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Schwartz correspondence for the complex motion group on $\mathbb{C}^2$

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A B S T R A C T

If (G, K) is a Gelfand pair, with G a Lie group of polynomial growth and K a compact subgroup of G, the Gelfand spectrum Σ of the bi-K-invariant algebra $L^1(K\backslash G/K)$ admits natural embeddings into $\mathbb{R}^\ell$ spaces as a closed subset.
For any such embedding, define $\mathcal{S}(\Sigma)$ as the space of restrictions to Σ of Schwartz functions on $\mathbb{R}^\ell$. We call *Schwartz correspondence* for (G, K) the property that the spherical transform is an isomorphism of $\mathcal{S}(K\backslash G/K)$ onto $\mathcal{S}(\Sigma)$.
In all the cases studied so far, Schwartz correspondence has been proved to hold true. These include all pairs with $G = K \ltimes H$ and K abelian and a large number of pairs with $G = K \ltimes H$ and H nilpotent.
We prove Schwartz correspondence for the pair $(U_2 \ltimes M_2(\mathbb{C}), U_2)$, where $M_2(\mathbb{C})$ is the complex motion group and $U_2 = K$ acts on it by conjugation. Our proof goes through a detailed analysis of $(M_2(\mathbb{C}), U_2)$ as a strong Gelfand pair and reduction of the problem to Schwartz correspondence for

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each K-type $\tau \in \widehat{K}$ with appropriate control of the estimates in terms of τ.

1. Introduction

Let (G, K) be a Gelfand pair, with G a connected Lie group and K a compact subgroup of it. By definition, this means that the convolution algebra $L^1(K\backslash G/K)$ of bi-K-invariant integrable functions on G is commutative, or, equivalently, that the composition algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K is commutative.

The Gelfand spectrum Σ of $L^1(K\backslash G/K)$ is the space of bounded spherical functions on G with the topology induced by the weak* topology on $L^\infty(K\backslash G/K)$. For each choice of a finite generating subset $\mathcal{D} = \{D_1, \dots, D_\ell\}$ of $\mathbb{D}(G/K)$, Σ can be homeomorphically embedded with closed image $\Sigma_{\mathcal{D}}$ into $\mathbb{C}^\ell$, by assigning to each bounded spherical function $\varphi = \varphi_\xi \in \Sigma$ the ℓ-tuple $\xi = (\xi_1, \dots, \xi_\ell)$ if $D_j\varphi = \xi_j\varphi$ for $j = 1, \dots, \ell$ [6]. We call $\Sigma_{\mathcal{D}}$ the *embedded spectrum* of $L^1(K\backslash G/K)$ relative to the generating system $\mathcal{D}$ and regard the spherical transform $\mathcal{G}f$ of $f \in L^1(K\backslash G/K)$ as a map defined on $\Sigma_{\mathcal{D}}$ by

$$\mathcal{G}f(\xi) = \int_G f(x)\, \varphi_\xi(x^{-1})\, dx \qquad \xi \in \Sigma_{\mathcal{D}}.$$

If G has polynomial volume growth and the generators $D_j \in \mathcal{D}$ are taken essentially self-adjoint, the eigenvalues are real, so that $\Sigma_{\mathcal{D}} \subset \mathbb{R}^\ell$. We refer to [3] for a presentation of Gelfand pairs of polynomial growth and the proofs of various preliminary results that will be needed in this paper.

We say that *Schwartz correspondence* holds for a Gelfand pair (G, K) of polynomial growth if the following property is satisfied:

(S) The spherical transform maps the bi-K-invariant Schwartz space $\mathcal{S}(K\backslash G/K)$ isomorphically onto the space $\mathcal{S}(\Sigma_{\mathcal{D}})$ of restrictions to $\Sigma_{\mathcal{D}}$ of Schwartz functions on $\mathbb{R}^\ell$.

This is an intrinsic property of the pair because it does not depend on the choice of the generating system $\mathcal{D}$ [2,3,8]. It has been proved to be satisfied by all Gelfand pairs (G, K) with polynomial growth on which it has been tested so far. These include compact pairs (i.e., with G compact) [3], various families of nilpotent pairs (i.e., with $G = K \ltimes N$ and N nilpotent) [1,2,7–9], and pairs with $G = K \ltimes H$ and K abelian [3].

In this paper we go one step ahead, considering an example of Gelfand pair of polynomial growth that is not of the above mentioned types.

In order to locate this example in a more general perspective, it is convenient to recall the notion of strong Gelfand pair and the general structure of Gelfand pairs with polynomial growth.

Given a Lie group G and $K \subset G$ compact, we denote by $\mathrm{Int}(K)$ the group of inner automorphisms of G induced by elements of K and say that a function is K-central if it is $\mathrm{Int}(K)$-invariant. The pair (G, K) is said to be a strong Gelfand pair if the algebra $L^1(G)^{\mathrm{Int}(K)}$ of K-central integrable functions is commutative.

Since $L^1(K\backslash G/K) \subset L^1(G)^{\mathrm{Int}(K)}$, a strong Gelfand pair is obviously a Gelfand pair. Moreover, (G, K) is strong Gelfand if and only if $\big(\mathrm{Int}(K) \ltimes G, \mathrm{Int}(K)\big)$ is Gelfand (see [3,15] and Section 2.2 for details).

When (G, K) is a strong Gelfand pair, we must distinguish between the different notions of spherical functions, Gelfand spectrum and Schwartz correspondence for the (non-strong) pairs (G, K) and $\big(\mathrm{Int}(K) \ltimes G, \mathrm{Int}(K)\big)$. In particular, we say that property (S) holds for the strong pair (G, K) if it holds for the associated pair $\big(\mathrm{Int}(K) \ltimes G, \mathrm{Int}(K)\big)$. Explicitly, this means that the spherical transform maps the K-central Schwartz space $\mathcal{S}(G)^{\mathrm{Int}(K)}$ isomorphically onto $\mathcal{S}(\Sigma_{\mathcal{D}})$, where $\Sigma_{\mathcal{D}}$ is an embedded spectrum of $L^1(G)^{\mathrm{Int}(K)}$.

It follows from Vinberg's structure theorem [18, Thm. 13.3.20] and classification [19] that the list of irreducible strong Gelfand pairs (G, K) of polynomial growth consists of the three families

$$(SO_n \ltimes \mathbb{R}^n, SO_n)\ , \qquad (U_n \ltimes \mathbb{C}^n, U_n)\ , \qquad (U_n \ltimes H_n, U_n)\ , \tag{1.1}$$

where H_n is the $(2n+1)$-dimensional Heisenberg group.

Irreducibility of (G, K) means that there is no nontrivial decomposition $G = G_1 \times G_2$, $K = K_1 \times K_2$ with $K_i \subset G_i$ for $i = 1, 2$. A simple adaptation of the proof of Prop. 3.2 in [9] allows us to deduce property (S) for a reducible strong Gelfand pair from its validity for each irreducible component. Moreover, the positive results for the Schwartz correspondence in [3] include the lowest dimensional cases in (1.1), where $K \cong U_1$ is abelian. So attention must be focused on the irreducible strong pairs in (1.1) with K nonabelian.

We will use the notation $M_n(\mathbb{R}) = SO_n \ltimes \mathbb{R}^n$ and $M_n(\mathbb{C}) = U_n \ltimes \mathbb{C}^n$, for the real and complex motion groups, respectively.

The main result of this paper is the following.

Theorem 1.1. *Property* (S) *holds for the strong Gelfand pair* $\big(M_2(\mathbb{C}), U_2\big)$.

The proof of Theorem 1.1 is based on decomposition into K-types, a notion that we now briefly explain.

If (G, K) is a strong Gelfand pair, the algebra $L^1(G)^{\mathrm{Int}(K)}$ splits as the direct sum of the subalgebras $L^1(G)^{\mathrm{Int}(K)}_\tau$, $\tau \in \widehat{K}$, of K-central functions f which are of K-type τ (see Section 2 for definitions). Correspondingly, the spectrum Σ of $L^1(G)^{\mathrm{Int}(K)}$ decomposes as the disjoint union of the spectra Σ^τ of $L^1(G)^{\mathrm{Int}(K)}_\tau$. It follows that the spherical transform $\mathcal{G}_\tau$ on $L^1(G)^{\mathrm{Int}(K)}_\tau$ is given by

$$\mathcal{G}_\tau f = (\mathcal{G} f)_{|_{\Sigma^\tau}}\ .$$

The following general principle allows us to reduce verification of property (S) for a strong Gelfand pair (G, K) to a Schwartz extension property for single K-type components.

Theorem 1.2 *([3, Prop. 5.2 and Thm. 7.1]). Property* (S) *holds for a strong Gelfand pair* (G, K) *of polynomial growth if and only if the following condition is satisfied:*

(S') *given* $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$ *and* $N \in \mathbb{N}$, *for each* K*-type component* f_τ *of* f, $\tau \in \widehat{K}$, $\mathcal{G}f_\tau$ *admits a Schwartz extension* g_τ^N *such that* $\|g_\tau^N\|_{(N)}$ *is rapidly decaying in* τ.

The notion of rapid decay in τ will be defined in Section 4.5, see Remark 4.7.

Restricting to our case, $(G, K) = \big(M_2(\mathbb{C}), U_2\big)$, and denoting by V_τ the representation space of a given $\tau \in \widehat{K}$, $\mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ is isomorphic, as a topological algebra, to $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_\tau)\big)^K$ consisting of K-equivariant $\mathrm{End}(V_\tau)$-valued Schwartz functions on $\mathbb{C}^2$. This isomorphism, introduced in Lemma 4.1, is quite standard in spherical analysis on semisimple groups [3,5,15,17].

The $\mathrm{End}(V_\tau)$-valued model has various advantages. Since functions are defined on $\mathbb{C}^2$ standard Fourier analysis becomes available. Moreover, exploiting the algebraic properties of the representation τ we can express any $F \in \mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_\tau)\big)^K$ as

$$F = \sum_{j=0}^{n} \mathbf{D}_n^j g_j \ , \tag{1.2}$$

where $n = \dim V_\tau$, $\mathbf{D}_n$ is the matrix valued differential operator defined in (4.9) and the g_j are K-invariant scalar-valued Schwartz functions on $\mathbb{C}^2$ (cf. Corollary 5.6).

Formula (1.2) together with property (S) for the ordinary (non-strong) Gelfand pair $(G, K) = \big(M_2(\mathbb{C}), U_2\big)$, proved in [2, Theorem 6.1] and applied here to the functions g_j, gives a Schwartz extension of the Gelfand transform of F (cf. Corollary 6.3). This argument proves that, given $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$, the spherical transform $\mathcal{G}_\tau f_\tau$ of each K-type f_τ admits a Schwartz extension.

However, the norm estimates that one can deduce do not guarantee the rapid decay requested by property (S'). To overcome this problem we use another method to extend each $\mathcal{G}_\tau f_\tau$ to a Schwartz function. This seems to make the previous part of the proof useless, but this is not the case, because there is a crucial point in the proof where one needs to know that certain linear systems are solvable (cf. Lemma 7.4), and this is guaranteed by *a priori* knowledge that a Schwartz extension exists.

The latter construction is based on a Whitney-type extension argument, in the spirit of the previous proofs of Schwartz correspondence [2,7–9].

Despite the fact that $\big(M_2(\mathbb{C}), U_2\big)$ is only a special case, the proof requires nonetheless a considerable machinery of both analytic and algebraic nature. The same tools of representation theory of SU_2 that are used here can be adapted to prove Schwartz correspondence for the pairs $\big(M_n(\mathbb{R}), SO_n\big)$ with $n = 3, 4$. However, even these simple cases require some extra arguments on the analytic side and we treat them in a forthcoming paper. We believe that the difficulties in extending our method to highter dimensional cases, such as $(M_n(\mathbb{C}), U_n)$ with $n \geq 3$, should mostly be of algebraic nature.

The paper is organized as follows.

In Sections 2 and 3 we establish the basic terminology, summarize the basic relations among the defined objects, set up notation for the representations $\tau = \tau_{m,n}$ of $K = U_2$, and introduce the complex motion group $G = M_2(\mathbb{C})$. We also describe the algebra $\mathbb{D}(G)^{\mathrm{Int}(K)}$ for this case, selecting a privileged system of four generators, $D_1, \dots, D_4$, the last two being generators of $\mathbb{D}(K)^{\mathrm{Int}(K)}$, the centre of $\mathbb{D}(K)$.

In Section 4 we establish the isomorphism A_τ from $L^1(G)_\tau^{\mathrm{Int}(K)}$ to $L^1(\mathbb{C}^2, \mathrm{End}(V_\tau))^K$ and describe the operator algebra $\big(\mathbb{D}(\mathbb{C}^2) \otimes \mathrm{End}(V_\tau)\big)^K$. In particular, under conjugation by A_τ, the operators D_3, D_4 are mapped into scalar multiples of the identity, so that the two operators $A_\tau D_i A_\tau^{-1}$, $i = 1, 2$, are sufficient to generate $\big(\mathbb{D}(\mathbb{C}^2) \otimes \mathrm{End}(V_\tau)\big)^K$. In Section 4.5, we describe, for $\mathcal{D} = \{D_1, \dots, D_4\}$, the embedded spectrum $\Sigma_{\mathcal{D}} \subset \mathbb{R}^4$. For $\tau = \tau_{m,n}$, the image $\Sigma_{\mathcal{D}}^\tau$ of Σ^τ under this embedding is the intersection of $\Sigma_{\mathcal{D}}$ with the affine two-dimensional subspace with equations $\xi_3 = n^2 + 2n$ and $\xi_4 = m$.

In Section 5, we describe $\Sigma_{\mathcal{D}}^\tau$ as the union of $n + 1$ half lines in the ξ_1, ξ_2 plane, exiting from the origin and only depending on n. Moreover, we study the structure of K-equivariant $\mathrm{End}(V_\tau)$-valued functions and obtain formula (1.2).

In Section 6 we prove Schwartz correspondence for $\mathcal{G}_\tau$ and that the inverse transforms $\mathcal{G}_\tau^{-1}$ satisfy norm estimates that grow at most polynomially in τ. These estimates are used in the subsequent section.

Finally, in Section 7 we produce our Whitney-type extension and prove condition (S').

We wish to thank the referee whose comments helped us to considerably improve the presentation of this work.

2. Gelfand pairs and strong Gelfand pairs

In this section we collect the main notation and conventions that will be used in the paper, recalling some basic facts at the same time.

2.1. Underlying notation and definitions

Let G be a Lie group, K a compact group and σ an action of K on G. We introduce the following notation.

- If $X(G)$ is a space of scalar-valued functions on G, $X(G)^{\sigma(K)}$ denotes the subspace of $\sigma(K)$-invariant elements. We simply write $X(G)^K$ if there is no ambiguity on the action σ.
- In particular, if $K \subseteq G$ and $\mathrm{Int}(K)$ denotes the group of conjugations by elements of K, $X(G)^{\mathrm{Int}(K)}$ is the space of K-*central* functions on G.

