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Extremal functions in AdS/CFT: black hole entropy, equivariant localization

DOCTORAL THESIS

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ABSTRACT

Extremization problems are frequently encountered in AdS/CFT. Field theory observables such as central charges and sphere partition functions can be computed from extremal functions of the dual supergravity solution. The latter can be expressed in terms of topological quantities that naturally arise in the context of equivariant localization. A particularly interesting example of extremal functions are the entropy functions of AdS supersymmetric black holes, whose Legendre transform reproduces the Bekenstein-Hawking entropy. Focusing on the case of Kerr-Newman black holes asymptotically $\mathrm{AdS}_5\times\mathrm{SE}_5$, the superconformal index of the dual four-dimensional $\mathcal{N}=1$ quiver theory can match the entropy function in the large-N limit.

In the first part of this thesis we study the superconformal index of $\mathcal{N}=1$ quiver theories at large-N for general values of electric charges and angular momenta, using both the Bethe Ansatz formulation and the more recent elliptic extension method. We are particularly interested in the case of unequal angular momenta, $J_1 \neq J_2$, which has only been partially considered in the literature. We revisit the previous computation with the Bethe Ansatz formulation with generic angular momenta and extend it to encompass a large class of competing exponential terms. In the process, we also provide a simplified derivation of the original result. We consider the newly-developed elliptic extension method as well; we apply it to the $J_1 \neq J_2$ case, finding a good match with the Bethe Ansatz results. We also investigate the relation between the two different approaches, finding in particular that for every saddle of the elliptic action there are corresponding terms in the Bethe Ansatz formula that match it at large-N.

In the second part of this thesis we study extremal functions of supergravity solutions through the lenses of equivariant localization. Recently it has been proposed that a vast class of gravitational extremization problems in holography can be formulated in terms of the equivariant volume of the internal geometry, or of the cone over it. We substantiate this claim by analysing supergravity solutions corresponding to branes partially or totally wrapped on a four-dimensional orbifold, both in M-theory as well as in type II supergravities. We show that our approach recovers the relevant gravitational central charges/free energies of several known supergravity solutions and can be used to compute these also for solutions that are not known explicitly. Moreover, we demonstrate the validity of previously conjectured gravitational block formulas for M5 and D4 branes. In the case of M5 branes we make contact with a recent approach based on localization of equivariant forms, constructed with Killing spinor bilinears.

PUBLICATIONS

This dissertation is based on the following PhD research projects [1, 2] :

- E. Colombo, *The large-N limit of 4d superconformal indices for general BPS charges*, *JHEP* **12** (2022) 013, [arXiv:2110.01911].
- E. Colombo, F. Faedo, D. Martelli, and A. Zaffaroni, *Equivariant volume extremization and holography*, arXiv:2309.04425.

Prior to the beginning of the PhD the author also contributed to the following publication [3], which will be frequently mentioned but is not part of this thesis:

- F. Benini, E. Colombo, S. Soltani, A. Zaffaroni, and Z. Zhang, *Superconformal indices at large N and the entropy of $AdS_5 \times SE_5$ black holes*, *Class. Quant. Grav.* **37** (2020), no. 21 215021, [arXiv:2005.12308].

Contents

Chapter 1

Introduction

In the AdS/CFT holography many interesting observables can be expressed in terms of extremization problems. In the conformal field theory side, a notable example is the way that the exact R-symmetry of the theory can be determined by extremizing a functional. The exact R-symmetry is the $U(1)_R$ symmetry whose current is in the same superconformal multiplet as the stress-energy tensor. In four dimensions, the exact R-symmetry can be found by a process called a-maximization [4]. Since the a central charge can be easily computed from the 't Hooft anomalies of the $U(1)_R$ symmetry, the idea is to define a *trial R-symmetry*, which is a generic linear combination of all the possible global abelian symmetries of the CFT, and then to compute a_{trial} from the trial R-symmetry the same way one would compute a from the exact R-symmetry. a-maximization states that a_{trial} is maximized when the trial R-symmetry matches the exact R-symmetry. A two-dimensional analogue of this process involves the central charge c and is called c-extremization [5]. Similar extremal problems can also be set up for odd-dimensional field theories, but instead of central charges the functional to be extremized is given by the sphere partition function (or its logarithm, the free energy), as is the case for F-maximization [6] and $\mathcal{I}$-extremization [7].

The gravitational dual of the extremization of central charges and free energies was first described in [8–12]. On the gravitational side we have *extremal functions* that depend on a set of parameters for the abelian isometries of the background and another set of parameters describing the geometry. The extremization with respect to these parameters gives the gravitational free energy of the supergravity solution, that is holographically equal to the central charge or free energy of the dual conformal field theory. Given the close relation between these field theory observables and quantities that are invariant under small deformation of the theory, such as 't Hooft anomalies and Witten indices, it should be of no surprise that the gravitational extremal functions can be built from basic topological objects of the internal geometry. Such is the case for the *Sasakian volume* [8, 9], dual to a and

F-maximization, and its generalization in GK geometry, the *master volume* [11, 12], dual to c and $\mathcal{I}$-extremization.

The extremal problems that we have mentioned up to this point are far from being an exhaustive list. One the most notorious and well-studied example of extremal functions are the *entropy functions* of supersymmetric black holes. The Bekenstein-Hawing formula [13–17] gives a semiclassical prediction for the black hole entropy in terms of the area of the event horizon. The entropy should just be a function of the conserved charges of the black hole, but the supergravity solution from which the horizon area can be computed typically also depends on the asymptotic moduli. This discrepancy is fixed by the *attractor mechanism*, which expresses the value of the moduli fields at the black hole horizon exclusively in terms of the conserved charges. The explicit realization of the attractor mechanism in four-dimensional $\mathcal{N} = 2$ gauged supergravity was found in [18–20]. A notable consequence of this mechanism is that horizon area can be determined as the extremal value of a functional. More in general, the entropy of extremal black holes in various dimensions can be expressed as the Legendre transform of the so-called entropy function [21].

We will focus on asymptotically-AdS black holes, whose microscopical entropy can be investigated by means of the AdS/CFT holography. The microstates of a black hole in the bulk correspond holographically to an ensemble of states of the dual CFT on the boundary. The black hole entropy can then be determined by counting these states, and the result should be compared with the semiclassical prediction given by the Bekenstein-Hawing formula. The first successful microstate counting of this type was obtained for a class of static dyonic BPS black holes in $\mathrm{AdS}_4 \times S^7$ [7, 22], and has been followed by an extensive literature. The BPS states in the dual ABJM theory were counted in [7] by means of a Witten index, the topologically twisted index $\mathcal{I}$ [23]. In the weak-gravity / large-N limit, the index $\mathcal{I}$ reproduces the entropy function of the black holes, whose Legendre transform correctly matches the value of the Bekenstein-Hawing entropy. Interestingly, if we consider the AdS_2 solution arising as the near-horizon limit of the black hole, its dual is a quantum mechanics obtained by dimensionally reducing the CFT. The index $\mathcal{I}$ is invariant under this process and can thus be interpreted as the Witten index of the quantum mechanics. The $\mathcal{I}$-extremization principle then suggests that the critical point of $\mathcal{I}$ selects the exact R-symmetry of the superconformal algebra, and the gravitational dual of this procedure can be understood using the master volume of GK geometry [24, 25].

Extremal functions of known black holes (and black strings) can be expressed in terms of *gravitational blocks*. Inspired by the holomorphic blocks in field theory [26], it was found in [27] using the attractor mechanism in 4d that the entropy function of various black holes and strings in AdS_4 and AdS_5[1] could be obtained by gluing

[1]For black holes and black strings in AdS_5 a dimensional reduction to 4d was performed.

two "blocks", each corresponding to the fixed points of the rotational symmetry of the sphere in the near horizon. This strongly suggested a connection to equivariant localization, given that fixed point formulas frequently appear in the latter. Recently this connection has been made more clear by the suggestion of [28] that extremal functions of a vast class of supergravity solutions can be expressed in terms of the equivariant volume [29] of the internal geometry, a universal topological quantity computable by means of the fixed point formulas of equivariant localization.

In the first part of this thesis we will focus on the case of supersymmetric Kerr-Newman black holes in AdS_5, whose entropy function can be reproduced by computing the large-N limit of the superconformal index [30, 31]. In the second part of this thesis we will generalize the proposal of [28] and show how the extremal function of various systems of branes wrapped around four-dimensional toric orbifolds (or two-cycles within them) can be expressed in terms of the equivariant volume.

1.1 Kerr-Newman black holes in AdS_5

Let us consider the asymptotically $\mathrm{AdS}_5 \times S^5$ supersymmetric black hole solutions of type IIB supergravity [32–36]. In the weak-gravity / large-N limit it should be possible to reproduce the entropy of these black holes by counting the 1/16 BPS states of the dual boundary theory, which is $\mathcal{N} = 4$ Super Yang-Mills on $S^3 \times \mathbb{R}$. These states are counted, with a $(-1)^F$ sign, by the superconformal index [30, 31]. The entropy function of these black holes was found in [37], and in [38] a more general expression for $\mathrm{AdS}_5 \times \mathrm{SE}_5$ black holes with toric Sasaki-Einstein SE_5 was conjectured, despite that black hole solutions of this type are not known for general values of the conserved charges. It would be natural then to expect that the logarithm of the superconformal index of $\mathcal{N} = 1$ quiver theories might be able to reproduce this entropy function in the large-N limit.

Early attempts to compute the superconformal index at large-N did not reproduce the $\mathcal{O}(N^2)$ growth expected from the entropy function, leading to the belief that large cancellations between fermions and bosons caused by the $(-1)^F$ sign made this approach non-viable [31, 39–41]. More recently a solution to this puzzle has been found: when the fugacities associated to electric charges and angular momenta are extended to complex values the cancellations between fermions and bosons states are obstructed [42, 43]. Then at the leading $\mathcal{O}(N^2)$ order the resulting expression for the superconformal index does indeed match the entropy function of $\mathrm{AdS}_5 \times S^5$ black holes [3, 43–45].

The large-N computation of the superconformal index simplifies considerably when the angular momenta are assumed to be equal, $J_1 = J_2$. This special case was the sole focus of the first large-N results [43–45], which all made use of the so-called *Bethe Ansatz formula* [46, 47]. The Bethe Ansatz formula recasts the standard integral representation of the index as a sum over the solutions of a set of transcendental equations, the Bethe Ansatz equations (BAE). When $J_1 \neq J_2$ the terms that contribute to the formula are also indexed by a vector of integers that takes values in the Cartan subalgebra of the gauge group, making the computation technically difficult. These obstacles were overcome in [3], in which we showed that there exist a term in Bethe Ansatz formula that can reproduce the entropy function of AdS_5 BPS black holes with generic values for the conserved charges. A notable shortcoming of [3] is that we only computed a single contribution to the index out of the many competing and exponentially growing ones.

The main goal of the work we will present in chapter 3, which is based on [1], is to seek a better understanding of the large-N limit of the index with generic charges (and especially $J_1 \neq J_2$) by computing a much wider class of competing exponential terms that contribute to the index. In order to achieve this, other than the Bethe Ansatz formula, we will also make use of the *elliptic extension method* [48–50], which consists on a saddle point analysis with the peculiarity that the integrand is not meromorphic, it is instead doubly periodic. We will provide the first application of this method to the case of $J_1 \neq J_2$, and compare the effective action of the large-N saddles with the analogous contributions to the Bethe Ansatz formula, finding a good match.

1.2 Equivariant volume extremization

Equivariant localization, and especially the Atiyah-Bott-Berline-Vergne formula [51, 52], can be used to compute the integral of large class of differential forms in terms of in terms of lower dimensional integrals over the fixed point set of the abelian symmetry of the geometry. If the fixed point set consists of isolated points, the integral reduces to a sum over the fixed points: this is the case for toric orbifolds, which will feature in all the examples we will consider in chapter 5, based on [2]. A very useful quantity that can be computed with this fixed point formula is the equivariant volume [29], which for compact orbifolds can be thought as the generator of all the possible integrals of equivariant Chern forms. In [28] it was proposed that a large class of extremal functions in supergravity can be formulated in terms of the equivariant volume: they showed that the Sasakian volume and the master volume of GK geometry could be extracted from the equivariant volume, and they successfully tested their approach on supergravity solutions that arise as the near-horizon limit of branes wrapped on the sphere or the spindle.

One of the main goal of our work in [2] was to bring further evidence to the proposal of [28] by studying various systems of branes wrapped around four-dimensional toric orbifolds or two-cycles within them. We find that in order to fully capture the quantization of the fluxes of the Ramond-Ramond or M theory forms a slight generalization of the equivariant volume is needed, one that also includes *higher times*. While the Kähler moduli couple linearly with the equivariant Chern forms, the higher times are additional parameters that couple with products of equivariant Chern forms. We find that the equivariant volume with higher times can reproduce the gravitational central charges and free energies of various known supergravity solutions, provided that it is extremized with respect to all the parameters that are not fixed by supersymmetry or the quantization of the fluxes. Our approach can also be applied to cases where the supergravity solution is not explicitly known and we can recover various previously conjectured gravitational block formulas.

We analyze AdS_3 and AdS_5 solutions to eleven dimensional supergravity corresponding to systems of M5 branes, solutions in AdS_2 and AdS_4 of massive type IIA corresponding to D4 branes in the presence of D8 and a O8 plane, and lastly AdS_3 solutions in type IIB arising as the near-horizon limit of D3 branes. In the case of M5 branes we will show that our approach is equivalent to the very recent method of [53], which constructed equivariant forms using Killing spinor bilinears as building blocks and subsequently applied the fixed point formulas of equivariant localization.

1.3 Structure of the thesis

This thesis is structured as follows. In chapter 2 we review the BPS Kerr-Newman black holes in AdS_5, their Bekenstein-Hawking entropy, their entropy function, and how the latter can be reproduced by the large-N limit of the superconformal index. We review the definition and key properties of the superconformal index, together with some useful formulae for its computation, namely the integral representation and the Bethe Ansatz formula.

In chapter 3 we compute the large-N limit of the superconformal index of $\mathcal{N} = 1$ quiver theories. This chapter is based on the paper [1]. First, we briefly review the elliptic extension method and explain how we generalized it to the case of unequal angular momenta. We describe the large-N saddles and compute their effective action. Then we study the contributions to the Bethe Ansatz formula at large-N and compare them with the results obtained with the elliptic extension method.

In chapter 4 we review equivariant localization, toric orbifolds and the equivariant volume. We review the the Atiyah-Bott-Berline-Vergne equivariant localization formula for orbifolds, the definition of toric orbifolds and some key aspects of them: moment maps, polytopes, toric-Käher metrics, the Chern classes associated to the toric divisors. Then we review the definition equivariant volume, its computation by means of a fixed point formula and some of its salient properties. At last we introduce the higher times generalization of the equivariant volume.

In chapter 5 we propose a prescription for extremal functions in supergravity based on the equivariant volume of the internal space, which generalizes the proposal presented in [28]. This chapter is mostly based on the paper [2]. We begin by briefly reviewing the Sasakian and master volumes. Then we present our prescription, which we use to study various systems of branes wrapped around four-dimensional toric orbifolds or two-cycles within them, reproducing the gravitational central charges/free energies and various previously conjectured gravitational block formulas. For systems of M5 branes we consider solutions in AdS_3 and AdS_5 and make contact with the approach of [53]. For systems of D4 branes in massive type IIA we consider solutions in AdS_2 and AdS_4. Lastly for systems of D3 branes in type IIB we consider solutions in AdS_3, generalizing to the orbifold case the results of [54].

In chapter 6, which contains unpublished material, we begin by reviewing the Molien-Weyl formula for the equivariant volume. Then we provide a direct derivation of the formula from the standard formulation of the equivariant volume and discuss the correspondence between residues and fixed points. We revisit the $\mathrm{AdS}_3 \times M_8$ solutions in M theory, reformulating the prescription for extremal functions that we advanced in [2] in terms of the Molien-Weyl formula, and discuss some interesting future directions.

At last, in chapter 7 we briefly summarize our main findings. For a more in-depth discussion of our results and interesting future directions we refer instead to sections 3.4, 5.4 and 6.3, each one pertaining to their respective chapters.

The appendices A and B are respectively the appendices of [1] and [2], the former containing technical aspects pertaining the large-N computation of the superconformal index, the latter providing more details about the parametrization of the Kähler moduli for extremal function and the computation of the gravitational free-energy of the solutions constructed in [55].

Chapter 2

Counting black hole microstates with the superconformal index

In this chapter we review the Bekenstein - Hawking entropy of asymptotically AdS_5 Kerr-Newman BPS black holes in type IIB supergravity, and how it is possible to reproduce such entropy with a microstate count in the dual CFT. The fundamental object used for this count is the superconformal index [30, 31].

This chapter is organized as follows. In section 2.1 we review the Bekenstein - Hawking entropy of $\mathrm{AdS}_5 \times S^5$ black holes. In section 2.2 we discuss how the picture changes when S^5 is substituted with a more general five-dimensional Sasaki-Einstein SE_5. Then in section 2.3 we review the definition of the superconformal index and how it can provide a count for the microstates of AdS_5 black holes. Lastly in section 2.4 we review two formulae for the computation of the index.

2.1 Entropy of BPS Kerr-Newman black holes in $\mathrm{AdS}_5 \times S^5$

A class of supersymmetric Kerr-Newman black holes asymptotic to AdS_5 has been found in [32–36]. They have an embedding in type IIB supergravity on $\mathrm{AdS}_5 \times S^5$, which means that we can probe their microstates using the AdS/CFT correspondence. Let us briefly review their key properties.

From the symmetries of $\mathrm{AdS}_5 \times S^5$ we can determine the conserved charges carried by the black holes. The group of rotations in AdS_5 corresponds to the maximal compact subgroup of the group of isometries of the anti-de Sitter space, which is $SO(4) \subset SO(2,4)$. The maximal torus of $SO(4)$ is $U(1)^2 \subset SO(4)$, which means that the AdS_5 black holes depend on two angular momenta $J_{1,2}$, each corresponding to a Cartan isometry of AdS_5. Furthermore the black holes carry three electric charges $Q_{1,2,3}$ associated to the Cartan isometries of the internal space S^5, considering that its symmetry group is $SO(6)$ which has maximal torus $U(1)^3$.

The black holes that we consider are supersymmetric, more precisely they are 1/16 BPS, since they preserve two real supercharges out of the 32 of type IIB supergravity. The black hole mass is related to the conserved charges by the following linear BPS condition:

$$M = \frac{1}{\ell_5}\Big(|J_1| + |J_2| + |Q_1| + |Q_2| + |Q_3|\Big)\,, \tag{2.1.1}$$

which also makes the black holes extremal. Here ℓ_5 denotes the curvature radius of the anti-de Sitter space.

In order to have regular solutions with no closed time-like curves other nonlinear constraints among the five charges are necessary. We will mention some of these constraints in section 2.1.1 when we review the Legendre transform of the entropy. A consequence of these conditions imposed on the charges is that the angular momenta $J_{1,2}$ cannot be seto to zero, meaning that all these supersymmetric black holes rotate and there is no static limit. Also at most one of the three electric charges $Q_{1,2,3}$ can be zero or negative, the other must be strictly positive.

The Bekenstein–Hawking entropy of these black holes can be expressed as a function of the charges as [56]

$$S_{BH} = \frac{A}{4G_N} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{\pi\ell_5^3}{4G_N}(J_1 + J_2)}\,. \tag{2.1.2}$$

For simplicity we will denote the coefficient that appears in front of the sum of the angular momenta as

$$\nu \equiv \frac{\pi\ell_5^3}{4G_N} = \frac{N^2}{2}\,, \tag{2.1.3}$$

where N is the color number of the dual field theory.

2.1.1 Black hole entropy from an extremization principle

In [37] it was shown that the Bekenstein-Hawking entropy (2.1.2) for the general class of supersymmetric AdS$_5$ black holes discussed in the previous section can be obtained as the Legendre transform of the quantity

$$E\big(X\,;\hat{\omega}\big) = -2\pi i\nu\,\frac{X_1X_2X_3}{\hat{\omega}_1\,\hat{\omega}_2}\,, \tag{2.1.4}$$

as long as the chemical potentials $X_{1,2,3}$ and $\hat{\omega}_{1,2}$ conjugated respectively to the electric charges $Q_{1,2,3}$ and to the angular momenta $J_{1,2}$ satisfy one of the two following choices for the constraint:

$$\sum_{a=1}^{3} X_a = \sum_{i=1}^{2}\hat{\omega}_i \pm 1\,. \tag{2.1.5}$$

$E\big(X\,;\hat{\omega}\big)$ goes by the name of entropy function. The constraint on the chemical potentials mirrors the fact that that the five conserved charges depend on only four parameters. We will now review a simple method of extracting the value of the entropy from $E\big(X\,;\hat{\omega}\big)$ which follows appendix B of [57].

The entropy S_{BH} is given by the extremal value of

$$\mathcal{S}\big(X\,;\hat{\omega}\big) = E\big(X\,;\hat{\omega}\big) - 2\pi i\bigg(\sum_{a=1}^{3} Q_a X_a + \sum_{i=1}^{2} J_i\,\hat{\omega}_i\bigg) - 2\pi i\Lambda\bigg(\sum_{a=1}^{3} X_a - \sum_{i=1}^{2}\hat{\omega}_i \mp 1\bigg)\,, \tag{2.1.6}$$

where Λ is a Lagrange multiplier introduced in order to impose the constraint (2.1.5). Then the critical point of $\mathcal{S}$ can be found as the solution to these equations:

$$Q_a + \Lambda = \frac{1}{2\pi i}\,\frac{\partial E}{\partial X_a} = -\nu\,\frac{X_1X_2X_3}{\hat{\omega}_1\,\hat{\omega}_2\,X_a}\,, \tag{2.1.7}$$

$$J_i - \Lambda = \frac{1}{2\pi i}\,\frac{\partial E}{\partial\,\hat{\omega}_i} = \nu\,\frac{X_1X_2X_3}{\hat{\omega}_1\,\hat{\omega}_2\,\hat{\omega}_i}\,. \tag{2.1.8}$$

Fixing the value of Λ would require using the constraint (2.1.5) afterwards.

However, instead of solving (2.1.7) and (2.1.8) to determine the critical point, we can use the fact that $E\big(X\,;\hat{\omega}\big)$ is a homogeneous function of degree one to write

$$\begin{aligned} E &= \sum_{a=1}^{3} X_a\,\frac{\partial E}{\partial X_a} + \sum_{i=1}^{2}\hat{\omega}_i\,\frac{\partial E}{\partial\,\hat{\omega}_i} = \\ &= 2\pi i\sum_{a=1}^{3} X_a\,\big(Q_a+\Lambda\big) + 2\pi i\sum_{i=1}^{2}\hat{\omega}_i\,\big(J_i-\Lambda\big)\,. \end{aligned} \tag{2.1.9}$$

The second identity is only valid at the critical point. Substituting this expression back into $\mathcal{S}$ we find the entropy S_{BH} in terms of the Lagrange multiplier:

$$S_{BH} = \mathcal{S}\Big|_{\text{crit.}} = \pm\,2\pi i\,\Lambda\,. \tag{2.1.10}$$

There is a simple way to determine the value of Λ. From (2.1.7) and (2.1.8) it is easy to see that the following quantity must be zero:

$$0 = \big(Q_1{+}\Lambda\big)\big(Q_2{+}\Lambda\big)\big(Q_3{+}\Lambda\big){+}\nu\big(J_1{-}\Lambda\big)\big(J_2{-}\Lambda\big) \equiv \Lambda^3 + p_2\,\Lambda^2 + p_1\,\Lambda + p_0\,, \tag{2.1.11}$$

which gives us a third degree equation for Λ. Its coefficients are

$$\begin{aligned} p_2 &= Q_1 + Q_2 + Q_3 + \nu\,,\\ p_1 &= Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \nu(J_1+J_2)\,,\\ p_0 &= Q_1Q_2Q_3 + \nu J_1J_2\,. \end{aligned} \tag{2.1.12}$$

From (2.1.10) we see that the entropy is real only for Λ purely imaginary. Since the coefficients (2.1.12) are real, the polynomial (2.1.11) has imaginary roots only if it can be factored as $(\Lambda^2+x_1)(\Lambda+x_2)$ for some $x_1, x_2 \in \mathbb{R}$, with $x_1 > 0$. The coefficients $p_{1,2,3}$ in terms of $x_{1,2}$ are given by $p_2 = x_2$, $p_1 = x_1$ and $p_0 = x_1 x_2$. This means that in order to have real value for the entropy we must impose that

$$p_0 = p_1 p_2 \,, \qquad p_1 > 0 \,, \tag{2.1.13}$$

which correspond to some of the already mentioned nonlinear constraints among the black hole charges required to avoid closed time-like curves. The imaginary solutions of $(\Lambda^2 + x_1)(\Lambda + x_2) = 0$ are $\Lambda = \pm i\sqrt{x_1}$, we have to chose the one that makes (2.1.10) positive. Either way we find that

$$S_{BH} = 2\pi\, \sqrt{p_1} = 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - \nu(J_1 + J_2)}\,, \tag{2.1.14}$$

which reproduces precisely (2.1.2).

2.2 Entropy of AdS$_5\times$SE$_5$ black holes

The black holes we have discussed so far are asymptotic to AdS$_5 \times S^5$. Solutions with general values for the electric charges for choices of the internal manifold other than S^5 (and its quotient spaces under discrete symmetries) are not known. However in [35] solutions of minimal gauged supergravity that are asymptotically AdS$_5$ are described, which can be embedded in type IIB supergravity on AdS$_5 \times$ SE$_5$ for more general Sasaki-Einstein manifolds other than S^5. These solutions only depend on a single electric charge Q, and their Bekenstein-Hawking entropy can be read from (2.1.2) just by identifying $Q_1 = Q_2 = Q_3 \equiv Q$.

In [38] the following expression for the entropy function of AdS$_5 \times$ SE$_5$ black holes with toric SE$_5$ was conjectured:

$$\begin{aligned}
\mathcal{S}\big(X\,;\hat\omega\big) &= E\big(X\,;\hat\omega\big) - 2\pi i \bigg(\sum_{a=1}^{d} Q_a X_a + \sum_{i=1}^{2} J_i\,\hat\omega_i\bigg) - 2\pi i\Lambda\bigg(\sum_{a=1}^{d} X_a - \sum_{i=1}^{2}\hat\omega_i \mp 1\bigg)\,,\\
E\big(X\,;\hat\omega\big) &= -\,\pi i N^2 \sum_{a,b,c=1}^{d} \frac{C_{abc}}{6}\,\frac{X_a X_b X_c}{\hat\omega_1\,\hat\omega_2}\,.
\end{aligned} \tag{2.2.1}$$

where $N^2 C_{abc} = \frac{1}{4}\,\mathrm{tr}\, R_a R_b R_c$ are 't Hooft anomaly coefficients of the dual theory. R_a gives charge 2 to the a-th chiral multiplet and zero to the others. The trace is over the fermions of the theory. In terms of toric data $C_{abc} = |\det(v^a, v^b, v^c)|$, where the v^a are the vectors that generate the fan.[1] For AdS$_5 \times S^5$ black holes this expression reduces to (2.1.4). Using known relations valid for holographic superconformal theories [58]. we can re-express (2.2.1) at leading $\mathcal{O}(N^2)$ order as

$$E = -\frac{\pi i}{24}\,\frac{(\hat\omega_1 + \hat\omega_2 \pm 1)^3}{\hat\omega_1\,\hat\omega_2}\,\mathrm{tr}\, R(\delta^\pm)^3 = -\frac{4\pi i}{27}\,\frac{(\hat\omega_1 + \hat\omega_2 \pm 1)^3}{\hat\omega_1\,\hat\omega_2}\,a(\delta^\pm)\,, \tag{2.2.2}$$

[1] We will review toric geometry in chapter 4.

where

$$\delta_a^{\pm} = \frac{X_a}{\hat{\omega}_1 + \hat{\omega}_2 \pm 1}\,, \qquad \sum_{a=1}^{d} \delta_a^{\pm} = 2\,. \tag{2.2.3}$$

For the universal black holes of [35] it is straight-forward to check that (2.2.2) reproduces their entropy (see e.g. [3]). Since the δ_a can be seen as parameterizing a trial R-charge, by a-maximization [4] $a(\delta)$ is extremized when δ_a matches the exact R-symmetry and $a(\delta) = \frac{\pi \ell_5^3}{8G_N}$. The rest of the computation is the same as subsection 2.1.1, leading to expression 2.1.2 with $Q_1 = Q_2 = Q_3 = Q$.

In [3] we provided further evidence that (2.2.1) is the correct entropy function by focusing in the case of $\mathrm{AdS}_5 \times T^{1,1}$ with equal angular momenta. $T^{1,1} = \frac{\mathrm{SU(2)}\times\mathrm{SU(2)}}{\mathrm{U(1)}}$ is the five-dimensional Sasaki-Einstein whose Kähler cone is the conifold Calabi-Yau three-fold, and whose dual is the Klebanov-Witten gauge theory [59]. We can use a consistent truncation from type IIB down to 5d $\mathcal{N} = 2$ gauged supergravity (the second one in section 7 of [60]). Even if an AdS_5 black hole solution with general electric charges and equal angula momenta is not known, assuming it exist it would have the topology of AdS_2 fibered over S^3 and we could reduce it along the Hopf fiber of S^3 down to a black hole in four dimensions. The four dimensional black hole would then have the same entropy as the AdS_5 one. Since in four dimensions the attractor mechanism is known [61–63], we can use it to compute the entropy, and the result we obtain is in accordance with (2.2.1).

In the rest of this chapter and in chapter 3 we will discuss how the entropy of supersymmetric Kerr-Newmann black holes can be reproduced from a microstate counting in the dual CFT. The field theory results corroborate (2.2.1).

2.3 Superconformal index and microstate counting

In order to provide a microscopic description of the Bekenstein-Hawking entropy of the AdS_5 black holes we have described in the previous sections we rephrase the problem in terms of counting the number of states of the dual CFT that holographically correspond to the black hole. Under the assumption that for large charges the contribution of single-center black holes dominates, this amounts to counting all BPS states of the dual theory with the appropriate value for the conserved charges. For the $\mathrm{AdS}_5 \times S^5$ black holes in particular we need to count 1/16 BPS states of $\mathcal{N} = 4$ super Yang-Mills.

The BPS states of a supersymmetric CFT transform under short representations of the superconformal algebra, since they are annihilated by some of the supercharges. The superconformal index [30, 31] counts these states with a sign $(-1)^F$, $-$ for fermions and $+$ for bosons. In this section we will review the definition of the index, and in subsections 2.3.1 and 2.3.2 how it is possible to extract the black hole entropy (2.1.2) from it.

Let us consider a generic four dimensional $\mathcal{N}=1$ superconformal theory. We work in radial quantization, in which the euclidean distance from the origin takes the role usually reserved to the time coordinate. Because of conformal invariance the radially quantized euclidean theory on $\mathbb{R}^4$ is equivalent to the same field theory defined on $S^3\times\mathbb{R}$, the conformal boundary of AdS$_5$. Given a supercharge $\mathcal{Q}$ we can define a supersymmetric index as

$$\mathcal{I}(\mu)=\mathrm{tr}\,(-1)^F\,e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\prod_j e^{2\pi i\,\mu_j\mathcal{M}_j}\ , \tag{2.3.1}$$

where $\mathcal{M}_j$ are conserved charges that are invariant under the action of $\mathcal{Q}$ and μ_j are their respective chemical potentials.

As in the case of the Witten index [64], the index (2.3.1) only receives contributions from BPS states that are annihilated by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$, thus it does not actually depend on the parameter β. Indeed from the relation $\{\mathcal{Q},\mathcal{Q}\}=0$ we can deduce that $\mathcal{Q}$ and $\{\mathcal{Q},\mathcal{Q}^\dagger\}$ commute, which combined with the fact that the charges $\mathcal{M}_i$ are $\mathcal{Q}$-closed by assumption and that any supercharge satisfies the relation $\{(-1)^F,\mathcal{Q}\}=0$, we come to the conclusion that the state $\mathcal{Q}|\Omega\rangle$, if not zero, gives a contribution to the trace in (2.3.1) that is opposite to the one coming from the state $|\Omega\rangle$. This means that unless $\mathcal{Q}|\Omega\rangle=0$ (and by similar logic $\mathcal{Q}^\dagger|\Omega\rangle=0$) the contribution of $|\Omega\rangle$ will cancel out. Therefore (2.3.1) can be written equivalently as

$$\mathcal{I}(\mu)=\mathrm{tr}\Big|_{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0}(-1)^F\prod_j e^{2\pi i\,\mu_j\mathcal{M}_j}\ , \tag{2.3.2}$$

where the trace is over the kernel of $\{\mathcal{Q},\mathcal{Q}^\dagger\}$, which coincides with the subspace of BPS states since

$$0=\langle\Omega|\{\mathcal{Q},\mathcal{Q}^\dagger\}|\Omega\rangle=\|\mathcal{Q}|\Omega\rangle\|^2+\left\|\mathcal{Q}^\dagger|\Omega\rangle\right\|^2\quad\Rightarrow\quad\mathcal{Q}|\Omega\rangle=\mathcal{Q}^\dagger|\Omega\rangle=0\,. \tag{2.3.3}$$

The reason why (2.3.2) is called an “index” is the fact that it doesn’t change under continuous deformations of the theory. Indeed under such continuous deformations the quantum numbers of a state with respect to the conserved charges $\mathcal{M}_i$ cannot change; the only thing that could affect the value of (2.3.2) is if a multiplet of states in a long representation of the algebra, not counted by (2.3.2), were to break down into two or more multiplets of BPS states with $\{\mathcal{Q},\mathcal{Q}^\dagger\}=0$, or viceversa. However this would lead to a jump in the value of (2.3.2), which is a continuous functions. We can conclude that continuous deformations of the theory do not affect the index (2.3.2).

To obtain the proper definition of the superconformal index we still need to make an adequate choice for the supercharge $\mathcal{Q}$ and the conserved charges $\mathcal{M}_i$ that appear in (2.3.2).

We denote the conserved charges associated to the Cartan generators of $SO(4)$, the isometry group of S^3, as $J_{1,2}$, mirroring the black hole angular momenta introduced at the beginning of this chapter. Their linear combinations $J_\pm = \frac{1}{2}\left(J_1 \pm J_2\right)$ are the Cartan generators of the subalgebra $\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_- \subset \mathfrak{su}(2,2) \simeq \mathfrak{so}(2,4)$.

It is convenient to assemble the supercharges into doublets of $SU(2)_+$ and $SU(2)_-$. If α and $\dot\alpha$ are the respective spinor indices associated to these two groups we can write the $\mathcal{N}=1$ supercharges as $\{Q^\alpha, \overline{Q}^{\,\dot\alpha}, S_\alpha, \overline{S}_{\dot\alpha}\}$, which amount to 8 real supercharges, since they are related by hermitian conjugation. For a radially quantized theory we need to be careful with the hermitian conjugate of operators, considering that in euclidean space-time hermitian conjugation is accompanied by a time reversal, which in radial quantization corresponds to the inversion $x \to x\,/\,|x|^2$. This leads to nontrivial relations like $P_\mu^\dagger \equiv K_\mu$, which implies that $(Q^\alpha)^\dagger \equiv S_\alpha$, $(\overline{Q}^{\,\dot\alpha})^\dagger \equiv \overline{S}_{\dot\alpha}$.

If we choose Q_2 as the supercharge $\mathcal{Q}$ that enters the definition of the index, then $\mathcal{Q}^\dagger \equiv S_1$ and their anticommutator is given by

$$\frac{1}{2}\,\{\mathcal{Q},\mathcal{Q}^\dagger\} = H - 2J_+ - \frac{3}{2}\,R\,. \tag{2.3.4}$$

Here H denotes the conformal Hamiltonian in radial quantization, which corresponds to the generator of dilatations, while R generates the exact $U(1)_R$ symmetry of the theory. Thus the superconfomal index will only count states for which the quantity on the right hand side of (2.3.4) vanishes.

One can check that the Cartan generators of the subalgebra commuting with $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ are given by $H+J_+$ and J_-. In the case of the latter it is evident since we've chosen supercharges that are in doublets of $SU(2)_+$ and not $SU(2)_-$. Therefore we can define the superconformal index as a function of the chemical potentials τ and σ conjugated to convenient linear combinations of $H+J_+$ and J_-:

$$\mathcal{I}(\tau,\sigma) = \mathrm{tr}\,(-1)^F\,\exp\left[-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}+2\pi i\,\tau\Big(\tfrac{1}{3}\big(H+J_+\big)+J_-\Big)+2\pi i\,\sigma\Big(\tfrac{1}{3}\big(H+J_+\big)-J_-\Big)\right]. \tag{2.3.5}$$

Considering that this expression does not depend on the value of β, we can shift the conserved charges conjugated to the chemical potentials τ and σ by a constant multiple of $\{\mathcal{Q},\mathcal{Q}^\dagger\}$ without affecting the index. In particular if we subtract $\frac{1}{3}\{\mathcal{Q},\mathcal{Q}^\dagger\}$ from them we obtain the much simpler $J_+ \pm J_- + \frac{R}{2} = J_{1,2} + \frac{R}{2}$.

If the flavor symmetry group of the theory is given by G_F then its Cartan generators $\{q_\alpha\}_{\alpha=1}^{\mathrm{rk}(G_F)}$ are $\mathcal{Q}$-closed conserved charges that we can add to the definition of the index as following:

$$\mathcal{I}(\xi,\tau,\sigma) = \mathrm{tr}\,(-1)^F\,\exp\left[-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}+2\pi i\,\tau\big(J_1+\tfrac{R}{2}\big)+2\pi i\,\sigma\big(J_2+\tfrac{R}{2}\big)+2\pi i\sum_{\alpha=1}^{\mathrm{rk}(G_F)}\xi_\alpha\,q_\alpha\right]. \tag{2.3.6}$$

This is the general definition of the four dimensional superconformal index that we will use from now on. The chemical potentials τ and σ are conjugated to the angular momenta $J_{1,2}$, up to a shift of $R/2$, while the flavor chemical potentials $\{\xi_\alpha\}_{\alpha=1}^{\mathrm{rk}(G_F)}$ are conjugated to the charges $\{q_\alpha\}_{\alpha=1}^{\mathrm{rk}(G_F)}$.

Had we chosen $\overline{Q}_{\dot{2}}$ to take the role of the supercharge $\mathcal{Q}$ instead of Q_2 we would have obtained the same expression as (2.3.6) but with a minus sign in front of the R symmetry, obtaining the "left-handed" index, whereas (2.3.6) gives the "right-handed" index. The difference between the two indices can be reabsorbed just by changing the sign of the potentials τ and σ (a change in sign of $J_\pm$ doesn't affect the index because of $SU(2)_\pm$ symmetry). The choice $\mathcal{Q} \equiv Q_1$ on the other hand produces an index that is equivalent to the right-handed one, and similarly $\mathcal{Q} \equiv \overline{Q}_{\dot{1}}$ produces an index equivalent to the left-handed one.

The superconformal index is related to the supersymmetric partition function of the theory on $S^3 \times S^1_\beta$ as follows [65, 66]:

$$\mathcal{Z}_{S^3\times S^1_\beta}(\xi,\tau,\sigma) = e^{-\beta E}\,\mathcal{I}(\xi,\tau,\sigma)\,, \tag{2.3.7}$$

where E is the supersymmetric Casmir energy, which can be expressed as a function of the central charges a, c. The supersymmetric partition function $\mathcal{Z}_{S^3\times S^1_\beta}$ can be computed by means of localization.

2.3.1 $\mathcal{N} = 4$ SYM and the microstates of AdS$_5 \times S^5$ black holes

Let us specialize (2.3.6) to the case of $\mathcal{N} = 4$ super Yang-Mills. The R-symmetry group is $SO(6)_R$, which has $U(1)^3$ for its maximal torus. A basis of Cartan generators is given by $\{R_a\}_{a=1,2,3}$, where R_a gives charge 2 to the chiral superfield Φ_a and zero charge to the other two. In terms of these generator we can write the $U(1)_R$ symmetry and flavor charges $q_{1,2}$ that enter the definition of the superconformal index (3.1.1) as

$$R = \frac{1}{3}\left(R_1 + R_2 + R_3\right)\,, \tag{2.3.8}$$

$$q_{1,2} = \frac{1}{2}\left(R_{1,2} - R_3\right)\,. \tag{2.3.9}$$

For a R-symmetry the charge of the fields in a multiplet lowers by one for progressively higher spins, while in the case of a flavor symmetry all the fields in the same multiplet have the same charge. Since $R_{1,2,3}$ are R-symmetries, from (2.3.8) and (2.3.9) we see respectively that R is a proper R-symmetry while $q_{1,2}$ are flavor charges.

Following [43], it is convenient to define the chemical potentials $\{\Delta_a\}_{a=1,2,3}$ as

$$\Delta_{1,2} = \xi_{1,2} + \frac{\tau+\sigma}{3}\,, \qquad \Delta_3 = \tau + \sigma - \Delta_1 - \Delta_2 \pm 1\,. \tag{2.3.10}$$

The definition of Δ_3 is such that the $\{\Delta_a\}$ satisfy a constraint similar to the one imposed on the $\{X_a\}$ in (2.1.5). If we introduce the fugacities

$$p = e^{2\pi i\tau}\,, \qquad q = e^{2\pi i\sigma}\,, \qquad y_a = e^{2\pi i\Delta_a}\,, \tag{2.3.11}$$

then in terms of them we can write the superconformal index (2.3.6) as

$$\mathcal{I}(p,q,y_1,y_2) = \operatorname{tr}\Big|_{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0} (-1)^F p^{J_1+\frac{R_3}{2}}\, q^{J_2+\frac{R_3}{2}}\, y_1^{q_1}\, y_2^{q_2} = \tag{2.3.12}$$

$$= \operatorname{tr}\Big|_{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0} p^{J_1}\, q^{J_2}\, y_1^{Q_1}\, y_2^{Q_2}\, y_3^{Q_3}\,. \tag{2.3.13}$$

Here $Q_a \equiv R_a/2$ correspond to the black hole charges, which were defined as the Cartan generators of the isometry group $SO(6)$ of the internal manifold S^5, the same way the $R_{1,2,3}$ are (with a different normalization) the Cartan generators of the R-symmetry group $SU(4)_R \simeq SO(6)$. To go from (2.3.12) to (2.3.13) we must notice that $(-1)^F = (-1)^{R_3}$. Indeed the fact that R_3 gives even charge to all the superfields (charge two to Φ_3 and V, charge zero to $\Phi_{1,2}$) means that all boson fields have even charge under R_3 while the fermions have odd charge.

