

Nonparametric needlet estimation for partial derivatives of a probability density function on the d -torus

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May 11, 2023

Abstract

This paper is concerned with the estimation of the partial derivatives of a probability density function of directional data on the d -dimensional torus within the local thresholding framework. The estimators here introduced are built by means of the toroidal needlets, a class of wavelets characterized by excellent concentration properties in both the real and the harmonic domains. In particular, we discuss the convergence rates of the L^p -risks for these estimators, investigating their minimax properties and proving their optimality over a scale of Besov spaces, here taken as nonparametric regularity function spaces.

- **Keywords and Phrases:** Local thresholding, needlets, directional data, nonparametric density estimation, Besov spaces, adaptivity.
- **AMS Classification:** 62G08, 62G20, 65T60

1 Introduction and background

The estimation of derivatives of the probability density function is related to several open problems in statistics. Estimators of the first order derivatives in the unidimensional framework are exploited to detect the modes of uni-modal distributions (see, among others, [Parzen \(1962\)](#); [Schuster \(1969\)](#)). The straightforward generalization to the multivariate case has led to the mean-shift algorithm (see, for example, [Fukunaga and Hostetler \(1975\)](#); [Silverman \(1986\)](#)), where the estimation of the gradient vector of the density function is exploited to cluster and filter data. This algorithm has become widely popular in several research fields, such as image analysis and segmentation (see, among others, [Cheng \(1995\)](#));

Comaniciu and Meer (2002)). In the same setting, estimators of second order derivatives of a probability density function are used to perform statistical tests for modes of the data density and to identify key characteristics of the distribution, such as local and global extrema, ridges or saddle points (see, for example, Genovese et al. (2016)). These estimators are also extensively used in other statistical problems, such as establishing the optimal bandwidth in the framework of kernel density estimation, Fisher information estimation, parameter estimation, regression problems, hypothesis testing and others (see, among others, Singh (1977)).

In the nonparametric setting, efforts have been made to exploit kernel estimation for the derivatives of a density function, even if the excellent properties of kernel density estimators are partially lost due to the problem of bandwidth/smoothing parameter selection (see, among others, Chacon et al. (2012)).

Several estimators for the derivatives of density functions on \mathbb{R} or \mathbb{R}^d have been built by means of wavelet systems and have already been proposed as alternatives to the kernel methods. This approach have been initially exploited to deal with the estimation of unknown density functions and regression functions (see, for example, Hardle et al. (1997); Tsybakov (2009)), to be then extended to the estimation of higher order derivatives of probability density functions. Among others, wavelet estimators have been defined first on \mathbb{R} in Prakasa Rao (1996) (see also Prakasa Rao (1990)), and then generalized to \mathbb{R}^d in Prakasa Rao (2000) (see also Di Marzio et al. (2011)). Several linear and nonlinear estimators for derivatives of probability density functions on \mathbb{R} and \mathbb{R}^d have been studied, among others, by Hosseinioun et al. (2011, 2012); Liu and Wang (2013); Prakasa Rao (2017, 2018); Xu (2020).

The so-called needlets, a second generation wavelet system, have already been extensively used in nonparametric statistics, in view of their extraordinary concentration properties in both the real and the frequency domains. Needlets have been initially built over the d -dimensional sphere in Narcowich et al. (2006b,a), while some of their stochastic properties, mostly related to spherical random fields, are discussed in Baldi et al. (2009b); Durastanti et al. (2013, 2014); Bourguin et al. (2016); Cammarota and Marinucci (2015); Shevchenko and Todino (2021). Needlets have also been established over general compact manifolds in Geller and Mayeli (2009); Kerkyacharian et al. (2012); Pesenson (2013), and over spin fiber bundles (see Geller and Marinucci (2010)), while a generalized construction of spherical needlets admitting additional flexibility in the harmonic domain has been introduced more recently in Durastanti et al. (2021).

The pioneering results concerning needlet frames in the setting of nonparametric statistics, described in Baldi et al. (2009a), have established minimax rates of convergence for the L^p -risk of needlet density estimators built by means of hard local thresholding techniques. Analogous results in the block and global thresholding framework were presented in Durastanti (2013, 2016). Nonparametric regression estimators of spin functions have been discussed in Durastanti et al. (2011). The reader is referred to Bourguin and Durastanti (2017); Gautier and Le Pennec (2013); Kerkyacharian et al. (2012) for other relevant applications in the nonparametric paradigm. As far as the d -dimensional torus \mathbb{T}^d is concerned, toroidal needlets have already been discussed and applied in the framework of the two sample problem, in Bourguin and Durastanti (2018).

The goal of this paper is to study the asymptotic optimality of adaptive nonlinear

wavelet estimators for the derivatives of a probability density function defined over the d -dimensional torus \mathbb{T}^d in the nonparametric setting. We combine local thresholding techniques and concentration properties of the needlets to construct estimators achieving optimality of the L_p -risk for probability density functions defined in some scales of Besov spaces. The rates here obtained are consistent with the results given in [Baldi et al. \(2009a\)](#); [Durastanti et al. \(2011\)](#), where local thresholding techniques were applied to estimate density of spherical data, as well as those related to the estimation of derivatives of densities on \mathbb{R} and \mathbb{R}^d (see, for instance, [Prakasa Rao \(2000\)](#); [Liu and Wang \(2013\)](#)). As already remarked in [Bourguin and Durastanti \(2017\)](#), the choice of the d -dimensional torus for the support of the probability density function is quite general: since \mathbb{T}^d is locally homeomorphic to \mathbb{R}^d , hence to any other manifold with the same dimension, the spatial localization property of the toroidal needlets ensures that the results here achieved remain valid when approximating locally any such manifold with \mathbb{T}^d . Furthermore, as a consequence of the strong concentration properties of the needlet frames, the estimators here proposed is asymptotically robust even if defined only on sub-regions of the torus (see for further details [Baldi et al. \(2009c\)](#); [Marinucci and Peccati \(2011\)](#)).

The importance of studying and modeling multivariate toroidal data naturally arises in many practical problems. Indeed, it is natural to describe the time evolution of $(d - 1)$ -toroidal variables as an additional dimension in a d -dimensional setting (see [Bott et al. \(2017\)](#) for $d = 2$). This fact leads to some applications of interest, for example, in meteorology, where wind directions can be modeled as toroidal data (see, among others, [Garcia-Portugues et al. \(2013\)](#)). Most actual wind speeds are shown to lie near the peaks of the predicted density functions (see, for example [Zhang et al. \(2020\)](#)). Then the study of the gradient of the corresponding density function by means of estimators with good concentration properties can provide an excellent method to identify those peaks. Some other applications can be found in astrophysics. Studying locally 3-dimensional density functions of galaxies and their geometric characteristics is crucial to understand morphology–density, color–density and color–concentration–density relations (see for example [Ferdosi, B. J. et al. \(2011\)](#) and references therein). A fascinating classification problem, introduced by [Sengupta and Ugwuowo \(2011\)](#) and mentioned by [Di Marzio et al. \(2011\)](#), concerns measurements on the skull of members from two different population groups represented by a front angle and a side angle. Identifying different locations for the peaks can be helpful to support their classification results. Another interesting example comes from econometrics. In [Grith et al. \(2018\)](#), methods from functional principal component analysis are used to estimate partial derivatives of multivariate curves and then applied to the study of state price density surfaces.

The structure of this paper is as follows. Section 2 introduces some preliminary results on the harmonic analysis on the torus, toroidal needlet frames, Besov spaces and related properties. In Section 3, we present the construction of our estimators and the statements of our main results. Section 4 contains some numerical evidence, while all the proofs are collected in Section 5.

2 Some preliminary results

In this section, we will provide the reader with some background concerning harmonic analysis on the torus, toroidal needlets, their derivatives, and Besov spaces. We begin by setting some necessary notation. Throughout this paper, given two real-valued sequences $\{x_k\}_{k \in \mathbb{N}}$ and $\{y_k\}_{k \in \mathbb{N}}$, we write $x_k \lesssim y_k$ or $x_k \gtrsim y_k$ if there exists an absolute positive constant $c \in \mathbb{R}$ such that, for any $k \in \mathbb{N}$, $x_k \leq cy_k$ or $x_k \geq cy_k$ respectively. The notation $x_k \approx y_k$ indicates that both $x_k \lesssim y_k$ and $x_k \gtrsim y_k$ hold. Furthermore, for any $z \in \mathbb{C}$, \bar{z} denotes its conjugate.

2.1 Harmonic analysis on the torus

As is well known in the literature (see, for example, Grafakos (2008)), the d -dimensional torus \mathbb{T}^d can be read as the direct product of d unit circles,

$$\mathbb{T}^d = \mathbb{S}^1 \times \dots \times \mathbb{S}^1 \subset \mathbb{C}^d.$$

From now on, we will denote the generic coordinates over \mathbb{T}^d by $\vartheta = (\vartheta_1, \dots, \vartheta_d)$, where $\vartheta_i \in [0, 2\pi)$ for $i = 1, \dots, d$. As a straightforward consequence, the uniform Lebesgue measure over \mathbb{T}^d will be given by

$$\rho(d\vartheta) = \prod_{i=1}^d d\vartheta_i,$$

where for $i = 1, \dots, d$, $d\vartheta_i$ is the Lebesgue measure over the unit circle.

Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product between d -dimensional vectors; the set of functions $S_\ell : \mathbb{T}^d \rightarrow \mathbb{C}$, where $\ell = (\ell_1, \dots, \ell_d) \in \mathbb{Z}^d$ is the frequency index, defined by

$$S_\ell(\vartheta) = (2\pi)^{-\frac{d}{2}} \exp(\langle \ell, \vartheta \rangle),$$

describes an orthonormal basis for $L^2(\mathbb{T}^d)$, the space of square-integrable functions over \mathbb{T}^d (see again Grafakos (2008)). Indeed, the one-dimensional torus \mathbb{T}^1 can be identified as an equivalence class of the quotient space \mathbb{R}/\mathbb{Z} , so that the canonical representation in $[0, 2\pi)^d$ describes a coordinate system on \mathbb{T}^d . The set $\{S_\ell : \ell \in \mathbb{Z}^d\}$ corresponds to the eigenfunctions of the Laplace-Beltrami operator on the torus $\nabla_{\mathbb{T}^d}$, defined by

$$\nabla_{\mathbb{T}^d} = \sum_{i=1}^d \frac{\partial^2}{\partial \vartheta_i^2},$$

so that

$$(\nabla_{\mathbb{T}^d} + \varepsilon_\ell^2) S_\ell(\vartheta) = 0,$$

where $\varepsilon_\ell = \sqrt{\sum_{i=1}^d |\ell_i|^2}$. Thus, the following orthonormality property holds

$$\langle S_\ell, S_{\ell'} \rangle_{\mathbb{T}^d} = \int_{\mathbb{T}^d} S_\ell(\vartheta) \overline{S_{\ell'}(\vartheta)} \rho(d\vartheta) = \delta_\ell^{\ell'},$$

where δ is the multivariate Kronecker delta. Any function $f \in L^2(\mathbb{T}^d)$ can be then represented by its harmonic expansion

$$f(\vartheta) = \sum_{\ell \in \mathbb{Z}^d} a_\ell S_\ell(\vartheta), \quad \vartheta \in \mathbb{T}^d,$$

where $\{a_\ell : \ell \in \mathbb{Z}^d\}$ is the set of the complex-valued Fourier coefficients, given by

$$a_\ell = \int_{\mathbb{T}^d} f(\vartheta) \overline{S_\ell(\vartheta)} \rho(d\vartheta).$$

Given the multi-index $m = (m_1, \dots, m_d) \in \mathbb{N}_0^d$ such that $|m| = \sum_{i=1}^d m_i$, for $f \in C^m(\mathbb{T}^d)$, the m -th order derivative of f is defined by

$$f^{(m)}(\vartheta) = D^m f(\vartheta) = \frac{\partial^{|m|}}{\partial \vartheta_1^{m_1} \dots \partial \vartheta_d^{m_d}} f(\vartheta),$$

where the differential operator D^m is given by

$$D^m = \frac{\partial^{|m|}}{\partial \vartheta_1^{m_1} \dots \partial \vartheta_d^{m_d}}.$$

2.2 Toroidal needlets

As already mentioned in Section 1, needlets have been originally introduced on the d -dimensional sphere in [Narcowich et al. \(2006b,a\)](#), and then generalized to compact manifolds (see [Bourguin and Durastanti \(2017\)](#); [Geller and Mayeli \(2009\)](#); [Kerkycharian et al. \(2012\)](#)). Needlet-like wavelets on \mathbb{T}^d have been already used in [Bourguin and Durastanti \(2018\)](#) in the framework of the two-sample problem. Their construction can be resumed as follows.

Fixed a resolution level $j \in \mathbb{N}$, by means of the Littlewood-Paley decomposition on \mathbb{T}^d (see [Narcowich et al. \(2006a\)](#)), there exists a set of cubature points and weights

$$\{(\xi_{j,k}, \lambda_{j,k}) : k = 1, \dots, K_j\},$$

where $\xi_{j,k} \in \mathbb{T}^d$, $\lambda_{j,k} \in \mathbb{R}^+$, and K_j is the cardinality of needlets at the level j . Loosely speaking, \mathbb{T}^d can be decomposed into a partition of K_j sub-regions, called pixels, centered on the corresponding $\xi_{j,k}$ and with area equal to $\lambda_{j,k}$. The d -dimensional toroidal needlets are defined by

$$\psi_{j,k}(\vartheta) = \sqrt{\lambda_{j,k}} \sum_{\ell \in \mathbb{Z}^d} b\left(\frac{\varepsilon_\ell}{B^j}\right) \overline{S_\ell(\xi_{j,k})} S_\ell(\vartheta),$$

where $B > 1$ is a scale parameter (typically, $B = 2$), and $b : \mathbb{R} \rightarrow \mathbb{R}^+$ is the so-called needlet weight or window function, which satisfies the following properties:

- (i) b has compact support in $[B^{-1}, B]$;
- (ii) $b \in C^\infty(\mathbb{R})$;

(iii) the partition of unity property holds, that is, for any $c > 1$,

$$\sum_{j \in \mathbb{N}} b^2 \left(\frac{c}{B^j} \right) = 1.$$

Therefore, needlets are characterized by the following properties. Following (i), for any $j \in \mathbb{N}$, $b(\frac{\varepsilon_\ell}{B^j})$ is not null only over a finite subset of \mathbb{Z}^d , that is $\Lambda_j^d = \{\ell : \varepsilon_\ell \in (B^{j-1}, B^{j+1})\}$. As a direct consequence, we can rewrite

$$\psi_{j,k}(\vartheta) = \sqrt{\lambda_{j,k}} \sum_{\ell \in \Lambda_j^d} b \left(\frac{\varepsilon_\ell}{B^j} \right) \overline{S_\ell(\xi_{j,k})} S_\ell(\vartheta).$$

In view of (ii), toroidal needlets are characterized by a quasi-exponential localization property in the spatial domain, which guarantees that each needlet $\psi_{j,k}$ is not-negligible almost only in the corresponding pixel. More rigorously, for any $M > 0$, there exists $c_M > 0$ such that, for any $\vartheta \in \mathbb{T}^d$,

$$|\psi_{j,k}(\vartheta)| \leq \frac{c_M B^{\frac{d}{2}j}}{(1 + B^j d(\vartheta, \xi_{j,k}))^M},$$

where $d(\cdot, \cdot)$ is the geodesic distance over \mathbb{T}^d . As a consequence, the following bounds on the L^p -norms of the toroidal needlets hold (see [Narcowich et al. \(2006a\)](#)): for any $p \in [1, \infty)$, there exist two positive constants c_p, C_p , which depend only on p , such that

$$c_p B^{jd(\frac{1}{2}-\frac{1}{p})} \leq \|\psi_{j,k}\|_{L^p(\mathbb{T}^d)} \leq C_p B^{jd(\frac{1}{2}-\frac{1}{p})}. \quad (2.1)$$