- A locally integrable function f on G is called *of K-type* τ, with $\tau \in \widehat{K}$, if $f = f *_K (d_\tau \overline{\chi_\tau})$, where d_τ and χ_τ are dimension and character of τ. The symbol $X(G)_\tau^{\mathrm{Int}(K)}$ denotes the subspace of K-central functions of K-type τ in $X(G)$.
- More generally, if V_π is a finite-dimensional representation space of K and $X(G, V_\pi)$ is a space of V_π-valued functions on G, we denote by $X(G, V_\pi)^K$ the space of K-equivariant elements F of $X(G, V_\pi)$, i.e., such that, for all $x \in G$ and $k \in K$, $F\big(\sigma(k)x\big) = \pi(k)F(x)$.
- By $\mathbb{D}(G)$ we denote the algebra of left-invariant differential operators on G. By $\mathbb{D}(G)^{\mathrm{Int}(K)} \cong \mathfrak{U}(\mathfrak{g})^{\mathrm{Ad}(K)}$ we denote the subalgebra of those which are also invariant under $\mathrm{Int}(K)$.
- By $\mathbb{D}(G/K) \cong \mathfrak{U}(\mathfrak{g})^{\mathrm{Ad}(K)}/\big(\mathfrak{U}(\mathfrak{g})^{\mathrm{Ad}(K)} \cap \mathfrak{U}(\mathfrak{g})\mathfrak{k}\big)$ we denote the algebra of G-invariant differential operators on G/K.

The following properties hold [15]:

(i) for each $\tau \in \widehat{K}$, $L^1(G)_\tau^{\mathrm{Int}(K)}$ is an algebra;
(ii) $\sum_\tau L^1(G)_\tau^{\mathrm{Int}(K)}$ is dense in $L^1(G)^{\mathrm{Int}(K)}$ and given $f = \sum\limits_\tau f_\tau$, $g = \sum\limits_\tau g_\tau$ with $f_\tau, g_\tau \in L^1(G)_\tau^{\mathrm{Int}(K)}$, then

$$f * g = \sum_\tau f_\tau * g_\tau \ ;$$

in particular, $f_\tau * g_{\tau'} = 0$ if $\tau \not\sim \tau'$;
(iii) with τ_0 denoting the trivial representation of K, $L^1(G)_{\tau_0}^{\mathrm{Int}(K)} = L^1(K\backslash G/K)$;
(iv) if $G = K \ltimes H$, the quotient G/K is naturally identified with H itself and, under this identification, $L^1(K\backslash G/K) \cong L^1(H)^K$, and $\mathbb{D}(G/K) \cong \mathbb{D}(H)^K$, the algebra of left- and K-invariant differential operators on H.

2.2. Gelfand pairs, strong Gelfand pairs

Definition 2.1. Let G be a Lie group and K a compact subgroup of G.

(a) (G, K) is called a *Gelfand pair* if the algebra $L^1(K\backslash G/K)$ is commutative;
(b) (G, K) is called a *strong Gelfand pair* if the algebra $L^1(G)^{\mathrm{Int}(K)}$ is commutative;
(c) for $\tau \in \widehat{K}$, (G, K, τ) is called *a commutative triple* if the algebra $L^1(G)_\tau^{\mathrm{Int}(K)}$ of K-central functions of K-type τ is commutative.

It follows from (iv) above that, if $K \subset G$ and acts on G as $\mathrm{Int}(K)$, then $\big(\mathrm{Int}(K) \ltimes G, \mathrm{Int}(K)\big)$ is a Gelfand pair if and only if (G, K) is a strong Gelfand pair.

In the literature it is more common to find the pair $\big(\mathrm{Int}(K) \ltimes G, \mathrm{Int}(K)\big)$ replaced by $(K \times G, \mathrm{diag}\, K)$. The two pairs are equivalent because the semi-direct product $K \ltimes G$ is isomorphic to the direct product $K \times G$ via the map $\iota : K \times G \to \mathrm{Int}(K) \ltimes G$ given by

$$\iota(k,g) = (k, gk^{-1}) ,$$

which identifies $\operatorname{diag} K \subset K \times G$ with $\operatorname{Int}(K) \ltimes \{e\}$.

As a consequence of (i)-(iii), we have the following implications:

$$\left.\begin{array}{c}(G,K) \text{ strong Gelfand pair}\\ \Updownarrow \\ (G,K,\tau) \text{ commutative triple for every } \tau\end{array}\right\} \Longrightarrow \left\{\begin{array}{c}(G,K) \text{ Gelfand pair}\\ \Updownarrow \\ (G,K,\tau_0) \text{ commutative triple}\end{array}\right. \tag{2.1}$$

Each of the three types of commutative structure listed in Definition 2.1 has its own kind of *spherical functions*, defined as the normalized joint eigenfunctions of the appropriate differential operators and with the appropriate invariance properties. Precisely,

(a') if (G,K) is a Gelfand pair, the bi-K-invariant eigenfunctions of all operators in $\mathbb{D}(G/K)$, taking value 1 at the identity element;
(b') if (G,K) is a strong Gelfand pair, the K-central eigenfunctions of all operators in $\mathbb{D}(G)^{\operatorname{Int}(K)}$, taking value 1 at the identity element;
(c') if (G,K,τ) is a commutative triple, the K-central eigenfunctions of K-type τ of all operators in $\mathbb{D}(G)^{\operatorname{Int}(K)}$, taking value 1 at the identity element.

2.3. Spherical transforms

We refer to [15] and the references therein for the material in this section. The *bounded* spherical functions defined in (a'), resp. (b'), (c'), determine the multiplicative functionals on the corresponding L^1 algebra in (a), resp. (b), (c), via the formula

$$f \longmapsto \int_G f(x)\varphi(x^{-1})\,dx \qquad (\varphi \text{ spherical}).$$

In each case, the bounded spherical functions form the *Gelfand spectrum* of the corresponding L^1 algebra. Each Gelfand spectrum is endowed with the weak* topology induced from $L^\infty(G)$, coinciding with the compact-open topology.

For given f, the map $\varphi \longmapsto \int_G f(x)\varphi(x^{-1})\,dx$, defined on the spectrum Σ, is the *spherical transform* of f in the given structure.

If $\mathcal{D} = \{D_1, \dots, D_\ell\}$ is a system of generators of the appropriate algebra of differential operators in (a')-(c'), we denote by $\Sigma_{\mathcal{D}} \subset \mathbb{C}^\ell$ the corresponding *embedded spectrum*, where each bounded spherical function is represented by the ℓ-tuple of its eigenvalues w.r. to the elements of $\mathcal{D}$.

Then Σ and $\Sigma_{\mathcal{D}}$ are homeomorphic and $\Sigma_{\mathcal{D}}$ is closed in $\mathbb{C}^\ell$ [6]. If G has polynomial growth and the ℓ generators are symmetric, then $\Sigma_{\mathcal{D}} \subset \mathbb{R}^\ell$ [3, Lemma 4.1].

Assume that (G,K) is a strong Gelfand pair, as we will do in the course of this paper. We denote by Σ its spectrum and, for $\tau \in \widehat{K}$, we denote by Σ^τ the Gelfand spectrum of the commutative triple (G, K, τ). In particular, Σ^{τ_0} is the Gelfand spectrum of the underlying (non-strong) Gelfand pair (G, K).

By [15, Proposition 7.3] each $\varphi \in \Sigma$ has a K-type, so that

$$\Sigma = \bigcup_{\tau \in \widehat{K}} \Sigma^\tau \ ,$$

where the union is disjoint and each term is open and closed.

By $\mathcal{G} : L^1(G)^{\mathrm{Int}(K)} \longrightarrow C_0(\Sigma)$ we denote the spherical transform of the strong Gelfand pair (G, K), i.e. of the (non-strong) pair $(\mathrm{Int}(K) \ltimes G, \mathrm{Int}(K))$. Then its restriction $\mathcal{G}_\tau$ from $L^1(G)^{\mathrm{Int}(K)}_\tau$ to $C_0(\Sigma^\tau)$ is the spherical transform of the commutative triple (G, K, τ), and $\mathcal{G}_{\tau_0}$ the spherical transform of the (non-strong) Gelfand pair (G, K).

2.4. Notation for U_2 and its irreducible representations

First of all, we denote by τ_n the irreducible representation of SU_2 of dimension $n + 1$ and by V_n the (abstract) representation space for τ_n. We will often use the realization of V_n as the space $\mathcal{P}^{(n,0)}(\mathbb{C}^2)$ of holomorphic polynomials on $\mathbb{C}^2$ that are homogeneous of degree n, with

$$\big[\tau_n(k)p\big](z) = p(k^{-1}z) \ , \qquad k \in SU_2 \ .$$

We then define, for $n \geq 0$ and $m \in n + 2\mathbb{Z}$, the representation $\tau_{m,n}$ of U_2 on V_n such that, for $k = e^{i\theta}k'$ with $k' \in SU_2$,

$$\tau_{m,n}(k) = e^{-im\theta}\tau_n(k') \ .$$

For $V_n = \mathcal{P}^{(n,0)}(\mathbb{C}^2)$, this takes the form

$$\big[\tau_{m,n}(k)p\big](z) = e^{-im\theta}p({k'}^{-1}z) = (\det k)^{(n-m)/2}\, p(k^{-1}z) \ , \qquad k \in U_2 \ . \tag{2.2}$$

In particular,

$$\big[\tau_{n,n}(k)p\big](z) = p(k^{-1}z) \ , \qquad k \in U_2 \ .$$

The set

$$E = \big\{(m, n) : n \geq 0 \ , \ n - m \in 2\mathbb{Z}\big\} \ , \tag{2.3}$$

parametrizes $\widehat{U_2}$ via (2.2).

We also fix the basis of $\mathfrak{su}_2$

$$X_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \ , \qquad X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ , \qquad X_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \ ,$$

and set $X_4 = iI$ to complete a basis of $\mathfrak{u}_2$.

Choosing $\mathfrak{t} = \mathbb{R}X_1$ as a maximal toral subalgebra of $\mathfrak{su}_2$, the elements $X_2 \pm iX_3$ of $\mathfrak{su}_2^{\mathbb{C}}$ are root vectors, relative to the roots $\mp 2i$ respectively.

Moreover, each monomial $z_1^j z_2^{n-j}$, $j = 0, \dots, n$ is a weight vector for the representation τ_n, with weight $(n-2j)i$. With respect to the Fischer inner product

$$\langle p, q \rangle_n = \frac{1}{n!}\, p(\partial_z) q^* \qquad \text{where} \quad q^*(z) = \overline{q(\bar{z})}\ ,$$

the normalized monomials

$$e_n^j(z) = \binom{n}{j}^{1/2} z_1^j z_2^{n-j} \qquad j = 0, \dots, n \tag{2.4}$$

form an orthonormal basis of V_n.

3. The complex motion group $M_2(\mathbb{C})$ and the strong Gelfand pair $\big(M_2(\mathbb{C}), U_2\big)$

The complex motion group $M_2(\mathbb{C})$ is the semidirect product $U_2 \ltimes \mathbb{C}^2$, where the action of U_2 on $\mathbb{C}^2$ is the natural one.

It is easy to check that $\big(M_2(\mathbb{C}), U_2, \tau_{m,n}\big)$ is a commutative triple for every m, n (see e.g. [15, Thm. 10.1, Cor. 10.4]). By (2.1), this gives

Proposition 3.1. *The pair $\big(M_2(\mathbb{C}), U_2\big)$ is a strong Gelfand pair.*

We write elements of $M_2(\mathbb{C})$ as pairs $(k, z) \in U_2 \times \mathbb{C}^2$ with product

$$(k, z)(k', z') = (kk', z + kz')\ .$$

The adjoint action of U_2 on the Lie algebra $\mathfrak{m}_2(\mathbb{C}) \cong \mathfrak{u}_2 \times \mathbb{C}^2$ is

$$\mathrm{Ad}(k)(U, z) = (kUk^{-1}, kz)\ ,$$

which splits $\mathfrak{m}_2(\mathbb{C})$ as $\mathfrak{su}_2 \times \mathbb{R}(iI) \times \mathbb{C}^2$. We decompose $U \in \mathfrak{u}_2$ as $X + itI$ with $X \in \mathfrak{su}_2$ and $2it = \operatorname{tr} U$ with $t \in \mathbb{R}$.

By [8, Thm. 7.5] we deduce the following.

Proposition 3.2.

(i) *The algebra of $\mathrm{Ad}(U_2)$-invariant polynomials on $\mathfrak{m}_2(\mathbb{C})$ is freely generated by the four polynomials*

$$p_1 = |z|^2\ , \qquad p_2 = z^* X z\ , \qquad p_3 = \det X = |X|^2\ , \qquad p_4 = t\ ,$$

where z is represented by the column vector $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

(ii) *The algebra* $\mathbb{D}(M_2(\mathbb{C}))^{\mathrm{Int}(U_2)}$ *is freely generated by*

$$D_1 = \Delta_z \ , \qquad D_3 = \Omega \ , \qquad D_4 = iX_4 \ ,$$

and

$$D_2 = i\big(\Delta_{z_2} - \Delta_{z_1}\big)X_1 - 4\big(\partial_{\bar z_1}\partial_{z_2} - \partial_{z_1}\partial_{\bar z_2}\big)X_2 + 4i\big(\partial_{z_1}\partial_{\bar z_2} + \partial_{\bar z_1}\partial_{z_2}\big)X_3 \ ,$$

where $\Omega = -X_1^2 - X_2^2 - X_3^3$ *is the Casimir operator on* SU_2 *and*

$$\Delta_z = \Delta_{z_1} + \Delta_{z_2} \ , \qquad \Delta_{z_i} = -4\partial_{z_i}\partial_{\bar z_i} \ .$$

Proof. For the proof of (i) see [8, Thm. 7.5].

In order to obtain generators D_j of $\mathbb{D}(M_2(\mathbb{C}))^{\mathrm{Int}(U_2)}$, we apply the symmetrization λ' in [15, formula (2.4)] to the p_j's. Allowing multiplication by scalar coefficients in order to obtain symmetric operators, positive when they have a sign, we set

$$D_1 = \Delta_z \ , \qquad D_3 = \Omega \ , \qquad D_4 = iX_4 \ ,$$

where $\Omega = -X_1^2 - X_2^2 - X_3^3$ is the Casimir operator on SU_2 and

$$\Delta_z = \Delta_{z_1} + \Delta_{z_2} \ , \qquad \Delta_{z_i} = -4\partial_{z_i}\partial_{\bar z_i} \ .$$

To obtain D_2, observe that, if $p(x,z) = \sum_j q_j(x)r_j(\mathrm{Re}\, z, \mathrm{Im}\, z)$, the symmetrization λ' on $M_2(\mathbb{C})$ is symmetrization on U_2 followed by symmetrization on $\mathbb{C}^2$ on each summand, i.e.,

$$\lambda'(p)f(k,z) = \sum_j r_j(\partial_{\mathrm{Re}\, w}, \partial_{\mathrm{Im}\, w})_{|w=0}\, q_j(\partial_x)_{|x=0}\, f\big((k,z)(e,w)(\exp_K x, 0)\big) \ . \qquad \square \tag{3.1}$$

Remark 3.3. It is worth noticing that $\big(SM_2(\mathbb{C}), SU_2\big)$, where $SM_2(\mathbb{C}) = SU_2 \ltimes \mathbb{C}^2$, is a Gelfand pair but not a strong one. To see this, one can use the representation theoretic argument in [15, Cor. 10.4] or, alternatively, observe that the polynomial

$$q(X,z) = {}^t z J X z \ , \qquad \text{where } J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

on $\mathfrak{su}_2 \times \mathbb{C}^2$ is SU_2-invariant, but its symmetrization

$$\lambda'(q) = -2i\partial_{z_1}\partial_{z_2}X_1 - (\partial_{z_1}^2 + \partial_{z_2}^2)X_2 + i(\partial_{z_1}^2 - \partial_{z_2}^2)X_3$$

does not commute with D_2.