From (2.3.12) we can see that the superconformal index is a single valued function of the fugacities, considering that the flavor charges $q_{1,2}$ are integers and so are $J_{1,2} + \frac{R_3}{2}$. As for the latter R_3 is odd only for fermions, in which case the spin statistic theorem says that $J_{1,2}$ are half an odd integer. On the other hand for bosons $J_{1,2}$ are integers and R_3 is even.

From (2.3.13) one can extract the sought after value of the entropy of $\mathrm{AdS}_5 \times S^5$ black holes as a Fourier coefficient. Schematically:

$$e^{S_{BH}(J,Q)} \approx \int d\tau\, d\sigma\, d\Delta_1\, d\Delta_2\, \mathcal{I}(\tau,\sigma,\Delta_1,\Delta_2)\, e^{-2\pi i\left(\tau J_1+\sigma J_2+\sum_{a=1,2,3}\Delta_a Q_a\right)}\,. \tag{2.3.14}$$

The saddle point approximation of this integral leads to expressing the entropy S_{BH} as the Legendre transform of $\log \mathcal{I}(\tau,\sigma,\Delta_1,\Delta_2,\Delta_3)$ with the constraint $\Delta_1 + \Delta_2 + \Delta_3 = \tau + \sigma \pm 1$. In the light of the extremization principle for the black hole entropy discussed in section 2.1.1, by identifying $\Delta_a \equiv X_a$, $\hat\omega_1 \equiv \tau$ and $\hat\omega_2 \equiv \sigma$ we conclude that in large N limit the logarithm of the superconformal index should reproduce the entropy function (2.1.4), which scales as $O(N^2)$.

The matching between the entropy function and the logarithm of the superconformal index at large N was first achieved in [43] for the case of $\tau = \sigma$, which corresponds to considering black holes with equal angular momenta. Previous attempts [31] to reproduce the black hole entropy had failed to obtain the expected $O(N^2)$ behavior, getting instead a $O(1)$ scaling for the large N limit of the superconformal index. This discrepancy was attributed to large cancellations between boson and fermions states due to the $(-1)^F$ factor in the index. The authors of [31] only considered real values for the fugacities; as pointed out in [42, 43, 57, 67], this corresponds to a Stokes line for the large N behavior of the index. The critical point of the entropy function corresponds to complex values for the fugacities, for which the logarithm of the superconformal index grows as $O(N^2)$.

In [3] we provided the first large-N computation of the index for general charges, including the case of unequal angular momenta $J_1 \neq J_2$, and we reproduced the entropy function (2.1.4).

We note that there is also another limit, other than the large-N one, for which the superconformal index has been show to reproduce the entropy function of AdS_5 black holes: the Cardy limit $\tau, \sigma \to 0$ [42].

2.3.2 Microstate counting for $\mathrm{AdS}_5 \times \mathrm{SE}_5$ black holes

A microstate counting for $\mathrm{AdS}_5 \times \mathrm{SE}_5$ black holes can be done in a similar manner, by computing the large-N limit of the dual field theory. The cone over a toric SE_5 is a Gorenstein toric singularity, and the dual CFT lives in the world-volume of the stack of N D3 branes probing the singularity. The $\mathcal{N} = 1$ quiver theories that in the IR flow in this CFT are called the toric phases and they are related by Seiberg dualities. Such toric phases have as a gauge group multiple copies of $\mathrm{SU}(N)$, and in terms of matter content only have chiral multiplets in the bi-fundamental representation of two of the $\mathrm{SU}(N)$ groups [68]. They can thus be represented with a quiver diagram, a directed graph where the $\mathrm{SU}(N)$ subgroups are the nodes and arrows between nodes represent bi-fundamental chiral multiplets. A prescription on how to extract important data such as the R-charges of a toric phase with minimal content has been proposed in [69].

In [70] it was shown that the Cardy limit of the logarithm of the superconformal index of toric models can reproduce the entropy function (2.2.1).

The first large-N computation of the superconformal index for toric model was done in [45] in the special case $\tau = \sigma$. In [3] we generalized this result to the case of general parameters, thus fully reproducing (2.2.1). We note that the quiver theories considered in these works can also include chiral multiplets in the adjoint representation, which are not needed for the minimal phase of toric models, but it is an easy generalization which is also useful to make contact with $\mathcal{N} = 4$ super Yang-Mills, whose three chiral multiplets are in the adjoint of $\mathrm{SU}(N)$. These quiver theories are also the ones we will focus in chapter 3 in our large-N limit analysis. In the next section we will review the necessary formulae for the computation of the superconformal index.

2.4 Formulae for the superconformal index

In this section we will review two formulae for the superconformal index of quiver theories, the integral representation [30, 31, 71] and the Bethe Ansatz formula [46, 47].

We will discuss how to extract the large-N limit of the index from them in chapter 3.

2.4.1 Integral representation

The index for $\mathcal{N}=1$ superconformal theories in four dimensions has the following integral representation [30, 31, 71]:

$$\mathcal{I}(\xi,\tau,\sigma)=\kappa_G\int_{[0,1]^{\mathrm{rk}(G)}}\mathcal{Z}(u;\xi,\tau,\sigma)\,d^{\,\mathrm{rk}(G)}u\,. \tag{2.4.1}$$

The integration variables are called *holonomies* and they parametrize the Cartan subalgebra of the gauge group G. The factor κ_G in front is defined as

$$\kappa_G=\frac{1}{|\mathcal{W}_G|}\left[\prod_{k=1}^{\infty}\Big(1-e^{2\pi i k\tau}\Big)\Big(1-e^{2\pi i k\sigma}\Big)\right]^{\mathrm{rk}(G)}\,, \tag{2.4.2}$$

where $|\mathcal{W}_G|$ is order of the the Weyl group that acts on the root system of G. The integrand on the other hand is given by the following product of elliptic gamma functions (see appendix A.1 for their definition and properties) :

$$\mathcal{Z}(u;\xi,\tau,\sigma)=\frac{\prod\limits_{I=1}^{n_\chi}\prod\limits_{\rho_I\in\mathcal{R}_I}\Gamma_e\Big(\rho_I(u)+\omega_I(\xi)+r_I\frac{\tau+\sigma}{2}\,;\tau,\sigma\Big)}{\prod\limits_{\alpha\in\Delta}\Gamma_e\Big(\alpha(u)\,;\tau,\sigma\Big)}\,. \tag{2.4.3}$$

Let us look at the numerator first. The index $I=1,\dots,n_\chi$ runs over all chiral superfields of the theory, each one transforming in the representation $\mathcal{R}_I$ of the gauge group G and carrying flavor weight ω_I with respect to the representation $\mathcal{R}_F$ of the flavor symmetry group G_F. The ρ_I denote the weights of $\mathcal{R}_I$ and act on the holonomies u, given how the latter parametrize the Cartan subalgebra of G. While the numerator of (2.4.3) accounts for the contribution of the matter content of the theory, the terms at the denominator come from the vector multiplets. Here Δ denotes the set of the roots of G; indeed the gauge fields are always in the adjoint representation, for which the nonzero weights are the roots of the group.

We will denote $\omega_I(\xi)\equiv\omega_I^\alpha\xi_\alpha$ simply as ξ_I. The $\{\xi_I\}$ are a more convenient parametrization of the flavor chemical potentials, but they are not linearly independent: for each superpotential term W in the Lagrangian they satisfy the constraint

$$\sum_{I\in W}\xi_I=0\,. \tag{2.4.4}$$

It will also be useful to define a new set of chemical potentials $\{\Delta_I\}$ as

$$\Delta_I\;\equiv\;\xi_I+r_I\,\frac{\tau+\sigma}{2} \tag{2.4.5}$$

in analogy to the already mentioned potentials (2.3.10) of $\mathcal{N}=4$ super Yang-Mills. Notably, the superconformal index as a function of τ, σ, Δ_I is invariant under integer shifts of its arguments.[2] For each superpotential term W in the Lagrangian we have that

$$\sum_{I\in W}\Delta_I=\tau+\sigma+n_W\;, \tag{2.4.6}$$

where $n_W\in\mathbb{Z}$ can be chosen arbitrarily, considering that the Δ_I are only defined up to integers. This constraint follows from (2.4.4) and the fact that each superpotential term must have R-charge 2:

$$\sum_{I\in W} r_I=2\,. \tag{2.4.7}$$

The integral representation (2.4.1) of the index is valid in the following domain:

$$\operatorname{Im}(\tau+\sigma)>\operatorname{Im}\Delta_I>0\,,\qquad \operatorname{Im}\tau>0\,,\qquad \operatorname{Im}\sigma>0\,. \tag{2.4.8}$$

Outside the above domain it would be necessary to change the integration contour to avoid the poles of the Γ_e.

2.4.2 Bethe Ansatz formula

In this subsection we will review the Bethe Ansatz formula for the superconformal index [46, 47].

First, we restrict ourselves to values of the chemical potentials τ and σ such that their ratio τ/σ is a rational number. By doing so we do not loose any relevant information: as observed in [47], the set $\{(\tau,\sigma)\in\mathbb{H}^2\mid\tau/\sigma\in\mathbb{Q}\}+\mathbb{Z}^2$ is dense in $\mathbb{H}^2$.[3] Considering that the index as a function of τ, σ, Δ_I is invariant under integer shifts and it is continuous, the value of the index for generic angular chemical potentials can be inferred from the $\tau/\sigma\in\mathbb{Q}$ case For $\tau/\sigma\in\mathbb{Q}_+$ we can then define ω, a and b such that

$$\tau=a\omega,\quad \sigma=b\omega,\qquad \operatorname{Im}\omega>0,\qquad \operatorname{gdc}(a,b)=1\,. \tag{2.4.9}$$

Then the Bethe Ansatz formula expresses the superconformal index as

$$\mathcal{I}(\xi,\tau,\sigma)=\kappa_G\sum_{\hat{u}\in\mathfrak{M}_{\text{BAE}}}\sum_{\{m_i\}=1}^{ab}\mathcal{Z}(\hat{u}-m\omega\,;\xi,\tau,\sigma)\,H^{-1}(\hat{u};\Delta,\omega)\,. \tag{2.4.10}$$

Here $\mathfrak{M}_{\text{BAE}}$ denotes the set of all the inequivalent solutions to the following transcendental equations:

$$Q_i(u;\Delta,\omega)\equiv\prod_{a=1}^{n_\chi}\prod_{\rho_a\in\mathcal{R}_a}P\Big(\rho_a(u)+\Delta_a\,;\omega\Big)^{(\rho_a)_i}=1,\qquad\forall i\in\{1,...,\text{rk}(G)\}\,, \tag{2.4.11}$$

[2] In general the same is not true when the index is written as a function of τ, σ and ξ_a. The reason is that the index is not a single-valued function of the fugacities p, q and v_a, unless all the R-charges of the theory are even.

[3] In our conventions $\mathbb{H}$ is the set of complex numbers with positive imaginary part.

where $P(v;\omega)$ is defined in terms of the θ_0 function (see appendix A.1) as

$$P(v;\omega) \equiv \frac{e^{\pi i\left(v-\frac{v^2}{\omega}\right)}}{\theta_0(v;\omega)} \,. \tag{2.4.12}$$

The equations (2.4.11) are called the Bethe Ansatz equations (BAE in short). The identifications

$$u_i \sim u_i + 1 \sim u_i + \omega \tag{2.4.13}$$

together with Weyl group transformations define the equivalence classes of solutions that constitute the elements of $\mathfrak{M}_{\text{BAE}}$. Indeed the "Bethe Ansatz operator" $Q_i(u;\Delta,\omega)$ can be show to be invariant under (2.4.13) and is trivially invariant under the Weyl group. Lastly, the quantity $H(u;\Delta,\omega)$ that appears in (2.4.10) is a Jacobian and it is given by

$$H(u;\Delta,\omega) = \det\left[\frac{1}{2\pi i}\,\frac{\partial Q_i(u;\Delta,\omega)}{\partial u_j}\right] \,. \tag{2.4.14}$$

The Bethe Ansatz formula (2.4.10) has been derived from the integral representation (2.4.1) in [47]. We will now very briefly review how it is proven. The general idea is to rewrite (2.4.1) in the following trivial way

$$\mathcal{I}(\xi,\tau,\sigma) = \kappa_G \int_{[0,1]^{\text{rk}(G)}} \mathcal{Z}(u;\xi,\tau,\sigma) \prod_{i=1}^{\text{rk}(G)} \frac{1-Q_i(u;\Delta,\omega)}{1-Q_i(u;\Delta,\omega)}\; d^{\,\text{rk}(G)}u \tag{2.4.15}$$

and then apply this shift formula for $\mathcal{Z}$ to the numerator:

$$Q_i(u;\Delta,\omega)\mathcal{Z}(u;\xi,a\omega,b\omega) = \mathcal{Z}(u-\delta_i ab\omega\,;\xi,a\omega,b\omega) \,. \tag{2.4.16}$$

The result can be written as a single contour integral of the function $\mathcal{Z}\cdot\Pi_i(1-Q_i)^{-1}$. The Bethe Ansatz formula (2.4.10) is then obtained by application of the residue theorem.

Chapter 3

Superconformal index at large-N

There are primarily two distinct methods that have been used to compute the superconformal index of $\mathcal{N} = 1$ quiver theories at large-N.[1] The first one makes use of the Bethe Ansatz formula [46, 47], which we reviewed in subsection 2.4.2. The Bethe Ansatz formula simplifies considerably in the particular case of equal angular momenta; for this reason the computations of [43, 73, 74] for $\mathcal{N} = 4$ Super Yang-Mills and [44, 45] for more generic quiver theories were restricted to $J_1 = J_2$. The $J_1 \neq J_2$ case was finally addressed in [3]: a particular contribution to the Bethe Ansatz formula for the index has been shown to reproduce the entropy of Kerr-Newman BPS black holes with arbitrary charges. However, a notable limitation of [3] is that we only computed a single exponentially growing term out of the many competing ones that contribute to the Bethe Ansatz formula.

The other approach to the large-N computation of the superconformal index is the *elliptic extension* method [48–50]. It consists of a saddle point analysis of the matrix integral representation of the index, with the peculiarity that the integrand is not extended analytically outside its contour of integration; instead, it is extended to a doubly periodic function. The action of the matrix integral is thus well-defined on a torus, and a large class of saddle point solutions can be found by taking advantage of its periodicity properties. This method was pioneered in [48] for $\mathcal{N} = 4$ Super Yang-Mills and later generalized to other quiver gauge theories in [49]; furthermore, a reformulation of this approach that that is exact even at finite values of N has been developed in [50]. So far this type of saddle point analysis has been employed only for the case of equal angular momenta; the reason behind this technical restriction is that the modulus of the torus is taken to be equal the chemical potential of the angular momentum $J \equiv J_1 = J_2$.

The primary motivation behind the work presented in this chapter is to better understand the large-N behavior of the superconformal index for general values

[1] There is also the approach of [72], which considered a truncated matrix model for the index and showed that higher order corrections are numerically small.

of BPS charges, especially in the case of unequal angular momenta, $J_1 \neq J_2$. We consider both approaches, the elliptic extension method and the Bethe Ansatz formalism, in order to provide an estimate of the large-N limit of the index. We also investigate the relation between the two methods, focusing in particular on what the saddles of the elliptic action correspond to in the Bethe Ansatz formalism.

First, we extend the saddle point analysis of [48, 49] to the $J_1 \neq J_2$ case. We achieve this by employing the same trick as [47]: we can assume without loss of generality that the angular chemical potentials are integer multiples of the same quantity, that is $\tau = a\omega$ and $\sigma = b\omega$, so that we can take advantage of the properties of the elliptic gamma functions [75] to rewrite the action as a function that is well-defined on a torus of modulus $ab\omega$. We find that the class of solutions to the saddle point equations described in [49] can be easily generalized to the $\tau \neq \sigma$ case, and we compute their effective action.

We then consider the Bethe Ansatz approach to the large-N computation of the index, proceeding as following.

- We revisit the computation of [3], which focused only on a single contribution to the Bethe Ansatz formula, and extend it to encompass a large class of competing exponential terms, finding a good match with the effective action of the elliptic saddles. Our large-N estimate of the superconformal index is thus verified in both formalisms.

- We provide a simplified derivation of the same large-N result of [3]. The most laborious step in the computation of [3] is proving that a particular simplification does not affect the large-N leading order of the index. We show how this step can be avoided altogether, provided that N and ab are coprime.

- We study the relation between the saddles of the elliptic extension method and the solutions to the Bethe Ansatz equations (BAE), with the intent to shed light on the connection between the two different approaches. We find that in the $J_1 = J_2$ case every elliptic saddle corresponds exactly to a BAE solution; however this is no longer true when $J_1 \neq J_2$, since the elliptic action and the Bethe Ansatz equations have different periodicities. Nonetheless, we show that matching elliptic saddles with holonomy configurations that contribute to the Bethe Ansatz formula is always possible, as long as the role of the auxiliary integer variables m_i present in the Bethe Ansatz formalism is taken into consideration. This matching is not always exact: sometimes the two differ by $\mathcal{O}(1/N)$ corrections, which can be shown to produce a negligible effect at leading order.

This chapter is organized as follows. In section 3.1 we introduce the integral representation of the superconformal index of $\mathcal{N} = 1$ quiver theories and we define the elliptic extension of the integrand. In section 3.2 we describe the saddles of the elliptic action and compute their effective action. In section 3.3 we switch to the Bethe Ansatz formalism: in subsection 3.3.2 we study the relation between solutions of the Bethe Ansatz equations and saddles of the elliptic action, while in subsection 3.3.3 we evaluate the large-N limit of the contributions to the Bethe Ansatz formula that correspond to holonomy distributions that match the saddles; lastly, in subsection 3.3.4 we elaborate on the relation between our results and the ones of [3]. In section 3.4 we provide a summary of our results and discuss some open questions.

3.1 The superconformal index of quiver theories

We are interested in computing the large-N limit of the superconformal index of a broad class of four dimensional $\mathcal{N} = 1$ quiver gauge theories. We will focus on theories whose gauge group can be written as the direct sum of $\mathrm{SU}(N)$ subgroups, and with matter fields that transform in either the adjoint or the bifundamental representation. The exact field content of these theories can be summarized in the quiver diagram, a directed graph with $|G|$ nodes and and n_χ arrows (oriented edges) between them, according to the following rules:

- Each node of the quiver denotes a $\mathrm{SU}(N)$ subgroup of the gauge group G.
- An arrow between two distinct nodes denotes a chiral multiplet in the bifundamental representation of the two $\mathrm{SU}(N)$ groups associated to the respective nodes.
- An arrow that has both ends attached to the same node denotes a chiral multiplet in the adjoint representation of the respective $\mathrm{SU}(N)$ subgroup.

Let us specialize the general formula for the integral representation of the superconformal index (2.4.1) to the case of the above class of quiver theories. We will label the nodes of the quiver diagram with the index $\alpha = 1, \ldots, |G|$. The index $I_{\alpha\beta}$ will run over all the arrows of the quiver that start from the node α and end on the node β. Then the integral representation (2.4.1) can be written as

$$\mathcal{I} = \kappa \int [D\underline{u}] \prod_{\alpha=1}^{|G|} \prod_{i\neq j=1}^{N} \Gamma_e\Big(u_{ij}^{\alpha} + \tau + \sigma\,; \tau, \sigma\Big) \cdot \prod_{\alpha,\beta=1}^{|G|} \prod_{I_{\alpha\beta}} \prod_{i,j=1}^{N} \Gamma_e\Big(u_{ij}^{\alpha\beta} + \Delta_I\,; \tau, \sigma\Big), \quad (3.1.1)$$

where the prefactor κ is given by

$$\kappa = \left[\prod_{k=1}^{\infty}\Big(1 - e^{2\pi i k\tau}\Big)\Big(1 - e^{2\pi i k\sigma}\Big)\right]^{|G|(N-1)} \prod_{\alpha=1}^{|G|} \prod_{I_{\alpha\alpha}} \Gamma_e\Big(\Delta_I\,; \tau, \sigma\Big)^{-1} \qquad (3.1.2)$$

and the integration measure is

$$[D\underline{u}\,] = \frac{1}{(N!)^{|G|}} \prod_{\alpha=1}^{|G|} \delta\Big(\sum_{j=1}^{N} u_j^\alpha \Big) \prod_{i=1}^{N} du_i^\alpha \,. \tag{3.1.3}$$

The above variables u_i^a are an over-parametrization of Cartan subalgebra of G: for any given α they parametrize the Cartan subalgebra of $\mathrm{U}(N)$. The delta functions in the measure then restrict them to the Cartan subalgebra of $\mathrm{SU}(N)$. For brevity in (3.1.1) we used the notation

$$u_{ij}^\alpha \,\equiv\, u_i^\alpha - u_j^\alpha \,, \qquad u_{ij}^{\alpha\beta} \,\equiv\, u_i^\alpha - u_j^\beta \,. \tag{3.1.4}$$

The domain of validity of the integral representation of the index is (2.4.8). Throughout the rest of this chapter we will assume that the value chemical potentials Δ_I, τ, σ is within this domain.

Since $\log \kappa = \mathcal{O}(N)$, in the large-$N$ limit this term gives a subleading contribution and can be neglected. Similarly the $\frac{1}{(N!)^{|G|}}$ factor in the measure is also subleading, being $\mathcal{O}(N \log N)$. In the following discussion we will ignore them, as we will only be interested in the leading order $\mathcal{O}(N^2)$ contributions.

In formula (3.1.1) the contour of integration for the holonomies u_i^α lies exclusively on the real axis. The integrand of (3.1.1) can be extended analytically to the rest of the complex plane, since it is a product of elliptic gamma functions, which are are meromorphic. However it is possible to consider different extensions to the complex plane; one of the key ideas behind the saddle-point approach of [48, 49] for the large-N limit of the index is to forgo the analytic extension of the integrand in favor of a doubly periodic one. Focusing exclusively on the $\tau = \sigma$ case, the authors of [48, 49] rewrote the integral representation of the index in terms of the function $Q_{c,d}(z\,;\tau)$, which is a doubly periodic function in z with periodicities $1\,,\ \tau$ that matches the elliptic gamma function on the real axis as following:

$$Q_{c,d}\big(x\,;\tau\big) = \Gamma_e\big(x + (c+1)\tau + d\,;\tau,\tau\big)^{-1} \,, \qquad \forall\, x \in \mathbb{R}\,. \tag{3.1.5}$$

For all $c, d \in \mathbb{R}$ the $Q_{c,d}$ function is defined by [48]

$$Q_{c,d}\,(z\,;\tau) = e^{\pi i \tau \left(\frac{c^3}{3} - \frac{c}{6}\right)} \frac{Q(z + c\tau + d\,;\tau)}{P(z + c\tau + d\,;\tau)^c} \,, \tag{3.1.6}$$

where the functions P and Q are defined by (A.1.17) and (A.1.19) respectively [76–78]. There is an ambiguity in the definition of the phase of P, Q which will play an important role in the discussion of section 3.2.1.

One of the goals of our work is to extend the computation of [48, 49] to the case of unequal angular momenta. We can take advantage of the same observation at the heart of the Bethe Ansatz formula for $\tau \neq \sigma$ [47], that is we can assume without loss of generality that the angular chemical potential τ, σ are such that τ/σ is a rational number (see the discussion at the beginning of subsection 2.4.2). Again, we will define $\omega \in \mathbb{H}$ and integers a, b so that

$$\tau = a\omega\,, \qquad \sigma = b\omega\,, \qquad \gcd(a,b) = 1\,. \tag{3.1.7}$$

We can now take advantage of the following gamma function identity [75]:

$$\Gamma_e\big(z\,; a\omega, b\omega\big) = \prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\Gamma_e\big(z + (as+br)\omega\,; ab\omega, ab\omega\big)\,, \tag{3.1.8}$$

which follows from (A.1.3) and it allows us to write an analogue of (3.1.5) valid for $\tau \neq \sigma$:

$$\Gamma_e\big(x+(c+1)ab\omega+d\,; a\omega, b\omega\big)^{-1} = \prod_{r=0}^{a-1}\prod_{s=0}^{b-1} Q_{\frac{r}{a}+\frac{s}{b}+c\,,\,d}\big(x\,; ab\omega\big)\,, \qquad \forall x \in \mathbb{R}\,. \tag{3.1.9}$$

We can use this relation to rewrite the integral representation of the index (3.1.1) in terms of a new integrand which is doubly periodic but not meromorphic:

$$\begin{aligned}\mathcal{I} = \kappa \int [D\underline{u}] \prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\Bigg[\prod_{\alpha=1}^{|G|}\prod_{i\neq j=1}^{N} Q_{\frac{r+1}{a}+\frac{s+1}{b}-1\,,\,0}\Big(u_{ij}^{\alpha}\,; ab\omega\Big)\cdot \\ \prod_{\alpha,\beta=1}^{|G|}\prod_{I_{\alpha\beta}}\prod_{i,j=1}^{N} Q_{\frac{r}{a}+\frac{s}{b}+(\Delta_I)_2-1\,,\,(\Delta_I)_1}\Big(u_{ij}^{\alpha\beta}\,; ab\omega\Big)\Bigg]^{-1}\,,\end{aligned} \tag{3.1.10}$$

where $(\Delta_I)_{1,2}$ are defined by

$$\Delta_I \;\equiv\; (\Delta_I)_1 + ab\omega(\Delta_I)_2\,, \qquad (\Delta_I)_{1,2} \in \mathbb{R}\,. \tag{3.1.11}$$

This integral representation will be the starting point of the saddle point analysis of section 3.2.

3.2 Large-N saddle points and the effective action

In this section we compute the large-N limit of quiver theories for general angular momenta by following the same saddle-point approach as [48, 49]. First, we write the matrix model (3.1.10) as

$$\mathcal{I} = \int [D\underline{u}]\,\exp\Big(-S(\underline{u})\Big)\,, \tag{3.2.1}$$

where the action $S(\underline{u})$ takes the following form:

$$S(\underline{u}) = S_0 + \sum_{\alpha=1}^{|G|}\sum_{i\neq j=1}^{N} V\big(u_{ij}^{\alpha}\,, \tau+\sigma\big) + \sum_{\alpha,\beta=1}^{|G|}\sum_{I_{\alpha\beta}}\sum_{i,j=1}^{N} V\big(u_{ij}^{\alpha\beta}\,, \Delta_I\big)\,. \tag{3.2.2}$$

Here S_0 is a constant that does not depend on the holonomies $\underline{u}$ and whose value is subleading at large-N, while the function V is defined as following:

$$V\big(z,\Delta\big) = \sum_{r=0}^{a-1}\sum_{s=0}^{b-1} \log Q_{\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1\,,\,(\Delta)_1}\big(z\,;ab\omega\big) \tag{3.2.3}$$

Since $Q_{c,d}(z;ab\omega)$ is doubly-periodic in the variable z with periodicities 1 and $ab\omega$, so is the function V.

The saddle point equations are obtained by varying the quantity

$$S(\underline{u},\underline{\bar{u}}) - \sum_{\alpha=1}^{|G|}\sum_{i=1}^{N}\Big(\lambda^\alpha u_i^\alpha + \widetilde{\lambda}^\alpha \overline{u_i^\alpha}\Big) \tag{3.2.4}$$

with respect to the holonomies $\{u_i^\alpha\}$ and their complex conjugates $\{\overline{u_i^\alpha}\}$. The quantities λ^α and $\widetilde{\lambda}^\alpha$ are Lagrange multipliers required to enforce the SU(N) constraint. We have denoted the action (3.2.2) as $S(\underline{u},\underline{\bar{u}})$ to stress the fact that it is not meromorphic and thus $\partial_{\overline{u_i^\alpha}} S \neq 0$. Varying with respect to u_i^α leads to the following equation:

$$\sum_{j=1}^{N}\Bigg(\partial V(u_{ij}^\alpha,\tau{+}\sigma){-}\partial V(u_{ji}^\alpha,\tau{+}\sigma){+}\sum_{\beta=1}^{|G|}\sum_{I_{\alpha\beta}}\partial V\big(u_{ij}^{\alpha\beta},\Delta_I\big){-}\sum_{\gamma=1}^{|G|}\sum_{I_{\gamma\alpha}}\partial V\big(u_{ji}^{\gamma\alpha},\Delta_I\big)\Bigg) = \lambda^\alpha. \tag{3.2.5}$$

Here ∂V is a shorthand for $\partial_z V(z,\bar{z},\Delta)$. A similar equation with $\bar{\partial}V$ and $\widetilde{\lambda}^\alpha$ replacing ∂V and λ^α is obtained when we vary with respect to $\overline{u_i^\alpha}$.

When $a = b = 1$ equation (3.2.5) and its analogue for $\bar{\partial}V$ match the saddle point equations discussed in [49]. A large class of solutions for the $a = b = 1$ case has been found in [48, 49] using only the periodicity properties of V. When $ab \neq 1$ the expression for V becomes more complicated, but it still remains a doubly periodic function and thus the solutions known for the $a = b = 1$ case can be easily generalized; we will now briefly review them.

Because of the periodicities of V the solutions to equation (3.2.5) live in the torus $E_T \equiv \mathbb{C}/(\mathbb{Z} + T\,\mathbb{Z})$, where $T \equiv ab\omega$. The solutions that we consider are such that $u_i^\alpha = u_i^\beta \equiv u_i$ for all α,β; the advantage of this ansatz is that equation (3.2.5) can now be solved simply by searching for configurations $\{u_i\}_{i=1}^N$ such that the sum

$$\sum_{j=1}^{N}\partial V\big(u_j - u_i,\Delta\big) \tag{3.2.6}$$

does not depend on the value of the index i. This can be achieved by taking $\{u_i\}_{i=1}^N = \mathcal{U}+\bar{u}$, where $\mathcal{U}$ is a finite subgroup of the torus E_T and $\bar{u}$ is some constant ($\bar{u}$ vanishes when we take the difference $u_j - u_i$). Indeed, for any $u_i \in \mathcal{U}$ we have that $\{u - u_i\}_{u\in\mathcal{U}}$

and $\mathcal{U}$ are the same set, and thus the following sum does not actually depend on the value of u_i:

$$\sum_{u\in\mathcal{U}} \partial V(u - u_i\,, \Delta) = \sum_{u\in\mathcal{U}} \partial V(u\,, \Delta)\,, \tag{3.2.7}$$

Thus equation (3.2.5) is solved by the choice $\{u_i\}_{i=1}^N = \mathcal{U} + \bar{u}$, and the same is true for the analogue equation for $\bar{\partial}V$. In particular this means that these solutions to the saddle point equations can be classified by homomorphisms of abelian groups of order N into the torus E_T [49].

Any abelian group G of order N is isomorphic to a product of cyclic groups:

$$G \;\cong\; (\mathbb{Z}\,/\,N_1\,\mathbb{Z}) \times \ldots \times (\mathbb{Z}\,/\,N_\ell\,\mathbb{Z})\,, \tag{3.2.8}$$

where $N_1 \ldots N_\ell = N$. Furthermore, we can assume without loss of generality that each N_i is a divisor of N_{i-1} [2], which we write compactly as $N_i \mid N_{i-1}$. The most general homomorphism of the cyclic group of order N into the torus can be written as

$$i \;\mapsto\; \frac{i}{N}\big(m\,T + n\big)\,, \qquad i \in \mathbb{Z}\,/\,N\,\mathbb{Z}\,, \tag{3.2.9}$$

for some $m, n \in \mathbb{Z}$. Hence, the most general saddle point configuration that corresponds to a finite group homomorphism in the torus takes the following form:

$$u^\alpha_{i_1\ldots i_\ell} = \frac{i_1}{N_1}\big(m_1\,T + n_1\big) + \ldots + \frac{i_\ell}{N_\ell}\big(m_\ell\,T + n_\ell\big) + \bar{u}\,, \tag{3.2.10}$$

where $N_1 \ldots N_\ell = N$ and $N_i \mid N_{i-1}$. The value for the constant $\bar{u}$ is chosen so that the $\mathrm{SU}(N)$ constraint is satisfied:

$$\sum_{i=1}^N u_i^\alpha = 0\,. \tag{3.2.11}$$

Since (3.2.2) only depends on differences between holonomies $\bar{u}$ ultimately cancels out in all the relevant equations. From now on we will omit $\bar{u}$ completely.

We note that different choices of integers $\{N_i, m_i, n_i\}_{i=1}^\ell$ may lead to equivalent solutions, that is solutions that match under the periodicities of the torus E_T or permutations of the index i of u_i^α. As an example, (3.2.10) is invariant under $u_i^\alpha \mapsto -u_i^\alpha$, or equivalently $\{m_i, n_i\}_{i=1}^\ell \mapsto \{-m_i, -n_i\}_{i=1}^\ell$. For this reason we can assume without loss of generality that $m_1 \geq 0$.

[2]This is due to the fact that $(\mathbb{Z}\,/\,m\,\mathbb{Z}) \times (\mathbb{Z}\,/\,n\,\mathbb{Z}) \;\cong\; (\mathbb{Z}\,/\,nm\,\mathbb{Z})$ if $\gcd(m,n) = 1$, hence there are multiple factorizations $N = N_1 \times \ldots \times N_\ell$ that, up to isomorphisms, define the same abelian group G, and it is always possible to find one that satisfies $N_i \mid N_{i-1}\ \forall i$ [49].

3.2.1 Contour deformation

It is not sufficient for the saddles (3.2.10) to be stationary points of the action (3.2.2) for them to contribute to the integral representation of the index (3.1.10); it is necessary for the contour of integration to pass through the saddle point as well. There is a problem: the integrand of (3.1.10) is not meromorphic, and thus it is not possible to use the Cauchy theorem to change contour. An alternative procedure for the deformation of the contour has been used in [48, 49] for the analysis of the $\tau = \sigma$ case; in this section we will show that it can be adapted to the $\tau \neq \sigma$ case as well.

In both integral representations of the superconformal index, (3.1.1) and (3.1.10), each holonomy variable u_i^α is integrated over the interval $[0, 1)$. In the case of (3.1.1) the integrand is meromorphic and we are free to deform the contour as long as we don't cross any poles; however the saddles (3.2.10) are only stationary points of the integral representation with a doubly periodic integrand (3.1.10). The key insight of [48, 49] is that the integrands of (3.1.1) and (3.1.10) are equal when evaluated on any given saddle, as long as the phase of the $Q_{c,d}$ function is chosen appropriately. The idea is to deform the contour of integration of the meromorphic integrand to one that passes thought the saddle point, and then show that the meromorphic integrand can be substituted with the doubly periodic one up to subleading corrections.

In order to show that the argument of [48, 49] can be adapted the $\tau \neq \sigma$ case we only need to check that the integrand of (3.1.1), which is a product of elliptic gamma functions Γ_e, and the integrand of (3.1.10), which depends on the $Q_{c,d}$ function and the choice of its phase, match when the holonomies u_i^α take (3.2.10) as their value.

When z is real the functions $Q_{c,d}(z;\tau)$ and $\Gamma_e(z + (c+1)\tau + d\,; \tau, \tau)^{-1}$ match exactly; otherwise their relation is given by the following formula [48], obtained by substituting (A.1.19) in (3.1.6):

$$Q_{c,d}(z;\tau) = e^{2\pi i\,\alpha_Q(z+c\tau+d)}\, e^{-2\pi i\tau A_c(z_2)}\, \frac{P(z + c\tau + d\,;\tau)^{z_2}}{\Gamma_e(z + (c+1)\tau + d\,;\tau,\tau)}\,. \tag{3.2.12}$$

Here $z_{1,2} \in \mathbb{R}$ are defined by $z \equiv z_1 + \tau\, z_2$. The phase of $Q_{c,d}$ depends on the particular choice for the real-valued function α_Q. Apart from the constraint $\alpha_Q(x) = 0\;\forall\, x \in \mathbb{R}$, α_Q can be chosen arbitrarily in the fundamental domain $0 \leq z_{1,2} < 1$; its value on the rest of the complex plain is then fixed by the requirement that $Q_{c,d}(z;\tau)$ must be doubly periodic in z with periods 1, τ.

The rest of this subsection will be dedicated to showing that the integrands of (3.1.1) and (3.1.10) are equal in absolute value when evaluated on any given saddle. It is then possible to choose α_Q appropriately so that the integrands match in phase as well, and thus the contour deformation argument of [48, 49] can also be applied to the $\tau \neq \sigma$ case.

The function A_c that appears in (3.2.12) denotes the following cubic polynomial:

$$A_c(x) = \tfrac{1}{6}x^3 + \tfrac{1}{2}cx^2 + \tfrac{1}{2}c^2x - \tfrac{1}{12}x\,. \tag{3.2.13}$$

We can show that the total contribution of A_c to the integrand of (3.1.10) vanishes when evaluated at the saddle points, that is

$$\sum_{r,s}\sum_{i,j}\left[|G|\,A_{\frac{r+1}{a}+\frac{s+1}{b}-1}\big((u_{ij})_2\big)+\sum_I A_{\frac{r}{a}+\frac{s}{b}+(\Delta_I)_2-1}\big((u_{ij})_2\big)\right]\Bigg|_{u_i\text{ as in (3.2.10)}}=0\,. \tag{3.2.14}$$

First, we note that the odd powers of x in $A_c(x)$ vanish when we sum over i,j since $(u_{ij})_2$ changes sign when i and j are exchanged. This leaves only the quadratic term in x, which is proportional to c; when we sum over r,s and all the multiplet contributions the c-terms vanish:

$$\begin{aligned}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}&\left[|G|\left(\frac{r+1}{a}+\frac{s+1}{b}-1\right)+\sum_I\left(\frac{r}{a}+\frac{s}{b}+(\Delta_I)_2-1\right)\right]=\\ &=\frac{a+b}{2}\left[|G|+\sum_I\left(\frac{2ab(\Delta_I)_2}{a+b}-1\right)\right]=0\,.\end{aligned} \tag{3.2.15}$$

The term in the square bracket in the second line can be shown to be vanishing by imposing the $\mathrm{U}(1)_\mathrm{R}$ - gauge^2 anomaly cancellation condition, which for the quiver theories that we are considering can be written as following[3]:

$$|G|+\sum_I\big(\widetilde{r}_I-1\big)=0\,. \tag{3.2.16}$$

This relation is valid for *any* R-symmetry. Then (3.2.15) follows from (3.2.16) if we consider the R-symmetry obtained by assigning the following charges to each chiral multiplet:

$$\widetilde{r}_I\equiv\frac{2ab(\Delta_I)_2}{a+b}\,. \tag{3.2.17}$$

Because of relation (2.4.6) this choice of R-charges does indeed satisfy

$$\sum_{I\in W}\widetilde{r}_I=2 \tag{3.2.18}$$

for every superpotential term W in the Lagrangian.

The contribution of $\log|P|$ to the integrand is vanishing as well:

$$\sum_{i,j=1}^{N}(u_{ij})_2\,\log\Big|P\big(u_{ij}+(c+1)+d\,;\tau\big)\Big|=0\,. \tag{3.2.19}$$

This relation can be derived from the double Fourier expansion of $\log|P|$ (A.1.18) and the fact that sums of the following type vanish:

$$\sum_{i_k,j_k=1}^{N_k}(i_k-j_k)\,e^{2\pi i\left(\frac{i_k-j_k}{N_k}(m_kn-n_km)\right)}=0\,. \tag{3.2.20}$$

[3] Condition (3.2.16) is equivalent to the statement that $\operatorname{tr}\widetilde{R}=\mathcal{O}(1)$ at large-N, for any R-symmetry $\widetilde{R}$. For more details we refer to appendix B of [49].

Since the contribution of the A_c and $\log|P|$ terms is overall zero, from (3.2.12) we see that the integrands of (3.1.1) and (3.1.10) are equal in absolute value on the saddles, which is what we needed to show.

We conclude this subsection by mentioning that in [50] a more rigorous framework for this type of saddle point analysis has been presented, based on Atiyah-Bott-Berline-Vergne equivariant integration formula [51, 52]. The method of [50] is also applicable at finite N, and it provides more solid evidence for the fact that the (3.2.10) saddles do indeed contribute to index.

3.2.2 Continuum limit

In the large-N limit the saddles (3.2.10) become uniform continuous distributions. We can make the substitutions

$$u_i^\alpha \longmapsto u^\alpha(x)\,, \qquad \sum_{i=1}^{N} \longmapsto N\int_0^1 dx \tag{3.2.21}$$

and replace the discrete action (3.2.2) with a large-N effective action $S_{\text{eff}}[u]$, which is a functional of the distribution $u^\alpha(x)$ and is given by

$$S_{\text{eff}}[u] = N^2\int_0^1 dx\int_0^1 dy\left[\sum_{\alpha=1}^{|G|} V\big(u^\alpha(x)-u^\alpha(y)\,,\tau+\sigma\big)+\sum_{\alpha,\beta=1}^{|G|}\sum_{I_{\alpha\beta}} V\big(u^\alpha(x)-u^\beta(y),\Delta_I\big)\right]. \tag{3.2.22}$$

The stationary points of this action can be found by extremising the functional

$$S_{\text{eff}}[u] - \sum_{\alpha=1}^{|G|}\int_0^1 dx\Big(\lambda^\alpha u^\alpha(x) + \widetilde{\lambda}^\alpha\overline{u^\alpha(x)}\Big)\,, \tag{3.2.23}$$

and correspond to the continuum limit of the discrete saddles (3.2.10). The superconformal index at large-N can then be written as a sum over these stationary points:

$$\mathcal{I} \sim \sum_{u\in\{\text{saddles}\}} \exp\big(-S_{\text{eff}}[u]\big)\,. \tag{3.2.24}$$

In order to take the continuum limit of the saddles (3.2.10) we need to distinguish between a few cases. Each saddle depends on a particular factorization of N, that is $N \equiv N_1\ldots N_\ell$ with $N_i \mid N_{i-1}\ \forall\, i:$

$$u^\alpha_{i_1\ldots i_\ell} = \frac{i_1}{N_1}\big(m_1 T + n_1\big) + \ldots + \frac{i_\ell}{N_\ell}\big(m_\ell T + n_\ell\big)\,. \tag{3.2.25}$$

Hence, the $N\to\infty$ limit can be realized in multiple ways.

