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Calculus of Variations

Perturbed eigenvalues of polyharmonic operators in domains with small holes

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Abstract

We study singular perturbations of eigenvalues of the polyharmonic operator on bounded domains under removal of small interior compact sets. We consider both homogeneous Dirichlet and Navier conditions on the external boundary, while we impose homogeneous Dirichlet conditions on the boundary of the removed set. To this aim, we develop a notion of capacity which is suitable for our higher-order context, and which permits to obtain a description of the asymptotic behaviour of perturbed simple eigenvalues in terms of a capacity of the removed set, in dependence of the respective normalized eigenfunction. Then, in the particular case of a subset which is scaling to a point, we apply a blow-up analysis to detect the precise convergence rate, which turns out to depend on the order of vanishing of the eigenfunction. In this respect, an important role is played by Hardy–Rellich inequalities in order to identify the appropriate functional space containing the limiting profile. Remarkably, for the biharmonic operator this turns out to be the same, regardless of the boundary conditions prescribed on the exterior boundary.

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1 Introduction

The aim of the present work is to study perturbations of the eigenvalues of the polyharmonic operator $(-\Delta)^m$, $m \geq 2$, when from a given bounded domain $\Omega \subset \mathbb{R}^N$ an interior compact set K is removed, thus introducing a singular perturbation. We focus on the case in which K is small, in the sense that its capacity is asymptotically near 0, with respect to a notion of capacity suitably developed for our higher-order setting. More specifically, for $m \geq 2$ we consider the eigenvalue problems

$$\begin{cases}(-\Delta)^m u = \lambda u & \text{in } \Omega,\\ u = \partial_n u = \cdots = \partial_n^{m-1} u = 0 & \text{on } \partial\Omega,\end{cases} \quad \text{resp.} \quad \begin{cases}(-\Delta)^m u = \lambda u & \text{in } \Omega,\\ u = \Delta u = \cdots = \Delta^{m-1} u = 0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

with Dirichlet and Navier boundary conditions (BCs) respectively, and, given a compact set $K \subset\subset \Omega$, we are interested in the corresponding eigenvalue problems in case K is removed from Ω, that is

$$\begin{cases}(-\Delta)^m u = \lambda u & \text{in } \Omega \setminus K,\\ u = \partial_n u = \cdots = \partial_n^{m-1} u = 0 & \text{in } \partial(\Omega \setminus K),\end{cases} \tag{1.2}$$

in the Dirichlet case, and

$$\begin{cases}(-\Delta)^m u = \lambda u & \text{in } \Omega \setminus K,\\ u = \Delta u = \cdots = \Delta^{m-1} u = 0 & \text{on } \partial\Omega,\\ u = \partial_n u = \cdots = \partial_n^{m-1} u = 0 & \text{on } \partial K,\end{cases} \tag{1.3}$$

where, instead, Navier BCs on $\partial\Omega$ are considered. Note that in both cases we always deal with Dirichlet BCs on ∂K. The goal is to investigate spectral stability and sharp asymptotic estimates for the eigenvalues of problems (1.2) and (1.3) when K vanishes in a capacitary sense.

Qualitative properties of solutions to higher-order problem are deeply related to the boundary conditions that one prescribes. The most common ones in the literature are Dirichlet BCs

$$u = \partial_n u = \cdots = \partial_n^{m-1} u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

and Navier BCs

$$u = \Delta u = \cdots = \Delta^{m-1} u = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

Indeed, from the point of view of the applications, they correspond to the simplest Kirchhoff-Love models of a thin plate, either clamped or hinged at the boundary, respectively in the Dirichlet and the Navier case.

While the existence and regularity theory for linear problems is essentially the same in both cases (see e.g. [17]), however solutions have relevant differences, even when Ω is a smooth domain. The most striking and famous one is regarding positivity. In the Navier case the solution inherits its sign from the data, since one can decouple the problem into a system of second-order equations, for which a maximum principle holds. Instead, positivity preserving is in general lost in the Dirichlet case, even for smooth and convex domains, except for peculiar situations in which one can rely on a global analysis of the Green function, such as for the case of the ball and its smooth deformations, see [17, 19]. On the other hand, functions which undergo Dirichlet BCs can be trivially extended by 0 outside the domain, so that the extension continues to belong to the same higher-order Sobolev space, while this is not true anymore for solutions of Navier problems because of possible jumps on $\partial\Omega$ of the normal derivative. Motivated by these arguments, we investigate perturbations of the eigenvalues of $(-\Delta)^m$ in both context of Dirichlet and Navier BCs on $\partial\Omega$. Since we rely more on extension properties rather than positivity issues, our analysis will be harder in the Navier case.

In the second-order case (i.e. $m = 1$), spectral stability under removal of small (condenser) capacity sets is proved in [10] in a very general context, see also [7] and [14]. More specifically, in [10] it is shown there that the function $\lambda(\Omega \setminus K) - \lambda(\Omega)$ is differentiable with respect to the capacity of the removed set K relative to Ω. A sharp quantification of the vanishing order of the variation of simple eigenvalues is given in [2], when concentrating families of compact sets are considered: the precise rate of convergence is asymptotic to the u-capacity associated to the limit eigenfunction u (see [7, Definition 2.1] and [10, (14)] for the notion of u-capacity) and sharp asymptotic estimates are given in terms of the diameter of the removed set if the limit set is a point in $\mathbb{R}^2$, and either if the eigenfunction does not vanish there, or in case of specific concentrating sets such as disks or segments. Asymptotic estimates of u-capacities and eigenvalues of the Dirichlet Laplacian, on bounded planar domains with small holes of the more general form $\varepsilon\overline{\omega}$ with ω a bounded domain and $\varepsilon \to 0$, are given in [1]. In both [2] and [1], a tool that helps to provide precise asymptotic estimates in dimension two is given by elliptic coordinates, which allow rewriting the equations satisfied by the capacitary potentials in a rather explicit way and which however do not have a simple analogue in higher dimensions. In the complementary case $N \geq 3$, an approach based on a blow-up argument is used in [13] to derive sharp asymptotic estimates of the u-capacity, and consequently of the eigenvalue variation, for general families of sets which may also concentrate at the boundary. This method has been applied also to fractional problems in [3].

For the higher order setting $m \geq 2$, asymptotic expansions of eigenvalues of biharmonic operators under removal of a family of sets which are uniformly vanishing to a point $\{x_0\}$ have been obtainsed in [8, 21, 22]. All these papers deal with the two-dimensional case and only Dirichlet boundary conditions, both on $\partial\Omega$ and on ∂K, are considered. The main difference with the corresponding two-dimensional second-order problem, is that the limiting problem involves the punctured domain $\Omega \setminus \{x_0\}$. In [8] formal recursive asymptotic expansions are found in the nondegenerate case, namely when the gradient of the corresponding eigenfunction does not vanish at x_0, as well as in the degenerate case. In the former case, these expansions are justified in a suitable functional setting which makes use of weighted Sobolev spaces, named after Kondrat'ev, in order to deal with the point constraint. On the other hand, motivated by the study of MEMS devices, in [21], the asymptotic behaviour of eigenpairs is formally obtained, using the method of matching asymptotic expansions. A more delicate situation is taken into account in [22], when both the removed subdomain is vanishing, as well as the biharmonic part of the operator, provided a second-order term is introduced in the equation. In all these works, the asymptotic expansions of the perturbed eigenvalues are of logarithmic kind, fact that recalls the expansion in the two-dimensional case for the Laplace operator given in [2, Theorem 1.7]. We note however that, unlike what happens for the second order problem, capacities cannot play there the role of perturbation parameters, since in dimension 2 the higher order capacity of a point (defined as in (1.11)) is different from zero; this is also the reason why the limiting problem is formulated in the punctured domain. We mention that the spectral behavior of higher-order elliptic operators upon domain perturbation is investigated also in [5] for Dirichlet, Neumann and intermediate boundary conditions.

The first aim of the present paper is a rigorous description of the asymptotic behaviour of the perturbed eigenvalues for polyharmonic operators $(-\Delta)^m$ for any $m \geq 2$ and for a large class of removed sets, in the spirit of [2, 3, 13]. Since we deal with sets of vanishing capacities, we are focused on the high dimensional case $N \geq 2m$. Furthermore, as second important objective, we investigate whether and how different boundary conditions on $\partial\Omega$ affect the analysis. As already remarked, in the present work we consider Dirichlet boundary conditions on ∂K. In order to have a complete picture of the influence of the boundary conditions, the complementary situation of Navier BCs on ∂K should be addressed. However, the techniques developed in the present work strongly rely on extension properties which are characteristic of the Dirichlet case, so that a different approach should be devised to treat the Navier case on ∂K. We plan to address this in a future work.

In order to give the precise statements of the main results, we first describe the functional setting and the notation we are going to use throughout the paper.

Notation We denote the normal derivative of the function u by $\partial_n u$. For a set $D \subset \mathbb{R}^N$, $\mathcal{U}(D)$ denotes some open neighbourhood of D, $C_0^\infty(D)$ is the space of the infinitely differentiable functions which are compactly supported in D, and $L^p(D)$ with $p \in [1, +\infty]$ is the space of p-integrable functions. The norm of $L^p(D)$ is denoted simply by $\|\cdot\|_p$ whenever the domain is clear from the context. For every $m \in \mathbb{N}$ and $u : D \to \mathbb{R}$ with $D \subset \mathbb{R}^N$, we denote as $D^m u$ the tensor of m-th order derivatives of u and define $|D^m u|^2 = \sum_{|\alpha|=m} |D^\alpha u|^2$, where $|\alpha|$ is the length of the multi-index α.

The symbol $\lesssim$ is used when an inequality is true up to an omitted structural constant, and we write $f = \mathcal{O}(g)$ (resp. $f = o(g)$) as $x \to x_0$ when there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ in a neighbourhood of x_0 (resp. $\frac{f(x)}{g(x)} \to 0$ as $x \to x_0$).

1.1 The functional setting

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. In order to treat at once different boundary conditions on $\partial\Omega$, i.e. the settings of problems (1.2) and (1.3), we introduce the following notation. For $m \geq 2$ the set $V^m(\Omega) \subset H^m(\Omega)$ is defined either as

$$V^m(\Omega) := H_0^m(\Omega)$$

in case Dirichlet boundary conditions (1.4) are prescribed on $\partial\Omega$, where $H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$, or by

$$V^m(\Omega) := H_\vartheta^m(\Omega)$$

if Navier boundary conditions (1.5) are assumed on $\partial\Omega$. Here $H_\vartheta^m(\Omega)$ is the closure in $H^m(\Omega)$ of the space

$$C_\vartheta^m(\overline{\Omega}) := \left\{ u \in C^m(\overline{\Omega}) \;\middle|\; \Delta^j u|_{\partial\Omega} = 0 \text{ for all } 0 \leq j < \tfrac{m}{2} \right\}$$

and it can be characterized as

$$H_\vartheta^m(\Omega) = \left\{ u \in H^m(\Omega) \;\middle|\; \Delta^j u|_{\partial\Omega} = 0 \text{ in the sense of traces for all } 0 \leq j < \tfrac{m}{2} \right\}.$$

Note that for $m = 2$ we have $H_\vartheta^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$. In both cases $V^m(\Omega)$ is a closed subspace of $H^m(\Omega)$. Moreover, for a bounded domain $\Omega \subset \mathbb{R}^N$, the norms

$$\|\cdot\|_{H^m(\Omega)} := \sum_{|\alpha| \leq m} \|D^\alpha \cdot\|_{L^2(\Omega)}$$

(with the multi-index notation) and

$$\|\nabla^m \cdot\|_{L^2(\Omega)}, \qquad \text{where} \quad \nabla^m f := \begin{cases} \Delta^{\frac{m}{2}} f & \text{for } m \text{ even}, \\ \nabla \Delta^{\frac{m-1}{2}} f & \text{for } m \text{ odd}, \end{cases}$$

are equivalent on both $H^m_0(\Omega)$ and $H^m_\vartheta(\Omega)$, see e.g. [17, Theorem 2.2] for the Dirichlet case and [18] for the Navier case. In particular, there exists a constant $C = C(N, m, \Omega) > 0$, depending only on N, m, and Ω, such that

$$\|u\|_{H^m(\Omega)} \leq C \|\nabla^m u\|_{L^2(\Omega)} \quad \text{for all } u \in V^m(\Omega). \tag{1.6}$$

Note also that in the Dirichlet case all boundary conditions are stable, and therefore they are all included in the definition of the space $H^m_0(\Omega)$; on the other hand, only the first half of the Navier conditions are stable, while the boundary conditions $\Delta^j u|_{\partial\Omega} = 0$ for $\frac{m}{2} \leq j \leq m-1$ are natural and thus do not appear in the definition of $H^m_\vartheta(\Omega)$. For a comprehensive discussion, see [17, Sec.2.4].

The next spaces are relevant when a "hole" is produced in the domain. For a compact set $K \subset \Omega$, we define

$$V^m_0(\Omega \setminus K) := \begin{cases} H^m_0(\Omega \setminus K) & \text{in the Dirichlet case}, \\ H^m_{\vartheta,0}(\Omega \setminus K) & \text{in the Navier case}. \end{cases}$$

Here $H^m_{\vartheta,0}(\Omega \setminus K)$ denotes the space suitable for Navier BCs on $\partial\Omega$ and Dirichlet BCs on ∂K. More precisely, $H^m_{\vartheta,0}(\Omega \setminus K)$ is the closure in $H^m_\vartheta(\Omega)$ of

$$C^m_{\vartheta,0}(\overline{\Omega} \setminus K) := \left\{ u \in C^m_\vartheta(\overline{\Omega}) \;\middle|\; \operatorname{supp} u \cap \mathcal{U}(K) = \emptyset \text{ for some } \mathcal{U}(K) \right\}.$$

In case ∂K is smooth, $u \in H^m_{\vartheta,0}(\Omega \setminus K)$ if and only if $u \in H^m(\Omega \setminus K)$ and

$$\Delta^j u|_{\partial\Omega} = 0 \ \text{ for all } \ 0 \leq j < \tfrac{m}{2} \quad \text{and} \quad \partial^h_n u|_{\partial K} = 0 \ \text{ for all } \ 0 \leq h \leq m-1$$

in the sense of L^2-traces. Note that we have the following chain of inclusions

$$H^m_0(\Omega \setminus K) \subsetneq H^m_{\vartheta,0}(\Omega \setminus K) \subsetneq H^m_\vartheta(\Omega) \subsetneq H^m(\Omega), \tag{1.7}$$

where the second inclusion holds by extending to 0 in K functions defined in $\Omega \setminus K$, thanks to the Dirichlet conditions imposed on ∂K. For the same reason, note also that, for any compact sets K_1, K_2 such that $K_1 \subset K_2 \subset \Omega$, one has

$$V^m(\Omega \setminus K_2) \subset V^m(\Omega \setminus K_1).$$

All such spaces are Hilbert spaces with scalar product[1] $q_m(u, v) := \int_\Omega \nabla^m u \, \nabla^m v$. Note that, unlike the general case, $q_m(\cdot, \cdot)$ does not involve boundary integrals, see [17, Sec.2.4]. By standard arguments [17, Theorem 2.15], the linear problem $(-\Delta)^m u = f$ in $\Omega \setminus K$, with $f \in L^2(\Omega \setminus K)$ and boundary conditions either (1.4) or (1.5), admits a unique weak solution $u \in V^m_0(\Omega \setminus K)$, in the sense that

$$\int_\Omega \nabla^m u \, \nabla^m \varphi = \int_\Omega f \varphi \qquad \text{for all } \ \varphi \in V^m_0(\Omega \setminus K).$$

Analogously, we define the eigenvalues of problems (1.2) and (1.3) in the weak sense. We say that (λ, u) is an eigenpair of (1.2) (resp. (1.3)) if $(\lambda, u) \in \mathbb{R} \times V^m_0(\Omega \setminus K)$ satisfies

$$u \not\equiv 0 \quad \text{and} \quad \int_\Omega \nabla^m u \, \nabla^m \varphi = \lambda \int_\Omega u \varphi \qquad \text{for all } \varphi \in V^m_0(\Omega \setminus K). \tag{1.8}$$

[1] We always omit to indicate the scalar product in $\mathbb{R}^N$ with $\cdot$.

By classical spectral theory, problems (1.2) and (1.3) admit a diverging sequence of positive eigenvalues

$$0 < \lambda_1(\Omega \setminus K) \le \cdots \le \lambda_j(\Omega \setminus K) \le \cdots \to +\infty,$$

where each one is repeated as many times as its multiplicity. Of course the same holds for the unperturbed problems (1.1), whose eigenvalues are denoted as $\big(\lambda_j(\Omega)\big)_{j\in\mathbb{N}}$. We recall that the eigenvalues may be variationally characterized as

$$\lambda_j(\Omega \setminus K) = \min_{\substack{\mathcal{X}_j \subset V_0^m(\Omega\setminus K)\\ \dim \mathcal{X}_j = j}} \max_{v\in\mathcal{X}_j} \frac{\int_{\Omega\setminus K} |\nabla^m v|^2}{\int_{\Omega\setminus K} |v|^2}. \tag{1.9}$$

Finally, for Ω and K as before, we define

$$X^m(\Omega) := \begin{cases} C_0^\infty(\Omega) & \text{in the Dirichlet case,}\\ C_\vartheta^m(\overline{\Omega}) & \text{in the Navier case,} \end{cases} \tag{1.10}$$

and

$$X_0^m(\Omega \setminus K) := \begin{cases} C_0^\infty(\Omega \setminus K) & \text{in the Dirichlet case,}\\ C_{\vartheta,0}^m(\overline{\Omega} \setminus K) & \text{in the Navier case,} \end{cases}$$

for the sake of a compact notation in some of the proofs.

1.2 Main results

In the spirit of the previously cited works [2, 3, 13], asymptotic expansions of eigenvalues under removal of small sets can be established treating as a perturbation parameter a suitable notion of capacity. Extending to the higher-order Sobolev framework the classical definition in the second-order case, for every compact set $K \subset \Omega$ we define the (condenser) V^m-*capacity* of K in Ω as

$$\mathrm{cap}_{V^m,\Omega}(K) := \inf\left\{ \int_\Omega |\nabla^m f|^2 \,\middle|\, f \in V^m(\Omega),\ f - \eta_K \in V_0^m(\Omega \setminus K) \right\}, \tag{1.11}$$

where η_K is a fixed smooth function such that $\operatorname{supp} \eta_K \subset \Omega$ and $\eta_K \equiv 1$ in a neighbourhood of K. The V^m-capacity of a set K gives an indication about its relevance for the higher-order Sobolev space V^m, in the sense that zero V^m-capacity sets do not affect the space $V^m(\Omega)$ when they are removed from Ω, and hence nor the spectrum of the polyharmonic operator (Proposition 2.1).

In our analysis, a notion of "weighted" capacity, which represents the higher order analogue of the u-capacity introduced in [7, Definition 2.1] and [10, (14)] for second order problems, will be significant too. Given a function $u \in V^m(\Omega)$, we define the (u, V^m)-*capacity* of K in Ω as

$$\mathrm{cap}_{V^m,\Omega}(K, u) := \inf\left\{ \int_\Omega |\nabla^m f|^2 \,\middle|\, f \in V^m(\Omega),\ f - u \in V_0^m(\Omega \setminus K) \right\}. \tag{1.12}$$

Note that u is relevant only in a neighbourhood of K. Hence,

$$\mathrm{cap}_{V^m,\Omega}(K, u) = \mathrm{cap}_{V^m,\Omega}(K, \eta_K u)$$

for any cut-off function η_K as before. This permits to extend the notion of (u, V^m)-capacity to functions $u \in H^m_{loc}(\mathbb{R}^N)$.