4. K-type subalgebras and $\mathrm{End}(V_n)$-valued spherical analysis

This section deals with the isomorphisms between K-type subalgebras and their matrix-valued realizations. We start recalling some general facts, which can be found in [17, vol. II, Ch. 6] and in [5,15]. Then we specialize them to the case of the complex motion group. The aforementioned isomorphisms are extended to the differential operators in $\mathcal{D}$ to produce "$\mathrm{End}(V_n)$-valued" differential operators on $\mathbb{C}^2$ with a suitable invariance. We then show how these isomorphisms relate spherical functions of the appropriate commutative L^1-algebras. Finally, we determine the embeddings of spectra Σ^τ in $\mathbb{R}^2$.

4.1. General facts

Given a Lie group G together with a compact subgroup K and a representation $\tau \in \widehat{K}$, there is a one-to-one correspondence between K-central scalar-valued functions f on G of K-type τ and bi-τ-equivariant integrable functions F from G to $\mathrm{End}(V_\tau)$, i.e., verifying the identity

$$F(k_1 x k_2) = \tau(k_2^{-1}) F(x) \tau(k_1^{-1}) , \qquad \forall\, k_1, k_2 \in K \quad \forall x \in G. \tag{4.1}$$

This correspondence is given by

$$f \longmapsto F(x) = \int_K f(xk)\tau(k)\, dk , \qquad F \longmapsto f(x) = d_\tau \mathrm{tr}\, F(x) , \tag{4.2}$$

preserves integrability and respects convolution, once this is defined on $\mathrm{End}(V_\tau)$-valued functions as

$$F_1 * F_2(x) = \int_G F_2(y^{-1}x)\, F_1(y)\, dy .$$

Assume now that $G = K \ltimes H$ and denote by $h \mapsto kh$ the action of $k \in K$ on the group H. If F satisfies (4.1), its restriction $F_\flat$ to H satisfies the identity

$$F_\flat(kh) = F(e, kh) = F((k,e)(e,h)(k^{-1},e)) = \tau(k) F_\flat(h) \tau(k^{-1}) , \qquad \forall\, k \in K , \tag{4.3}$$

and completely determines F via (4.1).

In the semidirect product case, the correspondence in (4.2) between f and F takes the following explicit form:

$$f(k,h) = d_\tau \sum_{i,j} f_{ij}(h) \overline{(\tau(k))_{ij}}, \tag{4.4}$$

where $f_{ij}(h)$ are the components of $F(e,h)$ and $(\tau(k))_{ij}$ the coefficients of τ in some orthonormal basis of V_τ.

4.2. The special case $G = M_2(\mathbb{C})$, $K = U_2$

For notational convenience, from now on we will use the symbol G for $M_2(\mathbb{C})$ and K for U_2. It follows from the definition (2.2) of $\tau_{m,n}$ that the equivariance condition (4.3) does not depend on m. This allows us to introduce the representation $\tilde{\tau}_n$ of K on $\mathrm{End}(V_n)$ defined by

$$\tilde{\tau}_n(k)A = \tau_{m,n}(k)A\,\tau_{m,n}(k^{-1}) \qquad \forall k \in K, \quad \forall A \in \mathrm{End}(V_n) \tag{4.5}$$

and to write the equivariance condition (4.3) for a function $F : \mathbb{C}^2 \longrightarrow \mathrm{End}(V_n)$ in the form

$$F(e^{i\theta}kz) = \tilde{\tau}_n(e^{i\theta}k)F(z) = \tau_n(k)F(z)\tau_n(k^{-1}) \qquad \forall \theta \in \mathbb{R} \quad \forall\, k \in SU_2 \quad \forall z \in \mathbb{C}^2\ . \tag{4.6}$$

We denote by $L^1\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$, and similarly for other function spaces, the subspace of K-equivariant functions.

Adapting (4.2) to the case of the complex motion group we obtain the following

Lemma 4.1. *The two maps*

$$\begin{array}{rl} A_{m,n}: f(k,z) \longmapsto & F(z) = \tau_{m,n}\big(f(\cdot,z)\big) = \int_K f(k,z)\tau_{m,n}(k)\,dk \\ A_{m,n}^{-1}: \quad F(z) \longmapsto & f(k,z) = (n+1)\mathrm{tr}\,\big(\tau_{m,n}(k^{-1})F(z)\big) \end{array}$$

establish a one-to-one correspondence between locally integrable K-central functions f on G of K-type $\tau_{m,n}$ and locally integrable K-equivariant $\mathrm{End}(V_n)$-valued functions on $\mathbb{C}^2$.

In particular, $A_{m,n}$ is an isomorphism of algebras from $L^1(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$ onto $L^1\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ and $\sqrt{n+1}A_{m,n}$ is unitary from $L^2(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$ onto $L^2\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$, where

$$\|F\|_2^2 = \int_{\mathbb{C}^2} \big\|F(z)\big\|_{HS}^2\,dz\ . \tag{4.7}$$

Remark 4.2. It follows from Lemma 4.1 that the algebras $L^1(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$ are all isomorphic to $L^1\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$, which is independent from m. This is coherent with the fact that, for every $m \in \mathbb{Z}$, the map

$$\mu_{m,n} : f(k,z) \longmapsto (\det k)^{(m-n)/2} f(k,z)$$

is an isomorphism from $L^1(G)^{\mathrm{Int}(K)}_{\tau_{n,n}}$ onto $L^1(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$ which intertwines the action of G and satisfies $A_{m,n} \circ \mu_{m,n} = A_{n,n}$.

Remark 4.3. For $n = 0$, and in particular for the trivial representation $\tau_{0,0}$, Lemma 4.1 establishes the trivial fact that K-invariant (scalar) functions on $\mathbb{C}^2$ coincide with restrictions to $\{e\} \times \mathbb{C}^2$ of bi-K-invariant functions on G and that, via this identification, $L^1(K\backslash G/K)$ is isomorphic to $L^1(\mathbb{C}^2)^K$.

4.3. Equivariant differential operators

We denote by $\big(\mathbb{D}(\mathbb{C}^2) \otimes \mathrm{End}(V_n)\big)^K$ the algebra of "End(V_n)-valued" differential operators on $\mathbb{C}^2$ which commute with translations and with the action of U_2 on smooth End(V_n)-valued functions F on $\mathbb{C}^2$ given by

$$k : F \longmapsto F^k(z) = \tau_{m,n}(k)F(k^{-1}z)\tau_{m,n}(k)^{-1} \ .$$

We recall the linear symmetrization $\lambda' : \mathcal{P}(\mathfrak{u}_2 \times \mathbb{C}^2) \longrightarrow \mathbb{D}(G)$ defined in (3.1) for polynomials in separate variables. The following statement, proved in [15, Cor. 2.3 and Prop. 2.4], gives the conjugation formula for $\lambda'(p)$ under $A_{m,n}$.

Lemma 4.4. *Let $p(x, z) = \sum_j q_j(x)r_j(z)$ be a polynomial on $\mathfrak{u}_2 \times \mathbb{C}^2$. Defining $\check{q}_j(x) = q_j(-x)$, we have the identity*

$$A_{m,n}\lambda'(p)A_{m,n}^{-1} = \sum_j r_j(\partial) \otimes d\tau_{m,n}\big(\lambda_K(\check{q}_j)\big) \ .$$

Conjugation by $A_{m,n}$ is a homomorphism of $\mathbb{D}(G)^{\mathrm{Int}(K)}$ onto $\big(\mathbb{D}(\mathbb{C}^2) \otimes \mathrm{End}(V_n)\big)^K$ and its kernel consists of the operators which vanish on functions of K-type $\tau_{m,n}$.

In particular, this lemma establishes the correspondence

$$\begin{aligned} D_1 &\longleftrightarrow \Delta_z \otimes I \\ D_2 &\longleftrightarrow i\big(\Delta_{z_2} - \Delta_{z_1}\big) \otimes d\tau_n(X_1) - 4\big(\partial_{\bar{z}_1}\partial_{z_2} - \partial_{z_1}\partial_{\bar{z}_2}\big) \otimes d\tau_n(X_2) \\ &\qquad\qquad + 4i\big(\partial_{z_1}\partial_{\bar{z}_2} + \partial_{\bar{z}_1}\partial_{z_2}\big) \otimes d\tau_n(X_3) \\ D_3 &\longleftrightarrow d\tau_{m,n}(\Omega) = 1 \otimes (n^2 + 2n)I \\ D_4 &\longleftrightarrow d\tau_{m,n}(iX_4) = 1 \otimes mI \ . \end{aligned} \tag{4.8}$$

Since the operators $D_1, \ldots, D_4$ generate $\mathbb{D}(G)^{\mathrm{Int}(K)}$, by Lemma 4.4 and (4.8), we conclude that $\big(\mathbb{D}(\mathbb{C}^2) \otimes \mathrm{End}(V_n)\big)^K$ is generated (as algebra) by the operators

$$A_{m,n}D_1A_{m,n}^{-1} = \Delta_z \ , \qquad A_{m,n}D_2A_{m,n}^{-1} \stackrel{\mathrm{def}}{=} \mathbf{D}_n \ . \tag{4.9}$$

Since $X_1, X_2, X_3 \in \mathfrak{su}_2$, the operator $A_{m,n}D_2A_{m,n}^{-1}$ does not depend on m.
For completeness we show that Δ_z and $\mathbf{D}_n$ are not free generators.

Lemma 4.5. *The operators* $\Delta_z^j \mathbf{D}_n^k$ *with* $j \in \mathbb{N}$ *and* $0 \le k \le n$ *form a basis of* $\big(\mathbb{D}(\mathbb{C}^2) \otimes \mathrm{End}(V_n)\big)^K$.

Proof. Taking Fourier transform in z, the symbol $\widehat{\mathbf{D}}_n$ of $\mathbf{D}_n$ can be expressed as an $(n+1) \times (n+1)$ matrix with polynomial entries in the dual variable ζ. The coefficients $q_{n,k}$ of the characteristic equation

$$\det\big(\lambda I - \widehat{\mathbf{D}}_n(\zeta)\big) = \lambda^{n+1} + \sum_{k=0}^{n} \lambda^k q_{n,k}(\zeta) = 0 \tag{4.10}$$

are K-invariant polynomials, hence polynomials $p_{n,k}$ in $|\zeta|^2$. Applying the Cayley-Hamilton theorem and undoing Fourier transform,

$$\mathbf{D}_n^{n+1} = \sum_{k=0}^{n} \mathbf{D}_n^k p_{n,k}(\Delta_z) \ .$$

It remains to prove that $\widehat{\mathbf{D}}_n$ does not solve any equation of smaller degree in λ than (4.10). This follows from the fact that, by (4.8),

$$\widehat{\mathbf{D}}_n(0,1) = id\tau_n(X_1) = \mathrm{diag}(-n, \dots, -n+2\ell, \dots, n) \ , \tag{4.11}$$

and the matrix has $n+1$ distinct eigenvalues. □

4.4. End(V_n)*-valued spherical functions*

As recalled in Section 2.3 the characters of $L^1(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$ are given by integration against the bounded spherical functions of K-type $\tau_{m,n}$.

Since, by Lemma 4.1, $A_{m,n}$ is an algebra isomorphism, one can see that the characters of $L^1\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ have the form

$$F \longrightarrow (n+1) \int_{\mathbb{C}^2} \mathrm{tr}\,\big(F(z) A_{m,n}\varphi(-z)\big)\, dz \ , \tag{4.12}$$

where φ is a bounded spherical function of the strong Gelfand pair (G, K) of K-type $\tau_{m,n}$.

It is easy to prove that $\varphi(k, 0) = \frac{1}{n+1} \overline{\chi_{\tau_{m,n}}(k)}$, so that $A_{m,n}\varphi(0) = \frac{1}{(n+1)^2} I$.

Definition 4.6. We call End(V_n)*-valued spherical functions* the functions

$$\Phi(z) = (n+1)^2 (A_{m,n}\varphi)(z) \ , \tag{4.13}$$

where φ is a spherical function of the strong Gelfand pair (G, K) of K-type $\tau_{m,n}$.

More frequently, cf. [17, vol. II, ch. 6], these are called $\tau_{m,n}$-spherical functions and the scalar-valued ones $\tau_{m,n}$-*spherical trace* functions.

By Remark 4.2, the definition does not depend on m. More precisely, φ is a spherical function of K-type $\tau_{n,n}$ if and only if $\mu_{m,n}\varphi$ is a spherical function of K-type $\tau_{m,n}$. In this case $A_{m,n}\mu_{m,n}\varphi = A_{n,n}\varphi$.

Moreover, comparing (4.12) and (4.13) we conclude that the bounded $\mathrm{End}(V_n)$-valued spherical functions are the K-equivariant functions Φ which define nontrivial multiplicative functionals on $L^1\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ through the formula

$$F \longmapsto \frac{1}{n+1}\int\limits_{\mathbb{C}^2} \mathrm{tr}\,\big(F(z)\Phi(-z)\big)\, dz\ . \tag{4.14}$$

Finally, by [15, Corollary 6.2], the $\mathrm{End}(V_n)$-valued spherical functions are characterized by the property of being the joint eigenfunctions Φ of Δ_z and $\mathbf{D}_n$ which are K-equivariant, with $\Phi(0) = I$.

4.5. Embedded spectra

We fix here the notation for the embeddings of spectra mentioned in the introduction. In Proposition 3.2, we have fixed a system $\mathcal{D} = \{D_1, D_2, D_3, D_4\}$ of generators of $\mathbb{D}(G)^K$. By [13], the D_j, that we have chosen to be symmetric, are essentially self-adjoint. As mentioned in the Introduction, this has the advantage that the eigenvalues of the spherical functions are all real.

Given a bounded spherical function φ of the strong Gelfand pair (G, K), the symbol $\xi(\varphi)$ stands for the quadruple $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$ of eigenvalues of φ with respect to D_1, D_2, D_3, D_4 respectively. Moreover we recall that $\Sigma_{\mathcal{D}}$ is the embedded spectrum

$$\Sigma_{\mathcal{D}} = \big\{\xi(\varphi) \in \mathbb{R}^4\ :\ \varphi \in \Sigma\big\}\ .$$

By (4.8), if φ is of K-type $\tau_{m,n}$ then

$$\xi(\varphi) = (\xi_1, \xi_2, n^2 + 2n, m)\ ;$$

moreover, by Lemma 4.4 the first two components ξ_1, ξ_2 are the eigenvalues of $\Phi = (n+1)^2 A_{m,n}\varphi$ under the action of Δ_z and $\mathbf{D}_n$ respectively. In accordance with [3, Sect. 6], we set

$$\Sigma^n_{\mathcal{D}} = \big\{(\xi_1, \xi_2)\ :\ (\xi_1, \xi_2, n^2 + 2n, m) \in \Sigma_{\mathcal{D}}\big\}\ ,$$

which is independent of m by Remark 4.2 and will be described in Proposition 5.4. Finally,

$$\Sigma_{\mathcal{D}} = \bigcup_{(m,n)\in E} \Sigma^n_{\mathcal{D}} \times \big\{(n^2 + 2n, m)\big\}\ , \tag{4.15}$$

where $E = \big\{(m,n) : n \geq 0\,,\, n-m \in 2\mathbb{Z}\big\}$ has been introduced in Section 2.4.