Let us consider the case of saddles with $\ell = 1$ first. After the the substitution $i_1/N_1 \mapsto x$ they become

$$u^\alpha(x) = x\,\big(mT + n\big)\,. \tag{3.2.26}$$

We omitted the subscript on m_1 and n_1 as it is no longer needed. The effective action for these saddles can be written as

$$\begin{aligned}
S_{\text{eff}}(m,n) &= \\
&= N^2 \int_0^1 dx\, dy \left[|G|\, V\Big((x-y)(mT+n), \tau+\sigma\Big) + \sum_I V\Big((x-y)(mT+n), \Delta_I\Big)\right] \\
&= N^2 \int_0^1 dx \left[|G|\, V\Big(x(mT+n), \tau+\sigma\Big) + \sum_I V\Big(x(mT+n), \Delta_I\Big)\right] .
\end{aligned} \tag{3.2.27}$$

The second equality follows from the fact that $mT+n$ is a period of V. Another consequence of the periodicity of V is that $S_{\text{eff}}(m,n) = S_{\text{eff}}(m/h, n/h)$, where $h \equiv \gcd(m,n)$. This is expected, considering that the (m,n) saddle describes a distribution of holonomies equivalent to the one of the $(m/h, n/h)$ saddle. Thus for the $\ell = 1$ "string-like" saddles we can assume that $\gcd(m,n) = 1$ without loss of generality. We postpone the computation of $S_{\text{eff}}(m,n)$ to section 3.2.3.

We consider the $\ell = 2$ saddles now. Let us first assume that $N_2 \sim \mathcal{O}(1)$ at large-N. We can make the substitution $i_1/N_1 \mapsto x$ and write the $\ell = 2$ saddles in the continuum limit as

$$u^{\alpha}_{i_2}(x) = x\,\big(m_1 T + n_1\big) + \frac{i_2}{N_2}\big(m_2 T + n_2\big)\,. \tag{3.2.28}$$

If we want to write the saddle without the extra index i_2 we can change variables to $x_{\text{new}} \equiv x/N_2 + i_2/N_2$ so that

$$u^{\alpha}(x) = \{N_2\, x\}\, \big(m_1 T + n_1\big) + \frac{\lfloor N_2\, x \rfloor}{N_2}\, \big(m_2 T + n_2\big)\,, \tag{3.2.29}$$

where $\{N_2\, x\} \equiv N_2\, x - \lfloor N_2\, x\rfloor$. It is straightforward to see that (3.2.29) extremises the effective action (3.2.22) for any value of N_2. For convenience we will use representation (3.2.28) and keep the index i_2; the effective action is then given by

$$\begin{aligned}
S_{\text{eff}}(m_1, n_1; m_2, n_2, N_2) = \frac{N^2}{N_2} \sum_{i_2=1}^{N_2} \int_0^1 dx \Bigg[|G|\, V\Big(x(m_1 T + n_1) + \frac{i_2}{N_2}(m_2 T + n_2), \tau+\sigma \Big) + \\
+ \sum_I V\Big(x(m_1 T + n_1) + \frac{i_2}{N_2}(m_2 T + n_2), \Delta_I \Big)\Bigg].
\end{aligned} \tag{3.2.30}$$

Again, without loss of generality we can assume that $\gcd(m_1, n_1) = \gcd(m_2, n_2) = 1$. We postpone the computation of (3.2.30) to section 3.2.4.

If we take $N_2 \to \infty$ in (3.2.28) we obtain the "surface" saddles:

$$u^{\alpha}(x,y) = x\,\big(m_1 T + n_1\big) + y\,\big(m_2 T + n_2\big)\,. \tag{3.2.31}$$

The effective action for these saddles is the same as (3.2.30), provided that the following substitutions are made:

$$\frac{i_2}{N_2} \longmapsto y\ , \qquad \frac{1}{N_2}\sum_{i_2=1}^{N} \longmapsto \int_0^1 dy\ . \tag{3.2.32}$$

Because of the periodicity of the potential V, as long as (m_1, n_1) and (m_2, n_2) are linearly independent the effective action of surface saddles does not depend on any of these integers:

$$\int_0^1 dx \int_0^1 dy\, V\Big(x(m_1 T + n_1) + y(m_2 T + n_2), \Delta\Big) = \int_0^1 dx \int_0^1 dy\, V\Big(x\, T + y\, , \Delta\Big)\,. \tag{3.2.33}$$

On the other hand if (m_1, n_1) and (m_2, n_2) are linearly dependent the saddle (3.2.31) is just equivalent to one of the "string-like" saddles (3.2.26).

The saddles with $\ell \geq 3$ in the continuum limit are always equivalent to one of the already discussed cases, (3.2.26), (3.2.28) or (3.2.31). To see why, let us fist assume that $m_1 \neq 0$. We can rewrite the $\ell = 2$ saddle (3.2.28) by shifting $x \mapsto x - (i_2/N_2)(m_2/m_1)$, obtaining the following equivalent expression:

$$u^{\alpha}_{i_2}(x) = x\left(m_1\, T + n_1\right) + \frac{i_2}{N_2}\,\frac{m_1 n_2 - m_2 n_1}{m_1}\,. \tag{3.2.34}$$

Similarly, a generic saddle with $\ell = 3$ and $m_1 \neq 0$ after the $i_1/N_1 \mapsto x$ substitution and analogue shifts can be written as

$$u^{\alpha}_{i_2,i_3}(x) = x\left(m_1\, T + n_1\right) + \frac{i_2}{N_2}\,\frac{m_1 n_2 - m_2 n_1}{m_1} + \frac{i_3}{N_3}\,\frac{m_1 n_3 - m_3 n_1}{m_1}\,. \tag{3.2.35}$$

Considering that $N_3 \mid N_2$ and $\gcd(m_1, n_1) = 1$, it is always possible to find $(\widetilde{m}_2, \widetilde{n}_2)$ such that the $\ell = 2$ saddle with $m_1, n_1, \widetilde{m}_2, \widetilde{n}_2, N_2$ is equivalent to (3.2.35). The $m_1 = 0$ case is similar: the saddle

$$u^{\alpha}_{i_2,i_3}(x) = x + \frac{i_2}{N_2}\big(m_2\, T + n_2\big) + \frac{i_3}{N_3}\big(m_3\, T + n_3\big) \tag{3.2.36}$$

can be rewritten as

$$u^{\alpha}_{i_2,i_3}(x) = x + \frac{i_2}{N_2}\, m_2\, T + \frac{i_3}{N_3}\, m_3\, T \tag{3.2.37}$$

by shifting $x \mapsto x - (i_2/N_2)n_2 - (i_3/N_3)n_3$; it is then always possible to find $\widetilde{N}_2$ such that the saddle is equivalent to

$$u^{\alpha}_{\widetilde{i}_2}(x) = x + \frac{\widetilde{i}_2}{\widetilde{N}_2}\, T\,. \tag{3.2.38}$$

In conclusion, there is no need to consider saddles with $\ell \geq 3$ in the continuum limit. This argument does not hold at finite N however; we will discuss the saddles (3.2.10) at finite N in more detail in section 3.3.2.

3.2.3 String-like saddles

In this section we focus of the saddles $u^\alpha(x) = x(mT+n)$ and compute their effective action $S_{\text{eff}}(m,n)$. Without loss of generality we can assume that $\gcd(m,n) = 1$ and $m \geq 0$. Given (3.2.3) and (3.2.27), the effective action of these saddles takes the following form:

$$S_{\text{eff}}(m,n) = N^2 \int_0^1 dx \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \Bigg[|G| \log Q_{\frac{r+1}{a}+\frac{s+1}{b}-1\,,\,0}\Big(x(mab\omega+n)\,;ab\omega\Big) + \\ + \sum_I \log Q_{\frac{r}{a}+\frac{s}{b}+(\Delta_I)_2-1\,,\,(\Delta_I)_1}\Big(x(mab\omega+n)\,;ab\omega\Big)\Bigg]\,. \tag{3.2.39}$$

When $m \neq 0$ the integral can be computed using formula (A.1.22), which we can write as

$$\int_0^1 dx \log Q_{c,d}\Big(x(m\tau{+}n)\,;\tau\Big) = -\frac{\pi i}{6}c\tau + \frac{\pi i}{3}\,\frac{B_3\big([m(c\tau+d)]'_{m\tau+n}\big)}{m(m\tau+n)^2} + \Big(\text{purely imaginary}\Big)\,, \tag{3.2.40}$$

where the function $[\,\cdot\,]'_\tau$ is defined as follows:

$$\big[x+y\tau\big]'_\tau = \begin{cases} x - \lfloor x \rfloor + y\tau & \text{for } x \in \mathbb{R}\setminus\mathbb{Z},\ y \in \mathbb{R} \\ \text{either }\ y\tau\ \text{ or }\ y\tau+1 & \text{for } x \in \mathbb{Z},\ y\in\mathbb{R} \end{cases}\,. \tag{3.2.41}$$

There is an ambiguity in the definition of $[z]'_\tau$ when $z \in \mathbb{Z} + \tau\,\mathbb{R}$; however, because of property (A.1.13) of the Bernoulli polynomials and the fact that $B_n(0) = B_n(1)$, one can see that equation (3.2.40) is unaffected by this ambiguity. The purely imaginary terms left out from equation (3.2.40) do not actually contribute to the large-N leading order of the effective action, considering that S_{eff} is defined up to multiples of $2\pi i$.

Using (3.2.40) we find the contribution of a single multiplet to the effective action:

$$N^2 \sum_{r=0}^{a-1}\sum_{s=0}^{b-1} \int_0^1 dx\ \log Q_{\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1\,,\,(\Delta)_1}\Big(x(mab\omega+n)\,;ab\omega\Big) = \tag{3.2.42}$$

$$= -\frac{\pi i}{6}\,abN^2\Big(ab\omega(\Delta)_2 - \frac{\tau+\sigma}{2}\Big) + \pi i N^2 \sum_{r=0}^{a-1}\sum_{s=0}^{b-1} \frac{B_3\big([m\Delta + m\omega(as+br-ab)]'_{mab\omega+n}\big)}{3m(mab\omega+n)^2}\,,$$

where $\Delta \equiv \tau+\sigma$ for vector multiplets and $\Delta \equiv \Delta_I$ for the I-th chiral multiplet. When we sum over all multiplet contributions the first term in the second line of (3.2.42) gives an overall null contribution because of anomaly cancellation relations; it is indeed the same (3.2.15) term that we discussed in section 3.2.1, up to a proportionality constant. The effective action for (m,n) saddles with $m \neq 0$ can thus be written as

$$S_{\text{eff}}(m,n) = \pi i N^2 \left(|G|\, \Psi_{m,n}(\tau+\sigma) + \sum_I \Psi_{m,n}(\Delta_I) \right) , \tag{3.2.43}$$

where $\Psi_{m,n}(\Delta)$ denotes the following quantity:

$$\Psi_{m,n}(\Delta) = \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \frac{B_3\big([m\Delta + m\omega(as+br-ab)]'_{mab\omega+n}\big)}{3m(mab\omega+n)^2} . \tag{3.2.44}$$

As a simple check, we notice that (3.2.44) is invariant under $(m,n) \mapsto (-m,-n)$, as expected. Using that $[-z]'_\tau = 1-[z]'_\tau$ and property (A.1.11) of the Bernoulli polynomials, we can see that under $(m,n) \mapsto (-m,-n)$ the numerator of the summand in (3.2.44) changes sign; since the denominator changes sign as well, (3.2.44) is indeed invariant.

When $a=b=1$ we have $\tau=\sigma=\omega$ and the effective action (3.2.43) matches the analogous result obtained in [49]. It is also in accord with the results [43–45, 73, 74] derived from the Bethe Ansatz formula. As for the $a \neq b$ case, the effective action of the $(m,n) = (1,0)$ saddle matches perfectly the contribution to the index we computed in [3] using the Bethe Ansatz formalism; we will discuss in more detail the relation between the saddle point and the Bethe Ansatz approaches in section 3.3.2. As we showed in [3] the contribution to the index corresponding to the $(m,n) = (1,0)$ saddle reproduces the entropy of supersymmetric Kerr-Newman AdS$_5$ black holes, as we will now briefly review.

3.2.3.1 The $(m,n) = (1,0)$ saddle

As we already noted in [3], expression (3.2.44) can be simplified significantly when $(m,n) = (1,0)$. Using the translation property of the Bernoulli polynomials (A.1.13) it is possible to write $\Psi_{1,0}(\Delta)$ as

$$\begin{aligned}\Psi_{1,0}(\Delta) &= \frac{1}{3(ab\omega)^2} \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} B_3\Big([\Delta]'_\omega + \omega(as+br-ab)\Big) = \\ &= \frac{1}{3} \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{k=0}^{3} \binom{3}{k} (ab\omega)^{k-2} \left(\frac{r}{a} + \frac{s}{b} + \frac{a+b}{2ab} - 1\right)^k B_{3-k}\Big([\Delta]'_\omega - \frac{\tau+\sigma}{2}\Big).\end{aligned} \tag{3.2.45}$$

The sum over r and s can now be easily computed by means of a simple trick; we consider the following power series

$$\sum_{k=0}^{\infty} \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \left(\frac{r}{a} + \frac{s}{b} + \frac{a+b}{2ab} - 1\right)^k \frac{(2t)^k}{k!} = \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} e^{2t\left(\frac{r}{a}+\frac{s}{b}+\frac{a+b}{2ab}-1\right)} = \frac{\sinh^2 t}{\sinh\frac{t}{a}\sinh\frac{t}{b}} =$$

$$= ab + \frac{t^2}{6}\left(2ab - \frac{a}{b} - \frac{b}{a}\right) + O(t^4)\,. \tag{3.2.46}$$

which gives us immediately the relations that we need:

$$\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\left(\frac{r}{a}+\frac{s}{b}+\frac{a+b}{2ab}-1\right)^k = \begin{cases} ab & \text{for } \; k=0 \\ 0 & \text{for } \; k=1 \\ \frac{1}{12}\left(2ab-\frac{a}{b}-\frac{b}{a}\right) & \text{for } \; k=2 \\ 0 & \text{for } \; k=3 \end{cases} \tag{3.2.47}$$

Substituting (3.2.47) in (3.2.45) we get

$$\Psi_{1,0}(\Delta) = \frac{1}{3\tau\sigma}\,B_3\Big([\Delta]'_\omega - \frac{\tau+\sigma}{2}\Big) + \frac{1}{12}\Big(2ab - \frac{a}{b} - \frac{b}{a}\Big)B_1\Big([\Delta]'_\omega - \frac{\tau+\sigma}{2}\Big). \tag{3.2.48}$$

When we sum over all the multiplets, the total contribution to the effective action $S_{\text{eff}}(1,0)$ coming from the B_1 terms is purely imaginary and at leading N^2 order can be neglected. Indeed the ω - dependent part of the B_1 term gives a total contribution proportional to the term in the second line of (3.2.15). Therefore, we can equivalently define the function $\Psi_{1,0}(\Delta)$ as

$$\Psi_{1,0}(\Delta) \,\equiv\, \frac{1}{3\tau\sigma}\,B_3\Big([\Delta]'_\omega - \frac{\tau+\sigma}{2}\Big)\,. \tag{3.2.49}$$

The disappearance of the term proportional to $2ab$ is not surprising considering that the index is ultimately a continuous function of $\tau = a\omega$ and $\sigma = b\omega$.

By using the explicit expression (A.1.12) for B_3 it is possible to write (3.2.49) as

$$\Psi_{1,0}(\Delta) = \frac{1}{24\tau\sigma}\Big[\big(2[\Delta]'_\omega - \tau - \sigma - 1\big)^3 - \big(2[\Delta]'_\omega - \tau - \sigma - 1\big)\Big]\,. \tag{3.2.50}$$

We can then introduce variables $\hat{\Delta}^\pm$ so that

$$2[\Delta]'_\omega - \tau - \sigma - 1 = (\tau + \sigma \pm 1)(\hat{\Delta}^\pm - 1)\,, \tag{3.2.51}$$

or explicitly

$$\hat{\Delta}^+ = \frac{2[\Delta]'_\omega}{\tau+\sigma+1}\,, \qquad \hat{\Delta}^- = \frac{2([\Delta]'_\omega - 1)}{\tau+\sigma-1}\,. \tag{3.2.52}$$

In light of constraint (2.4.6), it is natural to expect that there are regions of the parameter space where

$$\sum_{I\in W}\hat{\Delta}_I^\pm = 2\,, \tag{3.2.53}$$

for superpotential terms W and either $+$ sign or $-$ sign. In [3] we verified that in toric models such regions always exists. Then the $\hat{\Delta}_I^\pm$ can be interpreted as R-charges for a trial R-symmetry.

In the region of parameter space where the $(m,n)=(1,0)$ dominates and either $\hat{\Delta}^{+}$ or $\hat{\Delta}^{-}$ can be interpreted as a trial R-symmetry, the large-N limit of the index will be given by

$$\begin{aligned}\mathcal{I} &= \pi i N^2\Big(|G|\,\Psi_{1,0}(\tau+\sigma)+\sum_I \Psi_{1,0}(\Delta_I)\Big)+o(N^2)= \\ &= \frac{\pi i}{24}\,\frac{(\tau+\sigma\pm 1)^3}{\tau\sigma}\,\operatorname{tr} R(\hat{\Delta}^{\pm})^3+\frac{\pi i}{24}\,\frac{\tau+\sigma\pm 1}{\tau\sigma}\,\operatorname{tr} R(\hat{\Delta}^{\pm})+o(N^2)\end{aligned} \tag{3.2.54}$$

For theories with a holographic dual it is possible to show that $\operatorname{tr} R=\mathcal{O}(1)$, so the second term is actually vanishes at leading order. The entropy function of the dual $\mathrm{AdS}_5\times\mathrm{SE}_5$ black holes (2.2.2) is thus perfectly reproduced.

3.2.3.2 The $(m,n)=(0,1)$ saddle

So far we have assumed $m\neq 0$; let us now discuss the $m=0$ case. The requirement $\gcd(m,n)=1$ only leaves $n=\pm 1$ as possible choices, and they are equivalent; hence, there is only one saddle with $m\neq 0$. As we will now show, the effective action of this saddle is zero at the leading N^2 order, which is coherent with the results obtained in [43, 48, 49] for the $\tau=\sigma$ case.

For the $(m,n)=(0,1)$ saddle the $\{u_i^{\alpha}\}$ are all real and thus the doubly periodic function $Q_{c,d}$ simply coincides with the analytic elliptic gamma, and thus the action $S(\underline{u})$ given by (3.2.2) and (3.2.3) is just minus the logarithm of the integrand of (3.1.1). We find it easier in this case to work with the elliptic gamma functions directly rather than the $Q_{c,d}$.

First, let us look at the contribution to the effective action of the $(m,n)=(0,1)$ saddle coming from a chiral multiplet. Using the property (A.1.4) and the definition (A.1.1) of the elliptic gamma function we can write it as

$$\begin{aligned}&-\sum_{i,j=1}^{N}\log\Gamma_e\Big(\Delta_I+\frac{i-j}{N}\,;a\omega,b\omega\Big)=-N\log\Gamma_e\Big(N\Delta_I\,;Na\omega,Nb\omega\Big)= \\ &\quad= N\sum_{j,k=0}^{\infty}\left[\log\left(1-e^{2\pi i\,N\left(ja\omega+kb\omega+\Delta_I\right)}\right)-\log\left(1-e^{2\pi i\,N\left((j+1)a\omega+(k+1)b\omega-\Delta_I\right)}\right)\right].\end{aligned} \tag{3.2.55}$$

If either $\big((j+1)a\omega+(k+1)b\omega-\Delta_I\big)$ or $\big(ja\omega+kb\omega+\Delta_I\big)$ had a negative imaginary part the respective logarithm term would be $\mathcal{O}(N)$ and we would get nonzero contributions at the N^2 order. However in the domain (2.4.8) the imaginary part of these terms is always positive and at large-N all the logarithms are exponentially suppressed. Hence, the chiral multiplet contribution is null at the N^2 order.

The contribution to the effective action coming from the vector multiplets is subleading as well. We can write it as

$$-|G|\sum_{i\neq j=1}^{N}\log\Gamma_e\Big(a\omega+b\omega+\frac{i-j}{N}\,;a\omega,b\omega\Big)=-N|G|\sum_{\ell=1}^{N-1}\log\Gamma_e\Big(a\omega+b\omega+\frac{\ell}{N}\,;a\omega,b\omega\Big). \tag{3.2.56}$$

This term is of order $\mathcal{O}(N\log N)$. Indeed, if we substitute the definition (A.1.1) of the elliptic gamma function in the following product

$$\prod_{\ell=1}^{N-1}\Gamma_e\Big(a\omega+b\omega+\frac{i}{N}\,;a\omega,b\omega\Big)= \tag{3.2.57}$$

$$=\prod_{\ell=1}^{N-1}\left[\Big(1-e^{-2\pi i\frac{\ell}{N}}\Big)\Big(1-e^{2\pi i\left(a\omega-\frac{\ell}{N}\right)}\Big)\Big(1-e^{2\pi i\left(b\omega-\frac{\ell}{N}\right)}\Big)\prod_{j,k=1}^{\infty}\frac{1-e^{2\pi i\left(a\omega+b\omega-\frac{\ell}{N}\right)}}{1-e^{2\pi i\left(a\omega+b\omega+\frac{\ell}{N}\right)}}\right],$$

then we can use a slight modification of identity (A.1.5),

$$\prod_{\ell=1}^{N-1}\Big(1-e^{-2\pi i\frac{\ell}{N}}\,z\Big)=\frac{1-z^N}{1-z}=1+z+\ldots+z^{N-1}\,, \tag{3.2.58}$$

to conclude that

$$\prod_{\ell=1}^{N-1}\Gamma_e\Big(a\omega+b\omega+\frac{i}{N}\,;a\omega,b\omega\Big)=N\;\frac{1-e^{2\pi iNa\omega}}{1-e^{2\pi ia\omega}}\;\frac{1-e^{2\pi iNb\omega}}{1-e^{2\pi ib\omega}}=\mathcal{O}(N)\,, \tag{3.2.59}$$

and thus (3.2.56) does not contribute to the leading N^2 order either.

3.2.4 General saddles

In section 3.2.3 we considered the particular case of the $u^\alpha(x)=x(mT+n)$ saddles; we will now evaluate the effective action of the other saddles discussed in section 3.2.2. Other than the surface saddles (3.2.31), in the continuum limit the only type of saddles that we still need to account for are the "two-factor" saddles (3.2.28), whose effective action $S_{\text{eff}}(m_1,n_1;m_2,n_2,N_2)$ is given by (3.2.30). We start from the two-factor saddles and postpone the discussion about surface saddles at the end of this section.

We will assume that $m_1\neq 0$; without loss of generality we can take $m_1>0$ and $\gcd(m_1,n_1)=\gcd(m_2,n_2)=1$. The contribution to the effective action coming from a single multiplet is given by the following expression:

$$\frac{N^2}{N_2}\sum_{i_2=1}^{N_2}\int_0^1 dx\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\log Q_{\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1\,,\,(\Delta)_1}\Big(x(m_1ab\omega+n_1)+\frac{i_2}{N_2}(m_2ab\omega+n_2)\,;ab\omega\Big)\,, \tag{3.2.60}$$

where as usual Δ is equal to $\tau+\sigma$ for vector multiplets and to Δ_I for the I-th chiral multiplet. In order to compute (3.2.60) we first generalize formula (3.2.40) to include the sum over the new index i_2. Using (A.1.22) and ignoring purely imaginary terms we find that

$$
\begin{aligned}
&\frac{1}{N_2}\sum_{i_2=1}^{N_2}\int_0^1 dx \log Q_{c,d}\Big(x(m_1\tau+n_1)+\frac{i_2}{N_2}(m_2\tau+n_2)\,;\tau\Big) = \\
&\qquad = -\frac{\pi i}{6}\,c\tau+\frac{\pi i}{3}\,\frac{1}{N_2}\sum_{i_2=1}^{N_2}\frac{B_3\big(\{m_1 d-n_1 c+\frac{i_2}{N_2}(m_1n_2-m_2n_1)\}+c(m_1\tau+n_1)\big)}{m_1(m_1\tau+n_1)^2} = \\
&\qquad = -\frac{\pi i}{6}\,c\tau+\frac{\pi i}{3}\,\frac{B_3\big([m(c\tau+d)]'_{m\tau+n}\big)}{m(m\tau+n)^2}\,.
\end{aligned}
\tag{3.2.61}
$$

In the last equality we used formula (A.1.15) to simplify the sum of Bernoulli polynomials and we defined the integers m and n as following:

$$
\big(m,n\big)\equiv\frac{N_2}{\gcd(N_2,m_1n_2-m_2n_1)}\cdot\big(m_1,n_1\big)\,. \tag{3.2.62}
$$

Given the similarity between the last line of (3.2.60) and the right-hand side of (3.2.40), the rest of the computation is identical to the one in section 3.2.3.

In conclusion the effective action for the (3.2.28) saddles can also be expressed in terms of the $\Psi_{m,n}(\Delta)$ function (3.2.44) as

$$
S_{\text{eff}}(m_1,n_1;m_2,n_2,N_2)=\pi i N^2\bigg(|G|\,\Psi_{m,n}(\tau+\sigma)+\sum_I\Psi_{m,n}(\Delta_I)\bigg). \tag{3.2.63}
$$

The difference between this expression and (3.2.43) lies in the definition of the integers m,n: for the latter they could be any pair of coprime integers, $\gcd(m,n)=1$, while in the case of the former they are given by (3.2.62) and $\gcd(m,n)=N_2/\gcd(N_2,m_1n_2-m_2n_1)$. If we set $N_2=1$ the two-factor saddles (3.2.28) become simple string-like saddles (3.2.26); in this case the integers m,n in (3.2.62) simply match m_1,n_1, and expressions (3.2.43) and (3.2.63) are in agreement. Furthermore, in the particular case of $a=b=1$ the effective action (3.2.63) matches the one computed in [49].

An explanation for the similarity between (3.2.63) and (3.2.43) can be found by recasting the saddles (3.2.28) in a new form. Starting from expression (3.2.34), we can make the following manipulations:

$$
\begin{aligned}
u^{\alpha}_{i_2}(x) &= x\,(m_1\,T+n_1)+\frac{i_2}{N_2}\,\frac{m_1n_2-m_2n_1}{m_1} = \\
&= \{m_1x\}\left(T+\frac{n_1}{m_1}\right)+\lfloor m_1x\rfloor\frac{n_1}{m_1}+\frac{i_2}{N_2}\,\frac{m_1n_2-m_2n_1}{m_1}\quad \mod T\,.
\end{aligned}
\tag{3.2.64}
$$

If we set $x_{\text{new}} \equiv \{m_1 x\}$ and $j \equiv n_1 \lfloor m_1 x \rfloor (m/m_1) + i_2(m_1 n_2 - m_2 n_1)/\gcd(N_2, m_1 n_2 - m_2 n_1) \mod m$, we can thus rewrite the two-factor saddle as

$$u_j^\alpha(x) = \frac{j}{m} + x\left(T + \frac{n}{m}\right)\,, \tag{3.2.65}$$

where m, n are the same as in (3.2.62).

From result (3.2.63) we find the following estimate for the large-N limit of the superconformal index:

$$\log \mathcal{I} \gtrsim \max_{\substack{m,n\in\mathbb{Z} \\ m\neq 0}} \left[-\pi i N^2 \Big(|G|\, \Psi_{m,n}(\tau+\sigma) + \sum_I \Psi_{m,n}(\Delta_I) \Big) \right] + o(N^2)\,, \tag{3.2.66}$$

where the maximum is taken with respect to the real part. In regions of the parameter space where there is no maximum all the competing exponentially growing contributions to the index should be summed. In this case information about the phase of each term would be necessary to accurately compute the index, and that would require an analysis of the $o(N^2)$ terms. Hence, estimate (3.2.66) does not apply in these regions. The same can be said for the codimension-one surfaces where multiple contributions have have the same real component (i.e. Stokes lines).

We will not try to determine which contribution maximizes (3.2.66) in each region of the parameter space. The large-N phase structure of the index has been studied in the case of equal angular momenta in [43, 45, 48, 49, 73].

3.2.4.1 Surface saddles

The last type of saddles that we still need to account for are the surface saddles (3.2.31). Assuming that $(m_1, n_1) \neq (m_2, n_2)$, the following relation follows from formula (A.1.22):

$$\int_0^1 dx\, dy \log Q_{c,d}\Big(x(m_1\tau+n_1)+y(m_2\tau+n_2)\,;\tau\Big) = \pi i \tau\Big(\frac{c^3}{3}-\frac{c}{6}\Big)+\Big(\text{purely imaginary}\Big). \tag{3.2.67}$$

As expected there is no dependence on the specific value of the integers m_1, n_1, m_2, n_2. This formula can also be found by taking the $N_2 \to \infty$ limit of (3.2.61). In particular this means that surface saddles correspond to the $m, n \sim \mathcal{O}(N)$ terms in estimate (3.2.66).

Using relation (3.2.67) we can compute contribution to the effective action of the surface saddle coming from a single multiplet:

$$\begin{aligned}
&\frac{\pi i}{3} N^2\, ab\omega \sum_{r=0}^{a-1}\sum_{s=0}^{b-1} \left[\Big(\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1\Big)^3 - \frac{1}{2}\Big(\frac{r}{a}+\frac{s}{b}+(\Delta)_2-1\Big)\right] = \\
&\qquad = \frac{\pi i}{3} N^2\, a^2 b^2 \omega \Big((\Delta)_2 - \frac{a+b}{2ab}\Big)^3 - \frac{\pi i}{12} N^2\,(a^2+b^2)\,\omega\Big((\Delta)_2 - \frac{a+b}{2ab}\Big). \qquad (3.2.68)
\end{aligned}$$

The sums over r, s in the first line are calculated quickly with the help of relations (3.2.47). When we sum over all multiplet contributions the second term in the second line of (3.2.68) sums to zero: it is proportional to (3.2.15). If we define a set of trial R-charges $\widehat{\Delta}_{\text{trial}\,,I}$ as

$$\widehat{\Delta}_{\text{trial}\,,I} = \frac{2ab\omega(\Delta_I)_2}{\tau+\sigma}\,, \tag{3.2.69}$$

then the effective action of surface saddles can be expressed in terms of the cubic 't Hooft anomaly for this trial R-symmetry:

$$S_{\text{eff}} = \frac{\pi i}{24}\frac{(\tau+\sigma)^3}{\tau\sigma}\,\operatorname{tr}R^3(\widehat{\Delta}_{\text{trial}})\,, \tag{3.2.70}$$

where the trace is taken over the fermions of the theory. When $\tau = \sigma$ this result matches the one of [49].

3.3 The large-N limit with the Bethe Ansatz formula

In this section we will consider a different approach to the computation of the superconformal index at large-N. Our starting point will not be the matrix model (3.1.1), but rather the Bethe Ansatz formula [46, 47]. A contribution to the Bethe Ansatz formula that reproduces the entropy of black holes with unequal angular momenta was found in [3]; in this section we will revisit the computation of [3] and also expand it to include more contributions. The results we will find reaffirm estimate (3.2.66), thus providing a double check for the saddle-point analysis of section 3.2.

This section is organized as follows. We begin by briefly discussing in subsection 3.3.1 the Bethe Ansatz formula for the quiver theories that we are considering. Then in subsection 3.3.2 we study the relation between the holonomy distributions that contribute to the Bethe Ansatz formula and the saddles (3.2.10) found in [48, 49]. If the reader is not interested in the technical details of subsection 3.3.2 it is possible to skip directly to subsection 3.3.3, in which we evaluate the large-N limit of the index with the Bethe Ansatz formula. Lastly, in subsection 3.3.4 we elaborate on the relation between our results and the ones of [3].

3.3.1 The Bethe Ansatz formula for quiver theories

As always we assume that the angular chemical potentials τ and σ are integer multiples of the same quantity $\omega \in \mathbb{H}$, that is $\tau = a\omega$, $\sigma = b\omega$. The Bethe Anstatz formula (2.4.10) specializes to the quiver theories that we are considering as follows:

$$\mathcal{I} = \frac{\kappa}{(N!)^{|G|}} \sum_{\hat{u}\in\mathfrak{M}_{\text{BAE}}} \sum_{\{m_i^\alpha\}=1}^{ab} \widetilde{\mathcal{Z}}(\hat{u}-m\omega\,;\Delta,\tau,\sigma)\; H^{-1}(\hat{u}\,;\Delta,\omega)\,, \tag{3.3.1}$$

where $\widetilde{\mathcal{Z}}(u\,;\Delta,\tau,\sigma)$ denotes the integrand of matrix model (3.1.1), or more accurately its analytic continuation to the complex plane with respect to the holonomies $\{u_i^\alpha\}$, and it is given by

$$\widetilde{\mathcal{Z}}(u\,;\Delta,\tau,\sigma) = \prod_{\alpha=1}^{|G|}\prod_{i\neq j=1}^{N}\Gamma_e\Big(u_{ij}^\alpha+\tau+\sigma\,;\tau,\sigma\Big)\cdot\prod_{\alpha,\beta=1}^{|G|}\prod_{I_{\alpha\beta}}\prod_{i,j=1}^{N}\Gamma_e\Big(u_{ij}^{\alpha\beta}+\Delta_I\,;\tau,\sigma\Big)\,. \tag{3.3.2}$$

The first out of the two sums in formula (3.3.1) runs over the set of inequivalent solutions to the Bethe Ansatz equations (BAE), which for our case can be written as

$$1 = Q_i^\alpha(u\,;\Delta,\omega) \equiv e^{2\pi i\lambda^\alpha}\prod_{j=1}^{N}\frac{\prod_{\beta=1}^{|G|}\prod_{I_{\alpha\beta}}\exp\left(2\pi i u_i^\alpha\left(\frac{1}{2}-\frac{1}{\omega}\Delta_I\right)\right)\,\theta_0\big(-u_{ij}^{\alpha\beta}+\Delta_I\,;\omega\big)}{\prod_{\gamma=1}^{|G|}\prod_{I_{\gamma\alpha}}\exp\left(-2\pi i u_i^\alpha\left(\frac{1}{2}-\frac{1}{\omega}\Delta_I\right)\right)\,\theta_0\big(u_{ij}^{\alpha\gamma}+\Delta_I\,;\omega\big)}\,, \tag{3.3.3}$$

where the λ^α are Lagrange multipliers introduced for convenience: once the value of the λ^α are fixed, for example by solving the $1 = Q_N^\alpha$ equations, the remaining equations match the general BAE we gave in (2.4.11). Again, solutions to the BAE are equivalent if they match under the identifications $u_i \sim u_i+1 \sim u_i+\omega$ or differ by a Weyl group transformation, which in this case consists in permutations of the N holonomies associated to each $\mathrm{SU}(N)$ subgroup of the gauge group.

We will focus our attention on the class of solutions to the BAE found in [79], often referred to as Hong-Liu solutions. Given any choice of three integers $\{p,\,q,\,r\}$ such that $p\cdot q = N$ and $0\leq r<q$,[4] the following configuration of complex holonomies solves the Bethe Ansatz equations:

$$u_{jk}^\alpha = \frac{j}{p}+\frac{k}{q}\left(\omega+\frac{r}{p}\right)+\bar{u}\,, \tag{3.3.4}$$

where $j=0,\ldots,p-1$ and $k=0,\ldots,q-1$ constitute a new parametrization of the index $i=1,\ldots,N$, while $\bar{u}$ is a constant needed to satisfy the $\mathrm{SU}(N)$ constraint

$$\sum_{i=1}^{N}u_i^\alpha = 0\,. \tag{3.3.5}$$

We point out that the Hong-Liu solutions (3.3.4) are such that $u_{j_1k_1}^\alpha \neq u_{j_2k_2}^\alpha \mod 1,\,\omega$ whenever $(j_1,k_1)\neq(j_2,k_2)$, or in other words they are not invariant under nontrivial Weyl group transformations. As argued in [47], BAE solutions that do not fit this requirement give an overall null contribution to the superconformal index when plugged in the Bethe Anstatz formula (3.3.1).

[4]Taking into account identifications (2.4.13), we could substitute r with $r+nq$ in (3.3.4) for any $n\in\mathbb{Z}$ and the solution would be the same up to a redefinition of the index j, that is $j_{\text{new}}\equiv j+n \mod p$. For this reason the range of r can be limited to $0\leq r<q$.

Other than the discrete class of solutions (3.3.4) there is evidence in favor of the existence of other solutions to the BAE, either isolated or belonging to continuous families of solutions [73, 80, 81]. We will not account for the contribution of these "non-standard" solutions, we will instead focus on the standard Hong-Liu solutions exclusively.

The other sum that appears in formula (3.3.1) is a sum over a collection of integers $\{m_i^\alpha\}$. When $i \neq N$ the possible values that m_i^α can take range from 1 to ab; on the other hand m_N^α is fixed by the $\mathrm{SU}(N)$ constraint:

$$m_N^\alpha = -\sum_{i=1}^{N-1} m_i^\alpha \,. \tag{3.3.6}$$

However in the large-N limit we can ignore this constraint and set m_N^α to whatever is most convenient: the leading order of $\log \mathcal{Z}(u\,;\Delta,\tau,\sigma)$ is unaffected by a change in value of a single holonomy u_i^α, and thus changing m_N^α from (3.3.6) to something else entirely does not impact the computation of the index [3].

Lastly, the Jacobian $H(u\,;\Delta,\omega)$ can be written as

$$H(u\,;\Delta,\omega) = \det\left[\frac{1}{2\pi i}\,\frac{\partial\big(\log Q_1^1,\dots,\log Q_N^1,\dots,\log Q_1^{|G|},\dots,\log Q_N^{|G|}\big)}{\partial\big(u_1^1,\dots,u_{N-1}^1,\lambda^1,\dots,u_1^{|G|},\dots,u_{N-1}^{|G|},\lambda^{|G|}\big)}\right]. \tag{3.3.7}$$

In this expression the holonomies $\{u_N^\alpha\,|\,\alpha = 1,\dots,|G|\}$ are not considered independent variables, they are instead treated like functions of the other holonomies, $u_N^\alpha \equiv -\sum_{i=1}^{N-1} u_i^\alpha$. The Lagrange multipliers λ^α on the other hand are regarded as independent variables.

3.3.2 BAE solutions and saddle points of the elliptic action

For a direct comparison of the saddle point analysis with the Bethe Ansatz formula it is important to understand the relation between the saddles found in [48, 49] with the configurations that arise from the discrete solutions to the Bethe Ansatz equations; this will be the goal of this section. The bulk of the computation of the large-N limit of the index will be in section 3.3.3, and it is possible for the reader to skip ahead.

In the first half of this section we will show that the saddles given by (3.2.10) can always be written in a form similar to the Hong-Liu solutions (3.3.4), namely it is possible to find integers p, q and r and a new set of indices $j = 0,\dots,p-1$ and $k = 0,\dots,q-1$ such that[5]

$$u^\alpha_{i_1\dots i_\ell} \equiv \frac{i_1}{N_1}\big(m_1 T + n_1\big) + \dots + \frac{i_\ell}{N_\ell}\big(m_\ell T + n_\ell\big) \;=\; \frac{j}{p} + \frac{k}{q}\left(T + \frac{r}{p}\right) \quad \mod 1,\, T\,. \tag{3.3.8}$$

[5]The vice versa however does not hold: we will later provide an explicit example of a choice of integers p, q and r such that the right-hand side of (3.3.8) does not correspond to any of the saddles given by (3.2.10).

This expression generalizes relation (3.2.65), which is valid only in the continuum limit, to the case of finite N. The main difference between the right-hand side of (3.3.8) and the BAE solutions (3.3.4) is that the saddles of the doubly periodic action have $T \equiv ab\omega$ as their period, while the solutions to the Bethe Ansatz equations have periodicity ω. We will address this discrepancy in the second half of this section, where we will discuss the role played the vector of integers m that appears in the Bethe Ansatz formula (3.3.1).

We can ignore without loss of generality saddle point configurations that repeat values, or in other words saddles such that $u^{\alpha}_{i_1 \dots i_\ell} = u^{\alpha}_{j_1 \dots j_\ell} \mod 1, T$ for some $(i_1, \dots i_\ell) \neq (j_1, \dots, j_\ell)$. Since the saddles given by (3.2.10) can be thought as homomorphisms of finite abelian groups into the torus, repetitions occur only if the kernel is nontrivial. If the kernel contains n elements, then the image group in the torus is the same as the image group of a $\mathrm{SU}(N/n)$ saddle point configuration with no repetitions. Therefore (3.3.8) holds for these saddles as long as we take $p \cdot q = N/n$, assuming (3.3.8) is true for saddles that don't repeat values. Furthermore, we note that solutions to the Bethe Ansatz equations that repeat values give an overall null contribution to the index because they are not invariant under nontrivial Weyl group transformations. For these reasons we will only consider configurations without repetitions from now on.

For $\ell = 1$ the relation (3.3.8) has already been proven in [43]. The idea is to take $p = \gcd(m_1, N_1)$, $q = N_1/p$ and defining the new indices $k = 0, ..., q-1$, $\hat{\jmath} = 0, ..., p-1$ so that $i_1 = sk + q\hat{\jmath} \mod N_1$, where s is a positive integer such that $s\, m_1/p \mod q = 1$; such an integer must exist since m_1/p and q are coprime. Furthermore, s cannot have factors in common with q, and thus the set $\{sk + q\hat{\jmath}\,|\, k = 0, ..., q-1,\, \hat{\jmath} = 0, ..., p-1\}$ covers all residue classes modulo N_1 once. The saddle can then be written as

$$\frac{i_1}{N_1}\big(m_1 T + n_1\big) = \big(sk + q\hat{\jmath}\big)\left(\frac{m_1/p}{q} T + \frac{n_1}{N_1}\right) = \frac{n_1 \hat{\jmath}}{p} + \frac{k}{q}\left(T + \frac{n_1 s}{p}\right) \mod 1,\, T\,, \tag{3.3.9}$$

which matches the right-hand side of (3.3.8) for $r \equiv n_1 s \mod q$, $j \equiv n_1 \hat{\jmath} \mod p$.