Finally, it follows from (iii) that the needlet system $\{\psi_{j,k} : j \geq 0; k = 1, \dots, K_j\}$ is a tight frame over \mathbb{T}^d . Indeed, for any function $f \in L^2(\mathbb{T}^d)$, we can define the set of needlet coefficients $\{\beta_{j,k} : j \geq 0; k = 1, \dots, K_j\}$, each of those given by

$$\beta_{j,k} = \langle f, \psi_{j,k} \rangle_{\mathbb{T}^d} = \int_{\mathbb{T}^d} f(\vartheta) \overline{\psi_{j,k}(\vartheta)} \rho(d\vartheta).$$

Then, it holds that

$$\sum_{j \in \mathbb{N}_0} \sum_{k=1}^{K_j} |\beta_{j,k}|^2 = \|f\|_{L^2(\mathbb{T}^d)}^2,$$

and the following reconstruction formula holds in the L^2 -sense

$$f(\vartheta) = \sum_{j \in \mathbb{N}_0} \sum_{k=1}^{K_j} \beta_{j,k} \psi_{j,k}(\vartheta), \quad \vartheta \in \mathbb{T}^d.$$

2.3 Needlets and derivatives

The needlet expansion of the m -th order derivative of the density function, $f^{(m)} = D^m f$, is given by

$$f^{(m)}(\vartheta) = \sum_{j \in \mathbb{N}_0} \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k}(\vartheta), \quad \vartheta \in \mathbb{T}^d, \quad (2.2)$$

where $\{\beta_{j,k}^{(m)} : j \geq 0, k = 1, \dots, K_j\}$ is the collection of needlet coefficients associated to $f^{(m)}$, that is,

$$\beta_{j,k}^{(m)} = \int_{\mathbb{T}^d} f^{(m)}(\vartheta) \overline{\psi_{j,k}}(\vartheta) \rho(d\vartheta). \quad (2.3)$$

The m -derivative of a toroidal needlet is given by

$$\psi_{j,k}^{(m)}(\vartheta) = D^m \psi_{j,k}(\vartheta), \quad (2.4)$$

such that the following result holds.

Lemma 1. *Let $\psi_{j,k}^{(m)}$ be given by (2.4). Then, it holds that*

$$\psi_{j,k}^{(m)}(\vartheta) = \sqrt{\lambda_{j,k}} B^{j|m|} \sum_{\ell \in \Lambda_j^d} b_j^{(m)}(\ell) \overline{S_\ell(\xi_{j,k})} S_\ell(\vartheta), \quad \vartheta \in \mathbb{T}^d, \quad (2.5)$$

where

$$b_j^{(m)}(\ell) = (-1)^{|\ell|} \frac{\prod_{i=1}^d \ell_i^{m_i}}{B^{j|\ell|}} b\left(\frac{\varepsilon_\ell}{B^j}\right).$$

The proof of Lemma 1 is given in Section 5.1

Remark 2. *Observe that the function $b_j^{(m)} : \mathbb{R}^d \rightarrow \mathbb{R}$ preserves some of the properties of b . Indeed, it is $C^\infty(\mathbb{R}^d)$ and has compact support in Λ_j^d . On the other hand, the partition of unity property does not hold anymore.*

The next result is concerned with the localization property of the derivatives of a toroidal needlet.

Lemma 3. *Let $\psi_{j,k}^{(m)}$ be given by (2.5). Then, for any multi-index $m \in \mathbb{N}_0^d$ and for any $M > 0$, there exists $c_M > 0$ such that, for any $\vartheta \in \mathbb{T}^d$,*

$$\left| \psi_{j,k}^{(m)}(\vartheta) \right| \leq \frac{c_M B^{j(|m| + \frac{d}{2})}}{(1 + B^j d(\vartheta, \xi_{j,k}))^M}.$$

The following corollary establishes bounds for the L^p -norms of the needlet derivatives. These bounds correspond to the ones given by (2.1) for the toroidal needlets.

Corollary 4. *Let $\psi_{j,k}^{(m)}$ be given by (2.5). Then, for any multi-index $m \in \mathbb{N}_0^d$ and for any $p \in [1, \infty)$, there exist two constants c_p^*, C_p^* , depending only on p such that*

$$c_p^* B^{j(|m| + d(\frac{1}{2} - \frac{1}{p}))} \leq \left\| \psi_{j,k}^{(m)} \right\|_{L^p(\mathbb{T}^d)} \leq C_p^* B^{j(|m| + d(\frac{1}{2} - \frac{1}{p}))}.$$

The bound in Lemma 3 follows a general result in mathematical analysis, given by (Geller and Mayeli, 2009, Theorem 2.2), which states that the spatial concentration properties of needlet-like constructions over compact manifolds are conserved under the action of C^∞ -differential operators, up to a polynomial term which depends on the degree of the operator itself, in our case $|m|$ for D^m .

On the other hand, the proof of Corollary 4 follows strictly the one for the standard needlets given by Narcowich et al. (2006a)[Eqq. 3.12, 3.13] and for the Mexican needlets in Durastanti (2017)[Corollary 3.2]. Both the proofs are then omitted for the sake of the brevity.

Then, the following result holds

Lemma 5. Let $\psi_{j,k}^{(m)}$ and $\beta_{j,k}^{(m)}$ be given by (2.5) and (2.3) respectively. Then, it holds that

$$\beta_{j,k}^{(m)} = (-1)^{|m|} \langle f, \psi_{j,k}^{(m)} \rangle_{\mathbb{T}^d}. \quad (2.6)$$

Furthermore, for any $j \geq 0$, it holds that

$$\sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k}^{(m)}(\vartheta) = \sum_{k=1}^{K_j} \beta_{j,k} \psi_{j,k}^{(m)}(\vartheta), \quad \vartheta \in \mathbb{T}^d. \quad (2.7)$$

As a straightforward consequence, Equation (2.2) becomes

$$f^{(m)}(\vartheta) = \sum_{j \in \mathbb{N}_0} \sum_{k=1}^{K_j} \beta_{j,k} \psi_{j,k}^{(m)}(\vartheta), \quad \vartheta \in \mathbb{T}^d,$$

The next Lemma collects some results which can be seen as the counterpart in our setting of (Baldi et al., 2009a, Lemma 2, Lemma 18).

Lemma 6. For any $j \in \mathbb{N}$, let $\{a_k : k = 1, \dots, K_j\}$ be a finite real-valued sequence. Hence, for any $0 < p \leq \infty$, it holds

$$\left\| \sum_{k=1}^{K_j} a_k \psi_{j,k}^{(m)} \right\|_{L^p(\mathbb{T}^d)} \lesssim \begin{cases} B^j (|m| + d(\frac{1}{2} - \frac{1}{p})) \left(\sum_{k=1}^{K_j} |a_k|^p \right)^{\frac{1}{p}} & 0 < p < \infty \\ B^j (|m| + \frac{d}{2}) \left(\sup_{k=1, \dots, K_j} |a_k| \right) & p = \infty \end{cases}. \quad (2.8)$$

Furthermore, there exists a subset $A_j \subset \{1, \dots, K_j\}$, where

$$\text{card } A_j \gtrsim B^{jd},$$

such that

$$\left\| \sum_{k \in A_j} a_k \psi_{j,k}^{(m)} \right\|_{L^p(\mathbb{T}^d)} \gtrsim \begin{cases} B^j (|m| + d(\frac{1}{2} - \frac{1}{p})) \left(\sum_{k \in A_j} |a_k|^p \right)^{\frac{1}{p}} & 0 < p < \infty \\ B^j (|m| + \frac{d}{2}) \left(\sup_{k \in A_j} |a_k| \right) & p = \infty \end{cases}. \quad (2.9)$$

As discussed below (see Formula (2.11)), Equation (2.8) will be crucial to establish a suitable upper bound for the needlet expansion of toroidal density functions belonging to some Besov space. As far as a lower bound is concerned, if the toroidal needlets were orthonormal, then (2.8) could have been immediately reversed, such that $A_j = \{1, \dots, K_j\}$ in (2.9). Nevertheless, needlets, as well as their derivatives, are not orthogonal, but for $j \in \mathbb{N}_0$ and $k, k' \in \{1, \dots, K_j\}$ such that $\xi_{j,k}$ and $\xi_{j,k'}$ are distant enough, the scalar product between $\psi_{j,k}$ and $\psi_{j,k'}$ (and, consequently, between $\psi_{j,k}^{(m)}$ and $\psi_{j,k'}^{(m)}$) is almost negligible. This reasoning can be extended pairwise to all the needlets whose cubature points belong to A_j . The proof of Lemma 6 is very similar to the ones given in (Baldi et al., 2009a, Lemma 2, Lemma 18) and, then, here omitted for the sake of brevity.

2.4 Besov spaces

Before concluding this section, we recall the construction of the Besov spaces on \mathbb{T}^d and their excellent approximation properties for needlet coefficients. Further details can be found, among others, in [Geller and Mayeli \(2009\)](#), and in the references therein. Following [Geller and Pesenson \(2011\)](#) (see also [Baldi et al. \(2009a\)](#); [Donoho et al. \(1996\)](#); [Durastanti et al. \(2011\)](#); [Durastanti \(2016\)](#)), we consider \mathcal{P}_t the space of polynomial functions of degree t on \mathbb{T}^d . For any $f : \mathbb{T}^d \mapsto \mathbb{R}$, $f \in L^p(\mathbb{T}^d)$, the *approximation error* $G_t(f; p)$, obtained when we replace f by $g \in \mathcal{P}_t$, is defined by

$$G_t(f; p) = \inf_{g \in \mathcal{P}_t} \|f - g\|_{L^p(\mathbb{T}^d)}.$$

The Besov space $\mathcal{B}_{r,q}^s$ is the space of functions such that

$$f \in L^p(\mathbb{T}^d) \quad \text{and} \quad \left(\sum_{t=0}^{\infty} (t^s G_t(f; r))^q \right)^{\frac{1}{q}} < \infty.$$

Since $t \mapsto G_t(f, r)$ is decreasing, a standard condensation argument yields the following equivalent conditions:

$$f \in L^r(\mathbb{T}^d) \quad \text{and} \quad \left(\sum_{j=0}^{\infty} (B^{js} G_{B^j}(f; r))^q \right)^{\frac{1}{q}} < \infty.$$

The parameters of the Besov space $\mathcal{B}_{r,q}^s$ can be read as follows (see also [Bourgain and Durastanti \(2017\)](#)):

- r concerns the summability with respect to j . In particular, for any $j > 0$, $\{\beta_{j,k} : k = 1, \dots, K_j\}$ belongs to the set of r -summable sequences $\ell_r(\mathbb{T}^d)$.
- q is the controlling parameter related to the weighted q -norm along the scale of the needlet coefficients at the resolution level j ;
- s controls the smoothness of the decay rate for the weighted q -norm of the needlet coefficients.

Defining $s = s' + a$, where $s' \in \mathbb{N}$ and $a \in (0, 1)$, and following ([Kerkycharian and Picard, 1992](#), Definition 1) (see also [Geller and Pesenson \(2011\)](#)) yields

$$f \in \mathcal{B}_{r,q}^s \iff f^{(m)} \in \mathcal{B}_{r,q}^a, \text{ for } |m| \leq s.$$

As a straightforward consequence, we have that

$$f \in \mathcal{B}_{r,q}^{s+|m|} \iff f^{(m)} \in \mathcal{B}_{r,q}^s. \quad (2.10)$$

Then, we can prove the next result, analogous to ([Baldi et al., 2009a](#), Theorem 4) but properly adapted to derivative functions defined on \mathbb{T}^d .

Proposition 7. *Let $1 \leq r \leq \infty$, $s > 0$, $0 \leq q \leq \infty$. Let f be a measurable function on \mathbb{T}^d , associated to the needlet coefficients $\{\beta_{j,k} : j \in \mathbb{N}_0, k = 1, \dots, K_j\}$. Then, the following conditions are equivalent*

(i) $f \in \mathcal{B}_{r,q}^{s+|m|}$;

(ii) $f^{(m)} \in \mathcal{B}_{r,q}^s$;

(iii) for every $j \geq 1$,

$$\sum_{k=1}^{K_j} \beta_{j,k}^r \|\psi_{j,k}\|_{L^r(\mathbb{T}^d)}^r = B^{-j(s+|m|)} \delta_j,$$

where $(\delta_j : j \in \mathbb{N}_0)$ is a q -summable sequence;

(iv) for every $j \geq 1$,

$$\sum_{k=1}^{K_j} (\beta_{j,k}^{(m)})^r \|\psi_{j,k}\|_{L^r(\mathbb{T}^d)}^r = B^{-js} \delta_j,$$

where $(\delta_j : j \in \mathbb{N}_0)$ is a q -summable sequence.

(v) for every $j \geq 1$,

$$\sum_{k=1}^{K_j} \beta_{j,k}^r \|\psi_{j,k}^{(m)}\|_{L^r(\mathbb{T}^d)}^r = B^{-js} \delta_j.$$

where $(\delta_j : j \in \mathbb{N}_0)$ is a q -summable sequence.

The proof of this Proposition follows directly (2.10), (2.7) in Lemma 5, as well as (2.8) and the proof of (Baldi et al., 2009a, Theorem 4), so it is here omitted for the sake of brevity.

Using jointly (2.7) in Lemma 5, (2.8) in Lemma 6 and Proposition 7 leads to the following inequality, for any $j \in \mathbb{N}$,

$$\left\| \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)} \lesssim B^{j(|m|+d(\frac{1}{2}-\frac{1}{p}))} \left(\sum_{k=1}^{K_j} |\beta_{j,k}|^p \right)^{\frac{1}{p}}. \quad (2.11)$$

Following Baldi et al. (2009a), the Besov space $\mathcal{B}_{r,q}^s$ can be read as a Banach space with norm

$$\|f\|_{\mathcal{B}_{r,q}^s} = \left\| \left(B^{j[s+d(\frac{1}{2}-\frac{1}{r})]} \left\| (\beta_{j,k})_{k=1,\dots,K_j} \right\|_{\ell^r} \right)_{j \in \mathbb{N}_0} \right\|_{\ell^q} < \infty,$$

so that we can define the *Besov ball* of radius $L > 0$ as the following set:

$$\mathcal{B}_{r,q}^s(L) = \left\{ f \in \mathcal{B}_{r,q}^s : \|f\|_{\mathcal{B}_{r,q}^s} \leq L \right\}.$$

Finally, as stated in (Baldi et al., 2009a, Theorem 5) (see also Kerkyacharian and Picard (1993, 2004)), the following *Besov embeddings* hold for $p < r$

$$\begin{aligned} \mathcal{B}_{r,q}^s &\subset \mathcal{B}_{p,q}^s \\ \mathcal{B}_{p,q}^s &\subset \mathcal{B}_{r,q}^{s-d(\frac{1}{p}-\frac{1}{r})}, \end{aligned}$$

which can be equivalently stated as

$$\sum_{k=1}^{K_j} |\beta_{j,k}|^r \leq \sum_{k=1}^{K_j} |\beta_{j,k}|^p$$

$$\sum_{k=1}^{K_j} |\beta_{j,k}|^p \leq \left(\sum_{k=1}^{K_j} |\beta_{j,k}|^r \right) K_j^{1-\frac{p}{r}}.$$

A detailed proof is similar to the ones in (Baldi et al., 2009a, Theorem 5) and (Durastanti et al., 2011, Equation 8).