For those cases in which we need to distinguish the capacities according to the boundary conditions on $\partial\Omega$, we use the following notation:

$$\mathrm{cap}_{m,\,\Omega}(K) := \mathrm{cap}_{H^m_0,\Omega}(K) \qquad \text{and} \qquad \mathrm{cap}_{m,\vartheta,\,\Omega}(K) := \mathrm{cap}_{H^m_\vartheta,\Omega}(K),$$

for the Dirichlet and Navier BCs on $\partial\Omega$, respectively. Similarly we denote

$$\mathrm{cap}_{m,\,\Omega}(K,u) := \mathrm{cap}_{H^m_0,\Omega}(K,u) \qquad \text{and} \qquad \mathrm{cap}_{m,\vartheta,\,\Omega}(K,u) := \mathrm{cap}_{H^m_\vartheta,\Omega}(K,u). \tag{1.13}$$

We point out that the V^m-capacity as well as the (u, V^m)-capacity of a compact set K are attained by a unique minimizer, which is called capacitary potential, and which we denote by W_K and $W_{K,u}$ respectively. The proof of the attainment of both capacities, together with some basic properties which will be used throughout the paper, is presented in Sect. 2.

Our first result is about the stability of the spectrum of $(-\Delta)^m$, once a set of small V^m-capacity is removed.

Theorem 1.1 *Let $N \geq 2m$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Suppose one of the following:*

(D) $V^m(\Omega) = H^m_0(\Omega)$ and $K \subset \Omega$ is compact;
(N) $V^m(\Omega) = H^m_\vartheta(\Omega)$ and the exists $K_0 \subset \Omega$ compact such that K is compact and $K \subset K_0$.

Denote by $\lambda_j(\Omega)$ and $\lambda_j(\Omega \setminus K)$, $j \in \mathbb{N} \setminus \{0\}$, the eigenvalues respectively for (1.1) *and* (1.8)*. For all $j \in \mathbb{N} \setminus \{0\}$, there exist $\delta > 0$ and $C > 0$ (which depends on K_0 in the Navier case (N)) such that, if $\mathrm{cap}_{V^m,\,\Omega}(K) < \delta$, one has*

$$|\lambda_j(\Omega \setminus K) - \lambda_j(\Omega)| \leq C \left(\mathrm{cap}_{V^m,\,\Omega}(K)\right)^{1/2}.$$

In particular $\lambda_j(\Omega \setminus K) \to \lambda_j(\Omega)$ as $\mathrm{cap}_{V^m,\,\Omega}(K) \to 0$.

The proof of Theorem 1.1 is based on the variational characterization of the eigenvalues (1.9) and it is detailed in Sect. 3.1. We remark that spectral stability in a more general higher-order context was also established in [5] with a different approach. Here we propose a self-contained and simple proof for our Dirichlet and Navier-Dirichlet settings.

Aiming now at a more precise estimate of the convergence rate, we introduce the following notion of convergence of sets.

Definition 1.1 Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in Ω. We say that K_ε *is concentrating to a compact set* $K \subset \Omega$ as $\varepsilon \to 0$ if, for every open set $U \subseteq \Omega$ such that $U \supset K$, there exists $\varepsilon_U > 0$ such that $U \supset K_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_U)$.

An example is given by a decreasing family of compact sets, see e.g. [13, Example 3.7]. Note that this property alone is not sufficient to have the standard (i.e. metric) convergence of sets. For instance, the uniqueness of the limit set is not assured (e.g. if K_ε is concentrating to K then K_ε is concentrating also to $\widetilde{K}$ for any compact set $\widetilde{K}$ which contains K). However, as for second-order problems, in the case of a 0-capacity limit set, this concept of convergence of sets is enough to prove the continuity of the capacity (Proposition 3.1) and the Mosco convergence [11, 25] of the respective V^m-spaces (Proposition 3.2). These will be the tools needed for a sharp asymptotic expansion of a perturbed simple eigenvalue $\lambda_J(\Omega \setminus K_\varepsilon)$ in terms of the (u_J, V^m)-capacity of the vanishing compact sets K_ε, where u_J is a normalized eigenfunction relative to $\lambda_J(\Omega)$.

Theorem 1.2 *Let $N \geq 2m$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let $\lambda_J(\Omega)$ be a simple eigenvalue of* (1.1) *and $u_J \in V^m(\Omega)$ be a corresponding eigenfunction normalized in $L^2(\Omega)$. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets concentrating, as $\varepsilon \to 0$, to a compact set K with $\mathrm{cap}_{V^m,\Omega}(K) = 0$. Then, as $\varepsilon \to 0$,*

$$\lambda_J(\Omega \setminus K_\varepsilon) = \lambda_J(\Omega) + \mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J) + o(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)). \tag{1.14}$$

Theorem 1.2 is the higher-order counterpart of [2, Theorem 1.4] and its proof is presented in Sect. 3.2. In the expansion (1.14), the asymptotic parameter is the (u_J, V^m)-capacity of the vanishing set. The next aim is to quantify $\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)$ as a function of the diameter of K_ε. In this respect, we focus on the particular case in which the limit set K is a point $x_0 \in \Omega$ (which in dimension $N \geq 2m$ has zero V^m-capacity, see Proposition 2.3); without loss of generality, we consider $x_0 = 0$. We deal with a uniformly shrinking family of compact sets K_ε, the model case being $K_\varepsilon = \varepsilon\mathcal{K} \ni 0$ for some fixed compact set $\mathcal{K} \subset \mathbb{R}^N$. In this case, assuming 0 to be an interior point of Ω, and having the operator $(-\Delta)^m - \lambda$ constant coefficients, the eigenfunction u_J is analytic at 0, see [20], and hence it does not have infinite order of vanishing there. Therefore, there exist $\gamma \in \mathbb{N}$ and a γ-homogeneous polyharmonic polynomial $U_0 \in H^m_{loc}(\mathbb{R}^N)$ such that

$$U_\varepsilon := \frac{u_J(\varepsilon\,\cdot)}{\varepsilon^\gamma} \to U_0 \qquad \text{in } H^m(B_R(0)) \tag{1.15}$$

for all $R > 0$ as $\varepsilon \to 0$. This fact follows from a general result about elliptic equations by Bers [6, Sec.4 Theorem 1], see also [9, Theorem 2.1], provided—as in our case—one discards the possibility of an infinite order of vanishing.

In light of (1.15), our strategy to find an asymptotic expansion of $\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)$ is based on a blow-up argument: we rescale the boundary value problem defining the capacitary potential W_{K_ε,u_J}, find a limit equation on $\mathbb{R}^N \setminus \mathcal{K}$, and prove the convergence of the family of rescaled capacitary potentials to the one for the limiting problem. To this aim, a suitable notion of capacity in $\mathbb{R}^N$, involving homogeneous higher-order Sobolev spaces $\mathcal{D}^{m,2}_0(\mathbb{R}^N)$ and denoted by $\mathrm{cap}_{m,\mathbb{R}^N}$, will be needed, see Sect. 2.2. The asymptotic expansion of $\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)$ obtained by these arguments turns out to depend on the order of vanishing of u_J at the point 0. More precisely, we have the following results, which we state below for the model case $K_\varepsilon = \varepsilon\mathcal{K}$ and prove in more generality in Sect. 4.1. For the Dirichlet case we have the following:

Theorem 1.3 (Dirichlet case) *Let $N > 2m$ and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with $0 \in \Omega$. Let $\mathcal{K} \subset \mathbb{R}^N$ be a fixed compact set and, for all $\varepsilon > 0$, $K_\varepsilon = \varepsilon\mathcal{K}$. Let λ_J be an eigenvalue of* (1.1) *with Dirichlet boundary conditions and $u_J \in H^m_0(\Omega)$ be a corresponding eigenfunction normalized in $L^2(\Omega)$. Then*

$$\mathrm{cap}_{m,\Omega}(K_\varepsilon, u_J) = \varepsilon^{N-2m+2\gamma}\left(\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) + o(1)\right) \tag{1.16}$$

as $\varepsilon \to 0$, with γ and U_0 as in (1.15).

The dimensional restriction $N > 2m$ is mainly due to the possibility of characterizing higher-order homogeneous Sobolev spaces as concrete functional spaces satisfying Sobolev and Hardy-type inequalities (see Sects. 2.2.1 and 2.2.2). In the conformal case $N = 2m$ such spaces are instead made of classes of functions defined up to additive polynomials, see [15, II.6-7]. In the Navier setting, we need to restrict to the biharmonic case $m = 2$.

Theorem 1.4 (Navier case) *Let $N > 4$ and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with $0 \in \Omega$. Let $\mathcal{K} \subset \mathbb{R}^N$ be a fixed compact set and, for all $\varepsilon > 0$, $K_\varepsilon = \varepsilon\mathcal{K}$. Let λ_J be an* *eigenvalue of* (1.1) *with* $m = 2$ *and Navier boundary conditions and* $u_J \in H^2_\vartheta(\Omega)$ *be a corresponding eigenfunction normalized in* $L^2(\Omega)$*. Then*

$$\mathrm{cap}_{2,\vartheta,\Omega}(K_\varepsilon, u_J) = \varepsilon^{N-4+2\gamma}\left(\mathrm{cap}_{2,\mathbb{R}^N}(\mathcal{K}, U_0) + o(1)\right)$$

as $\varepsilon \to 0$*, with* γ *and* U_0 *as in* (1.15) *with* $m = 2$*.*

It is remarkable that the same asymptotic expansion (1.16) for $m = 2$ holds true for both Dirichlet and Navier BCs on $\partial\Omega$. As a consequence, imposing different conditions on the external boundary does not affect the first term of the asymptotic expansion of the perturbed eigenvalues. In the proof of Theorems 1.3 and 1.4 we will need to distinguish between the two settings. If in the case of Dirichlet BCs on $\partial\Omega$ the natural candidate as functional space for the limiting problem is $D^{m,2}_0(\mathbb{R}^N \setminus \mathcal{K})$, on the other hand, in the Navier case, because of the impracticability of the trivial extension of a function outside Ω, this is not evident and follows after a more involved analysis which makes use of suitable Hardy–Rellich inequalities. In Sect. 2.2.2 we give the precise statement and proofs of such inequalities. This is the main reason for the restriction to the case $m = 2$, see Sect. 4.1.

Braiding together Theorem 1.2 and Theorems 1.3–1.4, we obtain the following sharp asymptotic expansions of $\lambda_J(\Omega \setminus K_\varepsilon)$, stated here for the model case $K_\varepsilon = \varepsilon\mathcal{K}$.

Theorem 1.5 (Dirichlet case) *Let* $N > 2m$ *and* $\Omega \subset \mathbb{R}^N$ *be a bounded smooth domain containing* 0*. Let* $\mathcal{K} \subset \mathbb{R}^N$ *be a fixed compact set and, for all* $\varepsilon > 0$*,* $K_\varepsilon = \varepsilon\mathcal{K}$*. Let* λ_J *be a simple eigenvalue of* (1.1) *with Dirichlet boundary conditions and let* $u_J \in H^m_0(\Omega)$ *be a corresponding eigenfunction normalized in* $L^2(\Omega)$*. Then*

$$\lambda_J(\Omega \setminus K_\varepsilon) = \lambda_J(\Omega) + \varepsilon^{N-2m+2\gamma}\left(\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) + o(1)\right)$$

as $\varepsilon \to 0$*, with* γ *and* U_0 *as in* (1.15)*.*

Theorem 1.6 (Navier case) *Let* $N > 4$ *and* $\Omega \subset \mathbb{R}^N$ *be a bounded smooth domain containing* 0*. Let* $\mathcal{K} \subset \mathbb{R}^N$ *be a fixed compact set and, for all* $\varepsilon > 0$*,* $K_\varepsilon = \varepsilon\mathcal{K}$*. Let* λ_J *be a simple eigenvalue of* (1.1) *with* $m = 2$ *and Navier boundary conditions and let* $u_J \in H^2_\vartheta(\Omega)$ *be a corresponding eigenfunction normalized in* $L^2(\Omega)$*. Then*

$$\lambda_J(\Omega \setminus K_\varepsilon) = \lambda_J(\Omega) + \varepsilon^{N-4+2\gamma}\left(\mathrm{cap}_{2,\mathbb{R}^N}(\mathcal{K}, U_0) + o(1)\right)$$

as $\varepsilon \to 0$*, with* γ *and* U_0 *as in* (1.15) *with* $m = 2$*.*

Theorems 1.3–1.6 deal with the model case $K_\varepsilon = \varepsilon\mathcal{K}$. Section 4.1 will be devoted to the proof of their analogues for a more comprehensive setting of general families of concentrating compact sets $\{K_\varepsilon\}_{\varepsilon>0}$ which uniformly shrink to a point, see Theorems 4.4–4.7.

The asymptotic expansion provided by Theorems 1.5–1.6 detects the sharp vanishing rate of the eigenvalue variation whenever $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) \neq 0$. In Sect. 4.2 we establish sufficient conditions for this to hold. In particular, this will always be the case when the Lebesgue measure of $\mathcal{K}$ is positive (Proposition 4.8), or when either the eigenfunction u_J does not vanish at the point x_0 (Proposition 4.9), or it does vanish but the compactum $\mathcal{K}$ and the null-set of the limiting polynomial U_0 in (1.15) are "transversal enough" (Proposition 4.10).

The paper is then concluded by the short Sect. 5 which contains a discussion about questions which are left open by our analysis and possible directions in which our results may be extended.

2 Definition of higher-order capacity with Dirichlet and Navier BCs and first properties

The aim of this section is to introduce a notion of *capacity* which agrees with the higher-order framework of the problem and which turns out to be an important tool in order to study the asymptotics of the eigenvalues of the perturbed problems (1.2)–(1.3). The concept of (condenser) capacity, well-known for the second-order case, was first considered in the higher-order setting by Maz'ya for bounded domains on which Dirichlet boundary conditions are imposed, or for the whole space, see[2] e.g. [23, 24]. In Sect. 2.1 we propose an unified treatment for both Dirichlet and Navier settings and establish the main properties of the capacities defined by (1.11)–(1.12). In Sect. 2.2 we recall the main properties of the homogeneous Sobolev spaces and establish a Hardy–Rellich inequality with intermediate derivatives. Moreover we introduce the notion of capacity of a compact set in the whole space $\mathbb{R}^N$ for large dimensions $N > 2m$.

2.1 Higher-order capacities in V_0^m

Let $m \in \mathbb{N} \setminus \{0\}$, Ω be a bounded smooth domain in $\mathbb{R}^N$ and K be a compact subset of Ω. First, we observe that both capacities (1.11)–(1.12) are attained. Indeed, for any $u \in V^m(\Omega)$, we have that $S_u := \left\{g \in V^m(\Omega) \mid g - u \in V_0^m(\Omega \setminus K)\right\}$ is an affine hyperplane in $V^m(\Omega)$, so in particular a convex set. This implies that there exists a unique element in $V^m(\Omega)$ which minimizes the distance from the origin, i.e. the norm $\|\nabla^m \cdot\|_2$ in S_u, which is called *capacitary potential* and is denoted by $W_{K,u}$ (in case u is replaced by η_K, we simply denote it by W_K). This means that $W_{K,u}$ is such that

$$\operatorname{cap}_{V^m,\,\Omega}(K, u) = \int_\Omega |\nabla^m W_{K,u}|^2$$

and it is the unique (weak) solution of the problem

$$\begin{cases} (-\Delta)^m W_{K,u} = 0 \quad \text{in } \Omega \setminus K, \\ W_{K,u} \in V^m(\Omega), \\ W_{K,u} - u \in V_0^m(\Omega \setminus K), \end{cases} \tag{2.1}$$

in the sense that $W_{K,u} \in V^m(\Omega)$, $W_K - u \in V_0^m(\Omega \setminus K)$ and

$$\int_{\Omega \setminus K} \nabla^m W_{K,u} \nabla^m \varphi = 0 \qquad \text{for all } \varphi \in V_0^m(\Omega \setminus K).$$

In (2.1) we are in fact prescribing homogeneous Dirichlet or Navier boundary conditions on $\partial\Omega$ and, in case ∂K is smooth, an "m-Dirichlet-matching" between $W_{K,u}$ and u on ∂K, i.e. the m conditions $W_{K,u} = u$, $\partial_n W_{K,u} = \partial_n u, \ldots, \partial_n^{m-1} W_{K,u} = \partial_n^{m-1} u$ on ∂K.

In particular, the minimizer W_K of the V^m-capacity is such that

$$\operatorname{cap}_{V^m,\,\Omega}(K) = \int_\Omega |\nabla^m W_K|^2$$

[2] In these works the higher-order capacity is defined through the L^p-norm of the tensor of the m-th derivatives $D^m u$. However, the two norms are equivalent on any bounded smooth domain.

and it is the unique (weak) solution of the problem

$$\begin{cases} (-\Delta)^m W_K = 0 & \text{in } \Omega \setminus K, \\ W_K \in V^m(\Omega), \\ W_K - \eta_K \in V_0^m(\Omega \setminus K), \end{cases}$$

in the sense that $W_K \in V^m(\Omega)$, $W_K - \eta_K \in V_0^m(\Omega \backslash K)$ and

$$\int_{\Omega \backslash K} \nabla^m W_K \nabla^m \varphi = 0 \qquad \text{for all } \varphi \in V_0^m(\Omega \setminus K). \tag{2.2}$$

We observe that $\text{cap}_{V^m,\Omega}(K) = 0$ implies that $0 \in S_{\eta_K}$, i.e. $\eta_K \in V_0^m(\Omega \setminus K)$. Since $\eta_K \equiv 1$ on K, this can only hold true when the Sobolev space "does not see" K, i.e. when $V_0^m(\Omega \setminus K) = V^m(\Omega)$. As a consequence, the eigenvalues of problems (1.2) and (1.3) coincide with those of (1.1). More precisely, in the spirit of [10, Propositions 2.1 and 2.2] (see also [13, Proposition 3.3]), we prove the following.

Proposition 2.1 *The following statements are equivalent:*

i) $\text{cap}_{V^m,\Omega}(K) = 0$*;*
ii) $V_0^m(\Omega \setminus K) = V^m(\Omega)$*;*
iii) $\lambda_n(\Omega \setminus K) = \lambda_n(\Omega)$ *for all* $n \in \mathbb{N}$.

Proof To show $(i) \Rightarrow (ii)$, by density of $X^m(\Omega)$ in $V^m(\Omega)$, see (1.10), it is enough to prove that each $u \in X^m(\Omega)$ may be approximated by functions in $V_0^m(\Omega \setminus K)$ in the V^m-norm. Since $\text{cap}_{V^m,\Omega}(K) = 0$, there exists $(w_i)_i \subset V^m(\Omega)$ with $w_i - \eta_K \in V_0^m(\Omega \backslash K)$ so that $\|\nabla^m w_i\|_2^2 \to 0$ as $i \to +\infty$. Hence, defining $v_i := u(1 - \eta_K w_i)$, one has that $v_i \in V_0^m(\Omega \setminus K)$ and, in view of (1.6),

$$\begin{aligned} \|\nabla^m(u - v_i)\|_2^2 = \|\nabla^m(u\eta_K w_i)\|_2^2 &\lesssim \sum_{j=0}^m \int_\Omega |D^{m-j}(\eta_K u)|^2 |D^j w_i|^2 \\ &\le \|\eta_K u\|_{W^{m,\infty}(\Omega)}^2 \sum_{j=0}^m \int_\Omega |D^j w_i|^2 = \|\eta_K u\|_{W^{m,\infty}(\Omega)}^2 \|w_i\|_{H^m(\Omega)}^2 \\ &\le C^2 \|\eta_K u\|_{W^{m,\infty}(\Omega)}^2 \|\nabla^m w_i\|_2^2 \to 0 \end{aligned}$$

as $i \to +\infty$.

The reversed implication $(ii) \Rightarrow (i)$ is due to the fact that $\varphi = W_K$ may be used as a test function in (2.2) to obtain that $\|W_K\|_{V_0^m(\Omega \backslash K)} = \|W_K\|_{V^m(\Omega)} = 0$, which is equivalent to (i).

$(ii) \Rightarrow (iii)$ easily follows from the minimax characterization of the eigenvalues (1.9). The converse is implied by the spectral theorem, because by (iii) one is able to find an orthonormal basis of $V^m(\Omega)$ made of $V_0^m(\Omega \setminus K)$-functions. □

Remark 1 From Proposition 2.1, in particular from the implication $(i) \Rightarrow (ii)$, one derives the following equivalence:

$$\text{cap}_{V^m,\Omega}(K) = 0 \quad \Leftrightarrow \quad \text{cap}_{V^m,\Omega}(K, u) = 0 \text{ for all } u \in V^m(\Omega).$$

Next, we investigate some properties of the above defined capacities, in particular the monotonicity properties with respect to Ω and K, and the relation between the Dirichlet and the Navier capacities.