We can now prove (3.3.8) in the general case using induction. Let us assume that there are positive integers p_1, q_1 and r_1 such that $p_1 q_1 = N_1 \dots N_{\ell-1} = N/N_\ell$ and

$$\frac{i_1}{N_1}\big(m_1 T + n_1\big) + \dots + \frac{i_{\ell-1}}{N_{\ell-1}}\big(m_{\ell-1} T + n_{\ell-1}\big) = \frac{j_1}{p_1} + \frac{k_1}{q_1}\left(T + \frac{r_1}{p_1}\right) \mod 1,\, T\,. \tag{3.3.10}$$

The left-hand side of this identity is missing the following piece:

$$\frac{i_\ell}{N_\ell}\big(m_\ell T + n_\ell\big) \equiv \frac{j_2}{p_2} + \frac{k_2}{q_2}\left(T + \frac{r_2}{p_2}\right) \mod 1,\, T\,, \tag{3.3.11}$$

where the integers p_2, q_2 and r_2 are determined as in the $\ell = 1$ case. In this case p_2, q_2 satisfy $p_2 q_2 = N_\ell$; furthermore the condition $N_\ell \mid N_{\ell-1}$ implies that $p_2 q_2 \mid p_1 q_1$.

The left-hand side of (3.3.8) can thus be written as

$$\frac{j_1}{p_1}+\frac{k_1}{q_1}\left(T+\frac{r_1}{p_1}\right)+\frac{j_2}{p_2}+\frac{k_2}{q_2}\left(T+\frac{r_2}{p_2}\right)\mod 1,\, T\,. \tag{3.3.12}$$

We will now show that this expression doesn't repeat values only when p_1, p_2, q_1, q_2 satisfy $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$.

The necessity of the condition $\gcd(p_1, p_2) = 1$ can be inferred just from the $j_1/p_1 + j_2/p_2$ portion of (3.3.12), considering that the set

$$\left\{\frac{j_1 p_2 + j_2 p_1}{\gcd(p_1,p_2)}\ \middle|\ j_1 = 0,\ldots,p_1-1,\ j_2 = 0,\ldots,p_2-1\right\} \tag{3.3.13}$$

covers every residue class modulo $p_1p_2/\gcd(p_1, p_2)$ exactly $\gcd(p_1, p_2)$ times. Therefore if p_1 and p_2 were not coprime $(j_1/p_1 + j_2/p_2 \mod 1)$ would repeat values, and so would (3.3.12) for fixed k_1 and k_2.

It is easy to see that requiring $\gcd(q_1, q_2) = 1$ in addition to $\gcd(p_1, p_2) = 1$ is sufficient to ensure that (3.3.12) doesn't repeat values. Indeed if q_1 and q_2 were coprime $(k_1/q_1 + k_2/q_2)T$ modulo T wouldn't repeat and therefore all possible combinations of j_1, j_2, k_1, k_2 would give rise to unique values for expression (3.3.12). On the other hand it is a little trickier to show that the condition $\gcd(q_1, q_2) = 1$ is necessary, as we need to take in account the terms proportional to r_1 and r_2 as well. Since $j_1/p_1 + j_2/p_2$ covers all multiples of $1/p_1p_2$ modulo 1 once, if (3.3.12) doesn't have repetitions modulo $1, T$ then the same expression without $j_1/p_1 + j_2/p_2$ won't have repetitions modulo $1/p_1p_2\,, T$. Each possible value of $(k_1/q_1 + k_2/q_2)T$ modulo T is repeated $\gcd(q_1, q_2)$ times, which means that either $\gcd(q_1, q_2) = 1$ or the following expression doesn't have repetitions:

$$\frac{k_1 r_1}{p_1 q_1}+\frac{k_2 r_2}{p_2 q_2}\mod\frac{1}{p_1p_2}=\frac{1}{p_1q_1}\left(k_1r_1+k_2r_2\frac{p_1q_1}{p_2q_2}\mod\frac{q_1}{p_2}\right). \tag{3.3.14}$$

Considering that both p_1q_1/p_2q_2 and q_1/p_2 are integers[6] and that the pair (k_1, k_2) can take a total of q_1q_2 distinct values, the term in the parenthesis will take the same values multiple times unless $p_1 = q_1 = 1$, which cannot be possible as it would imply $N_1 = \ldots = N_{\ell-1} = 1$. Therefore we must have $\gcd(q_1, q_2) = 1$.

Let us now show that (3.3.12) can be written in the same form as the right-hand side of (3.3.8), assuming that $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$. First we define $k \equiv k_1q_2 + k_2q_1 \mod q_1q_2$; since q_1 and q_2 are coprime k is an index that runs from 0 to $q_1q_2 - 1$ once. Let us ignore for the moment terms that are integer multiples of $1/p_1p_2$; we can write the rest as

$$\frac{k_1}{q_1}\left(T+\frac{r_1}{p_1}\right)+\frac{k_2}{q_2}\left(T+\frac{r_2}{p_2}\right)=\frac{1}{q_1q_2}\left(kT+\frac{k_1r_1p_2q_2}{p_1p_2}\right)\mod\frac{1}{p_1p_2}\,,\,T\,. \tag{3.3.15}$$

[6]The condition $\gcd(p_1, p_2) = 1$ together with $p_2q_2 \mid p_1q_1$ implies that $p_2 \mid q_1$.

The term proportional to r_2 is a multiple of $1/p_1p_2$, considering that $p_2q_2 \mid p_1q_1$ and $\gcd(q_1, q_2) = 1$ imply that $q_2 \mid p_1$. Since q_1 and q_2 don't have factors in common it is possible to find an integer n such that $r_1p_2 + nq_1 = 0 \mod q_2$, which is going to help us rewrite (3.3.15) solely in terms of k:

$$\begin{aligned} k_1r_1p_2q_2 &= k_1q_2\big(r_1p_2 + nq_1\big) && \mod q_1q_2 \\ &= \big(k - k_2q_1\big)\big(r_1p_2 + nq_1\big) && \mod q_1q_2 \\ &= k\big(r_1p_2 + nq_1\big) && \mod q_1q_2\,. \end{aligned} \tag{3.3.16}$$

Defining $p \equiv p_1p_2$, $q \equiv q_1q_2$ and $r \equiv r_1p_2 + nq_1 \mod q$, equation (3.3.15) becomes

$$\begin{aligned} \frac{k_1}{q_1}\left(T + \frac{r_1}{p_1}\right) + \frac{k_2}{q_2}\left(T + \frac{r_2}{p_2}\right) &= \frac{k}{q}\left(T + \frac{r}{p}\right) \quad \mod \frac{1}{p}\,,\ T \equiv \\ &\equiv \frac{k}{q}\left(T + \frac{r}{p}\right) + \frac{n_k}{p} \quad \mod 1\,,\ T \end{aligned} \tag{3.3.17}$$

for some k-dependent integer n_k. At last we can define $j \equiv j_1p_2 + j_2p_1 + n_k \mod p$, so that

$$\frac{j_1}{p_1} + \frac{k_1}{q_1}\left(T + \frac{r_1}{p_1}\right) + \frac{j_2}{p_2} + \frac{k_2}{q_2}\left(T + \frac{r_2}{p_2}\right) = \frac{j}{p} + \frac{k}{q}\left(T + \frac{r}{p}\right) \quad \mod 1,\ T\,, \tag{3.3.18}$$

which concludes the proof of (3.3.8).

Vice versa, we can show that there exist some choices of integers p, q and r such that the set of points

$$\left\{\frac{j}{p} + \frac{k}{q}\left(T + \frac{r}{p}\right) \,\middle|\, j = 0, \ldots, p-1,\ k = 0, \ldots, q-1\right\} \tag{3.3.19}$$

does not match any of the saddles given by (3.2.10), modulo 1, T. One way to see this is to look at the greatest common divisor of p, q and r obtained by the procedure above.

First, in the case of saddles with $\ell = 1$ the steps outlined in (3.3.9) lead to values of p, q and r that don't have factors in common:

$$\gcd\big(p, q, r\big) = \gcd\big(m_1, N_1, q, n_1s\big) = 1\,, \tag{3.3.20}$$

where we used that $\gcd(m_1, n_1) = 1 = \gcd(q, s)$. For more general saddles on the other hand we find that

$$\gcd\big(p, q, r\big) = N_2 \ldots N_\ell\,. \tag{3.3.21}$$

We can show this by means of induction, by writing $\gcd(p, q, r)$ in terms of $\gcd(p_1, q_1, r_1)$. Considering that $kr = k_1r_1p_2q_2 \mod q$ for all possible values of k, we must have that $\gcd(q, r) = \gcd\big(q, r_1p_2q_2\big)$. The greatest common divisor of p, q, r is thus given by

$$\gcd\big(p, q, r\big) = \gcd\big(p, q, r_1p_2q_2\big) = p_2q_2 \gcd\left(\frac{p_1}{q_2}, \frac{q_1}{p_2}, r_1\right) = p_2q_2 \gcd\big(p_1, q_1, r_1\big)\,. \tag{3.3.22}$$

In the last step we used that $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$. Since $p_2 q_2 = N_\ell$, by induction we find formula (3.3.21).

Let us consider for example the case $p = 4$, $q = 4$, $r = 2$. We want to try to find a saddle point configuration that matches the set (3.3.22) for these values of p, q, r. Using formula (3.3.21) we find that such a saddle would have $N_2 \dots N_\ell = \gcd(p, q, r) = 2$, which implies $\ell = 2$ and $N_2 = 2$. Consequently, there are only two possible values that $\gcd(m_2, N_2)$ can take, either 1 or 2, and neither of them works: the former would lead to $q_1 = q_2 = 2$, while the latter would lead to $p_1 = p_2 = 2$, and in both cases $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$ is not satisfied. Hence there is no saddle that reproduces the configuration with $p = 4$, $q = 4$, $r = 2$.

3.3.2.1 The different periodicities

The most jarring difference between the saddles (3.2.10) of the doubly-periodic action and the Hong-Liu solutions (3.3.4) to the Bethe Ansatz equations lies in the different value for T, the modulus of the torus, which is $ab\omega$ for the former and just ω for the latter.

In the particular case of equal angular momenta we have $\tau = \sigma \equiv \omega$, which implies $a = b = 1$ and thus this discrepancy between the known saddles and the standard BAE solutions disappears.

When $ab \neq 1$ the solution to the problem comes from a key element that we haven't taken in consideration yet: the presence of the vector of integers m in the Bethe Ansatz formula (3.3.1). Each of its entries m_i^α takes values that range from 1 to ab, and its shifts the corresponding holonomy inside the argument of the integrand $\widetilde{\mathcal{Z}}$ as $\hat{u}_i^\alpha - m_i^\alpha \omega$, where $\hat{u}$ is the BAE solution that we are considering. Rather than trying to match the saddles (3.2.10) with the Hong-Liu solutions directly, it is more sensible to compare them with configurations of the type $\hat{u} - m\omega$. That is, given any choice of integers $\{p, q, r\}$, we search for a BAE solution $\hat{u}$ and a choice of vector m such that

$$\frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{r}{p}\right) = \hat{u}_i^\alpha - m_i^\alpha \omega + \text{constant} \mod 1,\, ab\omega\,. \tag{3.3.23}$$

The constant term ultimately vanishes because the integrand (3.3.2) only depends on differences between holonomies.

We point out that there is a large number of valid $\hat{u} - m\omega$ configurations other than the ones that satisfy (3.3.23). We won't try to account for all possible $(\hat{u}, m)$ combinations, especially considering that the number of possible values that the vector m can take is $(ab)^{|G|(N-1)}$, which grows exponentially with N.

In order to find a $(\hat{u}, m)$ combination that satisfies (3.3.23) for a given choice of $\{p, q, r\}$, we need to search for integers $\widetilde{p}$, $\widetilde{q}$ and $\widetilde{r}$ that satisfy $\widetilde{p}\,\widetilde{q} = pq$ and a new set of indices $\widetilde{\jmath} = 0, \ldots, \widetilde{p}-1$ and $\widetilde{k} = 0, \ldots, \widetilde{q}-1$ such that

$$\frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{r}{p}\right) = \frac{\widetilde{\jmath}}{\widetilde{p}} + \frac{\widetilde{k}}{\widetilde{q}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right) \quad \mathrm{mod}\ 1,\, \omega\,. \tag{3.3.24}$$

Unfortunately this isn't always possible; to see why, let us set $h \equiv \gcd(q, ab)$ and define a new parametrization of the index k in terms of new indices $k' = 0, \ldots, q/h-1$ and $k'' = 0, \ldots, h-1$ such that $k \equiv k' + (q/h)\, k''$. The left-hand side of (3.3.24) then becomes

$$\frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{r}{p}\right) = \frac{j}{p} + \frac{k'}{q}\left(ab\omega + \frac{r}{p}\right) + \frac{k'' r}{hp} \quad \mathrm{mod}\ 1,\, \omega\,. \tag{3.3.25}$$

If r and h are not coprime then the right hand side of (3.3.25) manifestly repeats values when k'' varies while j and k' are fixed. This means that unless $\gcd(q, r, ab) = 1$ it is not possible to match the right-hand side of (3.3.24), since the latter never repeats values modulo 1, ω as $\widetilde{\jmath}$ and $\widetilde{k}$ vary.

Even if it is not possible to find a $(\hat{u}, m)$ combination that satisfies (3.3.23) when q and r are such that $\gcd(q, r, ab) \neq 1$, it is always possible to find $\hat{u}$ and m that approximate the left-hand side of (3.3.23) well enough in the large-N limit; we will discuss this in more detail in appendix A.2.

Let us consider the case $\gcd(q, r, ab) = 1$ and show that it is indeed possible to obtain (3.3.24) starting from (3.3.25). Since ab/h and q/h are coprime $k'(ab/h)$ mod q/h takes all the values from 0 to $q/h - 1$ once; therefore if we set $\widetilde{q} \equiv q/h$ and $\widetilde{k} \equiv k'(ab/h)$ mod $\widetilde{q}$ we can match the ω-dependent portion of (3.3.24) and (3.3.25) as follows:

$$\frac{k'}{q}\, ab\omega = \frac{1}{q/h}\left(k'\,\frac{ab}{h}\right)\omega = \frac{\widetilde{k}}{\widetilde{q}}\,\omega \quad \mathrm{mod}\ \omega\,. \tag{3.3.26}$$

Since k' also appears in the ω-independent term proportional to r, we need to rewrite this term as well in terms of the new index $\widetilde{k}$; to do so, we will ignore for the moment the role of integer multiples of $1/hp$ so that we can write

$$\frac{k' r}{pq} = \frac{k'(r + nq/h)}{pq} \quad \mathrm{mod}\ \frac{1}{hp} = \left(\frac{1}{q/h}\, k'\,\frac{ab}{h}\right)\frac{1}{hp}\left(\frac{r + nq/h}{ab/h}\right) \quad \mathrm{mod}\ \frac{1}{hp}\,, \tag{3.3.27}$$

where n is an arbitrary integer that we have introduced. Once again we make use of the fact that ab/h and q/h are coprime: $r + n\, q/h$ for $n \in \mathbb{Z}$ covers all the residue classes modulo ab/h, which means that we can always choose n such that $\widetilde{r} \equiv (r + nq/h)/(ab/h)$ is an integer. Setting $\widetilde{p} \equiv hp$ and noticing that the other term inside parentheses is equal to $\widetilde{k}/\widetilde{q}$ mod 1, we find that

$$\frac{k' r}{pq} = \frac{\widetilde{k}\,\widetilde{r}}{\widetilde{p}\,\widetilde{q}} \quad \mathrm{mod}\ \frac{1}{\widetilde{p}} \equiv \frac{\widetilde{k}\,\widetilde{r}}{\widetilde{p}\,\widetilde{q}} + \frac{n_k}{\widetilde{p}} \quad \mathrm{mod}\ 1 \tag{3.3.28}$$

for some k-dependent integer n_k. We notice that the way $\widetilde{p}$ and $\widetilde{q}$ have been defined is such that $\widetilde{p}\,\widetilde{q}$ is equal to pq, as it should be. At last we make use of the fact that we are assuming that $\gcd(q,r,ab)=1$, which is equivalent to the statement that h and r are coprime, and thus $\widetilde{\jmath} \equiv k''r + hj + n_k \mod \widetilde{p}$ is a proper definition for an index that runs from 0 to $\widetilde{p}-1$ a single time; using (3.3.28) and the definition of the new index $\widetilde{\jmath}$ we can match the ω-independent portion of (3.3.24) and (3.3.25):

$$\frac{j}{p} + \frac{k'r}{pq} + \frac{k''r}{hp} = \frac{\widetilde{\jmath}}{\widetilde{p}} + \frac{\widetilde{k}\,\widetilde{r}}{\widetilde{p}\,\widetilde{q}} \mod 1\,. \tag{3.3.29}$$

This concludes the proof of the existence of integers $\{\widetilde{p},\widetilde{q},\widetilde{r}\}$ for which the rewrite (3.3.24) is possible, under the assumption that $\gcd(q,r,ab)=1$.

3.3.3 Evaluation of the index

In this section we will evaluate the contribution to the Bethe Ansatz formula (3.3.1) coming from distributions of holonomies of the following type:

$$(u-m\omega)^{\alpha}_{jk} = \frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{\widehat{r}}{p}\right) + \text{const}\,. \tag{3.3.30}$$

As usual $p\cdot q = N$ and $0 \leq \widehat{r} < q$; we have added the hat on $\widehat{r}$ in order to avoid confusion with the index r that appears in definition (3.2.44). For simplicity when we take the large-N limit we will keep p and $\widehat{r}$ fixed and send $q\to\infty$.

In section 3.3.2 we have shown that configurations like (3.3.30) are possible only when $\gcd(ab,q,\widehat{r})=1$. Throughout this section we will assume that q satisfies this condition; in appendix A.2 we will show how this restriction can be removed.

The quantity that we need to compute is the following:

$$\lim_{q\to\infty}\log\left(\kappa\,(N!)^{-|G|}\,\widetilde{\mathcal{Z}}(u-m\omega\,;\Delta,\tau,\sigma)\,H^{-1}(u\,;\Delta,\omega)\right)\Big|_{(3.3.30)}\,. \tag{3.3.31}$$

The prefactor κ is given by (3.1.2); as already mentioned in section 3.1, it is subleading at large-N, specifically $\log\kappa = \mathcal{O}(N)$. The factor $(N!)^{-|G|}$ is subleading as well, it is $\mathcal{O}(N\log N)$ by Stirling formula. We are left with the Jacobian H, given by (3.3.7), and the integrand $\widetilde{\mathcal{Z}}$, given by (3.3.2).

Let us start from the Jacobian: we can show that generically it gives a subleading contribution and can be neglected, using an argument similar to the one given in [43, 45]. The Jacobian H is the determinant of the matrix whose elements are the partials derivatives of the Bethe Ansatz operators Q_i^{α} defined in (3.3.3); let us examine these partial derivatives. First, the derivatives of Q_i^{α} with respect to the Lagrange multipliers are simply

$$\frac{\partial \log Q_i^{\alpha}}{\partial\lambda^{\beta}} = 2\pi i\,\delta_{\alpha\beta}\,. \tag{3.3.32}$$

We can ignore these terms as they are just $\mathcal{O}(1)$. On the other hand, the derivatives of with respect to the holonomies are given by

$$\frac{\partial \log Q_i^\alpha}{\partial u_j^\beta} = -\delta_{\alpha\beta}\left(\delta_{ij}-\delta_{iN}\right)\sum_{k=1}^{N}\sum_{\gamma=1}^{|G|}\left[\sum_{I_{\alpha\gamma}} F(-u_{ik}^{\alpha\gamma}+\Delta_I)+\sum_{I_{\gamma\alpha}} F(u_{ik}^{\alpha\gamma}+\Delta_I)\right]+$$
$$+\sum_{I_{\alpha\beta}}\left[F(-u_{ij}^{\alpha\beta}+\Delta_I)-F(-u_{iN}^{\alpha\beta}+\Delta_I)\right]+\sum_{I_{\beta\alpha}}\left[F(u_{ij}^{\alpha\beta}+\Delta_I)-F(u_{iN}^{\alpha\beta}+\Delta_I)\right], \tag{3.3.33}$$

where F is the following function:

$$F(u) \equiv \frac{2\pi i}{\omega}\,u - \pi i + \frac{\partial_u \theta_0(u;\omega)}{\theta_0(u;\omega)}\,. \tag{3.3.34}$$

This function becomes singular only at the zeros of the θ_0, that is when $u \in \mathbb{Z}+\omega\,\mathbb{Z}$. If the the chemical potentials Δ_I are such that the distribution of points $u_{ij}^{\alpha\beta}+\Delta_I$ doesn't accumulate around any of these poles in the limit $N\to\infty$, then $F(u_{ij}^{\alpha\beta}+\Delta_I)\sim\mathcal{O}(1)$ and

$$\frac{\partial \log Q_i^\alpha}{\partial u_j^\beta} = \delta_{\alpha\beta}\left(\delta_{ij}-\delta_{iN}\right)\cdot\mathcal{O}(N)\,+\,\mathcal{O}(1)\,. \tag{3.3.35}$$

In our case u_i^α is given by (3.3.30), and in the $q\to\infty$ limit the poles of F are provided that

$$\Delta_I \notin \frac{\gcd(ab,\widehat{r})}{pab}\,\mathbb{Z} + (pab\omega+\widehat{r})\,\mathbb{R}\,. \tag{3.3.36}$$

As long as this condition is satisfied for all chemical potentials only the diagonal elements and the $i=N$, $\alpha=\beta$ elements are of order $\mathcal{O}(N)$, while all the others are just $\mathcal{O}(1)$. In particular this means that the determinant (3.3.7) grows like $N^{(|G|N)}$, and thus $\log H = \mathcal{O}(N\log N)$.

Since the Jacobian is subleading as long as the chemical potentials satisfy (3.3.36), the large-N leading order of (3.3.31) is determined solely by the integrand $\widetilde{\mathcal{Z}}$. The computation boils down to the evaluation of the following quantity:

$$\Phi_{p,q,\widehat{r}}(\Delta) \equiv \sum_{j_1,j_2=0}^{p-1}\ \sum_{k_1\neq k_2=0}^{q-1} \log\Gamma_e\left(\Delta+\frac{j_1-j_2}{p}+\frac{k_1-k_2}{q}\left(ab\omega+\frac{\widehat{r}}{p}\right);a\omega,b\omega\right). \tag{3.3.37}$$

In terms of this function we can write $\log\widetilde{\mathcal{Z}}$ as[7]

$$\log\widetilde{\mathcal{Z}}(u-m\omega\,;\Delta,\tau,\sigma)\Big|_{(3.3.30)} = |G|\,\Phi_{p,q,\widehat{r}}(\tau+\sigma)+\sum_I \Phi_{p,q,\widehat{r}}(\Delta_I)+\mathcal{O}(N)\,. \tag{3.3.38}$$

[7]If p is fixed as $N\to\infty$ then the sum of all the terms that have $k_1=k_2$ is subleading: $q|G|\sum_{j_1\neq j_2=0}^{p-1}\log\Gamma_e\big(a\omega+b\omega+\frac{j_1-j_2}{p}\,;a\omega,b\omega\big)+q\sum_I\sum_{j_1,j_2=0}^{p-1}\log\Gamma_e\big(\Delta+\frac{j_1-j_2}{p}\,;a\omega,b\omega\big)=\mathcal{O}(N)$

A quick comparison with (3.2.63) tells us that in the large-N limit we should expect $\Phi_{p,q,\widehat{r}}(\Delta)$ to be related to the function $\Psi_{m,n}(\Delta)$ defined in (3.2.44). More specifically, because of (3.2.65) we expect the leading order of $\Phi_{p,q,\widehat{r}}(\Delta)$ to be equal to $-\pi i N^2\, \Psi_{p,\widehat{r}}(\Delta)$.

Using formula (A.1.4) we can take care of the sum over j_1, j_2:

$$\Phi_{p,q,\widehat{r}}(\Delta) = p \sum_{k_1 \neq k_2 = 0}^{q-1} \log \Gamma_e \left(p\Delta + \frac{k_1 - k_2}{q} \big(pab\omega + \widehat{r} \big) ; pa\omega, pb\omega \right) . \tag{3.3.39}$$

In order to compute the $q \to \infty$ limit of this sum we can take advantage of the following result found in [43]:

$$\sum_{i \neq j = 1}^{N} \log \Gamma_e \left(\Delta + \frac{i-j}{N}\, \omega \, ; \omega, \omega \right) = -\pi i N^2 \frac{B_3\big([\Delta - \omega]'_\omega\big)}{3\omega^2} + o(N^2) \, . \tag{3.3.40}$$

Here the subleading terms are of order $\mathcal{O}(N)$ when $\Delta \notin \mathbb{Z} + \mathbb{R} \cdot \omega$, and $\mathcal{O}(N \log N)$ for the $\Delta = 0$ case. We can recast (3.3.39) in a form similar to (3.3.40) by making use of the following identity:

$$\log \Gamma_e \Big(z \, ; pa\omega, pb\omega \Big) = \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \log \Gamma_e \Big(z + p\omega(as + br) \, ; pab\omega + \widehat{r}, pab\omega + \widehat{r} \Big) \, , \tag{3.3.41}$$

which follows from (A.1.3) and the invariance of the elliptic gamma under integer shifts of any of its arguments. Denoting $pab\omega + \widehat{r}$ as $\widetilde{\omega}$ for convenience, (3.3.39) becomes

$$\Phi_{p,q,\widehat{r}}(\Delta) = p \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{k_1 \neq k_2 = 0}^{q-1} \log \Gamma_e \left(p\Delta + p\omega(as + br) + \frac{k_1 - k_2}{q}\, \widetilde{\omega} \, ; \widetilde{\omega}, \widetilde{\omega} \right) . \tag{3.3.42}$$

Applying formula (3.3.40), we get at last the following expression for the large-N leading order of $\Phi_{p,q,\widehat{r}}(\Delta)$:

$$\begin{aligned} \Phi_{p,q,\widehat{r}}(\Delta) &= -\frac{\pi i N^2}{3p(pab\omega + \widehat{r})^2} \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} B_3 \Big(\big[p\Delta + p\omega(as + br - ab) \big]'_{pab\omega + \widehat{r}} \Big) + o(N^2) = \\ &= -\pi i N^2\, \Psi_{p,\widehat{r}}(\Delta) + o(N^2) \, , \end{aligned} \tag{3.3.43}$$

where as usual the function $\Psi_{m,n}(\Delta)$ is defined by (3.2.44).

This result is consistent with the ones we obtained in sections 3.2.3 and 3.2.4 with the elliptic extension approach. In particular, we find that the large-N estimate for the index (3.2.66) is verified by the Bethe Ansatz formalism as well.

As already mentioned in section 3.2.3, the $p = 1$, $\widehat{r} = 0$ case matches the contribution to the Bethe Ansatz formula computed in [3], which reproduces the entropy function of AdS_5 black holes. In the next subsection we will elaborate more on how result (3.3.43) and the one of [3] compare.

3.3.4 Relation with previous work and competing exponential terms

Let us discuss the relation between the computation of subsection 3.3.3 and the one of [3], which also estimated the large-N leading order of the index with $\tau \neq \sigma$ using the Bethe Ansatz formula.

In [3] we focused exclusively on a single contribution to the index, the one coming from the following distribution of holonomies:

$$(u - m\omega)^{\alpha}_{i} \;=\; \frac{i}{N}\,\omega + (i \bmod ab)\,\omega + \text{const}\,. \tag{3.3.44}$$

Up to $\mathcal{O}(1/N)$ terms, this configuration matches the right-hand side of (3.3.30) with $p = 1$, $\widehat{r} = 0$. For clarity, let us assume that N is a multiple of ab; we can then reparametrize the index i in terms of new indices $i' = 0, \ldots, N/ab - 1$ and $i'' = 0, \ldots, ab - 1$ such that $i \equiv i'ab + i''$, and rewrite (3.3.44) as following:

$$\frac{i}{N}\,\omega + (i \bmod ab)\,\omega \;=\; \frac{i' + i''(N/ab)}{N}\,ab\omega + \frac{i''}{N}\,\omega \;\equiv\; \frac{\widetilde{\imath}}{N}\,ab\omega \;+\; \mathcal{O}\left(\frac{1}{N}\right). \tag{3.3.45}$$

In the second step we defined a new index $\widetilde{\imath} = 0, \ldots, N-1$ as $\widetilde{\imath} \equiv i' + i''(N/ab)$. As argued in appendix A of [3], these $\mathcal{O}(1/N)$ terms can be neglected in the large-N limit, if we are only interested in the leading order. Accordingly, the contribution to the index coming from the distribution of holonomies (3.3.44) computed in [3] does indeed match the result that we have obtained for the $p = 1$, $\widehat{r} = 0$ case.

Proving that the $\mathcal{O}(1/N)$ terms in (3.3.45) do not affect the large-N leading order is possibly the most laborious step in the large-N computation of [3]. In this section we have shown that it is possible to avoid this step completely by choosing a different set up for $u - m\omega$, i. e. (3.3.30), at least as long as the assumption $\gcd(ab, q, \widehat{r}\,) = 1$ is valid. Since the superconformal index is a continuous function of $\tau = a\omega$ and $\sigma = b\omega$, it is natural to expect that the $\gcd(ab, q, \widehat{r}\,) = 1$ condition doesn't actually play a role in the large-N behavior of the index; rather, this condition should be a byproduct of focusing strictly on holonomy distributions that can be written as (3.3.30).[8]

We want to stress the fact that the distributions of holonomies (3.3.44) and (3.3.30) (with $p = 1$, $\widehat{r} = 0$) are distinct from one another, even if in the large-N limit they differ just by $\mathcal{O}(1/N)$ terms. This raises a problem: the contributions to the index coming from these two distributions are exponentially growing terms whose logarithms match at leading N^2 order, and it is easy to see that there are

[8] In appendix A.2 we will verify that this intuition is indeed correct: for any possible choice of integers p, q, $\widehat{r}$ there are contributions to the Bethe Anstatz formula that in the large-N limit give the same result as (3.3.43), even when the condition $\gcd(ab, q, \widehat{r}\,) = 1$ is not satisfied; the price to pay is that we will have to deal with the $\mathcal{O}(1/N)$ terms once again.

many other similar contributions;[9] all these competing exponential terms must be summed together since there is no guarantee that one of them clearly dominates over the others.

Let us first estimate how many competing exponential terms there are. Any possible choice of $u - m\omega$ that matches $\frac{i}{N}\, ab\omega$ up to $\mathcal{O}(1/N)$ terms must have $u_i \equiv \frac{i}{N}\,\omega$, since this is the only Hong-Liu solution (3.3.4) whose holonomies are strictly proportional to ω. There are then $(ab)^{|G|(N-1)}$ possible choices for the vector m, at most. If we were to assume that all the competing exponential terms interfere constructively, we would get at most a $|G|(N-1)\log(ab)$ correction to our previous estimate for the large-N limit, which is subleading and thus negligible. In other words the leading N^2 order does not receive corrections from the multiplicity of the competing exponentials. However it would be possible, albeit very unlikely, for all these terms to interfere destructively in such a way that they cancel completely. In order to determine whether this is the case, we would need to calculate the exact phase of all the competing contributions, which is unfeasible. Given that in the saddle point analysis of section 3.2 this problem does not occur at all, we are lead to believe that such a cancellation does not happen and the leading N^2 order is unaffected.

3.4 Summary and discussion

In this chapter we have estimated the large-N limit of the superconformal index of $\mathcal{N}=1$ quiver theories with adjoint and bifundamental matter for general values of BPS charges, using both the elliptic extension approach of [48–50] and the Bethe Ansatz formula [46, 47]. We have found a good accord between the two methods, resulting in the following estimate for the index:

$$\log \mathcal{I}\left(\{\Delta_I\};\tau,\sigma\right) \gtrsim \max_{\substack{m,n\,\in\,\mathbb{Z}\\ m\neq 0}} \Big[-\pi i N^2 \Big(|G|\,\Psi_{m,n}(\tau+\sigma) + \sum_I \Psi_{m,n}(\Delta_I)\Big)\Big] + o(N^2)\,, \tag{3.4.1}$$

where the function $\Psi_{m,n}(\Delta)$ is defined by

$$\Psi_{m,n}(\Delta) \equiv \sum_{r=0}^{a-1}\sum_{s=0}^{b-1} \frac{B_3\big([m\Delta + m\omega(as+br-ab)]'_{mab\omega+n}\big)}{3m(mab\omega+n)^2} \qquad (\tau\equiv a\omega\,,\ \sigma\equiv b\omega) \tag{3.4.2}$$

and the parentheses $[\,\cdot\,]'_T$ are such that $[\,x+y\,T\,]'_T = x - \lfloor x\rfloor + y\,T$ for real x, y.

[9] For example, as already argued in [3] changing the value of a single holonomy does not impact the large N leading order, and it is always possible to change the value of a single holonomy by by changing the value of one of the entries of the vector of integers m. Hence, for any contribution to the index that we have computed there are always many possible competing exponential terms.

Our results extend the saddle point analysis of [48, 49] to the case of unequal angular momenta ($\tau \neq \sigma$). They also extend the computation of [3] to include multiple competing exponentially-growing contributions to the Bethe Ansatz formula; the single contribution computed in [3] corresponds to the $m = 1$, $n = 0$ term in (3.4.1).

In section 3.3.2 we have shown that the saddles of the elliptic action found in [48, 49] can always be written in the form

$$\frac{j}{p} + \frac{k}{q}\left(T + \frac{r}{p}\right) + \text{const.} \tag{3.4.3}$$

for some integers p, q, r, with $T \equiv ab\omega$. This is the same form that the standard Hong-Liu solutions [79] to the Bethe Ansatz equations (BAE) take, with the only difference being that the latter are defined on a torus with a modulus $T \equiv \omega$. When $a = b = 1$ this means that each saddle has a matching BAE solution, and thus a corresponding term in the Bethe Ansatz equations; however for general a, b the different values of T cause a mismatch between saddles and BAE solutions. In this chapter we have shown how the two different pictures can be reconciled: we have to consider that each contribution to the Bethe Ansatz formula is labeled not only by the BAE solution u but also by the choice of value for the auxiliary integer parameters $\{m_i\}$ that shift the BAE solution as $u_i \mapsto u_i - m_i\,\omega$. We have found that for each saddle of the elliptic action there is a $(u, \{m_i\})$ combination that matches it, either exactly or up to $\mathcal{O}(1/N)$ corrections that are negligible at large-N.

There are still some open questions concerning the matching between the two approaches. Most notably, the number of $(u, \{m_i\})$ combinations that label each contribution to the Bethe Ansatz formula is exponentially bigger than the number of known saddles of the elliptic action. In this chapter we have computed only the contribution of the $(u, \{m_i\})$ combinations that match a saddle, but there are many other contributions that are unaccounted for. It is not feasible to try to evaluate all of them, given their exponentially large number: the integers $\{m_i\}$ can take $(ab)^{|G|(N-1)}$ different values. Furthermore, the formulas that we have used in section 3.3.3 would not apply in general. Nonetheless, trying to understand what role do all these terms play remains an interesting question. The simplest possible answer would be that only the $(u, \{m_i\})$ combinations that match one of the elliptic saddles up to negligible corrections give a contribution that at large-N dominates in some region of the space of parameters; further work is however still needed to test the correctness of such a conjecture.

In our work we have not analyzed which contribution maximizes (3.4.1) in each region of the parameter space. A detailed study of the phase structure of the index at large-N for general values of BPS charges is a possible direction for future research. In our analysis we focus exclusively on the $\mathcal{O}(N^2)$ leading order; an interesting generalization would be to compute some lower order corrections.

Chapter 4

Equivariant localization and equivariant volume

In this chapter we review some of the key mathematical aspects that we will need in chapter 5 to derive extremal functions for supergravity solutions with an holographic dual. In particular, we will review the Atiyah-Bott-Berline-Vergne equivariant localization formula [51, 52] for orbifolds [82, 83], and how it can be applied to compute the equivariant volume, which will be the fundamental object for our discussion in chapter 5.

We will focus on toric orbifolds, and although we will make ample use of the symplectic and Kähler structures constructed in [84–87], we stress the fact that the quantities that we will compute are topological in nature. In particular our results apply to geometries that are topologically symplectic and toric, regardless of whether the particular metric is compatible with a symplectic structure, which is the case for many of the supergravity solutions that we consider in chapter 5.

This chapter is organized as follows: first, in section 4.1 we briefly review equivariant localization, then in section 4.2 we review some of the key properties of toric orbifolds, their classification in terms of polytopes and their symplectic and Kähler structures. In section 4.3 we review the equivariant volume of toric orbifolds following the discussion and notations of [28]. At last in section 4.4 we generalize the equivariant volume with the addition of higher times.

4.1 Equivariant localization

In this section we briefly review the Atiyah-Bott-Berline-Vergne equivariant localization formula [51, 52] for orbifolds [82, 83]. Given a torus action on a compact orbifold $\mathbb{M}$, the formula allows to compute integrals of *equivariant forms* on $\mathbb{M}$ in terms of integrals on the fixed point set of the torus action. If the fixed point set only contains isolated points, then the integrals simply reduce to sums over the fixed points. The equivariant localization formula can be applied to non-compact orbifolds as well, provided that eventual contributions from infinity are accounted for.

Let us consider a torus $\mathbb{T}^{\mathfrak{m}} = \mathbb{R}^{\mathfrak{m}}/2\pi\mathbb{Z}^{\mathfrak{m}}$ action on an orbifold $\mathbb{M}$. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_{\mathfrak{m}})$ be a vector in the Lie algebra of $\mathbb{T}^{\mathfrak{m}}$, and let ξ be the corresponding vector field in $\mathbb{M}$ generated by the action of $\mathbb{T}^{\mathfrak{m}}$. We can define a differential operator acting on the space of mixed-degree differential forms as

$$d_\xi \alpha = d\alpha + 2\pi\, i_\xi \alpha \,. \tag{4.1.1}$$

Since $d_\xi^2 = 0$, the equivariant differential d_ξ defines a cohomology, the equivariant cohomology of $\mathbb{M}$. Mixed-degree forms that are closed under d_ξ are called equivariant forms, and can be though as functions of the parameters $\epsilon_1, \ldots, \epsilon_{\mathfrak{m}}$, which are called equivariant parameters. We will denote equivariant forms with as $\alpha^{\mathbb{T}^{\mathfrak{m}}}$, or more simply with $\alpha^{\mathbb{T}}$ if there is no confusion about the dimension of the torus.

We can now state the equivariant localization formula. If $\mathbb{M}$ is a compact, orientable, connected orbifold with a smooth $\mathbb{T}^{\mathfrak{m}}$ action, then the integral of an equivariant form $\alpha^{\mathbb{T}}$ is given by

$$\int_{\mathbb{M}} \alpha^{\mathbb{T}} = \sum_F \frac{1}{d_F} \int_F \frac{i_F^* \alpha^{\mathbb{T}}}{e^{\mathbb{T}}(\mathcal{N}F)} \,, \tag{4.1.2}$$

where F are the components of the fixed point set of the $\mathbb{T}^{\mathfrak{m}}$ action (which always have even co-dimension), d_F is the order of the orbifold singularity of $\mathbb{M}$ in F, $i_F^*\alpha^{\mathbb{T}}$ is the pullback of $\alpha^{\mathbb{T}}$ under the inclusion map $i_F : F \to \mathbb{M}$, and $e^{\mathbb{T}}(\mathcal{N}F)$ is the equivariant Euler form of the normal bundle $\mathcal{N}F$.[1]

If $\mathcal{N}F$ splits into a sum of invariant orbifold line bundles L_j,[2]

$$\mathcal{N}F = \bigoplus_{j=1}^{\mathrm{codim}(F)/2} L_j \,, \tag{4.1.3}$$

then the equivariant Euler form $e^{\mathbb{T}}(\mathcal{N}F)$ can be defined as the following polynomial of the Chern forms $c_1(L_j)$

$$e^{\mathbb{T}}(\mathcal{N}F) = \prod_{j=1}^{\mathrm{codim}(F)/2} \big(c_1(L_j) - w_j \cdot \epsilon\big) \tag{4.1.4}$$

where w_j are the orbifold weights of the $\mathbb{T}^{\mathfrak{m}}$ action on L_j.

In the rest of this thesis we will focus on $(2\,\mathfrak{m})$-dimensional toric orbifolds $\mathbb{M}_{2\mathfrak{m}}$. In this case the fixed point set of the $\mathbb{T}^{\mathfrak{m}}$ action is comprised of isolated fixed points y_a.

[1] For any point $p \in F$, the fiber of the normal bundle at p is by definition $\mathcal{N}_p F = T_p\mathbb{M}/T_pF$.

[2] This will be true for the cases relevant to us. The following discussion easily generalizes using the splitting principle.

Then the above equivariant integration formula simplifies to the fixed point formula

$$\int_{\mathbb{M}_{2\mathfrak{m}}} \alpha^{\mathbb{T}} = \sum_{\alpha} \frac{\alpha^{\mathbb{T}}|_{y_\alpha}}{d_\alpha \, e^{\mathbb{T}}|_{y_\alpha}} \,, \tag{4.1.5}$$

which we will frequently use.

4.2 Toric orbifolds

In this section we review the key properties of toric orbifolds that we will need. In subsection 4.2.1 and 4.2.2 we review their Kähler metrics and the explicit construction of the equivariant Chern classes respectively, while in subsection 4.2.3 we review the details of the construction of toric orbifolds as symplectic quotients.

