mathbb{Z}}) \subseteq G^{\mathbb{Z}} \setminus Z(G)^{\mathbb{Z}}$.

Proof: By Lemma 1 and Theorem 2 it holds that $h_i(Z(G)) = Z(G)$ and $Img(h_j) \subseteq Z(G)$ for every $j \neq i$. Therefore, $F(Z(G)^{\mathbb{Z}}) \subseteq Z(G)^{\mathbb{Z}}$. Consider now any configuration $c \in G^{\mathbb{Z}} \setminus Z(G)^{\mathbb{Z}}$ and let q be an integer such that $c(q) \notin Z(G)$. Set $g_1 = c(q - i + 1), \ldots, g_i = c(q), \ldots, g_k = c(q + k - i)$, and $g = f(g_1, \ldots, g_k) = h_1(g_1) \cdots h_k(g_k)$. It holds that $h_i(g_i) \notin Z(G)$ and $h_j(g_j) \in Z(G)$ for every $j \neq i$. Hence, g can be written as $g = xy$ for some $x \in Z(G)$ and $y \notin Z(G)$. Let $w \in G$ be such that $wy \neq yw$. We get $w(xy) = (wx)y = (xw)y = x(wy) \neq x(yw) = (xy)w$, proving that $g \notin Z(G)$. Therefore, $F(c) \notin Z(G)^{\mathbb{Z}}$. □

Lemma 3: Let F be any GCA on a finite non-abelian group G and let $f = (h_1, \ldots, h_k)$ be its local rule. If $Img(h_i) = G$ for some $i \in \{1, \ldots, k\}$ then $Ker(F) \subseteq Z(G)^{\mathbb{Z}}$.

Proof: It directly follows from Lemma 2. □

Theorem 4: Let F be any GCA on a finite non-abelian group G and let $f = (h_1, \ldots, h_k)$ be its local rule. If $Img(h_i) = G$ for some $i \in \{1, \ldots, k\}$ then the following facts hold:

(*i*) *F is surjective (resp., injective) iff F_Z is surjective (resp., injective), where F_Z is the GCA F restricted to $Z(G)^{\mathbb{Z}}$;*

(*ii*) *if at least one of the two following conditions holds then F is either a bijective shift-like or a bijective identity-like GCA:*
- every endomorphism of G is either surjective or trivial;
- $Z(G) = \{e\}$.

(*iii*) *F is neither strongly transitive nor positively expansive.*

Proof: (*i*): First of all, Lemma 2 ensures that F_Z is actually a GCA. It is well-known that any GCA is surjective iff its kernel is finite, and in particular it is injective iff its kernel is a singleton. By Lemma 3 it follows that $Ker(F) = Ker(F_Z)$ and, hence, the statement is true.

(*ii*): If every endomorphism of G is either surjective or trivial then, by Corollary 1 and since h_i is surjective for some $i \in \{1, \ldots, k\}$, it follows that $i = k = 1$, i.e., F is either a shift-like or an identity-like GCA. Moreover, F is also bijective since its local rule is defined by a unique surjective endomorphism of G. The same fact happens if $Z(G) = \{e\}$. Indeed, since h_i is surjective for some $i \in \{1, \ldots, k\}$, by Theorem 2, it follows again that $i = k = 1$ and, so, F is either a shift-like or an identity-like GCA. Hence (*ii*) is true.

(*iii*): Consider a cylinder containing only configurations from $G^{\mathbb{Z}} \setminus Z(G)^{\mathbb{Z}}$. By Lemma 2 we know that $F(Z(G)^{\mathbb{Z}}) \subseteq Z(G)^{\mathbb{Z}}$ and $F(G^{\mathbb{Z}} \setminus Z(G)^{\mathbb{Z}}) \subseteq G^{\mathbb{Z}} \setminus Z(G)^{\mathbb{Z}}$. Hence, for every configuration c belonging to that cylinder and any $m \in \mathbb{N}$ it holds that $F^m(c) \neq e^{\mathbb{Z}}$, i.e., F is not strongly transitive. As a consequence, F is neither positively expansive. □

We wish to emphasize that $Z(G)$ is an abelian group and then F restricted to $Z(G)^{\mathbb{Z}}$ is an Additive CA and as such it can be easily analyzed.

IV. GCAs ON SIMPLE GROUPS

Any simple group has no proper non-trivial normal subgroups. Simple finite groups are indecomposable (the converse is not true) and the finite simple abelian groups are exactly the cyclic groups of prime order.

The classification of finite simple groups is a monumental achievement in group theory. It refers to a theorem stating that every finite simple group belongs to one of a few specific families or is one of a finite number of exceptional cases called sporadic groups. That theorem, completed in the 20th century after decades of collaborative effort by numerous mathematicians, provides a comprehensive understanding of the structure of finite simple groups. Namely, according to the classification of finite simple groups, every finite simple group is either a cyclic group of prime order, or an alternating group of degree at least 5, or a group belonging to one of 16 infinite families of Lie type groups, or, finally, one of 26 sporadic groups.

Since any simple group G has no non-trivial proper normal subgroup, for every endomorphism h of G it holds that either $Ker(h) = \{e\}$ or $Ker(h) = G$, i.e., in other terms, every endomorphism of G is either surjective or trivial. As a consequence, GCAs on a finite non-abelian simple group behave as stated in the following.

Theorem 5: Let F be a GCA on a finite non-abelian simple group. The following facts hold

(*i*) *F is either a bijective shift-like or a bijective identity-like GCA;*

(*ii*) *F is topologically transitive iff the GCA neighbor vector is $\langle d\rangle$ with $d \neq 0$;*

(*iii*) *F is neither strongly transitive nor positively expansive.*

Proof: Let $f = (h_1, \ldots, h_k)$ be the local rule of F. Since every endomorphism of a simple group is either surjective or trivial and every h_i is not trivial, the thesis follows from Theorem 4 and the fact that identity-like CAs are not topologically transitive. □

V. GCAs ON DECOMPOSABLE GROUPS

We now deal with GCAs on a finite group G which is the internal direct product of two normal subgroups G_1 and G_2, or, equivalently, $G \cong G_1 \times G_2$. Let F be a GCA on G and let $f = (h_1, \ldots, h_k)$ be its local rule. Then, $F \cong F_1 \times F_2$, where F_1 and F_2 are the two GCAs of F on G_1 and G_2, induced by the projections of G on G_1 and G_2, respectively.

The dynamical behavior of F can be studied by separately examining the behaviors of F_1 and F_2. Namely, F turns out to be surjective/injective/topologically transitive/strongly transitive/positively expansive if and only if both F_1 and F_2 are, too, while F is sensitive to the initial conditions if and only if at least one between F_1 and F_2 is sensitive to the initial conditions.

As an example, consider now the Pyritohedral group T_h. T_h is isomorphic to the direct product of C_2 and A_4, or, equivalently, $T_h = C_2A_4$. A possible set of generators (given as permutations) for T_h is $\{(2, 3, 4), (1, 2)(3, 4)(5, 6)\}$. A more suitable alternative choice of generator set is $\{(1, 2, 3), (1, 3, 4), (5, 6)\}$. Indeed, the subgroup A of T_h generated by $\{(1, 2, 3), (1, 3, 4)\}$ is isomorphic to A_4 and the subgroup B of T_h generated by $\{(5, 6)\}$ is isomorphic to C_2. Moreover, A and B are normal subgroups of T_h and their intersection is equal to $\{e\}$. This ensures that $T_h = AB = BA$. We also stress that A and B are defined on two sets of distinct elements, namely $\{1, 2, 3, 4\}$ for A and $\{5, 6\}$ for B.

Hence, the dynamics of any GCA F on the pyritohedral group T_h can be decomposed into the dynamics of a GCA F_1 on A_4 and the dynamics of a GCA F_2 on C_2. The dynamical behavior of GCAs on A_4 is analyzed in Section VI while that of GCAs on C_2 has been deeply studied in the literature (see for example [24]). To make things clearer, let us consider the following specific example.