Proposition 2.2 (Monotonicity properties of the capacity) *The following properties hold.*

i) If $K_1 \subset K_2 \subset \Omega$, K_1, K_2 are compact, and $h \in V^m(\Omega)$, then

$$\mathrm{cap}_{V^m,\,\Omega}(K_1, h) \leq \mathrm{cap}_{V^m,\,\Omega}(K_2, h).$$

ii) If $K \subset \Omega_1 \subset \Omega_2$, K is compact, and $h \in H^m(\Omega_2)$, then

$$\mathrm{cap}_{m,\Omega_2}(K, h) \leq \mathrm{cap}_{m,\Omega_1}(K, h).$$

iii) For every $K \subset \Omega$ compact and $h \in H^m(\Omega)$, there holds

$$\mathrm{cap}_{m,\vartheta,\,\Omega}(K, h) \leq \mathrm{cap}_{m,\,\Omega}(K, h).$$

Proof *i*) It is enough to notice that, for $u \in V^m(\Omega)$, the condition $u - h \in V_0^m(\Omega \setminus K_2)$ is more restrictive than $u - h \in V_0^m(\Omega \setminus K_1)$.

ii) Any $u \in H_0^m(\Omega_1)$ can be extended by 0 to a function in $H_0^m(\Omega_2)$, so the minimization for $\mathrm{cap}_{m,\Omega_2}(K, h)$ takes into consideration a larger set of test functions than the one for $\mathrm{cap}_{m,\Omega_1}(K, h)$, and consequently the inf decreases.

iii) It follows directly from the inclusions in (1.7). □

Remark 2 Note that the argument used in the proof of (*ii*) for Dirichlet BCs is no more available in the case of Navier BCs on $\partial\Omega$.

As an example, which is also relevant for our purposes, we compute the capacity of a point in $\mathbb{R}^N$.

Proposition 2.3 (Capacity of a point) *Let $x_0 \in \Omega$. Then $\mathrm{cap}_{V^m,\,\Omega}(\{x_0\}) = 0$ if $N \geq 2m$, while $\mathrm{cap}_{V^m,\,\Omega}(\{x_0\}) > 0$ when $N \leq 2m - 1$.*

Proof It is not restrictive to assume that $x_0 = 0 \in \Omega$. If $N \leq 2m - 1$, then the embedding $V^m(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is continuous i.e. $\|\nabla^m u\|_2 \geq C(m, N, \Omega)\|u\|_\infty$ for all $u \in V^m(\Omega)$, with a constant $C(m, N, \Omega) > 0$ which does not depend on u. In particular for those functions in $V^m(\Omega)$ for which $u(0) = 1$, one has $\|\nabla^m u\|_2 \geq C(m, N, \Omega)$. Hence the infimum in the definition of $\mathrm{cap}_{V^m,\,\Omega}(K)$ is strictly positive.

In view of Proposition 2.2(iii), it is sufficient to prove the result for the Dirichlet case. Let $N \geq 2m+1$ and take a sequence of shrinking cut-off in the following way: let $\zeta \in C_0^\infty(B_2(0))$ such that $\zeta \equiv 1$ on $B_1(0)$ and consider $\zeta_k(x) := \zeta(kx)$. One has that $\zeta_k \in C_0^\infty(B_{\frac{2}{k}}(0))$ and $\zeta_k \equiv 1$ on $B_{\frac{1}{k}}(0)$, hence $\mathrm{supp}\,\zeta_k \subset \Omega$ for $k \geq k_0 = k_0(\mathrm{dist}(0, \partial\Omega))$. We compute

$$\int_\Omega |\nabla^m \zeta_k|^2 = \int_{B_{\frac{2}{k}}(0)} |k^m(\nabla^m\zeta)(kx)|^2 dx = k^{2m-N} \int_{B_2(0)} |\nabla^m\zeta|^2 \to 0$$

as $k \to \infty$ since $2m - N < 0$. Being such functions admissible for the minimization of $\mathrm{cap}_{m,\,\Omega}$, we deduce $\mathrm{cap}_{m,\,\Omega}(\{0\}) = 0$. The argument is similar for the case $N = 2m$, provided we choose accurately the sequence of cut-off functions, see [24, Proposition 7.6.1/2 and Proposition 13.1.2/2]. For the sake of completeness, we retrace here the proof. Let α denote a function in $C^\infty([0, 1])$ equal to zero near $t = 0$, to 1 near $t = 1$, and such that $0 \leq \alpha(t) \leq 1$. Define then $\zeta_\varepsilon := \alpha(v_\varepsilon)$, where

$$v_\varepsilon(x) := \begin{cases} 1 & \text{if } |x| \leq \varepsilon, \\ \frac{\log|x| - \log\sqrt{\varepsilon}}{\log\varepsilon - \log\sqrt{\varepsilon}} & \text{if } \varepsilon \leq |x| \leq \sqrt{\varepsilon}, \\ 0 & \text{if } |x| \geq \sqrt{\varepsilon}. \end{cases}$$

Notice that v_ε is continuous but not C^1; on the other hand $\zeta_\varepsilon \in C_0^\infty(B_{\sqrt{\varepsilon}}(0))$, since α is constant in a neighbourhood of 0 and in a neighbourhood of 1 by construction. Therefore $\zeta_\varepsilon \in H_0^m(B_1(0))$ for any $\varepsilon \in (0,1)$. Moreover $\zeta_\varepsilon \equiv 1$ in $B_\varepsilon(0)$ so that ζ_ε is an admissible test function in the minimization of $\mathrm{cap}_{m,\Omega}(\{0\})$. By direct calculations, there exists a constant $C = C(m) > 0$ (independent of ε) such that

$$|\nabla^m \zeta_\varepsilon(x)| \le \frac{C}{|\log \varepsilon|}\frac{1}{|x|^m} \quad \text{for all } \varepsilon < |x| < \sqrt{\varepsilon},$$

whereas

$$\nabla^m \zeta_\varepsilon(x) = 0 \quad \text{if either } |x| \le \varepsilon \text{ or } |x| \ge \sqrt{\varepsilon}.$$

Therefore

$$\int_\Omega |\nabla^m \zeta_\varepsilon|^2 \lesssim \frac{1}{\log^2 \varepsilon}\int_\varepsilon^{\sqrt{\varepsilon}} \frac{1}{r}\, dr = \frac{1}{2|\log \varepsilon|} \to 0$$

as $\varepsilon \to 0$. The argument is concluded as above. □

2.2 Homogeneous Sobolev spaces and capacities in $\mathbb{R}^N$

2.2.1 The homogeneous Sobolev spaces $D_0^{m,2}(\mathbb{R}^N)$

So far, we defined the notion of V^m-capacity for compact sets contained in an open bounded smooth domain $\Omega \subset \mathbb{R}^N$. An analogous definition can be given when $\Omega = \mathbb{R}^N$, provided the underlined space is of homogeneous kind. We introduce the homogeneous Sobolev spaces (sometimes referred to as Beppo Levi spaces) $D_0^{m,2}(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D_0^{m,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla^m u|^2\right)^{\frac{1}{2}}.$$

Actually, the spaces $D_0^{m,2}(\mathbb{R}^N)$ are more commonly defined as the completion with respect to the norm $\|D^m \cdot\|_2$, i.e. with respect to the full tensor of all highest derivatives. However, the two definitions are equivalent since, by integration by parts, $\|D^m \cdot\|_2$ and $\|\nabla^m \cdot\|_2$ are equivalent norms on $C_0^\infty(\mathbb{R}^N)$, see e.g. [17, Sec.2.2.1].

For large dimensions $N > 2m$, the following Sobolev inequalities are well-known: for every $0 \le j \le m$ there exists a constant $S(N,m,j) > 0$ (depending only on N, m and j) such that

$$S(N,m,j)\left(\int_{\mathbb{R}^N} |D^j u|^{2^*_{m,j}}\right)^{\frac{2}{2^*_{m,j}}} \le \|D^m u\|^2_{L^2(\mathbb{R}^N)} \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N), \tag{2.3}$$

where $2^*_{m,j} := \frac{2N}{N-2(m-j)}$. In particular, for $j = 0$, there exists a constant $S(N,m) > 0$ such that

$$S(N,m)\left(\int_{\mathbb{R}^N} |u|^{2^*_m}\right)^{\frac{2}{2^*_m}} \le \|u\|_{D_0^{m,2}(\mathbb{R}^N)} \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N),$$

where $2_m^* := 2_{m,0}^* = \frac{2N}{N-2m}$, see [17, Theorem 2.3]. In view of (2.3), if $N > 2m$, one may also characterize $D_0^{m,2}(\mathbb{R}^N)$ as

$$D_0^{m,2}(\mathbb{R}^N) = \left\{u \in L^{2_m^*}(\mathbb{R}^N) \,\middle|\, D^j u \in L^{2_{m,j}^*}(\mathbb{R}^N) \text{ for all } 0 < j \le m\right\}.$$

Analogously, for $K \subset \mathbb{R}^N$ compact, one may consider the exterior domain $\Omega = \mathbb{R}^N \setminus K$ and define $D_0^{m,2}(\mathbb{R}^N \setminus K)$ as the completion of $C_0^\infty(\mathbb{R}^N \setminus K)$ with respect to the norm $\|\nabla^m \cdot\|_2$, which is characterized, for $N > 2m$, as

$$D_0^{m,2}(\mathbb{R}^N \setminus K) = \left\{ u \in L^{2_m^*}(\mathbb{R}^N \setminus K) \,\middle|\, \begin{array}{l} D^j u \in L^{2_{m,j}^*}(\mathbb{R}^N \setminus K) \text{ for all } 0 < j \le m \\ \text{and } \psi u \in H_0^m(\mathbb{R}^N \setminus K) \text{ for all } \psi \in C_0^\infty(\mathbb{R}^N) \end{array} \right\},$$

see [15, Theorem II.7.6].

2.2.2 A Hardy–Rellich-type inequality with intermediate derivatives

Besides Sobolev inequalities, an important tool in the theory of Sobolev spaces in large dimensions $N > 2m$ is represented by Hardy–Rellich inequalities, which state that the Sobolev norm of the highest order derivatives controls a singularly weighted Sobolev norm of the function. We refer to [12] for such inequalities in $H_0^m(\Omega)$ and to [16, 18] for their extensions to $H_\vartheta^m(\Omega)$. In this section, inspired by [26], we prove a Hardy–Rellich-type inequality for the space $H_\vartheta^2(\Omega)$ including also the gradient term, which provides a further characterization of the space $D_0^{2,2}(\mathbb{R}^N)$ for $N > 4$. It will be needed in Sect. 4.1 to identify the functional space containing the limiting profile in the blow-up argument, when Navier BCs are imposed on $\partial\Omega$.

Theorem 2.4 *Let $N > 4$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Then, for every function $u \in H^2(\Omega) \cap H_0^1(\Omega)$, one has that $\frac{u}{|x|^2}, \frac{\nabla u}{|x|} \in L^2(\Omega)$ and*

$$(N-4)^2 \int_\Omega \frac{|u|^2}{|x|^4}\,dx + 2(N-4)\int_\Omega \frac{|\nabla u|^2}{|x|^2}\,dx \le \int_\Omega |\Delta u|^2\,dx. \tag{2.4}$$

Proof Let $u \in C^\infty(\overline{\Omega})$ be such that $u|_{\partial\Omega} = 0$. Let us assume that $0 \in \Omega$. Let us introduce a parameter λ to be fixed later and, for $\varepsilon > 0$ small, let us denote $\Omega_\varepsilon := \Omega \setminus B_\varepsilon(0)$. We have that

$$0 \le \int_{\Omega_\varepsilon} \left(\frac{x}{|x|}\Delta u + \lambda u \frac{x}{|x|^3}\right)^2 dx = \int_{\Omega_\varepsilon} (\Delta u)^2 + \lambda^2 \int_{\Omega_\varepsilon} \frac{u^2}{|x|^4}\,dx + 2\lambda \int_{\Omega_\varepsilon} \frac{u}{|x|^2}\Delta u\,dx. \tag{2.5}$$

We can rewrite the third term as

$$\begin{aligned}\int_{\Omega_\varepsilon} \frac{u}{|x|^2}\Delta u\,dx &= -\int_{\Omega_\varepsilon} \nabla u\left(\frac{\nabla u}{|x|^2} - 2u\frac{x}{|x|^4}\right)dx + \int_{\partial\Omega} \frac{u}{|x|^2}\partial_\nu u\,d\sigma - \frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon} u\nabla u \cdot \frac{x}{\varepsilon}\,d\sigma \\ &= -\int_{\Omega_\varepsilon} \frac{|\nabla u|^2}{|x|^2}\,dx + \int_{\Omega_\varepsilon} \nabla(u^2)\frac{x}{|x|^4}\,dx + \mathcal{O}(\varepsilon^{N-3}) \\ &= -\int_{\Omega_\varepsilon} \frac{|\nabla u|^2}{|x|^2}\,dx - (N-4)\int_{\Omega_\varepsilon} \frac{u^2}{|x|^4}\,dx + \int_{\partial\Omega} u^2\frac{x\cdot\nu}{|x|^4} \\ &\quad - \int_{\partial B_\varepsilon} \frac{u^2}{\varepsilon^3}\,d\sigma + \mathcal{O}(\varepsilon^{N-3})\end{aligned}$$

as $\varepsilon \to 0$. Since the third term vanishes and the second to last term is $\mathcal{O}(\varepsilon^{N-4})$ as $\varepsilon \to 0$, from (2.5) we get

$$0 \le \int_{\Omega_\varepsilon} (\Delta u)^2 + \lambda^2 \int_{\Omega_\varepsilon} \frac{u^2}{|x|^4}\, dx - 2\lambda \int_{\Omega_\varepsilon} \frac{|\nabla u|^2}{|x|^2}\, dx - 2\lambda(N-4) \int_{\Omega_\varepsilon} \frac{u^2}{|x|^4}\, dx + \mathcal{O}(\varepsilon^{N-4}).$$

Choosing now $\lambda = N-4$, we obtain

$$(N-4)^2 \int_{\Omega_\varepsilon} \frac{u^2}{|x|^4}\, dx + 2(N-4) \int_{\Omega_\varepsilon} \frac{|\nabla u|^2}{|x|^2}\, dx \le \int_{\Omega_\varepsilon} (\Delta u)^2 + \mathcal{O}(\varepsilon^{N-4}) \quad \text{as } \varepsilon \to 0.$$

Inequality (2.4) follows by letting $\varepsilon \to 0$ and by density of the set $\{u \in C^\infty(\overline{\Omega}) \mid u|_{\partial\Omega} = 0\}$ in $H^2(\Omega) \cap H^1_0(\Omega)$. If $0 \notin \Omega$ the above argument can be repeated by considering directly in (2.5) the integral on the whole Ω. □

We observe that (2.4) holds also for all functions in $C_0^\infty(\mathbb{R}^N)$ (since any $u \in C_0^\infty(\mathbb{R}^N)$ is contained in some $H^2_\vartheta(\Omega)$). Therefore, by density of $C_0^\infty(\mathbb{R}^N)$ in $D_0^{2,2}(\mathbb{R}^N)$ and Fatou's Lemma we easily deduce that, if $N > 4$, then

$$(N-4)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4}\, dx + 2(N-4) \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^2}\, dx \le \int_{\mathbb{R}^N} |\Delta u|^2\, dx$$

for all $u \in D_0^{2,2}(\mathbb{R}^N)$. In particular we have that $D_0^{2,2}(\mathbb{R}^N)$ is contained in the space

$$\mathcal{S}^2(\mathbb{R}^N) := \left\{ u \in H^2_{loc}(\mathbb{R}^N) \,\middle|\, \frac{\nabla^{2-k} u}{|x|^k} \in L^2(\mathbb{R}^N) \text{ for } k \in \{0,1,2\} \right\}.$$

We prove now that the two functional spaces coincide.

Proposition 2.5 $\mathcal{S}^2(\mathbb{R}^N) = D_0^{2,2}(\mathbb{R}^N)$ *for all* $N > 4$.

Proof We have already observed above that $\mathcal{S}^2(\mathbb{R}^N) \supseteq D_0^{2,2}(\mathbb{R}^N)$. Let now $u \in \mathcal{S}^2(\mathbb{R}^N)$, η be a cutoff function with support in $B_2(0)$ and which takes the value 1 in $B_1(0)$, and define $\eta_R := \eta\left(\frac{\cdot}{R}\right)$ for all $R > 0$. Hence $\eta_R u \in H^2_0(B_{2R}(0))$ and we claim that $\|\Delta(\eta_R u - u)\|_2 \to 0$ as $R \to +\infty$. Indeed,

$$\|\Delta\big((\eta_R - 1)\, u\big)\|_2^2 \lesssim \|(\Delta\eta_R) u\|_2^2 + \|\nabla\eta_R \nabla u\|_2^2 + \|(\eta_R - 1)\, \Delta u\|_2^2,$$

where

$$\|(\eta_R - 1)\, \Delta u\|_2^2 \le \int_{\mathbb{R}^N \setminus B_R} |\Delta u|^2 \to 0$$

as $R \to +\infty$, and for $k \in \{1,2\}$,

$$\begin{aligned} \|\nabla^k \eta_R \nabla^{2-k} u\|_2^2 &= \int_{R<|x|<2R} \frac{1}{R^{2k}} \left|(\nabla^k \eta)\left(\frac{x}{R}\right)\right|^2 |\nabla^{2-k} u|^2\, dx \\ &\lesssim 2^{2k} \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|\nabla^{2-k} u|^2}{|x|^{2k}}\, dx \to 0 \end{aligned}$$

as $R \to +\infty$. By density of $C_0^\infty(B_{2R}(0))$ in $H^2_0(B_{2R}(0))$, this implies that $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{S}^2(\mathbb{R}^N)$ in the $D_0^{2,2}$-norm, thus concluding the proof. □

2.2.3 Capacities in $\mathbb{R}^N$

Similarly to the case of a bounded set Ω described in Sect. 2.1, for any compact set $K \subset \mathbb{R}^N$ and any $u \in D_0^{m,2}(\mathbb{R}^N)$ with $N > 2m$, we define

$$\mathrm{cap}_{m,\mathbb{R}^N}(K,u) := \inf\left\{ \int_{\mathbb{R}^N} |\nabla^m f|^2 \,\middle|\, f \in D_0^{m,2}(\mathbb{R}^N),\ f - u \in D_0^{m,2}(\mathbb{R}^N \setminus K) \right\}, \quad (2.6)$$

which we simply denote by $\mathrm{cap}_{m,\mathbb{R}^N}(K)$ when $u = \eta_K$. The argument for the attainability of the capacity is easily adapted from the one for $\mathrm{cap}_{V^m,\Omega}(K,u)$. Analogous properties hold also in this setting, in particular it is true that

$$D_0^{m,2}(\mathbb{R}^N) = D_0^{m,2}(\mathbb{R}^N \setminus K) \quad \text{if and only if} \quad \mathrm{cap}_{m,\mathbb{R}^N}(K) = 0 \qquad (2.7)$$

which directly implies that

$$\mathrm{cap}_{m,\mathbb{R}^N}(K) = 0 \quad \Leftrightarrow \quad \mathrm{cap}_{m,\mathbb{R}^N}(K,u) = 0 \ \text{ for all } \ u \in D_0^{m,2}(\mathbb{R}^N).$$

The analogue of (2.7) in the case of a bounded domain Ω is contained in Proposition 2.1 and its proof relies on (1.6), which in turn is based on a Poincaré inequality, the latter being no longer valid in $\mathbb{R}^N$. However, if $N > 2m$, the role played by Poincaré inequalities can be replaced by the critical Sobolev embedding. Although known, here we retrace the proof of (2.7) for the sake of completeness. Let $u \in C_0^\infty(\mathbb{R}^N)$, set $\Sigma := \mathrm{supp}(u)$, and consider $(w_i)_i \subset D_0^{m,2}(\mathbb{R}^N)$ with $w_i - \eta_K \in D_0^{m,2}(\mathbb{R}^N \setminus K)$ such that $\|\nabla^m w_i\|_2^2 \to 0$ as $i \to +\infty$. Then $v_i := u(1-w_i) \in D_0^{m,2}(\mathbb{R}^N \setminus K)$ and, defining $q_j := 2^*_{m,j} = \frac{2N}{N-2(m-j)} \geq 2$ for $j \in \{0,\dots,m\}$, one has that

$$\begin{aligned}
\|\nabla^m(u - v_i)\|_2^2 &= \|\nabla^m(uw_i)\|_{L^2(\Sigma)}^2 \lesssim \|u\|_{W^{m,\infty}(\mathbb{R}^N)}^2 \sum_{j=0}^m \int_\Sigma |D^j w_i|^2 \\
&\lesssim \sum_{j=0}^m \left(\int_\Sigma |D^j w_i|^{q_j} \right)^{\frac{2}{q_j}} \leq \sum_{j=0}^m \|D^j w_i\|_{L^{q_j}(\mathbb{R}^N)}^2 \\
&\lesssim \|D^m w_i\|_{L^2(\mathbb{R}^N)}^2 \lesssim \|\nabla^m w_i\|_{L^2(\mathbb{R}^N)}^2 \to 0,
\end{aligned}$$

where in the last steps we used Hölder inequality, the Sobolev inequality (2.3), and the equivalence of the norms $\|D^m \cdot\|_2$ and $\|\nabla^m \cdot\|_2$.