Given a $(2\,\mathfrak{m})$-dimensional orbifold $\mathbb{M}_{2\mathfrak{m}}$ with a symplectic form ω, the action of a Lie group G on $\mathbb{M}_{2\mathfrak{m}}$ is Hamiltonian if there exist a G-equivariant[3] moment map $\mu_G : \mathbb{M}_{2\mathfrak{m}} \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G, such that

$$i_\xi \omega = -d\langle \mu_G, \epsilon \rangle \,, \tag{4.2.1}$$

where $\epsilon \in \mathfrak{g}$ and ξ is the corresponding vector field on $\mathbb{M}_{2\mathfrak{m}}$ generated by the action of G. The symplectic orbifold $\mathbb{M}_{2\mathfrak{m}}$ is said to be toric if it is equipped with a $\mathbb{T}^{\mathfrak{m}}$ Hamiltonian action. It is convenient to rewrite (4.2.1) as

$$i_\xi \omega = -\sum_{i=1}^{\mathfrak{m}} \epsilon_i \, d\mu^i \,, \tag{4.2.2}$$

where we have split $\epsilon \equiv (\epsilon_1, \ldots, \epsilon_{\mathfrak{m}})$ and $\mu^i : \mathbb{M}_{2\mathfrak{m}} \to \mathbb{R}$ are the respective components of the moment map of $\mathbb{T}^{\mathfrak{m}}$, each one being the moment map of a S^1 subgroup of $\mathbb{T}^{\mathfrak{m}}$. The moment maps μ^i are constant over the orbits of the $\mathbb{T}^{\mathfrak{m}}$ action.

A classic theorem of Delzant [88], generalized to the case of orbifolds in [89], states that the image under the moment map of a $(2\,\mathfrak{m})$-dimensional compact toric orbifold $\mathbb{M}_{2\mathfrak{m}}$ is a $\mathfrak{m}$-dimensional Delzant polytope.[4] A convex polytope $\mathcal{P}$ can always be written as

$$\mathcal{P} = \{\, y_i \in \mathbb{R}^{\mathfrak{m}} \mid y_i \, \hat{v}_i^a \, \geq \, \hat{\lambda}_a, \; a = 1, \ldots, d \,\} \tag{4.2.3}$$

where $\hat{v}_i^a$ is the vectors orthogonal to the facet $\mathcal{F}_a$ of the polytope and we are using Einstein notation on the index i. The above polytope is said to be Delzant if each vertex is the intersection of exactly $\mathfrak{m}$ facets[5] (i.e. it is *simple*) and the corresponding

[3] If L_g is the action of $g \in G$ on $\mathbb{M}_{2\mathfrak{m}}$, then $\mu_G \circ L_G = Ad_g^* \circ \mu_G$, for all $g \in G$.

[4] For non-compact orbifolds we will still use the term polytope to refer to the image of the moment map, even if in this case it is unbounded.

[5] It is not uncommon to encounter geometries with worse-than-orbifold singularities, such as the singularity at the center of the cone over a toric Sasaki-Einstein. In this cases the singularity corresponds to a vertex of the polytope which is the intersection of more than $\mathfrak{m}$ facets. As we will discuss later, such singularities can be dealt with by resolving them, which corresponds to cutting off the singularity at the cost of introducing new facets.

$\mathfrak{m}$ vectors $\hat{v}_i^a$ can be chosen to be a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^{\mathfrak{m}}$ (i.e. it is *rational*). Smooth compact toric manifolds are completely classified by their respective Delzant polytopes: from the polytope it is always possible to reconstruct the toric manifold, up to symplectomorphisms. For toric orbifolds additional information is needed: a generic compact toric orbifold can be reconstructed from its polytope and the *labels* d_{D_a}, which are the order of the toric singularity of $\mathbb{M}_{2\mathfrak{m}}$ at $D_a = \mu^{-1}(\mathcal{F}_a)$. This information can be baked inside the definition of the polytope by defining the non-primitive vectors $v_i^a = d_{D_a} \cdot \hat{v}_i^a$, which we will exclusively use from now on. The polytope can then be written as[6]

$$\mathcal{P} = \{ \, y_i \in \mathbb{R}^{\mathfrak{m}} \mid l_a(y) \geq 0, \, a = 1, \ldots, d \, \} \,, \qquad l_a(y) \equiv y_i \, v_i - \lambda_a \,. \tag{4.2.4}$$

In subsection 4.2.3 we will briefly review how the orbifold $\mathbb{M}_{2\mathfrak{m}}$ can be reconstructed from the above information.

The sets $D_a = \mu^{-1}(\mathcal{F}_a)$ are $2(\mathfrak{m}-1)$-cycles of $\mathbb{M}_{2\mathfrak{m}}$ and are called *toric divisors*. They are subject to the homological relations

$$\sum_{a=1}^{d} v_i^a D_a = 0 \,, \qquad i = 1, \ldots, \mathfrak{m} \,. \tag{4.2.5}$$

At each toric divisor a S^1 subgroup of the $\mathbb{T}^{\mathfrak{m}}$ degenerates. The intersection of q distinct divisors is thus a toric sub-orbifold of $\mathbb{M}_{2\mathfrak{m}}$ of codimension $2q$, if non-empty. In particular the fixed point of the $\mathbb{T}^{\mathfrak{m}}$ action correspond to the intersections of $\mathfrak{m}$ distinct toric divisors D_a, and the moment map provides a one-to-one correspondence between the fixed points of $\mathbb{M}_{2\mathfrak{m}}$ and the vertices of the polytope.

The vectors v_i^a are the generators of the *fan* of the orbifold $\mathbb{M}_{2\mathfrak{m}}$. The fan is a collection of cones spanned by the vectors v_i^a and encodes information about the intersections of the facets $\mathcal{F}_a$. More precisely, the cone spanned by $(v^{a_1}, \ldots, v^{a_k})$ is part of the fan if-and-only-if the intersection $\mathcal{F}_{a_1} \cap \ldots \cap \mathcal{F}_{a_k}$ is non-empty. In particular the fixed points correspond to cones in the fan spanned by $\mathfrak{m}$ vectors $(v^{a_1}, \ldots, v^{a_{\mathfrak{m}}})$. The order of the orbifold singularity at the fixed point y_α, $\alpha \equiv (v^{a_1}, \ldots, v^{a_{\mathfrak{m}}})$, which is needed for the fixed point formula (4.1.5), is given by

$$d_\alpha = | \det(v^{a_1}, \ldots, v^{a_{\mathfrak{m}}}) | \,. \tag{4.2.6}$$

Indeed we have

$$1 = | \det(\hat{v}^{a_1}, \ldots, \hat{v}^{a_{\mathfrak{m}}}) | = \frac{| \det(v^{a_1}, \ldots, v^{a_{\mathfrak{m}}}) |}{d_{D_{a_1}} \ldots d_{D_{a_{\mathfrak{m}}}}} = \frac{| \det(v^{a_1}, \ldots, v^{a_{\mathfrak{m}}}) |}{d_\alpha} \,. \tag{4.2.7}$$

From the fan it is possible to reconstruct the algebraic structure of $\mathbb{M}_{2\mathfrak{m}}$ as a toric variety, but we will not need the details of this construction in the following.

[6] Also $\lambda_a = d_{D_a} \cdot \hat{\lambda}_a$

The orbifold $\mathbb{M}_{2\mathfrak{m}}$ can be seen as a $\mathbb{T}^{\mathfrak{m}}$ fibration over the polytope $\mathcal{P}$, provided that we collapse to a point the appropriate one-cycle of $\mathbb{T}^{\mathfrak{m}}$ above each facet $\mathcal{F}_a$. In particular there is a convenient densely defined local chart for $\mathbb{M}_{2\mathfrak{m}}$, given by the cartesian coordinates y_i taking values in the interior of the polytope and the angular coordinates $\phi_i \sim \phi_i + 2\pi$ that parametrize the torus. This coordinate system is symplectic, in the sense that the symplectic form ω is given by the following simple expression:

$$\omega = \mathrm{d}y_i \wedge \mathrm{d}\phi_i \, . \tag{4.2.8}$$

We note that in this coordinate system the moment maps μ^i are just the projections onto the coordinates y_i.

4.2.1 Toric-Kähler metrics

In the symplectic (y, ϕ) coordinates the most general $\mathbb{T}^{\mathfrak{m}}$ invariant almost complex structure on $\mathbb{M}_{2\mathfrak{m}}$ compatible with ω given is [90]

$$J = \begin{pmatrix} 0 & -G^{ij}(y) \\ G_{ij}(y) & 0 \end{pmatrix} \, , \tag{4.2.9}$$

where $G^{ij}(y)$ is the inverse of $G_{ij}(y)$. The integrability of the complex structure is equivalent to the condition on $G_{ij}(y)$ being a Hessian matrix:

$$G_{ij}(y) = \frac{\partial^2 G(y)}{\partial y_i \partial y_j} \, . \tag{4.2.10}$$

The function $G(y)$ is called the symplectic potential. The holomorphic coordinates are then given by

$$z_j(y, \phi) = \frac{\partial G(y)}{\partial y_j} + i\, \phi_j \, , \tag{4.2.11}$$

while the toric-Käler metric obtained from ω and J is

$$\mathrm{d}s^2 = G_{ij}(y)\mathrm{d}y_i\mathrm{d}y_j + G^{ij}(y)\mathrm{d}\phi_i\mathrm{d}\phi_j \, . \tag{4.2.12}$$

In order to behave correctly at the facets of the polytope $G(y)$ must take the form

$$G(y) = \frac{1}{2}\sum_{a=1}^{d} l_a(y) \log l_a(y) + h(y) \, , \tag{4.2.13}$$

where $h(y)$ is a smooth function on the whole polytope $\mathcal{P}$ (not just on its interior). In addition it must satisfy

$$\det\left[G_{ij}(y)\right] \cdot \prod_{a=1}^{d} l_a(y) > 0 \qquad \forall y \in \mathcal{P} \tag{4.2.14}$$

in order to be positive defined.

If the function $h(y)$ in (4.2.13) is chosen to be $h(y)=0$ we obtain the *canonical* metric

$$G_{\text{can.}}(y)=\frac{1}{2}\sum_{a=1}^{d} l_a(y)\log l_a(y)\,,\qquad G^{\text{can.}}_{ij}=\frac{\partial^2 G_{\text{can.}}(y)}{\partial y_i\partial y_j}=\frac{1}{2}\sum_{a=1}^{d}\frac{v_i^a v_j^a}{l_a(y)}\,. \tag{4.2.15}$$

The canonical metric is the toric-Kähler metric obtained by constructing $\mathbb{M}_{2\mathfrak{m}}$ as a symplectic quotient $\mathbb{C}^d//G$, as we will discuss more in detail in subsection 4.2.3. Given that the quantities that we want to study are topological and do not depend on the specific choice of metric, in the following we will use the canonical metric and we will simply write G instead of $G_{\text{can.}}$.

4.2.2 Equivariant Chern classes

Every facet $\mathcal{F}_a$ of the polytope of a toric orbifold $\mathbb{M}_{2\mathfrak{m}}$ is associated to a toric divisor D_a by inverse image of the moment map. There extist a holomorphic line bundle L_a over $\mathbb{M}_{2\mathfrak{m}}$ which is canonically associated with the divisor D_a. The bundle L_a is uniquely defined by requiring that its restriction on D_a is the normal bundle $\mathcal{N}D_a$, and that its restriction on $\mathbb{M}_{2\mathfrak{m}}\smallsetminus D_a$ is trivial. The Chern class of the line bundle L_a has been computed in [91, 92]:

$$c_1(L_a)=-\frac{i}{2\pi}\left[\partial\bar\partial\log l_a(y)\right]\,, \tag{4.2.16}$$

where $[\,\cdot\,]$ denotes the cohomology class. A key property of $c_1(L_a)$ is that it is the Poincaré dual of the divisor D_a, that is

$$\int_{\mathbb{M}_{2\mathfrak{m}}} c_1(L_a)\wedge\alpha=\int_{D_a}\alpha\,, \tag{4.2.17}$$

for any $(2\mathfrak{m}-2)$-form α on $\mathbb{M}_{2\mathfrak{m}}$. From the homological relation among the divisors (4.2.5) we can deduce its cohomological counterpart:

$$\sum_{a=1}^{d} v_i^a\, c_1(L_a)=0\,,\qquad i=1,\dots,\mathfrak{m}\,. \tag{4.2.18}$$

We will now introduce equivariant analogue of the Chern forms, following the discussion in [28]. First, we define the functions $\mu_a^i(y)$ as

$$\mu_a^i(y)=-\frac{1}{4\pi}\frac{G^{ij}(y)\,v_j^a}{l_a(y)} \tag{4.2.19}$$

We can then give the explicit expression for another representative of the Chern class,

$$c_1(L_a)=\left[\mathrm{d}\big(\mu_a^i\mathrm{d}\phi_i\big)\right]. \tag{4.2.20}$$

In the following, with an abuse of notation, we will be indicating this specific representative of the Chern class the same way as the Chern class itself, $c_1(L_a)$. From $c_1(L_a) = \mathrm{d}\big(\mu_a^i \mathrm{d}\phi_i\big)$ it is easy to verify the relation

$$i_{\partial_{\phi_i}} c_1(L_a) = -\,\mathrm{d}\mu_a^i\,. \tag{4.2.21}$$

We can now define the equivariant Chern forms as

$$c_1^{\mathbb{T}}(L_a) = c_1(L_a) + 2\pi\,\epsilon_i\,\mu_a^i\,, \tag{4.2.22}$$

which are easily verified to be closed under the equivariant differential:

$$d_\xi\, c_1^{\mathbb{T}}(L_a) = \big(d + 2\pi\, i_\xi\big)\big(c_1(L_a) + 2\pi\,\epsilon_i\,\mu_a^i\big) = 0\,. \tag{4.2.23}$$

The equivarant counterpart of the cohomological relation (4.2.18) is

$$\sum_{a=1}^{d} v_i^a\, c_1^{\mathbb{T}}(L_a) = -\,\epsilon_i\,, \qquad i = 1, \dots, \mathfrak{m}\,, \tag{4.2.24}$$

where we have used that

$$\sum_{a=1}^{d} v_i^a\,\mu_a^j = -\frac{\delta_i^j}{2\pi}\,, \tag{4.2.25}$$

which is a simple consequence of $G^{ij}G_{jk} = \delta_j^i$ and (4.2.15).

The cohomology class of the symplectic form ω can be expressed in terms of the Chern classes as [91]

$$[\omega] = -2\pi \sum_{a=1}^{d} \lambda_a\, c_1(L_a)\,. \tag{4.2.26}$$

The λ_a are thus an over-parametrization of the Kähler moduli. The equivariant analogue of the above formula is

$$\big[\omega^{\mathbb{T}}\big] = -2\pi \sum_{a=1}^{d} \lambda_a\, c_1^{\mathbb{T}}(L_a)\,, \tag{4.2.27}$$

where this time $[\,\cdot\,]$ denotes the equivariant cohomology class and

$$\omega^{\mathbb{T}} = \omega + 2\pi\, H(\epsilon_i) = \omega + 2\pi\,\epsilon_i\, y_i\,. \tag{4.2.28}$$

This form is equivariantly closed since $H(\epsilon_i)$ is the Hamiltonian of the vector ξ and thus by definition

$$i_\xi \omega = -\mathrm{d}H\,. \tag{4.2.29}$$

The relations (4.2.26) and (4.2.27) can be derived from the observation that the moment maps $\mu^i(y) = y_i$ can be written as

$$\begin{aligned} y_i &= G^{ij}\, G_{jk}\, y_k = G^{ij} \sum_a \frac{v_j^a\, v_k^a\, y_k}{2\, l_a} = G^{ij} \sum_a \frac{v_j^a (l_a + \lambda_a)}{2\, l_a} = \\ &= -\,2\pi \sum_a \lambda_a \mu_i^a + \frac{1}{2} \sum_a G^{ij} v_j^a\,. \end{aligned} \tag{4.2.30}$$

Then we find

$$\omega = \mathrm{d}y_i \wedge \mathrm{d}\phi_i = -2\pi \sum_a \lambda_a \, \mathrm{d}\big(\mu_i^a \mathrm{d}\phi_i\big) + \mathrm{d}\left(\frac{1}{2}\sum_a G^{ij} v_j^a \, \mathrm{d}\phi_i\right) , \tag{4.2.31}$$

which implies (4.2.26), and the other relation easily follows.

4.2.3 Toric orbifolds as symplectic quotients

In this section we review how toric orbifolds can be reconstructed from polytopes (and labels at the facets, which in our notations are baked inside the vectors that define the polytope). For compact toric orbifolds this provides a complete classification, according to the generalization to orbifolds [89] of the classic theorem of Delzant [88]. This construction will be particularly useful to us when in chapter 6 we will discuss the Molien-Weyl formula for the equivariant volume.

Our starting point is a generic simple rational convex polytope

$$\mathcal{P} = \{ y_i \in \mathbb{R}^m \,|\, y_i v_i^a \geq \lambda_a \} \, . \tag{4.2.32}$$

Let us consider the space $\mathbb{C}^d$, where we have a complex coordinate z_a for each vector $v^a \in \mathbb{R}^{\mathfrak{m}}$ of the fan. The standard $\mathbb{T}^d$ torus action on this space simply shifts the phases of each z_a. Then we can define the following subgroup of $\mathbb{T}^d$:

$$G = \left\{ \big(e^{2\pi i \mathcal{Q}_1}, \ldots, e^{2\pi i \mathcal{Q}_d}\big) \in \mathbb{T}^d \,\middle|\, \sum_a \mathcal{Q}_a v^a \in \mathbb{Z}^{\mathfrak{m}} \right\} . \tag{4.2.33}$$

The idea is to construct the toric orbifold $\mathbb{M}_{2\mathfrak{m}}$ as the zero level set of the moment map of the action of G, quotiented by G: in short $\mathbb{M}_{2\mathfrak{m}} = \mu_G^{-1}(0)/G$. In the following we will review this procedure step by step.

It is always possible to define the "GLSM charges" $Q_a^m \in \mathbb{Z}$ by requiring that

$$\sum_{a=1}^{d} Q_a^m v_i^a = 0 \, , \qquad m = 1, \ldots, d - \mathfrak{m} \, . \tag{4.2.34}$$

These charges generate the $\mathbb{T}^{d-\mathfrak{m}}$ continuous component of G. More precisely we have

$$G = \Gamma \oplus \left\{ \big(e^{2\pi i Q_1^m \theta_m}, \ldots, e^{2\pi i Q_d^m \theta_m}\big) \in \mathbb{T}^d \,\middle|\, \theta_1, \ldots, \theta_{d-\mathfrak{m}} \in \mathbb{R} \right\} , \tag{4.2.35}$$

where for simplicity we are using Einstein notation on the index $m = 1, \ldots, d - \mathfrak{m}$. Here Γ is a discrete group, which we will call the "torsion group". Without loss of generality we will assume that the GLSM charges Q_a^m are chosen so that the map $\mathbb{T}^{d-\mathfrak{m}} \to \mathbb{T}^d$ that they define is injective.

In order to construct the moment map of G, we first construct the moment map of $\mathbb{T}^d$. The standard symplectic form on $\mathbb{C}^d$ is given by

$$\omega_0 = \frac{1}{2i} \sum_{a=1}^{d} dz_a \wedge d\bar{z}_a \,. \tag{4.2.36}$$

Then the components of the moment map of $\mathbb{T}^d$ are

$$\mu_0^a(z, \bar{z}) = \frac{|z_a|^2}{2} + \lambda_a \,, \tag{4.2.37}$$

where for later convenience we have introduced the Kähler moduli λ_a, since μ_0^a is only defined up to a constant. It is easy to check that the above μ_0^a satisfy the defining property of the moment maps:

$$i_{\partial_{\varphi_a}} \omega_0 = -d\mu_0^a \,, \tag{4.2.38}$$

where φ_a is the phase of z_a.

The moment map of G can be found from the moment map of $\mathbb{T}^d$ by contracting the components of the latter with the GLSM charges:

$$\mu_G^m(z, \bar{z}) = \sum_{a=1}^{d} Q_a^m \mu_0^a(z, \bar{z}) \,. \tag{4.2.39}$$

This trivially follows from the fact that the infinitesimal generators of G are $\sum_a Q_a^m \partial_{\varphi_a}$, and thus from (4.2.38) we immediately get

$$i_{\partial_{\sum_a Q_a^m \varphi_a}} \omega_0 = -d\mu_G^m \,. \tag{4.2.40}$$

As already mentioned, the zero level set of the moment map of G

$$\mu_G^{-1}(0) = \left\{ z \in \mathbb{C}^d \,\middle|\, \mu_G^m(z, \bar{z}) = 0 \,,\ m = 1, \dots, d - \mathfrak{m} \right\} \tag{4.2.41}$$

quotiented by the action of G reconstructs the toric orbifold $\mathbb{M}_{2\mathfrak{m}}$:

$$\mathbb{M}_{2\mathfrak{m}} = \mu_G^{-1}(0) \,/\, G \,. \tag{4.2.42}$$

The symplectic form ω on $\mathbb{M}_{2\mathfrak{m}}$ is then constructed by imposing that

$$p^* \omega = i^* \omega_0 \,, \tag{4.2.43}$$

where $p : \mu_G^{-1}(0) \to \mathbb{M}_{2\mathfrak{m}}$ is the projection into the quotient, $i : \mu_G^{-1} \to \mathbb{C}^d$ is the inclusion map, and p^*, i^* are the respective pullbacks. This procedure of constructing a symplectic orbifold $(\mathbb{M}_{2\mathfrak{m}}, \omega)$ from the Hamiltonian action of a group G on a higher dimensional symplectic space ($\mathbb{C}^d$ in this case) is called symplectic reduction and is usually denoted by

$$\mathbb{M}_{2\mathfrak{m}} = \mathbb{C}^d // G \,. \tag{4.2.44}$$

Let us quickly comment the toricity and the Kähler structure of $\mathbb{M}_{2\mathfrak{m}}$ thus constructed. The standard $\mathbb{T}^d$ action on $\mathbb{C}^d$ induces a $\mathbb{T}^d$ action on $\mu_G^{-1}(0)$ which reduces to $\mathbb{T}^{\mathfrak{m}}$ action on $\mathbb{M}_{2\mathfrak{m}}$. It is then possible to show that this $\mathbb{T}^{\mathfrak{m}}$ action is Hamiltonian and the image of its moment map is precisely the polytope (4.2.32) that we started with. Furthermore, since $\mathbb{C}^d$ is a Kähler manifold and the action of G preserves its complex structure, it follows that the symplectic quotient $\mathbb{M}_{2\mathfrak{m}} = \mathbb{C}^d//G$ inherits a complex structure that is compatible with ω. Hence $\mathbb{M}_{2\mathfrak{m}}$ has a canonical Kähler structure, and the respective Kähler metric is precisely the one given by (4.2.15).

The toric divisors of $\mathbb{M}_{2\mathfrak{m}}$ are the $2(\mathfrak{m}-1)$ cycles where a $U(1)$ subgroup of $\mathbb{T}^{\mathfrak{m}}$ acts trivially. In $\mathbb{C}^d$ the sets in which the $U(1)$ subgroups act trivially are of the form $\{(z_1,\dots,z_d)\in\mathbb{C}^d\,|\,z_a=0\}$. Thus the divisors are given by

$$D_a = \Big(\mu_G^{-1}(0)\cap\{(z_1,\dots,z_d)\in\mathbb{C}^d\,|\,z_a=0\}\Big)\,/\,G\,. \tag{4.2.45}$$

4.3 Equivariant volume

In this section we review the definition and the basic properties of the equivariant volume. We will review the fixed point formula, the special case of four-dimensional orbifolds and the Calabi-Yau case respectively in subsections 4.3.1, 4.3.2 and 4.3.3. We will mostly follow the discussion and the conventions of [28] throughout the entirety of this section. In section 4.4 we will generalize the equivariant volume by including *higher times*.

Given a toric orbifold $\mathbb{M}_{2\mathfrak{m}}$ and a vector $\xi = \epsilon_i\,\partial_{\phi_i}$ with Hamiltonian $H(\epsilon_i)$, then the equivariant volume $\mathbb{V}$ is defined as [29]

$$\mathbb{V}(\lambda_a,\epsilon_i) = \frac{1}{(2\pi)^{\mathfrak{m}}}\int_{\mathbb{M}_{2\mathfrak{m}}} \mathrm{e}^{-H}\,\frac{\omega^{\mathfrak{m}}}{\mathfrak{m}!}\,. \tag{4.3.1}$$

In the symplectic coordinates we have $\omega = \mathrm{d}y_i\wedge\mathrm{d}\phi_i$ and $H(\epsilon_i)=\epsilon_i\,y_i$, which we can use to write the equivariant volume as an integral over the polytope (4.2.4):

$$\mathbb{V}(\lambda_a,\epsilon_i) = \int_{\mathcal{P}} \mathrm{e}^{-\epsilon_i\,y_i}\,\mathrm{d}^{\mathfrak{m}}y\,, \tag{4.3.2}$$

where the factors of 2π at the denominator have been canceled by similar factors coming from the integration over the angles ϕ_i. The equivariant volume can be computed using the equivariant localization fixed point formula (4.1.5). Indeed, the integrand of (4.3.1) can be expressed in terms of the equivariant form $\omega^{\mathbb{T}} = \omega + 2\pi H$ as follows:

$$\mathbb{V}(\lambda_a,\epsilon_i) = (-1)^{\mathfrak{m}}\int_{\mathbb{M}_{2\mathfrak{m}}} \mathrm{e}^{-\frac{1}{2\pi}\,\omega^{\mathbb{T}}}\,. \tag{4.3.3}$$

In the case of compact toric orbifolds we can use (4.2.27) to write

$$\mathbb{V}(\lambda_a,\epsilon_i) = (-1)^{\mathfrak{m}}\int_{\mathbb{M}_{2\mathfrak{m}}} \mathrm{e}^{\sum_{a=1}^d \lambda_a\, c_1^{\mathbb{T}}(L_a)}\,. \tag{4.3.4}$$

For non-compact orbifolds we cannot ignore the exact form in (4.2.27) because it plays a key role in ensuring the convergence of the integral (there are also constraints on the values of the equivariant parameters ϵ_i which are needed for convergence, as evident for formula (4.3.2)). We will discuss some of the caveats regarding the non-compact case in subsection 4.3.3 when examining toric Calabi-Yau $\mathfrak{m}$-folds.

In the case of compact orbifolds the equivariant volume is the generating functional of equivariant intersection numbers, which are topological quantities that can be defined as integrals of the equivariant Chern classes:

$$D_{a_1 \dots a_k} = \int_{\mathbb{M}_{2\mathfrak{m}}} c_1^{\mathbb{T}}(L_{a_1}) \cdots c_1^{\mathbb{T}}(L_{a_k}) \,. \tag{4.3.5}$$

Indeed we can expand the equivariant integral as

$$\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{k=0}^{\infty} \frac{(-1)^{\mathfrak{m}}}{k!} \sum_{a_1,\dots,a_k=1}^{d} \lambda_{a_1} \cdots \lambda_{a_k} \int_{\mathbb{M}_{2\mathfrak{m}}} c_1^{\mathbb{T}}(L_{a_1}) \cdots c_1^{\mathbb{T}}(L_{a_k}) \,, \tag{4.3.6}$$

so that

$$D_{a_1 \dots a_k} = (-1)^{\mathfrak{m}} \frac{\partial^k \mathbb{V}(\lambda_a, \epsilon_i)}{\partial \lambda_{a_1} \cdots \partial \lambda_{a_k}}\Big|_{\lambda_a=0} \,. \tag{4.3.7}$$

The equivariant intersection numbers are zero for $k < \mathfrak{m}$ since in that case the integrand is a mixed degree form with a zero top degree ($2\,\mathfrak{m}$) form. When $k \geq \mathfrak{m}$ the top degree form is homogeneous of degree $\mathfrak{m} - k$ in ϵ_i. Hence, the intersection $D_{a_1 \dots a_k}$ is a homogeneus polynomial of degree $\mathfrak{m} - k$ in ϵ_i.

When the orbifold is non-compact it is still possible to derive the equivariant intersection numbers of compact intersections of divisors from the equivariant volume. This has been done in [93] using the Molien-Weyl formula. Regardless of intersection numbers, we will need the expansion of the equivariant volume in powers of λ_a, which we write as

$$\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{k=0}^{\infty} \mathbb{V}^{(k)}(\lambda_a, \epsilon_i) \,, \tag{4.3.8}$$

where $\mathbb{V}^{(k)}$ is homogeneus of degree k in the λ_a. In the compact case $\mathbb{V}^{(k)}$ is given by the k-th term in the sum of (4.3.6).

From the equivariant cohomological relations (4.2.24) we find the following formula for the shift of the λ_a: [7]

$$\mathbb{V}(\lambda_a + \beta_i v_i^a, \epsilon_i) = \mathrm{e}^{-\beta_i \epsilon_i} \mathbb{V}(\lambda_a, \epsilon_i) \,. \tag{4.3.9}$$

When $\beta_i \epsilon_i = 0$ this can be regarded as a "gauge transformation". The infinitesimal version of the (4.3.9) is

$$\sum_{a=1}^{d} v_i^a \frac{\partial \mathbb{V}}{\partial \lambda_a} = -\epsilon_i \mathbb{V} \,, \tag{4.3.10}$$

[7] The transformation $\lambda \to \lambda_a + \beta_i v_i^a$ correspond to a translation of the polytope, so this formula can also be easily derived from (4.3.2).

and in terms of homogeneous components

$$\sum_{a=1}^{d} v_i^a \frac{\partial \mathbb{V}^{(k)}}{\partial \lambda_a} = -\epsilon_i \mathbb{V}^{(k-1)} \,. \tag{4.3.11}$$

4.3.1 Fixed point formula

In this subsection we review the fixed point formula for the equivariant volume, focusing on the case of compact toric orbifolds. We postpone the non-compact case to subsection 4.3.3.

If we apply the fixed point formula (4.1.5) to the integral (4.3.4) we find

$$\mathbb{V}(\lambda_a, \epsilon_i) = (-1)^{\mathfrak{m}} \sum_{\alpha} \frac{\mathrm{e}^{\tau^{\mathbb{T}}|_{y_\alpha}}}{d_\alpha \, e^{\mathbb{T}}|_{y_\alpha}} \,, \tag{4.3.12}$$

where α runs over the fixed points of $\mathbb{M}_{2\mathfrak{m}}$, which are in a one-to-one correspondence with the $\mathfrak{m}$-dimensional cones of the of the fan, $\alpha \equiv (v^{a_1}, \dots, v^{a_\mathfrak{m}})$, and we have defined the equivariant form $\tau^{\mathbb{T}}$ as

$$\tau^{\mathbb{T}} = \sum_{a=1}^{d} \lambda_a \, c_1^{\mathbb{T}}(L_a) \,. \tag{4.3.13}$$

The restriction of the equivariant Chern classes $c_1^{\mathbb{T}}(L_a)$ to the fixed point y_α are computed by simply evaluating the zero-form component of $c_1^{\mathbb{T}}(L_a)$ in y_α, which is given by (4.2.22), and thus

$$c_1^{\mathbb{T}}(L_a)\Big|_{y_\alpha} = 2\pi \, \epsilon_i \, \mu_i^a\Big|_{y_\alpha} \,. \tag{4.3.14}$$

In order to evaluate the functions μ_i^a (4.2.19) in y_α we need to study the behavior of G^{ij} in the neighborhood of y_α. If we specialize (4.2.15) around the fixed point y_α we can write

$$G_{ij} = \frac{1}{2} \sum_{k=1}^{\mathfrak{m}} \frac{v_i^{a_k} v_j^{a_k}}{l_{a_k}} + (\text{terms that are regular at } y_\alpha) \,. \tag{4.3.15}$$

The above matrix can be easily inverted if we first define the vectors $u_\alpha^{a_i}$ by requiring that they satisfy

$$u_\alpha^{a_i} \cdot v^{a_j} = d_\alpha \, \delta_{ij} \,. \tag{4.3.16}$$

The above relation is equivalent to stating that $(u_\alpha^{a_i})^j$ is d_α times the inverse matrix of the $v_i^{a_j}$ square $\mathfrak{m} \times \mathfrak{m}$ matrix. Therefore we must also have that

$$\sum_{k=1}^{\mathfrak{m}} (u_\alpha^{a_k})^i \, v_j^{a_k} = d_\alpha \, \delta_j^i \,. \tag{4.3.17}$$

The vectors $u_\alpha^{a_i}$ are the inward normals to the facets of the cone $\alpha = (v^{a_1}, \dots, v^{a_\mathfrak{m}})$ (or, equivalently, the vectors along the edges of the polytope that meet at the vertex

y_α) and their components $(u_\alpha^{a_i})^j$ are integers. We can then use relations (4.3.16) and (4.3.17) to find the inverse of G_{ij} in the neighborhood of y_α:

$$G^{ij} \sim \frac{2}{(d_\alpha)^2} \sum_{k=1}^{\mathfrak{m}} (u_\alpha^{a_k})^i \, (u_\alpha^{a_k})^j \, l_{a_k} \,. \tag{4.3.18}$$

We can now finally compute the restrictions of the μ_i^a at the fixed point y_α

$$\mu_i^a\Big|_{y_\alpha} = -\frac{1}{4\pi} \frac{G^{ij}\, v_j^a}{l_a}\Big|_{y_\alpha} = \begin{cases} -\frac{1}{2\pi}\frac{(u_\alpha^a)^i}{d_\alpha} & \text{if } v^a \in \alpha \\ 0 & \text{if } v^a \notin \alpha \end{cases} , \tag{4.3.19}$$

from which we find the restriction of the equivariant Chern forms:

$$c_1^{\mathbb{T}}(L_a)\Big|_{y_\alpha} = 2\pi\, \epsilon_i\, \mu_i^a\Big|_{y_\alpha} = \begin{cases} -\frac{\epsilon \cdot u_\alpha^a}{d_\alpha} & \text{if } v^a \in \alpha \\ 0 & \text{if } v^a \notin \alpha \end{cases} . \tag{4.3.20}$$

The normal bundle over the fixed point y_α is just the tangent space $T_{y_\alpha}\mathbb{M}_{2\mathfrak{m}}$, and it factorizes as the direct sum of the $\mathfrak{m}$ line bundles L_{a_i} associated to the divisors that intersect into the fixed point. In particular formula (4.1.4) for the equivariant Euler class $e^{\mathbb{T}}$ at y_α is just the product of the restrictions of the equivariant forms $c_1^{\mathbb{T}}(L_{a_i})$, and thus

$$e^{\mathbb{T}}|_{y_\alpha} = \prod_{i=1}^{\mathfrak{m}} c_1^{\mathbb{T}}(L_{a_i})\Big|_{y_\alpha} = (-1)^{\mathfrak{m}} \prod_{i=1}^{\mathfrak{m}} \frac{\epsilon \cdot u_\alpha^{a_i}}{d_\alpha} \,. \tag{4.3.21}$$

Putting everything together, we find that the fixed point formula for the equivariant volume can be expressed as

$$\mathbb{V}\big(\lambda_a, \epsilon_i\big) = \sum_{\alpha=(v^{a_1},\dots,v^{a_\mathfrak{m}})} \frac{\mathrm{e}^{\tau_\alpha}}{d_\alpha \prod_{i=1}^{\mathfrak{m}} \frac{\epsilon \cdot u_\alpha^{a_i}}{d_\alpha}} \,, \tag{4.3.22}$$

where

$$\tau_\alpha = -\sum_{i=1}^{\mathfrak{m}} \lambda_{a_i} \Big(\frac{\epsilon \cdot u_\alpha^{a_i}}{d_\alpha} \Big) \tag{4.3.23}$$

is the restriction of the equivariant form (4.3.13) to the fixed point y_α.

4.3.2 Four-dimensional compact toric orbifolds

In this section we specialize to the case of four-dimensional compact toric orbifolds, and review some of their salient features. The formulas of this section will be heavily used in chapter 5, where we will mostly consider geometries that are fibrations over four-dimensional compact toric orbifolds $\mathbb{M}_4$.

The polytope of a four-dimensional compact orbifold is just a convex polygon. The fan is generated by the two-dimensional integer vectors v^a, $a = 1, \dots, d$ that are normal to the sides of the polygon. The fixed points are associated with the cones (v^a, v^{a+1}), where we take a counter-clockwise order for the vector and identify cyclically $v^{a+d} = v^a$. Notice that in the compact four-dimensional case the number of fixed points is equal to the number of vectors in the fan and we can use the index a to label both. It is convenient to define the quantities

$$\epsilon_1^a = \frac{\epsilon \cdot u_1^a}{d_{a,a+1}}\,, \qquad \epsilon_2^a = \frac{\epsilon \cdot u_2^a}{d_{a,a+1}}\,, \tag{4.3.24}$$

where u_1^a and u_2^a are the inward normals to the cones (v^a, v^{a+1}). Explicitly

$$\epsilon_1^a = -\frac{\det(v^{a+1}, \epsilon)}{\det(v^a, v^{a+1})}\,, \qquad \epsilon_2^a = \frac{\det(v^a, \epsilon)}{\det(v^a, v^{a+1})}\,, \tag{4.3.25}$$

where $\epsilon \equiv (\epsilon_1, \epsilon_2)$. In particular, the equivariant Euler class of the tangent bundle at a fixed point y_a reads

$$e^{\mathbb{T}}\big|_{y_a} = \epsilon_1^a\, \epsilon_2^a\,, \tag{4.3.26}$$

and the order of the local orbifold singularity is

$$d_{a,a+1} = \det(v^a, v^{a+1})\,. \tag{4.3.27}$$

The restriction to the fixed points of the equivariant Chern classes $c_1^{\mathbb{T}}(L_a)$ can be written as

$$c_1^{\mathbb{T}}(L_a)\big|_{y_b} = -(\delta_{a,b}\epsilon_1^b + \delta_{a,b+1}\epsilon_2^b)\,. \tag{4.3.28}$$

The fixed point formula (4.3.22) for the equivariant volume is thus given by the expression[8]

$$\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{a=1}^{d} \frac{\mathrm{e}^{-\lambda_a \epsilon_1^a - \lambda_{a+1}\epsilon_2^a}}{d_{a,a+1}\,\epsilon_1^a \epsilon_2^a}\,. \tag{4.3.29}$$

From the equivariant volume we can derive the intersections numbers by using formula (4.3.7). The expression for the intersections D_{ab} and D_{abc} will be particularly useful in chapter 5. Since $D_{a_1\dots a_k}$ is a homogeneous polynomial of degree $k-2$, the intersections D_{ab} are just numbers. Explicitly they are

$$D_{ab} = \int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_a)\, c_1^{\mathbb{T}}(L_b) = \begin{cases} \frac{1}{d_{a-1,a}} & \text{if } b = a-1\,, \\ \frac{1}{d_{a,a+1}} & \text{if } b = a+1\,, \\ -\frac{d_{a-1,a+1}}{d_{a-1,a} d_{a,a+1}} & \text{if } b = a\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.3.30}$$

The expression for D_{aa} is obtained by application of the useful identity

$$\frac{\epsilon_1^a}{d_{a,a+1}\epsilon_2^a} + \frac{\epsilon_2^{a-1}}{d_{a-1,a}\epsilon_1^{a-1}} = -\frac{d_{a-1,a+1}}{d_{a-1,a}d_{a,a+1}}\,, \tag{4.3.31}$$

[8]Notice that there is no summation on a in the exponent.

where $d_{a-1,a+1} = \det(v^{a-1}, v^{a+1})$.

The triple intersections $D_{a_1a_2a_3}$ are linear in ϵ_i and are given by

$$
\begin{aligned}
D_{a_1a_2a_3} &= \int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_{a_1}) c_1^{\mathbb{T}}(L_{a_2}) c_1^{\mathbb{T}}(L_{a_3}) = \\
&= \begin{cases} -\frac{\epsilon_2^{a-1}}{d_{a-1,a}} & \text{if } a_i = a_j = a_k + 1 \equiv a\,, \\ -\frac{\epsilon_1^{a}}{d_{a,a+1}} & \text{if } a_i = a_j = a_k - 1 \equiv a\,, \\ \frac{d_{a-1,a+1}(2d_{a-1,a}\epsilon_1^a + d_{a-1,a+1}\epsilon_2^a)}{d_{a-1,a}^2 d_{a,a+1}} & \text{if } a_1 = a_2 = a_3 \equiv a\,, \\ 0 & \text{otherwise}\,. \end{cases}
\end{aligned}
\tag{4.3.32}
$$

We note that from the vanishing of $D_{a_1\dots a_k}$ for $k < 2$ it is possible to derive useful relations:

$$
0 = \int_{\mathbb{M}_4} 1 = \sum_a \frac{1}{d_{a,a+1}\epsilon_1^a \epsilon_2^a}\,, \tag{4.3.33}
$$

$$
0 = \int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_a) = -\frac{1}{d_{a,a+1}\epsilon_2^a} - \frac{1}{d_{a-1,a}\epsilon_1^{a-1}}\,. \tag{4.3.34}
$$

4.3.3 Toric Calabi-Yau $\mathfrak{m}$-folds

In this subsection we briefly review the Calabi-Yau case, which will be relevant for many of the supergravity solutions that we will study in chapter 5. Given that all toric Calabi-Yau are non-compact we will also discuss some of the subtleties of the definition and computation of the equivariant volume of non-compact geometries.

Let us consider a toric Calabi-Yau $\mathfrak{m}$-fold. The Calabi-Yau condition is equivalent to the requirement that all the vectors v^a that generate the fan lie in the same hyperplane. Up to an $\mathrm{SL}(\mathfrak{m}, \mathbb{Z})$ transformation[9] we can choose one component, say $i = 1$,[10] to be one for all the vectors, that is $v_1^a = 1$ for all $a = 1, \dots, d$. An immediate consequence is that all toric Calabi-Yau are non-compact. The large y_i approximation of the polytope $\mathcal{P}$ is then a cone $\mathcal{P}'$:[11]

$$
\mathcal{P}' = \{ y \in \mathbb{R}^{\mathfrak{m}} \mid y_i\, v_i^{a_k} \geq 0\,,\ k = 1, \dots, d' \}\,. \tag{4.3.35}
$$

As evident from formula (4.3.2) for the equivariant volume as an integral over $\mathcal{P}$, the equivariant parameters ϵ_i must lie in th cone $\mathcal{P}'$ for the equivariant volume to be well defined. The fixed point formula can be applied to Calabi-Yau geometries with only orbifold singularities, considering that equivariant localization fails for non-compact orbifolds only if there are contributions from infinity, which there are none.