Example 1: Denote by 0_G the trivial endomorphism of a group G. Let h_1 and h_2 be any two endomorphisms of $A \cong A_4$ and $B \cong C_2$, respectively. Let F_1 and F_2 be the GCAs on A and B having $f_1 = (h_1, 0_A)$ and $f_2 = (0_B, h_2)$ as local rules, respectively, and with the same neighbor vector $v = \langle -1, 1\rangle$ (0_A and 0_B have been artificially added to h_1 and h_2 just in such a way that F_1 and F_2 have the same neighbor vector).

Since every element of $g \in T_h$ can be written in a unique way as product g_1g_2 where $g_1 \in A$ and $g_2 \in B$, we can combine f_1 and f_2 as follows in order to obtain the local rule f of a GCA on T_h. Let $\pi_1 : T_h \to A$ and $\pi_2 : T_h \to B$ be the two projection maps defined as usual by $\forall g_1 \in A, \forall g_2 \in B, \pi_1(g_1g_2) = g_1$ and $\pi_2(g_1g_2) = g_2$. Consider the endomorphism $f : T_h^2 \to T_h$ such that for any pair $(a, b) \in T_h^2$

$$\begin{aligned} f(a, b) &= f_1(\pi_1(a), \pi_1(b))f_2(\pi_2(a), \pi_2(b)) \\ &= h_1(\pi_1(a))\, h_2(\pi_2(b)) \\ &= h_1'(a)\, h_2'(b) \end{aligned}$$

where h_1' and h_2' are the two endomorfisms of T_h defining the local rule f. Let F be the GCA on T_h having f as local rule. Clearly, $F \cong F_1 \times F_2$ and the dynamics of F can be decomposed into the dynamics of F_1 and F_2.

Making reference to Example 1, we stress that if both h_1 and h_2 are surjective endomorphisms of A and B, respectively, then F is surjective. Note that h_1' and h_2' are not surjective on T_h but F is a surjective GCA. Indeed, this is the first example of a GCA local rule only defined by non surjective endomorphisms, but giving rise to a surjective GCA. The way used in Example 1 can be exploited in general to build a GCA satisfying a given property, even though the endomorphisms defining its local rule, when considered individually, give rise to a GCA which does not satisfy it.

The same reasoning applied to T_h can be repeated for a number of other decomposable finite groups. To name a few, the Octahedral group O_h is isomorphic to $S_4 \times C_2$, the

TABLE 1. Basic information about quaternion, pyritohedral, octahedral and icosahedral groups and their direct factors C_2, S_4, A_4 and A_5.

	Order	*Abelian*	*Simple*	*Center*	Decomposition
Quaternion	8	No	No	isomorphic to C_2	indecomposable
Pyritohedral	24	No	No	isomorphic to C_2	$A_4 \times C_2$
Octahedral	48	No	No	isomorphic to C_2	$S_4 \times C_2$
Icosahedral	120	No	No	isomorphic to C_2	$A_5 \times C_2$
C_2	2	Yes	Yes	isomorphic to C_2	indecomposable
S_4	24	No	No	Trivial	indecomposable
A_4	12	No	No	Trivial	indecomposable
A_5	60	No	Yes	Trivial	indecomposable

Icosahedral group I_h is isomorphic to $A_5 \times C_2$ and Dihedral groups D_{2k} with $k \geq 3$ and k odd are isomorphic to $D_k \times C_2$ (see Table 1).

VI. GCAs ON ALTERNATING GROUPS A_N

An alternating group is the group of the even permutations of a finite set. When the latter has n elements the alternating group is denoted by A_n. It is well-known that A_n is abelian if and only if $n \leq 3$ and A_n is simple if and only if $n = 3$ or $n \geq 5$. Hence, the normal subgroups of A_n are $\{e\}$ and A_n for $n = 3$ or $n \geq 5$. The normal subgroups of A_4 are $\{e\}$, A_4, and $\{e, (12)(34), (13)(24), (14)(23)\}$ (isomorphic to $C_2 \times C_2$). Regarding the center of the alternating groups, $Z(A_n) = A_n$ if $n \leq 3$, and $Z(A_n) = \{e\}$, otherwise. We address the reader to Table 2 for the basic properties of alternating groups.

Theorem 6: Let F be a GCA on a A_n with $n \geq 5$. The following facts hold

(*i*) *F is either a bijective shift-like or a bijective identity-like GCA;*

(*ii*) *F is topologically transitive iff the GCA neighbor vector is $\langle d \rangle$ with $d \neq 0$;*

(*iii*) *F is neither strongly transitive nor positively expansive.*

Proof: Since A_n is simple the thesis immediately follows from Theorem 5. □

Theorem 7: Let F be a GCA on A_4. The following facts hold:

(*i*) *F is surjective iff F is injective iff F is either a bijective shift-like or a bijective identity-like GCA;*

(*ii*) *F is topologically transitive iff F is a bijective shift-like GCA;*

(*iii*) *F is neither strongly transitive nor positively expansive.*

Proof: Let $f = (h_1, \ldots, h_k)$ be the local rule of F. By Corollary 1, at most one among the k endomorphisms $h_1, \ldots, h_k$ is surjective. We deal with the following two mutually exclusive cases. If h_i is surjective for some (unique) i, i.e., $Img(h_i) = A_4$, since $Z(A_4) = \{e\}$, by Theorem 4, it follows that (*i*) is true. Let us now consider the case in which all the endomorphisms $h_1, \ldots, h_k$ are not surjective. We know that for every $1 \leq i \leq k$, $Ker(h_i)$ is equal to either $\{e\}$ or A_4 or $\{e, (12)(34), (13)(24), (14)(23)\}$. Since h_i is neither surjective nor trivial, necessarily it holds that $Ker(h_i) = \{e, (12)(34), (13)(24), (14)(23)\}$. Hence, by Theorem 3, it follows that F is not surjective and then (*i*) is true.
(*ii*): Since identity-like GCAs are not topologically transitive and topologically transitive CAs are surjective, (*ii*) follows from (*i*).
(*iii*): it follows from (*i*) and the fact that strongly transitive CAs are surjective but not injective. □

We now give an example of a GCA on A_4 that is neither shift-like nor identity-like GCA and then, in view of Theorem 7, it is not surjective.

Example 2: Let h_1 and h_2 be the two non trivial endomorphisms of A_4 defined by setting the images of the generators of A_4 as follows.

$$h_1((1, 2, 3)) = (2, 3, 4) \quad h_1((2, 3, 4)) = (2, 4, 3)$$
$$h_2((1, 2, 3)) = (2, 4, 3) \quad h_2((2, 3, 4)) = (2, 3, 4)$$

It is not hard to verify that property (2) is satisfied as far as h_1 and h_2 are concerned, or, in other words, we can state that $f = (h_1, h_2)$ is the local rule of a GCA F on A_4. Clearly, is F neither a shift-like nor an identity-like GCA.

VII. GCAs ON SYMMETRIC GROUPS S_N

A symmetric group, denoted by S_n, is a group consisting of all permutations of a set with n elements. In other words, it is the group of all bijective functions from a set of n elements to itself, where the operation is function composition.

Symmetric groups are fundamental objects in group theory and have applications in various fields, including cryptography (e.g., in the design of cryptographic algorithms), combinatorics (e.g., in counting and enumeration problems), and theoretical computer science (e.g., in the analysis of algorithms and computational complexity).

If S_n is a symmetric group then its order is $n!$ and it is generated by $(1, 2, \ldots, n)$ and $(1, 2)$. Every group is isomorphic to a subgroup of a symmetric group. This result, known as Cayley's theorem, provides a significant connection between arbitrary groups and symmetric groups. See Table 3 for the basic properties of symmetric groups.