For later use, we also recall the right continuity of the capacity, see [24, Sec. 13.1.1].

Lemma 2.6 *Let K be a compact subset of $\Omega \subset \mathbb{R}^N$. For any $\varepsilon > 0$ there exists a neighbourhood $\mathcal{U}(K) \subset \Omega$ such that for any compact set $\widetilde{K}$ with $K \subset \widetilde{K} \subset \mathcal{U}(K)$, there holds*

$$\mathrm{cap}_{m,\Omega}(\widetilde{K}) \leq \mathrm{cap}_{m,\Omega}(K) + \varepsilon.$$

Although the notion of capacity needed for the blow-up analysis in Sect. 4.1 is the one given in (2.6), sometimes it is useful to consider a second one defined as

$$\mathrm{Cap}^{\geq}_{m,\mathbb{R}^N}(K) := \inf\left\{ \int_{\mathbb{R}^N} |\nabla^m f|^2 \,\middle|\, f \in D_0^{m,2}(\mathbb{R}^N),\ f \geq 1 \text{ a.e. on } K \right\}, \qquad (2.8)$$

which is well-defined for $N > 2m$, and similarly $\mathrm{Cap}^{\geq}_{m,\Omega}$ for $\Omega \subset\subset \mathbb{R}^N$, see [23, 24]. One of the advantages in this approach is that the capacitary potential associated to $\mathrm{Cap}^{\geq}_{m,\mathbb{R}^N}$ is positive, see [17, Sec. 3.1.2]. Note that for all $\Omega \subset \mathbb{R}^N$ one has $\mathrm{Cap}^{\geq}_{m,\Omega}(K) \leq \mathrm{cap}_{m,\Omega}(K)$

because the class of test functions considered in (2.8) includes the one considered for the minimization in (2.6). Actually it turns out that the two definitions are equivalent, in the sense that the two capacities estimate each other, as stated below. We report here the result for $\Omega = \mathbb{R}^N$, referring to [23] for the general case $\Omega \subsetneq \mathbb{R}^N$.

Lemma 2.7 ([24], Theorem 13.3.1) *Let $m \in \mathbb{N} \setminus \{0\}$ and $N > 2m$. There exists a constant $c > 0$ such that*

$$c\,\mathrm{cap}_{m,\mathbb{R}^N}(K) \le \mathrm{Cap}^{\ge}_{m,\,\mathbb{R}^N}(K) \le \mathrm{cap}_{m,\mathbb{R}^N}(K)$$

for any compact set $K \subset \mathbb{R}^N$.

Remark 3 The constant c appearing in Lemma 2.7 can be taken 1 in the second-order case $m = 1$, so the two definitions coincide, see e.g. [24, Sec. 13.3]. Whether this is the case also for the higher-order case it is still an open question.

Remark 4 As an extension of Proposition 2.3, it is known that a regular manifold of dimension d has zero capacity in the sense of (2.8) if and only if $d \le N - 2m$, see [4, Corollary 5.1.15]. By Lemma 2.7, this result holds also for the notion (2.6) of capacity.

3 Convergence and asymptotic expansion of the perturbed eigenvalues

In this section we study stability and asymptotic expansion of the perturbed eigenvalues of (1.2) and (1.3), when from a bounded domain $\Omega \subset \mathbb{R}^N$ one removes a compact set K of small V^m-capacity. The main goal is to extend the results obtained in the second-order framework (in particular [2, Theorem 1.4]) to the higher-order settings described in the introduction. The first part is devoted to the proof of the stability result of Theorem 1.1, which applies for rather general domains, while in the second part we focus on the asymptotic expansion of simple eigenvalues contained in Theorem 1.2, for which we require the notion of concentrating family of compact sets.

3.1 Spectral stability: Proof of Theorem 1.1

We present here a simple and self-contained proof of the stability of the point spectrum of the polyharmonic operator with respect to the capacity of the removed compactum, in both Dirichlet and Navier settings described in Sect. 1.1. It is essentially based on the variational characterization of the eigenvalues (1.9) and on the properties of the capacitary potentials, and it follows some ideas exploited for the same question in the second-order case in [3, Theorem 1.2].

Proof of Theorem 1.1 Denote by $(u_i)_{i=1}^{\infty}$ an orthonormal basis of $L^2(\Omega)$ such that each u_i is an eigenfunction of problem (1.1) associated to the eigenvalue $\lambda_i(\Omega)$. By classical elliptic regularity theory (see e.g. [17, Section 2.5]), the smoothness of $\partial\Omega$ yields $u_i \in C^m(\overline{\Omega})$ for all $i \in \mathbb{N}$. In order to deal at once with both cases (D) and (N), we introduce the function H defined by $H \equiv 1$ in the Dirichlet case, and by $H = \eta_{K_0}$ in the Navier case. Here η_{K_0} is a cutoff function which is equal to 1 in a neighbourhood on K_0 and with support contained in some compact set $\widetilde{K_0}$ such that $K_0 \subset \widetilde{K_0} \subset \Omega$. The cutoff η_{K_0} is introduced in order to enforce the boundary conditions on $\partial\Omega$ in the Navier case.

Fix $j \in \mathbb{N} \setminus \{0\}$. For any $\ell \in \{1, \dots, j\}$, we define $\Phi_\ell := u_\ell(1 - HW_K)$ and introduce the subspace $X_j := \operatorname{span}\{\Phi_\ell\}_{\ell=1}^j$. Note that $\Phi_\ell \in V_0^m(\Omega \setminus K)$ by definition of the capacitary potential W_K, so $X_j \subset V_0^m(\Omega \setminus K)$. The aim is to prove that X_j is a j-dimensional subspace of $V_0^m(\Omega \setminus K)$ so that the right hand side of (1.9) is smaller than the maximum of the Rayleigh quotient over X_j. Note that, by trivially extending the functions $\{\Phi_\ell\}_{\ell=1}^j$ in K, the integrals may be evaluated on Ω. First,

$$\int_\Omega \Phi_h \Phi_\ell = \int_\Omega u_h u_\ell - 2 \int_\Omega u_h u_\ell \, HW_K + \int_\Omega u_h u_\ell \, H^2 W_K^2,$$

therefore, by orthonormality of $\{u_\ell\}_{\ell=1}^j$ in $L^2(\Omega)$ and (1.6),

$$\begin{aligned} \left| \int_\Omega \Phi_h \Phi_\ell - \delta_{h,\ell} \right| &\le \max_{1 \le h \le j} \|u_h\|^2_{L^\infty(\Omega)} \left(2|\Omega|^{1/2} \|W_K\|_2 + \|W_K\|_2^2 \right) \\ &\lesssim \left(\operatorname{cap}_{V^m,\Omega}(K) \right)^{1/2} + \operatorname{cap}_{V^m,\Omega}(K), \end{aligned} \tag{3.1}$$

where $\delta_{h,\ell}$ stands for the Kroenecker delta. Let now $(W_n)_n \subset X^m(\Omega)$, see (1.10), be a sequence of smooth functions which approximates in the V^m-norm the capacitary potential W_K and satisfying $W_n = 1$ in $\mathcal{U}(K)$. The existence of such a sequence is guaranteed by the definition of W_K. Define moreover $\Phi_n^\ell := u_\ell(1 - HW_n)$ for $n \in \mathbb{N}$. Note that $\Phi_n^\ell \to \Phi_\ell$ in $V^m(\Omega)$ as $n \to +\infty$. We get

$$\begin{aligned} \int_\Omega \nabla^m \Phi_n^h \nabla^m \Phi_n^\ell &= \int_\Omega \nabla^m \left(u_h(1 - HW_n) \right) \nabla^m \left(u_\ell(1 - HW_n) \right) \\ &= \int_\Omega \nabla^m u_h \nabla^m u_\ell (1 - HW_n)^2 + T_m(u_h, u_\ell, W_n), \end{aligned} \tag{3.2}$$

where the term T_m contains all remaining products between the derivatives of u_h, u_ℓ, and $1 - HW_n$. To deal with the first term on the right in (3.2), consider $u_\ell(1 - HW_n)^2 \in V^m(\Omega)$ by regularity of the factors, as a test function for the eigenvalue problem (1.8) for $\lambda_h(\Omega)$. One obtains

$$\begin{aligned} \lambda_h(\Omega) \int_\Omega \Phi_n^h \Phi_n^\ell = \lambda_h(\Omega) \int_\Omega u_h u_\ell (1 - HW_n)^2 &= \int_\Omega \nabla^m u_h \nabla^m \left(u_\ell (1 - HW_n)^2 \right) \\ &= \int_\Omega \nabla^m u_h \nabla^m u_\ell (1 - HW_n)^2 + \int_\Omega \nabla^m u_h S_m(u_\ell, W_n), \end{aligned}$$

where again all remaining products involving intermediate derivatives of u_ℓ and $1 - HW_n$ are collected in the term S_m (which is a vector if m is odd). Isolating the first term on the right hand-side, and substituting it into (3.2), we get

$$\int_\Omega \nabla^m \Phi_n^h \nabla^m \Phi_n^\ell - \lambda_h(\Omega) \int_\Omega \Phi_n^h \Phi_n^\ell = -\int_\Omega \nabla^m u_h S_m(u_\ell, W_n) + T_m(u_h, u_\ell, W_n). \tag{3.3}$$

Moreover,

$$
\begin{aligned}
&\Big|\int_\Omega \nabla^m u_h S_m(u_\ell, W_n)\Big| \\
&\quad \le \sum_{i=1}^m \sum_{\tau=0}^i \int_\Omega |\nabla^m u_h||D^{m-i}u_\ell||D^{i-\tau}(1-HW_n)||D^\tau(1-HW_n)\Big| \\
&\quad \le \|\nabla^m u_h\|_\infty \|u_\ell\|_{W^{m,\infty}(\Omega)} \sum_{i=1}^m \sum_{\tau=0}^i \|D^{i-\tau}(1-HW_n)\|_2 \|D^\tau(1-HW_n)\|_2 \\
&\quad \le \max_{1\le k\le j} \|u_k\|^2_{W^{m,\infty}(\Omega)} \sum_{i=1}^m \Big(2\|D^i(HW_n)\|_2\|1-HW_n\|_2 \\
&\qquad + \sum_{\tau=1}^{i-1} \|D^{i-\tau}(HW_n)\|_2 \|D^\tau(HW_n)\|_2\Big) \\
&\quad \le C(\Omega, j, m)\Big(\|H\|_{W^{m,\infty}(\Omega)}\|W_n\|_{H^m(\Omega)}(|\Omega|^{1/2} + \|W_n\|_2) \\
&\qquad + \|H\|^2_{W^{m,\infty}(\Omega)}\|W_n\|^2_{H^m(\Omega)}\Big) \\
&\quad \lesssim C(\Omega, j, m)\Big(\|H\|_{W^{m,\infty}(\Omega)}\left(\mathrm{cap}_{V^m,\Omega}(K) + o_n(1)\right)^{1/2} \\
&\qquad + \|H\|^2_{W^{m,\infty}(\Omega)}(\mathrm{cap}_{V^m,\Omega}(K) + o_n(1))\Big) \quad \text{as } n \to \infty,
\end{aligned}
\tag{3.4}
$$

having used the equivalence of the norms $\|\cdot\|_{H^m(\Omega)}$ and $\|\nabla^m \cdot\|_2$ in $V^m(\Omega)$. Here $o_n(1)$ denotes a real sequence converging to 0 as $n \to +\infty$. Analogously one may estimate the last term in (3.3):

$$
\begin{aligned}
&|T_m(u_h, u_\ell, W_n)| \le \sum_{\substack{i,\tau\in\{0,\dots,m\}\\ (i,\tau)\ne(0,0)}} \int_\Omega |D^{m-i}u_h||D^i(1-HW_n)||D^{m-\tau}u_\ell||D^\tau(1-HW_n)| \\
&\quad \le \max_{1\le h\le j} \|u_h\|^2_{W^{m,\infty}(\Omega)} \Big(2\sum_{\tau=1}^m \|D^\tau(HW_n)\|_2\|1-HW_n\|_2 \\
&\qquad + \sum_{i,\tau\in\{1,\dots,m\}} \|D^i(HW_n)\|_2\|D^\tau(HW_n)\|_2\Big) \\
&\quad \le C(\Omega, j, m)\Big(\|H\|_{W^{m,\infty}(\Omega)}\|W_n\|_{H^m(\Omega)}\left(|\Omega|^{1/2} + \|W_n\|_2\right) \\
&\qquad + \|H\|^2_{W^{m,\infty}(\Omega)}\|W_n\|^2_{H^m(\Omega)}\Big) \\
&\quad \lesssim C(\Omega, j, m)\Big(\|H\|_{W^{m,\infty}(\Omega)}\left(\mathrm{cap}_{V^m,\Omega}(K) + o_n(1)\right)^{1/2} \\
&\qquad + \|H\|^2_{W^{m,\infty}(\Omega)}(\mathrm{cap}_{V^m,\Omega}(K) + o_n(1))\Big) \quad \text{as } n \to \infty.
\end{aligned}
\tag{3.5}
$$

All in all, from (3.3)–(3.5), one concludes

$$
\begin{aligned}
&\Big|\int_\Omega \nabla^m \Phi_n^h \nabla^m \Phi_n^\ell - \lambda_h(\Omega)\int_\Omega \Phi_n^h\Phi_n^\ell\Big| \\
&\quad \le \widetilde{C}\left(\left(\mathrm{cap}_{V^m,\Omega}(K) + o_n(1)\right)^{1/2} + \mathrm{cap}_{V^m,\Omega}(K) + o_n(1)\right),
\end{aligned}
$$

where $\widetilde{C}$ depends on K_0 in the Navier case. Letting now $n \to +\infty$ in both sides of the inequality, and taking into account (3.1), one infers

$$\left| \int_\Omega \nabla^m \Phi_h \nabla^m \Phi_\ell - \lambda_h(\Omega)\delta_{h,\ell} \right| \le \widetilde{C} \left(\left(\mathrm{cap}_{V^m,\Omega}(K)\right)^{1/2} + \mathrm{cap}_{V^m,\Omega}(K) \right). \tag{3.6}$$

Hence, from (3.1) and (3.6) one sees that, when $\mathrm{cap}_{V^m,\Omega}(K)$ is small enough, the functions $\{\Phi_\ell\}_{\ell=1}^j$ are linearly independent in $V_0^m(\Omega \setminus K)$, and so the subspace X_j has dimension j. Therefore, recalling that $\lambda_h(\Omega) \le \lambda_j(\Omega)$ for all $h \in \{1, \dots, j\}$, again from (3.1) and (3.6) one finally infers that

$$\begin{aligned} \lambda_j(\Omega \setminus K) &\le \max_{\substack{(\alpha_1,\dots,\alpha_j)\in\mathbb{R}^j \\ \sum_{i=1}^j \alpha_i = 1}} \frac{\sum\limits_{h,\ell=1}^{j} \alpha_h\alpha_\ell \int_\Omega \nabla^m\Phi_h \nabla^m\Phi_\ell}{\sum\limits_{h,\ell=1}^{j} \alpha_h\alpha_\ell \int_\Omega \Phi_h\Phi_\ell} \\ &\le \max_{\substack{(\alpha_1,\dots,\alpha_j)\in\mathbb{R}^j \\ \sum_{i=1}^j \alpha_i = 1}} \frac{\sum\limits_{h=1}^{j} \alpha_h^2 \lambda_h(\Omega) + \mathcal{O}\left(\left(\mathrm{cap}_{V^m,\Omega}(K)\right)^{1/2}\right)}{\sum\limits_{h=1}^{j} \alpha_h^2 + \mathcal{O}\left(\left(\mathrm{cap}_{V^m,\Omega}(K)\right)^{1/2}\right)} \\ &\le \frac{\lambda_j(\Omega) + \mathcal{O}\left(\left(\mathrm{cap}_{V^m,\Omega}(K)\right)^{1/2}\right)}{1 + \mathcal{O}\left(\left(\mathrm{cap}_{V^m,\Omega}(K)\right)^{1/2}\right)} = \lambda_j(\Omega) + \mathcal{O}\left(\left(\mathrm{cap}_{V^m,\Omega}(K)\right)^{1/2}\right) \end{aligned}$$

as $\mathrm{cap}_{V^m,\Omega}(K) \to 0$. □

3.2 Asymptotic expansion of eigenvalues: Proof of Theorem 1.2

Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact subsets of Ω and denote by $\lambda_J(\Omega \setminus K_\varepsilon)$ the J-th eigenvalue of $(-\Delta)^m$ in $V_0^m(\Omega \setminus K_\varepsilon)$, i.e. of problem (1.8) with $K = K_\varepsilon$. If there exists a limiting set K for which $\mathrm{cap}_{V^m,\Omega}(K_\varepsilon) \to \mathrm{cap}_{V^m,\Omega}(K) = 0$, Theorem 1.1 and Proposition 2.1 guarantee that $\lambda_J(\Omega\setminus K_\varepsilon) \to \lambda_J(\Omega\setminus K) = \lambda_J(\Omega)$, if we denote by $\lambda_J(\Omega \setminus K)$ the corresponding eigenvalue of the limiting problem in $V_0^m(\Omega\setminus K) = V_0^m(\Omega)$. Moreover, Theorem 1.1 gives us a first estimate on the eigenvalue convergence rate in terms of the V^m-capacity of the removed set K_ε. Inspired by [2], we are now going to sharpen this result, by detecting the first term of the asymptotic expansion of $\lambda_J(\Omega \setminus K_\varepsilon)$, provided the family of compact sets $\{K_\varepsilon\}_{\varepsilon>0}$ converges to K as specified in Definition 1.1. Indeed, as the next two propositions show, this definition of convergence, although very general, is enough to prove the stability of the (u, V^m)-capacity in case $\mathrm{cap}_{V^m,\Omega}(K) = 0$, as well as the Mosco convergence of the functional spaces.

Proposition 3.1 *Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in $\Omega \subset \mathbb{R}^N$ concentrating to a compact set $K \subset \Omega$ with $\mathrm{cap}_{V^m,\Omega}(K) = 0$ as $\varepsilon \to 0$. Then, for every function $u \in V^m(\Omega)$, one has that $W_{K_\varepsilon,u} \to W_{K,u} = 0$ strongly in $V^m(\Omega)$ and*

$$\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u) \to \mathrm{cap}_{V^m,\Omega}(K, u) = 0 \quad \textit{as} \quad \varepsilon \to 0.$$

Proof It is analogous to the one for the case $m = 1$ given in [2, Proposition B.1]. It is in fact essentially based on the fact that $V_0^m(\Omega \setminus K) = V^m(\Omega)$ for sets of null V^m-capacity, as shown in Proposition 2.1, and on the consequent Remark 1. □

Definition 3.1 Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets compactly contained in a bounded domain Ω. We say that $\Omega \setminus K_\varepsilon$ converges to $\Omega \setminus K$ *in the sense of Mosco* in V_0^m if the following two conditions are satisfied:

(*i*) the weak limit points in $V^m(\Omega)$ of every family of functions $u_\varepsilon \in V_0^m(\Omega \setminus K_\varepsilon)$ belong to $V_0^m(\Omega \setminus K)$;
(*ii*) for every $u \in V_0^m(\Omega \setminus K)$, there exists a family of functions $\{u_\varepsilon\}_{\varepsilon>0}$ such that, for every $\varepsilon > 0$, $u_\varepsilon \in V_0^m(\Omega \setminus K_\varepsilon)$ and $u_\varepsilon \to u$ in $V^m(\Omega)$.