3 Local thresholding via toroidal needlets

3.1 The construction of the estimators

Let X_1, \dots, X_n be independent and identically distributed random vectors on \mathbb{T}^d with unknown density $f : \mathbb{T}^d \mapsto [0, \infty)$, $f \in C^m(\mathbb{T}^d)$, whose upper bound is given by

$$M = \sup_{\vartheta \in \mathbb{T}^d} |f(\vartheta)|.$$

We assume that f has derivatives of order $\mu \in \mathbb{N}$. Our goal is to produce an estimator $\hat{f}^{(m)}$ for $f^{(m)}$, for $m \in \mathbb{N}^d$ such that $|m| \leq \mu$. The first step will consist in defining empirical estimators for the needlet coefficients; the second step will be focused on the thresholding procedure which yields the construction of the target estimator. Analogously to Prakasa Rao (2000) (cf. also Prakasa Rao (1996, 1990)), for any $j \geq 0$ and $k \in \{1, \dots, K_j\}$, we can define an *empirical estimator* for the m -th derivative needlet coefficients.

$$\hat{\beta}_{j,k}^{(m)} = \frac{(-1)^{|m|}}{n} \sum_{i=1}^n \overline{\psi_{j,k}^{(m)}}(X_i). \quad (3.1)$$

We exploit the empirical needlet coefficients (3.1) to define nonlinear estimators for derivatives of a density function as follows. First, we apply a selection procedure to each coefficient, and then only the selected ones will be used to construct the estimator of $f^{(m)}$ given by the following formula:

$$\hat{f}^{(m)}(\vartheta) = \sum_{j=0}^{J_{n,m}-1} \sum_{k=1}^{K_j} \eta(\hat{\beta}_{j,k}^{(m)}, \tau_{j,m,n}) \psi_{j,k}(\vartheta), \quad (3.2)$$

where $J_{n,m}$ and $\tau_{j,m,n}$ are the so-called *truncation bandwidth* and *threshold* respectively, while $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is the *thresholding function*.

The truncation bandwidth corresponds to the higher resolution level on which the empirical coefficients are computed. The optimal choice for the truncation bandwidth is defined as follows:

$$J_{n,m} = \left\lfloor \frac{1}{d+2|m|} \log_B \frac{n}{\log n} \right\rfloor. \quad (3.3)$$

Loosely speaking, (3.3) allows us to control, in an optimal way, the error due to the approximation of the infinite sum $f^{(m)}$ by the finite sum $\hat{f}^{(m)}$. Further details concerning choice of $J_{n,m}$ will be provided by Remark 12 below.

The thresholding function is aimed to control the selection of the empirical needlet coefficients (see, for example, Donoho et al. (1996)). In the framework of *local thresholding*, where each empirical coefficient is examined separately, it is standard to choose between

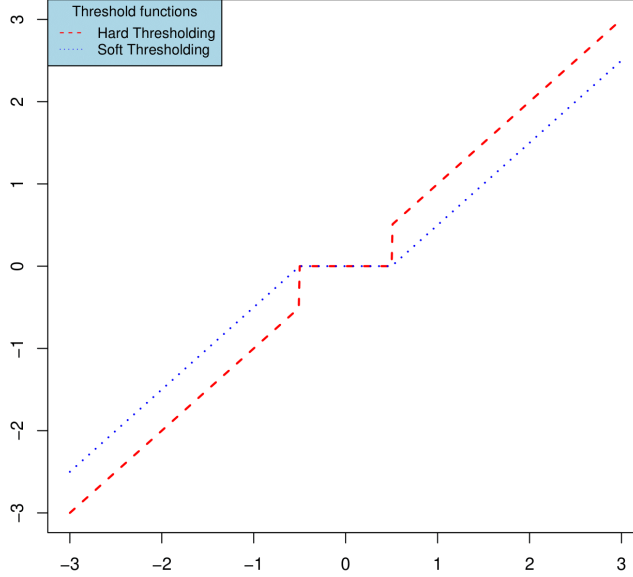


Figure 1: Comparison between hard (red line) and soft (blue line) threshold functions ($a = 0.5$).

hard and *soft* thresholding (see, for example, (Härdle et al., 1997, Chapters 10 and 11)). Hard thresholding either keeps or discards the empirical coefficients. The soft thresholding, also known as *wavelet shrinkage*, “shrinks” towards zero the values for the empirical coefficients. The two thresholding functions $\eta_{\text{hard}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\eta_{\text{soft}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given respectively by

$$\eta_{\text{hard}}(u, a) = \begin{cases} u & \text{if } |u| \geq a \\ 0 & \text{otherwise} \end{cases}, \quad (3.4)$$

$$\eta_{\text{soft}}(u, a) = \max(|u| - a, 0) \operatorname{sign}(u), \quad (3.5)$$

see Figure 1. The *hard* and the *soft thresholding needlet estimators for the m -th derivative of the density f* , at every $\vartheta \in \mathbb{T}^d$, are then respectively defined by

$$\hat{f}_{\text{hard}}^{(m)}(\vartheta) = \sum_{j=0}^{J_{n,m}-1} \sum_{k=1}^{K_j} \eta_{\text{hard}}(\hat{\beta}_{j,k}^{(m)}, \tau_{j,m,n}) \psi_{j,k}(\vartheta),$$

and

$$\hat{f}_{\text{soft}}^{(m)}(\vartheta) = \sum_{j=0}^{J_{n,m}-1} \sum_{k=1}^{K_j} \eta_{\text{soft}}(\hat{\beta}_{j,k}^{(m)}, \tau_{j,m,n}) \psi_{j,k}(\vartheta),$$

where the tuning parameters are described as follows. The *threshold* $\tau_{j,m,n}$ is the product of three objects:

$$\tau_{j,m,n} = \kappa B^{j|m|} \sqrt{\frac{\log n}{n}}, \quad (3.6)$$

where

- the *threshold constant* κ , which depends on the Besov parameters. Our theoretical setting will not answer the question how to choose κ (we will show only that κ should be large enough). However, in Section 4 we will provide some examples;
- the *derivative balancing parameter* $B^{j|m|}$, which depends on the order of the differential operator D^m ;
- the *sample size-dependent scaling factor* $\sqrt{\frac{\log n}{n}}$.

Remark 8. Another common choice in the literature concerning the sample size-dependent scaling factor is $\sqrt{j/n}$, rather than $\sqrt{\log n/n}$ (see, among others, [Donoho et al. \(1996\)](#); [Kerkycharian and Picard \(1992\)](#); [Tribouley \(1995\)](#)). As shown for example by ([Härdle et al., 1997, Proof of Proposition 10.3](#)), the two factors are equivalent, and, hence, another possible choice for the threshold is

$$\tau'_{j,m,n} = \kappa' \sqrt{j} B^{j|m|} n^{-\frac{1}{2}}.$$

3.2 Some probabilistic results on the empirical needlet coefficients

First, observe that the estimator (3.1) is *unbiased*,

$$\mathbb{E} [\widehat{\beta}_{j,k}^{(m)}] = \frac{(-1)^{|m|}}{n} \sum_{i=1}^n \mathbb{E} [\overline{\psi}_{j,k}^{(m)}(X_i)] = \beta_{j,k}^{(m)}.$$

Using Corollary 4 with $p = 2$ yields the following upper bound for its variance:

$$\text{Var} (\widehat{\beta}_{j,k}^{(m)}) \leq M \int_{\mathbb{T}^d} |\psi_{j,k}^{(m)}(\vartheta)|^2 \rho(d\vartheta) \leq C_2^* M B^{2j|m|}. \quad (3.7)$$

Additional probabilistic bounds on the empirical coefficients are given in the following Lemma 9, which can be read as the counterpart in our setting of ([Baldi et al., 2009a, Lemma 16](#)).

Lemma 9. Let $\widehat{\beta}_{j,k}^{(m)}$ be given by (3.1). Hence, for any $j \geq 0$ such that $B^{j(d+2|m|)} \leq \sqrt{n}$ and for $k = 1, \dots, K_j$, it holds that

$$\Pr \left(\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq x \right) \leq 2 \exp \left(- \frac{nx^2}{2B^{j|m|} \left(C_2^* M + \frac{1}{3} C_\infty^* \sqrt{nx} \right)} \right) \quad \text{for } x > 0; \quad (3.8)$$

$$\mathbb{E} \left[\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^\eta \right] \lesssim n^{-\frac{\eta}{2}} B^{j|m|\frac{\eta}{2}} \quad \text{for } \eta \geq 1; \quad (3.9)$$

$$\mathbb{E} \left[\sup_{k=1, \dots, K_j} \left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^\eta \right] \lesssim (j+1)^\eta n^{-\frac{\eta}{2}} B^{j|m|\frac{\eta}{2}} \quad \text{for } \eta \geq 1. \quad (3.10)$$

The proof of Lemma 9 can be found in Section 5.1.

Finally, using Equation (3.8) in Lemma 9 leads to the following result.

Lemma 10. Let $\tau_{j,m,n}$ be given by (3.6). For $\kappa \geq 6 \frac{C_2^* M}{C_\infty^*}$, under the hypotheses given in Lemma 9, there exists $\gamma \leq \left(\frac{3}{4C_\infty^*}\right) \kappa$, such that

$$\Pr \left(\left| \hat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq \frac{\tau_{j,m,n}}{2} \right) \lesssim n^{-\gamma}. \quad (3.11)$$

3.3 The main results

We will show that $\hat{f}^{(m)}$ achieves the optimal rates of convergence up to some logarithmic factors with respect to $L^p(\mathbb{T}^d)$ -loss functions in both the hard and soft thresholding frameworks. In other words, our goal is to evaluate the global error measure for the estimator $\hat{f}^{(m)}$, by studying the worst possible performance of the L^p -risk $\mathbb{E} \left[\left\| \hat{f}^{(m)} - f^{(m)} \right\|_{L^p(\mathbb{T}^d)} \right]$ over a given nonparametric regularity class $\left\{ \mathcal{B}_{r,q}^{s+|m|} : 1 < r < \infty, 1 \leq q \leq \infty, s > 0 \right\}$ of function spaces, that is, the minimax rate of convergence

$$\mathcal{R}_{p,n} \left(\mathcal{B}_{r,q}^s(R) \right) = \inf_{\hat{f}^{(m)}} \sup_{f \in \mathcal{B}_{r,q}^{s+|m|}(R)} \mathbb{E} \left[\left\| \hat{f}^{(m)} - f^{(m)} \right\|_{L^p(\mathbb{T}^d)} \right],$$

where the infimum is computed over all the possible estimators, $1 \leq p \leq \infty$ and $0 < R < \infty$ is the radius of the Besov ball on which f is defined. For $r > 0$, we will show that $\hat{f}^{(m)}$ is adaptive for the L^p -risk and for the scale of classes of Besov balls $\left\{ \mathcal{B}_{r,q}^{s+|m|}(R) : 1 < r < \infty, 1 \leq q \leq \infty, s > 0, 0 < R < \infty \right\}$, that is, for every choice of the parameters r, s, q and the radius R , there exists a constant $c_{r,s,q,R} > 0$, such that

$$\mathbb{E} \left[\left\| \hat{f}^{(m)} - f^{(m)} \right\|_{L^p(\mathbb{T}^d)} \right] \leq c_{r,s,q,R} \mathcal{R}_{p,n} \left(\mathcal{B}_{r,q}^s(R) \right),$$

see, e.g., (Hardle et al., 1997, Definition 11.1). Furthermore, we will prove that $\hat{f}^{(m)}$ attains the optimal rate of convergence, that is,

$$\sup_{f \in \mathcal{B}_{r,q}^{s+|m|}(R)} \mathbb{E} \left[\left\| \hat{f}^{(m)} - f^{(m)} \right\|_{L^p(\mathbb{T}^d)} \right] \approx \mathcal{R}_{p,n} \left(\mathcal{B}_{r,q}^s(R) \right),$$

see (Hardle et al., 1997, Definition 10.1). Our achievements are described by two theorems. The first is concerned with the upper bound for the $L^p(\mathbb{T}^d)$ -risk.

Theorem 11 (Upper bound). Let $f \in \mathcal{B}_{r,q}^{s+|m|}(R)$, where $s - \frac{d}{r} > 0$, let $f^{(m)} = D^m f$, and let $\hat{f}^{(m)}$ be defined by (3.2), with η given by (3.4)-(3.5). Then, for any $1 \leq p < \infty$, there exists a threshold constant $\kappa > 0$ such that it holds

$$\sup_{f \in \mathcal{B}_{r,q}^{s+|m|}(R)} \mathbb{E} \left[\left\| \hat{f}^{(m)} - f^{(m)} \right\|_{L^p(\mathbb{T}^d)}^p \right] \lesssim (\log n)^p \left[\frac{n}{\log n} \right]^{-\alpha(s,|m|,p,r)},$$

where

$$\alpha(s, |m|, p, r) = \begin{cases} \frac{ps}{2(s+|m|)+d} & \text{for } r \geq \frac{(2|m|+d)p}{2(s+|m|)+d} \quad (\text{regular zone}) \\ \frac{p(s+d(\frac{1}{p}-\frac{1}{r}))}{2[(s+|m|)+d(\frac{1}{2}-\frac{1}{r})]} & \text{for } r < \frac{(2|m|+d)p}{2(s+|m|)+d} \quad (\text{sparse zone}) \end{cases}. \quad (3.12)$$

Moreover, for $p = \infty$,

$$\sup_{f \in \mathcal{B}_{r,q}^s(\mathbb{R})} E \left[\left\| \widehat{f}^{(m)} - f^{(m)} \right\|_{L^\infty(\mathbb{T}^d)} \right] \lesssim \left[\frac{n}{\log n} \right]^{-\alpha(s, |m|, \infty, r)},$$

where

$$\alpha(s, |m|, \infty, r) = \frac{s - \frac{d}{r}}{2 \left[(s + |m|) - d \left(\frac{1}{r} - \frac{1}{2} \right) \right]}.$$

The names “regular” and “sparse” zones, standard in the literature (see, again, [Hardle et al. \(1997\)](#)), can be motivated as follows. In the regular zones, the hardest functions to be estimated are the ones characterized by a regular oscillatory behavior, that is, they are of a saw-teeth form. In the sparse zone the hardest function to estimate are those which are very regular everywhere but in small subsets of the domain, where they present strong irregularities. In this case, just few needlet coefficients $\beta_{j,k}^{(m)}$ are not null, and that justifies the name “sparse”. Observe that $p \leq 2$ corresponds always to the regular zone (see [Baldi et al. \(2009a\)](#)). For further details and discussions, the reader is referred to [Donoho et al. \(1996\)](#); [Hardle et al. \(1997\)](#).

As usual in the nonparametric setting, we can rewrite the L^p -risk as follows

$$\begin{aligned} \mathbb{E} \left[\left\| \widehat{f}^{(m)} - f^{(m)} \right\|_{L^p(\mathbb{T}^d)}^p \right] &= \mathbb{E} \left[\left\| \sum_{j=0}^{J_{n,m}-1} \eta \left(\widehat{\beta}_{j,k}^{(m)}, \tau_{j,m,n} \right) \psi_{j,k}(\vartheta) \psi_{j,k} - \sum_{j \in \mathbb{N}_0} \beta_{j,k} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \right] \\ &= \mathbb{E} \left[\left\| \sum_{j=0}^{J_{n,m}-1} \left[\eta \left(\widehat{\beta}_{j,k}^{(m)}, \tau_{j,m,n} \right) - \beta_{j,k} \right] \psi_{j,k} - \sum_{j \geq J_{n,m}} \beta_{j,k} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \right], \end{aligned}$$

so that it can be bounded as follows

$$\begin{aligned} &\mathbb{E} \left[\left\| \widehat{f}^{(m)} - f^{(m)} \right\|_{L^p(\mathbb{T}^d)}^p \right] \\ &\leq 2^{p-1} \left[\mathbb{E} \left[\left\| \sum_{j=0}^{J_{n,m}-1} \left[\eta \left(\widehat{\beta}_{j,k}^{(m)}, \tau_{j,m,n} \right) - \beta_{j,k} \right] \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \right] + \left\| \sum_{j \geq J_{n,m}} \beta_{j,k} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \right] \\ &= 2^{p-1} (\Sigma_p + D_p), \end{aligned}$$

with the natural extension for $p = \infty$.