It is well-known that a symmetric group S_n is non-abelian if and only if $n \geq 3$ and, as far as the center is concerned, it holds that $Z(S_n) = S_n = \{e, (1, 2)\}$ if $n = 2$, and $Z(S_n) = \{e\}$, otherwise. The normal subgroups of S_n are $\{e\}$, A_n and S_n for all natural $n \geq 3$ with $n \neq 4$, while those of S_4 are $\{e\}$, A_4, S_4, and $\{e, (12)(34), (13)(24), (14)(23)\}$.

Theorem 8: Let F be a GCA on S_n with $n \geq 3$. The following facts hold:

(*i*) *F is surjective iff F is injective iff F is either a bijective shift-like or a bijective identity-like GCA;*

TABLE 2. Basic information about alternating groups: A_n.

	Order	*Abelian*	*Simple*	*Center*	Decomposition
$n = 1$	1	Yes	No	isomorphic to C_1	indecomposable
$n = 2$	1	Yes	No	isomorphic to C_1	indecomposable
$n = 3$	3	Yes	Yes	isomorphic to C_3	indecomposable
$n = 4$	12	No	No	Trivial	indecomposable
$n \geq 5$	$\frac{n!}{2}$	No	Yes	Trivial	indecomposable

TABLE 3. Basic information about symmetric groups: S_n.

	Order	*Abelian*	*Simple*	*Center*	Decomposition
$n = 1$	1	Yes	No	Trivial	indecomposable
$n = 2$	2	Yes	Yes	isomorphic to C_2	indecomposable
$n \geq 3$	$n!$	No	No	Trivial	indecomposable

(ii) *F is topologically transitive iff F is a bijective shift-like GCA;*

(iii) *F is neither strongly transitive nor positively expansive.*

Proof: We deal with the two cases $n \neq 4$ and $n = 4$. If $n \neq 4$, since the normal subgroups of S_n are $\{e\}$, A_n and S_n, for any endomorphism h of S_n which is nether surjective nor trivial it holds that $Ker(h) = A_n$. If $n = 4$, for any endomorphism h of S_4 which is nether surjective nor trivial it holds that either $Ker(h) = A_4$ or $Ker(h) = \{e, (12)(34), (13)(24), (14)(23)\}$, a condition ensuring that the intersection of all the kernels of such endomorphisms contains $\{e, (12)(34), (13)(24), (14)(23)\}$. Hence, in both cases the same arguments used in the proof of Theorem 7 lead to the thesis. □

There also exist non transitive but sensitive GCAs on S_4. We now exhibit an example of such a GCA on S_4 that, in view of Theorem 8 and since identity-like CAs are not sensitive, is not surjective.

Example 3: Let h_1 and h_2 be the two endomorphisms of S_4 defined by setting the images of the generators of S_4 as follows.

$$h_1((1, 2, 3, 4)) = h_1((1, 2)) = (1, 4)(2, 3)$$
$$h_2((1, 2, 3, 4)) = h_2((1, 2)) = (1, 4)$$

It holds that property (2) is satisfied as far as h_1 and h_2 are concerned. Let F be the GCA on S_4 based on (f, v), where $f = (h_1, h_2)$ and $v = \langle -2, -1\rangle$. It is clear that F is neither a shift-like nor an identity-like GCA. We prove that F is sensitive to initial conditions. For any integer ℓ, let $c^*_\ell \in S_4^{\mathbb{Z}}$ be the configuration defined as follows:

$$c^*_\ell(i) = \begin{cases} (2,3) & \textit{if } i < \ell - 1, \\ (1,4)(2,3) & \textit{if } i = \ell - 1, \\ e & \textit{otherwise.} \end{cases}$$

Since $h_1((1, 4)(2, 3)) = h_2((1, 4)(2, 3)) = e$, $h_1((2, 3)) = (1, 4)(2, 3)$ and $h_2((2, 3)) = (1, 4)$, it holds that $F(c^*_\ell) = c^*_{\ell+1}$ (see Table 4), a condition ensuring that F is sensitive to the initial conditions. Indeed, for any $c \in S_4^{\mathbb{Z}}$ and any natural m, consider the configuration $c' \in S_4^{\mathbb{Z}}$ such that $c' = c^*_{-m} c$. Clearly, $c'(-m - 1) \neq c(-m - 1)$, while $c'(i) = c(i)$ for all integers $i \geq -m$. So, $0 < d(c', c) < 2^{-m}$. Furthermore, it holds that $F^{m+1}(c')(0) = [F^{m+1}(c^*_{-m}) F^{m+1}(c)](0) = [c^*_1 F^{m+1}(c)](0) \neq F^{m+1}(c)(0)$, and, hence, $d(F^{m+1}(c'), F^{m+1}(c)) \geq 1$.

VIII. GCAs ON DIHEDRAL GROUPS D_N

A dihedral group is the group of symmetries of a regular polygon, which includes rotations and reflections. Dihedral groups are among the simplest examples of finite groups, and they play an important role in group theory, geometry, and chemistry. The notation for the dihedral group differs in geometry and abstract algebra. In geometry, D_n refers to the symmetries of the n-gon, a group of order $2n$. In abstract algebra, D_{2n} refers to this same dihedral group. Here we use the geometric notation. It holds that $D_1 \cong C_2$, $D_2 \cong D_1 \times D_1 \cong C_2 \times C_2$, and D_3, D_4, and D_5 are indecomposable. For $n \geq 6$, D_n is decomposable if and only if $n = 2k$ where k is odd and $D_n \cong C_2 \times D_{\frac{n}{2}}$ or, equivalently, $D_{2k} \cong C_2 \times D_k$. Regarding the center, it holds that $Z(D_n) = \{e\}$, if n is odd, $Z(D_n) \cong C_2$, otherwise. See Table 5 for the basic properties of dihedral groups.

Fact 1: Let n be any odd integer. Let $g \in D_n$ such that $g^2 = e$ (flip element). Then $C_{D_n}(g) = \{e, g\}$.

Theorem 9: Let F be a GCA F on D_n where n is any odd natural with $n > 1$ and let $f = (h_1, \ldots, h_k)$ be its local rule. One of the following two facts happen:

(F1) *$k = 1$ and F is a bijective shift-like or a bijective identity-like GCA;*

(F2) *$k > 1$, $h_1 = \cdots = h_k$, and $Img(h_i) \cong D_1$ for each endomorphism h_i.*

Proof: If h_i is surjective for some i then, since $Z(D_n)$ is trivial, by Theorem 4, it follows that fact (F1) is true.

Otherwise, i.e., if every endomorphism h_i is neither surjective neither trivial, it holds that $Ker(h_i) \neq D_n$. Now, it is well known that if $N \neq D_n$ is a normal subgroup of D_n then D_n/N is isomorphic to the dihedral group $D_{n/|N|}$. Hence, we get that $D_n/Ker(h_i) \cong D_{n/|Ker(h_i)|}$. By the first isomorphism theorem, it follows that $Img(h_i) \cong D_n/Ker(h_i)$ and so $Img(h_i) \cong D_{n/|Ker(h_i)|}$. Since n is odd, $n/|Ker(h_i)|$ is odd, too, and clearly either $n/|Ker(h_i)| = 1$ (i.e., $|Ker(h_i)| = n$) or $n/|Ker(h_i)| \geq 3$ holds. We are now going to show that

TABLE 4. Evolution of c^*_ℓ under iterations of the GCA on S_4 with local rule $f = (h_1, h_2)$.

$c^*_\ell =$	$\cdots$	$(2,3)$	$(2,3)$	$(1,4)(2,3)$	e	e	e	$\cdots$
$F(c^*_\ell) =$	$\cdots$	$(2,3)$	$(2,3)$	$(2,3)$	$(1,4)(2,3)$	e	e	$\cdots$
$F^2(c^*_\ell) =$	$\cdots$	$(2,3)$	$(2,3)$	$(2,3)$	$(2,3)$	$(1,4)(2,3)$	e	$\cdots$

TABLE 5. Basic information about dihedral groups: D_n.