In order to stress the underlined functional space, we also say that $V_0^m(\Omega \setminus K_\varepsilon)$ converges to $V_0^m(\Omega \setminus K)$ in the sense of Mosco.

Lemma 3.2 *Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets concentrating to a compact set $K \subset \Omega$ with $\mathrm{cap}_{V^m,\Omega}(K) = 0$ as $\varepsilon \to 0$. Then $V_0^m(\Omega \setminus K_\varepsilon)$ converges to $V_0^m(\Omega \setminus K)$ as $\varepsilon \to 0$ in the sense of Mosco.*

Proof Verification of (*i*). Let $\{u_\varepsilon\}_\varepsilon \subset V^m(\Omega)$ be such that $u_\varepsilon \in V_0^m(\Omega \setminus K_\varepsilon)$ and $u_\varepsilon \rightharpoonup u$ in $V^m(\Omega)$. Since $\mathrm{cap}_{m,\Omega}(K) = 0$, we have that $V^m(\Omega) = V_0^m(\Omega \setminus K)$ by Proposition 2.1, hence u belongs to $V_0^m(\Omega \setminus K)$.

Verification of (*ii*). Let $u \in V_0^m(\Omega \setminus K) = V^m(\Omega)$. For every $k \in \mathbb{N} \setminus \{0\}$, by density there exists $\chi_k \in X_0^m(\Omega \setminus K)$ such that $\|\nabla^m(\chi_k - u)\|_2 < \frac{1}{k}$. Note that, if K_ε is concentrating to K in the sense of Definition 1.1, for a chosen cutoff function $\eta_K \in C_0^\infty(\Omega)$ such that $\eta_K \equiv 1$ in a neighbourhood of K, one has that $\eta_K \equiv 1$ in a neighbourhood of K_ε for ε small enough. By definition of W_K, one may find $(W_n)_n \subset V^m(\Omega)$ and a sequence $(\varepsilon_n)_n \searrow 0$ such that $\|\nabla^m W_n\|_2 < \frac{1}{n}$ and $W_n \equiv 1$ in a neighbourhood of K_ε for all $\varepsilon \in (0, \varepsilon_n]$. Defining, for all $n, k \in \mathbb{N} \setminus \{0\}$, $Z_n^k := \chi_k (1 - \eta_K W_n)$, one has that $Z_n^k \in V_0^m(\Omega \setminus K_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_n]$ and

$$\|\nabla^m \left(Z_n^k - \chi_k\right)\|_2 \lesssim \|\eta_K\|_{W^{m,\infty}(\Omega)} \|W_n\|_{V^m(\Omega)} \|\chi_k\|_{W^{m,\infty}(\Omega)} \leq \frac{C_k}{n}$$

for some $C_k > 0$ depending on k. Hence, for each $k \in \mathbb{N} \setminus \{0\}$, there exists $n_k \in \mathbb{N}$ such that $n_k \nearrow \infty$ as $k \to \infty$ and $\|\nabla^m \left(Z_{n_k}^k - \chi_k\right)\|_2 < \frac{1}{k}$. In order to construct the family required for the Mosco convergence, for any $\varepsilon \in (0, \varepsilon_{n_1})$ it is sufficient to define $u_\varepsilon := Z_{n_k}^k$, choosing k such that $\varepsilon \in (\varepsilon_{n_{k+1}}, \varepsilon_{n_k}]$. Indeed, for any $\delta > 0$, letting $k \in \mathbb{N} \setminus \{0\}$ be such that $\frac{2}{k} < \delta$, we have that, for all $\varepsilon \in (0, \varepsilon_{n_k}]$, $u_\varepsilon = Z_{n_j}^j$ for some $j \geq k$, so that

$$\|\nabla^m (u_\varepsilon - u)\|_2 \leq \|\nabla^m \left(Z_{n_j}^j - \chi_j\right)\|_2 + \|\nabla^m \left(\chi_j - u\right)\|_2 < \frac{2}{j} \leq \frac{2}{k} < \delta,$$

thus proving that $u_\varepsilon \to u$ in $V^m(\Omega)$ as $\varepsilon \to 0$. □

Remark 5 Note that the Mosco convergence of sets implies the convergence of the spectra of the polyharmonic operators, see [5]. For the Dirichlet case, in particular this can be seen combining [5, Proposition 2.9, footnote 2 p.8, and Theorem 4.3].

Lemma 3.3 *Let $K \subset \Omega$ be a compact set and $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact subsets of Ω concentrating to K as $\varepsilon \to 0$. If $\mathrm{cap}_{V^m,\Omega}(K) = 0$, then, for every $f \in V^m(\Omega)$, we have that $\|W_{K_\varepsilon,f}\|^2_{H^{m-1}(\Omega)} = o(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, f))$ as $\varepsilon \to 0$.*

Proof The proof is inspired by [2, Lemma A.1]. Suppose by contradiction that there exist $C > 0$ and a sequence $\varepsilon_n \to 0$ such that

$$\|W_{K_{\varepsilon_n},f}\|^2_{H^{m-1}(\Omega)} \geq C\, \mathrm{cap}_{V^m,\Omega}(K_{\varepsilon_n}, f) \quad \text{for all } n. \tag{3.7}$$

Let us consider

$$Z_n := \frac{W_{K_{\varepsilon_n},f}}{\|W_{K_{\varepsilon_n},f}\|_{H^{m-1}(\Omega)}}.$$

We have

$$\|Z_n\|_{H^{m-1}(\Omega)} = 1 \qquad \text{and} \qquad \|\nabla^m Z_n\|_2^2 = \frac{\|\nabla^m W_{K_{\varepsilon_n},f}\|_2^2}{\|W_{K_{\varepsilon_n},f}\|_{H^{m-1}(\Omega)}^2} \le \frac{1}{C}$$

with $C > 0$ as in (3.7).

Then one may find a subsequence (still denoted by Z_n) and $Z \in V^m(\Omega)$, so that $Z_n \rightharpoonup Z$ in $V^m(\Omega)$. By the compact embedding $H^m(\Omega) \hookrightarrow\hookrightarrow H^{m-1}(\Omega)$, Z is also the strong limit in the $H^{m-1}(\Omega)$ topology. This implies that $\|Z\|_{H^{m-1}(\Omega)} = 1$. However, by the Mosco convergence of Lemma 3.2 one may show that

$$\int_{\Omega\setminus K} \nabla^m Z\, \nabla^m \varphi = 0 \qquad \text{for all } \varphi \in V_0^m(\Omega \setminus K), \tag{3.8}$$

and hence for all $\varphi \in V^m(\Omega)$ by Proposition 2.1, since we assumed $\mathrm{cap}_{V^m,\Omega}(K) = 0$. Indeed, given $\varphi \in V_0^m(\Omega \setminus K)$ there exists a sequence $\{\varphi_{\varepsilon_n}\}_n$ so that $\varphi_{\varepsilon_n} \in V_0^m(\Omega\setminus K_{\varepsilon_n})$ for each $n \in \mathbb{N}$ and $\varphi_{\varepsilon_n} \to \varphi$ in $V^m(\Omega)$, for which then

$$\int_{\Omega\setminus K_{\varepsilon_n}} \nabla^m Z_n\, \nabla^m \varphi_{\varepsilon_n} = 0$$

for all $n \in \mathbb{N}$ by definition of Z_n as a multiple of the capacitary potential $W_{K_{\varepsilon_n},f}$. Then, (3.8) follows by weak-strong convergence in $V^m(\Omega)$, yielding $Z = 0$, a contradiction. □

We are now in the position to prove the asymptotic expansion of the perturbed eigenvalues. The suitable asymptotic parameter turns out to be the (u_J, V^m)-capacity of the removed set, where u_J is an eigenfunction normalized in $L^2(\Omega)$ associated to the eigenvalue λ_J.

In the following, $(-\Delta)_\varepsilon^m$ stands for the polyharmonic operator acting on $V_0^m(\Omega\setminus K_\varepsilon)$. Similarly, to shorten notation, we write $\lambda_\varepsilon := \lambda_J(\Omega \setminus K_\varepsilon)$ and the corresponding (u_J, V^m)-capacitary potential is denoted by $W_\varepsilon := W_{K_\varepsilon,u_J} \in V^m(\Omega)$; we also write $\lambda_J := \lambda_J(\Omega)$.

Proof of Theorem 1.2 First note that the simplicity of λ_J, i.e. of $\lambda_J(\Omega\setminus K)$ by Proposition 2.1, together with the convergence of the perturbed eigenvalues given by Theorem 1.1, implies the simplicity of λ_ε for ε sufficiently small.

Let $\psi_\varepsilon := u_J - W_\varepsilon \in V_0^m(\Omega\setminus K_\varepsilon)$ and $\varphi \in V_0^m(\Omega\setminus K_\varepsilon)$. Then

$$\int_\Omega \nabla^m \psi_\varepsilon \nabla^m \varphi - \lambda_J \int_\Omega \psi_\varepsilon \varphi = \int_{\Omega\setminus K_\varepsilon} \nabla^m u_J \nabla^m \varphi - \lambda_J \int_\Omega \psi_\varepsilon \varphi = \lambda_J \int_\Omega W_\varepsilon \varphi.$$

This means that ψ_ε satisfies weakly in $V_0^m(\Omega \setminus K_\varepsilon)$ the equation

$$\big((-\Delta)^m - \lambda_J\big)\, \psi_\varepsilon = \lambda_J W_\varepsilon. \tag{3.9}$$

Since by Lemma 3.3 with $f = u_J$ one has $\|W_\varepsilon\|_2 = \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2})$ as $\varepsilon \to 0$, we infer

$$\mathrm{dist}(\lambda_J, \sigma((-\Delta)_\varepsilon^m)) \le \frac{\|((-\Delta)^m - \lambda_J)\psi_\varepsilon\|_2}{\|\psi_\varepsilon\|_2} = \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2})$$

as $\varepsilon \to 0$. Since we know by Theorem 1.1 and Proposition 3.1 that the spectrum of $(-\Delta)^m$ in $V_0^m(\Omega \backslash K_\varepsilon)$ varies continuously with respect to ε and that the eigenvalue λ_ε is simple for ε small enough, one first deduces

$$|\lambda_\varepsilon - \lambda_J| = \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2}) \quad \text{as } \varepsilon \to 0.$$

Denote now by Π_ε the projector (with respect to the scalar product in L^2) onto the eigenspace related to λ_ε and take $u_\varepsilon := \frac{\Pi_\varepsilon \psi_\varepsilon}{\|\Pi_\varepsilon \psi_\varepsilon\|_2}$ as normalized eigenfunction. The first goal is to estimate the difference of the two eigenfunctions u_J and u_ε:

$$\begin{aligned}\|u_J - u_\varepsilon\|_2 &\le \|u_J - \psi_\varepsilon\|_2 + \|\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon\|_2 + \left\| \Pi_\varepsilon \psi_\varepsilon - \frac{\Pi_\varepsilon \psi_\varepsilon}{\|\Pi_\varepsilon \psi_\varepsilon\|_2} \right\|_2 \\ &= \|W_\varepsilon\|_2 + \|\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon\|_2 + \left|1 - \|\Pi_\varepsilon \psi_\varepsilon\|_2^{-1}\right| \|\Pi_\varepsilon \psi_\varepsilon\|_2.\end{aligned}$$

Note that Lemma 3.3 yields $\|W_\varepsilon\|_2 = \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2})$ and, moreover, we have

$$\|\Pi_\varepsilon \psi_\varepsilon\|_2 \le \|\psi_\varepsilon\|_2 \le \|u_J\|_2 + \|W_\varepsilon\|_2 = \mathcal{O}(1) \quad \text{as } \varepsilon \to 0.$$

Hence, we need to estimate $\|\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon\|_2$ and $\left|1 - \|\Pi_\varepsilon \psi_\varepsilon\|_2^{-1}\right|$. We claim that both quantities are $\mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2})$, obtaining thus

$$\|u_J - u_\varepsilon\|_2 = \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2}) \quad \text{as } \varepsilon \to 0, \tag{3.10}$$

and postpone the proof of such claim to the end of the proof. Then we have

$$\begin{aligned}\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J) &= \int_\Omega |\nabla^m W_\varepsilon|^2 = \int_\Omega \nabla^m (u_J - \psi_\varepsilon) \nabla^m W_\varepsilon = \int_\Omega \nabla^m u_J \nabla^m W_\varepsilon \\ &= \lambda_J \int_\Omega u_J W_\varepsilon = \lambda_J \int_\Omega u_\varepsilon W_\varepsilon + \lambda_J \int_\Omega (u_J - u_\varepsilon) W_\varepsilon \\ &\overset{(3.9)}{=} \int_\Omega \nabla^m \psi_\varepsilon \nabla^m u_\varepsilon - \lambda_J \int_\Omega u_\varepsilon \psi_\varepsilon + \lambda_J \int_\Omega (u_J - u_\varepsilon) W_\varepsilon \\ &= (\lambda_\varepsilon - \lambda_J) \int_\Omega u_\varepsilon \psi_\varepsilon + \lambda_J \int_\Omega (u_J - u_\varepsilon) W_\varepsilon,\end{aligned}$$

and therefore

$$(\lambda_\varepsilon - \lambda_J) \int_\Omega u_\varepsilon \psi_\varepsilon = \mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J) - \lambda_J \int_\Omega (u_J - u_\varepsilon)\, W_\varepsilon. \tag{3.11}$$

Since now

$$\int_\Omega u_\varepsilon \psi_\varepsilon = \|u_\varepsilon\|_2^2 + \int_\Omega u_\varepsilon\, (\psi_\varepsilon - u_\varepsilon) = 1 + \int_\Omega u_\varepsilon\, (\psi_\varepsilon - u_\varepsilon)$$

and

$$\left| \int_\Omega u_\varepsilon\, (\psi_\varepsilon - u_\varepsilon) \right| \le \|u_\varepsilon\|_2 \|\psi_\varepsilon - u_\varepsilon\|_2 = \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2}),$$

where the last equality is again due to the claims above, from (3.11) and (3.10), we infer

$$\lambda_\varepsilon - \lambda_J = \frac{\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J) + \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J))}{1 + \mathcal{O}(1)} = \mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)\, (1 + \mathcal{O}(1))$$

as $\varepsilon \to 0$, as desired. To conclude, we prove the claims above. Since λ_ε is a simple eigenvalue, denoting by T_ε the restriction of $(-\Delta)^m_\varepsilon$ on $\ker \Pi_\varepsilon$, we have that $\sigma(T_\varepsilon) = \sigma((-\Delta)^m_\varepsilon) \setminus \{\lambda_\varepsilon\}$ and, by simplicity, $\mathrm{dist}(\lambda_\varepsilon, \sigma(T_\varepsilon)) \geq \delta$ for some $\delta > 0$, uniformly with respect to ε. Hence,

$$\begin{aligned}\|\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon\|_2 &\leq \frac{1}{\delta}\|(T_\varepsilon - \lambda_\varepsilon)(\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon)\|_2 \lesssim \|\big((-\Delta)^m - \lambda_\varepsilon\big)\psi_\varepsilon\|_2 \\ &\leq \|\big((-\Delta)^m - \lambda_J\big)\psi_\varepsilon\|_2 + |\lambda_J - \lambda_\varepsilon|\|\psi_\varepsilon\|_2 = |\lambda_J|\|W_\varepsilon\|_2 + |\lambda_J - \lambda_\varepsilon|\|\psi_\varepsilon\|_2 \\ &= \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2}).\end{aligned}$$

Since, by definition of ψ_ε and Lemma 3.3, $\|\psi_\varepsilon\|_2 = 1 + \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon,u_J)^{1/2})$ as $\varepsilon \to 0$, one thus finds that $\|\Pi_\varepsilon\psi_\varepsilon\|_2 = 1 + \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2})$, which in particular yields the desired estimate $1 - \|\Pi_\varepsilon\psi_\varepsilon\|_2^{-1} = \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2})$. This concludes the proof. □

Remark 6 We observe that, in the proof of Theorem 1.2, the following estimate for the normalized eigenfunction $u_\varepsilon \in V^m_0(\Omega\setminus K_\varepsilon)$ of $(-\Delta)^m$ relative to $\lambda_J(\Omega\setminus K_\varepsilon)$ was established:

$$\|u_\varepsilon - u_J\|_2 = \mathcal{O}(\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)^{1/2}) \quad \text{as } \varepsilon \to 0.$$

4 Sharp asymptotic expansions of perturbed eigenvalues: the case of uniformly shrinking holes.

4.1 A blow-up analysis

In Theorem 1.2 we obtained an asymptotic expansion of a perturbed simple eigenvalue in terms of $\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J)$, in case the limiting removed set has zero V^m-capacity. However, in view of possible applications, the dependence on the removed set K_ε is quite implicit using such an asymptotic parameter. Therefore, we aim to understand how this quantity behaves with respect to the diameter of the hole, in the case of a uniformly shrinking family of compact sets which concentrate to a point, a set with zero V^m-capacity in large dimensions by Proposition 2.3.

First, we only suppose that $\{K_\varepsilon\}_{\varepsilon>0}$ uniformly shrinks to a point, which is assumed to be 0 in the following, in the sense that

$$K_\varepsilon \subset \overline{B_{C\varepsilon}(0)} \tag{4.1}$$

for some constant $C > 0$ and ε small enough. The following is a generalization of [2, Lemma 2.2] to the higher-order setting.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain such that $0 \in \Omega$ and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets satisfying (4.1). Let $h \in H^m(\Omega)$ be such that*

$$|D^k h(x)| = \mathcal{O}(|x|^{\gamma-k}) \quad \text{as} \quad |x| \to 0$$

for some $\gamma \in \mathbb{N}$ and all $k \in \{0, \ldots, m\}$. Then

$$\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, h) = \mathcal{O}(\varepsilon^{N-2m+2\gamma}) \quad \text{as } \varepsilon \to 0. \tag{4.2}$$

Proof By Proposition 2.2(*iii*), it is sufficient to prove (4.2) for the Dirichlet case $V^m(\Omega) = H^m_0(\Omega)$. Let $\varphi \in C^\infty_0(\mathbb{R}^N)$ with $\mathrm{supp}\,\varphi \subset B_2(0)$ and $\varphi \equiv 1$ in a neighbourhood of $\overline{B_1(0)}$, and define $\varphi_\varepsilon(x) := \varphi((C\varepsilon)^{-1}x)$ for all $\varepsilon > 0$ small. Then $h_\varepsilon := \varphi_\varepsilon h$ coincides with h in a neighbourhood of $\overline{B_{C\varepsilon}(0)}$. By monotonicity

$$
\begin{aligned}
\operatorname{cap}_{m,\Omega}(K_\varepsilon, h) \le \operatorname{cap}_{m,\Omega}(\overline{B_{C\varepsilon}(0)}, h) &\le \int_\Omega |\nabla^m h_\varepsilon|^2 \\
&\lesssim \sum_{k=0}^m \int_{B_{2C\varepsilon}(0)} |D^{m-k}\varphi_\varepsilon(x)|^2 |D^k h(x)|^2\,dx \\
&\lesssim \sum_{k=0}^m (C\varepsilon)^{2k-2m} \int_{B_{2C\varepsilon}(0)} \left|D^{m-k}\varphi\left(\frac{x}{C\varepsilon}\right)\right|^2 |D^k h(x)|^2\,dx \\
&\lesssim \sum_{k=0}^m (C\varepsilon)^{2k-2m+N} \int_{B_2(0)} |D^{m-k}\varphi(y)|^2 |D^k h(C\varepsilon y)|^2\,dy \\
&\lesssim \varepsilon^{N-2m+2\gamma} \sum_{k=0}^m \int_{B_2(0)} |D^{m-k}\varphi(y)|^2\,dy \lesssim \varepsilon^{N-2m+2\gamma},
\end{aligned}
$$

having used the assumption that $\|D^k h\|_\infty \lesssim \varepsilon^{\gamma-k}$ in $B_{2C\varepsilon}(0)$. □

Next, having in mind the model case $K_\varepsilon := \varepsilon\mathcal{K}$ for a fixed compactum $\mathcal{K}$, we consider families of compact sets which uniformly shrink to $\{0\}$ as in (4.1) but enjoying a more specific structure. To this aim we assume

(M1) there exists $M \subset \mathbb{R}^N$ compact such that $\varepsilon^{-1}K_\varepsilon \subseteq M$ for all $\varepsilon \in (0, 1)$;
(M2) there exists $\mathcal{K} \subset \mathbb{R}^N$ compact such that $\mathbb{R}^N \setminus \varepsilon^{-1}K_\varepsilon \to \mathbb{R}^N \setminus \mathcal{K}$ in the sense of Mosco as $\varepsilon \to 0$.