Remark 4.7. Since the type τ of a given spherical function is identified by (ξ_3, ξ_4), rapid decay in $\tau = \tau_{m,n}$ is to be understood as rapid decay in $(\xi_3, \xi_4) = (n^2 + 2n, m)$, therefore in (n, m).

Coherently with the notation in Section 2.3, we shall consider the spherical transform $\mathcal{G}_{\tau_{m,n}} f$, of $f \in L^1(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$, as a function defined on $\Sigma^n_{\mathcal{D}} \subset \mathbb{R}^2$. For better clarity, we will use the slightly different notation $\mathcal{G}_n$ for the spherical transform of $L^1\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ defined by (4.14).

In order to prove Schwartz correspondence for each K-type spherical transform $\mathcal{G}_{\tau_{m,n}}$, it is convenient to adopt the $\mathrm{End}(V_n)$-valued model, replacing $\mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$ with $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$. This allows us to take advantage of the algebraic structure of $\mathrm{End}(V_n)$ and to completely disregard the parameter m.

5. Gelfand spectrum of $L^1\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ and decomposition of $\mathrm{End}(V_n)$-valued functions

In this section we study $\mathrm{End}(V_n)$-valued K-equivariant functions, i.e., functions $F : \mathbb{C}^2 \longrightarrow \mathrm{End}(V_n)$ satisfying (4.6). A crucial rôle will be played by the operator $\mathbf{D}_n$ defined in (4.9).

The main results are an explicit formula for the spherical functions and their eigenvalues (Proposition 5.4) and the decomposition (1.2) of a function $F \in \mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ as the sum of terms of the form $\mathbf{D}_n^j g_j$ where the scalar valued functions g_j are in $\mathcal{S}(\mathbb{C}^2)^K$ (Corollary 5.6).

To achieve these results, the first ingredient will be the decomposition (5.3) of $\mathrm{End}(V_n)$ into its invariant irreducible subspaces relative to the representation $\tilde{\tau}_n$ defined in (4.5).

Denoting by

$$B_n^1 = \widehat{\mathbf{D}}_n(0,1) = id\tau_n(X_1) = \mathrm{diag}(-n, \dots, -n+2\ell, \dots, n) \tag{5.1}$$

the matrix introduced in (4.11), we will determine which polynomials in B_n^1 belong to the irreducible subspaces of $\mathrm{End}(V_n)$.

Via Fourier transform, we will then be able to describe the equivariant Schwartz functions taking values in each irreducible subspace of $\mathrm{End}(V_n)$.

Throughout this section we exploit the following basic fact, which explains the importance of diagonal matrices in our analysis.

Recall the diffeomorphism

$$z = (z_1, z_2) \longmapsto k_z = \begin{bmatrix} \bar{z}_2 & z_1 \\ -\bar{z}_1 & z_2 \end{bmatrix} \tag{5.2}$$

which identifies the unit sphere S^3 with the group SU_2 and denote by $\mathbf{o}$ the base point $(0,1)$.

Then we can write any $z \in \mathbb{C}^2$ as $z = rz'$ with $r \in \mathbb{R}$ and $z' \in S^3$ and when $F : \mathbb{C}^2 \longrightarrow \mathrm{End}(V_n)$ is a K-equivariant function, by (4.6)

$$F(z) = F(rk_{z'}\mathbf{o}) = \tau_n(k_{z'})\,F(r\mathbf{o})\,\tau_n(k_{z'}^{-1}).$$

Since $\mathbf{o}$ is stable under the action of matrices of the form $\begin{pmatrix} e^{2i\theta} & 0 \\ 0 & 1 \end{pmatrix} = e^{i\theta}\exp(\theta X_1)$, we also have that

$$\begin{aligned} F(r\mathbf{o}) &= F(r\,e^{i\theta}\exp(\theta X_1)\mathbf{o}) \\ &= \tau_n(\exp(\theta X_1))\,F(r\mathbf{o})\,\tau_n(\exp(-\theta X_1)) \end{aligned} \qquad \forall r,\theta \in \mathbb{R}\ .$$

Differentiating with respect to θ we obtain that the matrix $F(r\mathbf{o})$ commutes with the matrix $d\tau_n(X_1)$, which is diagonal with distinct eigenvalues. We conclude that the matrix $F(r\mathbf{o})$ is diagonal. Moreover, by equivariance,

$$F(-r\mathbf{o}) = F(r\mathbf{o})$$

and the matrices $F(r\mathbf{o})$ with $r \geq 0$ determine F uniquely.

5.1. *Decomposition of* End(V_n)

Identifying $\mathrm{End}(V_n)$ with $V_n' \otimes V_n$, the representation $\tilde{\tau}_n$ in (4.5) is equivalent to the tensor product $\tau_{m,n}' \otimes \tau_{m,n}$, where $\tau_{m,n}' \sim \tau_{-m,n}$ is the contragredient representation of $\tau_{m,n}$.

Since the centre of $K = U(2)$ acts trivially on $\mathrm{End}(V_n)$, we restrict $\tilde{\tau}_n$ to SU_2. We have

$$\tilde{\tau}_{n\,|_{SU_2}} \sim \tau_n' \otimes \tau_n \ \sim\ \tau_{2n} \oplus \tau_{2n-2} \oplus \cdots \oplus \tau_2 \oplus \tau_0\ ,$$

and, correspondingly,

$$\mathrm{End}(V_n) = \mathcal{W}_n^n \oplus \mathcal{W}_n^{n-1} \oplus \cdots \oplus \mathcal{W}_n^0\ , \tag{5.3}$$

where $\dim \mathcal{W}_n^\ell = 2\ell + 1$.

Since the matrix B_n^1 defined in (5.1) has distinct eigenvalues, any diagonal matrix $B = \mathrm{diag}(b_0, b_1, \ldots, b_n)$, can be written as a polynomial in B_n^1. Indeed, there is a unique polynomial p of degree at most n such that $p(-n+2\ell) = b_\ell$, $\ell = 0, \ldots, n$. Then $B = p\big(B_n^1\big)$.

Lemma 5.1. *For every $\ell = 0, \ldots, n$, let $\mathcal{B}_n^\ell$ be the subspace of diagonal matrices in $\mathcal{W}_n^\ell$.*

(i) The subspace $\mathcal{B}_n^\ell$ is one-dimensional.
(ii) The subspace $\mathcal{B}_n^1$ consists of the scalar multiples of B_n^1.
(iii) For general ℓ, there is a monic polynomial q_n^ℓ of degree ℓ such that $\mathcal{B}_n^\ell$ consists of the scalar multiples of the matrix

$$B_n^\ell = q_n^\ell(B_n^1). \tag{5.4}$$

(iv) For a polynomial p of degree at most n, $p(B_n^1) \in \sum_{\ell\le j} \mathcal{W}_n^\ell$ if and only if $\deg(p) \le j$.

Proof. Since $d\tau_n(X_1)$ has distinct eigenvalues, diagonal matrices are those that commute with $d\tau_n(X_1)$. Since $\tilde{\tau}_n$ restricted to $\mathcal{W}_n^\ell$ is equivalent to $\tau_{2\ell}$, it contains the null weight with multiplicity one, and this proves (i).

Consider now the $\tilde{\tau}_n$-invariant subspace $\mathcal{W}$ generated by $d\tau_n(X_1)$. Then

$$\mathcal{W} = \operatorname{span}_{\mathbb{C}}\{\tau_n(k)d\tau_n(X_1)\tau_n(k)^{-1} \,:\, k \in SU_2\} = d\tau_n(\mathfrak{su}_2^{\mathbb{C}})\ ,$$

which is a 3-dimensional invariant subspace. So it must coincide with the component $\mathcal{W}_n^1$ in (5.3). This proves (ii).

The statements (iii) and (iv) are trivial for $\ell = 0$ and have been proved above for $\ell = 1$. For $\ell \ge 2$, item (iv) follows by induction from the fact that $\tilde{\tau}_n|_{\mathcal{W}_n^{\ell-1}} \sim \tau_{2\ell-2}$ and the decomposition

$$\tau_{2\ell-2} \otimes \tau_2 \sim \tau_{2\ell} \oplus \tau_{2\ell-2} \oplus \tau_{2\ell-4}$$

in irreducible summands with multiplicity one for $\tau_{2\ell}$. Finally, if $p(t) = t\, q_n^{\ell-1}(t)$, then p is monic, has degree ℓ and $p(B_n^1)$ is a diagonal matrix in $\mathcal{W}_n^\ell \oplus \cdots \oplus \mathcal{W}_n^0$ with a nontrivial component in $\mathcal{W}_n^\ell$, that we call $q_n^\ell(B_n^1)$. Hence $(p - q_n^\ell)(B_n^1)$ is in $\sum_{j\in L} \mathcal{W}_n^j$, which, by the inductive hypothesis, is a polynomial in B_n^1 of degree at most $\ell - 1$. Then q_n^ℓ has degree ℓ and its leading term is the same as p, so it is monic. □

5.2. Equivariant polynomials

Suppose now $P : \mathbb{C}^2 \to \operatorname{End}(V_n)$ is a K-equivariant polynomial. Then the homogeneous component of P of bi-degree (d_1, d_2) is also equivariant, and trivial if $d_1 \ne d_2$.

Assume therefore that $P \in \mathcal{P}^{d,d}$, i.e., homogeneous of bi-degree (d, d). By homogeneity, P is uniquely determined by its restriction to the unit sphere. Moreover, for what we have seen at the beginning of this section, the matrix $B = P(\mathbf{o})$ is diagonal and

$$P(z) = |z|^{2d}\tau_n(k_{z'})B\tau_n(k_{z'}^{-1}) \tag{5.5}$$

where $z' = z/|z|$ for $z \ne (0,0)$ and the matrix $k_{z'}$ is defined in formula (5.2).

Conversely, given $B = \operatorname{diag}(b_0, b_1, \ldots, b_n)$, we will determine for what values of d formula (5.5) defines a polynomial. The answer to this question goes together with the issue of describing the equivariant polynomials taking values in a given $\mathcal{W}_n^\ell$. In this respect the following remarks are quite obvious, after Lemma 5.1, for an $\mathrm{End}(V_n)$-valued equivariant polynomial P:

- P can be uniquely decomposed as the sum of $\mathcal{W}_n^\ell$-valued ones;
- P takes values in $\mathcal{W}_n^\ell$ if and only if $P(\mathbf{o}) \in \mathcal{W}_n^\ell$.

For $B = \mathrm{diag}(b_0, b_1, \dots, b_n)$ we denote by $d(B) \le n$ the degree of the polynomial p such that $B = p(B_n^1)$. From Lemma 5.1, it follows that $d(B) \le j$ if and only if $B \in \sum_{\ell \le j} \mathcal{W}_n^\ell$.

For B as above, and d in $\mathbb{N}$, we set

$$Q_B^d(z) = |z|^{2d}\tau_n(k_{z'})B\tau_n(k_{z'}^{-1}) \ , \qquad z = |z|z' \ . \tag{5.6}$$

Lemma 5.2.

(i) For a diagonal B, the function Q_B^d can be continued to an $\mathrm{End}(V_n)$-valued equivariant polynomial if and only if $d \ge d(B)$. In this case, the polynomial Q_B^d is homogeneous of bi-degree (d, d).

(ii) Every $\mathcal{W}_n^\ell$-valued equivariant polynomial has the form $p\big(|z|^2\big)Q_{B_n^\ell}^\ell(z)$, where p is a scalar-valued polynomial in one variable.

Proof. (i) Since any diagonal matrix B is a linear combination of the matrices B_n^ℓ, it is enough to treat the case where $B = B_n^\ell = q_n^\ell(B_n^1)$.

Assume that, for a given d, $Q_{B_n^\ell}^d$ extends to a polynomial. Recalling that $\mathcal{W}_n^\ell \sim V_{2\ell}$ and denoting by $\mathcal{I}_n^\ell : \mathcal{W}_n^\ell \to V_{2\ell}$ a unitary operator intertwining $\tilde{\tau}_n$ and $\tau_{2\ell}$, let $\tilde{Q}_d$ be the $V_{2\ell}$-valued polynomial defined by the rule

$$\tilde{Q}_d(z) = \mathcal{I}_n^\ell(Q_{B_n^\ell}^d(z)) \qquad \forall z \in \mathbb{C}^2 \ .$$

Then $\tilde{Q}_d$ is $\tau_{2\ell}$-equivariant and it suffices to prove the necessity of the condition $d \ge \ell$ for $\tilde{Q}_d$. Since $Q_{B_n^\ell}^d(\mathbf{o}) = \tilde{\tau}_n(\exp tX_1)Q_{B_n^\ell}^d(\mathbf{o})$ we have

$$\tilde{Q}_d(\mathbf{o}) = \mathcal{I}_n^\ell(\tilde{\tau}_n(\exp tX_1)Q_{B_n^\ell}^d(\mathbf{o})) = \tau_{2\ell}(\exp tX_1)\tilde{Q}_d(\mathbf{o}) \ ,$$

so that $\tilde{Q}_d(\mathbf{o})$ is a 0-weight vector for $\tau_{2\ell}$. In the polynomial model of Section 2.4, $\tilde{Q}_d(\mathbf{o}) \in V_{2\ell}$ has then the form

$$\big[\tilde{Q}_d(\mathbf{o})\big](w) = c\, w_1^\ell\, w_2^\ell$$

for some constant c. It follows that, if $|z'| = 1$,

$$\big[\tilde{Q}_d(z')\big](w) = \big[\tau_{2\ell}(k_{z'})\tilde{Q}_d(\mathbf{o})\big](w) = c\,(z_2'w_1 - z_1'w_2)^\ell(\bar{z}_1'w_1 + \bar{z}_2'w_2)^\ell \ ,$$

and the homogeneous extension of bi-degree (d, d) with z in $\mathbb{C}^2$ is

$$\big[\tilde{Q}_d(z)\big](w) = c|z|^{2(d-\ell)}(z_2w_1 - z_1w_2)^\ell(\bar{z}_1w_1 + \bar{z}_2w_2)^\ell \ .$$

It is a polynomial in z if and only if $d \geq \ell$.