The term Σ_p can be read as the stochastic error to the replacement of the true needlet coefficients with the selected empirical ones, and D_p is the deterministic error which arises when we select only a finite set of empirical coefficients (see also [Baldi et al. \(2009a\)](#)). While the bias term D_p does not affect the rate of convergence for $s > \frac{d}{p}$, the asymptotic behavior of the stochastic error Σ_p is established thanks to the so-called *optimal bandwidth selection*, that is, the resolution level $J_{s,m}$ defined by

$$J_{s,m} : B^{J_{s,m}} \approx \begin{cases} \left(\frac{n}{\log n} \right)^{\frac{1}{2(s+|m|)+d}} & \text{(regular zone)} \\ \left(\frac{n}{\log n} \right)^{\frac{1}{2(s+|m|+d(\frac{1}{2}-\frac{1}{2}))}} & \text{(sparse zone)} \end{cases} \quad (3.13)$$

Remark 12. Following [Baldi et al. \(2009a\)](#), but also [Efromovich \(1985\)](#); [Kerkyacharian et al. \(1996\)](#), both in the regular and in the sparse zones for any $k = 1, \dots, K_j$, $|\beta_{j,k}| \geq \tau_{j,m,n}$ implies that $j \leq J_{s,m}$. Contrarily, the converse implication is not true. Anyway, in this case, the coefficient $\widehat{\beta}_{j,k}^{(m)}$ should be discarded, since its error would be of order $B^{j|m|}n^{-\frac{1}{2}}$, as shown in (3.7) (see also [Durastanti \(2016\)](#)), and this consideration motivates the choice of the threshold $\tau_{j,m,n}$ in (3.4). The true value of s is unknown and, then, establishing explicitly $J_{s,m}$ is not possible. Nevertheless, the sum (3.2) truncated at $J_{n,m}$ includes all the terms up to $J_{s,m}$, since

$$J_{s,m} \leq J_{n,m}.$$

Finally, the next result establishes a lower bound for the rate of convergence, yielding optimality.

Theorem 13 (Lower bound). *Let $f \in \mathcal{B}_{r,q}^{s+|m|}(G)$, where $s - \frac{d}{r} > 0$, let $\widehat{f}^{(m)}$ be defined by (3.2), with η given by (3.4) or (3.5). Hence, if $1 \leq p \leq \infty$*

$$\sup_{f \in \mathcal{B}_{r,q}^{s+|m|}(G)} \mathbb{E} \left[\left\| \widehat{f}^{(m)} - f^{(m)} \right\|_{L^p(\mathbb{T}^d)}^p \right] \gtrsim n^{-\alpha(s,|m|,p,r)},$$

where

$$\alpha(s, |m|, p, r) = \begin{cases} \frac{ps}{2(s+|m|)+d} & \text{for } r \geq \frac{(2|m|+d)p}{2(s+|m|)+d} \quad (\text{regular zone}) \\ \frac{p(s+d(\frac{1}{p}-\frac{1}{r}))}{2[(s+|m|)+d(\frac{1}{2}-\frac{1}{r})]} & \text{for } r < \frac{(2|m|+d)p}{2(s+|m|)+d} \quad (\text{sparse zone}) \end{cases},$$

as given by (3.12).

4 Numerical evidence

In this section, we produce the results of some numerical experiments for $d = 1, 2$. We present the empirical evaluation of the L^p -risks ($p = 2, 3, 5, \infty$), computed thanks to local thresholding techniques for the first and the second derivatives of some test density functions with respect to the choice of several values of the truncation bandwidth $J_{n,m}$, the cardinality of cubature points K_j for any $j = 1, \dots, J_{n,m}$, and the threshold constant κ . The weight function b has been defined as the suitably rescaled primitive of the function $u \mapsto \exp(-(1-x^2)^{-1})$ (cf. [Figure 2](#) and see also [Marinucci and Peccati \(2011\)](#)). Note that, as the simulations are produced over finite samples, they should be read as a reasonable hint.

First, fixed $n \in \mathbb{N}$, (3.3) hints a value for truncation bandwidth $J_{n,m}$. For any fixed n, m , let L_{\max} denote the highest eigenvalue for frequency indexes admitted within the frame $\{\psi_{j,k}^{(m)}; j = 1, \dots, J_{n,m}, k = 1, \dots, K_j\}$ without being annihilated by the weight function b . For $n = 2000, 10000$, we have $J_{2000,1} = 3$ ($L_{\max} = 15$), $J_{2000,2} = 2$ ($L_{\max} = 7$), for the first and the second derivatives respectively, and $J_{10000,1} = 4$ ($L_{\max} = 31$), $J_{10000,2} = 3$ ($L_{\max} = 15$).

Finally, concerning the thresholding constant κ , for which this paper only establishes a

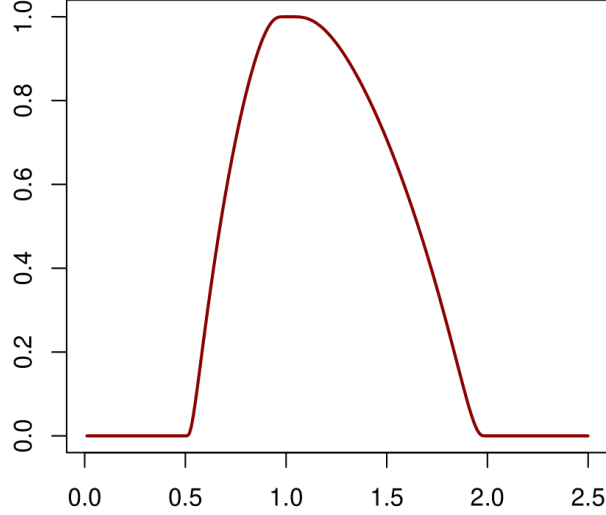


Figure 2: The needlet weight function b .

lower bound, we follow the lead given by (Baldi et al., 2009a, Section 6). For $q \in \mathbb{N}$, let us now define

$$I_q = \int_{\frac{1}{B}}^B u^q b^2(u) du;$$

Chosen the function b , $I_{|m|}/B^{j|m|}$ provides an indication about the size of the square of the L^2 -norm of $\psi_{j,k}^{(m)}$, since

$$\lim_{j \rightarrow \infty} \|\psi_{j,k}^{(m)}\|_{L^2(\mathbb{T}^d)}^2 = I_{|m|}.$$

In (Baldi et al., 2009a, Section 6), where for $m = 0$, the authors have chosen on the sphere the analogous of $\kappa = \kappa_0 M I_0$, with $k_0 > 0$ and $M = \|f\|_{L^\infty(\mathbb{T}^d)}$, since using Cauchy-Schwarz inequality yields

$$|\widehat{\beta}_{j,k}^{(0)}| = \int_{\mathbb{T}^d} f(\vartheta) \bar{\psi}_{j,k}(\vartheta) \lesssim M \|\psi_{j,k}\|_{L^2(\mathbb{T}^d)} \simeq M I_0.$$

In our case, for $d \in 1, 2$ and $m \in \mathbb{N}_0^d$ such that $|m| = 1, 2$, we choose

$$\kappa = \kappa_0 M I_{|m|},$$

trying different values for k_0 .

The uniform density. We define on the unit circle ($d = 1$) the following uniform density:

$$f_{\text{test};1}(\vartheta) = \frac{1}{2\pi}, \quad \vartheta \in \mathbb{T}^1,$$

L^p -risks (power p)		\tilde{f} and $k_0 = 0$	$k_0 = 1$	$k_0 = 2$	$k_0 = 3$	$k_0 = 4$
$n = 2000$						
$p = 2$	$m = 1$	0.00841	0.00722	0.00296	0.00249	0.00120
	$m = 2$	0.07470	0.07350	0.06129	0.05879	0.05643
$p = 3$	$m = 1$	$1.23 \cdot 10^{-3}$	$1.04 \cdot 10^{-3}$	$2.91 \cdot 10^{-4}$	$2.43 \cdot 10^{-4}$	$8.30 \cdot 10^{-5}$
	$m = 2$	0.03366	0.03282	0.02850	0.02798	0.02755
$p = 5$	$m = 1$	$4.19 \cdot 10^{-5}$	$3.37 \cdot 10^{-5}$	$3.82 \cdot 10^{-6}$	$2.99 \cdot 10^{-6}$	$5.09 \cdot 10^{-7}$
	$m = 2$	0.00947	0.00908	0.00822	0.00814	0.00779
$p = \infty$	$m = 1$	0.26254	0.26142	0.25438	0.20800	0.06324
	$m = 2$	0.28402	0.26142	0.17695	0.15665	0.09593
$n = 10000$						
$p = 2$	$m = 1$	0.01128	0.00900	0.00211	0.00012	0
	$m = 2$	0.11253	0.09892	0.07370	0.01987	0
$p = 3$	$m = 1$	$1.67 \cdot 10^{-3}$	$1.28 \cdot 10^{-3}$	$1.94 \cdot 10^{-4}$	$4.07 \cdot 10^{-7}$	0
	$m = 2$	0.05723	0.04964	0.03610	0.00555	0
$p = 5$	$m = 1$	$4.97 \cdot 10^{-5}$	$3.63 \cdot 10^{-5}$	$2.49 \cdot 10^{-6}$	$5.84 \cdot 10^{-9}$	0
	$m = 2$	0.02165	0.01757	0.01256	0.00059	0
$p = \infty$	$m = 1$	0.26721	0.25019	0.14390	0.04776	0
	$m = 2$	0.28172	0.26501	0.08387	0.13909	0

Table 1: Test density function $f_{\text{test},1}$: some L^p -risk evaluations (p -th powers) for $m = 1, 2$.

so that

$$\beta_{j,k}^{(m)} = 0 \quad \text{for any } j \in \mathbb{N}, k = 1, \dots, K_j.$$

In this case, counting the number of coefficients surviving the thresholding procedure gives an overview of the performance of our proposal. A measure of the goodness of the fit is obtained by estimating the p -powers of the L^p -risks, by means of the sums of the p -th powers of the selected coefficients.

For $n = 2000, 10000$, all the empirical needlet coefficients ($m = 0, 1, 2$) are discarded for $k_0 = 5.1$ and $k_0 = 3.8$. Table 1 provides some evidence on the rate, collecting the estimates of the p powers of the L^p -norms, $p = 2, 3, 5, \infty$ for $k_0 = 0, 1, 2, 3, 4$, showing that the L^p -risk decreases along with p and n . Note that the case $k_0 = 0$ corresponds to the so-called linear estimator, where no selection on the needlet coefficients is performed. Finally, a comparison is provided with the estimator $\tilde{f}^{(m)}$, built by means of the standard harmonic basis on \mathbb{T}^d and defined by $\tilde{f}(\vartheta) = \sum_{\ell \in L_n} \tilde{a}_\ell^{(m)} S_\ell(\vartheta)$, where

$$\tilde{a}_\ell^{(m)} = \frac{(-1)^m}{n} \sum_{i=1}^n \bar{S}_\ell^{(m)}(X_i)$$

and $L_n = \{\ell \in \mathbb{Z}^d : |\ell| \leq B^{J_{n,m+1}}\}$. Observe that, this choice of L_n the \tilde{f} and the needlet linear estimator are characterized by the same $L^p(\mathbb{T}^d)$ -risk, the latter being a linear combination of the former. As expected, for the highest value of k_0 , no coefficient survives the thresholding procedure and then we find a higher value for all the risks.

L^p -risks		\tilde{f} and $k_0 = 0$	$k_0 = 2$	$k_0 = 4$	$k_0 = 8$	$k_0 = 25$
$n = 2000$						
$p = 2$	$m = 1$	$2.05 \cdot 10^{-2}$	$1.20 \cdot 10^{-2}$	$2.04 \cdot 10^{-3}$	$3.66 \cdot 10^{-5}$	$1.26 \cdot 10^{-2}$
	$m = 2$	0.03015	0.01800	0.00019	0.00019	0.05055
$p = 3$	$m = 1$	$4.89 \cdot 10^{-3}$	$2.40 \cdot 10^{-3}$	$2.70 \cdot 10^{-4}$	$3.01 \cdot 10^{-7}$	$1.71 \cdot 10^{-3}$
	$m = 2$	$7.41 \cdot 10^{-3}$	$3.59 \cdot 10^{-3}$	$3.74 \cdot 10^{-6}$	$3.74 \cdot 10^{-6}$	$1.36 \cdot 10^{-2}$
$p = 5$	$m = 1$	$4.27 \cdot 10^{-4}$	$1.42 \cdot 10^{-4}$	$6.22 \cdot 10^{-6}$	$2.52 \cdot 10^{-11}$	$3.47 \cdot 10^{-5}$
	$m = 2$	$5.60 \cdot 10^{-4}$	$1.84 \cdot 10^{-4}$	$1.80 \cdot 10^{-9}$	$1.80 \cdot 10^{-9}$	$1.11 \cdot 10^{-3}$
$p = \infty$	$m = 1$	0.40043	0.31739	0.18048	0.01147	0.15915
	$m = 2$	0.35177	0.28271	0.02717	0.02717	0.31830
$n = 10000$						
$p = 2$	$m = 1$	$1.82 \cdot 10^{-2}$	$2.95 \cdot 10^{-3}$	$4.17 \cdot 10^{-6}$	$4.17 \cdot 10^{-6}$	$1.26 \cdot 10^{-2}$
	$m = 2$	$1.49 \cdot 10^{-1}$	$4.01 \cdot 10^{-2}$	$3.72 \cdot 10^{-5}$	$3.72 \cdot 10^{-5}$	$1.27 \cdot 10^{-2}$
$p = 3$	$m = 1$	$3.92 \cdot 10^{-3}$	$3.08 \cdot 10^{-4}$	$1.19 \cdot 10^{-8}$	$1.19 \cdot 10^{-8}$	$1.71 \cdot 10^{-3}$
	$m = 2$	$8.71 \cdot 10^{-2}$	$2.26 \cdot 10^{-2}$	$3.18 \cdot 10^{-7}$	$3.18 \cdot 10^{-7}$	$1.36 \cdot 10^{-2}$
$p = 5$	$m = 1$	$2.62 \cdot 10^{-4}$	$4.57 \cdot 10^{-6}$	$1.22 \cdot 10^{-13}$	$1.22 \cdot 10^{-13}$	$3.47 \cdot 10^{-5}$
	$m = 2$	$4.52 \cdot 10^{-2}$	$9.82 \cdot 10^{-3}$	$2.89 \cdot 10^{-11}$	$2.89 \cdot 10^{-11}$	$1.10 \cdot 10^{-3}$
$p = \infty$	$m = 1$	0.38401	0.15848	0.00400	0.00400	0.15915
	$m = 2$	1.03924	0.82377	0.01186	0.01186	0.31830

Table 2: Test density function $f_{\text{test};2}$: some L^p -risk evaluations, for $m = 1$ (first entry) and $m = 2$ (second entry)

The trigonometric density. The next density function on the unit circle $d = 1$ is defined as follows,

$$f_{\text{test};2}(\vartheta) = \frac{1}{2\pi} \left(1 + \frac{1}{2} \cos(2\vartheta) \right), \quad \vartheta \in \mathbb{T}^1.$$

This density features two local maxima (in $\pi/2$ and $3/2\pi$) and two local minima (in 0 and π). Using (2.5) yields

$$\beta_{j,k}^{(m)} = \lambda_{j,k} (-1)^m 2^m b \left(\frac{2}{B^j} \right) \cos(2\xi_{j,k}) \quad \text{for any } j \in \mathbb{N}, k = 1, \dots, K_j,$$

which are not null only if $j = 1$, in view of our choice of b . In this case, we aim to find an optimal threshold which allows to discard all the empirical coefficients with $j \neq 1$ but, at the same time, to keep the most relevant one with $j = 1$ (corresponding to the peaks). In this sense, choosing a too high value for k_0 produces a worse L^p -risk estimate.