In our context (M2) means the following:

(i) if $u_\varepsilon \in D_0^{m,2}(\mathbb{R}^N \setminus \varepsilon^{-1}K_\varepsilon)$ is so that $u_\varepsilon \rightharpoonup u$ in $D_0^{m,2}(\mathbb{R}^N)$ as $\varepsilon \to 0$, then we have that $u \in D_0^{m,2}(\mathbb{R}^N \setminus \mathcal{K})$;
(ii) if $u \in D_0^{m,2}(\mathbb{R}^N \setminus \mathcal{K})$, then there exists a family of functions $\{u_\varepsilon\}_{\varepsilon>0}$ such that $u_\varepsilon \in D_0^{m,2}(\mathbb{R}^N \setminus \varepsilon^{-1}K_\varepsilon)$ for all $\varepsilon > 0$ and $u_\varepsilon \to u$ in $D_0^{m,2}(\mathbb{R}^N)$ as $\varepsilon \to 0$.

In this case we also say that $D_0^{m,2}(\mathbb{R}^N \setminus \varepsilon^{-1}K_\varepsilon)$ converges to $D_0^{m,2}(\mathbb{R}^N \setminus \mathcal{K})$ in the sense of Mosco.

Remark 7 Assumption (M1) is actually equivalent to the condition (4.1), since $M \subset B_C(0)$ for some $C > 0$.

Lemma 4.2 *Let $N > 2m$. Under the assumption (M1) the following are equivalent:*

1. *$D_0^{m,2}(\mathbb{R}^N \setminus \varepsilon^{-1}K_\varepsilon)$ converges to $D_0^{m,2}(\mathbb{R}^N \setminus \mathcal{K})$ in the sense of Mosco;*
2. *$H_0^m(B_R(0) \setminus \varepsilon^{-1}K_\varepsilon)$ converges to $H_0^m(B_R(0) \setminus \mathcal{K})$ in the sense of Mosco for all $R > r(M)$, where $r(M) := \inf\{\rho > 0 \mid B_\rho(0) \supset M\}$.*

We denote by (M_R2.i) and (M_R2.ii) the correspondent conditions (M2.i) and (M2.ii) which enter in the definition of the Mosco convergence relative to the space $H_0^m(B_R(0))$. In the following we use the shorter notation $B_R := B_R(0)$.

Proof **1) ⇒ 2)**. *Verification of (M_R2.i).* Let $\{u_\varepsilon\}_{\varepsilon>0} \subset H_0^m(B_R)$ be a family of functions such that $u_\varepsilon \in H_0^m(B_R \setminus \varepsilon^{-1}K_\varepsilon)$ and $u_\varepsilon \rightharpoonup u$ in $H_0^m(B_R)$. We show that $u \in H_0^m(B_R \setminus \mathcal{K})$. Denoting by u_ε^E and u^E the trivial extension of u_ε and u outside B_R respectively, then $u_\varepsilon^E \in D_0^{m,2}(\mathbb{R}^N \setminus \varepsilon^{-1}K_\varepsilon)$ and $u_\varepsilon^E \rightharpoonup u^E$ in $D_0^{m,2}(\mathbb{R}^N)$. Hence condition (M2.i) guarantees that $u^E \in D_0^{m,2}(\mathbb{R}^N \setminus \mathcal{K})$, which by construction implies that $u \in H_0^m(B_R \setminus \mathcal{K})$.

Verification of (M_R2.ii). Let $v \in C_0^\infty(B_R \setminus \mathcal{K})$ and $\Lambda_1, \Lambda_2 \subset \Omega$ be two open sets such that $\operatorname{supp} v \subset\subset \Lambda_1 \subset\subset \Lambda_2 \subset\subset B_R$. Take $\eta \in C_0^\infty(\Lambda_2)$ with $\eta \equiv 1$ on Λ_1. Since $v^E \in D_0^{m,2}(\mathbb{R}^N \setminus \mathcal{K})$, then by (M2.ii) there exists a family $\{v_\varepsilon\}_{\varepsilon>0} \subset D_0^{m,2}(\mathbb{R}^N)$ with $v_\varepsilon \in D_0^{m,2}(\mathbb{R}^N \setminus \varepsilon^{-1} K_\varepsilon)$ for all $\varepsilon > 0$ such that $v_\varepsilon \to v^E$ in $D_0^{m,2}(\mathbb{R}^N)$, i.e.

$$\|\nabla^m (v_\varepsilon - v^E)\|_{L^2(\mathbb{R}^N)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

By construction, $\eta v_\varepsilon \in H_0^m(B_R \setminus \varepsilon^{-1} K_\varepsilon)$. We claim that $\eta v_\varepsilon \to v$ in $H_0^m(B_R)$. Indeed, denoting by $q_j := 2^*_{m,j} = \frac{2N}{N-2(m-j)} \geq 2$ and $p_j := 2\left(\frac{q_j}{2}\right)' = \frac{N}{m-j}$ for $j \in \{0, \dots, m\}$, one has

$$\begin{aligned}
\|\nabla^m(\eta v_\varepsilon - v)\|_{L^2(B_R)} &= \|\nabla^m(\eta(v_\varepsilon - v))\|_{L^2(B_R)} \leq \sum_{j=0}^m \|D^{m-j}\eta \, D^j(v_\varepsilon - v)\|_{L^2(B_R)} \\
&\leq \sum_{j=0}^m \|D^{m-j}\eta\|_{L^{p_j}(\operatorname{supp}\eta)} \|D^j(v_\varepsilon - v^E)\|_{L^{q_j}(\mathbb{R}^N)} \\
&\leq \sum_{j=0}^m |\operatorname{supp}\eta|^{\frac{1}{p_j}} \|D^{m-j}\eta\|_\infty \|D^m(v_\varepsilon - v^E)\|_{L^2(\mathbb{R}^N)} \\
&\leq C(m, N, R)\|\eta\|_{W^{m,\infty}(\mathbb{R}^N)} \|\nabla^m(v_\varepsilon - v^E)\|_{L^2(\mathbb{R}^N)} \to 0.
\end{aligned} \tag{4.3}$$

The last steps are due to the critical Sobolev embedding on $\mathbb{R}^N$ (for which it is fundamental that $N > 2m$), see (2.3), and to the equivalence of the norms $\|D^m \cdot\|_2$ and $\|\nabla^m \cdot\|_2$, see e.g. [17, Chp. 2.2].

The above argument and the density of $C_0^\infty(B_R \setminus \mathcal{K})$ in $H_0^m(B_R \setminus \mathcal{K})$ imply that, fixing any $v \in H_0^m(B_R \setminus \mathcal{K})$, for every $\delta > 0$ there exists a family $\{v_{\delta,\varepsilon}\}_{\varepsilon>0}$ such that $v_{\delta,\varepsilon} \in H_0^m(B_R \setminus K_\varepsilon)$ and $\|v_{\delta,\varepsilon} - v\|_{H^m(B_R)} < \delta$ for all $\varepsilon \in (0, \bar{\varepsilon}_\delta]$ for some $\bar{\varepsilon}_\delta > 0$. Therefore there exists a vanishing sequence $(\varepsilon_n)_n \searrow 0$ such that $\left\|v_{\frac{1}{k},\varepsilon} - v\right\|_{H^m(B_R)} < \frac{1}{k}$ for all $\varepsilon \in (0, \varepsilon_k]$. Defining $v_\varepsilon = v_{\frac{1}{n},\varepsilon}$ for $\varepsilon \in (\varepsilon_{n+1}, \varepsilon_n]$, we have that, for all $\varepsilon \in (0, \varepsilon_1]$, $v_\varepsilon \in H_0^m(B_R \setminus K_\varepsilon)$ and $v_\varepsilon \to v$ in $H_0^m(B_R)$ as $\varepsilon \to 0$.

2) ⇒ 1). *Verification of (M2.i)*. Let $\{u_\varepsilon\}_{\varepsilon>0} \subset D_0^{m,2}(\mathbb{R}^N)$ be a family of functions such that $u_\varepsilon \in D_0^{m,2}(\mathbb{R}^N \setminus \varepsilon^{-1} K_\varepsilon)$ and $u_\varepsilon \rightharpoonup u$ in $D_0^{m,2}(\mathbb{R}^N)$. Taking $\eta \in C^\infty(\mathbb{R}^N)$ and $R > 0$ such that $\operatorname{supp}\eta \subset B_R(0)$, due to the continuity of the map $D_0^{m,2}(\mathbb{R}^N) \to H_0^m(B_R)$, $u \mapsto \eta u$, which can be easily proved arguing as in (4.3), one has that $\eta u_\varepsilon \rightharpoonup \eta u$ in $H_0^m(B_R)$. Hence, (M_R2.i) implies $\eta u \in H_0^m(B_R \setminus \mathcal{K})$. Hence $\eta u \in D_0^{m,2}(\mathbb{R}^N \setminus \mathcal{K})$ for every $\eta \in C_0^\infty(\mathbb{R}^N)$. Let us now take $\eta_1 \in C_0^\infty(\mathbb{R}^N)$ with $0 \leq \eta_1 \leq 1$, $\eta_1 \equiv 1$ on $B_{\frac{1}{2}}$ and $\operatorname{supp}\eta_1 \subset B_1$, and define $\eta_k := \eta_1\left(\frac{\cdot}{k}\right)$, so that $\operatorname{supp}\eta_k \subset B_k$. We are going to prove that $\eta_k u \to u$ in $D_0^{m,2}(\mathbb{R}^N)$ as $k \to +\infty$, in order to conclude that $u \in D_0^{m,2}(\mathbb{R}^N \setminus \mathcal{K})$. We estimate as follows:

$$\begin{aligned}\|\nabla^m(\eta_k u-u)\|^2_{L^2(\mathbb{R}^N)} &\le \int_{\mathbb{R}^N}|\eta_k-1|^2|\nabla^m u|^2+\sum_{j=0}^{m-1}\int_{B_k\setminus B_{\frac{k}{2}}}|D^{m-j}\eta_k|^2|D^j u|^2\\ &\le \int_{\mathbb{R}^N\setminus B_{\frac{k}{2}}}|\nabla^m u|^2+\sum_{j=0}^{m-1}\|D^{m-j}\eta_k\|^2_{L^{p_j}(\mathbb{R}^N)}\|D^j u\|^2_{L^{q_j}(\mathbb{R}^N\setminus B_{\frac{k}{2}})}\\ &= \int_{\mathbb{R}^N\setminus B_{\frac{k}{2}}}|\nabla^m u|^2+\sum_{j=0}^{m-1}\|D^{m-j}\eta_1\|^2_{L^{p_j}(\mathbb{R}^N)}\|D^j u\|^2_{L^{q_j}(\mathbb{R}^N\setminus B_{\frac{k}{2}})},\end{aligned} \tag{4.4}$$

where we have used the fact that supp $(D^{m-j}\eta_k)\subset B_k\setminus B_{\frac{k}{2}}$ if $j\le m-1$ and

$$\begin{aligned}\|D^{m-j}\eta_k\|^2_{L^{p_j}(\mathbb{R}^N)} &= k^{-2(m-j)}\left(\int_{\mathbb{R}^N}|D^{m-j}\eta_1(x/k)|^{\frac{N}{m-j}}\,dx\right)^{\frac{2(m-j)}{N}}\\ &=\left(\int_{\mathbb{R}^N}|D^{m-j}\eta_1(y)|^{\frac{N}{m-j}}\,dy\right)^{\frac{2(m-j)}{N}}.\end{aligned}$$

Since $u\in D^{m,2}_0(\mathbb{R}^N)$, the first term at the right-hand side of (4.4) tends to 0 as $k\to+\infty$; moreover, the critical Sobolev embedding (2.3) implies that $D^j u\in L^{q_j}(\mathbb{R}^N)$ for all $0\le j\le m$, so that also the second term goes to 0. We conclude that $\eta_k u\to u$ in $D^{m,2}_0(\mathbb{R}^N)$ as $k\to+\infty$, which yields $u\in D^{m,2}_0(\mathbb{R}^N\setminus\mathcal{K})$.

Verification of (M2.ii). Let $u\in D^{m,2}_0(\mathbb{R}^N\setminus\mathcal{K})$. Let $\delta>0$. By density, there exists a function $v\in C^\infty_0(\mathbb{R}^N\setminus\mathcal{K})$ such that $\|\nabla^m(u-v)\|_{L^2(\mathbb{R}^N)}<\frac{\delta}{2}$. Take $R>0$ so that supp $v\subset B_R$. Then $v\in H^m_0(B_R\setminus\mathcal{K})$ and by (M$_R$2.ii) there exist $\bar\varepsilon_\delta>0$ and a family of functions $\{\varphi^\delta_\varepsilon\}_{\varepsilon\in(0,\bar\varepsilon_\delta)}$ such that $\varphi^\delta_\varepsilon\in H^m_0(B_R\setminus\varepsilon^{-1}K_\varepsilon)$ and $\|\nabla^m(v-\varphi^\delta_\varepsilon)\|_{L^2(B_R)}<\frac{\delta}{2}$ for all $\varepsilon\in(0,\bar\varepsilon_\delta)$. Hence, for all $\varepsilon\in(0,\bar\varepsilon_\delta)$, $(\varphi^\delta_\varepsilon)^E\in D^{m,2}_0(\mathbb{R}^N\setminus\varepsilon^{-1}K_\varepsilon)$ and $\|\nabla^m(u-(\varphi^\delta_\varepsilon)^E)\|_{L^2(\mathbb{R}^N)}<\delta$.

As a consequence, there exists a strictly decreasing and vanishing sequence $\{\varepsilon_n\}_n$ such that, for every $n\in\mathbb{N}\setminus\{0\}$, there exists a family of functions $\{u^n_\varepsilon\}_{\varepsilon\in(0,\varepsilon_n)}$ such that

$$u^n_\varepsilon\in D^{m,2}_0(\mathbb{R}^N\setminus\varepsilon^{-1}K_\varepsilon)\quad\text{and}\quad\|\nabla^m(u-u^n_\varepsilon)\|_{L^2(\mathbb{R}^N)}<\frac{1}{n}$$

for all $\varepsilon\in(0,\varepsilon_n)$. For every $\varepsilon\in(0,\varepsilon_1)$, we define $u_\varepsilon:=u^n_\varepsilon$ if $\varepsilon_{n+1}\le\varepsilon<\varepsilon_n$. It is easy to verify that, by construction, $u_\varepsilon\in D^{m,2}_0(\mathbb{R}^N\setminus\varepsilon^{-1}K_\varepsilon)$ for all $\varepsilon\in(0,\varepsilon_1)$ and $\|\nabla^m(u-u_\varepsilon)\|_{L^2(\mathbb{R}^N)}\to0$ as $\varepsilon\to0$. (M2.ii) is thereby verified. □

Before stating the main results of the section, we prepose a lemma about the stability of the (h,V^m)-capacitary potential with respect to the function h.

Lemma 4.3 *Let $K\subset\Omega\subset\mathbb{R}^N$, K compact, $\{h_n\}_{n\in\mathbb{N}}\subset H^m_{loc}(\Omega)$ and $h\in H^m_{loc}(\Omega)$. Let us suppose that, for some $\mathcal{U}(K)\subset\Omega$ open neighbourhood of K, $h_n\to h$ in $H^m(\mathcal{U}(K))$ as $n\to\infty$ and denote by W_{K,h_n} (resp. $W_{K,h}$) the capacitary potential for* $\mathrm{cap}_{V^m,\Omega}(K,h_n)$ *(resp.* $\mathrm{cap}_{V^m,\Omega}(K,h)$*). Then $W_{K,h_n}\to W_{K,h}$ in $V^m(\Omega)$ and* $\mathrm{cap}_{V^m,\Omega}(K,h_n)\to\mathrm{cap}_{V^m,\Omega}(K,h)$.

Proof Being capacitary potentials, the functions W_{K,h_n} and $W_{K,h}$ satisfy

$$\int_\Omega\nabla^m W_{K,h}\nabla^m\varphi=0\quad\text{and}\quad\int_\Omega\nabla^m W_{K,h_n}\nabla^m\varphi=0\quad\text{for all } n\in\mathbb{N}\text{ and }\varphi\in V^m_0(\Omega\setminus K).$$

Let $\eta_K \in C^\infty(\mathbb{R}^N)$ be a cutoff function such that $0 \le \eta_K \le 1$, $\operatorname{supp} \eta_K \subset \mathcal{U}(K)$ and $\eta_K \equiv 1$ in a neighbourhood of K. Hence, by construction, one has that

$$W_{K,h_n} - \eta_K h_n \in V_0^m(\Omega \setminus K) \qquad \text{and} \qquad W_{K,h} - \eta_K h \in V_0^m(\Omega \setminus K).$$

Therefore

$$\begin{aligned}
&\|\nabla^m (W_{K,h_n} - W_{K,h})\|_2^2 \\
&= \int_\Omega \left(\nabla^m W_{K,h_n} - \nabla^m W_{K,h}\right)\left(\nabla^m W_{K,h_n} - \nabla^m W_{K,h}\right) \\
&= \int_\Omega \nabla^m W_{K,h_n} \nabla^m \left(W_{K,h_n} - \eta_K h_n\right) + \int_\Omega \nabla^m W_{K,h_n} \nabla^m \left(\eta_K h_n - \eta_K h\right) \\
&\quad + \int_\Omega \nabla^m W_{K,h_n} \nabla^m \left(\eta_K h - W_{K,h}\right) - \int_\Omega \nabla^m W_{K,h} \nabla^m \left(W_{K,h_n} - \eta_K h_n\right) \\
&\quad - \int_\Omega \nabla^m W_{K,h} \nabla^m \left(\eta_K h_n - \eta_K h\right) - \int_\Omega \nabla^m W_{K,h} \nabla^m \left(\eta_K h - W_{K,h}\right) \\
&= \int_\Omega \nabla^m \left(W_{K,h_n} - W_{K,h}\right) \nabla^m \left(\eta_K h_n - \eta_K h\right) \\
&\le \|\nabla^m W_{K,h_n} - \nabla^m W_{K,h}\|_2 \|\nabla^m \left(\eta_K \left(h_n - h\right)\right)\|_2 .
\end{aligned}$$

This yields

$$\|\nabla^m W_{K,h_n} - \nabla^m W_{K,h}\|_2 \le \|\nabla^m \left(\eta_K \left(h_n - h\right)\right)\|_2 \lesssim \|h_n - h\|_{H^m(\mathcal{U}(K))} \to 0,$$

i.e. $W_{K,h_n} \to W_{K,h}$ in $V^m(\Omega)$, directly implying that $\operatorname{cap}_{V^m,\Omega}(K, h_n) \to \operatorname{cap}_{V^m,\Omega}(K, h)$ as $n \to \infty$. □

Remark 8 In case $\Omega = \mathbb{R}^N$ the same result holds with $W_{K,h_n} \to W_{K,h}$ in $D_0^{m,2}(\mathbb{R}^N)$.