[9] Choosing a different set of generators of the torus $\mathbb{T}^{\mathfrak{m}}$ action corresponds to an $\mathrm{SL}(\mathfrak{m}, \mathbb{Z})$ transformation.

[10] The Calabi-Yau examples that we will consider in this thesis are all either three-folds or four-folds, and for reason that will be more apparent later we will choose to set the third component to one.

[11] Vectors v^a associated to compact divisors D_a can be ignored in the large y_i approximation.

There many interesting cases where worse-than-orbifold singularities are present. The prototypical example are toric Calabi-Yau cones over Sasaki-Einstein $\mathrm{SE}_{2\mathfrak{m}-1}$, for which all the facets meet in a single vertex, which means that excluding the case with minimal number of facets the cone is not a simple polytope. In such cases the extremal functions of the supergravity solutions are found in terms of homogeneous components of the equivariant volume $\mathbb{V}^{(k)}$ of degree $k < \mathfrak{m}$ [28]. The equivariant volume can then be computed by resolving the singularity first, by adding vectors v^a to the fan in such a way that the polytope stays the same at large y_i but there are no longer any worse-than-orbifold singularities. This can be done without changing the final result because the components $\mathbb{V}^{(k)}$ of degree $k < \mathfrak{m}$ do not depend on the parameters λ_a associated to compact divisors.[12]

4.4 Equivariant volume with higher times

In this section we discuss a generalization of the equivariant volume, obtained by introducing *higher times*. The equivariant volume with higher times has appeared only recently in the literature [93][13] and is still poorly studied. As we will discuss in chapter 5, the equivariant volume with higher times contains all the information needed to fully capture the topological properties and the quantization of fluxes for a very large class of supergravity solutions.

The equivariant volume with higher times is defined by

$$\mathbb{V}\big(\{\lambda_{a_1\ldots a_k}\}_{k=1}^K, \epsilon_i\big) = (-1)^{\mathfrak{m}} \int_{\mathbb{M}_{2\mathfrak{m}}} \mathrm{e}^{-\frac{\omega^{\mathbb{T}}}{2\pi} + \sum_{k=2}^K \lambda_{a_1\ldots a_k} c_1^{\mathbb{T}}(L_{a_1})\ldots c_1^{\mathbb{T}}(L_{a_k})}\,, \tag{4.4.1}$$

where $\lambda_{a_1\ldots a_k}$ are symmetric tensors and a sum over repeated indices a is understood. K can be any integer greater than one. The above expression has a large gauge invariance and many parameters are redundant, as we will later discuss.

We can compute the equivariant volume with higher times (4.4.1) with the fixed point formula (4.1.5):

$$\mathbb{V}\big(\{\lambda_{a_1\ldots a_k}\}_{k=1}^K, \epsilon_i\big) = (-1)^{\mathfrak{m}} \sum_{\alpha} \frac{\mathrm{e}^{\tau^{\mathbb{T}}|_{y_\alpha}}}{d_\alpha\, e^{\mathbb{T}}|_{y_\alpha}}\,, \tag{4.4.2}$$

where y_α, d_α, and $e^{\mathbb{T}}$ are defined as in the previous section, whereas $\tau^{\mathbb{T}}$ is now defined as

$$\tau^{\mathbb{T}} = \sum_{k=1}^K \lambda_{a_1\ldots a_k} c_1^{\mathbb{T}}(L_{a_1})\ldots c_1^{\mathbb{T}}(L_{a_k})\,. \tag{4.4.3}$$

[12]The components of degree $\mathbb{V}^{(k)}$ of degree $k < \mathfrak{m}$ vanish for compact orbifolds. Any non-compact polytope can be made into a compact one by adding facets that only intersect the previously non-compact facets. This operation does not change the contribution of the original compact facets, which must then be zero for $\mathbb{V}^{(k)}$ with $k < \mathfrak{m}$.

[13]More precisely, in [93] they introduced a higher times analogue of the Molien-Weyl formula for the equivariant volume. We will discuss this in more detail in chapter 6.

Using the identities (4.3.20) and (4.3.21) we find

$$\mathbb{V}\big(\{\lambda_{a_1\ldots a_k}\}_{k=1}^K,\epsilon_i\big) = \sum_{\alpha=(v^{a_1},\ldots,v^{a_\mathtt{m}})} \frac{\mathrm{e}^{\tau_\alpha}}{d_\alpha \prod_{i=1}^{\mathtt{m}} \frac{\epsilon\cdot u_\alpha^{a_i}}{d_\alpha}}\,, \tag{4.4.4}$$

where τ_α is the restriction of the equivariant form (4.4.3) to the fixed point y_α and explicitly reads

$$\tau_\alpha = -\sum_{i=1}^{\mathtt{m}} \lambda_{a_i}\Big(\frac{\epsilon\cdot u_\alpha^{a_i}}{d_\alpha}\Big) + \sum_{i,j=1}^{\mathtt{m}} \lambda_{a_i a_j}\Big(\frac{\epsilon\cdot u_\alpha^{a_i}}{d_\alpha}\Big)\Big(\frac{\epsilon\cdot u_\alpha^{a_j}}{d_\alpha}\Big) + \ldots \tag{4.4.5}$$

As an example, we give the expression for the equivariant volume with *second times* (that is, with $K=2$) for a four-dimensional toric orbifold, thus generalizing expression (4.3.29):

$$\mathbb{V}(\lambda_a,\lambda_{ab},\epsilon_i) = \sum_{a=1}^{d} \frac{\mathrm{e}^{-\lambda_a\epsilon_1^a-\lambda_{a+1}\epsilon_2^a+\lambda_{a,a}(\epsilon_1^a)^2+2\lambda_{a,a+1}\epsilon_1^a\epsilon_2^a+\lambda_{a+1,a+1}(\epsilon_2^a)^2}}{d_{a,a+1}\,\epsilon_1^a\epsilon_2^a}\,. \tag{4.4.6}$$

We note that among the second times λ_{ab} only the ones of the form $\lambda_{a,a}$ and $\lambda_{a,a+1}$ effectively appear in $\mathbb{V}$. We will use this observation a few times in chapter 5.

The equivariant volume can be expanded in power series of the higher times

$$\mathbb{V}\big(\{\lambda_{a_1\ldots a_k}\}_{k=1}^K,\epsilon_i\big) = \sum_{n=0}^{\infty} \mathbb{V}^{(n)}\big(\{\lambda_{a_1\ldots a_k}\}_{k=1}^K,\epsilon_i\big)\,, \tag{4.4.7}$$

where we denote with $\mathbb{V}^{(n)}$ the homogeneous component of degree n in the set of higher times $\lambda_{a_1\ldots a_k}$ for all k. $\mathbb{V}^{(n)}$ is a polynomial in ϵ_i in the compact case, while it can be a rational function of ϵ_i when $\mathbb{M}_{2\mathtt{m}}$ is non-compact.

Notice that there is a large redundancy in the description with higher times. This should be clear if we consider the fact that $\mathbb{V}$ is a function of ϵ_I and τ_α, so the number of independent λ parameters is at most equal to the number of fixed points. The relations between the equivalent values of the λ are in general non-linear, but there is an interesting subset of linear *gauge transformations* which we will focus on. Due to the relation (4.2.24), $\tau^{\mathbb{T}}$ is invariant under the gauge transformations

$$\lambda_{a_1\ldots a_{k+1}} \to \lambda_{a_1\ldots a_{k+1}} + \beta_i^{(a_1\ldots a_k}v_i^{a_{k+1})}\,, \qquad \lambda_{a_1\ldots a_k} \to \lambda_{a_1\ldots a_k} + \epsilon_i\beta_i^{a_1\ldots a_k}\,, \tag{4.4.8}$$

where $\beta_i^{a_1\ldots a_k}$ is symmetric in the indices $a_1\ldots a_k$. Notice that the subgroup with $\epsilon_i\beta_i^{a_1\ldots a_k}=0$ acts only on $\lambda_{a_1\ldots a_{k+1}}$ without mixing times of different degree and it is the only transformation allowed for single times. In the Calabi-Yau case, where the vectors in the fan lie on a plane identified by the direction $i=CY$, say $v_{CY}^a=1$,[14] this subgroup can also be written as

$$\lambda_{a_1\ldots a_k} \to \lambda_{a_1\ldots a_k} + \gamma_i^{(a_1\ldots a_{k-1}}w_i^{a_k)}\,, \qquad w_i^a = \epsilon_{CY}v_i^a - \epsilon_i\,, \tag{4.4.9}$$

[14]In all the examples that we will discuss ϵ_{CY} will be identified with ϵ_3.

generalizing the results in [28]. Many times can be therefore gauge-fixed to zero.

We note that the fact that $\tau^{\mathbb{T}}$ is invariant under the transformation (4.4.8) does not always imply that $\mathbb{V}$ is also invariant under the same transformation. The issue is that the single times λ_a determine the shape of the polytope: as the λ_a are varied the position of the vertices of the poytope shifts, and when multiple vertices converge the fixed point structure of $\mathbb{M}_{2\mathfrak{m}}$ can transition into a different one. Therefore the the equivariant volume is invariant under (4.4.8) only as long as the shift in the single times λ_a does not cross into region of the moduli space with different fixed point structure.[15] In the following chapter we will often gauge fix the λ_a inside the equivariant form $\tau^{\mathbb{T}}$ to zero for simplicity. Whenever we do this it should be understood that we are not actually changing the fixed point structure, even if setting λ_a to zero at the polytope level would correspond to collapsing all the fixed points into one.

[15]This was not an issue for equation (4.3.9), since the shift $\lambda \to \lambda + \beta_i v_i^a$ simply corresponds to a translation of the polytope.

Chapter 5

Equivariant volume extremization and holography

In this chapter we propose a general prescription to write extremal functions for supergravity solutions with a holographic dual, widening the scope of the approach presented in [28]. In [28] it was suggested that it should be possible to express the extremal functions in supergravity in terms of the equivariant volume, which we reviewed in the previous chapter, and their proposal was checked by studying systems of branes wrapped around either a sphere or a spindle. In this chapter we will focus primarily on systems of branes wrapped around four-dimensional orbifolds (or wrapped around two-cycles inside of them). We will argue that in order to parameterize all the fluxes of Ramond-Ramond forms or M theory forms supported by a given geometry it is necessary to include in the definition of the equivariant volume the *higher times*, which we reviewed in subsection 4.4.

This chapter is organized as follows. In section 5.1 we introduce our prescription for the extremal functions in supergravity and explain how it relates to previous works in the literature.

In section 5.2 we analyse M theory solutions with M5 brane flux. In subsection 5.2.1 we consider solutions associated with M5 branes wrapped over a four-dimensional orbifold $\mathbb{M}_4$. We show that the free energy can be obtained by extremizing the appropriate term in the equivariant volume and that the result agrees with the field theory computation in [28], obtained by integrating the anomaly of the M5 brane theory over $\mathbb{M}_4$. In subsection 5.2.2 we consider solutions that are potentially related to M5 branes wrapped on a two-cycle in $\mathbb{M}_4$. By extremizing the appropriate term in the equivariant volume, we reproduce known results in the literature and extend them to predictions for solutions still to be found. In subsection 5.2.3 we compare our prescription with the recent approach based on Killing spinor bilinears in M theory [53].

In section 5.3 we consider solutions in type II string theory with geometries that are fibrations over a four-dimensional orbifold $\mathbb{M}_4$. In subsection 5.3.1 we consider massive type IIA solutions associated with D4 branes wrapped around a four-dimensional toric orbifold $\mathbb{M}_4$ and derive the free energy proposed in [85]. In subsection 5.3.2 we consider massive type IIA solutions associated with a system of D4/D8 branes, with the former wrapped on a two-cycle in $\mathbb{M}_4$. Extremizing the appropriate term in the equivariant volume we are able to reproduce the gravitational free energy computed from the explicit solution.

In section 5.3.3 we consider type IIB solutions with D3 flux associated with $S^3/\mathbb{Z}_p$ fibrations over $\mathbb{M}_4$, which could potentially arise as the near-horizon limit of a system of D3 branes wrapped on a two-cycle of the four-dimensional orbifold $\mathbb{M}_4$. This example can be covered by the formalism of GK geometry, that we here extend to the case of fibrations over orbifolds, using the equivariance with respect to the full four-torus $\mathbb{T}^4$. In this and other previous examples with M5 branes, we observe that, in order to obtain the correct critical point, one should allow all the equivariant parameters not fixed by symmetries to vary, thus rectifying some previous results in the literature.

Lastly, in section 5.4 we summarize our results and discuss open problems and future perspectives. The appendices B contain technical aspects of some computations.

5.1 Extremal functions from the equivariant volume

The equivariant volume of toric orbifolds is a basic topological object, sensitive only to the degenerations of the torus $\mathbb{T}^m$ near the fixed points, as can be seen from its fixed point furmula (4.3.22). In the applications to holography one encounters metrics that are not Kähler and not even symplectic, but with underlying spaces that are in fact symplectic toric orbifolds and one can nevertheless define $\mathbb{V}$ and use it to compute topological quantities that ultimately will not depend on the metric. Given these properties, the equivariant volume is the gravitational analogue of quantum field theory quantities like 't Hooft anomalies and supersymmetric indices that are invariant under small deformations of the theory once symmetries and matter content are fixed. In [28] it was argued therefore that all extremization problems in gravity can be reformulated in terms of the equivariant volume. It was shown that this is true for volume minimization [8, 9] (dual to a [4] and F-maximization [6]) and the formalism of GK geometry [11, 12] (dual to c [5] and $\mathcal{I}$-extremization [7]). In [28] it was proposed that this should be true more generally, and as a partial check of their proposal they showed that all known extremization problems for branes wrapped over a sphere or a spindle in type II and M theory can indeed be reformulated in terms of the equivariant volume.

In this chapter we will corroborate and generalize the proposal of [28] by analyzing systems of branes partially or totally wrapped around four-dimensional *toric* orbifolds. The toric assumption is not essential, but is made for two reasons. Firstly, if a geometry has a symmetry group that contains $\mathbb{T}^{\mathfrak{m}} = U(1)^{\mathfrak{m}}$, we need to extremize over the corresponding $\mathfrak{m} - 1$ equivariant parameters not fixed by supersymmetry, otherwise the critical point found would not be a *bona fide* extremum of the gravitational action. Secondly, in this case the fixed point theorem simplifies to a sum of contributions at isolated fixed points; more generally, it would be straightforward to proceed assuming a $\mathbb{T}^k = U(1)^k$ Hamiltonian action, with $1 < k < \mathfrak{m}$. Furthermore, in many examples when the underlying geometry is not strictly symplectic or toric but has a $\mathbb{T}^{\mathfrak{m}}$ isometry we can also define a natural generalization of $\mathbb{V}$ by a sort of analytical continuation, as is the case for geometries where the fan is not strictly convex or geometries involving S^4.

This section is organized as follows. We will begin in subsection 5.1.1 by briefly reviewing the the extremal functions in supergravity that have been known in the literature for quite a while [8, 9, 11, 12] and how they relate to the equivariant volume. Then in subsection 5.1.2 we review the proposal of [28] and explain how we generalized it.

5.1.1 Sasakian volume, master volume and equivariant volume

In this section we briefly review the Sasakian volume and its generalization, the *master volume*, mostly following the discussion of [8] and [12, 25]. We will review why these quantities are extremal functions and how they relate to the homogeneous components of the equivariant volume, as observed in [28].

Let $Y_{2\mathfrak{m}-1}$ be a toric Sasaki-Einstein of dimension $2\,\mathfrak{m} - 1$, and let $X_{2\mathfrak{m}}$ be the cone over $Y_{2\mathfrak{m}-1}$, namely $X_{2\mathfrak{m}} = \mathbb{R}_{\geq 0} \times Y_{2\mathfrak{m}-1}$ with metric

$$\mathrm{d}s^2_{X_{2\mathfrak{m}}} = \mathrm{d}r^2 + r^2 \mathrm{d}s^2_{Y_{2\mathfrak{m}-1}} \,. \tag{5.1.1}$$

Since $Y_{2\mathfrak{m}-1}$ is a Sasaki-Einstein, the above metric is Ricci-flat and Kähler, with respect to a compatible integrable complex structure J. Then $X_{2\mathfrak{m}}$ is a toric Calabi-Yau $\mathfrak{m}$-fold singularity, whose polytope is the cone

$$\mathcal{P} = \{ y \in \mathbb{R}^{\mathfrak{m}} \mid y_i \, v_i^a \geq 0 \} \,. \tag{5.1.2}$$

We impose the Calabi-Yau condition by setting $v_1^a = 1$ for all $a = 1, \dots, d$.

The killing vector of interest is the Reeb vector field of $X_{2\mathfrak{m}}$, defined as

$$\xi = J(r\partial_r) = \epsilon_i \partial_{\phi_i} \,, \tag{5.1.3}$$

where the ϵ_i play the role of equivariant parameters. The Reeb vector ξ has unit norm on $Y_{2\mathfrak{m}-1}$ and defines the foliation

$$\mathrm{d}s^2_{Y_{2\mathfrak{m}-1}} = \eta^2 + \mathrm{d}s^2_{2\mathfrak{m}-2} \,, \tag{5.1.4}$$

where η is the dual one-form of ξ (that is $\eta(\xi) = 1$) and the transverse metric $\mathrm{d}s^2_{2\mathfrak{m}-2}$ is conformally Kähler with Kähler form ω_{Sasakian}. We can find the subset of the cone $\mathcal{P}$ that correspond to $Y_{2\mathfrak{m}-1} \equiv X_{2\mathfrak{m}}|_{r=1}$ by imposing that the norm of ξ is one:

$$1 = \epsilon_i \epsilon_j G^{ij} = 2\, \epsilon_i G_{jk} y_k G^{ij} = 2\, \epsilon_i\, y_i \tag{5.1.5}$$

where G_{ij} and G^{ij} are the ones defined in section 4.2.1 and we have used that $\epsilon_j = 2\, G_{jk} y_k$, which follows from $r\partial_r = 2y_i\partial_{y_i}$ and expression (4.2.9) for the complex structure J. Then the subset of $\mathcal{P}$ that correspond to $Y_{2\mathfrak{m}-1}$ is $\mathcal{P} \cap \mathcal{H}_\epsilon$, where $\mathcal{H}_\epsilon$ is the hyperplane

$$\mathcal{H}_\epsilon = \{ y \in \mathbb{R}^{\mathfrak{m}} \mid \epsilon_i\, y_i = \tfrac{1}{2} \} \, . \tag{5.1.6}$$

The Sasakian volume is defined as the volume of $Y_{2\mathfrak{m}-1}$ and it can be determined as following:

$$\text{Vol}(Y_{2\mathfrak{m}-1})(\epsilon_i) = \int_{Y_{2\mathfrak{m}-1}} \eta \wedge \frac{\omega^{\mathfrak{m}-1}_{\text{Sasakian}}}{(\mathfrak{m}-1)!} = \frac{(2\pi)^{\mathfrak{m}}}{|\vec{\epsilon}\,|} \text{Vol}(\mathcal{P} \cap \mathcal{H}_\epsilon) \, . \tag{5.1.7}$$

Let us consider $\text{AdS}_5 \times Y_5$ solutions of type IIB supergravity, $Y_5 \equiv SE_5$. The Einstein-Hilbert action on the space of Sasakian metrics of Y_5 is proportional to the Sasakian volume [8]. Remarkably, the action only depends on the parameters ϵ_i, with ϵ_1 fixed to a constant value. The value of the ϵ_i at the critical point of the action determines the Reeb vector ξ of the Sasaki-Einstein metric. In the dual CFT this procedure mirrors how the R-symmetry can be determined by maximizing a [4] with respect to a trial R-symmetry. The value of a for the CFT can also be found in terms of the minimum of the Sasakian volume by using that [94, 95]

$$\frac{a_{Y_5}}{a_{S^5}} = \frac{\text{Vol}(S^5)}{\text{Vol}(Y_5)} \, . \tag{5.1.8}$$

With a suitable parametrization of the trial R-charge this identity is not only valid at the critical point but also for a generic value of ϵ_i (off-shell) [8, 69].

Similarly, for $\text{AdS}_4 \times SE_7$ solutions in 11d supergravity the Sasakian volume minimization is the dual of the maximization of the free energy F [6], intended as (minus) the logarithm of the supersymmetric partition function on S^3.

In [11, 12] the Sasakian volume has been generalized to the so-called master volume by allowing the transverse Kähler class in the foliation (5.1.4) to vary and no longer be fixed to $[\omega_{\text{Sasakian}}]$.[1] A generic transverse Kähler class can be written as

$$[\omega_B] = -2\pi \sum_{a=1}^{d} \lambda_a\, c_a \, , \tag{5.1.9}$$

[1] We stress that the metric of $Y_{2\mathfrak{m}-1}$ is no longer Sasakian after varying the transverse Kähler class. $Y_{2\mathfrak{m}-1}$ itself is still topologically Sasakian.

where the c_a are cohomology classes in the foliation of $Y_{2\mathfrak{m}-1}$ that uplift to $c_1(L_a)$ in $X_{2\mathfrak{m}}$. In general the above differs from the Sasakian class, which is given by

$$[\omega_{\text{Sasakian}}] = \frac{\pi}{\epsilon_1} \sum_{a=1}^{d} c_a \,, \tag{5.1.10}$$

and thus corresponds to (5.1.9) with $\lambda_a = -\frac{1}{2\epsilon_1}$. The master volume is then defined as

$$\mathcal{V}(\lambda_a, \epsilon_i) = \int_{Y_{2\mathfrak{m}-1}} \eta \wedge \frac{\omega_B^{\mathfrak{m}-1}}{(\mathfrak{m}-1)!} = \frac{(2\pi)^{\mathfrak{m}}}{|\vec{\epsilon}|} \operatorname{Vol}\big(\mathcal{P}(\lambda_a, \epsilon_i)\big) \,, \tag{5.1.11}$$

where $\mathcal{P}(\lambda_a, \epsilon_i)$ generalizes $\mathcal{P} \cap \mathcal{H}_\epsilon$ with the introduction of the λ_a:

$$\mathcal{P}(\lambda_a, \epsilon_i) = \{ y \in \mathcal{H}_\epsilon \mid (y_i - \tfrac{\delta_{i,1}}{2\epsilon_1})\, v_i^a \geq \lambda_a \} \,. \tag{5.1.12}$$

In order to discuss the connection between master volume and extremal functions, for concreteness let us focus on the case of $\mathrm{AdS}_3 \times Y_7$ solutions of type IIB supergravity described by GK geometry [96, 97]. These solutions can be taken off-shell: we can consider supersymmetric geometries that admit the required Killing spinors but do not solve the five-form equation of motion. Putting these geometries back on-shell can be shown to be equivalent to solving the equations of motions that come from varying the following supersymmetric action:

$$S_{SUSY} = \int_{Y_7} \eta \wedge \rho \wedge \frac{J_{2\mathfrak{m}-2}^2}{2} = -\sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} \,, \tag{5.1.13}$$

where ρ and $J_{2\mathfrak{m}-2}$ are the Ricci form and complex structure of the transverse Kähler class of the foliation of Y_7. Additionally we need to impose the quantization of the fluxes of the five-form on all independent five-cicles:

$$\frac{4\pi(2\pi\ell_s)^4 g_s}{L^4} M_a = \frac{L^4}{4\pi} \int_{D_a} F_5 = \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} = -\frac{\partial S_{SUSY}}{\partial \lambda_a} \,, \tag{5.1.14}$$

where the M_a are integers and since the toric divisors are not independent neither are the N_a: form $\sum_a v_i^a D_a = 0$ we find $\sum_a v_i^a N_a = 0$. There is also a topological constraint

$$0 = \int_{Y_7} \eta \wedge \rho^2 \wedge J_{2\mathfrak{m}-2} \tag{5.1.15}$$

which is equivalent to

$$0 = \sum_{a,b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} = -\sum_{a=1}^{d} \frac{\partial S_{SUSY}}{\partial \lambda_a} \,. \tag{5.1.16}$$

When $S_{SUSY}(\lambda_a, \epsilon_i)$ is extremized under the constraints (5.1.14) and (5.1.16) while keeping ϵ_1 fixed to a constant value,[2] the value of the ϵ_i at the critical point fixes the Killing vector ξ and the extremal value of S_{SUSY} can be used to determine the central charge c of the dual CFT:

$$c = \frac{3L^8}{(2\pi)^6 g_s^2 \ell_s^8} \, S_{SUSY}\big|_{\text{on-shell}} \, . \tag{5.1.17}$$

This process is the gravitational dual of c-extremization [5].

In [28] it has been shown that the master volume is proportional to the homogeneous component of degree $\mathfrak{m} - 1$ of the equivariant volume of the toric cone $X_{2\mathfrak{m}}$:

$$\mathcal{V}(\lambda_a, \epsilon_i) = (2\pi)^{\mathfrak{m}} \, \mathbb{V}^{(\mathfrak{m}-1)}(\lambda_a, \epsilon_i) \, . \tag{5.1.18}$$

Using the property (4.3.11) and the fact that $v_1^a = 1$ for $a = 1, \dots, d$, the supersymmetric action (5.1.13) can also be expressed as a homogeneous component of $\mathbb{V}$:

$$S_{SUSY} = \epsilon_1 \, (2\pi)^{\mathfrak{m}} \, \mathbb{V}^{(\mathfrak{m}-2)}(\lambda_a, \epsilon_i) \, . \tag{5.1.19}$$

Equations (5.1.14) and (5.1.16) will be naturally incorporated in the prescription [28] that we will generalize in subsection 5.1.2.

We conclude this subsection by commenting that the master volume formalism of GK geometry can also be applied to AdS solutions whose internal spaces are fibrations with a Kähler base manifold and toric topologically-Sasakian fiber [54]. We will not review the details here since all solutions in GK geometry that can studied with the master volume can also be studied with the prescription we present in the following subsection, provided that the geometry is toric.[3] A concrete example of this are the $\text{AdS}_3 \times M_7$ solutions in type IIB that we analyze in subsection 5.3.3, which generalize to the orbifold case the ones that have been studied in [54] with the master volume. Lastly, let us mention how $\text{AdS}_2 \times M_9$ solutions in GK geometry relate to $\mathcal{I}$-extremization [7, 24, 25]. If M_9 is a fibration of toric Y_7 over a Riemann surface, these solutions can arise as the near-horizon limit of asymptotically $\text{AdS}_4 \times Y_7$ black holes. Then the supersymmetric action S_{SUSY} is proportional to the entropy function of the black holes and reproduces their Bekenstein-Hawking entropy when extremized. The field theory quantity dual to the entropy function is the topologically-twisted index $\mathcal{I}$ [23], closely related to the superconformal index that we reviewed in chapter 2.

5.1.2 A general prescription

It has been shown in [28] that all known extremization problems for branes wrapped over a sphere or a spindle can be formulated in terms of an extremal function which

[2] ϵ_1 is fixed to a constant by the requirement that the Killing vector ξ has the appropriate R-charge.

[3] As already mentioned, it would be possible to relax toric assumption.

matches one of the homogeneous components (4.3.8) of the equivariant volume:

$$F \;=\; \mathbb{V}^{(\alpha)}(\lambda_A,\epsilon_I)\,, \tag{5.1.20}$$

where $\mathbb{V}^{(\alpha)}$ is homogeneous of degree α in λ_A. The parameters of F are subject to a set of flux constraints which can also be expressed in terms of a homogeneous component of $\mathbb{V}$ as follows:

$$\nu M_A = -\frac{\partial \mathbb{V}^{(\beta)}}{\partial \lambda_A}\,, \tag{5.1.21}$$

where M_A are the integer fluxes of the relevant Ramond-Ramond or M theory anti-symmetric form, obeying

$$\sum_A V_I^A M_A \;=\; 0\,. \tag{5.1.22}$$

ν is a normalization constant[4] that depends on the type of brane and the dimension of the internal geometry. We also note that from (5.1.21) and (5.1.22) it follows, using the property (4.3.11) of $\mathbb{V}$, that the constraint

$$\mathbb{V}^{(\beta-1)} \;=\; 0 \tag{5.1.23}$$

must be satisfied. Although it is not an independent relation, one can regard this as a topological constraint necessary in order to impose the flux quantization. Formulae (5.1.21) and (5.1.23) are the analogous of the master volume formulae (5.1.14) and (5.1.16) respectively. The integers α and β depend on the type of brane. By a simple scaling argument, it was found that

$$\begin{array}{lll}
\text{D3 branes in type IIB:} & \alpha=2\,, & \beta=2\\
\text{M2 branes in M theory:} & \alpha=3\,, & \beta=3\\
\text{M5 branes in M theory:} & \alpha=3\,, & \beta=2\\
\text{D4 branes in massive type IIA:} & \alpha=5\,, & \beta=3\\
\text{D2 branes in massive type IIA:} & \alpha=5\,, & \beta=4\,.
\end{array} \tag{5.1.24}$$

The extremal function F can be normalized such that its extremum reproduces the central charge of the dual field theory in even dimensions and the logarithm of the sphere partition function in odd dimensions and we will use this convention in the following.

In this chapter we show that this construction also holds for known extremization problems for branes (partially or totally) wrapped over four-dimensional toric

[4] *A priori* there is also an overall normalization constant in the definition of F, again depending on the type of brane and the dimension of the internal geometry, however this can always be absorbed in a rescaling of the λ_A, using the homogeneity of $\mathbb{V}^{(\gamma)}$. For simplicity, in the examples we will indicate only the type of brane as a subscript in ν, omitting the dependence on the dimension of the internal geometry.

orbifolds. As we will later argue, for this kind of geometries the equivariant volume with just single times $\mathbb{V}(\lambda_A, \epsilon_I)$ is not sufficient: we need to include higher times. For all the examples that we will consider in this chapter single and double times (λ_A and λ_{AB}) will be sufficient. Only in appendix B.2 we will show an example of computation where we intentionally over-parametrize the system by using triple times λ_{ABC} as well. As a general rule, to fully capture the parameters of the supergravity solution, we need a number of independent parameters at least equal to the number of fixed points. Indeed from equation (4.4.4) we can see that functionally $\mathbb{V}$ is a function of ϵ_I and τ_α only. If the number of higher times $\lambda_{A_1 \dots A_k}$ is too big to be fixed by flux constraints and gauge transformations, we will argue that the correct procedure is to *extremize* with respect to the excess parameters. As we will show in the rest of this chapter, the equivariant volume with higher times contains all the information needed to fully capture the topological properties and the quantization of fluxes for a very large class of supergravity solutions. Depending on the system the flux constraint (5.1.21) may need to be modified to include a derivative in the λ_{AB} in place of the derivative in λ_A. We will discuss this on a case-by-case basis.

The above construction relies on even-dimensional toric orbifolds. For supergravity backgrounds $\mathrm{AdS}_d \times M_k$ with odd-dimensional internal space M_k the geometry to consider is the cone over M_k, as familiar from holography. This cone is often a non-compact toric Calabi-Yau, or, in the case of supersymmetry preserved with anti-twist, a non-convex generalization.[5] When M_k is even-dimensional, we consider the equivariant volume of the compact M_k itself. Some M5 brane solutions have a $\mathbb{Z}_2$ symmetry that allows to cut into half the number of fixed point and consider an equivalent problem for a non-compact Calabi-Yau (half of the manifold). This was done in [28] for M5 branes wrapped on a spindle.

Since all the geometries that we will consider in this chapter are fibrations over the four-dimensional toric orbifold $\mathbb{M}_4$, for clarity of notation we will use capital letters (V_I^A, A, I) for the higher-dimensional geometry and lower-case letters (v_i^a, a, i) for $\mathbb{M}_4$.

Our approach naturally incorporates the GMS construction based on GK geometry [11, 12] as well as the recent localization technique based on Killing spinor bilinears in M theory [53]. Indeed, we will show that, for M5 solutions with even-dimensional M_6 or M_8, our approach is effectively equivalent to the one in [53]. In particular, all the geometrical constraints that must be imposed on a case-by-case analysis in order to find the free energy in [53] appear naturally in our construction as an extremization with respect to all the parameters that are not fixed by the flux quantization conditions. On the one hand, this is a nice confirmation of our prescription. On the other hand, our approach for the toric case is more gen-

[5] See for example [98]. Whenever we will encounter polytopes and polyhedral cones that are not convex, we will obtain results by performing a suitable extrapolation from the convex case.

eral, it covers in a simple and universal way the even and odd-dimensional cases, it naturally extends to massive type IIA solutions, which are not yet covered by the previous techniques, and expresses everything in terms of the extremization of a universal quantity, the equivariant volume of the associated geometry, without referring to supergravity quantities. We are confident that when the explicit case-by-case supergravity analysis will be performed for the missing backgrounds it will confirm our general prescription.

5.2 AdS_3 and AdS_5 solutions in M theory

We start by analysing M theory solutions with M5 brane flux and show that the free energy can be obtained by extremizing the appropriate term in the equivariant volume. The case of M5 branes wrapped on a spindle have been already studied in [28]. Here we focus on geometries that are fibrations over a four-dimensional toric orbifold $\mathbb{M}_4$.

5.2.1 $\mathrm{AdS}_3 \times M_8$ solutions

In this section we consider $\mathrm{AdS}_3 \times M_8$ solutions in M theory, where[6] M_8 is an S^4 fibration over the four-dimensional orbifold $\mathbb{M}_4$. Examples of this form have been found in [84] and further discussed in [85, 86, 99]. They are obtained by uplifting $\mathrm{AdS}_3 \times \mathbb{M}_4$ solutions of $D = 7$ maximal gauged supergravity to eleven dimensions. These $\mathrm{AdS}_3 \times M_8$ solutions can be interpreted as the near-horizon geometry of a system of M5 branes wrapped around $\mathbb{M}_4$.

We need first to identify the topological structure of the underlying geometry. We will focus on the case of toric $\mathbb{M}_4$. The eight-dimensional geometry M_8 is not strictly toric, but it admits an action of $\mathbb{T}^4 = U(1)^4$. If d is the dimension of the fan of $\mathbb{M}_4$, there are $2d$ fixed points of the torus action obtained by selecting a fixed point on $\mathbb{M}_4$ and combining it with the North and South pole of S^4. We will assume that there is a $\mathbb{Z}_2$ symmetry of the fibration that identifies the North and South pole contributions to the fixed point formula. In this situation we can consider half of the geometry, a $\mathbb{C}^2$ fibration over $\mathbb{M}_4$ with the geometry of a non-compact toric CY_4. One can understand the appearance of the fibre $\mathbb{C}^2$ from the transverse geometry of the brane system, which is $\mathbb{C}^2 \times \mathbb{R}$, with S^4 embedded inside. We then consider a CY_4 with fan generated by the vectors

$$V^a = (v^a, 1, \mathfrak{t}_a)\,, \qquad V^{d+1} = (0, 0, 1, 0)\,, \qquad V^{d+2} = (0, 0, 1, 1)\,, \tag{5.2.1}$$

where v^a, $a = 1, \dots, d$, are the vectors of the fan of $\mathbb{M}_4$ and $\mathfrak{t}_a$ are integers specifying the twisting of $\mathbb{C}^2$ over $\mathbb{M}_4$. When supersymmetry is preserved with anti-twist [100], the toric diagram is not convex and it does not strictly define a toric geometry. We

[6] In general, M_8 is itself an orbifold.

will nevertheless proceed also in this case, considering it as an extrapolation from the twist case. The non-convex case is obtained from the formulas in this chapter by sending $v^a \to \sigma^a v^a$, where $\sigma^a = \pm 1$.

In addition to the metric, the supergravity solution is specified by the integer fluxes of the M theory four-form along all the non-trivial four-cycles. The toric four-cycles of the geometry are $\mathbb{M}_4$ itself, the sphere S^4 and $\mathbb{P}^1$ fibrations over the toric two-cycles $\Sigma_a \subset \mathbb{M}_4$. In our half-geometry, the sphere S^4 and $\mathbb{P}^1 \subset S^4$ are replaced with copies of $\mathbb{C}^2$ and $\mathbb{C}$. All together, the toric four-cycles correspond to all the possible intersections of the toric divisors $D_A \cap D_B$ and we can therefore introduce a matrix of fluxes M_{AB}. As usual, not all toric divisors are inequivalent in co-homology. The relations $\sum_A V_I^A D_A = 0$ imply that the matrix of fluxes satisfy

$$\sum_A V_I^A M_{AB} = 0\,. \tag{5.2.2}$$

We are now ready to formulate our prescription for the extremal function. For M5 branes in M theory, as discussed in [28] and in the introduction, we define the free energy to extremize as

$$F = \mathbb{V}^{(3)}(\lambda_A, \lambda_{AB}, \epsilon_I)\,, \tag{5.2.3}$$

and impose the flux constraints[7]

$$\overline{\nu}_{M5}\,(2-\delta_{AB})\,M_{AB} = -\frac{\partial}{\partial \lambda_{AB}}\mathbb{V}^{(2)}(\lambda_A, \lambda_{AB}, \epsilon_I)\,. \tag{5.2.4}$$

Here the index $A = 1, \dots, d+2$ runs over all the vectors of the fan of the CY_4, whereas we reserve the lower-case index $a = 1, \dots, d$ for the vectors of the fan of the base $\mathbb{M}_4$. On the other hand, the index $I = 1, 2, 3, 4$ runs over the equivariant parameters of the CY_4 and we will use $i = 1, 2$ for the directions inside $\mathbb{M}_4$. We have added a $(2-\delta_{AB})$ factor in the equation for the fluxes for convenience. It is easy to see using (4.4.4) that this equation can be equivalently rewritten as

$$\overline{\nu}_{M5}\,M_{AB} = -\frac{\partial^2}{\partial \lambda_A \partial \lambda_B}\mathbb{V}^{(3)}(\lambda_A, \lambda_{AB}, \epsilon_I)\,, \tag{5.2.5}$$

and one may wonder if we really need higher times. The answer is yes. As we will discuss later, with only single times the previous equation cannot be solved.[8]

In the rest of this section we will show that F reproduces the expected extremal function and its factorization in gravitational blocks discussed in [28, 85].

[7] We put a bar on top of ν_{M5} to stress that we are using a half-geometry. To have the correct normalization of the free energy when using half of the geometry, the parameter ν_{M5} must be rescaled as in formula (5.2.47), as we will discuss more extensively in section 5.2.2.

[8] One would need to restrict the $\mathfrak{t}_a$ in order to find solutions.

5.2.1.1 The equivariant volume with double times

The $\mathbb{T}^4$ torus action on the CY_4 has d fixed points, each one corresponding to a cone in the fan with generators $(V^a, V^{a+1}, V^{d+1}, V^{d+2})$, $a = 1, \dots, d$. In particular, there is a one-to-one correspondence between these fixed points and the ones of the base orbifold $\mathbb{M}_4$; for the latter the fixed points correspond to two-dimensional cones of the form (v^a, v^{a+1}) and they can be labelled by the index a. The order of the orbifold singularities associated with the fixed points of CY_4 and $\mathbb{M}_4$ also match:

$$d_{a,a+1,d+1,d+2} = \left|\det(V^a, V^{a+1}, V^{d+1}, V^{d+2})\right| = \left|\det(v^a, v^{a+1})\right| = d_{a,a+1}\,. \tag{5.2.6}$$

Therefore, the fixed point formula for the equivariant volume with higher times of CY_4 takes the following form:

$$\mathbb{V}(\lambda_A, \lambda_{AB}, \epsilon_I) = \sum_a \frac{\mathrm{e}^{\tau_a}}{d_{a,a+1}\, e^{\mathbb{T}^4}|_a}\,. \tag{5.2.7}$$

Here, τ_a is the restriction to the fixed point a of the form (4.4.3)

$$\tau_a = \left(\sum_A \lambda_A\, c_1^{\mathbb{T}^4}(L_A) + \sum_{A,B} \lambda_{AB}\, c_1^{\mathbb{T}^4}(L_A)\, c_1^{\mathbb{T}^4}(L_B) \right)\Bigg|_a\,, \tag{5.2.8}$$

while at the denominator we have the restriction of the Euler class $e^{\mathbb{T}^4}$

$$e^{\mathbb{T}^4}\big|_a = \left(c_1^{\mathbb{T}^4}(L_a)\, c_1^{\mathbb{T}^4}(L_{a+1})\, c_1^{\mathbb{T}^4}(L_{d+1})\, c_1^{\mathbb{T}^4}(L_{d+2}) \right)\Big|_a\,. \tag{5.2.9}$$

The restrictions of the Chern classes can be computed using (4.3.20). The inward normals to the faces of the cone generated by $(V_a, V_{a+1}, V_{d+1}, V_{d+2})$ are

$$\begin{aligned}
U^a &= (u_1^a, 0, 0)\,, \\
U^{a+1} &= (u_2^a, 0, 0)\,, \\
U^{d+1} &= \big((\mathfrak{t}_a - 1)u_1^a + (\mathfrak{t}_{a+1} - 1)u_2^a\,, d_{a,a+1}, -d_{a,a+1}\big)\,, \\
U^{d+2} &= (-\mathfrak{t}_a u_1^a - \mathfrak{t}_{a+1} u_2^a\,, 0, d_{a,a+1})\,,
\end{aligned} \tag{5.2.10}$$

where u_1^a and u_2^a are the two-dimensional normals to the cone (v^a, v^{a+1}). Using the notations introduced in (4.3.24) we find

$$\begin{aligned}
c_1^{\mathbb{T}^4}(L_a)\big|_a &= -\frac{\epsilon_i\, (u_1^a)_i}{d_{a,a+1}} = -\epsilon_1^a\,, \\
c_1^{\mathbb{T}^4}(L_{a+1})\big|_a &= -\frac{\epsilon_i\, (u_2^a)_i}{d_{a,a+1}} = -\epsilon_2^a\,, \\
c_1^{\mathbb{T}^4}(L_b)\big|_a &= 0\,, \qquad b \neq a, a+1\,, \\
c_1^{\mathbb{T}^4}(L_{d+1})\big|_a &= -(\mathfrak{t}_a - 1)\epsilon_1^a - (\mathfrak{t}_{a+1} - 1)\epsilon_2^a - \epsilon_3 + \epsilon_4\,,
\end{aligned} \tag{5.2.11}$$

$$c_1^{\mathbb{T}^4}(L_{d+2})\big|_a = \mathfrak{t}_a\epsilon_1^a + \mathfrak{t}_{a+1}\epsilon_2^a - \epsilon_4\,,$$

where for simplicity we have used Einstein notation for the sums over the index $i = 1, 2$.