Table 2 shows the estimates of the p -powers of the L^p -norms, $p = 2, 3, 5, \infty$, for $k_0 = 2, 4, 8, 25$ (again, the case $k_0 = 0$ is the linear estimator). Table 3 collects the percentage of surviving coefficients. As expected, the better estimates for L^p -risks correspond to estimators where only coefficients with $j = 1$ survive.

Surviving coefficients (%)		$k_0 = 0$	$k_0 = 2$	$k_0 = 4$	$k_0 = 8$	$k_0 = 25$
$n = 2000$						
$j = 0$	$m = 1$	100	50	0	0	0
	$m = 2$	100	50	0	0	0
$j = 1$	$m = 1$	100	50	50	50	0
	$m = 2$	100	50	50	50	0
$j = 2$	$m = 1$	100	12.5	0	0	0
	$m = 2$	100	31.25	0	0	0
$j = 3$	$m = 1$	100	21.875	3.125	0	0
	$m = 2$	NA	NA	NA	NA	NA
$n = 10000$						
$j = 0$	$m = 1$	100	50	0	0	0
	$m = 2$	100	50	0	0	0
$j = 1$	$m = 1$	100	50	50	50	0
	$m = 2$	100	62.50	50	50	0
$j = 2$	$m = 1$	100	0	0	0	0
	$m = 2$	100	12.50	0	0	0
$j = 3$	$m = 1$	100	6.25	0	0	0
	$m = 2$	100	6.25	0	0	0
$j = 4$	$m = 1$	100	6.25	0	0	0
	$m = 2$	NA	NA	NA	NA	NA

Table 3: Test density function $f_{\text{test};2}$: percentage of surviving coefficients, for first and second derivatives at $k_0 = 0, 2, 4, 8, 25$.

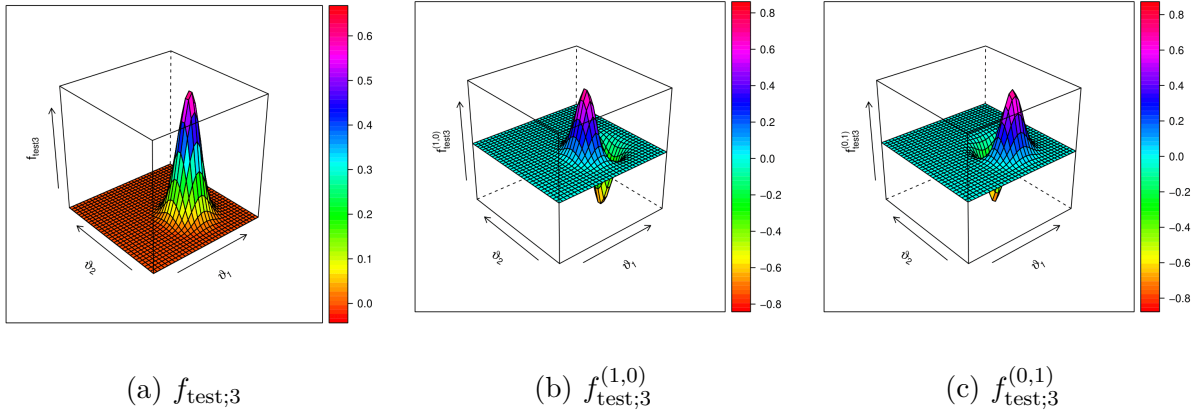


Figure 3: The test density $f_{\text{test};3}$ and its first derivatives

L^p -risks, $n = 10000$		\tilde{f} and $k_0 = 0$	$k_0 = 0.5$	$k_0 = 1$	$k_0 = 8$	$k_0 = 15$
$p = 2$	$m = (1, 0)$	0.00806	0.00699	0.00485	0.00601	0.00781
	$m = (0, 1)$	0.00889	0.00711	0.00493	0.00621	0.00801
$p = 3$	$m = (1, 0)$	0.00312	0.00227	0.00100	0.00102	0.00216
	$m = (0, 1)$	0.00343	0.00271	0.00103	0.00110	0.00221
$p = 5$	$m = (1, 0)$	$1.05 \cdot 10^{-3}$	$5.65 \cdot 10^{-4}$	$9.53 \cdot 10^{-5}$	$5.12 \cdot 10^{-5}$	$3.42 \cdot 10^{-4}$
	$m = (0, 1)$	$2.07 \cdot 10^{-3}$	$6.24 \cdot 10^{-4}$	$1.20 \cdot 10^{-4}$	$9.81 \cdot 10^{-5}$	$4.11 \cdot 10^{-4}$
$p = \infty$	$m = (1, 0)$	0.94465	0.80713	0.36937	0.43902	0.57350
	$m = (0, 1)$	0.95493	0.82432	0.36937	0.47981	0.57483

Table 4: Test density function $f_{\text{test};3}$: some L^p -risk evaluations for the first derivatives

Wrapped normal distribution. Here we consider the density function of a 2-d wrapped normal distribution, defined as:

$$f_{\text{test};3}(\vartheta_1, \vartheta_2) = \frac{1}{2\pi\sigma_1\sigma_2} \sum_{k_1=-100}^{100} \sum_{k_2=-100}^{100} e^{-\frac{(\vartheta_1-\mu_1+2\pi k_1)^2}{2\sigma_1^2} - \frac{(\vartheta_2-\mu_2+2\pi k_2)^2}{2\sigma_2^2}}, \quad (\vartheta_1, \vartheta_2) \in \mathbb{T}^2.$$

We choose $\mu_1 = 3/4\pi$, $\mu_2 = 5/4\pi$, $\sigma_1 = \sigma_2 = 0.5$. Figure 3 plots the graphs of $f_{\text{test};3}$ and both its derivatives. Our goal is to study the two first derivatives of $f_{\text{test};3}$ testing several values of k_0 . The goodness of our choice is evaluated thanks to the estimates of the L^p -risks as described above for $n = 10000$. Table 4 contains the estimates of the L^p -norm, $p = 2, 3, 5, \infty$ for $k_0 = 0, 0.5, 1, 8, 15$. Some of the estimators here obtained are shown in Figure 4. The optimal choice of k lies around $k_0 = 1$, for which we obtain better L^p -risks. As shown in Figures 4b,4e the graph is evidently smoother and less perturbed than the linear estimator ($k_0 = 0$) (see Figures 4a,4b). For $k_0 = 15$, the peak is lower than the theoretical one, so that smaller fluctuations appear more evidently also in the graph (Figures 4c,4f).

The interaction between multiresolution and the threshold technique shows here its advantages: finer levels of resolution are non discarded only at locations where they the theoretical curve is less regular. Hence, the multiresolution properties of the thresholding needlet estimator allow for local adaptation if the considered derivative presents peaks and different slopes.

5 Proofs

5.1 Auxiliary results on needlet derivatives and Besov spaces

Proof of Lemma 1. First of all, observe that, for any $i = 1, \dots, d$ and for any $n \in \mathbb{Z}$, it holds that

$$\frac{\partial^{m_i}}{\partial \vartheta_i^{m_i}} S_\ell(\vartheta) = (-1)^{m_i} \ell_i^{m_i} S_\ell(\vartheta).$$

Thus,

$$D^m S_\ell(\vartheta) = (-1)^{|m|} \prod_{i=1}^d \ell_i^{m_i} S_\ell(\vartheta).$$

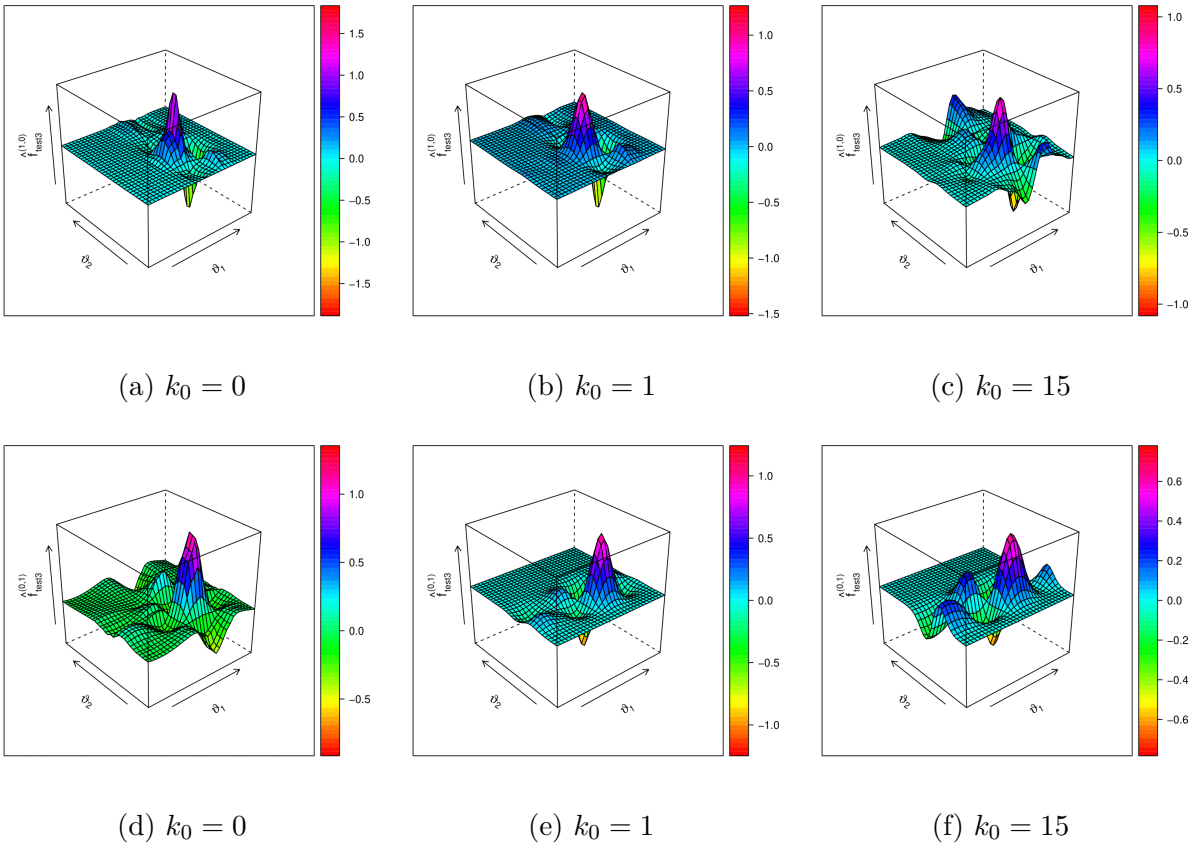


Figure 4: The density estimators $\hat{f}_{\text{test};3}^{(1,0)}$ (top panels), and $\hat{f}_{\text{test};3}^{(0,1)}$ (bottom panels), for $k_0 = 0, 1, 15$

Using (2.5) yields

$$\begin{aligned}
D^m \psi_{j,k}^{(m)}(\vartheta) &= \sqrt{\lambda_{j,k}} \sum_{\ell \in \Lambda_j^d} b\left(\frac{\varepsilon_\ell}{B^j}\right) \overline{S}_\ell(\xi_{j,k}) D^m S_\ell(\vartheta) \\
&= \sqrt{\lambda_{j,k}} \sum_{\ell \in \Lambda_j^d} b\left(\frac{\varepsilon_\ell}{B^j}\right) (-1)^{|m|} \prod_{i=1}^d \ell_i^{m_i} \overline{S}_\ell(\xi_{j,k}) S_\ell(\vartheta) \\
&= \sqrt{\lambda_{j,k}} B^{j|m|} \sum_{\ell \in \Lambda_j^d} b_j^{(m)}(\ell) \overline{S}_\ell(\xi_{j,k}) S_\ell(\vartheta),
\end{aligned}$$

as claimed. \square

Proof of Lemma 5. First of all, observe that (2.6) is obtained by integrating iteratively (2.2) by parts, thanks to the periodicity of the function f over \mathbb{T}^d . As far as (2.7) is concerned, for any $\vartheta \in \mathbb{T}^d$, observe first of all that

$$\begin{aligned}
\sum_{k=1}^{K_j} \psi_{j,k}(\vartheta) \overline{\psi_{j,k}^{(m)}}(\vartheta') &= \sum_{k=1}^{K_j} \lambda_{j,k} B^{j|m|} \sum_{\ell \in \Lambda_j^d} \sum_{\ell' \in \Lambda_j^d} b_j^{(m)}(\ell) b_j^{(0)}(\ell') \overline{S}_\ell(\vartheta') S_\ell(\xi_{j,k}) \overline{S}_{\ell'}(\xi_{j,k}) S_{\ell'}(\vartheta) \\
&= (-1)^{|m|} \sum_{\ell \in \Lambda_j^d} \left(\prod_{i=1}^d \ell_i^{m_i} \right) b^2\left(\frac{\varepsilon_\ell}{B^j}\right) S_\ell(\vartheta - \vartheta').
\end{aligned}$$

Thus, it holds that

$$\begin{aligned}
\sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k}(\vartheta) &= \int_{\mathbb{T}^d} f(\vartheta') \sum_{k=1}^{K_j} \overline{\psi_{j,k}^{(m)}}(\vartheta') \psi_{j,k}(\vartheta) \rho(d\vartheta') \\
&= (-1)^{|m|} \sum_{\ell \in \Lambda_j^d} \left(\prod_{i=1}^d \ell_i^{m_i} \right) b^2\left(\frac{\varepsilon_\ell}{B^j}\right) S_\ell(\vartheta) \int_{\mathbb{T}^d} f(\vartheta') \overline{S}_\ell(\vartheta') \rho(d\vartheta') \\
&= \sum_{k=1}^{K_j} \lambda_{j,k} B^{j|m|} \sum_{\ell \in \Lambda_j^d} \sum_{\ell' \in \Lambda_j^d} b_j^{(m)}(\ell) b_j^{(0)}(\ell') \langle f, S_{\ell'} \rangle_{L^2(\mathbb{T}^d)} S_{\ell'}(\xi_{j,k}) \overline{S}_\ell(\xi_{j,k}) S_\ell(\vartheta) \\
&= \sum_{k=1}^{K_j} \beta_{j,k} \psi_{j,k}^{(m)}(\vartheta),
\end{aligned}$$

as claimed. \square

5.2 Auxiliary probabilistic results

Proof of Lemma 9. This proof follows strictly the one of (Baldi et al., 2009a, Lemma 16), see also Durastanti et al. (2011).