We are now in a position to prove the main results of this section, namely a generalized version of Theorems 1.3–1.6, which take into account families of domains which satisfy (M1)–(M2), rather than just the model case $K_\varepsilon = \varepsilon\mathcal{K}$.

Motivated by the asymptotic scaling properties of the eigenfunctions (1.15), we apply a blow-up argument to a rescaled problem, in order to find a limit equation on $\mathbb{R}^N \setminus \mathcal{K}$ and to prove the convergence of the family of scaled capacitary potentials to the one for the limiting problem. The capacity $\operatorname{cap}_{V^m,\Omega}(K_\varepsilon, u_J)$ will behave then as the limit capacity on $\mathbb{R}^N \setminus \mathcal{K}$ multiplied by a suitable power of ε given by the scaling. In this argument, we work with the homogeneous Sobolev spaces and, in particular, for the Navier case the characterization via Hardy–Rellich inequalities of Sect. 2.2.2 will be needed. This is the main reason for the restriction to the fourth-order case in the Navier setting, since, up to our knowledge, the extension of Proposition 2.5 to the full generality $m \ge 2$ is an open problem.

Theorem 4.4 (Asymptotic expansion of the capacity, Dirichlet case) *Let $N > 2m$ and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with $0 \in \Omega$. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets uniformly concentrating to $\{0\}$ satisfying (M1)–(M2) for some compact set $\mathcal{K}$. Let λ_J be an eigenvalue of (1.1) with Dirichlet boundary conditions and $u_J \in H_0^m(\Omega)$ be a corresponding eigenfunction normalized in $L^2(\Omega)$. Then*

$$\operatorname{cap}_{m,\Omega}(K_\varepsilon, u_J) = \varepsilon^{N-2m+2\gamma}\left(\operatorname{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) + o(1)\right) \tag{4.5}$$

as $\varepsilon \to 0$, with γ and U_0 as in (1.15).

Theorem 4.5 (Asymptotic expansion of the capacity, Navier case) *Let $N > 4$ and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with $0 \in \Omega$. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets uniformly concentrating to $\{0\}$ satisfying (M1)–(M2) for some compact set $\mathcal{K}$. Let λ_J be an eigenvalue of (1.1) with $m = 2$ and Navier boundary conditions and $u_J \in H^2_\vartheta(\Omega)$ be a corresponding eigenfunction normalized in $L^2(\Omega)$. Then*

$$\operatorname{cap}_{2,\vartheta,\Omega}(K_\varepsilon, u_J) = \varepsilon^{N-4+2\gamma}\left(\operatorname{cap}_{2,\mathbb{R}^N}(\mathcal{K}, U_0) + o(1)\right) \tag{4.6}$$

as $\varepsilon \to 0$, with γ and U_0 as in (1.15) with $m = 2$.

As a direct consequence, braiding together Theorem 1.2 and Theorems 4.4–4.5 respectively, and recalling that for $N \geq 2m$ the point has null V^m-capacity by Proposition 2.3, we obtain Theorems 4.6 and 4.7 below.

Theorem 4.6 (Asymptotic expansion of perturbed eigenvalues, Dirichlet case) *Let $N > 2m$ and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain containing 0. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets uniformly concentrating to $\{0\}$ satisfying (M1)–(M2) for some compact set $\mathcal{K}$. Let λ_J be a simple eigenvalue of (1.1) with Dirichlet boundary conditions and let $u_J \in H^m_0(\Omega)$ be a corresponding eigenfunction normalized in $L^2(\Omega)$. Then*

$$\lambda_J(\Omega \setminus K_\varepsilon) = \lambda_J(\Omega) + \varepsilon^{N-2m+2\gamma}\left(\operatorname{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) + o(1)\right) \tag{4.7}$$

as $\varepsilon \to 0$, with γ and U_0 as in (1.15).

Theorem 4.7 (Asymptotic expansion of perturbed eigenvalues, Navier case) *Let $N > 4$ and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain containing 0. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets uniformly concentrating to $\{0\}$ satisfying (M1)–(M2) for some compact set $\mathcal{K}$. Let λ_J be a simple eigenvalue of (1.1) with $m = 2$ and Navier boundary conditions and let $u_J \in H^2_\vartheta(\Omega)$ be a corresponding eigenfunction normalized in $L^2(\Omega)$. Then*

$$\lambda_J(\Omega \setminus K_\varepsilon) = \lambda_J(\Omega) + \varepsilon^{N-4+2\gamma}\left(\operatorname{cap}_{2,\mathbb{R}^N}(\mathcal{K}, U_0) + o(1)\right) \tag{4.8}$$

as $\varepsilon \to 0$, with γ and U_0 as in (1.15) with $m = 2$.

The proofs of Theorems 4.4 and 4.5 follow a similar structure. We proceed hence to prove them at once using the introduced unifying notation, detailing the differences when needed.

Proof of Theorems 4.4–4.5 Motivated by (1.15), we define the analogously scaled potentials

$$\widetilde{W}_\varepsilon := \frac{W_\varepsilon(\varepsilon\cdot)}{\varepsilon^\gamma}, \qquad \text{where} \quad W_\varepsilon := W_{K_\varepsilon, u_J}.$$

It is easy to verify that $\widetilde{W}_\varepsilon$ is the capacitary potential for U_ε in $\varepsilon^{-1}\Omega \setminus \varepsilon^{-1}K_\varepsilon$, i.e.

$$\begin{cases} (-\Delta)^m \widetilde{W}_\varepsilon = 0 \quad \text{in } \varepsilon^{-1}\Omega \setminus \varepsilon^{-1}K_\varepsilon, \\ \widetilde{W}_\varepsilon \in V^m(\varepsilon^{-1}\Omega), \\ \widetilde{W}_\varepsilon - U_\varepsilon \in V^m_0(\varepsilon^{-1}\Omega \setminus \varepsilon^{-1}K_\varepsilon), \end{cases} \tag{4.9}$$

where $m = 2$ in the Navier case. The first goal now is to prove that the so-rescaled capacitary potentials weakly converge to some function $\widetilde{W}$ and prove that $\widetilde{W}$ is a capacitary potential in $\mathbb{R}^N \setminus \mathcal{K}$. To this aim, we need to distinguish between Dirichlet and Navier conditions on $\partial\Omega$. Indeed, by extension by zero outside the rescaled domains, in the first case it is rather natural to prove that the limit functional space is $D^{m,2}_0(\mathbb{R}^N \setminus \mathcal{K})$; that the same holds true in the Navier case is not evident and requires a finer analysis. A fundamental role in this second case is played by the Hardy–Rellich inequality discussed in Sect. 2.2.2, which is however available just for $m = 2$. The two cases will converge then in the final step where the asymptotic expansions (4.5)–(4.6) are proved.

Step 1 (Dirichlet case $V^m = H^m_0$). By (M1) there exists $R > 0$ such that, for ε small enough, $\varepsilon^{-1}K_\varepsilon \subset M \subset B_R(0) \subset \varepsilon^{-1}\Omega$, and hence, in view of Proposition 2.2,

$$\mathrm{cap}_{m,\varepsilon^{-1}\Omega}(\varepsilon^{-1}K_\varepsilon, U_\varepsilon) \le \mathrm{cap}_{m,B_R(0)}(M, U_\varepsilon).$$

Since $U_\varepsilon \to U_0$ in $H^m(B_R(0))$ by (1.15), applying Lemma 4.3 in $B_R(0)$, we infer that

$$\mathrm{cap}_{m,B_R(0)}(M, U_\varepsilon) \to \mathrm{cap}_{m,B_R(0)}(M, U_0) \quad \text{as } \varepsilon \to 0.$$

This yields in particular that $\|\nabla^m \widetilde{W}_\varepsilon\|^2_{L^2(\varepsilon^{-1}\Omega)} = \mathrm{cap}_{m,\varepsilon^{-1}\Omega}(\varepsilon^{-1}K_\varepsilon, U_\varepsilon)$ is bounded uniformly with respect to ε. Letting $\widetilde{W}^E_\varepsilon$ be the extension by 0 of $\widetilde{W}_\varepsilon$ outside $\varepsilon^{-1}\Omega$, we have thus that $\|\widetilde{W}^E_\varepsilon\|_{D^{m,2}_0(\mathbb{R}^N)} \le C$. Since $D^{m,2}_0(\mathbb{R}^N)$ is a Hilbert space, and so reflexive, for every sequence $\varepsilon_n \to 0^+$ there exist a subsequence ε_{n_k} and $\widetilde{W} \in D^{m,2}_0(\mathbb{R}^N)$ such that

$$\widetilde{W}^E_{\varepsilon_{n_k}} \rightharpoonup \widetilde{W} \quad \text{weakly in } D^{m,2}_0(\mathbb{R}^N) \text{ as } k \to \infty. \tag{4.10}$$

We claim now that $\|\nabla^m \widetilde{W}\|^2_2 = \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0)$.

Let $\varphi \in C^\infty_0(\mathbb{R}^N \backslash \mathcal{K})$ and $R > r(M)$ be such that $\operatorname{supp}\varphi \subset B_R$, then by (M$_R$2-*ii*) of Lemma 4.2, one may find a family $\{\varphi_\varepsilon\}_{\varepsilon>0} \subset H^m_0(B_R)$ such that $\varphi_\varepsilon \in H^m_0(B_R \setminus \varepsilon^{-1}K_\varepsilon)$ and $\varphi_\varepsilon \to \varphi$ in $H^m_0(B_R)$ as $\varepsilon \to 0$. In particular, for ε small enough, one has that $B_R \subset \varepsilon^{-1}\Omega$, so φ_ε may be taken as test function for the capacitary potential $\widetilde{W}_\varepsilon$. Hence,

$$\begin{aligned} 0 &= \int_{\varepsilon_{n_k}^{-1}\Omega\setminus\varepsilon_{n_k}^{-1}K_{\varepsilon_{n_k}}} \nabla^m \widetilde{W}_{\varepsilon_{n_k}}\, \nabla^m \varphi_{\varepsilon_{n_k}} \\ &= \int_{\mathbb{R}^N\setminus\varepsilon_{n_k}^{-1}K_{\varepsilon_{n_k}}} \nabla^m \widetilde{W}^E_{\varepsilon_{n_k}}\, \nabla^m \varphi_{\varepsilon_{n_k}} \to \int_{\mathbb{R}^N\setminus\mathcal{K}} \nabla^m \widetilde{W}\, \nabla^m\varphi \quad \text{as } k \to \infty \end{aligned}$$

by weak-strong convergence in $D^{m,2}_0(\mathbb{R}^N)$. We are left to show that $\widetilde{W} - \eta U_0 \in D^{m,2}_0(\mathbb{R}^N\backslash\mathcal{K})$, for some cutoff function η which is equal to 1 in a neighbourhood of $\mathcal{K}$. Let $\eta \in C^\infty_0(\mathbb{R}^N)$ be equal to 1 on an open set $\mathcal{U}$ with $\mathcal{K} \cup M \subset \mathcal{U}$; hence η is also equal to 1 on neighbourhoods of each $\varepsilon^{-1}K_\varepsilon$ by (M1). Then $\widetilde{W}^E_\varepsilon - \eta U_\varepsilon \in D^{m,2}_0(\mathbb{R}^N\backslash\varepsilon^{-1}K_\varepsilon)$ and $\widetilde{W}^E_{\varepsilon_{n_k}} - \eta U_{\varepsilon_{n_k}} \rightharpoonup \widetilde{W} - \eta U_0$ in $D^{m,2}_0(\mathbb{R}^N)$ as $k \to \infty$, and so by (M2-*i*) one infers that $\widetilde{W} - \eta U_0 \in D^{m,2}_0(\mathbb{R}^N \setminus \mathcal{K})$. All in all, we deduce that $\widetilde{W}$ is the capacitary potential relative to $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0)$, i.e.

$$\|\nabla^m \widetilde{W}\|^2_{L^2(\mathbb{R}^N)} = \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0). \tag{4.11}$$

Since the limit $\widetilde{W}$ in (4.10) depends neither on the sequence $\{\varepsilon_n\}$ nor on the subsequence $\{\varepsilon_{n_k}\}$, we conclude that

$$\widetilde{W}^E_\varepsilon \rightharpoonup \widetilde{W} \quad \text{weakly in } D^{m,2}_0(\mathbb{R}^N) \text{ as } \varepsilon \to 0. \tag{4.12}$$

Step 1 (Navier case $V^2 = H^2_\vartheta$). We recall that here we are assuming $m = 2$. The boundedness of $\|\Delta\widetilde{W}_\varepsilon\|_{L^2(\varepsilon^{-1}\Omega)}$ with a constant independent of ε follows from the Dirichlet case, by recalling Proposition 2.2(*iii*). However, unlike the former case, one cannot now extend $\widetilde{W}_\varepsilon$ to 0 outside $\varepsilon^{-1}\Omega$ and still obtain a function in $D^{2,2}_0(\mathbb{R}^N)$. To overcome this problem we rely on the Hardy–Rellich inequality proved in Theorem 2.4. In fact, we have

$$\int_{\varepsilon^{-1}\Omega} \frac{|\widetilde{W}_\varepsilon|^2}{|x|^4}\,dx + \int_{\varepsilon^{-1}\Omega} \frac{|\nabla\widetilde{W}_\varepsilon|^2}{|x|^2}\,dx \lesssim \int_{\varepsilon^{-1}\Omega} |\Delta\widetilde{W}_\varepsilon|^2 \le C$$

and therefore, by a diagonal process of extracted subsequences, for every sequence $\varepsilon_n \to 0^+$ there exist a subsequence ε_{n_j} and $\widetilde{W} \in H^2_{loc}(\mathbb{R}^N)$ for which

$$\frac{\nabla^{2-k}\widetilde{W}_{\varepsilon_{n_j}}}{|x|^k} \rightharpoonup \frac{\nabla^{2-k}\widetilde{W}}{|x|^k} \quad \text{in } L^2(B_R) \tag{4.13}$$

as $j \to \infty$ for any $R > 0$ and $k \in \{0, 1, 2\}$. By weak lower semicontinuity of the norm, we infer that

$$\int_{B_R} \frac{|\nabla^{2-k}\widetilde{W}|^2}{|x|^{2k}}\,dx \le \liminf_{j\to\infty} \int_{B_R} \frac{|\nabla^{2-k}\widetilde{W}_{\varepsilon_{n_j}}|^2}{|x|^{2k}}\,dx \le C,$$

so that, letting $R \to +\infty$,

$$\int_{\mathbb{R}^N} \frac{|\nabla^{2-k}\widetilde{W}|^2}{|x|^{2k}}\,dx \le C \quad \text{for all } k \in \{0, 1, 2\}.$$

By Proposition 2.5, this is equivalent to $\widetilde{W} \in D^{2,2}_0(\mathbb{R}^N)$.

It remains to prove that $\widetilde{W}$ is the capacitary potential relative to $\mathrm{cap}_{2,\mathbb{R}^N}(\mathcal{K}, U_0)$. Let η be as in the former case. Let $\varphi \in C^\infty_0(B_1)$ be such that $\varphi \equiv 1$ in $B_{1/2}(0)$ and consider the scaled functions $\varphi_R := \varphi\big(\frac{\cdot}{R}\big)$ with $R > r(M)$. Then $\varphi_R\big(\widetilde{W}_{\varepsilon_{n_j}} - \eta U_{\varepsilon_{n_j}}\big) \rightharpoonup \varphi_R\big(\widetilde{W} - \eta U_0\big)$ weakly in $H^2_0(B_R)$ and $\varphi_R(\widetilde{W}_\varepsilon - \eta U_\varepsilon\big) \in H^2_0(B_R\setminus\varepsilon^{-1}K_\varepsilon)$. By (M$_R$2-*i*) we know then that $\varphi_R\big(\widetilde{W} - \eta U_0\big) \in H^2_0(B_R\setminus\mathcal{K})$. Now we claim that $\varphi_R\big(\widetilde{W} - \eta U_0\big) \to \widetilde{W} - \eta U_0$ as $R \to +\infty$ in $D^{2,2}_0(\mathbb{R}^N)$, thus concluding that $\widetilde{W} - \eta U_0 \in D^{2,2}_0(\mathbb{R}^N\setminus\mathcal{K})$. Indeed,

$$\begin{aligned}\|\Delta\big((\varphi_R - 1)\big(\widetilde{W} - \eta U_0\big)\big)\|_2^2 \lesssim \|\Delta\varphi_R\big(\widetilde{W} - \eta U_0\big)\|_2^2 + \|\nabla\varphi_R\nabla\big(\widetilde{W} - \eta U_0\big)\|_2^2 \\ + \|(\varphi_R - 1)\,\Delta\big(\widetilde{W} - \eta U_0\big)\|_2^2,\end{aligned}$$

where

$$\|(\varphi_R - 1)\,\Delta\big(\widetilde{W} - \eta U_0\big)\|_2^2 \le \int_{\mathbb{R}^N\setminus B_{R/2}} |\Delta\big(\widetilde{W} - \eta U_0\big)|^2 \to 0$$

as $R \to +\infty$, and, for any $k \in \{1, 2\}$,

$$\begin{aligned}\|\nabla^k\varphi_R\nabla^{2-k}\big(\widetilde{W} - \eta U_0\big)\|_2^2 &= \int_{\frac{R}{2}<|x|<R} \frac{1}{R^{2k}} \left|\big(\nabla^k\varphi\big)\left(\frac{x}{R}\right)\right|^2 |\nabla^{2-k}\big(\widetilde{W} - \eta U_0\big)|^2\,dx \\ &\lesssim \int_{\mathbb{R}^N\setminus B_R(0)} \frac{|\nabla^{2-k}\big(\widetilde{W} - \eta U_0\big)|^2}{|x|^{2k}}\,dx \to 0\end{aligned}$$

as $R \to +\infty$ since $\widetilde{W} - \eta U_0 \in D^{2,2}_0(\mathbb{R}^N)$ together with Proposition 2.5.