	Order	*Abelian*	*Simple*	*Center*	Decomposition
$n = 1$	2	Yes	No	Trivial	indecomposable
$n = 2$	4	Yes	No	isomorphic to C_2	$D_2 \cong C_2 \times C_2$
$n = 3$	6	No	No	Trivial	indecomposable
$n = 4$	8	No	No	isomorphic toC_2	indecomposable
$n = 5$	10	No	No	Trivial	indecomposable
$n \geq 6$ and $n \; mod \; 4 = 0$	$2n$	No	No	isomorphic to C_2	indecomposable
$n \geq 6$ and $n \; mod \; 4 = 1$	$2n$	No	No	Trivial	indecomposable
$n \geq 6$ and $n \; mod \; 4 = 2$	$2n$	No	No	isomorphic to C_2	$D_n \cong C_2 \times D_k$
$n \geq 6$ and $n \; mod \; 4 = 3$	$2n$	No	No	Trivial	indecomposable

necessarily $Img(h_i) \cong D_1$ for every i and this fact implies that all the endomorphisms h_i coincide, i.e., (F2) is true.

First of all, we can state that the images of all the endomorphisms coincide. Indeed, if this does not happen, i.e., if $Img(h_j) \neq Img(h_l)$ for some distinct j and l then, by Fact 1, the flip elements of $Img(h_j)$ and $Img(h_l)$ do not commute, and this contradicts property (2). Moreover, if for some h_i it holds that $Img(h_i) \cong D_\alpha$ with $\alpha \geq 3$, then $Img(h_i) \cong D_\alpha$ holds for every h_i since the images of all the endomorphisms coincide. So, being D_α non abelian, we get that $Img(h_i)$ does not commute with itself and this violates property (2). Therefore, $Img(h_i) \cong D_1$ for every h_i.

To conclude, we prove that $Img(h_j) = Img(h_l)$ implies $h_j = h_l$. Assume that $Img(h_j) = Img(h_l)$ for any pair of distinct j and l and set $A = Img(h_j) = Img(h_l) = \{e, a\}$, where a is a flip element. Since every element of D_n has either order 2 or odd order and $ord(h(g))|ord(g)$ for any group endomorphism h and any group element g, for every $g \in D_n$ we get that $h_i(g) = h_j(g) = a$ if $ord(g) = 2$, and $h_i(g) = h_j(g) = e$ if $ord(g)$ is odd, i.e., $h_j = h_l$. Therefore, all the endomorphisms h_i coincide, i.e., (F2) is true. □

Theorem 10: Let F be a GCA on D_n where n is any odd natural with $n \geq 3$. The following facts hold:

(*i*) *F is surjective iff F is injective iff F is either a bijective shift-like or a bijective identity-like GCA;*

(*ii*) *F is topologically transitive iff F is a bijective shift-like GCA;*

(*iii*) *F is neither strongly transitive nor positively expansive.*

Proof: By Theorem 9, when fact (F1) happens it trivially follows that (*i*) is true. The same holds also when (F2) occurs since F is not surjective. Furthermore, (*ii*) and (*iii*) follows from (*i*). □

Regarding GCAs over D_n where n is even, Theorems 4 and 9 are no longer true, as illustrated in the following example.

Example 4: Let h_1 and h_2 be the two non trivial endomorphisms of D_6 defined by setting the images of the generators of D_6 as follows.

$$h_1((1,2,3,4,5,6)) = (1,6,5,4,3,2)$$
$$h_1((1,6)(2,5)(3,4)) = (2,6)(3,5)$$

$$h_2((1,2,3,4,5,6)) = e$$
$$h_1((1,6)(2,5)(3,4)) = (1,4)(2,5)(3,6)$$

It is not hard to verify that $f = (h_1, h_2)$ is the local rule of a GCA F on D_6. Moreover, $Img(h_1) = D_6$ and $Img(h_2) \neq \{e\}$.

Consider now any GCA on D_{2k} where k is an odd natural with $k \geq 3$. We know that $D_{2k} \cong C_2 \times D_k$ and then F can be decomposed as the product of two GCAs F_1 on C_2 and F_2 on D_k. In this way, set theoretic and dynamical properties of GCA on D_{2k} can be derived from the analysis of the two GCAs F_1 and F_2 (see Section V for details).

IX. GCAs ON QUATERNION GROUP Q_8

The quaternion group denoted by Q_8 has the unusual property of being Hamiltonian, i.e., Q_8 is not abelian, but every subgroup of Q_8 is normal. Q_8 has order 8 and a generator set is $\{(1,2,5,6)(3,4,7,8), (1,3,5,7)(2,8,6,4)\}$. Let $\pi = (1,5)(2,6)(3,7)(4,8)$. It holds that $Z(Q_8) = \{e, \pi\}$.

There are 28 endomorphisms of Q_8, 4 of them are non-surjective. If h is any non-surjective endomorphism of Q_8 then $h(\pi) = e$ and $Img(h)$ is either $\{e\}$ or $\{e, \pi\}$. Regarding any surjective endomorphisms h of Q_8 h it holds that $h(\pi) = \pi$.

Let $h_1, \ldots, h_k$ be k any endomorphisms of Q_8. It is easy to show that they satisfy property (2) and so $f = (h_1, \ldots, h_k)$ is a local rule of a GCA on Q_8.

Lemma 4: Let F be a GCA on Q_8. Let $f = (h_1, \ldots, h_k)$ and $v = \langle v_1, \ldots, v_k \rangle$ be its local rule and neighbor vector, respectively, such that there exists a surjective h_i. The following facts hold:

(1) *F is bijective and F_Z is either a CA shift or the CA identity;*

(2) *if $v_i = 0$ then for every finite configuration c there exists $m \geq 1$, such that $F^m(c) = c$.*

$c =$	$\cdots$	e	e	e	e	e	●	$\cdots$	●	e	e	e	e	e	$\cdots$
$F(c) =$	$\cdots$	e	e	e	○	○	●	$\cdots$	●	○	e	e	e	e	$\cdots$
$F^2(c) =$	$\cdots$	e	e	e	○	○	●	$\cdots$	●	○	e	e	e	e	$\cdots$
$F^3(c) =$	$\cdots$	e	e	e	○	○	●	$\cdots$	●	○	e	e	e	e	$\cdots$
$F^4(c) =$	$\cdots$	e	e	e	○	○	●	$\cdots$	●	○	e	e	e	e	$\cdots$
$F^5(c) =$	$\cdots$	e	e	e	○	○	●	$\cdots$	●	○	e	e	e	e	$\cdots$

FIGURE 1. Evolution of a GCA F on Q_8 having $f = (h_1, h_2, h_3)$ as local rule and $v = \langle -1, 0, 2\rangle$ as neighbor vector. The endomorphism h_2 is surjective, while the endomorphisms h_1 and h_3 are not. The initial configuration c is finite and in it ● represents any element of Q_8 while ∘ represents any element of $Z(Q_8)$.

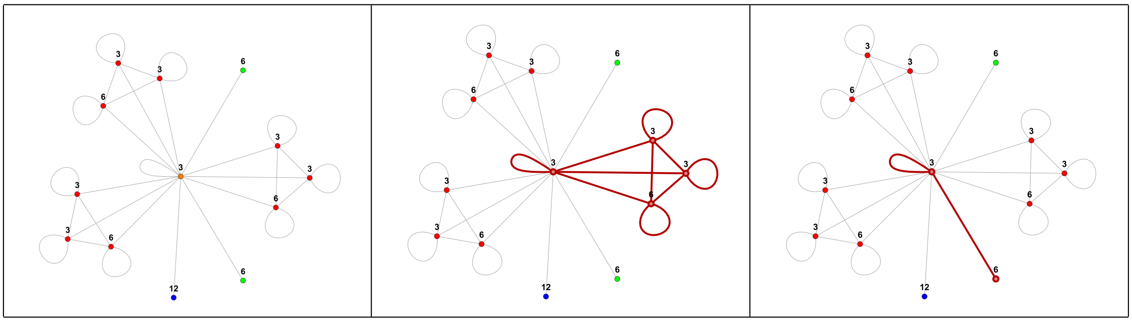

FIGURE 2. The image on the left shows the images graph $IG(D_6)$. The images in the center and on the right show two cliques (red subgraphs) of $IG(D_6)$.