Observe that (3.8) is obtained by means of the Bernstein's inequality (see, for example, Hardle et al. (1997)), applied to the set $\left\{ \overline{\psi_{j,k}^{(m)}}(X_1), \dots, \overline{\psi_{j,k}^{(m)}}(X_n) \right\}$, observing additionally that

$$\text{Var}\left(\overline{\psi_{j,k}^{(m)}}(X_i)\right) \leq \mathbb{E}\left[\left(\overline{\psi_{j,k}^{(m)}}(X_i)\right)^2\right] \leq M \|\psi_{j,k}^{(m)}\|_{L^2(\mathbb{T}^d)}^2 \leq M C_2^* B^{2j|m|},$$

while

$$\left| \overline{\psi_{j,k}^{(m)}}(X_i) \right| \leq \left\| \psi_{j,k}^{(m)} \right\|_{L^\infty(\mathbb{T}^d)} \leq C_\infty^* B^{j(|m|+\frac{d}{2})} \leq C_\infty^* B^{j|m|} \sqrt{n},$$

since $B^{dj} \leq n$.

Now, in order to prove (3.9) and (3.10), first observe that, for $x > 0$,

$$\Pr \left(\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq x \right) \leq 2 \left(\exp \left(-\frac{nx^2}{4B^{j|m|}C_2^*M} \right) \right) + \exp \left(-\frac{3\sqrt{nx}}{4B^{j|m|}C_\infty^*} \right).$$

Hence, on the one hand, for $\eta \geq 1$, we have

$$\begin{aligned} \mathbb{E} \left[\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^\eta \right] &= \int_0^\infty x^{\eta-1} \Pr \left(\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq x \right) dx \\ &\leq 2 \int_0^\infty x^{\eta-1} \left(\exp \left(-\frac{nx^2}{4B^{j|m|}C_2^*M} \right) \right) + \exp \left(-\frac{3\sqrt{nx}}{4B^{j|m|}C_\infty^*} \right) dx \\ &\lesssim n^{-\frac{\eta}{2}} B^{j|m|\frac{\eta}{2}} \end{aligned}$$

using $B^{j|m|} \leq n$. On the other hand, let us preliminarily fix

$$a = \frac{1}{(\log B)} \max \left(\frac{8}{3} d C_\infty^*, 2\sqrt{2} d C_2^* M \right)$$

and then write

$$\begin{aligned} \mathbb{E} \left[\sup_{k=1, \dots, K_j} \left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^\eta \right] &\leq \int_0^{\frac{a_j B^{j\frac{|m|}{2}}}{\sqrt{n}}} x^{\eta-1} dx \\ &\quad + 2 \int_{\frac{a_j B^{j\frac{|m|}{2}}}{\sqrt{n}}}^\infty x^{\eta-1} \Pr \left(\sup_{k=1, \dots, K_j} \left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq x \right) dx \\ &\leq 2 \int_0^\infty x^{\eta-1} \left(\exp \left(-\frac{nx^2}{4B^{j|m|}C_2^*M} \right) \right) + \exp \left(-\frac{3\sqrt{nx}}{4B^{j|m|}C_\infty^*} \right) dx. \end{aligned}$$

If $x \geq a_j B^{j\frac{|m|}{2}} / \sqrt{n}$, we have

$$B^{jd} e^{-\frac{-nx^2}{4B^{j|m|}C_2^*M}} = e^{-\frac{-nx^2}{8B^{j|m|}C_2^*M} - \frac{-nx^2}{8B^{j|m|}C_2^*M} + (\log B)jd} \leq e^{-\frac{-nx^2}{8B^{j|m|}C_2^*M}},$$

and

$$B^{jd} e^{-\frac{-3\sqrt{nx}}{4B^{j|m|}C_\infty^*}} = e^{-\frac{-3\sqrt{nx}}{8B^{j|m|}C_\infty^*} - \frac{-3\sqrt{nx}}{8B^{j|m|}C_\infty^*} + (\log B)jd} \leq e^{-\frac{-3\sqrt{nx}}{8B^{j|m|}C_\infty^*}},$$

so that

$$\mathbb{E} \left[\sup_{k=1, \dots, K_j} \left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^\eta \right] \lesssim \left(\frac{j+1}{\sqrt{n}} \right)^\eta,$$

as claimed. \square

Proof of Lemma 10. The proof follows directly (3.8), choosing $x = \tau_{j,m,n}$ and observing that

$$\begin{aligned} \exp\left(-\frac{3\kappa^2 B^{j|m|} n}{4C_\infty^* \left(6\frac{C_2^* M}{C_\infty^*} + \kappa\right)}\right) &\leq \exp\left(-\frac{3\kappa B^{j|m|} n}{8C_\infty^*}\right) \\ &\lesssim n^{-\frac{3}{8C_\infty^*} \kappa} \\ &\lesssim n^{-\frac{\gamma}{2}}, \end{aligned}$$

given that

$$\gamma \leq \left(\frac{3}{8C_\infty^*}\right) \kappa,$$

as claimed. \square

5.3 Main results

Let us start by proving the upper bound.

Proof of Theorem 11. Let us consider $p < \infty$. As far as Σ is concerned, properly adapting to our problem the procedure presented in Baldi et al. (2009a) (see also Donoho et al. (1996); Durastanti et al. (2011)) and using (3.4) yield

$$\begin{aligned} \Sigma / J_{n,m}^{p-1} &\leq \sum_{j=0}^{J_{n,m}-1} \left\| \sum_{k=1}^{K_j} \mathbb{E} \left[\left(\eta \left(\widehat{\beta}_{j,k}^{(m)}, \tau_{j,m,n} \right) - \beta_{j,k}^{(m)} \right) \psi_{j,k} \right] \right\|_{L^p(\mathbb{T}^d)}^p \\ &= \sum_{j=0}^{J_{n,m}-1} \mathbb{E} \left[\left\| \sum_{k=1}^{K_j} \left(\eta \left(\widehat{\beta}_{j,k}^{(m)}, \tau_{j,m,n} \right) - \beta_{j,k}^{(m)} \right) \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \mathbb{1}_{\left\{ \left| \widehat{\beta}_{j,k}^{(m)} \right| \geq \tau_{j,m,n} \right\}} \mathbb{1}_{\left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\}} \right] \\ &+ \sum_{j=0}^{J_{n,m}-1} \mathbb{E} \left[\left\| \sum_{k=1}^{K_j} \left(\eta \left(\widehat{\beta}_{j,k}^{(m)}, \tau_{j,m,n} \right) - \beta_{j,k}^{(m)} \right) \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \mathbb{1}_{\left\{ \left| \widehat{\beta}_{j,k}^{(m)} \right| \geq \tau_{j,m,n} \right\}} \mathbb{1}_{\left\{ \left| \beta_{j,k}^{(m)} \right| < \frac{1}{2} \tau_{j,m,n} \right\}} \right] \\ &+ \sum_{j=0}^{J_{n,m}-1} \mathbb{E} \left[\left\| \sum_{k=1}^{K_j} \left(\eta \left(\widehat{\beta}_{j,k}^{(m)}, \tau_{j,m,n} \right) - \beta_{j,k}^{(m)} \right) \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \mathbb{1}_{\left\{ \left| \widehat{\beta}_{j,k}^{(m)} \right| < \tau_{j,m,n} \right\}} \mathbb{1}_{\left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\}} \right] \\ &+ \sum_{j=0}^{J_{n,m}-1} \mathbb{E} \left[\left\| \sum_{k=1}^{K_j} \left(\eta \left(\widehat{\beta}_{j,k}^{(m)}, \tau_{j,m,n} \right) - \beta_{j,k}^{(m)} \right) \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \mathbb{1}_{\left\{ \left| \widehat{\beta}_{j,k}^{(m)} \right| < \tau_{j,m,n} \right\}} \mathbb{1}_{\left\{ \left| \beta_{j,k}^{(m)} \right| < \frac{1}{2} \tau_{j,m,n} \right\}} \right] \\ &=: Aa + Au + Ua + Uu, \end{aligned}$$

where the regions considered are the analogous of Bb , Bs , Sb , Ss in Baldi et al. (2009a). As in Durastanti et al. (2011); Durastanti (2016), we prefer to modify this notation since B , b , and s are already used in the current work. While the upper bounds for Ua and Uu are the same for hard and soft thresholding, we have to follow two slightly different approach to bound Au and Ua . Furthermore, by the heuristic point of view, the bounds two cross/terms Au and Ua depend on the inequality (3.11) in Lemma 10. Indeed, in the

hard thresholding settings, using Cauchy–Schwarz inequality and Equations (3.9) and (3.11) leads to

$$\begin{aligned}
Au &\lesssim \sum_{j=0}^{J_{n,m}-1} \mathbb{E} \left[\left\| \sum_{k=1}^{K_j} (\widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)}) \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \mathbb{1} \left\{ \left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq \frac{\tau_{j,m,n}}{2} \right\} \right] \\
&\lesssim \sum_{j=0}^{J_{n,m}-1} B^{jd(\frac{p}{2}-1)} \sum_{k=1}^{K_j} \mathbb{E} \left[\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^p \mathbb{1} \left\{ \left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq \frac{\tau_{j,m,n}}{2} \right\} \right] \\
&\lesssim \sum_{j=0}^{J_{n,m}-1} B^{jd(\frac{p}{2}-1)} \sum_{k=1}^{K_j} \mathbb{E}^{\frac{1}{2}} \left[\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^{2p} \right] \Pr^{\frac{1}{2}} \left(\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq \frac{\tau_{j,m,n}}{2} \right) \\
&\lesssim J_{n,m} n^{-\frac{p}{2}} n^{-\frac{\gamma}{2}} \sum_{j=0}^{J_{n,m}-1} B^{\frac{pJ_{n,m}}{2}(d+2|m|)} \\
&\leq n^{-\frac{p+\gamma}{2}} \left(\frac{n}{\log n} \right)^{\frac{p}{2}} \leq (\log n)^{-\frac{p}{2}} n^{-\frac{\gamma}{2}}, \tag{5.1}
\end{aligned}$$

while in the soft thresholding framework, using additionally the triangle inequality yields

$$\begin{aligned}
Au &\lesssim \sum_{j=0}^{J_{n,m}-1} B^{jd(\frac{p}{2}-1)} \sum_{k=1}^{K_j} \mathbb{E}^{\frac{1}{2}} \left[\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} - \tau_{j,m,n} \right|^{2p} \right] \Pr^{\frac{1}{2}} \left(\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq \frac{\tau_{j,m,n}}{2} \right) \\
&\lesssim \sum_{j=0}^{J_{n,m}-1} B^{jd(\frac{p}{2}-1)} \sum_{k=1}^{K_j} \left(\mathbb{E} \left[\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^{2p} \right] + \tau_{j,m,n}^{2p} \right)^{\frac{1}{2}} \Pr^{\frac{1}{2}} \left(\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| \geq \frac{\tau_{j,m,n}}{2} \right) \\
&\lesssim J_{n,m} n^{-\frac{p}{2}} n^{-\frac{\gamma}{2}} \sum_{j=0}^{J_{n,m}-1} B^{\frac{pJ_{n,m}}{2}(d+2|m|)} \\
&\leq n^{-\frac{p+\gamma}{2}} \left(\frac{n}{\log n} \right)^{\frac{p}{2}} \leq (\log n)^{-\frac{p}{2}} n^{-\frac{\gamma}{2}}. \tag{5.2}
\end{aligned}$$

Note that, in both the cases, for (5.1) and (5.2) we have derived the same bound. As far as the other cross/term is considered, in both the cases, it holds that

$$\begin{aligned}
Ua &\lesssim \sum_{j=0}^{J_{n,m}-1} \left\| \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \Pr \left(\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right| > \tau_{j,m,n} \right) \\
&\lesssim \left\| f^{(m)} \right\|_{L^p(\mathbb{T}^d)}^p n^{-\gamma}. \tag{5.3}
\end{aligned}$$

As far as Aa , and Uu are concerned, we derive their upper bounds by using the tail behavior in the Besov balls $\mathcal{B}_{r,q}^s(G)$, the crucial role of the optimal bandwidth selection $J_{s,m}$ defined by (3.13), and the bound on the centered moments of $\widehat{\beta}_{j,k}^{(m)}$ given in Lemma 9. Indeed, in the hard thresholding settings, using (3.9), we obtain that

$$Aa \lesssim \sum_{j=0}^{J_{n,m}-1} \mathbb{E} \left[\left\| \sum_{k=1}^{K_j} (\widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)}) \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \right] \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\}$$

$$\begin{aligned}
&= \sum_{j=0}^{J_{n,m-1}} \sum_{k=1}^{K_j} \mathbb{E} \left[\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^p \right] \|\psi_{j,k}\|_{L^p(\mathbb{T}^d)}^p \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\} \\
&\lesssim n^{-\frac{p}{2}} \sum_{j=0}^{J_{n,m-1}} B^{jp|m|} \sum_{k=1}^{K_j} \|\psi_{j,k}\|_{L^p(\mathbb{T}^d)}^p \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\}
\end{aligned}$$

In the soft thresholding framework, similarly to (5.2), using the triangular inequality leads to

$$\begin{aligned}
Aa &\lesssim \sum_{j=0}^{J_{n,m-1}} \sum_{k=1}^{K_j} \mathbb{E} \left[\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} - \tau_{j,m,n} \right|^p \right] \|\psi_{j,k}\|_{L^p(\mathbb{T}^d)}^p \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\} \\
&\lesssim \sum_{j=0}^{J_{n,m-1}} \sum_{k=1}^{K_j} \left(\mathbb{E} \left[\left| \widehat{\beta}_{j,k}^{(m)} - \beta_{j,k}^{(m)} \right|^p \right] + \tau_{j,m,n}^p \right) \|\psi_{j,k}\|_{L^p(\mathbb{T}^d)}^p \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\} \\
&\lesssim n^{-\frac{p}{2}} \sum_{j=0}^{J_{n,m-1}} B^{jp|m|} \sum_{k=1}^{K_j} \|\psi_{j,k}\|_{L^p(\mathbb{T}^d)}^p \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\}
\end{aligned}$$

Now, using the definition of the optimal bandwidth selection (3.13), we can rewrite in both the cases Aa as the sum of two finite series, that is,

$$\begin{aligned}
Aa &\leq \sum_{j=0}^{J_{s,m-1}} \sum_{k=1}^{K_j} n^{-\frac{p}{2}} \|\psi_{j,k}^{(m)}\|_{L^p(\mathbb{T}^d)}^p \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\} \\
&\quad + \sum_{j=J_{s,m}}^{J_{n,m-1}} \sum_{k=1}^{K_j} n^{-\frac{p}{2}} \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\} \|\psi_{j,k}^{(m)}\|_{L^p(\mathbb{T}^d)}^p \\
&=: Aa_1 + Aa_2.
\end{aligned} \tag{5.4}$$

Finally, in both the hard and soft thresholding frameworks, we have that

$$\begin{aligned}
Uu &= \sum_{j=0}^{J_{n,m-1}} \left\| \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| < 2\tau_{j,m,n} \right\} \\
&\leq \sum_{j=0}^{J_{s,m-1}} \sum_{k=1}^{K_j} \left| \beta_{j,k}^{(m)} \right|^p B^{jd(\frac{p}{2}-1)} \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| < 2\tau_{j,m,n} \right\} + \sum_{j=J_{s,m}}^{J_{n,m-1}} \left\| \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \\
&=: Uu_1 + Uu_2.
\end{aligned} \tag{5.5}$$