As for ii), if P is a polynomial, then P is $\mathcal{W}_n^\ell$-valued if and only if $P(\mathbf{o})$ is in $\mathcal{W}_n^\ell$. Therefore $P(\mathbf{o})$ is a constant multiple of B_n^ℓ. □

Lemma 5.2 gives a recipe to find a new basis for $\mathcal{P}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ and, at the same time, it proves that the operators Δ_z and $\mathbf{D}_n$ generate the algebra $\big((\mathbb{D}(\mathbb{C}^2)\otimes\mathrm{End}(V_n)\big)^K$, independently of [8]. We state these facts in Corollary 5.3, where we use the following notation. Let $\widehat{F}$ be the Fourier transform of $F \in L^1\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$, defined componentwise by

$$\widehat{F}(\zeta) = \int\limits_{\mathbb{C}^2} F(z)\, e^{-i\langle z,\zeta\rangle}\, dz$$

and let P be an $\mathrm{End}(V_n)$-valued polynomial on $\mathbb{C}^2$. Then $P(\partial)$ is the operator defined by the rule

$$\widehat{P(\partial)F}(\zeta) = P(\zeta)\widehat{F}(\zeta) \qquad \zeta \in \mathbb{C}^2.$$

In particular, if $P(z) = z^\alpha\, \bar{z}^\beta I$, then $P(\partial) = (-2i\partial_{\bar{z}})^\alpha\, (-2i\partial_z)^\beta I$.

Corollary 5.3.

(i) *The polynomials $Q^\ell_{B_n^\ell}$, where $\ell = 0, 1, \ldots, n$, form a basis of $\mathcal{P}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ as $\mathcal{P}(\mathbb{C}^2)^K$-module.*

(ii) *A set of generators of the algebra $\big(\mathbb{D}(\mathbb{C}^2) \otimes \mathrm{End}(V_n)\big)^K$ is*

$$\mathcal{D} = \Big\{\Delta_z = Q^1_{B_n^0}(\partial), \quad \mathbf{D}_n = Q^1_{B_n^1}(\partial)\Big\}.$$

5.3. $\mathrm{End}(V_n)$*-valued spherical functions*

For $\xi \geq 0$ we denote by φ_ξ the spherical function of (G, K), understood as a Gelfand pair, with eigenvalue ξ relative to Δ_z. It is well known that it can be expressed in terms of the Bessel function J_1 but, for our purposes, it will be enough to consider its expression as

$$\varphi_\xi(z) = \int\limits_{S^3} e^{-i\sqrt{\xi}\,\langle z,\zeta\rangle}\, d\sigma(\zeta) \qquad \forall z \in \mathbb{C}^2\ , \tag{5.7}$$

where σ is the normalized surface measure of the unit sphere S^3.

Proposition 5.4. *The spectrum $\Sigma^n_{\mathcal{D}}$ is the union of $n+1$ half-lines,*

$$\Sigma^n_{\mathcal{D}} = \left\{ \big(\xi, (-n+2j)\xi\big) \,:\, \xi \geq 0\,,\ j = 0, \ldots, n \right\} .$$

If $\xi = 0$, the only pair of eigenvalues $(0,0)$ is attained by the constant spherical function $\Phi_{0,0}(z) = I$.

For $\xi > 0$, the spherical function corresponding to the pair of eigenvalues $\big(\xi, (-n+2j)\xi\big)$ is

$$\Phi_{\xi,j}(z) = (n+1) \int\limits_{S^3} e^{-i\sqrt{\xi}\,\langle z,\zeta\rangle} Q^n_{E_{jj}}(\zeta)\, d\sigma(\zeta), \tag{5.8}$$

where E_{jj} is the matrix with null entries except the jj-entry which equals 1.

Proof. By [15, (11.2) and Thm. 11.1], the bounded spherical function of the triple $(G, K, \tau_{n,n})$ can be constructed according to the following recipe.

Fix $r\mathbf{o}$, with $r = \sqrt{\xi} \geq 0$, as base point in the K-orbit rS^3 in $\mathbb{C}^2$, let K_r be the stabilizer of $r\mathbf{o}$ in K, decompose V_n into its (inequivalent) irreducible components $W_{r,j}$ under K_r, and define

$$\Phi_{\xi,j}(z) = \frac{n+1}{\dim W_{r,j}} \int\limits_K e^{-ir\langle kz,\mathbf{o}\rangle} \tau_{n,n}(k^{-1}) P_{r,j} \tau_{n,n}(k)\, dk\ ,$$

where $P_{r,j}$ is the orthogonal projection onto $W_{r,j}$.

If $r = 0$, $K_r = K$ and we obtain constant function, $\Phi_0(z) = I$.

If $r > 0$, K_r is the torus of diagonal matrices $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}$, so each $W_{r,j}$ is the one-dimensional span of e^j_n in (2.4), for $j = 0, \ldots, n$.

Consequently, $P_{r,j}$ is represented by the matrix E_{jj} in the basis $\{e^j_n\}$ and, by (5.5) and (5.7) for $r = \sqrt{\xi}$ we obtain

$$\begin{aligned} \Phi_{\xi,j}(z) &= (n+1) \int\limits_K e^{-ir\langle kz,\mathbf{o}\rangle} \tau_{n,n}(k^{-1}) E_{jj} \tau_{n,n}(k)\, dk \\ &= (n+1) \int\limits_K e^{-ir\langle z,k^{-1}\mathbf{o}\rangle} Q^n_{E_{jj}}(k^{-1}\mathbf{o})\, dk \\ &= (n+1) \int\limits_{S^3} e^{-ir\langle z,\zeta\rangle} Q^n_{E_{jj}}(\zeta)\, d\sigma(\zeta)\ . \end{aligned}$$

This proves that $\Phi_{\xi,0}, \ldots \Phi_{\xi,n}$ are the bounded spherical functions whose eigenvalue relative to Δ_z is ξ. It remains to determine the eigenvalue relative to $\mathbf{D}_n$ for each of them. Noting that

$$\Phi_{\xi,j}(z) = (n+1)\big(Q^n_{E_{jj}}(\partial)\varphi_1\big)(\sqrt{\xi}z) = (n+1)\,\xi^{-n}\,\big(Q^n_{E_{jj}}(\partial)\varphi_\xi\big)(z)$$

and taking into account that $B^1_n E_{jj} = (-n+2j)E_{jj}$, we have

$$\begin{aligned}
\mathbf{D}_n \Phi_{\xi,j} &= (n+1)\,\xi^{-n}\,Q^1_{B^1_n}(\partial)\,Q^n_{E_{jj}}(\partial)\varphi_\xi \\
&= (n+1)\,\xi^{-n}\,Q^{n+1}_{B^1_n E_{jj}}(\partial)\varphi_\xi \\
&= (n+1)\,\xi^{-n}\,(-n+2j)\,Q^{n+1}_{E_{jj}}(\partial)\varphi_\xi \\
&= (n+1)\,\xi^{-n}\,(-n+2j)\,Q^{n}_{E_{jj}}(\partial)\Delta_z\varphi_\xi \\
&= (-n+2j)\xi\,(n+1)\,\xi^{-n}\,Q^{n}_{E_{jj}}(\partial)\varphi_\xi \\
&= (-n+2j)\xi\,\Phi_{\xi,j}\ . \quad \square
\end{aligned}$$

5.4. End(V_n)–*valued equivariant functions as derivatives of scalar valued functions*

In this subsection we obtain the decomposition (1.2) for functions in $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$.

Suppose that F is an End(V_n)–valued equivariant function. Then we can decompose F into the sum

$$F = \sum_{\ell=0}^{n} F_\ell$$

where each F_ℓ is $\mathcal{W}^\ell_n$–valued. We are going to prove that $\mathcal{W}^\ell_n$-valued functions turn out to be of a special form.

For our purposes it is convenient to consider on $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ the following family of norms

$$\|F\|_{(M)} = \max_{0\le q\le M} \|(1+|\cdot|^2)^M\,\Delta^q_z F\|_2, \tag{5.9}$$

where the L^2 norm of End(V_n)–valued functions is defined in (4.7).

For the sake of brevity, we denote by Q^ℓ_n the polynomial $Q^\ell_{B^\ell_n}$ defined in (5.6).

Proposition 5.5. *Let F be in $\mathcal{S}(\mathbb{C}^2, \mathcal{W}^\ell_n)^K$. Then there exists a unique scalar valued $g \in \mathcal{S}(\mathbb{C}^2)^K$ such that*

$$F = Q^\ell_n(\partial)g = q^\ell_n(\mathbf{D}_n)g\ .$$

Moreover, for any M there exists $M' \ge M+\ell$ such that

$$\|g\|_{(M)} \le C_M\,\|F\|_{(M')}\ .$$

Proof. As explained at the beginning of this section, for every $r \in \mathbb{R}$ the matrix $F(r\mathbf{o})$ is diagonal and equal to $F(-r\mathbf{o})$.

As $F(r\mathbf{o})$ is in $\mathcal{W}_n^\ell$, by Lemma 5.1(i) we can write $F(r\mathbf{o}) = f(r)\, B_n^\ell$ for some scalar $f(r)$. Clearly the so-obtained function f is in $\mathcal{S}(\mathbb{R})$ and even.

Suppose that P_d is the homogeneous term of degree d in the Taylor expansion of F centred at the origin. Then P_d is K-equivariant and $\mathcal{W}_n^\ell$-valued. It follows from Lemma 5.2 (i) that $P_d = 0$ if $d < 2\ell$ and $P_{2\ell}$ is a constant multiple of Q_n^ℓ. Hence

$$f(r) = c_\ell\, r^{2\ell} + o(r^{2\ell}) \qquad r \to 0 \ .$$

By Hadamard's division Lemma [14] there exists an even smooth function h_0 on $\mathbb{R}$ such that $f(r) = r^{2\ell}\, h_0(r)$ and $h_0(0) = c_\ell$, therefore, for $z = |z|z'$,

$$F(z) = h_0(|z|)\, |z|^{2\ell}\, \tau_n(k_{z'})\, B_n^\ell \tau_n(k_{z'})^* = h(z)\, Q_n^\ell(z) \ ,$$

where h is a scalar invariant Schwartz function on $\mathbb{C}^2$ and for any M'

$$\|h\|_{(M')} \le C_{M'}\, \|h_0\|_{(M')} \le C_{M'}\, \|f\|_{(M'+\ell)} \le C_{M'}\, \|F\|_{(M'+\ell)} \ .$$

Since the Fourier transform commutes with the action of K, the same kind of result holds for the Fourier transform of F. Therefore there exists an invariant Schwartz function γ on $\mathbb{C}^2$ such that

$$\widehat{F}(\zeta) = Q_n^\ell(\zeta)\, \gamma(\zeta) \ ,$$

and, taking inverse Fourier transforms,

$$F(z) = Q_n^\ell(\partial)\, g(z),$$

with

$$\|g\|_{(M')} \le C_{M'}\, \|\gamma\|_{(M')} \le C_{M'}\, \|\hat{F}\|_{(M'+\ell)} \le C_{M'}\, \|F\|_{(M'+\ell)}.$$

Uniqueness of the function g follows by K-invariance and the identity

$$\widehat{F}(r\mathbf{o}) = B_n^\ell\, r^{2\ell}\, \widehat{g}(r\mathbf{o}) \qquad \forall r \in \mathbb{R}. \quad \square$$

Corollary 5.6. *Let F be in $\mathcal{S}(\mathbb{C}^2, \mathrm{End}(V_n))^K$. Then F can be expressed in a unique way as*

$$F = \sum_{i=0}^{n} \mathbf{D}_n^i g_i \ ,$$

with scalar valued functions $g_i \in \mathcal{S}(\mathbb{C}^2)^K$. *Moreover, for any* M *there exists* $M' \geq M+n$ *such that, for every* i,

$$\|g_i\|_{(M)} \leq C_M \, \|F\|_{(M')} \ .$$

6. Schwartz correspondence for $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$

According to formula (4.14), we denote by $\mathcal{G}_n F$ the spherical transform of a function F in $L^1\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ given by

$$\mathcal{G}_n F(\xi, \xi(-n+2j)) = \frac{1}{n+1} \int_{\mathbb{C}^2} \mathrm{tr}\,\big(F(z)\Phi_{\xi,j}(-z)\big)\, dz \qquad \forall \xi \geq 0, \quad j = 0, 1, \ldots, n \ .$$

The first two subsections will provide the proof of the following theorem.

Theorem 6.1. *The map* $\mathcal{G}_n$ *is an isomorphism of* $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ *onto* $\mathcal{S}(\Sigma_{\mathcal{D}}^n)$.

This theorem proves the Schwartz extension property for single K-type components, but the estimates that come along with this method do not imply the rapid τ-decay requested by condition (S').

Actually, *a posteriori*, once property (S) for the strong Gelfand pair (G, K) will be established, we will obtain better estimates for $\mathcal{G}_n$, see Corollary 7.9.

At this stage we can prove, however, that the inverse transform $\mathcal{G}_n^{-1}$ satisfies Schwartz norm estimates that grow at most polynomially in τ. This fact is established in Section 6.3 and is needed in the final construction (see Theorem 7.8).

6.1. Schwartz extensions of $\mathcal{G}_n F$

We begin by proving that $\mathcal{G}_n$ maps $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ into $\mathcal{S}(\Sigma_{\mathcal{D}}^n)$ and that it is continuous. For $j = 0, 1, \ldots, n$ denote $t_j = -n + 2j$.

Lemma 6.2. *Let* F *be in* $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$. *Then the following hold.*

(i) $\widehat{F}(\sqrt{\xi}\, \mathbf{o})\, e_n^j = \mathcal{G}_n F\big(\xi, t_j \xi\big)\, e_n^j \qquad \forall \xi \geq 0, \quad j = 0, 1, \ldots, n.$

(ii) There exist $\gamma_0, \ldots, \gamma_n$ *in* $\mathcal{S}(\mathbb{R})$ *such that*

$$\mathcal{G}_n F\big(\xi, t_j \xi\big) = \sum_{k=0}^{n} (\xi\, t_j)^k \, \gamma_k(\xi) \qquad \forall \xi \geq 0, \quad j = 0, 1, \ldots, n. \tag{6.1}$$

Proof. Let F be in $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$. By (5.8), we have

$$\begin{aligned}\mathcal{G}_n F(\xi, \xi t_j) &= \frac{1}{n+1} \int\limits_{\mathbb{C}^2} \operatorname{tr}\big(F(z)\Phi_{\xi,j}(-z)\big)\, dz \\ &= \int\limits_{\mathbb{C}^2}\int\limits_{S^3} e^{-i\sqrt{\xi}\langle z,\zeta\rangle} \operatorname{tr}\big(F(z) Q^n_{E_{jj}}(\zeta)\big)\, d\sigma(\zeta)\, dz \\ &= \int\limits_{S^3} \operatorname{tr}\big(\widehat{F}(\sqrt{\xi}\,\zeta) Q^n_{E_{jj}}(\zeta)\big)\, d\sigma(\zeta) \\ &= \operatorname{tr}\big(\widehat{F}(\sqrt{\xi}\mathbf{o})\, E_{jj}\big) \\ &= \big(\widehat{F}(\sqrt{\xi}\mathbf{o})\big)_{jj}\end{aligned}$$

and (i) follows.

Next, by Corollary 5.6 there exist $g_0, \ldots, g_n$ in $\mathcal{S}(\mathbb{R})$ such that $F = \sum_{k=0}^{n} \mathbf{D}_n^k g_k$.