Proof: Since h_i is surjective for some i, by Corollary 1 it follows that for every $j \neq i$, h_j is not surjective and so $h_j(\pi) = e$. In other terms, it holds that for every $j \neq i$ the endomorphism h_j is trivial when restricted to $Z(Q_8)$, while h_i is the identity over Q_8. Thus, the GCA F_Z is just either the CA shift map or the CA identity. In particular, F_Z is bijective and so, by Theorem 4, F is bijective, too. Hence, (1) is true.

Now, also suppose that $v_i = 0$ and consider any finite configuration c. Let $r > 0$ be such that for every $j \in \mathbb{Z}$ with $|j| > r$, $c(j) = e$. Then, the following claims are true for every $m \geq 1$ (see Figure 1 for an example):

1. for every $j \in \{-r - v_k, \ldots, -r - 1\}$ it holds that $F^m(c)(j) \in \mathbb{Z}(Q_8)$ and, hence, it follows that $F^m(c)(j) = e$ for every $j < -r - v_k$;
2. for every $j \in \{r + 1, \ldots, r - v_1\}$ it holds that $F^m(c)(j) \in \mathbb{Z}(Q_8)$ and, hence, it follows that $F^m(c)(j) = e$ for every $j > r - v_1$.

Since by (1) F is injective, claims 1. and 2. imply that there exists $m \geq 1$ such that $F^m(c) = c$, i.e., (2) is true. □

Theorem 11: Let F be a GCA on Q_8. Let $f = (h_1, \ldots, h_k)$ and $v = \langle v_1, \ldots, v_k\rangle$ be its local rule and neighbor vector, respectively. The following facts hold:

(i) F is surjective iff F is injective iff at least one h_i is surjective iff F_Z is either a CA shift or the CA identity;
(ii) F is topologically transitive iff F is bijective and there exists at least one surjective h_i such that $v_i \neq 0$;
(iii) F is neither strongly transitive nor positively expansive.

Proof: (i): By Corollary 1, at most one among the k endomorphisms $h_1, \ldots, h_k$ is surjective. We deal with the following two mutually exclusive cases. If h_i is surjective for some i, by Lemma 4 it follows that (i) is true. On the other hand, if all the endomorphisms $h_1, \ldots, h_k$ are not surjective, since $Ker(h) \supseteq \{e, \pi\}$ for any non surjective endomorphism h of Q_8, by Theorem 3, it follows that F is not surjective and then (i) is true.

(ii): Let F' be the GCA obtained by F just adding $-v_i$ to each component of the neighbor vector v. By Lemma 4, every finite configuration is periodic for F'. This implies that F is topologically transitive.

(iii): it follows from (i) and the fact that strongly transitive CAs are surjective but not injective. □

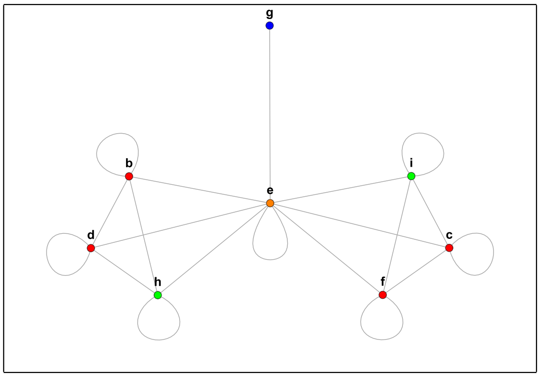

FIGURE 3. Images graph of the Dihedral group D_4.

X. IMAGES GRAPHS

We now introduce the following formal tool, namely a graph, allowing one to identify the possible local

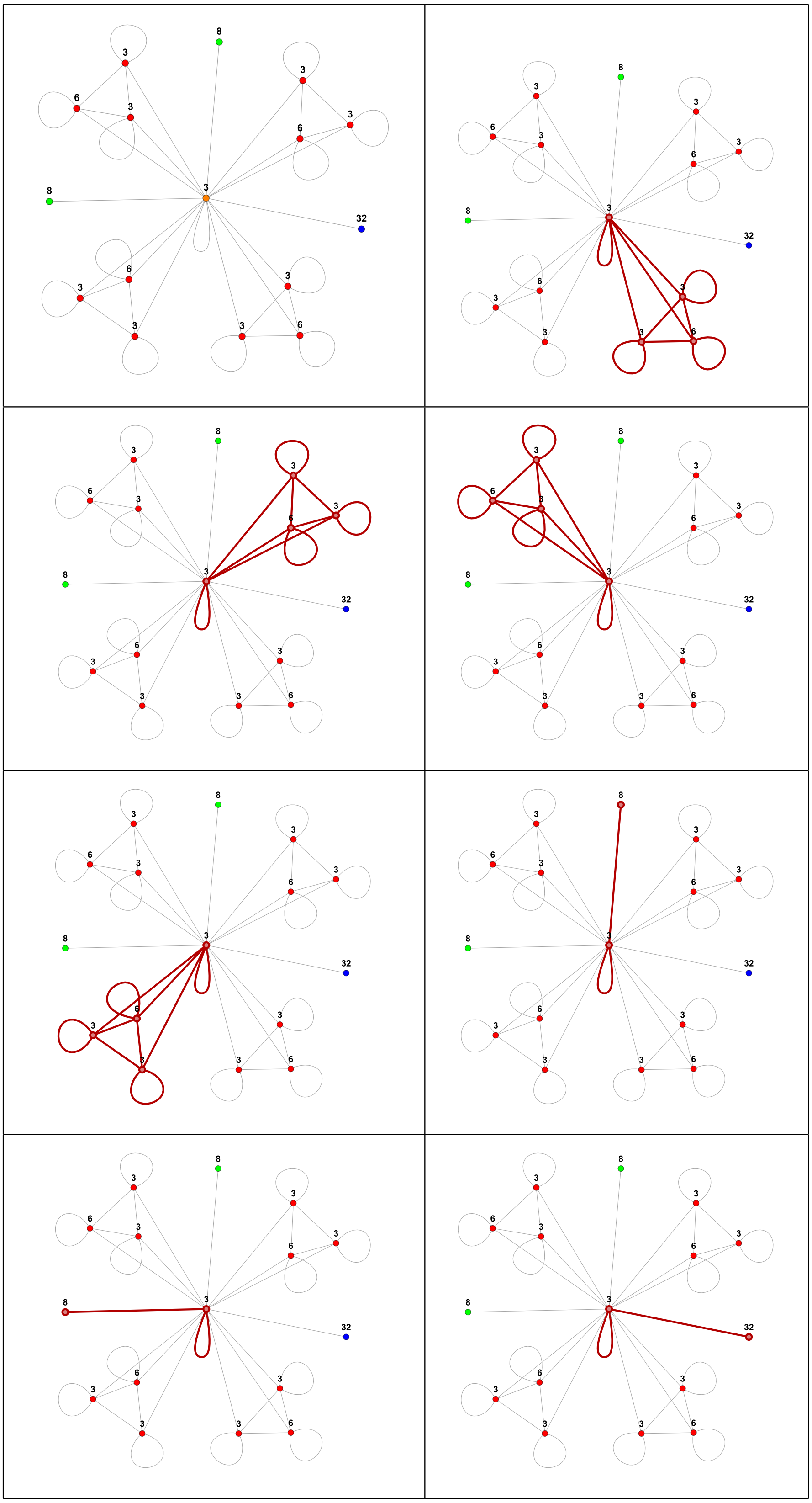

FIGURE 4. The top left image represents $IG(D_8)$. The other seven images highlight all maximal cliques of $IG(D_8)$. The product of all the vertices (images) of cliques 1 to 6 is strictly contained in G. While the product of all the vertices of bottom right clique is equal to G.