Regular zone. First of all, combining (5.1) and (5.3) we choose γ such that

$$\gamma \geq \frac{2sp}{2(s+|m|)+d}.$$

and, then,

$$n^{-\frac{\gamma}{2}} \leq n^{-\frac{sp}{2(s+|m|)+d}}.$$

Observe now (5.4). We have that

$$Aa_1 = \sum_{j=0}^{J_{s,m-1}} \sum_{k=1}^{K_j} n^{-\frac{p}{2}} \|\psi_{j,k}^{(m)}\|_{L^p(\mathbb{T}^d)}^p$$

$$\begin{aligned}
&\lesssim n^{-\frac{p}{2}} \sum_{j=0}^{J_{s,m}-1} \sum_{k=1}^{K_j} B^{j(p|m|+d(\frac{p}{2}-1))} \\
&\lesssim n^{-\frac{p}{2}} \sum_{j=0}^{J_{s,m}-1} B^{\frac{jp}{2}(2|m|+d)} \\
&\lesssim n^{-\frac{p}{2}} B^{\frac{J_{s,m}p}{2}(2|m|+d)} \\
&\lesssim (\log n)^{\frac{p(2|m|+d)}{2(2(s+|m|)+d)}} n^{-\frac{ps}{2(s+|m|)+d}},
\end{aligned}$$

since

$$\begin{aligned}
\frac{p(2|m|+d)}{2(2(s+|m|)+d)} - \frac{p}{2} &= \frac{p(2|m|+d) - p(2(s+|m|)+d)}{2(2(s+|m|)+d)} \\
&= \frac{-sp}{2(s+|m|)+d}
\end{aligned} \tag{5.6}$$

On the other hand,

$$\begin{aligned}
Aa_2 &= \sum_{j=J_{s,m}}^{J_{n,m}-1} \sum_{k=1}^{K_j} n^{-\frac{p}{2}} \|\psi_{j,k}^{(m)}\|_{L^p(\mathbb{T}^d)}^p \mathbb{1}\left\{|\beta_{j,k}^{(m)}| \geq \frac{1}{2}\tau_{j,m,n}\right\} \\
&\lesssim n^{-\frac{p}{2}} \sum_{j=J_{s,m}}^{J_{n,m}-1} \sum_{k=1}^{K_j} B^{j|m|p} \|\psi_{j,k}\|_{L^p(\mathbb{T}^d)}^p \frac{|\beta_{j,k}^{(m)}|^p}{\tau_{j,m,n}^p} \\
&\lesssim (\log n)^{-\frac{p}{2}} \sum_{j=J_{s,m}}^{J_{n,m}-1} \sum_{k=1}^{K_j} |\beta_{j,k}^{(m)}|^p \|\psi_{j,k}\|_{L^p(\mathbb{T}^d)}^p \\
&\lesssim (\log n)^{-\frac{p}{2}} \sum_{j=J_{s,m}}^{J_{n,m}-1} B^{-jsp} \\
&\lesssim (\log n)^{-\frac{p}{2}} B^{-J_{s,m}sp}.
\end{aligned}$$

Then, it holds that

$$Aa \lesssim (\log n)^{\frac{p(2|m|+d)}{2(2(s+|m|)+d)}} n^{-\frac{ps}{2(s+|m|)+d}}$$

Consider now (5.5). It holds that

$$\begin{aligned}
Uu_1 &= \sum_{j=0}^{J_{s,m}-1} \sum_{k=1}^{K_j} |\beta_{j,k}^{(m)}|^p B^{jd(\frac{p}{2}-1)} \mathbb{1}\left\{|\beta_{j,k}^{(m)}| < 2\tau_{j,m,n}\right\} \\
&\lesssim \sum_{j=0}^{J_{s,m}-1} 2^p \tau_{j,m,n}^p \sum_{k=1}^{K_j} B^{jd(\frac{p}{2}-1)} \\
&\lesssim \sum_{j=0}^{J_{s,m}-1} \left(\frac{\log n}{n}\right)^{\frac{p}{2}} B^{\frac{jp}{2}(2|m|+d)} \\
&\lesssim \left(\frac{\log n}{n}\right)^{\frac{p}{2}} B^{\frac{J_{s,m}p}{2}(2|m|+d)}
\end{aligned}$$

$$\lesssim (\log n)^{\frac{p}{2}} n^{-\frac{sp}{2(s+|m|)+d}},$$

in view of (5.6), while

$$\begin{aligned} Uu_2 &= \sum_{j=J_{s,m}}^{J_{n,m}-1} \left\| \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p \\ &\lesssim \sum_{j=J_{s,m}}^{J_{n,m}-1} B^{-jps} \\ &\lesssim B^{-J_{s,m}ps} \end{aligned}$$

Then,

$$Uu \lesssim (\log n)^{\frac{p}{2}} n^{-\frac{ps}{2(s+|m|)+d}}.$$

As far as D is concerned, observe that

$$\begin{aligned} D^{\frac{1}{p}} &\leq \sum_{j \geq J_{n,m}} \left\| \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)} \\ &= \sum_{j \geq J_{n,m}} \left\| \sum_{k=1}^{K_j} \beta_{j,k} \psi_{j,k}^{(m)} \right\|_{L^p(\mathbb{T}^d)} \\ &\leq \sum_{j \geq J_{n,m}} B^{j(|m|+d(\frac{1}{2}-\frac{1}{p}))} \|\beta_{j,\cdot}\|_{\ell^p} \\ &\leq \sum_{j \geq J_{n,m}} B^{-js} \|f\|_{\mathcal{B}_{r,q}^{s+|m|}} \\ &\leq B^{-J_{n,m}s} \|f\|_{\mathcal{B}_{r,q}^{s+|m|}}. \end{aligned}$$

Thus $D \lesssim \left(\frac{\log n}{n}\right)^{\frac{sp}{d+2|m|}}$ As in Baldi et al. (2009a); Durastanti et al. (2011), first of all, notice that for $p \leq r$,

$$\mathcal{B}_{r,q}^s \subseteq \mathcal{B}_{p,q}^s,$$

so that we can always use $r = p$; consider then the case $p \geq r$, where the following embedding holds

$$\mathcal{B}_{r,q}^s \subseteq \mathcal{B}_{p,q}^{s-d(\frac{1}{r}-\frac{1}{p})}.$$

Because in the regular zone

$$r \geq \frac{(2|m|+d)p}{2(s+|m|)+d},$$

it follows that

$$\frac{s}{2(s+|m|)+d} \leq \frac{sr}{(2|m|+d)p}.$$

Thus, since $s > \frac{d}{r}$, it holds that

$$\frac{s}{(2|m|+d)} - \frac{d}{(2|m|+d)} \left(\frac{1}{r} - \frac{1}{p}\right) - \frac{s}{2(s+|m|)+d}$$

$$\begin{aligned}
&\geq \frac{s}{(2|m|+d)} - \frac{d}{(2|m|+d)} \left(\frac{1}{r} - \frac{1}{p} \right) - \frac{sr}{(2|m|+d)p} \\
&= \frac{d}{(2|m|+d)} \left(\frac{1}{r} - \frac{1}{p} \right) \left(\frac{sr}{d} - 1 \right) \\
&\geq 0,
\end{aligned}$$

since $s > \frac{d}{r}$.

Sparse zone. This proof is similar to the one above, so it is properly shortened for the sake of the brevity. As far as (5.1) and (5.3) are concerned, in order to choose γ such that

$$Au + Ua \lesssim n^{-\frac{\gamma}{2}} \lesssim n^{-\frac{p(s+d(\frac{1}{p}-\frac{1}{r}))}{2[(s+|m|)+d(\frac{1}{2}-\frac{1}{r})]}}.$$

Observe now (5.4): note that

$$Aa_2 = 0,$$

since

$$\mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\} \text{ for } j \geq J_{s,m},$$

see also Baldi et al. (2009a); Durastanti (2016).

Then, we have that

$$\begin{aligned}
Aa_1 &= \sum_{j=0}^{J_{s,m}-1} \sum_{k=1}^{K_j} n^{-\frac{p}{2}} \left\| \psi_{j,k}^{(m)} \right\|_{L^p(\mathbb{T}^d)}^p \mathbb{1} \left\{ \left| \beta_{j,k}^{(m)} \right| \geq \frac{1}{2} \tau_{j,m,n} \right\} \\
&\lesssim n^{-\frac{p}{2}} \sum_{j=0}^{J_{s,m}-1} \sum_{k=1}^{K_j} \left| \beta_{j,k}^{(m)} \right|^r \tau_{j,m,n}^{-r} \left\| \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)}^p B^{j|m|p} \\
&\lesssim \frac{n^{\frac{r-p}{2}}}{(\log n)^{\frac{r}{2}}} \sum_{j=0}^{J_{s,m}-1} B^{j|m|(p-r)} B^{\frac{jd}{2}(p-r)} \sum_{k=1}^{K_j} \left| \beta_{j,k}^{(m)} \right|^r \left\| \psi_{j,k} \right\|_{L^r(\mathbb{T}^d)}^r \\
&\lesssim \frac{n^{\frac{r-p}{2}}}{(\log n)^{\frac{r}{2}}} \sum_{j=0}^{J_{s,m}-1} B^{j|m|(p-r)} B^{\frac{jd}{2}(p-r)} B^{-jsr} \\
&\lesssim \frac{n^{\frac{r-p}{2}}}{(\log n)^{\frac{r}{2}}} B^{J_{s,m}[\frac{p-r}{2}(2|m|+d)-sr]} \\
&\lesssim \frac{n^{\frac{-p(s-d(\frac{1}{r}-\frac{1}{p}))}{2(s+|m|+d(\frac{1}{2}-\frac{1}{r}))}}}{(\log n)^\delta},
\end{aligned}$$

since

$$\frac{(p-r)(2|m|+d)-2sr}{4(s+|m|+d(\frac{1}{2}-\frac{1}{r}))} + \frac{r-p}{2} = \frac{-p(s-d(\frac{1}{r}-\frac{1}{p}))}{2(s+|m|+d(\frac{1}{2}-\frac{1}{r}))}, \quad (5.7)$$

and

$$\delta = \frac{(p-r)(2|m|+d) - 2sr}{4\left(s + |m| + d\left(\frac{1}{2} - \frac{1}{r}\right)\right)} + \frac{r}{2} = \frac{\frac{p}{2}(2|m|+d) - d}{2\left(s + |m| - d\left(\frac{1}{r} - \frac{1}{2}\right)\right)}.$$

Consider now (5.5); on the one hand, it holds that

$$\begin{aligned} Uu_1 &= \sum_{j=0}^{J_{s,m}-1} \sum_{k=1}^{K_j} |\beta_{j,k}^{(m)}|^p B^{jd(\frac{p}{2}-1)} \mathbb{1}\{|\beta_{j,k}^{(m)}| < 2\tau_{j,m,n}\} \\ &\lesssim \sum_{j=0}^{J_{s,m}-1} B^{jd(\frac{p}{2}-1)} \tau_{j,m,n}^{p-r} \sum_{k=1}^{K_j} |\beta_{j,k}^{(m)}|^r \\ &\lesssim \left(\frac{n}{\log n}\right)^{\frac{r-p}{2}} \sum_{j=0}^{J_{s,m}-1} B^{\frac{j(p-r)}{2}(2|m|+d)} \sum_{k=1}^{K_j} |\beta_{j,k}^{(m)}|^r B^{jd(\frac{r}{2}-1)} \\ &\lesssim \left(\frac{n}{\log n}\right)^{\frac{r-p}{2}} \sum_{j=0}^{J_{s,m}-1} B^{\frac{j(p-r)}{2}(2|m|+d)} B^{-jsr} \\ &\lesssim \frac{n^{\frac{r-p}{2}}}{(\log n)^{\frac{r}{2}}} B^{J_{s,m}[\frac{p-r}{2}(2|m|+d) - sr]} \\ &\lesssim \frac{n^{\frac{-p(s-d(\frac{1}{r}-\frac{1}{p}))}{2(s+|m|-d(\frac{1}{r}-\frac{1}{2}))}}}{(\log n)^\delta}, \end{aligned}$$

in view of (5.7). On the other hand, analogously to in Baldi et al. (2009a); Durastanti et al. (2011), we define

$$g = \frac{p|m| + d\left(\frac{p}{2} - 1\right)}{s + |m| - d\left(\frac{1}{r} - \frac{1}{2}\right)}.$$

In the sparse zone we have

$$g - r = \frac{\frac{1}{2}[p(2|m|+d) - r(2s + 2|m|+d)]}{s + |m| - d\left(\frac{1}{r} - \frac{1}{2}\right)} > 0,$$

so that the embedding $\mathcal{B}_{r,q}^s \subseteq \mathcal{B}_{g,q}^{s-d(\frac{1}{r}-\frac{1}{g})}$ holds. Furthermore, we easily obtain that

$$p - g = \frac{p\left(s - d\left(\frac{1}{r} - \frac{1}{p}\right)\right)}{s + |m| - d\left(\frac{1}{r} - \frac{1}{2}\right)} > 0.$$

Thus,

$$\begin{aligned} Uu_2 &= \sum_{j=J_{s,m}}^{J_{n,m}-1} B^{jd(\frac{p}{2}-1)} \sum_{k=1}^{K_j} |\beta_{j,k}^{(m)}|^p \mathbb{1}\{|\beta_{j,k}^{(m)}| < 2\tau_{j,m,n}\} \\ &\lesssim \left(\frac{\log n}{n}\right)^{\frac{p-g}{2}} \sum_{j=J_{s,m}}^{J_{n,m}-1} B^{j(2|m|+d)(\frac{p-g}{2})} \sum_{k=1}^{K_j} |\beta_{j,k}^{(m)}|^g B^{jd(\frac{g}{2}-1)} \end{aligned}$$

$$\begin{aligned}
&\lesssim \left(\frac{\log n}{n}\right)^{\frac{p-g}{2}} \sum_{j=J_{s,m}}^{J_{n,m}-1} B^{j(2|m|+d)\left(\frac{p-g}{2}\right)} B^{-jg\left(s-d\left(\frac{1}{r}-\frac{1}{g}\right)\right)}, \\
&\lesssim J_{n,m} \left(\frac{\log n}{n}\right)^{\frac{p\left(s-d\left(\frac{1}{r}-\frac{1}{p}\right)\right)}{2\left(s+|m|-d\left(\frac{1}{r}-\frac{1}{2}\right)\right)}},
\end{aligned}$$

since

$$(2|m|+d)\left(\frac{p-g}{2}\right) - g\left(s-d\left(\frac{1}{r}-\frac{1}{g}\right)\right) = \left(|m|+\frac{d}{2}\right)(p-g) - g\left(s-\frac{d}{r}\right) - d = 0$$

As far as D is concerned, observe that

$$\begin{aligned}
D^{\frac{1}{p}} &\leq \sum_{j \geq J_{n,m}} \left\| \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k} \right\|_{L^p(\mathbb{T}^d)} \\
&= \sum_{j \geq J_{n,m}} \left\| \sum_{k=1}^{K_j} \beta_{j,k} \psi_{j,k}^{(m)} \right\|_{L^p(\mathbb{T}^d)} \\
&\lesssim B^{-J_{n,m} s - d\left(\frac{1}{r}-\frac{1}{p}\right)} \\
&\lesssim \left(\frac{n}{\log n}\right)^{-\frac{s-d\left(\frac{1}{r}-\frac{1}{p}\right)}{2|m|+d}}.
\end{aligned}$$

Recalling that, in the sparse zone, $r \leq p$, it is straightforward to prove that

$$\frac{s-d\left(\frac{1}{r}-\frac{1}{p}\right)}{2|m|+d} \geq \frac{s-d\left(\frac{1}{r}-\frac{1}{p}\right)}{2\left(s+|m|-d\left(\frac{1}{r}-\frac{1}{2}\right)\right)},$$

since, for $s > \frac{d}{r}$,

$$2\left(s+|m|-d\left(\frac{1}{r}-\frac{1}{2}\right)\right) \geq d+2|m|$$

Thus $D \lesssim \left(\frac{\log n}{n}\right)^{\frac{sp}{d+2|m|}}$, see also [Baldi et al. \(2009a\)](#); [Durastanti et al. \(2011\)](#).