Next, we verify that

$$\int_{\mathbb{R}^N\setminus\mathcal{K}} \Delta\widetilde{W}\,\Delta\varphi = 0 \quad \text{for all } \varphi \in D^{2,2}_0(\mathbb{R}^N\setminus\mathcal{K}). \tag{4.14}$$

By density, it is enough to prove (4.14) for all $\varphi \in C^\infty_0(\mathbb{R}^N\setminus\mathcal{K})$. Letting $\varphi \in C^\infty_0(\mathbb{R}^N\setminus\mathcal{K})$, there exist $R > r(M)$ and $\varepsilon_0 > 0$ such that $\mathrm{supp}\,\varphi \subset B_R \subset \varepsilon^{-1}\Omega$ for all $\varepsilon < \varepsilon_0$, so that $\varphi \in H^2_0(B_R\setminus\mathcal{K})$. By (M$_R$2-*ii*) there exists a family $\{\varphi_\varepsilon\}_\varepsilon \subset H^2_0(B_R)$ such that $\varphi_\varepsilon \in H^2_0(B_R\setminus\varepsilon^{-1}K_\varepsilon)$ and $\varphi_\varepsilon \to \varphi$ in $H^2_0(B_R)$. Hence,

$$0 = \int_{\varepsilon_{n_j}^{-1}\Omega} \Delta\widetilde{W}_{\varepsilon_{n_j}}\,\Delta\varphi_{\varepsilon_{n_j}} = \int_{B_R} \Delta\widetilde{W}_{\varepsilon_{n_j}}\,\Delta\varphi_{\varepsilon_{n_j}} \to \int_{B_R} \Delta\widetilde{W}\,\Delta\varphi = \int_{\mathbb{R}^N\setminus\mathcal{K}} \Delta\widetilde{W}\,\Delta\varphi$$

as $j \to \infty$, by weak-strong convergence in $H_0^2(B_R)$. We have thereby proved the claim that $\widetilde{W}$ is the capacitary potential relative to $\mathrm{cap}_{2,\mathbb{R}^N}(\mathcal{K}, U_0)$. Since the limit $\widetilde{W}$ in (4.13) depends neither on the sequence $\{\varepsilon_n\}$ nor on the subsequence $\{\varepsilon_{n_k}\}$, we conclude that the convergences in (4.13) actually hold as $\varepsilon \to 0$, i.e.

$$\frac{\nabla^{2-k}\widetilde{W}_\varepsilon}{|x|^k} \rightharpoonup \frac{\nabla^{2-k}\widetilde{W}}{|x|^k} \quad \text{in } L^2(B_R) \quad \text{as } \varepsilon \to 0 \quad \text{for all } R > 0 \text{ and } k \in \{0, 1, 2\}. \tag{4.15}$$

Step 2. ($m = 2$ in the Navier case, $m \geq 2$ in the Dirichlet case). We aim now to prove the asymptotic expansions (4.5)–(4.6). As above, let $\eta \in C_0^\infty(\mathbb{R}^N)$ be equal to 1 on an open set $\mathcal{U}$ with $\mathcal{K} \cup M \subset \mathcal{U}$. Let $R > 0$ be such that $\mathrm{supp}\,\eta \subset B_R$. Since $\widetilde{W}_\varepsilon - \eta U_\varepsilon \in V_0^m(\varepsilon^{-1}\Omega \setminus \varepsilon^{-1}K_\varepsilon)$, by (4.9) and (4.11), together with (4.12) or (4.15), we obtain that

$$\begin{aligned} \|\nabla^m \widetilde{W}_\varepsilon\|^2_{L^2(\varepsilon^{-1}\Omega)} &= \int_{\varepsilon^{-1}\Omega} \nabla^m \widetilde{W}_\varepsilon \, \nabla^m (\eta U_\varepsilon) = \int_{B_R} \nabla^m \widetilde{W}_\varepsilon \, \nabla^m (\eta U_\varepsilon) \\ &\to \int_{\mathbb{R}^N} \nabla^m \widetilde{W} \, \nabla^m (\eta U_0) = \|\nabla^m \widetilde{W}\|^2_{L^2(\mathbb{R}^N)} \end{aligned} \tag{4.16}$$

as $\varepsilon \to 0$ by weak-strong convergence. On the other hand, by rescaling one has that

$$\begin{aligned} \|\nabla^m \widetilde{W}_\varepsilon\|^2_{L^2(\varepsilon^{-1}\Omega)} &= \frac{1}{\varepsilon^{2\gamma}} \int_{\varepsilon^{-1}\Omega} \left|\nabla^m (W_\varepsilon(\varepsilon x))\right|^2 dx = \varepsilon^{-N+2m-2\gamma} \int_\Omega |\nabla^m W_\varepsilon(y)|^2 \, dy \\ &= \varepsilon^{-N+2m-2\gamma} \mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J). \end{aligned} \tag{4.17}$$

Hence, from (4.11) and (4.16)–(4.17) we finally infer that

$$\mathrm{cap}_{V^m,\Omega}(K_\varepsilon, u_J) = \varepsilon^{N-2m+2\gamma} \|\nabla^m \widetilde{W}_\varepsilon\|^2_{L^2(\varepsilon^{-1}\Omega)} = \varepsilon^{N-2m+2\gamma} \left(\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) + o(1)\right)$$

as $\varepsilon \to 0$. □

4.2 Sufficient conditions for a sharp asymptotic expansion

Looking at the asymptotic expansions we have found in Theorems 4.6–4.7, one may ask whether the results are sharp, in the sense that the vanishing rate of the eigenvalue variation $\lambda_J(\Omega \setminus K_\varepsilon) - \lambda_J(\Omega)$ is equal to $N - 2m + 2\gamma$. The next results provide sufficient conditions on $\mathcal{K}$ and U_0 in order to ensure that $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) \neq 0$.

Proposition 4.8 *Under the assumptions of Theorems 4.6 or 4.7, suppose that the Lebesgue measure of $\mathcal{K}$ is positive. Then*

$$\lim_{\varepsilon \to 0} \frac{\lambda_J(\Omega \setminus K_\varepsilon) - \lambda_J(\Omega)}{\varepsilon^{N-2m+2\gamma}} = \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) > 0.$$

Proof Denote by $W_{\mathcal{K}}^{(0)}$ the capacitary potential for $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0)$ and suppose by contradiction that $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) = 0$. Then, by the Hardy inequality for the polyharmonic operator (see [12, Theorem 12]), there exists a constant $c = c(N, m)$ such that

$$0 = \|\nabla^m W_{\mathcal{K}}^{(0)}\|^2_{L^2(\mathbb{R}^N)} \geq c \int_{\mathbb{R}^N} \frac{|W_{\mathcal{K}}^{(0)}|^2}{|x|^{2m}} \, dx \geq c \int_{\mathcal{K}} \frac{|W_{\mathcal{K}}^{(0)}|^2}{|x|^{2m}} \, dx = c \int_{\mathcal{K}} \frac{|U_0|^2}{|x|^{2m}} \, dx$$

since $W_{\mathcal{K}}^{(0)} \equiv U_0$ on $\mathcal{K}$. Since $|\mathcal{K}| > 0$, this readily implies that U_0 vanishes a.e. on $\mathcal{K}$. However, by construction, U_0 is a polyharmonic polynomial on $\mathbb{R}^N$ which is not identically zero (see [6, Sec.4 Theorem 1]), so it cannot vanish on a set of positive measure (since nontrivial analytic functions cannot vanish on positive measure sets). This, together with Theorems 4.6 and 4.7, concludes the proof. □

The next results apply to some specific situations in which, although K has vanishing Lebesgue measure, one may anyway have that $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) \neq 0$.

Proposition 4.9 *Let $N > 2m$ and $\mathcal{K} \subset \mathbb{R}^N$ be a compactum with $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}) > 0$. Suppose moreover that $u_J(0) \neq 0$. Then, in the setting of Theorems 4.6 or 4.7, we have that*

$$\lambda_J(\Omega \setminus K_\varepsilon) = \lambda_J(\Omega) + \varepsilon^{N-2m} u_J^2(0)\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}) + o(\varepsilon^{N-2m}) \tag{4.18}$$

as $\varepsilon \to 0$.

We mention that an expansion of type (4.18) was obtained in [10, Theorem 1.4] and [2, Theorem 1.7] in the case $N = 2m = 2$, in which the vanishing rate of the eigenvalue variation is logarithmic.

Proof of Proposition 4.9 Since $\{K_\varepsilon\}_{\varepsilon>0}$ is concentrating at $\{0\}$ and $u_J(0) \neq 0$, then the degree γ of the polynomial U_0 is 0, and $U_0 = u_J(0)$. It is then easy to see that

$$\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, u_J(0)) = u_J^2(0)\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}) > 0,$$

so that (4.7) and (4.8) can be rewritten as in (4.18). □

In the case $u_J(0) = 0$, the next result, inspired by [13, Lemma 3.11], may be useful. It tells that, if the compactum $\mathcal{K}$ and the null-set of the polynomial U_0 are "transversal enough", then again $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) > 0$.

Proposition 4.10 *Let $N > 2m$ and $\mathcal{K} \subset \mathbb{R}^N$ be a compactum with $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}) > 0$. Letting $f \in C^\infty(\mathbb{R}^N)$, let us consider the set*

$$Z_f^{\mathcal{K}} := \{x \in \mathcal{K} \mid f(x) = 0\}.$$

If $\mathrm{cap}_{m,\mathbb{R}^N}(Z_f^{\mathcal{K}}) < \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K})$, then $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, f) > 0$.

Proof Let $\{\mathcal{U}_n\}$ be a sequence of nested open sets in $\mathbb{R}^N$ so that $Z_f^{\mathcal{K}} \subset \mathcal{U}_n$ for all $n \in \mathbb{N}$ and $Z_f^{\mathcal{K}} = \bigcap_{n\in\mathbb{N}} \overline{\mathcal{U}_n}$ and let $\mathcal{K}_n := \mathcal{K} \setminus \mathcal{U}_n$, which is a sequence of compact sets. By subadditivity and monotonicity of the capacity (see e.g. [24]) one has that

$$\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}) \le \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}_n) + \mathrm{cap}_{m,\mathbb{R}^N}(\overline{\mathcal{U}_n}).$$

Moreover, fixing $0 < \delta < \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}) - \mathrm{cap}_{m,\mathbb{R}^N}(Z_f^{\mathcal{K}})$, one may find a neighbourhood $\mathcal{U}(Z_f^{\mathcal{K}})$ such that one has $\overline{\mathcal{U}_n} \subset \mathcal{U}(Z_f^{\mathcal{K}})$ by construction and $\mathrm{cap}_{m,\mathbb{R}^n}(\overline{\mathcal{U}_n}) \le \mathrm{cap}_{m,\mathbb{R}^n}(Z_f^{\mathcal{K}}) + \delta$ by Lemma 2.6, provided n is large enough. This implies

$$\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}_n) \ge \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}) - \mathrm{cap}_{m,\mathbb{R}^N}(Z_f^{\mathcal{K}}) - \delta > 0 \tag{4.19}$$

for n large enough. We define $\mathcal{K}_n^+ := \{x \in \mathcal{K}_n : f(x) > 0\}$ and $\mathcal{K}_n^- := \{x \in \mathcal{K}_n : f(x) < 0\}$ for all $n \in \mathbb{N}$. Noticing that $\mathcal{K}_n$ is the union of $\mathcal{K}_n^+$ and $\mathcal{K}_n^-$, we necessarily have that either $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}_n^+) > 0$ or $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}_n^-) > 0$; let us e.g. consider the case $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}_n^+) > 0$. By regularity of f and since $\mathcal{K}_n^+$ is compact, then $c_n^+ := \inf_{\mathcal{K}_n^+} f$ is attained and strictly positive.

Take now any $u_n \in D_0^{m,2}(\mathbb{R}^N)$ so that $u_n - \eta_{\mathcal{K}_n^+} f \in D_0^{m,2}(\mathbb{R}^N \setminus \mathcal{K}_n^+)$ and define $v_n := \frac{u_n}{c_n^+}$. Then it is clear that $v_n \in D_0^{m,2}(\mathbb{R}^N)$ and $v_n \geq 1$ a.e. on $\mathcal{K}_n^+$. Hence,

$$\mathrm{Cap}^{\geq}_{m,\,\mathbb{R}^N}(\mathcal{K}_n^+) \leq \int_{\mathbb{R}^N} |\nabla^m v_n|^2 = \frac{1}{\left(c_n^+\right)^2} \int_{\mathbb{R}^N} |\nabla^m u_n|^2.$$

By arbitrariness of u_n this yields $\left(c_n^+\right)^2 \mathrm{Cap}^{\geq}_{m,\,\mathbb{R}^N}(\mathcal{K}_n^+) \leq \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}_n^+, f) \leq \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, f)$, since $\mathcal{K}_n^+ \subset \mathcal{K}$ for all $n \in \mathbb{N}$. Using now the equivalence of the capacities in $\mathbb{R}^N$ stated in Lemma 2.7, one infers that

$$\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, f) \geq c\left(c_n^+\right)^2 \mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}_n^+) > 0$$

by (4.19). This concludes the proof. □

Remark 9 In view of Remark 4 it is immediate to see that, if $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}) > 0$ and $Z_{U_0}^{\mathcal{K}}$ has dimension $d \leq N - 2m$, then the assumptions of Proposition 4.10 are fulfilled, thus ensuring that $\mathrm{cap}_{m,\mathbb{R}^N}(\mathcal{K}, U_0) > 0$.

5 Open problems

We finally discuss possible generalizations and questions which are left open by our analysis and which we believe of interest.

Higher-order Navier setting. The results in Sect. 3 for the Navier setting are obtained in the general case $m \geq 2$. On the other hand, Theorem 4.5 and its consequent Theorem 4.7 are established only for $m = 2$. The main difficulty in their extension to higher orders relies in the characterization of homogeneous Sobolev spaces via Hardy–Rellich inequalities. In our argument this was necessary to compensate for the lack of a trivial extension, which is instead available in the Dirichlet setting. Although we envision that a generalization of the Hardy–Rellich inequality of Proposition 2.4 is reachable, the extension of the characterization contained in Proposition 2.5 seems to be a non trivial problem. Indeed, for $m = 2$ the only intermediate derivative is the gradient and $\nabla u = Du$; on the other hand for $m \geq 3$ the Hardy–Rellich inequality would provide a weighted estimate on the derivatives $\nabla^k u$, $k \in \{1, \dots, m-1\}$, while one would need to estimate the full tensor of the derivatives $D^k u$ to be able to conclude that $u \in D_0^{m,2}(\mathbb{R}^N)$.

Small dimensions. Most of our results deal with the high dimensional case $N \geq 2m$, because the concentration of the family of sets $\{K_\varepsilon\}_{\varepsilon>0}$ to a zero V^m-capacity compact set was needed. Recall that for $N < 2m$ all compact sets are of positive capacity, see Proposition 2.3. Nevertheless, in order to prove that the asymptotic expansions given by Theorem 1.2 are sharp, in Theorems 1.3 and 1.4 we have to restrict to $N > 2m$. It seems that the conformal case $N = 2m$ cannot be treated by blow-up analysis, not only due to the different characterization of the spaces $D_0^{m,2}(\mathbb{R}^N)$ and the use of Hardy–Rellich inequalities, but also because the m-capacity $\mathrm{cap}_{m,\mathbb{R}^{2m}}(K)$ of any compact set K in $\mathbb{R}^{2m}$ is null (see [23]). A different approach for conformal (and smaller!) dimensions should be in fact developed and we expect that the expansion involves the logarithm of the diameter of the shrinking sets, in analogy with the results in [2] for the case $m = 1$. We remark in particular that, when our results are applied to the biharmonic operator, i.e. $m = 2$, we cover the case $N \geq 5$, while the two-dimensional case, from a completely different point of view, is studied in [8, 21, 22]. The case $N = 3$ is left open and $N = 4$ only partially answered by Theorem 1.2.

Equivalent definitions of capacities. As described in Sect. 2, both $\mathrm{cap}_{m,\mathbb{R}^N}$ in (2.6) and $\mathrm{Cap}^{\geq}_{m,\mathbb{R}^N}$ in (2.8) are good definitions of capacity, and they are also equivalent for $N > 2m$, see Lemma 2.7. In the second order case, it is not difficult to prove that the two coincide, while—up to our knowledge—this is still unknown in the higher-order setting. A weaker question would be to ask whether the two capacities are asymptotic for families of shrinking domains, e.g. for $K_\varepsilon = \varepsilon\mathcal{K}$ as considered in Sect. 4.1. It would be also interesting to understand whether the equivalence remains true for the weighted capacities $\mathrm{cap}_{m,\mathbb{R}^N}(\cdot, h)$ and the analogue $\mathrm{Cap}^{\geq}_{m,\mathbb{R}^N}(\cdot, h)$ for some class of nonconstant functions h.
Boundary conditions. As mentioned in the introduction, it would be interesting to investigate the complementary cases of prescribing Navier BCs on the removed set and either Navier or Dirichlet BCs on the external boundary $\partial\Omega$. Because of the lack of an extension by zero in case of Navier BCs, which has consequences on the mutual relations between the spaces $V^m(\Omega \setminus K)$, a different argument would be needed. More in general, it would be challenging to consider more general types of BCs, which yield a different quadratic form associated to the polyharmonic operator, involving possibly also boundary integrals. An interesting case in the biharmonic setting, related to the physical model of hinged thin plates, is for example given by Steklov BCs $u = \Delta u - d\kappa\partial_n u = 0$, where $d \in \mathbb{R}$ and κ is the signed curvature of the boundary.

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References

1. Abatangelo, L., Bonnaillie-Noël, V., Léna, C., Musolino, P.: Asymptotic behavior of u-capacities and singular perturbations for the Dirichlet–Laplacian. ESAIM Control Optim. Calc. Var. **27**, 25 (2021)
2. Abatangelo, L., Felli, V., Hillairet, L., Léna, C.: Spectral stability under removal of small capacity sets and applications to Aharonov–Bohm operators. J. Spectr. Theory **9**(2), 379–427 (2019)
3. Abatangelo, L., Felli, V., Noris, B.: On simple eigenvalues of the fractional Laplacian under removal of small fractional capacity sets. Commun. Contemp. Math. **22**(8), 1950071 (2020)
4. Adams, D.R., Hedberg, L.I.: Function Spaces and Potential Theory. Grundlehren der mathematischen Wissenschaften, vol. 314. Springer, Berlin (1996)
5. Arrieta, J.M., Lamberti, P.D.: Higher order elliptic operators on variable domains. Stability results and boundary oscillations for intermediate problems. J. Differ. Equ. **263**(7), 4222–4266 (2017)
6. Bers, L.: Local behavior of solutions of general linear elliptic equations. Commun. Pure Appl. Math. **8**, 473–496 (1955)
7. Bertrand, J., Colbois, B.: Capacité et inégalité de Faber-Krahn dans $\mathbb{R}^n$. J. Funct. Anal. **232**(1), 1–28 (2006)
8. Campbell, A., Nazarov, S.: Asymptotics of eigenvalues of a plate with small clamped zone. Positivity **5**(3), 275–295 (2001)

9. Chen, X.-Y.: Polynomial asymptotics near zero points of solutions of general elliptic equations. Commun. Part. Differ. Equ. **22**(7–8), 1227–1250 (1997)
10. Courtois, G.: Spectrum of manifolds with holes. J. Funct. Anal. **134**(1), 194–221 (1995)
11. Daners, D.: Dirichlet problems on varying domains. J. Differ. Equ. **188**(2), 591–624 (2003)
12. Davies, E.B., Hinz, A.M.: Explicit constants for Rellich inequalities in $L^p(\Omega)$. Math. Z. **227**(3), 511–523 (1998)
13. Felli, V., Noris, B., Ognibene, R.: Eigenvalues of the Laplacian with moving mixed boundary conditions: the case of disappearing Dirichlet region. Calc. Var. Part. Differ. Equ. **60**(1), 12 (2021)
14. Flucher, M.: Approximation of Dirichlet eigenvalues on domains with small holes. J. Math. Anal. Appl. **193**(1), 169–199 (1995)
15. Galdi, G.P.: An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Steady-State Problems. Springer Monographs in Mathematics, 2nd edn. Springer, New York (2011)
16. Gazzola, F., Grunau, H.-C., Mitidieri, E.: Hardy inequalities with optimal constants and remainder terms. Trans. Am. Math. Soc. **356**(6), 2149–2168 (2004)
17. Gazzola, F., Grunau, H.-C., Sweers, G.: Polyharmonic Boundary Value Problems. Springer Lecture Notes in Mathematics 1991 (2010)
18. Gazzola, F., Grunau, H.-C., Sweers, G.: Optimal Sobolev and Hardy–Rellich constants under Navier boundary conditions. Ann. Mat. Pura Appl. **189**(3), 475–486 (2010)
19. Grunau, H.-C., Robert, F.: Positivity and almost positivity of biharmonic Green's functions under Dirichlet boundary conditions. Arch. Ration. Mech. Anal. **195**(3), 865–898 (2010)
20. John, F.: The fundamental solution of linear elliptic differential equations with analytic coefficients. Commun. Pure Appl. Math. **3**, 273–304 (1950)
21. Kropinski, M.C., Lindsay, A.E., Ward, M.J.: Asymptotic analysis of localized solutions to some linear and nonlinear biharmonic eigenvalue problems. Stud. Appl. Math. **126**(4), 347–408 (2011)
22. Lindasy, A.E., Ward, M.J., Kolokolnikov, T.: The transition to a point constraint in a mixed biharmonic eigenvalue problem. SIAM J. Appl. Math. **75**(3), 1193–1224 (2015)
23. Maz'ya, V.: On (p, l)-capacity, imbedding theorems and the spectrum of a selfadjoint elliptic operator. Izv. Akad. Nauk SSSR Ser. Mat. **37**, 356–385 (**in Russian**). (English translation: Math. USSR Izv. **7**(1973), 357–387 (1973))
24. Maz'ya, V.: Sobolev Spaces with Applications to Elliptic Partial Differential Equations. Second, revised and augmented edition. Grundlehren der Mathematischen Wissenschaften, vol 342. Springer, Heidelberg (2011)
25. Mosco, U.: Convergence of convex sets and of solutions of variational inequalities. Adv. Math. **3**, 510–585 (1969)
26. Pérez-Llanos, M., Primo, A.: Semilinear biharmonic problems with a singular term. J. Differ. Equ. **257**(9), 3200–3225 (2014)