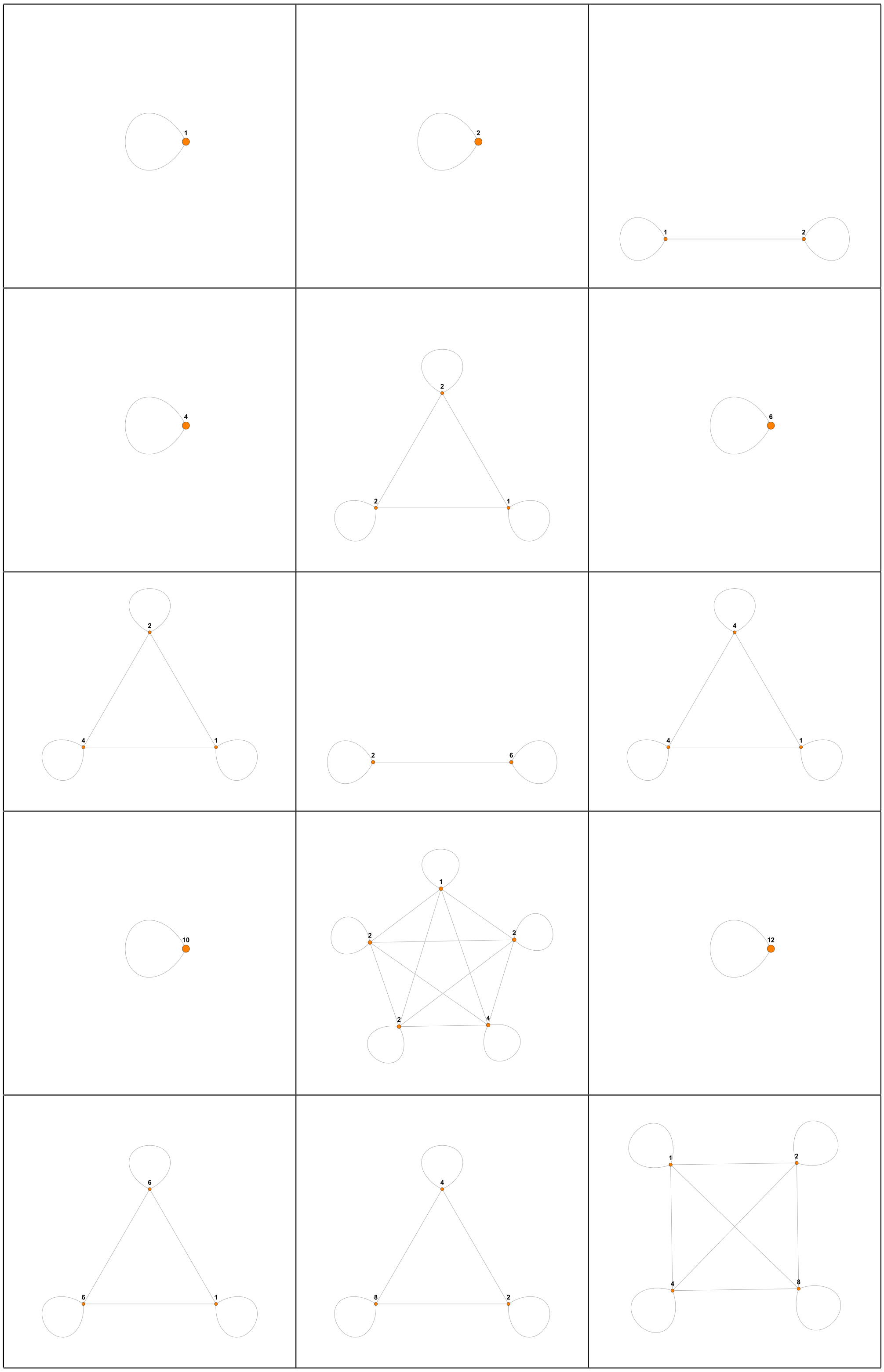

FIGURE 5. Images graphs of the cyclic groups C_2 to C_{16}.

rules of a GCA on a certain group that can be built on the basis of all the endomorphisms of that group itself.

Definition 3 (Images Graph): Let G be a finite group. The images graph $IG(G)$ of G is the labelled graph (V, E) where the vertex and edges sets V and E are defined as follows:

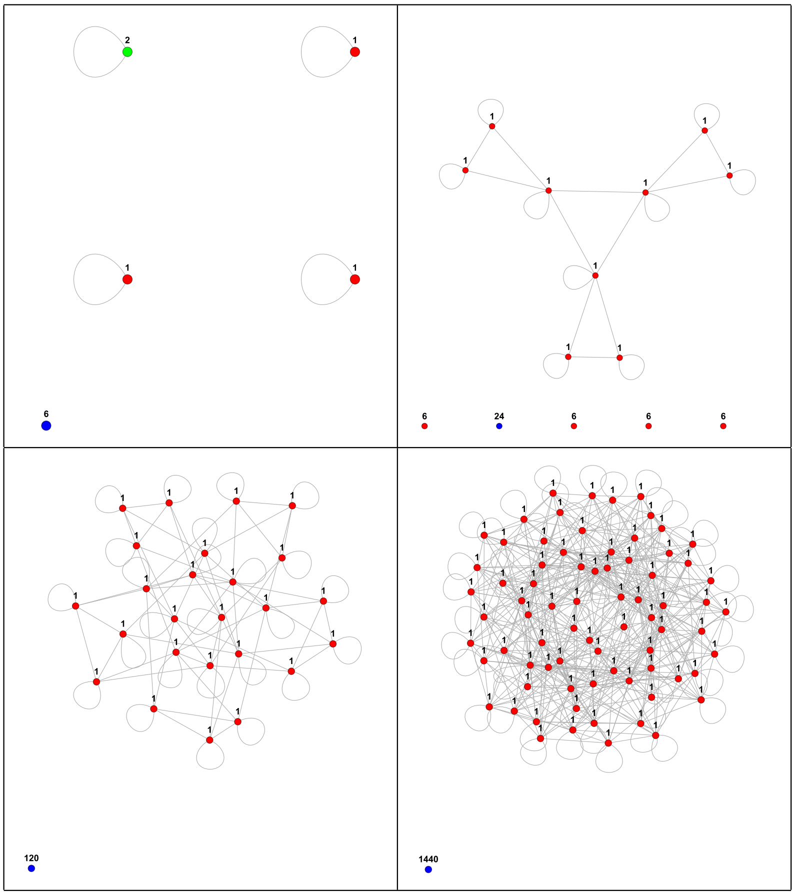

FIGURE 6. Images graphs of the symmetric groups S_3 to S_6.

- $V = \{Img_1, \ldots, Img_n\}$ *is the set of all distinct images of the non-trivial endomorphisms of G. Each vertex* Img_i *has a numeric label indicating the number of endomorphisms having* Img_i *as image (the set of such endomorphisms is denoted by* $H(Img_i)$*).*
- *for every* $i, j \in \{1, \ldots, n\}$, $(Img_i, Img_j) \in E$ *if and only if* Img_i *and* Img_j *commute (see property (1)).*

Moreover, each vertex is associated with a colour in such a way that the colour of Img_i *is*
- orange if and only if Img_i *commutes with* Img_j*, for every* $j \in \{1, \ldots, n\}$ *(in this way if G is abelian then all the vertices are orange);*
- blue if and only if its colour is not orange and $Img_i = G$*;*
- green if and only if its colour is neither orange nor blue and Img_i *is a normal subgroup of G;*
- red if and only if its colour is neither orange nor blue and Img_i *is not a normal subgroup of G.*

The image on the left of Figure 2 shows the images graph of the dihedral group D_6. The following notion allows identifying the endomorphisms defining a GCA local rule by means of the images graph.