Applying the standard Fourier transform on $\mathbb{C}^2$ to this equality we obtain

$$\big(\widehat{F}(\sqrt{\xi}\mathbf{o})\big)_{jj} = \sum_{k=0}^{n} (\xi t_j)^k\, \widehat{g}_k(\sqrt{\xi}\mathbf{o}).$$

By the Schwartz correspondence for the ordinary Gelfand pair $(M_2(\mathbb{C}^2), U_2)$ there exist $\gamma_0, \gamma_1, \ldots, \gamma_n \in \mathcal{S}(\mathbb{R})$ such that $\gamma_k(\xi) = \widehat{g}_k(\sqrt{\xi}\mathbf{o})$, $\xi \geq 0$, and item (ii) follows. □

Corollary 6.3. *Let F be in $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$. Then there exists $g \in \mathcal{S}(\mathbb{R}^2)$ such that $g_{|\Sigma^n_{\mathcal{D}}} = \mathcal{G}_n F$. Moreover, for every M there exist $M' > M + n$ and a constant $C_{M,n}$ such that*

$$\|g\|_{(M)} \leq C_{M,n} \|F\|_{(M')}.$$

Proof. Notice that $\Sigma^n_{\mathcal{D}}$ is contained in $C_n = \big\{(\xi_1, \xi_2) \in \mathbb{R}^2 \;:\; |\xi_2| \leq n\xi_1, \quad \xi_1 \geq 0\big\}$. Let η_n be a smooth function on $\mathbb{R}^2$ with bounded derivatives of any order which takes value 1 on C_n and vanishes outside $C_n - (\varepsilon, 0)$, for some $\varepsilon > 0$.

Let F be in $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ and let $\gamma_0, \ldots, \gamma_n$ in $\mathcal{S}(\mathbb{R})$ be as in (6.1). Then the function g defined on $\mathbb{R}^2$ by

$$g(\xi_1, \xi_2) = \eta_n(\xi_1, \xi_2) \sum_{k=0}^{n} \xi_2^k\, \gamma_k(\xi_1) \qquad \forall (\xi_1, \xi_2) \in \mathbb{R},$$

satisfies the required properties. □

6.2. Surjectivity of $\mathcal{G}_n$ and Schwartz correspondence for $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$

We conclude the proof of Theorem 6.1 by proving that the continuous linear map $\mathcal{G}_n$: $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K \longrightarrow \mathcal{S}(\Sigma_{\mathcal{D}}^n)$ is surjective. This fact could be deduced from a general result in [12, Prop. 4.2.1] for weighted subcoercive systems of left-invariant differential operators on Lie groups with polynomial growth. However, we give an independent and relatively simple proof, well adapted to our case.

Proposition 6.4. *Let g be in $\mathcal{S}(\mathbb{R}^2)$. Then there exists F in $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ such that $\mathcal{G}_n F = g_{|_{\Sigma_{\mathcal{D}}^n}}$.*

Proof. For $j = 0, 1, \dots, n$ let $t_j = (-n + 2j)$ and fix $\xi > 0$. Denote by p_ξ the polynomial such that $p_\xi(t_j) = g\big(\xi, \xi t_j\big)$, $j = 0, 1, \dots, n$... Using Newton's interpolation formula, we write

$$p_\xi(t) = \mu_0(\xi) + \xi\, \mu_1(\xi)(t - t_0) + \xi^2 \mu_2(\xi)(t - t_0)(t - t_1) + \cdots + \xi^n \mu_n(\xi)(t - t_0) \cdots (t - t_{n-1}).$$

Then by the Hermite–Genocchi formula [4], which we express in the equivalent form for equidistant points, we have

$$\mu_\ell(\xi) = \frac{1}{\ell!} \int_0^1 \int_0^1 \cdots \int_0^1 \partial_2^{(\ell)} g(\xi, \xi(-n + 2u_1 + \cdots + 2u_\ell))\, du_\ell \cdots du_2\, du_1 \quad \forall \xi > 0,$$

$\ell = 0, 1, \dots, n$.

Via this formula we extend $\mu_0, \mu_1 \dots, \mu_n$ to Schwartz functions on $\mathbb{R}$ and we can write

$$g(\xi, \xi t_j) = \sum_{0 \le k \le \ell \le n} b_{k,\ell}\, (t_j \xi)^k \xi^{\ell - k}\, \mu_\ell(\xi), \qquad \forall \xi \ge 0, \quad j = 0, 1, \dots, n,$$

for some complex numbers $b_{k,\ell}$, where $0 \le k \le \ell \le n$.

Define

$$f_\ell(z) = \frac{1}{(2\pi)^4} \int_{\mathbb{C}^2} \mu_\ell(|\zeta|^2)\, e^{i\langle z, \zeta \rangle}\, d\zeta,$$

then $\mu_\ell = \mathcal{G}_0 f_\ell$, $\ell = 0, 1, \dots, n$ and the function

$$F = \sum_{0 \le k \le \ell \le n} b_{k,\ell} \mathbf{D}_n^k \Delta_z^{\ell - k} f_\ell$$

satisfies the required properties. □

Proof of Theorem 6.1. By Corollary 6.3 $\mathcal{G}_n$ maps $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ into $\mathcal{S}(\Sigma_n)$ continuously and, by Proposition 6.4, it is surjective. It follows from the open mapping theorem for Fréchet spaces [16] that also $\mathcal{G}_n^{-1}$ is continuous. □

6.3. Norm estimates for $\mathcal{G}_n^{-1}$ with polynomial growth

In this subsection we prove that the inverse transform $\mathcal{G}_n^{-1}$ satisfies norm estimates with polynomial growth in n. These will be needed in the proof of Theorem 7.8.

Because of our choice (5.9) of Schwartz norms on $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$, in order to estimate the norms of $\mathcal{G}_n^{-1} \;:\; \mathcal{S}(\Sigma_{\mathcal{D}}^n) \longrightarrow \mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ we need to study the action of the Laplacian Δ_z on K-equivariant $\mathrm{End}(V_n)$-valued functions.

Identifying the unit sphere S^3 with the group SU_2 as in (5.2), the expression of the Laplacian in polar coordinates takes the form

$$\Delta_z = -\partial_r^2 - \frac{3}{r}\partial_r + \frac{1}{r^2}\Omega, \tag{6.2}$$

where $\Omega = -X_1^2 - X_2^2 - X_3^3$ is the Casimir operator on SU_2. In the next lemma we determine the action of the operator Ω on smooth, equivariant $\mathrm{End}(V_n)$–valued functions.

Lemma 6.5. *Let F be in $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ with $F(0,\cdot) = \mathrm{diag}(f_0, \dots f_n)$, then*

$$(\Omega F)(r\mathbf{o})e_n^j = \big((t_j^2 - n^2 - 2n)(f_{j+1} - 2f_j + f_{j-1}) + 2t_j(f_{j+1} - f_{j-1})\big)(r)e_n^j$$

where $t_j = -n + 2j$, $j = 0, 1, \dots, n$.

Proof. Let F be in $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$. By K–equivariance, we have

$$\Omega\Big(\tau_n(k)F(r\mathbf{o})\tau_n(k)^*\Big) = \tau_n(k)\; CF(r\mathbf{o})\; \tau_n(k)^* \qquad \forall r > 0, \quad \forall k \in SU_2\ .$$

Notice that

$$\begin{aligned} \Omega = -X_1^2 - X_2^2 - X_3^3 &= -X_1^2 - 2iX_1 - (X_2 + iX_3)(X_2 - iX_3), \\ &= -X_1^2 + 2iX_1 - (X_2 - iX_3)(X_2 + iX_3) \end{aligned}$$

and that, for every $X \in \mathfrak{su}_2$, we have

$$X\tau_n(k) = \tau_n(k)\, d\tau_n(X) \qquad \text{and} \qquad X\tau_n^*(k) = -d\tau_n(X)\,\tau_n^*(k).$$

Hence we obtain

$$\Omega\tau_n(k) = \tau_n(k)\, d\tau_n(\Omega), \qquad \Omega\tau_n(k)^* = d\tau_n(\Omega)\tau_n^*(k).$$

Moreover $d\tau_n(\Omega)$ and $id\tau_n(X_1) = B_n^1$ commute with $\Omega F(r\mathbf{o})$ so that

$$(\Omega\tau_n(k))\; F(r\mathbf{o})\; \tau_n(k)^* + \tau_n(k)\; F(r\mathbf{o})\; (\Omega\tau_n(k)^*) = 2\,\tau_n(k)\, d\tau_n(\Omega)\, F(r\mathbf{o})\, \tau_n(k)^*$$

and

$$(X_1\tau_n(k))\ F(r\mathbf{o})\ (X_1\tau_n(k)^*) = \tau_n(k)(B_n^1)^2\, F(r\mathbf{o})\tau_n(k)^*.$$

Therefore, by Leibniz rule,

$$(\Omega F)(r\mathbf{o}) = 2\left(d\tau_n(\Omega) - (B_n^1)^2\right) F(r\mathbf{o}) + \Lambda F(r\mathbf{o})\Lambda^* + \Lambda^* F(r\mathbf{o})\Lambda$$

where

$$\Lambda = d\tau_n(X_2 + iX_3) \quad \text{ so that } \quad \Lambda^* = -d\tau_n(X_2 - iX_3).$$

Recalling that $d\tau_n(\Omega) = (n^2+2n)I$ and $B_n^1 e_n^j = t_j\, e_n^j$, we only need to evaluate $\Lambda F(r\mathbf{o})\Lambda^* + \Lambda^* F(r\mathbf{o})\Lambda$. Since

$$\Lambda e_n^j = 2\sqrt{(j+1)(n-j)}\, e_n^{j+1}\ , \qquad \Lambda^* e_n^j = 2\sqrt{\ell(n-j+1)}\, e_n^{j-1}\ ,$$

then

$$\begin{aligned}
&\big(\Lambda F(r\mathbf{o})\Lambda^* + \Lambda^* F(r\mathbf{o})\Lambda\,\big) e_n^j \\
&\quad = -4j(n-j+1) f_{j-1}(r) e_n^j - 4(j+1)(n-j) f_{j+1}(r) e_n^j \\
&\quad = \Big(\big(t_j^2 - 2t_j - n^2 - 2n\big) f_{j-1} + \big(t_j^2 + 2t_j - n^2 - 2n\big) f_{j+1}\Big)(r) e_n^j\ ,
\end{aligned}$$

and this proves the lemma. □

Proposition 6.6. *For every $M \in \mathbb{N}$ there exists $N_M \in \mathbb{N}$ such that, for every n and every g in $\mathcal{S}(\mathbb{R}^2)$,*

$$\|\mathcal{G}_n^{-1}(g_{|_{\Sigma_{\mathcal{D}}^n}})\|_{(M)} \le C_{M,n} \|g\|_{(N_M)},$$

where the constant $C_{M,n}$ has polynomial growth in n.

Proof. Let g be in $\mathcal{S}(\mathbb{R}^2)$. By Proposition 6.4 and Lemma 6.2 we know that $\mathcal{G}_n$ is bijective and that the function $F = \mathcal{G}_n^{-1}(g_{|_{\Sigma_{\mathcal{D}}^n}}) \in \mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ satisfies the equality

$$\widehat{F}(\sqrt{\xi}\,\mathbf{o}) e_n^j = g\big(\xi, (-n+2j)\xi\big) e_n^j \qquad \forall \xi \ge 0 \quad 0 \le j \le n\ .$$

We have

$$\begin{aligned}\|(1+|\cdot|^2)^M\,\Delta_z^q\widehat{F}\|_2^2 &= \int\limits_{\mathbb{C}^2}(1+|\zeta|^2)^{2M}\,\left\|\Delta_z^q\widehat{F}(\zeta)\right\|_{HS}^2\,d\zeta \\ &= |S^3|\int\limits_0^{+\infty}\int\limits_K\left\|\Delta_z^q\widehat{F}(k\cdot r\mathbf{o})\right\|_{HS}^2 dk\,(1+r^2)^{2M}\,r^3\,dr \\ &= |S^3|\int\limits_0^{+\infty}(1+r^2)^{2M}\sum_{j=0}^n\left|\left(\Delta_z^q\widehat{F}(r\mathbf{o})\right)_{jj}\right|^2 r^3\,dr.\end{aligned}$$

We now compute $\Delta_z\widehat{F}(r\mathbf{o})$ using the polar decomposition (6.2) and Lemma 6.5. The action of $\partial_r^2+\frac{3}{r}\partial_r$ on the function

$$r\longmapsto\widehat{F}(r\mathbf{o})=\mathrm{diag}\Big(g(r^2,r^2t_0),\dots g(r^2,r^2t_n)\Big)$$

is given by

$$\left(\partial_r^2+\frac{3}{r}\partial_r\right)g(r^2,r^2t_j)=\big(8(\partial_1+t_j\partial_2)g+4r^2(\partial_1+t_j\partial_2)^2g\big)\,(r^2,r^2t_j).$$

In order to compute the action of the Casimir operator Ω, we apply formula (6.5) with $f_j(r)=g(r^2,r^2t_j)$. The Taylor expansion in the second variable of g gives, with $\xi=r^2$,

$$\begin{aligned}f_{j\pm k}(\sqrt{\xi}) &= g(\xi,\xi(t_j\pm 2k)) \\ &= \sum_{s=0}^p\frac{\partial_2^s g\,(\xi,\xi t_j)}{s!}\,(\pm 2k\xi)^s+(\pm 2k\xi)^{p+1}\int\limits_0^1\frac{\partial_2^{p+1}g\,(\xi,\xi(t_j\pm 2ku))}{p!}\,(1-u)^p\,du\ ,\end{aligned}$$

so that

$$\begin{aligned}(f_{j+1}-2f_j+f_{j-1})(\sqrt{\xi}) &= g(\xi,\xi t_j+2\xi)-g(\xi,\xi t_j)+g(\xi,\xi t_j-2\xi) \\ &= (2\xi)^2\int\limits_0^1\big(\partial_2^2 g\,(\xi,\xi(t_j+2u))+\partial_2^2 g\,(\xi,\xi(t_j-2u))\big)\,(1-u)\,du,\end{aligned}$$

and

$$\begin{aligned}(f_{j+1}-f_{j-1})(\sqrt{\xi}) &= g(\xi,\xi t_j+2\xi)-g(\xi,\xi t_j-2\xi) \\ &= 2\xi\int\limits_0^1\left(\partial_2 g\,(\xi,\xi(t_j+2u))+\partial_2 g\,(\xi,\xi(t_j-2u))\right)\,du\ .\end{aligned}$$

Therefore, by (6.2) and Lemma 6.5,

$$
\begin{aligned}
-\left(\Delta_z \widehat{F}(r\mathbf{o})\right)_{jj} &= \left(\partial_r^2 + \frac{3}{r}\partial_r\right)(g(r^2, r^2 t_j)) - \frac{1}{r^2}(\Omega \widehat{F})(r\mathbf{o}) \\
&= \left(4r^2(\partial_1 + t_j\partial_2)^2 g + 8(\partial_1 + t_j\partial_2)g\right)(r^2, r^2 t_j) \\
&\quad - 4r^2(t_j^2 - n^2 - 2n)\int_0^1 \left(\partial_2^2 g\left(r^2, r^2(t_j+2u)\right) + \partial_2^2 g\left(r^2, r^2(t_j-2u)\right)\right)(1-u)\,du \\
&\quad - 4t_j \int_0^1 \left(\partial_2 g\left(r^2, r^2(t_j+2u)\right) + \partial_2 g\left(r^2, r^2(t_j-2u)\right)\right)du\ .
\end{aligned}
$$

Since $|t_j| \le n$, $j = 0, 1, \ldots, n$, by iteration, we obtain

$$
\left|\left(\Delta_z^q \widehat{F}(r\mathbf{o})\right)_{jj}\right| \le C_q\,(1 + r^{2q})\, n^{2q} \sum_{s=1}^{2q} \sup_{u\in\mathbb{R}} \left|\partial^s g(r^2, r^2 u)\right|.
$$

Therefore

$$
\|(1 + |\cdot|^2)^M\, \Delta_z^q \widehat{F}\|_2 \le C_{q,M}\, n^{2q+1} \max_{0\le|\beta|\le 2q} \|(1 + |\cdot|^2)^{M'}\, \partial^\beta g\|_\infty
$$

for some $M' > M$. Therefore for every $M \in \mathbb{N}$ there exists $N_M \in \mathbb{N}$ such that

$$
\|F\|_{(M)} \le C_M \|\widehat{F}\|_{(M)} \le C_{M,n}\|g\|_{(N_M)}\ ,
$$

where $C_{M,n}$ has polynomial growth in n. $\square$

7. Schwartz correspondence for $\mathcal{S}\big(M_2(\mathbb{C})\big)^{\mathrm{Int}(U_2)}$

In this section we prove property (S') of Theorem 1.2, which implies Schwartz correspondence for the strong Gelfand pair (G, K).