We can write the equivariant volume of the CY_4 as an integral over the base orbifold $\mathbb{M}_4$ of four-dimensional equivariant forms with ϵ_3 and ϵ_4 as parameters. Let us denote with $\mathbb{T}$ the two-dimensional torus associated with ϵ_1 and ϵ_2, and let $c_1^{\mathbb{T}}(L_a)$ be the equivariant Chern classes associated to the restrictions of the line bundles L_a to the base $\mathbb{M}_4$. We can then take advantage of the one-to-one correspondence between fixed point of the CY_4 and fixed points of $\mathbb{M}_4$ and, using (4.3.28), we can rewrite (5.2.7) as

$$\mathbb{V}(\lambda_A, \lambda_{AB}, \epsilon_I) = \int_{\mathbb{M}_4} \frac{\mathrm{e}^{\tau^{\mathbb{T}}}}{\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}}\,, \tag{5.2.12}$$

where

$$\begin{aligned} \tau^{\mathbb{T}} &= \sum_A \lambda_A\,\mathcal{C}_A + \sum_{A,B} \lambda_{AB}\,\mathcal{C}_A\,\mathcal{C}_B\,, \\ \mathcal{C}_a \;&=\; c_1^{\mathbb{T}}(L_a)\,, \qquad a = 1, \dots, d\,, \\ \mathcal{C}_{d+1} \;&=\; -\epsilon_3 + \epsilon_4 + \sum_a (\mathfrak{t}_a - 1) c_1^{\mathbb{T}}(L_a)\,, \\ \mathcal{C}_{d+2} \;&=\; -\epsilon_4 - \sum_a \mathfrak{t}_a c_1^{\mathbb{T}}(L_a)\,. \end{aligned} \tag{5.2.13}$$

Notice the relations $\sum_a v_i^a\, c_1^{\mathbb{T}}(L_a) = -\epsilon_i$ and $\sum_A V_I^A\,\mathcal{C}_A = -\epsilon_I$, following from (**??**).[9]

The homogeneous component of degree α of the equivariant volume with higher times can be expressed as

$$\mathbb{V}^{(\alpha)}(\lambda_A, \lambda_{AB}, \epsilon_I) = \int_{\mathbb{M}_4} \frac{(\tau^{\mathbb{T}})^\alpha}{\alpha!\,\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}} = \sum_a \frac{B_a^{(\alpha)}}{d_{a,a+1}\,\epsilon_1^a\,\epsilon_2^a}\,, \tag{5.2.14}$$

where we have defined $B_a^{(\alpha)}$ to be the restriction over the a-th fixed point of $\mathbb{M}_4$ of the following equivariant form:

$$B^{(\alpha)} = \frac{(\tau^{\mathbb{T}})^\alpha}{\alpha!\,\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}}\,. \tag{5.2.15}$$

For later reference we derive the relation between $B_a^{(\alpha)}$ and $B_a^{(\beta)}$

$$B_a^{(\beta)} = \frac{(\tau_a)^\beta}{\beta!\,\big(\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}\big)|_a} = \frac{(\alpha!)^{\frac{\beta}{\alpha}}}{\beta!}\left[\frac{(\tau_a)^\alpha}{\alpha!\,\big(\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}\big)|_a}\right]^{\frac{\beta}{\alpha}}\Big[\big(\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}\big)|_a\Big]^{\frac{\beta}{\alpha}-1} \tag{5.2.16}$$

[9] The second relation, which can be checked by direct computation, is obviously the restriction of $\sum_A V_I^A\, c_1^{\mathbb{T}^4}(L_A) = -\epsilon_I$ to $\mathbb{M}_4$.

$$= \frac{(\alpha!)^{\frac{\beta}{\alpha}}}{\beta!} \big(B_a^{(\alpha)}\big)^{\frac{\beta}{\alpha}} \big((1-\mathfrak{t}_a)\epsilon_1^a + (1-\mathfrak{t}_{a+1})\epsilon_2^a - \epsilon_3 + \epsilon_4\big)^{\frac{\beta}{\alpha}-1} \big(\mathfrak{t}_a\epsilon_1^a + \mathfrak{t}_{a+1}\epsilon_2^a - \epsilon_4\big)^{\frac{\beta}{\alpha}-1} .$$

When α is even this formula holds in terms of absolute values and the signs must be fixed separately. This will not be the case for the computation of this section, so we postpone the discussion about the signs to section 5.3.1.

5.2.1.2 Solving the flux constraints

The flux constraints (5.2.4) reads

$$\overline{\nu}_{M5}\,(2-\delta_{AB})\,M_{AB} = -\frac{\partial \mathbb{V}^{(2)}}{\partial \lambda_{AB}} = -(2-\delta_{AB}) \int_{\mathbb{M}_4} \frac{\mathcal{C}_A\,\mathcal{C}_B\,\tau^{\mathbb{T}}}{\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}} \,, \tag{5.2.17}$$

or, equivalently

$$\overline{\nu}_{M5}\,M_{AB} = -\int_{\mathbb{M}_4} \frac{\mathcal{C}_A\,\mathcal{C}_B\,\tau^{\mathbb{T}}}{\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}} = -\sum_a \frac{B_a^{(1)} \cdot (\mathcal{C}_A\,\mathcal{C}_B)|_a}{d_{a,a+1}\,\epsilon_1^a\,\epsilon_2^a} \,. \tag{5.2.18}$$

Let us focus on the $A, B \in \{1, \dots, d\}$ sector. Using (4.3.28) we find

$$\begin{aligned}
\overline{\nu}_{M5}\,M_{a,a+1} &= -\frac{B_a^{(1)}}{d_{a,a+1}} \,, \\
\overline{\nu}_{M5}\,M_{a,a} &= -\frac{B_a^{(1)}\,\epsilon_1^a}{d_{a,a+1}\,\epsilon_2^a} - \frac{B_{a-1}^{(1)}\,\epsilon_2^{a-1}}{d_{a-1,a}\,\epsilon_1^{a-1}} \,, \\
\overline{\nu}_{M5}\,M_{ab} &= 0 \qquad \text{when } b \neq a,\ a+1,\ a-1 \,.
\end{aligned} \tag{5.2.19}$$

These equations give constraints on the fluxes but they have a very simple solution

$$\begin{aligned}
B_a^{(1)} &= -\overline{\nu}_{M5}\,N \,, \\
M_{ab} &= N\,D_{ab} \,,
\end{aligned} \tag{5.2.20}$$

where D_{ab} is the intersection matrix of divisors (4.3.30) and N is any integer that is a multiple of all the products $d_{a-1,a}\,d_{a,a+1}$.

This can be seen as follows. By combining the first two equations we obtain

$$M_{a,a} = M_{a,a+1}\,\frac{\epsilon_1^a}{\epsilon_2^a} + M_{a,a-1}\,\frac{\epsilon_2^{a-1}}{\epsilon_1^{a-1}} \,, \tag{5.2.21}$$

and using the relation (4.3.31)

$$\frac{\epsilon_1^a}{d_{a,a+1}\,\epsilon_2^a} + \frac{\epsilon_2^{a-1}}{d_{a-1,a}\,\epsilon_1^{a-1}} = -\frac{d_{a-1,a+1}}{d_{a-1,a}\ d_{a,a+1}} \,, \tag{5.2.22}$$

we can rewrite this as

$$M_{a,a}\,d_{a,a+1} + M_{a,a-1}\,d_{a-1,a+1} = \frac{\epsilon_1^a}{\epsilon_2^a}\big(M_{a,a+1}\,d_{a,a+1} - M_{a,a-1}\,d_{a-1,a}\big) \,. \tag{5.2.23}$$

Given that the fluxes M_{AB} and the orders of the orbifold singularity $d_{a,a+1}$ are just integers, the only way that this equation can be true for general values of ϵ is for both sides to vanish. This implies that M_{ab} is proportional to the intersections D_{ab} given in (4.3.30). We can then conclude that the only solution to equations (5.2.19) is (5.2.20). Notice that there is just one independent flux associated with the M_{ab} components of the flux matrix. This was to be expected since this corresponds to the M theory four-form flux on S^4.

The values of the remaining entries of the matrix of fluxes M_{AB} are related to the fibration parameters. By substituting $B_a^{(1)} = -\overline{\nu}_{M5}\, N$ in (5.2.18) we find

$$M_{AB} = N \sum_a \frac{(\mathcal{C}_A\, \mathcal{C}_B)|_a}{d_{a,a+1}\, \epsilon_1^a\, \epsilon_2^a} = N \int_{\mathbb{M}_4} \mathcal{C}_A\, \mathcal{C}_B = N \sum_{c,d} \mathfrak{t}_A^c\, \mathfrak{t}_B^d\, D_{cd}\,. \tag{5.2.24}$$

In the last step we have used (4.3.30) and for convenience we have defined $\mathfrak{t}_A^c$ as

$$\mathfrak{t}_A^c = \begin{cases} \delta_A^c & A \in \{1, \dots, d\} \\ \mathfrak{t}_c - 1 & A = d+1 \\ -\mathfrak{t}_c & A = d+2 \end{cases}\,. \tag{5.2.25}$$

Given that the $\mathfrak{t}_a$ are integers, the fluxes M_{AB} in (5.2.24) are all integers.

We note that the expression (5.2.24) for M_{AB} satisfies the relation required to be considered a matrix of fluxes,

$$\sum_A V_I^A M_{AB} = 0\,. \tag{5.2.26}$$

This can easily be verified by noting that

$$\sum_A V_I^A\, \mathfrak{t}_A^c = \begin{cases} v_i^c & I \equiv i = 1,2 \\ 0 & I = 3,4 \end{cases}\,, \qquad \sum_a v_i^c D_{cd} = 0\,. \tag{5.2.27}$$

The simplest solution to the equations

$$B_a^{(1)} \equiv \frac{\tau_a(\lambda_A(\epsilon_I), \lambda_{AB}(\epsilon_I), \epsilon_I)}{(\mathcal{C}_{d+1}\, \mathcal{C}_{d+2})|_a} = -\overline{\nu}_{M5}\, N \tag{5.2.28}$$

is to set $\lambda_{d+1,d+2} = -\frac{1}{2}\,\overline{\nu}_{M5} N$ while setting all the other λ_A and λ_{AB} to zero. We note that in general there exist no solutions to these equations with $\lambda_{AB} = 0$ for all A, B, meaning that the inclusion of the higher times to the equivariant volume is necessary. This stems from the fact that when $\lambda_{AB} = 0$ only $d-1$ of the τ_a are independent: using the gauge invariance (4.4.9)

$$\lambda_A \to \lambda_A + \sum_{I=1}^{4} \gamma_I (\epsilon_3 V_I^A - \epsilon_I)\,, \tag{5.2.29}$$

three out of the $d+2$ Kähler moduli λ_A can be set to zero.

5.2.1.3 The extremal function and c-extremization

We are now ready to compute the extremal function

$$F(\epsilon_I) = \mathbb{V}^{(3)}(\lambda_A(\epsilon_I), \lambda_{AB}(\epsilon_I), \epsilon_I)\,. \tag{5.2.30}$$

The dual field theory is supposed to be the two-dimensional SCFT obtained compactifying on $\mathbb{M}_4$ the $(2,0)$ theory living on a stack of N M5 branes. The gravitational extremization problem should correspond to c-extremization in the dual two-dimensional SCFT.

A general comment that applies to all the examples in this chapter is the following. The free energy must be extremized with respect to *all but one* of the parameters ϵ_I in order to find the critical point. The value of the remaining parameter must be instead fixed by requiring the correct scaling of the supercharge under the R-symmetry vector field ξ. This is familiar from the constructions in [8, 9, 11, 12]. In our case, we extremize with respect to ϵ_4, ϵ_1 and ϵ_2 with ϵ_3 fixed to a canonical value.[10]

Using relations (5.2.14) and (5.2.16) we find

$$\begin{aligned}
F &= \sum_a \frac{B_a^{(3)}}{d_{a,a+1}\,\epsilon_1^a\,\epsilon_2^a}\,,\\
B_a^{(3)} &= \frac{1}{6}(-\overline{\nu}_{M5}\,N)^3\,\big((1-\mathfrak{t}_a)\epsilon_1^a + (1-\mathfrak{t}_{a+1})\epsilon_2^a - \epsilon_3 + \epsilon_4\big)^2\big(\mathfrak{t}_a\epsilon_1^a + \mathfrak{t}_{a+1}\epsilon_2^a - \epsilon_4\big)^2\,,
\end{aligned} \tag{5.2.31}$$

which matches the form of the conjectured formula of [85] in terms of gravitational blocks [27].[11]

To make contact with the dual field theory, we can also write our result in terms of an integral of equivariant forms over the base $\mathbb{M}_4$ as follows:

$$\begin{aligned}
F &= -\frac{1}{6}\,\overline{\nu}_{M5}^3\,N^3\int_{\mathbb{M}_4}\mathcal{C}_{d+1}^2\,\mathcal{C}_{d+2}^2\\
&= -\frac{1}{6}\,\overline{\nu}_{M5}^3\,N^3\int_{\mathbb{M}_4}\Big(\epsilon_3 - \epsilon_4 + \sum_a(1-\mathfrak{t}_a)c_1^{\mathbb{T}}(L_a)\Big)^2\Big(\epsilon_4 + \sum_a \mathfrak{t}_a c_1^{\mathbb{T}}(L_a)\Big)^2\,.
\end{aligned} \tag{5.2.32}$$

This expression correctly reproduces the M5 brane anomaly polynomial integrated over the four-dimensional orbifold $\mathbb{M}_4$ as computed in [28].[12]

[10] Notice that the free energy is homogeneous of degree two in the parameters ϵ_I, so it makes no sense to extremize with respect to all parameters. The specific numerical value of the equivariant parameter fixed by supersymmetry depends on the setup considered as well as on conventions. In this chapter we will not fix the numerical values of this parameter from first principles, but rather we will show that this can be absorbed by the parameter ν.

[11] The convention for the sign of the free energy in [85] is the opposite of ours.

[12] Attention must be paid when performing the comparison since the symbol F refers to the central charge here, while it refers to the integral of the anomaly polynomial in [28] (see also (5.2.35)).

Let us briefly review the comparison with field theory, referring to [28] for details. The anomaly polynomial of the 2d SCFT is obtained by integrating the eight-form anomaly polynomial of the six-dimensional theory over $\mathbb{M}_4$, which, at large N, gives

$$\mathcal{A}_{\rm 2d} = \int_{\mathbb{M}_4} \mathcal{A}_{\rm 6d} = \frac{N^3}{24} \int_{\mathbb{M}_4} c_1(F_1)^2 c_1(F_2)^2 \,, \tag{5.2.33}$$

where F_I are the generators of the $U(1) \times U(1) \subset SO(5)_R$ Cartan subgroup of the $(2,0)$ theory R-symmetry. The $c_1(F_I)$ can be decomposed as

$$c_1(F_I) = \Delta_I c_1(F_R^{\rm 2d}) - \mathfrak{p}_I^a \Big(c_1(L_a) + 2\pi \mu_a^i c_1(\mathcal{J}_i) \Big) \,, \tag{5.2.34}$$

where $F_R^{\rm 2d}$, $\mathcal{J}_1, \mathcal{J}_2$ are line bundles associated with the 2d R-symmetry and the two global symmetries coming from the isometries of $\mathbb{M}_4$. They correspond to background fields for the two-dimensional theory with no legs along $\mathbb{M}_4$. Substituting (5.2.34) in (5.2.33) and setting $c_1(\mathcal{J}_i) = \epsilon_i c_1(F_R^{\rm 2d})$, leads to the equivariant integral

$$\mathcal{A}_{\rm 2d} = \frac{c_r}{6} c_1(F_R^{\rm 2d})^2 = \frac{N^3}{24} c_1(F_R^{\rm 2d})^2 \int_{\mathbb{M}_4} (\Delta_1 - \mathfrak{p}_1^a c_1^{\mathbb{T}}(L_a))^2 (\Delta_2 - \mathfrak{p}_2^a c_1^{\mathbb{T}}(L_a))^2 \,. \tag{5.2.35}$$

Preserving supersymmetry with a twist requires $c_1(F_1) + c_1(F_2) = 2c_1(F_R^{\rm 2d}) - \sum_a c_1(L_a)$ which gives [28]

$$\Delta_1 + \Delta_2 = 2 + \det(W, \epsilon) \,, \qquad \mathfrak{p}_1^a + \mathfrak{p}_2^a = 1 + \det(W, v^a) \,, \tag{5.2.36}$$

where $\epsilon = (\epsilon_1, \epsilon_2)$ and $W \in \mathbb{R}^2$ is a two-dimensional constant vector.[13] The two-dimensional central charge c_r is extracted from (5.2.35) and should be extremized with respect to ϵ_i and Δ_I subject to the previous constraint. We then see that the extremization of the gravitational free energy is equivalent to c-extremization under the identifications

$$\Delta_1 = \epsilon_4 \,, \qquad \Delta_2 = \epsilon_3 - \epsilon_4 \,, \qquad \mathfrak{p}_1^a = \mathfrak{t}_a \,, \qquad \mathfrak{p}_2^a = 1 - \mathfrak{t}_a \,, \qquad W = 0 \,, \tag{5.2.37}$$

where we set $\epsilon_3 = 2$ for convenience. The free energy F is actually homogeneous of degree two in ϵ_I. To match the free energy with the central charge we have to set $\epsilon_3^2 \overline{\nu}_{M5}^3 = -6$. The case of anti-twist is similar and can be discussed by taking a non-convex fan for $\mathbb{M}_4$. The most general supersymmetry condition is now $c_1(F_1) + c_1(F_2) = 2c_1(F_R^{\rm 2d}) - \sum_a \sigma^a c_1(L_a)$ where $\sigma_a = \pm 1$ as discussed in [85] and requires

$$\Delta_1 + \Delta_2 = 2 + \det(W, \epsilon) \,, \qquad \mathfrak{p}_1^a + \mathfrak{p}_2^a = \sigma_a + \det(W, v^a) \,. \tag{5.2.38}$$

This case can be just obtained by *formally* sending $v^a \to \sigma^a v^a$ everywhere, implying $\epsilon_1^a \to \sigma^a \epsilon_1^a$ and $\epsilon_2^a \to \sigma^{a+1} \epsilon_2^a$.

[13] W can be gauged away, see [28].

5.2.2 $\mathrm{AdS}_5 \times M_6$ solutions

In this section we consider a generalization of the family of M theory solutions found in [101] and further studied in [102]. Their geometry is $\mathrm{AdS}_5 \times M_6$ where M_6 is a manifold obtained as a $\mathbb{P}^1$ bundle over a four-dimensional compact manifold B_4, that can be either a Kähler-Einstein manifold ($B_4 = \mathrm{KE}_4$) or the product of two KE_2 ($B_4 = \Sigma_1 \times \Sigma_2$). The bundle is the projectivization of the canonical bundle over B_4, $\mathbb{P}(K \oplus \mathcal{O})$. Here we consider the case where B_4 is replaced by a generic four-dimensional toric orbifold $\mathbb{M}_4$. Notice that generically M_6 can be an orbifold,[14] like in the solutions discussed in [103]. In addition to recovering the gravitational central charges of the existing solutions, we give a prediction for these more general backgrounds that are still to be found. These solutions are potentially interpreted as M5 branes wrapped over a two-cycle in $\mathbb{M}_4$ (see for example [104, 105]).

The topological structure of the underlying geometry can be encoded in the fan

$$V^a = (v^a, 1)\,, \qquad V^{d+1} = (0,0,1)\,, \qquad V^{d+2} = (0,0,-1)\,, \qquad a = 1,\dots,d\,, \tag{5.2.39}$$

where v^a are the two-dimensional vectors in the fan of $\mathbb{M}_4$. We will use a capital index A to run over $a = 1,\dots,d$, $d+1$ and $d+2$. That this is the right geometry can be seen looking at the symplectic reduction presentation $\mathbb{C}^{d+2}//G$ of M_6. Here G is the subgroup of the torus $\mathbb{T}^{d+2} = U(1)^{d+2}$ generated by the GLSM charges

$$\sum_A Q^k_A V^A_I = 0\,, \qquad\qquad k = 1,\dots,d-1\,. \tag{5.2.40}$$

We can choose the following basis of GLSM charges

$$(q^p_a, -\sum_a q^p_a, 0)\,, \qquad (0,\dots,0,1,1)\,, \tag{5.2.41}$$

where q^p_a are the $d-2$ charges for $\mathbb{M}_4$, $\sum_a q^p_a v^a_i = 0$. The first $d-2$ vectors define the canonical bundle K of $\mathbb{M}_4$ with an extra copy of $\mathbb{C}$. The final charge vector projectivizes it and gives indeed the geometry we are interested in:

$$\mathbb{P}(K \oplus \mathcal{O})\,. \tag{5.2.42}$$

We need also to specify the integer fluxes of the M theory four-form along all the non-trivial four-cycles. There are $d+2$ toric four-cycles in the geometry, associated with the divisors D_A. The divisors D_a are $\mathbb{P}^1$ fibrations over the toric two-cycles $\Sigma_a \subset \mathbb{M}_4$, while D_{d+1} and D_{d+2} are copies of $\mathbb{M}_4$ sitting at the North and South pole of $\mathbb{P}^1$, respectively. All together, they define a vector of fluxes M_A. The relations

[14] Using our formalism, we could easily study the case that M_6 is a generic toric six-dimensional orbifold. It would be interesting to understand what kinds of orbifold admit an holographic interpretation.

$\sum_A V_I^A D_A = 0$ imply that not all toric divisors are inequivalent and that the vector of fluxes satisfies

$$\sum_A V_I^A M_A = 0 \,. \tag{5.2.43}$$

Since we are dealing with M5 branes in M theory, we define the free energy as in section 5.2.1

$$F = \mathbb{V}^{(3)}(\lambda_A, \lambda_{AB}, \epsilon_I) \,, \tag{5.2.44}$$

and, since now we have a vector of fluxes, we impose the flux constraints

$$\nu_{M5}\, M_A = -\frac{\partial}{\partial \lambda_A} \mathbb{V}^{(2)}(\lambda_A, \lambda_{AB}, \epsilon_I) \,. \tag{5.2.45}$$

Differently from the case discussed in the previous section, for these geometries there is no general field theory result for the central charge of the dual four-dimensional SCFTs. Our results here can therefore be seen as a prediction for the general form of the off-shell central charge, which presumably can be obtained by integrating the M5 brane anomaly polynomial on a suitable two-cycle inside M_6, or using the method of [104]. In order to compare with the existing literature, we will therefore consider in some detail a number of explicit examples of $\mathbb{M}_4$, including KE_4 and $\Sigma_1 \times \Sigma_2$, but also other examples for which there is no known supergravity solution, nor field-theoretic understanding. The equations to be solved in the extremization problem typically lead to finding the zeroes of simultaneous polynomials of high degree and are therefore not manageable. For this reason, we will proceed by making different technical assumptions to simplify the algebra. One such general assumption is the existence of a $\mathbb{Z}_2$ symmetry acting on the $\mathbb{P}^1$ fibre, as we discuss below. Furthermore, we will occasionally restrict to non-generic fluxes in order to simplify the otherwise unwieldy expressions.

If we restrict to a class of geometries with a $\mathbb{Z}_2$ symmetry that exchanges the North and South poles of $\mathbb{P}^1$, we can consider a simplified geometry obtained by cutting $\mathbb{P}^1$ into half. We thus obtain a non-compact Calabi-Yau geometry given by the canonical bundle over $\mathbb{M}_4$. The corresponding fan is obtained by dropping V^{d+2}:

$$V^a = (v^a, 1) \,, \qquad V^{d+1} = (0,0,1) \ , \qquad a = 1, \dots, d \,. \tag{5.2.46}$$

Notice that this is a (partial) resolution of a CY_3 cone where V^{d+1} is associated with a compact divisor. Supergravity solutions with $\mathbb{Z}_2$ symmetry have been considered in [101, 102] where they correspond to set the parameter called c to zero. Effectively, the $\mathbb{Z}_2$ symmetry reduces by one the number of independent fluxes we can turn on, thus simplifying the calculations. Notice that the on-shell equivariant volume $\mathbb{V}$ for the half-geometry is half of the one for the total geometry. The relation between the parameters to use in the two cases, in order to have the same normalization for the free energy, is the following

$$\overline{\nu}_{M5} = 2^{-2/3} \nu_{M5} \,, \tag{5.2.47}$$

where $\overline{\nu}_{M5}$ is the correct one for half-geometries.

Notice that we introduced single and double times in (5.2.45). We can immediately understand the need for higher times. In a compact geometry, $\mathbb{V}^{(2)}(\lambda_A)$ with only single times would vanish identically.[15] As we will discuss later, the double times are generically necessary also when imposing the $\mathbb{Z}_2$ symmetry in order to have enough parameters to solve the equations.[16]

5.2.2.1 Geometries with $\mathbb{Z}_2$ symmetry

We consider first geometries with $\mathbb{Z}_2$ symmetry. Cutting M_6 into half we consider the non-compact CY_3 specified by the fan (5.2.46). The $I=3$ condition in (5.2.43) gives

$$M_{d+1} = -\sum_a M_a\,, \tag{5.2.48}$$

thus fixing the flux along $\mathbb{M}_4$ in terms of the other fluxes. The $I=1,2$ conditions in (5.2.43) give two linear relations among the M_a, leaving a total number $d-2$ of independent fluxes. Notice that geometries without $\mathbb{Z}_2$ symmetry have one additional independent flux, as we discuss later.

The fan is the union of d cones (V^a, V^{a+1}, V^{d+1}) and we see that the number of fixed points is the same of that of the base $\mathbb{M}_4$. It is then easy to write the equivariant volume with higher times as a sum over the fixed points of $\mathbb{M}_4$

$$\mathbb{V} = \sum_a \frac{\mathrm{e}^{\tau_a}}{d_{a,a+1}\epsilon_1^a\epsilon_2^a(\epsilon_3-\epsilon_1^a-\epsilon_2^a)}\,, \tag{5.2.49}$$

where

$$\tau_a = \left(\sum_A \lambda_A\, c_1^{\mathbb{T}^3}(L_A) + \sum_{A,B}\lambda_{AB}\, c_1^{\mathbb{T}^3}(L_A)\, c_1^{\mathbb{T}^3}(L_B)\right)\Bigg|_a \tag{5.2.50}$$

or, more explicitly,

$$\begin{aligned}\tau_a = &-\lambda_a\epsilon_1^a - \lambda_{a+1}\epsilon_2^a - \lambda_{d+1}(\epsilon_3-\epsilon_1^a-\epsilon_2^a) + \lambda_{aa}(\epsilon_1^a)^2 + 2\lambda_{a,a+1}\epsilon_1^a\epsilon_2^a + \lambda_{a+1,a+1}(\epsilon_2^a)^2\\ &+ 2(\lambda_{a,d+1}\epsilon_1^a + \lambda_{a+1,d+1}\epsilon_2^a)(\epsilon_3-\epsilon_1^a-\epsilon_2^a) + \lambda_{d+1,d+1}(\epsilon_3-\epsilon_1^a-\epsilon_2^a)^2\,.\end{aligned} \tag{5.2.51}$$

Notice that the equations (5.2.45) are not solvable with only single times. $M_{d+1} = -\sum_a M_a \neq 0$ while, for $\lambda_{AB}=0$,

$$\begin{aligned}-\frac{\partial\mathbb{V}^{(2)}}{\partial\lambda_{d+1}} &= \sum_a \frac{-\lambda_a\epsilon_1^a - \lambda_{a+1}\epsilon_2^a - \lambda_{d+1}(\epsilon_3-\epsilon_1^a-\epsilon_2^a)}{d_{a,a+1}\epsilon_1^a\epsilon_2^a}\\ &= \int_{\mathbb{M}_4}\left(\sum_a \lambda_a c_1^{\mathbb{T}}(L_a) - \lambda_{d+1}\left(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)\right)\right) = 0\,,\end{aligned} \tag{5.2.52}$$

[15] For a compact geometry $\mathbb{V}^{(2)}(\lambda_A) = -\frac{1}{2}\sum_{AB}\lambda_A\lambda_B\int_{M_6} c_1^{\mathbb{T}}(L_A)c_1^{\mathbb{T}}(L_B) = 0$ since it is the integral of a four-form at most on a six-dimensional manifold. In the non-compact case, this condition is evaded and $\mathbb{V}^{(2)}(\lambda_A)$ is a rational function of ϵ_I. See [28] for details.

[16] In the case of compactification on a spindle they are not necessary [28].

being the integral of a two-form at most.

The equations (5.2.45) explicitly read

$$
\begin{aligned}
(I) \qquad & \overline{\nu}_{M5} M_a = \frac{\epsilon_1^a \tau_a}{d_{a,a+1}\epsilon_1^a\epsilon_2^a(\epsilon_3 - \epsilon_1^a - \epsilon_2^a)} + \frac{\epsilon_2^{a-1}\tau_{a-1}}{d_{a-1,a}\epsilon_1^{a-1}\epsilon_2^{a-1}(\epsilon_3 - \epsilon_1^{a-1} - \epsilon_2^{a-1})} \,, \\
(II) \qquad & -\overline{\nu}_{M5} \sum_a M_a = \sum_a \frac{\tau_a}{d_{a,a+1}\epsilon_1^a\epsilon_2^a} \,.
\end{aligned} \tag{5.2.53}
$$

These equations are not independent. In particular, (II) follows from (I).[17] The equations (I) can be written as

$$
B^{(1)}_{a-1} - B^{(1)}_a = d_{a,a+1}\epsilon_2^a \overline{\nu}_{M5} M_a \,, \qquad\qquad B^{(1)}_a = -\frac{\tau_a}{\epsilon_3 - \epsilon_1^a - \epsilon_2^a} \,. \tag{5.2.55}
$$

It is then clear that these equations can be solved for τ_a, but one "time", say τ_1, remains undetermined. Our prescription is to *extremize* the free energy with respect to *all* parameters that are left undetermined after imposing the flux constraints. In this case then we extremize

$$
\mathbb{V}^{(3)}(\epsilon_i, \tau_1) \tag{5.2.56}
$$

with respect to ϵ_1, ϵ_2 and τ_1, with ϵ_3 set to some canonical value, fixed by the scaling of the supercharge under the R-symmetry vector field. In the next subsection we will parameterize the free energy in a more convenient way.

5.2.2.2 The extremal function for geometries with $\mathbb{Z}_2$ symmetry

We can write the general form of the extremal function for geometries with $\mathbb{Z}_2$ symmetry. Let us define

$$
\tau^{\mathbb{T}^3}_{CY_3} = \sum_A \lambda_A \, c_1^{\mathbb{T}^3}(L_A) + \sum_{A,B} \lambda_{AB} \, c_1^{\mathbb{T}^3}(L_A) \, c_1^{\mathbb{T}^3}(L_B) \,, \tag{5.2.57}
$$

the equivariant form with restriction τ_a at the fixed points. By restricting the form to $\mathbb{M}_4$ and considering ϵ_3 as a parameter, we obtain

$$
\begin{aligned}
\tau^{\mathbb{T}} &= \sum_A \lambda_A \, \mathcal{C}_A + \sum_{A,B} \lambda_{AB} \, \mathcal{C}_A \, \mathcal{C}_B \,, \\
\mathcal{C}_a &= c_1^{\mathbb{T}}(L_a) \,, \qquad a = 1, \dots, d \,, \\
\mathcal{C}_{d+1} &= -\Big(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)\Big) \,,
\end{aligned} \tag{5.2.58}
$$

[17] Using $\sum_a v^a M_a = 0$ and the vector identity $v_i^a \epsilon_1^a + v_i^{a+1}\epsilon_2^a = \epsilon_i$, one derives $\epsilon_i \sum_a \frac{\tau_a}{d_{a,a+1}\epsilon_1^a\epsilon_2^a(\epsilon_3-\epsilon_1^a-\epsilon_2^a)} = 0$ from (I). Then, summing over a in (I), and using the previous identity:

$$
\overline{\nu}_{M5} \sum_a M_a = \sum_a \frac{(\epsilon_1^a + \epsilon_2^a)\tau_a}{d_{a,a+1}\epsilon_1^a\epsilon_2^a(\epsilon_3 - \epsilon_1^a - \epsilon_2^a)} = -\sum_a \frac{\tau_a}{d_{a,a+1}\epsilon_1^a\epsilon_2^a} \,, \tag{5.2.54}
$$

valid for $\epsilon_i \neq 0$. For $\epsilon_i = 0$ one should pay more attention and we will see in section 5.3.2 one instance where a similar subtlety is important. In the present case we will check explicitly that both (I) and (II) are valid.

where $c_1^{\mathbb{T}}(L_a)$ are the restrictions of the line bundles L_a to the base $\mathbb{M}_4$ and $\mathbb{T}$ is the two-dimensional torus spanned by ϵ_1 and ϵ_2. From now on, unless explicitly said, all classes will refer to the base $\mathbb{M}_4$. In terms of $\tau^{\mathbb{T}}$ the quadratic piece of the equivariant volume can be written as

$$\mathbb{V}^{(2)}(\lambda_A, \lambda_{AB}, \epsilon_I) = \frac{1}{2}\int_{\mathbb{M}_4} \frac{(\tau^{\mathbb{T}})^2}{\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)} \,. \tag{5.2.59}$$

The flux constraints (5.2.45) give

$$\begin{aligned} (I) \qquad & -\overline{\nu}_{M5} M_a = \int_{\mathbb{M}_4} \frac{c_1^{\mathbb{T}}(L_a)\, \tau^{\mathbb{T}}}{\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)} \,, \\ (II) \qquad & -\overline{\nu}_{M5} \sum_a M_a = \int_{\mathbb{M}_4} \tau^{\mathbb{T}} \,. \end{aligned} \tag{5.2.60}$$

For a generic fan, using the gauge transformations (4.4.8) and (4.4.9) we can set all $\lambda_a = \lambda_{a,a} = \lambda_{a,a+1} = 0$.[18] We will show more formally in appendix B.1 that $\mathbb{V}^{(3)}$ has a critical point at $\lambda_a = \lambda_{a,a} = \lambda_{a,a+1} = 0$. Then condition (I) becomes

$$\begin{aligned} -\overline{\nu}_{M5} M_a &= \int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_a)\Big(-\lambda_{d+1} - 2\sum_b \lambda_{b,d+1} c_1^{\mathbb{T}}(L_b) + \lambda_{d+1,d+1}\Big(\epsilon_3 + \sum_b c_1^{\mathbb{T}}(L_b)\Big)\Big) \\ &= \sum_b D_{ab}(-2\lambda_{b,d+1} + \lambda_{d+1,d+1}) \,. \end{aligned} \tag{5.2.61}$$

We can similarly compute (II) as an integral

$$-\overline{\nu}_{M5} \sum_a M_a = \int_{\mathbb{M}_4} \tau^{\mathbb{T}} = \sum_{a,b} D_{ab}(-2\lambda_{b,d+1} + \lambda_{d+1,d+1}) \,, \tag{5.2.62}$$

and see that it is automatically satisfied if (I) is. Since $\sum_a v_i^a M_a = 0$, the flux constraints fix the $\lambda_{b,d+1}$ only up to the ambiguities

$$\begin{aligned} \lambda_{a,d+1} &\to \lambda_{a,d+1} + \sum_{i=1}^{2} \delta_i v_i^a + \gamma \,, \\ \lambda_{d+1,d+1} &\to \lambda_{d+1,d+1} + 2\gamma \,, \end{aligned} \tag{5.2.63}$$

where δ_i and γ are free parameters. However, these free parameters can be all reabsorbed in a redefinition of

$$\lambda_{d+1} \to \lambda_{d+1} + 2\gamma\epsilon_3 + 2\sum_{i=1}^{2} \delta_i \epsilon_i \,, \tag{5.2.64}$$

[18] For special symmetric fans, like $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, and other simple examples with low d, one of the single times λ_a remains unfixed. However, from the combined conditions $(I) + (II)$ we obtain $\sum_{ab} \lambda_a \int_{M_4} \frac{c_1^{\mathbb{T}}(L_a)\, c_1^{\mathbb{T}}(L_b)}{\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)} = 0$ which implies that the remaining single time must vanish.

since $\sum_a v_i^a c_1^{\mathbb{T}}(L_a) = -\epsilon_i$ and they are not really independent.

The free energy is then given by

$$\mathbb{V}^{(3)} = \frac{1}{6} \int_{\mathbb{M}_4} \frac{(\tau^{\mathbb{T}})^3}{\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)} \,, \tag{5.2.65}$$

which explicitly gives

$$\mathbb{V}^{(3)} = \frac{1}{6} \int_{\mathbb{M}_4} \Big(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)\Big)^2 \Big(\bar\lambda_{d+1} + \sum_b \bar\lambda_{b,d+1} c_1^{\mathbb{T}}(L_b)\Big)^3 \,, \tag{5.2.66}$$

where we defined

$$\bar\lambda_{d+1} = -\lambda_{d+1} + \lambda_{d+1,d+1}\epsilon_3 \,, \qquad \bar\lambda_{a,d+1} = -2\lambda_{a,d+1} + \lambda_{d+1,d+1} \,, \tag{5.2.67}$$

which are subject to the constraints

$$-\,\overline{\nu}_{M5} M_a = \sum_b D_{ab} \bar\lambda_{b,d+1} \,. \tag{5.2.68}$$

Substituting the solution of the flux constraints, $\mathbb{V}^{(3)}$ becomes a function of ϵ_i and the extra parameter $\bar\lambda_{d+1}$. Indeed, as we have seen, the ambiguities (5.2.63) can be reabsorbed in a redefinition of $\bar\lambda_{d+1}$. A direct evaluation gives

$$\begin{aligned}
6\mathbb{V}^{(3)} = {} & \bar\lambda_{d+1}^3 \sum_{ab} D_{ab} + \bar\lambda_{d+1}^2 \Big(6\epsilon_3 \sum_{ab} D_{ab} \bar\lambda_{a,d+1} + 3 \sum_{abc} D_{abc} \bar\lambda_{a,d+1}\Big) \\
& + 3\bar\lambda_{d+1} \Big(\epsilon_3^2 \sum_{ab} D_{ab} \bar\lambda_{a,d+1}\bar\lambda_{b,d+1} + 2\epsilon_3 \sum_{abc} D_{abc} \bar\lambda_{a,d+1}\bar\lambda_{b,d+1} + \sum_{abcd} D_{abcd} \bar\lambda_{a,d+1}\bar\lambda_{b,d+1}\Big) \\
& + \Big(\epsilon_3^2 \sum_{abc} D_{abc} \bar\lambda_{a,d+1}\bar\lambda_{b,d+1}\bar\lambda_{c,d+1} + 2\epsilon_3 \sum_{abcd} D_{abcd} \bar\lambda_{a,d+1}\bar\lambda_{b,d+1}\bar\lambda_{c,d+1} \\
& + \sum_{abcde} D_{abcde} \bar\lambda_{a,d+1}\bar\lambda_{b,d+1}\bar\lambda_{c,d+1}\Big) \,,
\end{aligned} \tag{5.2.69}$$

where the generalized intersection numbers are defined by

$$D_{a_1 \dots a_p} = \int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_{a_1}) \dots c_1^{\mathbb{T}}(L_{a_p}) \,. \tag{5.2.70}$$

Notice that D_{ab} is ϵ-independent, while $D_{a_1 \cdots a_p}$ is a homogeneous function of degree $p-2$ in ϵ_1 and ϵ_2. $\mathbb{V}^{(3)}$ need to be extremized with respect to ϵ_1, ϵ_2 and $\bar\lambda_{d+1}$, with ϵ_3 set to the canonical value.

The critical point is generically at a non-zero value of ϵ_1 and ϵ_2. We can expect a critical point[19] at $\epsilon_1 = \epsilon_2 = 0$ only if the background and the fluxes have some extra

[19]In the opposite direction, of course one would have as critical point $\epsilon_1 = \epsilon_2 = 0$ if the base B_4 has *no* continuous symmetries. This is the case for examples for del Pezzo surfaces dP_k with $k > 3$, which we do not treat here. This would lead one to suspect that all KE_4 have $\epsilon_1 = \epsilon_2 = 0$ as critical point, but this is actually incorrect, as the example of the toric dP_3 will show.

symmetry, as for examples in the cases where all $U(1)$ isometries are enhanced to a non-abelian group. In these particular cases, we can further simplify the expression

$$\begin{aligned} 6\mathbb{V}^{(3)} &= \bar{\lambda}_{d+1}^3 \sum_{ab} D_{ab} + 6\epsilon_3 \bar{\lambda}_{d+1}^2 \sum_{ab} D_{ab}\bar{\lambda}_{a,d+1} + 3\epsilon_3^2 \bar{\lambda}_{d+1} \sum_{ab} D_{ab}\bar{\lambda}_{a,d+1}\bar{\lambda}_{b,d+1} + O(\epsilon_i^2) \\ &= \bar{\lambda}_{d+1}^3 \sum_{ab} D_{ab} - 6\overline{\nu}_{M5} \sum_{a} M_a \epsilon_3 \bar{\lambda}_{d+1}^2 + 3\epsilon_3^2 \bar{\lambda}_{d+1} \sum_{ab} D_{ab}\bar{\lambda}_{a,d+1}\bar{\lambda}_{b,d+1} + O(\epsilon_i^2)\,, \end{aligned} \tag{5.2.71}$$

and extremize it with respect to $\bar{\lambda}_{d+1}$.