Definition 4: Let $IG(G) = (\{Img_1, \ldots, Img_n\}, E)$ *be the images graph of a finite group G. A multiset* $C = \{c_1, \ldots, c_k\}$ *is an F-multiset of* $IG(G)$ *if and only if the following three conditions are satisfied:*
- for all $i \in \{1, \ldots, k\}$ $c_i \in \{Img_1, \ldots, Img_n\}$*;*
- for all $i \in \{1, \ldots, k\}$ *it holds that if* $(c_i, c_i) \notin E$ *then the multiplicity of* c_i *in C is one;*
- $(c_i, c_j) \in E$ *for all pairs* c_i, c_j *of distinct elements of C.*

At this point we can state that the following result holds

Proposition 1: Let G be a finite group and let $h_1, \ldots, h_k$ *be k endomorphisms of G. Let* $f : G^k \to G$ *be the homomorphism defined by the multiplication of all these endomorphisms. It holds that f is the local rule of a GCA on G if and only if* $\{Img(h_1), \ldots, Img(h_k)\}$ *is an F-multiset of* $IG(G)$*.*

Proof: We know that f is the local rule of a GCA on G if and only if both the conditions (1) and (2) from Theorem 1 hold. Since condition (1) is ensured by hypothesis, it turns out that f is the local rule of a GCA on G if and only if condition (2) holds, i.e., if and only if $\{Img(h_1), \ldots, Img(h_k)\}$ is an F-multiset of $IG(G)$. □

In other words, the images of all the endomorphisms defining any local rule f of a GCA on G must belong to the same clique of the images graph $IG(G)$ and each subset of the endomorphisms having a same image Img must have cardinality one when Img does not have a self-loop in $IG(G)$. We stress that, due to the presence of self-loops in

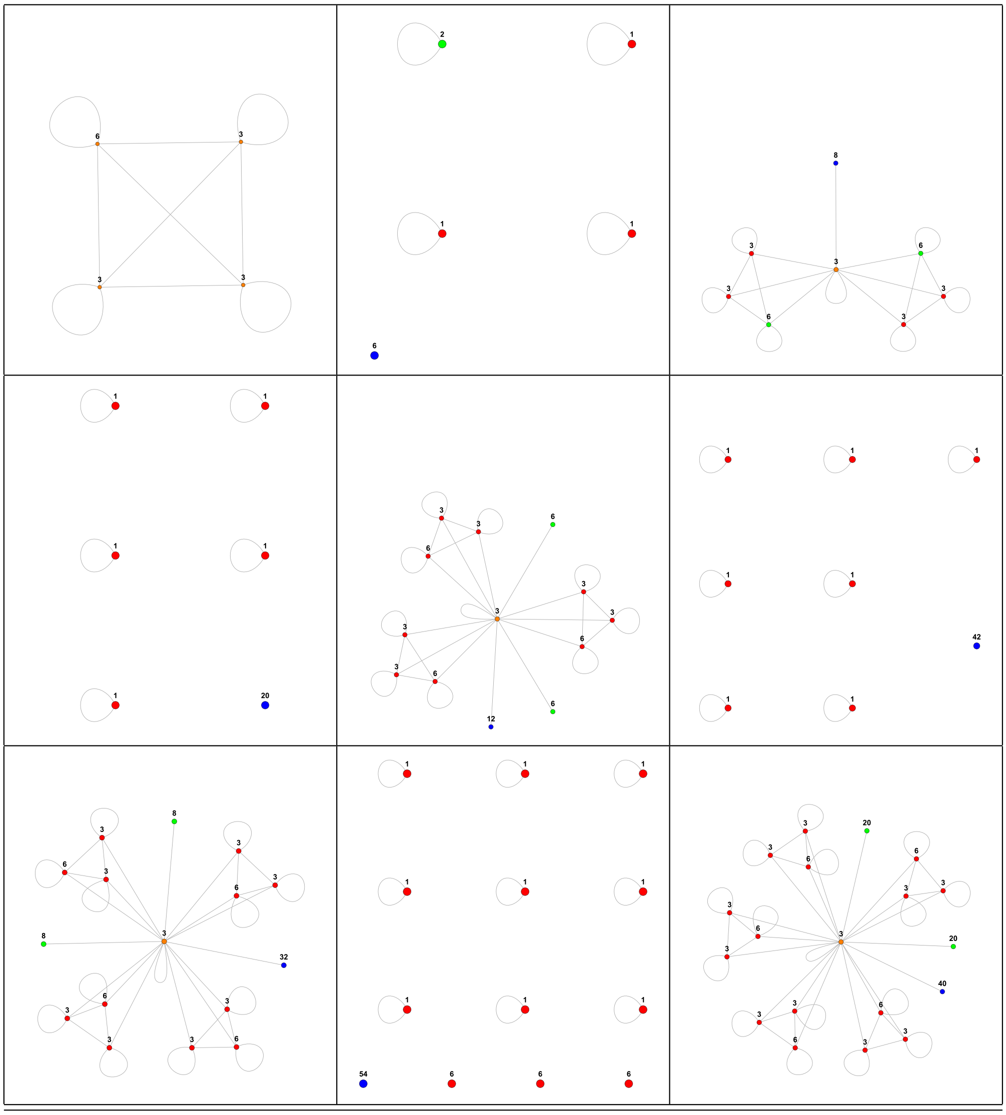

FIGURE 7. Images graphs of the dihedral groups D_i with $i \in \{2, \ldots, 10\}$.

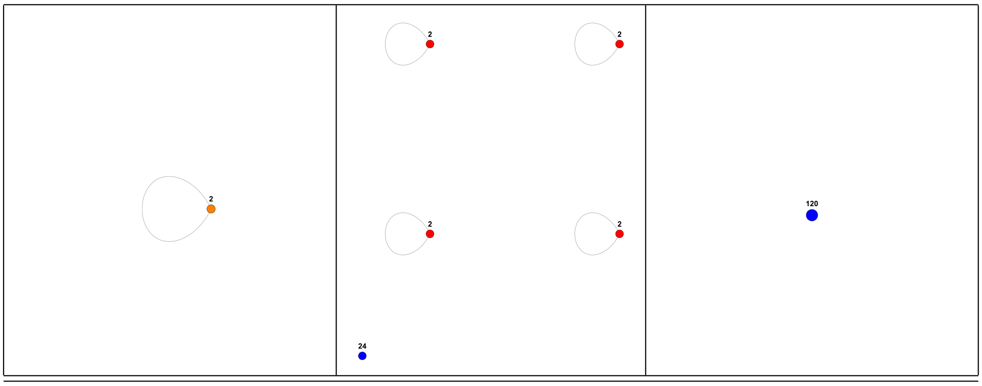

FIGURE 8. Images graphs of the Alternating groups A_3, A_4 and A_5.

$IG(G)$, the number of endomorphisms of a local rule can be arbitrarily large. The images in the center and on the right of Figure 2 show two cliques of the images graph of the dihedral group D_6.