Let us now consider the case $p = \infty$, where we have

$$\begin{aligned}
\mathbb{E} \left[\left\| \widehat{f}^{(m)} - f^{(m)} \right\|_{L^\infty(\mathbb{T}^d)} \right] &\lesssim \mathbb{E} \left[\left\| \sum_{j=0}^{J_{n,m}-1} \left[\eta\left(\widehat{\beta}_{j,k}^{(m)}, \tau_{j,m,n}\right) - \beta_{j,k} \right] \psi_{j,k} \right\|_{L^\infty(\mathbb{T}^d)} \right] \\
&\quad + \left\| \sum_{j \geq J_{n,m}} \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k} \right\|_{L^\infty(\mathbb{T}^d)} \\
&=: \Sigma_\infty + D_\infty.
\end{aligned}$$

Consider preliminarily $q = r = \infty$. Arguments similar to the one discussed above yield

$$\Sigma_\infty \lesssim \sum_{j=0}^{J_{s,m}-1} B^{\frac{j}{2}(2|m|+d)} \mathbb{E} \left[\sup_{k=1, \dots, K_j} \left| \widehat{\beta}_{j,k} - \beta_{j,k} \right| \right] + \sum_{j=J_{s,m}}^{J_{n,m}-1} B^{\frac{j}{2}+d} \sup_{k=1, \dots, K_j} \left| \beta_{j,k}^{(m)} \right| + n^{-\frac{1}{2}}$$

$$\begin{aligned}
&\lesssim J_{s,m} B^{\frac{J_{s,m}}{2}(2|m|+d)} n^{-\frac{1}{2}} + B^{-sJ_{s,m}} + n^{-\frac{1}{2}} \\
&\lesssim n^{-\frac{s}{2(s+|m|)+d}},
\end{aligned}$$

As far as the deterministic error is concerned, it is easy to prove that

$$\begin{aligned}
D_\infty &\lesssim \sum_{j \geq J_{n,m}} \left\| \sum_{k=1}^{K_j} \beta_{j,k}^{(m)} \psi_{j,k} \right\|_{L^\infty(\mathbb{T}^d)} \\
&\lesssim B^{-sJ_{n,m}} \\
&\lesssim \left(\frac{n}{\log n} \right)^{-\frac{s}{2(s+|m|)+d}}.
\end{aligned}$$

Now, as in Baldi et al. (2009a); Durastanti et al. (2011), for arbitrary q, r , the result holds since $\mathcal{B}_{r,q}^s \subseteq \mathcal{B}_{\infty,\infty}^{s-\frac{d}{r}}$. \square

The idea of proof of Theorem 13 comes from (Baldi et al., 2009a, Theorem 11), see also Liu and Wang (2013) and it is based on two crucial results, namely, the so-called Fano's lemma and Varshnov-Gilbert lemma (see, for example, Tsybakov (2009)). Given two probability measures P and Q , defined on some probability space, their Kullback-Leibler divergence is given by

$$KL(P, Q) = \begin{cases} \int \log \frac{dP}{dQ} dP = \int \frac{dP}{dQ} \log \frac{dP}{dQ} dQ & \text{if } P \ll Q \\ \infty & \text{otherwise} \end{cases},$$

Let P and Q be two probability measures on the d -torus with densities f, g with respect to $\rho(d\vartheta)$. If g is bounded below by some positive constant, it holds that

$$KL(P, Q) \lesssim \|f - g\|_{L^2(\mathbb{T}^d)}^2,$$

see, for example, (Baldi et al., 2009a, Equation (33)).

Lemma 14 (Fano's lemma). *For $t = 1, \dots, T$, let $(\Omega, \mathcal{F}, P_t)$ be a set of probability spaces, and $A_k \in \mathcal{F}$. Let, furthermore,*

$$\mathcal{L}_T = \inf_{t=1, \dots, T} \frac{1}{T} \sum_{t' \neq t} KL(P_t, P_{t'}).$$

If, for $t \neq t'$, $A_t \cap A_{t'} = \emptyset$, then

$$\sup_{t=1, \dots, T} P_t(A_t^c) \geq \min \left\{ \frac{1}{2}, \sqrt{T} e^{-\frac{3}{e}} e^{-\mathcal{L}_T} \right\}.$$

Lemma 15 (Varshnov-Gilbert lemma). *Let $\mathcal{E} = \{0, 1\}^T$. Then, there exists a subset $\{\varepsilon^0, \dots, \varepsilon^U\} \subseteq \mathcal{E}$, where $\varepsilon^0 = (0, \dots, 0)$, such that $U \geq 2^{T/8}$ and*

$$\sum_{t=1}^T |\varepsilon_t^u - \varepsilon_t^{u'}| \geq \frac{T}{8}, \quad 0 \leq u \neq u' \leq U.$$

Proof of Theorem 13. As aforementioned, this proof follows strictly the approach developed by (Baldi et al., 2009a, Theorem 11).

Let us fix $j \geq 0$ and consider the family \mathcal{A}_j of densities taking the form

$$f_\varepsilon = \frac{1}{(2\pi)^d} + \zeta \sum_{k \in A_j} \varepsilon_{j,k} \psi_{j,k},$$

where $A_j \subseteq \{1, \dots, K_j\}$ is chosen so that (2.9) in Lemma 6 holds, $\varepsilon_k \in \{0, 1\}$, and $\zeta > 0$ must ensure that all the densities in \mathcal{A}_j are positive. It is sufficient that $\zeta \lesssim B^{-j\frac{d}{2}}$, since

$$\begin{aligned} |f_\varepsilon| &\geq \frac{1}{(2\pi)^d} - |\zeta| \left| \sum_{k \in A_j} \varepsilon_{j,k} \psi_{j,k} \right| \\ &\geq \frac{1}{(2\pi)^d} - |\zeta| c_\infty B^{j\frac{d}{2}}, \end{aligned}$$

see again Baldi et al. (2009a). Using now (2.7) in Lemma 5 yields

$$f_\varepsilon^{(m)}(\vartheta) = \zeta \sum_{k \in A_j} \varepsilon_{j,k} \psi_{j,k}^{(m)}(\vartheta), \quad \vartheta \in \mathbb{T}^d.$$

To ensure that $f_\varepsilon \in \mathcal{B}_{r,q}^{s+|m|}(G)$, we impose that

$$|\zeta| \leq GB^{-j(s+|m|+\frac{d}{2})}.$$

Indeed, since

$$\left(\sum_{k=1}^{K_j} |\varepsilon_{j,k}|^r \right)^{\frac{1}{r}} \leq \left(\sum_{k=1}^{K_j} 1 \right)^{\frac{1}{r}} \lesssim B^{\frac{jd}{r}}$$

$$\begin{aligned} \|f_\varepsilon^{(m)}\|_{\mathcal{B}_{r,q}^{s+|m|}} &= |\zeta| B^{j(s+|m|+d(\frac{1}{2}-\frac{1}{r}))} \left(\sum_{k=1}^{K_j} |\varepsilon_{j,k}|^r \right)^{\frac{1}{r}} \\ &\lesssim |\zeta| B^{j(s+|m|+\frac{d}{2})}. \end{aligned}$$

Now, for $f_\varepsilon, f_{\varepsilon'} \in \mathcal{A}_j$,

$$\begin{aligned} \|f_\varepsilon^{(m)} - f_{\varepsilon'}^{(m)}\|_{L^2(\mathbb{T}^d)}^2 &\lesssim \zeta^2 \sum_{k \in A_j} |\varepsilon_{j,k} - \varepsilon'_{j,k}|^2 B^{2j|m|} \\ &\lesssim B^{-2js}. \end{aligned}$$

Using (2.9) in Lemma 6 yields

$$\|f_\varepsilon^{(m)} - f_{\varepsilon'}^{(m)}\|_{L^p(\mathbb{T}^d)} \geq |\zeta| \left(\sum_{k \in A_j} |\varepsilon_{j,k} - \varepsilon'_{j,k}|^p \|\psi_{j,k}^{(m)}\|_{L^p(\mathbb{T}^d)}^p \right)^{\frac{1}{p}}. \quad (5.8)$$

According to Lemma 15, there exists a finite subset of \mathcal{A}_j , whose elements are given by

$$f_{\varepsilon^u} = \frac{1}{(2\pi)^d} + \zeta \sum_{k \in A_j} \varepsilon_{j,k}^u \psi_{j,k},$$

where $\{\varepsilon_{j,\cdot}^u : u = 1, \dots, U\}$ is such that $U \geq 2^{cB^{dj}}$ and

$$\sum_{k \in A_j} |\varepsilon_{j,k}^u - \varepsilon_{j,k}^v| \gtrsim B^{jd}.$$

Thus, using (5.8) yields

$$\|f_{\varepsilon^m}^{(m)} - f_{\varepsilon^s}^{(m)}\|_{L^p(\mathbb{T}^d)} \gtrsim |\zeta| B^{j|m|+d(\frac{1}{2}-\frac{1}{p})} B^{\frac{jd}{p}} \approx B^{-js},$$

which implies that the events

$$A_{\varepsilon^u} = \left\{ \|\widehat{f}^{(m)} - f_{\varepsilon^u}^{(m)}\| < \frac{1}{2} B^{-js} \right\}, \quad u = 1, \dots, U,$$

are disjoint.

Fixed a density function f , consider now the probability measure P_f^n , corresponding to the density

$$f^n(x) = f(x_1) \cdot \dots \cdot f(x_n).$$

Using Lemma 14 leads to

$$\sup_{u=1, \dots, U} P_{f_{\varepsilon^u}^n}^n(A_{\varepsilon^u}^c) \geq \min \left\{ \frac{1}{2}, \sqrt{U} e^{-\frac{3}{e}} e^{-\mathcal{L}_U} \right\},$$

such that

$$\sup_{u=1, \dots, U} \mathbb{E} \left[\|\widehat{f}^{(m)} - f_{\varepsilon^u}^{(m)}\|_{L^p(\mathbb{T}^d)}^p \right] \geq \frac{B^{-jps}}{2} \sup_{u=1, \dots, U} P_{f_{\varepsilon^u}^n}^n(A_{\varepsilon^u}^c) \gtrsim B^{-jps} \min \left\{ \frac{1}{2}, \sqrt{U} e^{-\frac{3}{e}} e^{-\mathcal{L}_U} \right\}.$$

Observing that

$$KL(P_1^n, P_2^n) = \sum_{i=1}^N \int f_1(x_i) \log \frac{f_1(x_i)}{f_2(x_i)} dx_i = nKL(f_1, f_2),$$

it holds that

$$\mathcal{L}_U \lesssim \inf_{u=1, \dots, U} \frac{n}{U} \sum_{u' \neq u} KL(f_{\varepsilon^u}, f_{\varepsilon^{u'}}) \lesssim \frac{n}{U} \sum_{u=1}^U KL(f_{\varepsilon^u}, f_{\varepsilon^0}),$$

where

$$\begin{aligned} KL(f_{\varepsilon^u}, f_{\varepsilon^0}) &= \int \frac{1}{f_{\varepsilon^0}(\vartheta)} |f_{\varepsilon^u}(\vartheta) - f_{\varepsilon^0}(\vartheta)|^2 \rho(d\vartheta) \\ &= (2\pi)^d |\zeta|^2 \left\| \sum_{k \in A_j} \varepsilon_{j,k}^u \psi_{j,k} \right\|_{L^2(\mathbb{T}^d)}^2 \end{aligned}$$

$$\begin{aligned} &\leq G (2\pi)^d B^{-2j(s+|m|+\frac{d}{2})} B^{dj} \\ &\lesssim B^{-2j(s+|m|)}, \end{aligned}$$

and, then,

$$\mathcal{L}_U \lesssim nB^{-2j(s+|m|)}$$

while

$$\sqrt{U} \approx 2^{cB^{jd}},$$

Then, we choose j such that $nB^{-2js} \approx B^{jd}$, that is, $j \in \mathbb{N}$ such that $B^j \approx n^{\frac{1}{2(s+|m|)+d}}$. Hence, it holds that

$$\sup_{u=1,\dots,U} \mathbb{E} \left[\left\| \widehat{f}^{(m)} - f^{(m)}_{\varepsilon^u} \right\|_{L^p(\mathbb{T}^d)}^p \right] \gtrsim B^{-jsp} \approx n^{-\frac{sp}{2(s+|m|)+d}}.$$

Consider now two densities

$$f_0 = \frac{1}{(2\pi)^d} + \zeta \psi_{j,k}; \quad f_1 = \frac{1}{(2\pi)^d} + \zeta \psi_{j,k'},$$

where $|\zeta| \lesssim B^{-\frac{jd}{2}}$ ensures that the two densities are positive. If, additionally, $|\zeta| \leq GB^{-j(s+|m|+d(\frac{1}{2}-\frac{1}{r}))}$, then both f_0 and f_1 belong to the Besov ball $\mathcal{B}_{r,q}^{s+|m|}(G)$. Again,

$$KL(f_0, f_1) \approx \zeta^2,$$

while, if P_0 e P_1 denote the probability measures whose densities are given by the n -products of f_0 and f_1 respectively, it holds that

$$KL(P_0, P_1) \approx n\zeta^2.$$

using again Lemma 6, we have that

$$\begin{aligned} \left\| f_0^{(m)} - f_1^{(m)} \right\|_{L^p(\mathbb{T}^d)} &= |\zeta| \left\| \psi_{j,k}^{(m)} - \psi_{j,k'}^{(m)} \right\|_{L^p(\mathbb{T}^d)} \\ &\gtrsim \zeta B^{j|m|} B^{jd(\frac{1}{2}-\frac{1}{p})} \\ &\gtrsim B^{-j(s+d(\frac{1}{p}-\frac{1}{r}))}. \end{aligned}$$

Thus, the events $\left\{ \left\| \widehat{f}^{(m)} - f_i^{(m)} \right\|_{L^p(\mathbb{T}^d)} \geq B^{-j(s+d(\frac{1}{p}-\frac{1}{r}))} \right\}$ are disjoint. As in Baldi et al. (2009a), choosing $\zeta = n^{-\frac{1}{2}}$, so that $KL(f_0, f_1) \approx n$, implies that $j \approx \frac{1}{2(s+|m|+d(\frac{1}{2}-\frac{1}{r}))}$. Hence, using Fano's lemma yields

$$\begin{aligned} \sup_{i=0,1} \mathbb{E} \left[\left\| \widehat{f}^{(m)} - f_i^{(m)} \right\|_{L^p(\mathbb{T}^d)} \right] &\gtrsim B^{-j(s+d(\frac{1}{p}-\frac{1}{r}))} \\ &\approx n^{-\frac{(s+d(\frac{1}{p}-\frac{1}{r}))}{2(s+|m|+d(\frac{1}{2}-\frac{1}{r}))}}. \end{aligned}$$

Finally, combining these results and checking for which sets of the Besov parameters one rate is larger than the other one completes the proof of the theorem. \square

Acknowledgments

The authors would like to thank the Editor and the anonymous referees for their comments and constructive suggestions. Claudio Durastanti acknowledges the research project RM120172B7A31FFA “Costruzione di basi multiscala e trasformate wavelet per applicazioni in ambito numerico e statistico”, funded by Sapienza Università di Roma. Nicola Turchi was supported by FNR OPEN FoRGES (R-AGR-3376-10-C) at Université du Luxembourg.

Disclosure statement

No potential conflict of interest was reported by the authors.

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