We first rephrase our result for $\mathcal{G}_n$ on $\mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$ in terms of the Gelfand transform $\mathcal{G}_{\tau_{m,n}}$ on $\mathcal{S}(G)^{\mathrm{Int}(K)}$. Then for each K-type $f_{m,n} = f_{\tau_{m,n}}$ of $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$, we obtain the infinite jet at the origin for the transform $\mathcal{G}_{\tau_{m,n}} f_{m,n}$ and prove, in Lemma 7.4 that its coefficients have at most polynomial growth in m, n.

Next, in Lemma 7.6, we obtain by Whitney extension a Schwartz function on $\mathbb{R}^2$ which has infinite order of contact with $\mathcal{G}_{\tau_{m,n}} f_{m,n}$ at the origin. In Proposition 7.7 we give an explicit formula for a Schwartz extension of this difference.

Finally we assemble all these steps proving property (S') in Theorem 7.8. Since the Schwartz norms of the $f_{m,n}$ have rapid decay in (m, n), it is important to verify that each operation brings in multiplicative factors that grow polynomially in n. One of them involves the Schwartz norm estimates for the inverse spherical transform $\mathcal{G}_n^{-1}$ for matrix-valued equivariant functions.

7.1. Schwartz correspondence for $\mathcal{G}_{\tau_{m,n}}$

In order to relate our previous result for matrix-valued equivariant functions with the corresponding result for functions of a given type (m,n), we quantify the relation between the Schwartz norms of a function f of a given type (m,n) and of the corresponding matrix valued function $A_{m,n}f$.

As M-order Schwartz norm of a function f in $\mathcal{S}(G)^{\mathrm{Int}(K)}$ we take

$$\|f\|_{(M)} = \max_{q,r,s=0\dots M} \|(1+|z|^2)^M\, D_4^s D_3^r D_1^q f\|_2$$

where $D_1 = \Delta_z$, $D_3 = \Omega$ and $D_4 = iX_4$.

Lemma 7.1. *Let (m,n) be in the set E defined in (2.3). The following estimates hold*

$$\|A_{m,n}f\|_{(M)} \le \frac{1}{\sqrt{n+1}}\,\|f\|_{(M)}\ , \qquad \forall f \in \mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}\ ,$$

and conversely,

$$\|A^{-1}_{m,n}F\|_{(M)} \le (1+|m|)^M\,(1+n)^{2M+1/2}\,\|F\|_{(M)} \qquad \forall F \in \mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K\ .$$

Proof. Note that when f is of type (m,n),

$$D_3 f = (n^2+2n)f \qquad D_4 f = mf.$$

The estimates follow easily from the fact that $\sqrt{n+1}A_{m,n}$ is an isometry on the corresponding L^2-spaces. $\square$

By (4.15), $\Sigma_{\mathcal{D}}$ decomposes as the union of

$$\Sigma^{m,n}_{\mathcal{D}} = \Sigma^n_{\mathcal{D}} \times \big\{(n^2+2n, m)\big\}\ , \qquad (m,n)\in E\ .$$

At this stage we abandon the $\mathrm{End}(V_n)$-valued picture, and reinterpret Corollary 6.3 and Propositions 6.4, 6.6 in the following form, using the fact that $\mathcal{G}_{\tau_{m,n}} = \mathcal{G}_n \circ A_{m,n}$.

Corollary 7.2.

(i) Given $\tau_{m,n} \in \widehat{K}$ and $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$, the spherical transform $\mathcal{G}f$, which is supported on $\Sigma^n_{\mathcal{D}} \times \big\{(n^2+2n,m)\big\}$, admits a Schwartz extension to $\mathbb{R}^2 \times \big\{(n^2+2n,m)\big\}$, and hence a Schwartz extension to $\mathbb{R}^4$ which vanishes on the other components of $\Sigma_{\mathcal{D}}$.

(ii) For every $(m,n)\in E$, the transform $\mathcal{G}_{\tau_{m,n}}$ is an isomorphism from $\mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$ to $\mathcal{S}(\Sigma^n_{\mathcal{D}})$.

(iii) For every $M \in \mathbb{N}$ there exists $N_M \in \mathbb{N}$ such that, for every $(m,n) \in E$ and every g in $\mathcal{S}(\mathbb{R}^2)$,

$$\|\mathcal{G}^{-1}_{\tau_{m,n}}(g_{|_{\Sigma^n_{\mathcal{D}}}})\|_{(M)} \le C_{M,m,n}\|g\|_{(N_M)},$$

where the constants $C_{M,m,n}$ have polynomial growth in (m,n).

If we consider now a general $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$,

$$f = \sum_{(m,n)\in E} f_{m,n} \ ,$$

we cannot prove, on the basis of the results in Section 6, that the Schwartz extensions to $\mathbb{R}^4$ given in Corollary 7.2(i) for the individual $\mathcal{G}f_{m,n}$ add up to give a Schwartz function.

In order to do so, we need to proceed to a new construction of Schwartz extensions, possibly different from those already available, which gives, for any finite number of Schwartz norms, rapid decay as n goes to infinity (rapid decay in m for fixed n is trivial).

As we already noticed, this new construction does not replace the work done in Section 6 because it requires to know in advance that a Schwartz extension whatsoever exists for each (m,n).

7.2. Jets with polynomial growth for each K-type

For $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$, we have a simple estimate on the directional derivatives of $\mathcal{G}_{\tau_{m,n}}f$ in the $n+1$ directions of the half-lines forming $\Sigma^n_{\mathcal{D}}$.

Lemma 7.3. *For $d \in \mathbb{N}$, there exist constants C_d and N_d independent of (m,n) such that*

$$\big|(d/d\xi_1)^d \mathcal{G}_{\tau_{m,n}} f\big(\xi_1, (-n+2j)\xi_1\big)\big| \le C_d \,\|f\|_{(N_d)} \ , \qquad \forall f \in \mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$$

for all $\xi_1 \ge 0$.

Proof. From (4.12) it follows that

$$\mathcal{G}_{\tau_{m,n}} f\big(\xi_1, (-n+2j)\xi_1\big) = \mathcal{G}_n(A_{m,n}f)\big(\xi_1, (-n+2j)\xi_1\big).$$

Since $\Phi_{\xi_1,j}$ is even in z, we obtain from (5.8) that, for $\xi_1 > 0$,

$$\Phi_{\xi_1,j}(z) = (n+1)\int_{S^3} \cos(\sqrt{\xi}\langle z,\zeta\rangle)\, Q^n_{E_{jj}}(\zeta)\, d\sigma(\zeta) \ ,$$

and therefore

$$\begin{aligned}\left\|(d/d\xi_1)^d\Phi_{\xi,j}(z)\right\|_{HS} &\leq (n+1)\,|z|^{2d}\sup_{x\geq 0}|(d/dx)^d\cos\sqrt{x}|\left\|\int_{S^3}Q^n_{E_{jj}}(\zeta)\,d\sigma(\zeta)\right\|_{HS}\\ &\leq C_d\,(n+1)\,|z|^{2d}\ .\end{aligned}$$

The conclusion follows taking N_d sufficiently large and by Lemma 7.1. $\square$

Assume now that $g = \mathcal{G}_{\tau_{m,n}}f$ admits a smooth extension u on $\mathbb{R}^2$ with Taylor series in $(0,0)$

$$\sum_{p,q}\frac{a_{p,q}}{p!q!}\xi_1^p\xi_2^q\ .$$

Letting

$$c_{d,j} = (d/d\xi_1)^d_{|\xi_1=0}\mathcal{G}_{\tau_{m,n}}f\big(\xi_1,(-n+2j)\xi_1\big)\ ,$$

the following relations must hold for all $d\in\mathbb{N}$ and $j=0,\dots,n$:

$$c_{d,j} = \sum_{p+q=d}(-n+2j)^q\binom{d}{q}a_{p,q}\ .$$

For each d we obtain an $(n+1)\times(d+1)$ linear system $B_da_d = c_d$, where

$$\begin{aligned}a_d &= \Big(a_{d,0},\dots,\tbinom{d}{q}a_{d-q,q},\dots,a_{0,d}\Big)\ ,\\ c_d &= (c_{d,0},\dots,c_{d,n})\ ,\\ B_d &= (b_{j,q}) = \big((-n+2j)^q\big)\ .\end{aligned}$$

Lemma 7.4. *For every $d\in\mathbb{N}$ the system $B_da_d = c_d$ admits a solution a_d such that*

$$\binom{d}{q}|a_{d-q,q}| \leq C_d\,(1+n)^{1+d/2}\,\|f\|_{(N_d)}\ ,$$

with C_d independent of n.

Proof. Assume $d\geq n$. Observing that all $(n+1)\times(n+1)$ minors of consecutive columns of B_d are essentially Vandermonde determinants, we have that the matrix B_d has rank $n+1$ and the system is solvable, with infinite solutions if $d>n$. This case however can be reduced to the case $d=n$ by looking for a solution a_d with $a_{d-q,q}=0$ for $q>n$. By Cramer's rule,

$$\binom{d}{q}|a_{d-q,q}| \leq \sum_{j=0}^{n}|c_{d,j}|\left|\frac{V_{j,q}}{V}\right|\ , \tag{7.1}$$

where V is the full Vandermonde determinant with nodes $t_j = -n + 2j$ and $V_{j,q}$ are its cofactors. Expressing $V_{j,q}$ in terms of Schur polynomials, cf. [10], we have

$$\left|\frac{V_{j,q}}{V}\right| = \left|\sum_{k_1<k_2<\cdots<k_{n-q}\,,\,k_\ell\neq j} \frac{t_{k_1}t_{k_2}\cdots t_{k_{n-q}}}{\prod_{i\neq j}(t_i-t_j)}\right| \leq \sum_{k_1<k_2<\cdots<k_{n-q}\,,\,k_\ell\neq j} \left|\frac{t_{k_1}t_{k_2}\cdots t_{k_{n-q}}}{\prod_{i\neq j}(t_i-t_j)}\right| .$$

The numerator is controlled by the product of all the nonvanishing terms, i.e.

$$\left|t_{k_0}t_{k_1}\cdots t_{k_{n-q}}\right| \leq \prod_{j\neq\frac{n}{2}} |t_j| \quad = \prod_{j\neq\frac{n}{2}} |-n+2j| = (n!!)^2 .$$

The smallest denominator occurs for j central, i.e., $j = n/2$ when n is even and $j = (n-1)/2$ when n is odd, so that

$$\prod_{i\neq j}\left|t_i - t_j\right| = \prod_{i=0}^{i=j-1}(2j-2i)\prod_{i=j+1}^{i=n}(2i-2j) = (2j)!!\,(2n-2j)!! \geq (n!!)^2$$

and

$$\left|\frac{t_{k_1}t_{k_2}\cdots t_{k_{n-q}}}{\prod_{i\neq j}(t_i-t_j)}\right| \leq 1 .$$

Therefore $|V_{j,q}/V| \leq \binom{n}{q} \leq (n+1)^{n/2}$.

Finally, by formula (7.1) and Lemma 7.3, when $d \geq n$,

$$\binom{d}{q}|a_{d-q,q}| \leq (n+1)^{1+n/2}\max_{0\leq j\leq n}|c_{d,j}| \leq C_d\,(n+1)^{1+d/2}\,\|f\|_{(N_d)} .$$

Assuming now $d < n$, Theorem 6.1 guarantees that the system $B_d a_d = c_d$ is solvable. Since all the maximal minors of B_d are nonvanishing Vandermonde determinants, the solution is unique and we can apply Cramer's rule to the square submatrix formed by the $d+1$ central rows of B_d.

If d and n have the same parity, the system is exactly the same considered above, only with d in place of n. Therefore, since $d < n$,

$$\binom{d}{q}|a_{d-q,q}| \leq C_d\,(d+1)^{1+d/2}\,\|f\|_{(N_d)} \leq C_d\,(n+1)^{1+d/2}\,\|f\|_{(N_d)} .$$

If d and n have different parities, the system is slightly different, but a repetition of the previous arguments leads to the same conclusion. $\square$

Combining together the two Lemmas 7.3 and 7.4, we obtain the following asymptotic expansion

Corollary 7.5. *Let $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$. For every $(m,n) \in E$ and $d \in \mathbb{N}$, there exist coefficients $a_{d-q,q}$, $q = 0, \ldots, d$, and $N_d \in \mathbb{N}$ such that, for all $j = 0, \ldots, n$,*

(i) $\binom{d}{q}|a_{d-q,q}| \le C_d\, n^{1+d/2}\, \|f\|_{(N_d)}$

(ii) $(d/d\xi_1)^d_{|\xi_1=0}\mathcal{G}_{\tau_{m,n}} f\big(\xi_1, (-n+2j)\xi_1\big) = \sum_{q=0}^{d}(-n+2j)^q \binom{d}{q} a_{d-q,q}$*; equivalently, for* $\xi = (\xi_1, \xi_2) \in \Sigma^n_{\mathcal{D}}$,

$$\mathcal{G}_{\tau_{m,n}} f(\xi) \underset{\xi\to 0}{\sim} \sum_{d=0}^{\infty} \frac{1}{d!} \sum_{q=0}^{d} \binom{d}{q} a_{d-q,q} \xi_1^{d-q} \xi_2^{q} \ . \tag{7.2}$$

7.3. Jets and smooth extensions on the full spectrum

We consider now a single K-type and construct smooth functions on $\mathbb{R}^2$, supported on the unit disk and with Taylor development (7.2) at 0.