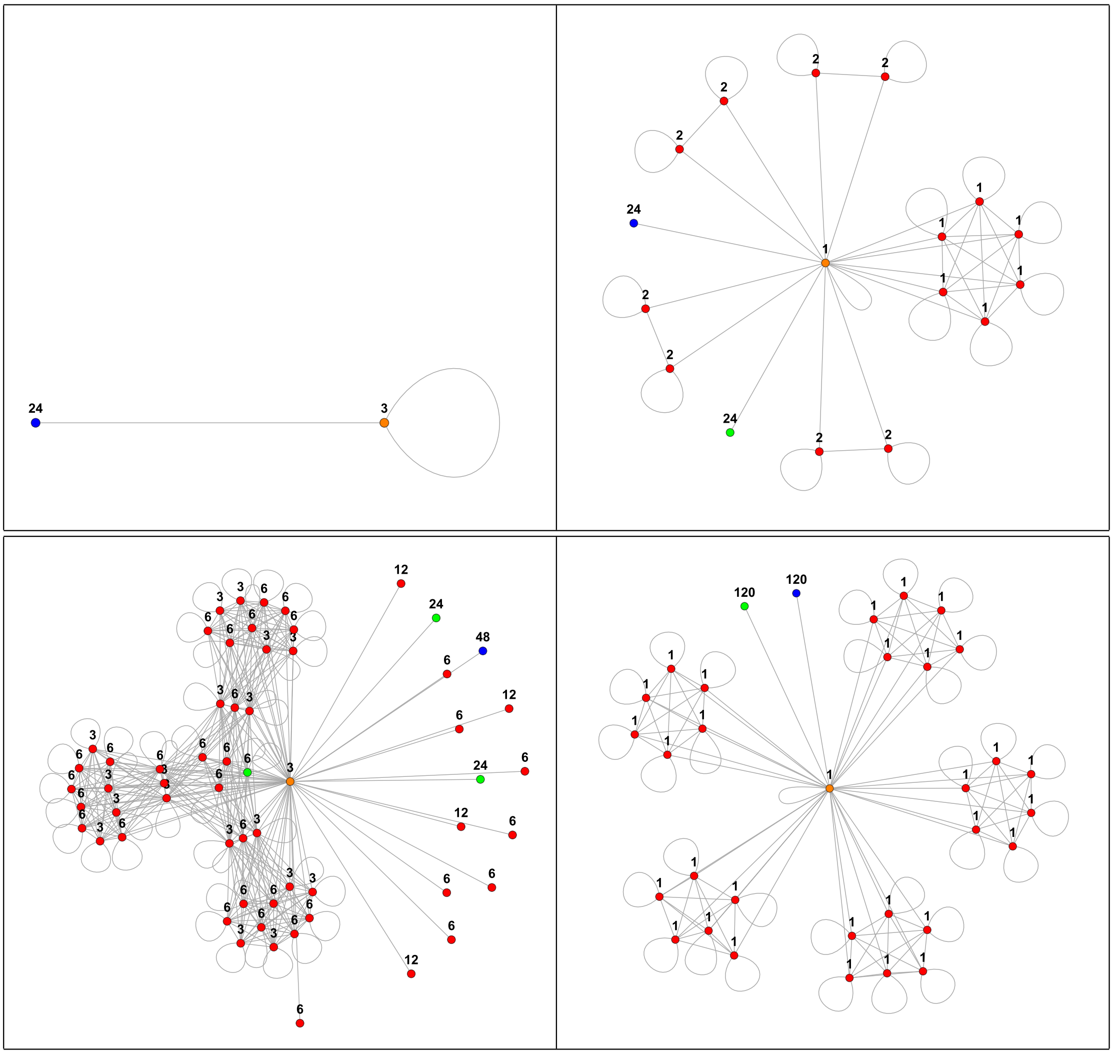

FIGURE 9. Images graphs of the quaternion group Q_8, pyritohedral group T_h, octahedral O_h and icosahedral group I_h.

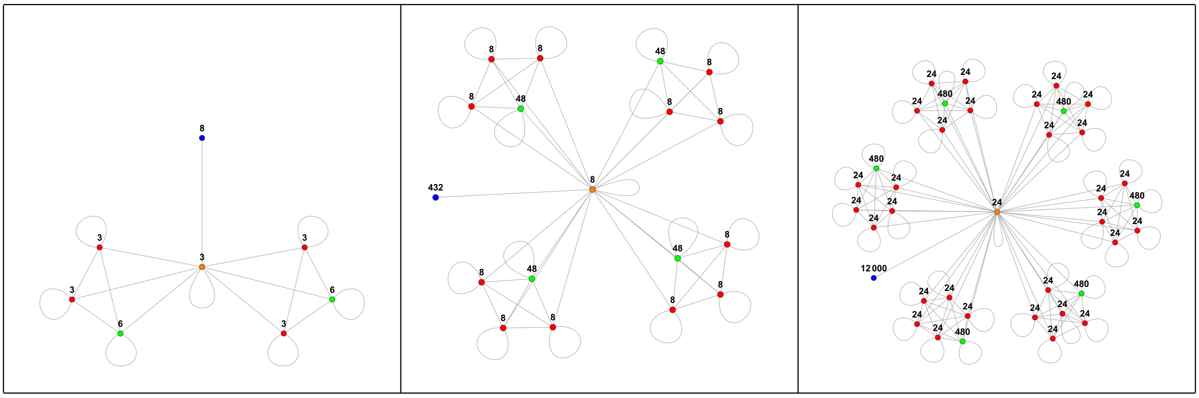

FIGURE 10. Images graphs of the Heisenberg groups H_2, H_3 and H_5.

Images graphs are very useful for visualizing and understanding the structure of local rules. Let us introduce the following example to make things even clearer.

Example 5: Consider the images graph $IG(D_4)$ in Figure 3 (vertices are identified by letters for convenience). Any local rule f of a CGA on D_4 can be built by selecting the

endomorphisms of D_4 as follows:
- any multiset of $H(e)$ together with at most one element of $H(g)$, or
- any multiset of $H(e) \cup H(b) \cup H(d) \cup H(h)$, or
- any multiset of $H(e) \cup H(i) \cup H(c) \cup H(f)$.

The following result provides a condition on the images graph in order to build a local rule giving rise to a GCA with a certain property.

Theorem 12: Let $IG(G) = (\{Img_1, \ldots, Img_n\}, E)$ be the images graph of a finite group G. Let $C = (V_C, E_C)$ be a complete subgraph of $IG(G)$ and let $f = (h_1, \ldots, h_k)$ be the local rule of a GCA F such that $Img(h_i) \in V_C$ for every $i \in \{1, \ldots, k\}$. If the product of all the elements of V_C is different from G then F is not surjective.

Proof: Assume that $\Pi \neq G$, where Π is the product of all the elements of V_C. Consider any k elements $g_1, \ldots, g_k \in G$ and let $g = f(g_1, \ldots, g_k) = h_1(g_1) \cdots h_k(g_k)$. Since, by hypothesis, $Img(h_i) \in V_C$, it follows that each $h_i(g_i)$ belongs to some element of V_C. Since the product of two subgroups A and B is a subgroup containing both A and B, we get that $g = h_1(g_1) \cdots h_k(g_k) \in \Pi \neq G$. This implies that f is not surjective and, hence, neither is F. □

Dihedral group D_8 admits 100 endomorphisms that generate 17 distinct images represented in the images graph shown in the top left picture of Figure 4. The other seven pictures of Figure 4 highlight all maximal cliques of $IG(D_8)$. It is not hard to verify that the product of all the vertices of cliques 1 to 6 is strictly contained in G. Instead, the product of all the vertices of the bottom right clique is equal to G. Hence, Theorem 12 ensures that surjective GCAs F must have local rules that consist of endomorphisms sharing the images represented by the vertices contained in the bottom right clique. Precisely, D_8 has 3 non surjective endomorphisms and 32 surjective endomorphisms whose images are the vertices of the bottom right clique.

A necessary condition to get a surjective GCA is then building the local rule by selecting one of the 32 surjective endomporphisms and any multiset of the 3 non surjective endomorphisms. The same argument also works for D_{12},D_{16} and D_{20}. We conjecture that this property remains true for every D_n with n multiple of 4.

In Figures 5 to 10, we show the images graphs of a number of small graphs.

XI. CONCLUSION AND OPEN QUESTIONS

One of the most interesting and at the same time challenging problem in CA theory is to make explicit the connection between the global behavior of a given CA and the local rule on which it is based. In particular, it is of great importance to understand which properties of the local rule influence the global behavior of the entire CA. In this paper, we have addressed this issue within a broad class of CAs, namely the GCAs, with a special focus on the non-abelian scenarios, and since not all group endomorphisms can be used to build a GCA local rule, we have characterized the set of those defining a GCA local rule by a condition on the images graph. However, due to the inherent complexity of finite groups structure, we have only managed to scratch the surface, leaving numerous compelling questions unanswered, some of them are reported here.

Namely, can indecomposability assist in characterizing local rules that give rise to GCAs with specific properties? Are there classes of finite groups, besides those examined in this paper, for which the task of characterizing set-theoretic and dynamical properties of GCAs on them is achievable? What properties must two endomorphisms of a finite group satisfy in such a way that their images commute? Do there exist strongly transitive or positively expansive GCAs on indecomposable groups?

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