The standard way to do so, cf. [11, Theorem 1.2.6], consists in defining

$$h(\xi) = \sum_{d\in\mathbb{N}} \varphi(\xi/\varepsilon_d) \frac{1}{d!} \sum_{q=0}^{d} \binom{d}{q} a_{d-q,q} \xi_1^{d-q} \xi_2^{q} = \sum_{d\in\mathbb{N}} h_d(\xi) \ , \tag{7.3}$$

where $\varphi \in C^\infty_c$ is supported for $|\xi| \le 1$ and is equal to 1 for $|\xi| \le 1/2$ and the coefficients $\varepsilon_d \in (0,1]$ are so chosen that the series in the right-hand side converges normally in every C^N-norm.

We follow this procedure keeping track at the same time of the norm estimates and of their dependence on the parameters m, n.

Lemma 7.6. *Let $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$ with Taylor development (7.2) and $M \in \mathbb{N}$. There exists a function $h = h_{m,n,M} \in C^\infty_c(\mathbb{R}^2)$ as in (7.3) supported in the unit disc and such that, for every $k \le M$,*

$$\|h\|_{C^k} \le A_M\, n^{1+M/2}\, \|f\|_{(N_M)} + r_{m,n} \ , \tag{7.4}$$

where $A_M > 0$ is independent of m, n and $r_{m,n}$ is independent of f and rapidly decaying in (m,n).

Proof. In (7.3) let $\psi_{d,q}(\xi) = \varphi(\xi)\xi_1^{d-q}\xi_2^q$, so that $h_d(\xi) = \frac{1}{d!}\sum_{q=0}^{d}\binom{d}{q} a_{d-q,q}\, \varepsilon_d^d\, \psi_{d,q}(\xi/\varepsilon_d)$.

By Corollary 7.5, for every $k \in \mathbb{N}$,

$$\|h_d\|_{C^k} \le \frac{C_d}{d!} n^{1+d/2} \|f\|_{(N_d)} \varepsilon_d^{d-k} \sum_{q=0}^{d} \|\psi_{d,q}\|_{C^k} \ .$$

With $\alpha_d = C_d \sum_{q=0}^{d} \|\psi_{d,q}\|_{C^{d-1}}$, we choose

$$\varepsilon_{d,m,n,M} = \begin{cases} 1 & \text{if } d \le M \\ \frac{1}{(n+|m|)!(1+\alpha_d \|f\|_{(N_d)})} & \text{if } d > M \end{cases} .$$

Then,

$$\sum_{d>M} \|h_d\|_{C^{d-1}} \le \sum_{d \ge M} \frac{n^{1+d/2}}{d!(n+|m|)!} \le \frac{ne^{\sqrt{n}}}{(n+|m|)!} \stackrel{\text{def}}{=} r_{m,n} .$$

This implies that the series $\sum_{d\in\mathbb{N}} \|h_d\|_{C^k}$ converges for every k, so that $h \in C^\infty$.

Notice that in Lemma 7.3 the sequence $\{N_d\}$ can be chosen to be increasing. Then, for $k \le M$,

$$\begin{aligned} \sum_{d\le M} \|h_d\|_{C^k} \le \sum_{d\le M} \|h_d\|_{C^M} &\le \sum_{d\le M} n^{1+d/2} \|f\|_{(N_d)} \frac{C_d}{d!} \sum_{q=0}^{d} \|\psi_{d,q}\|_{C^M} \\ &\le A_M\, n^{1+M/2} \|f\|_{(N_M)}, \end{aligned}$$

where $A_M = \sum_{d\le M} \frac{C_d}{d!} \sum_{q=0}^{d} \|\psi_{d,q}\|_{C^M}$. This implies (7.4). Rapid decay of $r_{m,n}$ is trivial. □

7.4. Extension of spherical transforms rapidly vanishing at 0

From the previous Lemma we obtain, for each $f_{m,n} \in \mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$, Schwartz functions $\{h_{m,n,M}\}_M$ on the spectrum $\Sigma^n_{\mathcal{D}}$ with infinite order of contact with $\mathcal{G}_{\tau_{m,n}} f_{m,n}$ at the origin. The following proposition gives an explicit formula for a Schwartz extension to $\mathbb{R}^2$ of the difference $\mathcal{G}_{\tau_{m,n}} f_{m,n} - h_{m,n,M}$.

Proposition 7.7. *Suppose that u in $\mathcal{S}(G)^{\mathrm{Int}(K)}_{\tau_{m,n}}$ is such that*

$$\left(\frac{d}{d\xi_1}\right)^q_{|\xi_1=0} \mathcal{G}_{\tau_{m,n}} u(\xi_1, \xi_1(-n+2j)) = 0 \qquad \forall j = 0, \dots, n, \quad \forall q \ge 0.$$

Then there exists $v_{m,n}$ in $\mathcal{S}(\mathbb{R}^2)$ such that

$$v_{m,n}(\xi_1, \xi_1(-n+2j)) = \mathcal{G}_{\tau_{m,n}} u(\xi_1, \xi_1(-n+2j)) , \qquad \forall \xi_1 \ge 0 ,$$

and for every $N \ge 0$ there exist constants C_N, N' depending only on N such that

$$\|v_{m,n}\|_{(N)} \le C_N \|u_{\tau_{m,n}}\|_{(N')}.$$

Proof. Let η be a bump function in $C_c^\infty(\mathbb{R})$ supported in $\left[-\frac{1}{2},\frac{1}{2}\right]$ and equal to 1 in a neighbourhood of the origin. Define the function $v = v_{m,n}$ on $\mathbb{R}^2$ by the rule

$$v(\xi_1,\xi_2) = \begin{cases} \displaystyle\sum_{j=0}^{n} \mathcal{G}_{\tau_{m,n}} u(\xi_1,\xi_1(-n+2j))\,\eta\left(\tfrac{\xi_2-\xi_1(-n+2j)}{\xi_1}\right) & \xi_1 > 0 \\ 0 & \xi_1 \leq 0. \end{cases}$$

It is straightforward to show that v extends $\mathcal{G}_{\tau_{m,n}} u$ to $\mathbb{R}^2$. We now check the required norm estimates.

For every $j = 0, 1, \cdots, n$, define

$$\eta_j(x) = \eta\left(x - (-n+2j)\right), \qquad \forall x \in \mathbb{R}$$

$$g_j(\xi_1) = g_{j,m,n}(\xi_1) = \mathcal{G}_{\tau_{m,n}} u(\xi_1,\xi_1(-n+2j)), \qquad \forall \xi_1 \geq 0$$

and note that for every $\xi_1 > 0$

$$\begin{aligned} \partial_{\xi_1}^p \partial_{\xi_2}^q v(\xi_1,\xi_2) &= \sum_{j=0}^{n} \partial_{\xi_1}^p \left(g_j(\xi_1)\,\xi_1^{-q}\,\eta_j^{(q)}\left(\tfrac{\xi_2}{\xi_1}\right)\right) \\ &= \sum_{j=0}^{n}\sum_{s=0}^{p} \binom{p}{s} g_j^{(p-s)}(\xi_1)\,\partial_{\xi_1}^s \left(\xi_1^{-q}\,\eta_j^{(q)}\left(\tfrac{\xi_2}{\xi_1}\right)\right). \end{aligned}$$

Moreover, one can check by induction that, for appropriate coefficients $c_{r,s}$ depending only on p, q,

$$\partial_{\xi_1}^s \left(\xi_1^{-q}\,\eta_j^{(q)}\left(\tfrac{\xi_2}{\xi_1}\right)\right) = \sum_{r=0}^{s} c_{r,s}\,\xi_1^{-s-q}\,\left(\tfrac{\xi_2}{\xi_1}\right)^r \eta_j^{(q+r)}\left(\tfrac{\xi_2}{\xi_1}\right),$$

so that

$$\partial_{\xi_1}^p \partial_{\xi_2}^q v(\xi_1,\xi_2) = \sum_{j=0}^{n}\sum_{s=0}^{p}\sum_{r=0}^{s} \binom{p}{s} c_{r,s}\,\xi_1^{-s-q}\,g_j^{(p-s)}(\xi_1)\,\left(\tfrac{\xi_2}{\xi_1}\right)^r \eta_j^{(q+r)}\left(\tfrac{\xi_2}{\xi_1}\right).$$

Since $\mathcal{G}_{\tau_{m,n}} u$ vanishes rapidly at the origin, for any integer $q \geq 0$ there exists $\theta_q \in (0,1)$ such that for any $\xi_1 \geq 0$

$$\xi_1^{-q}\,g_{j,m,n}^{(p)}(\xi_1) = \frac{1}{q!}\,g_{j,m,n}^{(p+q)}(\theta_q \xi_1).$$

Since for $j = 0, 1, \ldots$ the function $t \longmapsto t^r\,\eta_j^{(q+r)}(t)$ is still a bump function, in view of Lemma 7.3, $v \in C^\infty(\mathbb{R}^2)$ and the required norm estimates follow. $\square$

We can now conclude that property (S') in Theorem 1.2 is satisfied, so proving our main result Theorem 1.1.

Theorem 7.8. *Let f be in $\mathcal{S}(G)^{\mathrm{Int}(K)}$ and N in $\mathbb{N}$. Then, for every $(m,n) \in E$, $\mathcal{G}_{\tau_{m,n}} f_{m,n}$ admits a Schwartz extension $u^N_{m,n}$ from $\Sigma^n_{\mathcal{D}}$ to $\mathbb{R}^2$ such that $\|u^N_{m,n}\|_{(N)}$ is rapidly decaying in (m,n).*

Proof. For M to be chosen afterwards, let $h_{m,n,M}$ be the function defined in Lemma 7.6. Since any Schwartz norm of $f_{m,n}$ is rapidly decaying in (m,n), the M-Schwartz norm of $h_{m,n,M}$ is also rapidly decaying in (m,n). Moreover, let $g_{m,n} = g_{m,n,M} = \mathcal{G}^{-1}_{\tau_{m,n}}\big(h_{m,n,M}{}_{|_{\Sigma^n_{\mathcal{D}}}}\big)$. Then $g_{m,n}$ is a Schwartz function on G of type $\tau_{m,n}$ and

$$\left(\frac{d}{d\xi_1}\right)^q_{|_{\xi_1=0}} \mathcal{G}_{\tau_{m,n}}(f_{m,n} - g_{m,n})(\xi_1, \xi_1(-2j+n)) = 0 \qquad \forall j = 0, \dots, n, \quad \forall q \geq 0.$$

By Proposition 7.7, there exists $v_{m,n} = v_{m,n,M}$ in $\mathcal{S}(\mathbb{R}^2)$ such that

$$v_{m,n}(\xi_1, \xi_1(-n+2j)) = \mathcal{G}_{\tau_{m,n}}(f_{m,n} - g_{m,n})(\xi_1, \xi_1(-n+2j)) \qquad \forall \xi_1 \geq 0, \quad j = 0, \dots, n.$$

Applying Corollary 7.2 (iii), we obtain

$$\begin{aligned} \|v_{m,n,M}\|_{(N)} &\leq C_N \, \|f_{m,n} - g_{m,n,M}\|_{(N')} \\ &\leq C_N \left(\|f_{m,n}\|_{(N')} + \|\mathcal{G}^{-1}_{\tau_{m,n}}\big(h_{m,n,M}{}_{|_{\Sigma^n_{\mathcal{D}}}}\big)\|_{(N')} \right) \\ &\leq C_N \, \|f_{m,n}\|_{(N')} + C_{m,n,N} \|h_{m,n,M}\|_{(N'')} \end{aligned}$$

where N', N'' depend only on N and the constant $C_{m,n,N}$ has polynomial growth in (m,n).

Choosing M bigger than N and N'' and letting

$$u^N_{m,n} = v_{m,n,M} + h_{m,n,M}$$

we obtain a Schwartz extension to $\mathbb{R}^2$ whose N-Schwartz norm is rapidly decaying in (m,n). $\square$

For completeness, we derive from Theorem 1.1 that the transforms $\mathcal{G}_n$ satisfy norm estimates that grow at most polynomially in n.

Corollary 7.9. *For every $M \in \mathbb{N}$ there exist $N_M, Q_M \in \mathbb{N}$ and $C_M > 0$ such that, for every $n \in \mathbb{N}$ and $F \in \mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$, the spherical transform $\mathcal{G}_n F$ extends to $g_{M,n} \in \mathcal{S}(\mathbb{R}^2)$ satisfying*

$$\|g_{M,n}\|_{(M)} \leq C_M \, (n+1)^{Q_M} \, \|F\|_{(N_M)} \, . \tag{7.5}$$

Proof. To simplify the reading of the proof, we denote Schwartz norms as $\| \ \|_{(N,\mathbb{R}^2)}$, $\| \ \|_{(N,\mathbb{R}^4)}$, etc., with explicit mention of the domain. It is also convenient to replace the L^2-based Schwartz norms used so far, cf. (5.9), with the L^∞-based ones. This equivalence is a well-known fact and it is also easy to prove that, in the case of $\mathrm{End}(V_n)$-valued functions, the constants involved in the equivalence have polynomial growth in n. Therefore the validity of the statement and of the other inequalities can be referred to any choice of Schwartz norms.

Given $F \in \mathcal{S}\big(\mathbb{C}^2, \mathrm{End}(V_n)\big)^K$, let $f = A_{n,n}^{-1} F \in \mathcal{S}(G)_{\tau_{n,n}}^{\mathrm{Int}(K)}$. By Theorem 1.1, given $M \in \mathbb{N}$, its (strong) spherical transform $\mathcal{G}f$ defined on $\Sigma_{\mathcal{D}}$ admits a Schwartz extension h_M on $\mathbb{R}^4$ such that

$$\|h_M\|_{(M,\mathbb{R}^4)} \leq C_M \, \|f\|_{(N_M,G)} \ ,$$

with N_M and C_M independent of n. Decompose $\xi \in \mathbb{R}^4$ as (ξ', ξ'') with $\xi' = (\xi_1, \xi_2)$ and define $\xi''_n = \big(n(n+2), n\big)$.

Since $\mathcal{G}f_{\,|\Sigma_{\mathcal{D}}^n} = \mathcal{G}_{\tau_{n,n}} f = \mathcal{G}_n F$, the function $g_{M,n}$ defined by

$$g_{M,n}(\xi') = h_M(\xi', \xi''_n) \qquad \forall \xi' \in \mathbb{R}^2$$

extends $\mathcal{G}_n F$ and

$$\|g_{M,n}\|_{(M,\mathbb{R}^2)} \leq \|h_M\|_{(M,\mathbb{R}^4)} \ .$$

On the other hand, by Lemma 7.1, there exists Q_M such that

$$\|f\|_{(N_M,G)} = \|A_{n,n}^{-1} F\|_{(N_M,G)} \leq (n+1)^{Q_M} \, \|F\|_{(N_M,\mathbb{C}^2)} \ .$$

Therefore

$$\|g_{M,n}\|_{(M,\mathbb{R}^2)} \leq C_M \, (n+1)^{Q_M} \, \|F\|_{(N_M,\mathbb{C}^2)} \ . \quad \square$$

Data availability

No data was used for the research described in the article.

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