As a check of our expression, we can reproduce the central charge of the existing solutions with Kähler-Einstein metrics and fluxes all equal [102]. The only toric four-manifolds that are also KE are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_3, with fans

$$\begin{aligned} &\mathbb{P}^2 : \quad v^1 = (1,1)\,,\ v^2 = (-1,0)\,,\ v^3 = (0,-1)\,, \\ &\mathbb{P}^1 \times \mathbb{P}^1 : \ v^1 = (1,0)\,,\ v^2 = (0,1)\,,\ v^3 = (-1,0)\,,\ v^4 = (0,-1)\,, \\ &\mathrm{dP}_3 : \ v^1 = (1,0)\,,\ v^2 = (1,1)\,,\ v^3 = (0,1)\,,\ v^4 = (-1,0)\,,\ v^5 = (-1,-1)\,,\ v^6 = (0,-1)\,, \end{aligned} \tag{5.2.72}$$

and intersection matrices

$$\begin{aligned} &\mathbb{P}^2 : && D_{ab} = 1\,, \\ &\mathbb{P}^1 \times \mathbb{P}^1 : && D_{ab} = 1 \text{ if } |a-b| = 1 \,(\mathrm{mod}\, 2) \quad \text{and} \quad \text{zero otherwise}\,, \\ &\mathrm{dP}_3 : && D_{aa} = -1\,, \quad D_{a,a\pm1} = 1 \quad \text{and} \quad \text{zero otherwise}\,, \end{aligned} \tag{5.2.73}$$

where the indices are cyclically identified. To compare with the KE_4 solutions, we set all $M_a \equiv N$. We can then choose all $\bar{\lambda}_{a,d+1}$ equal and we find

$$\sum_{abc} D_{abc}\bar{\lambda}_{a,d+1} = \sum_{abc} D_{abc}\bar{\lambda}_{a,d+1}\bar{\lambda}_{b,d+1} = \sum_{abc} D_{abc}\bar{\lambda}_{a,d+1}\bar{\lambda}_{b,d+1}\bar{\lambda}_{c,d+1} = 0\,, \tag{5.2.74}$$

thus ensuring that the linear terms in ϵ_1 and ϵ_2 in $\mathbb{V}^{(3)}$ vanish, and that there is indeed a critical point at $\epsilon_1 = \epsilon_2 = 0$. Extremizing (5.2.71) we get

$$\mathbb{V}^{(3)} = \epsilon_3^3 \overline{\nu}_{M5}^3 (-5 + 3\sqrt{3}) N^3 \{\frac{1}{9}, \frac{1}{3}, 2\}\,, \tag{5.2.75}$$

for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_3, respectively, which agrees with (2.16) in [102] for $\epsilon_3\overline{\nu}_{M5} = 3$.[20]

For $\mathbb{P}^1 \times \mathbb{P}^1$ we can introduce a second flux. The general solution to $\sum_A V_I^A M_A = 0$ is indeed

$$M_A = (N_1, N_2, N_1, N_2, -2N_1 - 2N_2)\,. \tag{5.2.76}$$

The background has an expected $SU(2) \times SU(2)$ symmetry that it is realized in the supergravity solution [102], that now is not in the KE class. Using the gauge

[20] N_{C_N} in [102] can be identified with $M_{d+1} = -\sum M_a$, so that $N_{there} = -hN_{C_N}/M = h\sum M_a/M$ where (h, M) are defined in [102] and they have value $(3, 9), (4, 8)$ and $(2, 6)$ for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_3, respectively.

transformation (4.4.8) we can set $\bar{\lambda}_{a+2,d+1} = \bar{\lambda}_{a,d+1}$. A simple computation then shows that the free energy extremized has a critical point in $\epsilon_1 = \epsilon_2 = 0$, consistently with the non-abelian isometry of the solution, with critical value

$$\mathbb{V}^{(3)} = \frac{\epsilon_3^3 \bar{\nu}_{M5}^3}{6} \left(2(N_1^2 + N_1 N_2 + N_2^2)^{3/2} - (2N_1^3 + 3N_1^2 N_2 + 3N_1 N_2^2 + 2N_2^3)\right) , \tag{5.2.77}$$

which should be compared with (2.29) in [102] with $N_1 = pN$ and $N_2 = qN$. This looks superficially different, but it can be rewritten in the form above (*cf.* for example (F.14) in [104]).

5.2.2.3 Examples of geometries with non-zero critical ϵ

So far, in all the explicit examples we have discussed we found that $\epsilon_1 = \epsilon_2 = 0$ is a critical point. However, we have already pointed out that for generic toric $\mathbb{M}_4$ and/or with generic fluxes this will not be the case. In this subsection we will investigate situations in which at least one of ϵ_1, ϵ_2 is different from zero at the critical point, by considering geometries with $SU(2) \times U(1)$ symmetry, as well as the case of dP_3 with generic fluxes. Interestingly, it turns out that for dP_3 there exist two special configurations of fluxes (different from the case where they are all equal) where the critical point is again $\epsilon_1 = \epsilon_2 = 0$, but the corresponding supergravity solutions are not known. For four independent generic fluxes, instead, $\epsilon_1 = \epsilon_2 = 0$ is not a critical point.

dP_3 with unequal fluxes

The symmetry of dP_3 is just $U(1) \times U(1)$ and the existence of the critical point $\epsilon_1 = \epsilon_2 = 0$ of the extremization problem is not obviously implied by the fact that there exists a KE metric on dP_3. In the basis of the fan as in (5.2.72), the general assignment of fluxes compatible with $\sum_A V_I^A M_A = 0$ can be parameterized as

$$M_A = (N_1, N_2, N_3, N_4, N_5, N_6, -2N_1 - 3N_2 - 2N_3 + N_5) , \tag{5.2.78}$$

where we choose N_1, N_2, N_3, N_5 as independent, with $N_4 = N_1 + N_2 - N_5$ and $N_6 = N_2 + N_3 - N_5$. Upon setting $\lambda_a = \lambda_{a,a} = \lambda_{a,a+1} = 0$ using the gauge freedom, as discussed before, the constraint (5.2.68) on $\bar{\lambda}_{a,d+1}$ can be solved by taking, for example

$$\begin{aligned}
\bar{\lambda}_{1,d+1} &= -\bar{\nu}_{M5} \frac{N_2 + N_3}{2} , & \bar{\lambda}_{2,d+1} &= -\bar{\nu}_{M5} \frac{N_3 + N_1}{2} , \\
\bar{\lambda}_{3,d+1} &= -\bar{\nu}_{M5} \frac{N_1 + N_2}{2} , & \bar{\lambda}_{4,d+1} &= -\bar{\nu}_{M5} \frac{N_2 + N_3}{2} , \\
\bar{\lambda}_{5,d+1} &= -\bar{\nu}_{M5} \frac{N_3 + N_1 + 2(N_2 - N_5)}{2} , & \bar{\lambda}_{6,d+1} &= -\bar{\nu}_{M5} \frac{N_1 + N_2}{2} .
\end{aligned} \tag{5.2.79}$$

Writing out the free energy (5.2.69), up to linear order in ϵ_1, ϵ_2, we have

$$\mathbb{V}^{(3)} = \mathbb{V}^{(3)}\big|_{\epsilon_i=0} + \partial_{\epsilon_1} \mathbb{V}^{(3)}\big|_{\epsilon_i=0} \epsilon_1 + \partial_{\epsilon_2} \mathbb{V}^{(3)}\big|_{\epsilon_i=0} \epsilon_2 + O(\epsilon_i^2) \tag{5.2.80}$$

where the constant term is not particularly interesting and

$$\begin{aligned}
\partial_{\epsilon_1}\mathbb{V}^{(3)}\big|_{\epsilon_i=0} &= \bar{\nu}_{M5}\frac{N_5-N_2}{2}\Big[6\bar{\lambda}_{d+1}^2 - 12(N_2+N_3)\bar{\nu}_{M5}\epsilon_3\bar{\lambda}_{d+1} \\
&\quad + \big(N_2^2-2N_5^2+3N_3(N_2+N_5)+N_5N_2\big)\bar{\nu}_{M5}^2\epsilon_3^2\Big]\,, \\
\partial_{\epsilon_2}\mathbb{V}^{(3)}\big|_{\epsilon_i=0} &= \bar{\nu}_{M5}\frac{N_5-N_2}{2}\Big[6\bar{\lambda}_{d+1}^2 - 12(N_2+N_1)\bar{\nu}_{M5}\epsilon_3\bar{\lambda}_{d+1} \\
&\quad + \big(N_2^2-2N_5^2+3N_1(N_2+N_5)+N_5N_2\big)\bar{\nu}_{M5}^2\epsilon_3^2\Big]\,.
\end{aligned} \tag{5.2.81}$$

We see that for generic values of the fluxes the expressions above cannot be zero simultaneously, implying that $\epsilon_i = 0$ is not a critical point of the extremization. The complete extremization equations are unwieldy, so in the following we will instead concentrate on two special configurations of fluxes, with enhanced symmetry, for which $\epsilon_1 = \epsilon_2 = 0$ turns out to be a critical point.

The first special value of fluxes is clearly obtained for $N_5 = N_2$, that leaves three fluxes N_1, N_2, N_3 free. In this case the parameters (5.2.79) acquire the cyclic symmetry $\bar{\lambda}_{a,d+1} = \bar{\lambda}_{a+3,d+1}$, analogously to the $\mathbb{P}^1\times\mathbb{P}^1$ discussed in the previous section and indeed the linear terms in $\mathbb{V}^{(3)}$ manifestly vanish, so that $\epsilon_i = 0$ is a critical point. The fluxes display an enhanced symmetry:

$$M_A = (N_1, N_2, N_3, N_1, N_2, N_3, -2N_1-2N_2-2N_3)\,. \tag{5.2.82}$$

Extremizing $\mathbb{V}^{(3)}$ with respect to $\bar{\lambda}_{d+1}$ yields

$$\begin{aligned}
\bar{\lambda}_{d+1}^* &= \frac{2\bar{\nu}_{M5}\epsilon_3}{3}(N_1+N_2+N_3) \\
&\quad - \frac{\bar{\nu}_{M5}\epsilon_3}{3}\sqrt{4(N_1^2+N_2^2+N_3^2)+5(N_1N_2+N_2N_3+N_3N_1)}\,,
\end{aligned} \tag{5.2.83}$$

and the corresponding value of the on-shell central charge is

$$\begin{aligned}
\mathbb{V}^{(3)} &= \frac{2\bar{\nu}_{M5}^3\epsilon_3^3}{27}\Big[\big(4(N_1^2+N_2^2+N_3^2)+5(N_1N_2+N_2N_3+N_3N_1)\big)^{3/2} \\
&\quad - (N_1+N_2+N_3)\big(8(N_1^2+N_2^2+N_3^2)+7(N_1N_2+N_2N_3+N_3N_1)\big)\Big]\,.
\end{aligned} \tag{5.2.84}$$

It can be checked that this expression agrees precisely with the central charge given in eq. (3.79) of [106] and it correctly reduces to (5.2.75) upon setting $N_1 = N_2 = N_3 = N$.

Notice that while the expression of $\bar{\lambda}_{d+1}^*$ depends on the specific gauge chosen for the parameters $\bar{\lambda}_{a,d+1}$, the critical values $\epsilon_i^* = 0$ and the central charge (5.2.84) do not rely on this.

The second special value of fluxes that we found is $N_1 = N_3 = N_5$, which implies $N_a = N_{a+2}$, so that the fluxes have again an enhanced symmetry:

$$M_A = (N_1, N_2, N_1, N_2, N_1, N_2, -3N_1-3N_2)\,. \tag{5.2.85}$$

In this case, notice that the two expressions in (5.2.81) coincide, so that it is possible that both linear terms vanish, for a particular value of $\bar{\lambda}^*_{d+1}$, despite $N_2 \neq N_5$. However, the parameters in (5.2.79) do not enjoy this new symmetry, so it is better to look for a different gauge, where the parameters respect the additional symmetry, namely $\bar{\lambda}_{a,d+1} = \bar{\lambda}_{a+2,d+1}$. This can be achieved choosing

$$\bar{\lambda}_{1,d+1} = -\bar{\nu}_{M5}\frac{N_1+2N_2}{3}\,, \qquad \bar{\lambda}_{2,d+1} = -\bar{\nu}_{M5}\frac{2N_1+N_2}{3}\,, \tag{5.2.86}$$

and cyclic permutations. In this gauge, we can now check that $\mathbb{V}^{(3)}$ has no linear terms in ϵ_1 and ϵ_2. Therefore, extremizing $\mathbb{V}^{(3)}$ with respect to $\bar{\lambda}_{d+1}$, ϵ_1 and ϵ_2, we obtain the critical values $\epsilon^*_{1,2} = 0$ and

$$\bar{\lambda}^*_{d+1} = \frac{6(N_1+N_2) - \sqrt{6(5N_1^2+8N_1N_2+5N_2^2)}}{6}\,\bar{\nu}_{M5}\epsilon_3\,, \tag{5.2.87}$$

and the corresponding value of the on-shell central charge is

$$\mathbb{V}^{(3)} = \frac{\bar{\nu}_{M5}^3\epsilon_3^3}{4}\Big[\frac{\big(6(5N_1^2+8N_1N_2+5N_2^2)\big)^{3/2}}{27} - 2(N_1+N_2)\big(3N_1^2+4N_1N_2+3N_2^2\big)\Big]\,. \tag{5.2.88}$$

It can be checked that this expression agrees precisely with the central charge given in eq. (3.79) of [106] and it correctly reduces to (5.2.75) upon setting $N_1 = N_2 = N$.

It would be interesting to construct explicit supergravity solutions corresponding to the two special configurations of fluxes we found. If they exist, they should lie outside the KE class considered in [101].

$\mathbb{M}_4 = S^2 \ltimes \Sigma$

We now consider the toric orbifold $\mathbb{M}_4 = S^2 \ltimes \Sigma$, namely a spindle $\Sigma = \mathbb{WP}^1_{[n_+,n_-]}$ fibred over a two-sphere, which is a case with only an $SU(1)\times U(1)$ symmetry. We take the following fan

$$v^1 = (n_-,0)\,, \qquad v^2 = (-k,1)\,, \qquad v^3 = (-n_+,0)\,, \qquad v^4 = (0,-1)\,, \tag{5.2.89}$$

and refer to [85] for more details about this orbifold. The total fan is as in (5.2.46) and the constraint (5.2.43) is solved by

$$M_a = \left(\frac{N_1}{n_-}, N_2, \frac{N_1 - k\,N_2}{n_+}, N_2\right)\,, \qquad M_{d+1} = -\sum_a M_a\,, \tag{5.2.90}$$

where N_1, N_2 parameterize the two independent fluxes, and notice that $N_2 = N_4$ is implied by the $SU(2)$ symmetry acting on the base S^2. The constraint (5.2.68) on $\bar{\lambda}_{a,d+1}$ is solved by taking

$$\bar{\lambda}_{3,d+1} = -n_+\left(\bar{\nu}_{M5}N_2 + \frac{\bar{\lambda}_{1,d+1}}{n_-}\right)\,, \quad \bar{\lambda}_{4,d+1} = -\left(\bar{\nu}_{M5}N_1 + \frac{k\,\bar{\lambda}_{1,d+1}}{n_-} + \bar{\lambda}_{2,d+1}\right)\,, \tag{5.2.91}$$

and we can choose a gauge in which

$$\bar{\lambda}_{2,d+1} = -\frac{1}{2}\left(\bar{\nu}_{M5} N_1 + \frac{k\,\bar{\lambda}_{1,d+1}}{n_-}\right). \tag{5.2.92}$$

After using the remaining gauge freedom to fix $\bar{\lambda}_{1,d+1}$, we are then left to extremize $\mathbb{V}^{(3)}$ with respect to $\epsilon_1, \epsilon_2, \bar{\lambda}_{d+1}$. One can show that the combination

$$k\frac{\partial \mathbb{V}^{(3)}}{\partial \epsilon_1} - 2\frac{\partial \mathbb{V}^{(3)}}{\partial \epsilon_2} = 0 \tag{5.2.93}$$

implies $\epsilon_2^* = 0$, as expected from the $SU(2)$ symmetry, while generically $\epsilon_1^* \neq 0$. In particular, ϵ_1^* is determined solving a quartic equation, which takes about half a page to be written, so we will refrain from reporting this. The on-shell central charge can then written in terms of the parameters N_1, N_2, k, n_+, n_- and ϵ_1^*. For simplicity we shall present the results in three special cases, where the equations are qualitatively unchanged, but simpler to write.

Firstly, let us set $k = 0$. This leads to the direct product $\mathbb{M}_4 = S^2 \times \Sigma$ and in this case, defining

$$\chi \equiv \frac{n_+ + n_-}{n_+ n_-}\,, \qquad \mu \equiv \frac{n_+ - n_-}{n_+ + n_-}\,, \tag{5.2.94}$$

it is convenient to use the remaining gauge freedom to set

$$\bar{\lambda}_{1,d+1} = -\bar{\nu}_{M5}\frac{2(1-\mu)N_2 - \mu\chi N_1}{2(1+\mu)\chi}\,. \tag{5.2.95}$$

Upon extremizing we find that indeed $\epsilon_2^* = 0$ and

$$\bar{\lambda}_{d+1}^* = \frac{\bar{\nu}_{M5}\epsilon_3}{4\chi}\left[2(\chi N_1 + 2N_2) \pm \mathtt{s}_1^{1/2}\right], \tag{5.2.96}$$

where we defined the quantity

$$\mathtt{s}_1 = N_1^2\chi^2(\mu\chi\hat{\epsilon}_1^* - 2)^2 - 2N_2(\chi N_1 + 2N_2)\big((1-\mu)\chi\hat{\epsilon}_1^* + 2\big)\big((1+\mu)\chi\hat{\epsilon}_1^* - 2\big)\,. \tag{5.2.97}$$

Here, $\hat{\epsilon}_1^*$ is solution to the quartic equation

$$\begin{aligned}3\big[N_1^2\mu\chi^2(2-\mu\chi\hat{\epsilon}_1) + 2N_2(\chi N_1 + 2N_2)(2\mu + (1-\mu^2)\chi\hat{\epsilon}_1)\big]\mathtt{s}_1^{1/2} - 3N_1^3\mu\chi^3(2-\mu\chi\hat{\epsilon}_1)^2 \\ - N_2(3\chi^2N_1^2 + 6\chi N_1N_2 + 8N_2^2)\big(12\mu + 4(1-3\mu^2)\chi\hat{\epsilon}_1 - 3\mu(1-\mu^2)\chi^2\hat{\epsilon}_1^2\big) = 0\,,\end{aligned} \tag{5.2.98}$$

the critical value of ϵ_1 is given by $\epsilon_1^* = \hat{\epsilon}_1^*\epsilon_3$ and the on-shell central charge reads

$$\begin{aligned}\mathbb{V}^{(3)} = \frac{\bar{\nu}_{M5}^3\epsilon_3^3}{48\chi^2}\Big\{\mathtt{s}_1^{3/2} - (2-\mu\chi\hat{\epsilon}_1^*)\big[N_1^3\chi^3(2-\mu\chi\hat{\epsilon}_1^*)^2 \\ + N_2(3\chi^2N_1^2 + 6\chi N_1N_2 + 8N_2^2)\big(2 + (1-\mu)\chi\hat{\epsilon}_1^*\big)\big(2 - (1+\mu)\chi\hat{\epsilon}_1^*\big)\big]\Big\}\,.\end{aligned} \tag{5.2.99}$$

Notice that setting $n_+ = n_- = 1$ in the above expressions we get $\epsilon_1^* = 0$ and reproduce the expression (5.2.77) for the central charge of the $\mathbb{P}^1 \times \mathbb{P}^1$ case.

Following the reasoning in [104], the total space M_6 may be also viewed as an $\mathbb{F}_2$ fibred over the spindle $\mathbb{\Sigma}$ and we therefore interpret the corresponding putative $\mathrm{AdS}_5 \times M_6$ solution as arising from a stack of M5 branes at $\mathbb{C}^2/\mathbb{Z}_2$ singularity, further wrapped on the spindle $\mathbb{\Sigma}$. It would be very interesting to reproduce the above central charge from an anomaly computation, or to construct the explicit $\mathrm{AdS}_5 \times M_6$ supergravity solution.

A second sub-case is obtained setting $n_+ = n_- = 1$, with $k > 0$, and corresponds to the Hirzebruch surfaces $\mathbb{F}_k$. Using the remaining gauge freedom now we can set

$$\bar{\lambda}_{1,d+1} = -\bar{\nu}_{M5}\frac{(2-k)N_2}{4} \tag{5.2.100}$$

and we find that the remaining two extremization equations are solved by

$$\bar{\lambda}^*_{d+1} = \frac{\bar{\nu}_{M5}\epsilon_3}{4}\left[(2N_1 + (2-k)N_2) \pm \mathtt{s}_2^{1/2}\right], \tag{5.2.101}$$

where

$$\mathtt{s}_2 = 4N_1^2 - 4N_1N_2(\hat{\epsilon}_1^*+1)(\hat{\epsilon}_1^*-1+k) - N_2^2(\hat{\epsilon}_1^*+1)\big((4-2k-k^2)\hat{\epsilon}_1^* - 4 + 2k - k^2\big) \tag{5.2.102}$$

and $\hat{\epsilon}_1^*$ is the solution to the quartic equation

$$\begin{aligned}
&12N_1^2(2\hat{\epsilon}_1 + k) - 6N_1N_2\big(3k\hat{\epsilon}_1^2 - 2(2-2k-k^2)\hat{\epsilon}_1 - k(1-2k)\big)\\
&\quad - N_2^2\big(3k(2-3k-k^2)\hat{\epsilon}_1^2 - 2(8-6k+3k^2+3k^3)\hat{\epsilon}_1 - k(2-3k+3k^2)\big)\\
&\quad - 3\big[2N_1(2\hat{\epsilon}_1 + k) + N_2\big((4-2k-k^2)\hat{\epsilon}_1 - k^2\big)\big]\mathtt{s}_2^{1/2} = 0\,.
\end{aligned} \tag{5.2.103}$$

The critical value of ϵ_1 is again given by $\epsilon_1^* = \hat{\epsilon}_1^*\epsilon_3$ and the on-shell central charge reads

$$\begin{aligned}
\mathbb{V}^{(3)} = \frac{\bar{\nu}_{M5}^3\epsilon_3^3}{24}\Big\{&\mathtt{s}_2^{3/2} - 8N_1^3 + 12N_1^2N_2(\hat{\epsilon}_1^*+1)(\hat{\epsilon}_1^* - 1 + k)\\
&- 6N_1N_2^2(\hat{\epsilon}_1^*+1)\big(k\hat{\epsilon}_1^{*2} - (2-k-k^2)\hat{\epsilon}_1^* + 2 - 2k - k^2\big)\\
&- N_2^3(\hat{\epsilon}_1^*+1)\big(k(2-3k-k^2)\hat{\epsilon}_1^{*2} - 2(4-2k+k^3)\hat{\epsilon}_1^* + (8-6k+3k^2-k^3)\big)\Big\}\,.
\end{aligned} \tag{5.2.104}$$

Again, setting $k = 0$ in the above expressions we get $\epsilon_1^* = 0$ and reproduce the expression (5.2.77) for the central charge of the $\mathbb{P}^1 \times \mathbb{P}^1$ case.

This result is manifestly not in agreement with the central charge given in eq. (3.79) of [106], where by construction $\epsilon_1^* = \epsilon_2^* = 0$. In fact, we can reproduce this result if we impose by hand that $\epsilon_1^* = \epsilon_2^* = 0$ so that

$$\mathbb{V}^{(3)}(\bar{\lambda}_{d+1}) = \frac{\bar{\lambda}_{d+1}}{6}\left[8\bar{\lambda}_{d+1}^2 - 6(2N_1 + (2-k)N_2)\bar{\nu}_{M5}\epsilon_3\bar{\lambda}_{d+1} + 3N_2(2N_1 - k\,N_2)\bar{\nu}_{M5}^2\epsilon_3^2\right] \tag{5.2.105}$$

and then extremizing this with respect to the remaining parameter $\bar{\lambda}_{d+1}$ yields

$$\bar{\lambda}^*_{d+1} = \frac{2N_1 + (2-k)N_2 - \sqrt{4N_1^2 + 4(1-k)N_1N_2 + (4-2k+k^2)N_2^2}}{4}\,\bar{\nu}_{M5}\epsilon_3\,, \tag{5.2.106}$$

giving the on-shell central charge

$$\begin{aligned}\mathbb{V}^{(3)}(\bar{\lambda}^*_{d+1}) = \frac{\bar{\nu}^3_{M5}\epsilon_3^3}{24}\Big[&\big(4N_1^2 + 4(1-k)N_1N_2 + (4-2k+k^2)N_2^2\big)^{3/2} \\ &- (2N_1 + (2-k)N_2)\big(4N_1^2 + 2(1-2k)N_1N_2 + (4-k+k^2)N_2^2\big)\Big]\,,\end{aligned} \tag{5.2.107}$$

coinciding with the expression given in eq. (3.79) of [106]. This, however, does not correspond to a true extremum of $\mathbb{V}^{(3)}$and therefore it is unlikely that there exist corresponding supergravity solutions, nor dual SCFTs.

Finally, let us also present the results for the particular configuration of fluxes $N_1 = \frac{k}{2}N$, $N_2 = N$ implying that[21]

$$M_a = \left(\frac{k\,N}{2n_-}, N, -\frac{k\,N}{2n_+}, N\right)\,, \tag{5.2.108}$$

without any assumption on k and n_+, n_-. In order to simplify the expressions, we make use of the remaining gauge freedom to set

$$\bar{\lambda}_{1,d+1} = -\bar{\nu}_{M5}\frac{((1-\mu)(4+k\mu\chi) - k\chi)N}{(1+\mu)\chi(4+k\mu\chi)}\,. \tag{5.2.109}$$

The extremization problem is then solved by

$$\bar{\lambda}^*_{d+1} = \frac{\bar{\nu}_{M5}\epsilon_3 N}{2\chi\xi}\big(2\xi \pm \mathtt{s}_3^{1/2}\big)\,, \tag{5.2.110}$$

where we defined $\xi = 4 + k\mu\chi$ and

$$\mathtt{s}_3 = 4\xi^2 - 4\mu\chi\xi^2\hat{\epsilon}_1^* - \chi^2(8(2+k\mu\chi)(1-\mu^2) - (1-\mu^2+\mu^4)k^2\chi^2)\hat{\epsilon}_1^{*2}\,. \tag{5.2.111}$$

Here, $\hat{\epsilon}_1^*$ is solution to the quartic equation

$$\begin{aligned}2\xi^2(k\chi + 6\mu\xi) + 4\chi\xi^2(4-3\mu^2\xi)\hat{\epsilon}_1 - 3\chi^2\Big(\frac{\xi^2(1-\mu^2)(k\chi+2\mu\xi)}{2} - k^3\chi^3\Big)\hat{\epsilon}_1^2 \\ - 3\big(2\mu\xi^2 + \chi(8(2+k\mu\chi)(1-\mu^2) - (1-\mu^2+\mu^4)k^2\chi^2)\hat{\epsilon}_1\big)\mathtt{s}_3^{1/2} = 0\,,\end{aligned} \tag{5.2.112}$$

and the critical value of ϵ_1 is $\epsilon_1^* = \hat{\epsilon}_1^*\epsilon_3$. The central charge in terms of $\hat{\epsilon}_1^*$ is given by

$$\begin{aligned}\mathbb{V}^{(3)} = \frac{\bar{\nu}^3_{M5}\epsilon_3^3 N^3}{24\chi^2\xi^2}\Big\{&\mathtt{s}_3^{3/2} - \chi^3\Big(\frac{\xi^2(1-\mu^2)(k\chi+2\mu\xi)}{2} - k^3\chi^3\Big)\hat{\epsilon}_1^{*3} \\ &+ 2\chi^2\xi^2(4-3\mu^2\xi)\hat{\epsilon}_1^{*2} + 2\chi\xi^2(k\chi+6\mu\xi)\hat{\epsilon}_1^* - 8\xi^3\Big\}\,.\end{aligned} \tag{5.2.113}$$

[21] We take k to be even in this case.

5.2.2.4 General geometries

We now discuss the general $\mathrm{AdS}_5 \times M_6$ solution with no $\mathbb{Z}_2$ symmetry. The fan (5.2.39) corresponds now to a compact geometry. The fan is the union of $2d$ cones (V^a, V^{a+1}, V^{d+1}) and (V^a, V^{a+1}, V^{d+2}) corresponding to the fixed points of the torus action, that are specified by selecting a fixed point on $\mathbb{M}_4$ and simultaneously the North or South pole of the fibre $\mathbb{P}^1$.

The equivariant volume is now given by

$$\mathbb{V} = \sum_a \frac{\mathrm{e}^{-\lambda_a \epsilon_1^a - \lambda_{a+1}\epsilon_2^a - \lambda_{d+1}(\epsilon_3 - \epsilon_1^a - \epsilon_2^a) + \dots}}{d_{a,a+1}\epsilon_1^a \epsilon_2^a (\epsilon_3 - \epsilon_1^a - \epsilon_2^a)} - \sum_a \frac{\mathrm{e}^{-\lambda_a \epsilon_1^a - \lambda_{a+1}\epsilon_2^a + \lambda_{d+2}(\epsilon_3 - \epsilon_1^a - \epsilon_2^a) + \dots}}{d_{a,a+1}\epsilon_1^a \epsilon_2^a (\epsilon_3 - \epsilon_1^a - \epsilon_2^a)} \,, \tag{5.2.114}$$

where the dots at the exponents contain the higher times.

This expression can also be written as an integral over $\mathbb{M}_4$

$$\mathbb{V}(\lambda_A, \lambda_{AB}, \epsilon_I) = \int_{\mathbb{M}_4} \frac{\mathrm{e}^{\tau_N^{\mathbb{T}}} - \mathrm{e}^{\tau_S^{\mathbb{T}}}}{\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)} \,, \tag{5.2.115}$$

where we have defined the North pole equivariant form $\tau_N^{\mathbb{T}}$ and South pole equivariant form $\tau_S^{\mathbb{T}}$ as

$$\begin{aligned}
\tau_N^{\mathbb{T}} &= \sum_A \lambda_A \, \mathcal{C}_A^N + \sum_{A,B} \lambda_{AB} \, \mathcal{C}_A^N \, \mathcal{C}_B^N \,, \\
\tau_S^{\mathbb{T}} &= \sum_A \lambda_A \, \mathcal{C}_A^S + \sum_{A,B} \lambda_{AB} \, \mathcal{C}_A^S \, \mathcal{C}_B^S \,, \\
\mathcal{C}_a^N &= \mathcal{C}_a^S = c_1^{\mathbb{T}}(L_a) \,, \qquad a = 1, \dots, d \,, \\
\mathcal{C}_{d+2}^S &= -\mathcal{C}_{d+1}^N = \epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a) \,, \\
\mathcal{C}_{d+2}^N &= \mathcal{C}_{d+1}^S = 0 \,.
\end{aligned} \tag{5.2.116}$$

The flux equations are the following:

$$-\,\nu_{M5} M_A = \partial_{\lambda_A} \mathbb{V}^{(2)} = \int_{\mathbb{M}_4} \frac{\mathcal{C}_A^N \, \tau_N^{\mathbb{T}} - \mathcal{C}_A^S \, \tau_S^{\mathbb{T}}}{\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)} \,. \tag{5.2.117}$$

For a generic fan, using the gauge transformations (4.4.8) and (4.4.9) we can set all $\lambda_a = \lambda_{a,a} = \lambda_{a,a+1} = 0$. However, as already mentioned, for special fans, including $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, one of the single times λ_a remains unfixed. As a difference with the the $\mathbb{Z}_2$ symmetric case, an arbitrary λ_a solves trivially the flux equations. Therefore we set $\lambda_{a,a} = \lambda_{a,a+1} = 0$ and keep λ_a with the understanding that the latter can be partially or totally gauged fixed to zero. The forms $\tau_N^{\mathbb{T}}$ and $\tau_S^{\mathbb{T}}$ with all variables can then be written as

$$\begin{aligned}
\tau_N^{\mathbb{T}} &= \sum_a \lambda_a \, c_1^{\mathbb{T}}(L_a) + \Big(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)\Big)\Big(\overline{\lambda}_{d+1} + \sum_a \overline{\lambda}_{a,d+1} \, c_1^{\mathbb{T}}(L_a)\Big) \,, \\
\tau_S^{\mathbb{T}} &= \sum_a \lambda_a \, c_1^{\mathbb{T}}(L_a) - \Big(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)\Big)\Big(\overline{\lambda}_{d+2} + \sum_a \overline{\lambda}_{a,d+2} \, c_1^{\mathbb{T}}(L_a)\Big) \,,
\end{aligned} \tag{5.2.118}$$

where we have defined the $\overline{\lambda}$ variables as

$$\begin{aligned}
\overline{\lambda}_{d+1} &= \epsilon_3 \lambda_{d+1,d+1} - \lambda_{d+1} \, , & \overline{\lambda}_{d+2} &= -\epsilon_3 \lambda_{d+2,d+2} - \lambda_{d+2} \, , \\
\overline{\lambda}_{b,d+1} &= \lambda_{d+1,d+1} - 2\lambda_{b,d+1} \, , & \overline{\lambda}_{b,d+2} &= -\lambda_{d+2,d+2} - 2\lambda_{b,d+2} \, .
\end{aligned} \tag{5.2.119}$$

Then equations (5.2.117) become

$$\begin{aligned}
-\nu_{M5} M_a &= \sum_b D_{ab} (\bar{\lambda}_{b,d+1} + \bar{\lambda}_{b,d+2}) \, , \\
-\nu_{M5} M_{d+1} &= -\sum_{ab} D_{ab} \, \overline{\lambda}_{b,d+1} \, , \\
-\nu_{M5} M_{d+2} &= \sum_{ab} D_{ab} \, \overline{\lambda}_{b,d+2} \, .
\end{aligned} \tag{5.2.120}$$

The expression for $\mathbb{V}^{(3)}$ is

$$\begin{aligned}
\mathbb{V}^{(3)} = & \frac{1}{6} \int_{\mathbb{M}_4} \Big(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a) \Big)^2 \Big[\big(\Lambda^N\big)^3 + \big(\Lambda^S\big)^3 \Big] \\
& + \frac{1}{2} \int_{\mathbb{M}_4} \Big(\sum_a \lambda_a \, c_1^{\mathbb{T}}(L_a) \Big) \Big(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a) \Big) \Big[\big(\Lambda^N\big)^2 - \big(\Lambda^S\big)^2 \Big] \\
& + \frac{1}{2} \int_{\mathbb{M}_4} \Big(\sum_a \lambda_a \, c_1^{\mathbb{T}}(L_a) \Big)^2 \Big[\Lambda^N + \Lambda^S \Big] \, ,
\end{aligned} \tag{5.2.121}$$

where we defined

$$\Lambda^N = \overline{\lambda}_{d+1} + \sum_a \overline{\lambda}_{a,d+1} \, c_1^{\mathbb{T}}(L_a) \, , \qquad \Lambda^S = \overline{\lambda}_{d+2} + \sum_a \overline{\lambda}_{a,d+2} \, c_1^{\mathbb{T}}(L_a) \, . \tag{5.2.122}$$

The flux constraints are not enough to fix all the $\overline{\lambda}$, so the idea is again to extremize $\mathbb{V}^{(3)}$ with respect to the remaining variables. It is convenient to define $\overline{\lambda}_{b,+}$ and $\overline{\lambda}_{b,-}$ as

$$\overline{\lambda}_{b,\pm} = \overline{\lambda}_{b,d+1} \pm \overline{\lambda}_{b,d+2} \, , \tag{5.2.123}$$

so that all the $\overline{\lambda}_{b,+}$ are fixed (up to gauge transformations) by

$$-\nu_{M5} M_a = \sum_b D_{ab} \, \overline{\lambda}_{b,+} \, , \tag{5.2.124}$$

whereas the $\overline{\lambda}_{b,-}$ are only subject to the following constraint:

$$\nu_{M5} (M_{d+1} + M_{d+2}) = \sum_{ab} D_{ab} \, \overline{\lambda}_{b,-} \, . \tag{5.2.125}$$

The extremization conditions then are

$$0 = \frac{\partial \mathbb{V}^{(3)}}{\partial \lambda_a} = \frac{\partial \mathbb{V}^{(3)}}{\partial \overline{\lambda}_{d+1}} = \frac{\partial \mathbb{V}^{(3)}}{\partial \overline{\lambda}_{d+2}} = \sum_a \rho^a \, \frac{\partial \mathbb{V}^{(3)}}{\partial \overline{\lambda}_{a,-}} \, , \quad \forall \, \rho^a \text{ such that } \sum_{ab} D_{ab} \, \rho^b = 0 \, . \tag{5.2.126}$$

In general these equations do not look easy, but in the special case $M_{d+1}+M_{d+2}=0$ there is a simple solution: we can set $\lambda_a=0$, $\overline{\lambda}_{d+1}=\overline{\lambda}_{d+2}$ and $\overline{\lambda}_{b,d+1}=\overline{\lambda}_{b,d+2}$ so that the equations with ρ^a are trivially solved. The rest of the computation reduces to that of section 5.2.2.2 for $\mathbb{Z}_2$ symmetric geometries. When $M_{d+1}+M_{d+2}\neq 0$ this simple solution is not possible because the constraint (5.2.125) would not be satisfied.

5.2.2.5 Examples of general geometries

In this section we consider again the examples based on $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and dP_3 and compare with the supergravity solutions in [102], where the additional parameter c is turned on. Since the solutions in [102] all correspond to a critical point at $\epsilon_1=\epsilon_2=0$, for simplicity in this section we restrict again to configurations with this feature, which as we discussed requires a special choice of fluxes for the case of dP_3, while it is automatic for generic fluxes for $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$. The explicit value of the central charge has been written in [104–106].

The free energy (5.2.121) can be expanded in a sum of integrals of equivariant Chern classes. Since the multiple intersections (5.2.70) are homogeneous function of degree $p-2$ in ϵ_1 and ϵ_2, all the terms involving $D_{a_1,\dots a_p}$ with $p>2$ in (5.2.121) vanish for $\epsilon_1=\epsilon_2=0$, and the free energy simplifies to

$$\begin{aligned}6\mathbb{V}^{(3)} &= \bar{\lambda}_{d+1}^3\sum_{ab}D_{ab}+3\bar{\lambda}_{d+1}^2\sum_{ab}D_{ab}(2\epsilon_3\bar{\lambda}_{a,d+1}+\lambda_a)\\ &+3\bar{\lambda}_{d+1}\sum_{ab}D_{ab}(\epsilon_3\bar{\lambda}_{a,d+1}+\lambda_a)(\epsilon_3\bar{\lambda}_{b,d+1}+\lambda_b)\\ &+\bar{\lambda}_{d+2}^3\sum_{ab}D_{ab}+3\bar{\lambda}_{d+2}^2\sum_{ab}D_{ab}(2\epsilon_3\bar{\lambda}_{a,d+2}-\lambda_a)\\ &+3\bar{\lambda}_{d+2}\sum_{ab}D_{ab}(\epsilon_3\bar{\lambda}_{a,d+2}-\lambda_a)(\epsilon_3\bar{\lambda}_{b,d+2}-\lambda_b)\,.\end{aligned}\tag{5.2.127}$$

Using the flux constraints we can also write

$$\begin{aligned}\mathbb{V}^{(3)} &= \bar{\lambda}_{d+1}^3\sum_{ab}D_{ab}+3\bar{\lambda}_{d+1}^2\epsilon_3\nu_{M5}M_{d+1}+3\bar{\lambda}_{d+1}^2\sum_{ab}D_{ab}\lambda_a^+ +3\bar{\lambda}_{d+1}\sum_{ab}D_{ab}\lambda_a^+\lambda_b^+\\ &+\bar{\lambda}_{d+2}^3\sum_{ab}D_{ab}-3\bar{\lambda}_{d+2}^2\epsilon_3\nu_{M5}M_{d+2}+3\bar{\lambda}_{d+2}^2\sum_{ab}D_{ab}\lambda_a^- +3\bar{\lambda}_{d+2}\sum_{ab}D_{ab}\lambda_a^-\lambda_b^-\,,\end{aligned}\tag{5.2.128}$$

where

$$\lambda_a^+=\epsilon_3\bar{\lambda}_{a,d+1}+\lambda_a\,,\qquad \lambda_a^-=\epsilon_3\bar{\lambda}_{a,d+2}-\lambda_a\,,\tag{5.2.129}$$

are constrained variables.

We consider first the general case ($\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and dP_3) with all fluxes associated to the fan of the KE_4 set equal to N. The fan and intersection matrix are given in (5.2.72) and (5.2.73). There are, in principle, $d-2$ independent fluxes on $\mathbb{M}_4$ that we can turn on but in the supergravity solution with KE metric they are equal and we first restrict to this case. The relations $\sum_A V_I^A M_A = 0$ require $M_A = (N, \dots, N, N_N, N_S)$ with $dN + N_N - N_S = 0$ and we can parameterize $N_N = M - \frac{d}{2}N$ and $N_S = M + \frac{d}{2}N$, possibly allowing an half-integer M. Given the symmetry of the problem, we take all λ_a to be equal, and similarly for the $\bar\lambda_{a,d+1}$ and $\bar\lambda_{a,d+2}$. The condition $\sum_{abc} D_{abc} = 0$ holds for these models, and therefore all linear terms in ϵ_1, ϵ_2 in $\mathbb{V}^{(3)}$ vanish, guaranteeing a critical point at $\epsilon_1 = \epsilon_2 = 0$. The flux conditions are solved by

$$\bar\lambda_{a,d+1} = \nu_{M5}\frac{\left(M - \frac{dN}{2}\right)}{dm_k}\,, \qquad \bar\lambda_{a,d+2} = -\nu_{M5}\frac{\left(M + \frac{dN}{2}\right)}{dm_k}\,, \tag{5.2.130}$$

where $\sum_{ab} D_{ab} = dm_k$ so that $m_k = 3, 2, 1$ for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_3, respectively. Extremizing with respect to λ_a and defining $\bar\lambda_{d+1} = \nu_{M5}(H+K)$ and $\bar\lambda_{d+2} = \nu_{M5}(H-K)$ we find

$$6\nu_{M5}^{-3}\mathbb{V}^{(3)} = 2dm_k H^3 - \frac{3}{2}d\epsilon_3^2 N^2 \frac{K^2}{m_k H} - 6d\epsilon_3 N H^2 + 3\epsilon_3 H(4KM + \frac{d}{2m_k}\epsilon_3 N^2)\,, \tag{5.2.131}$$

which after extremization gives[22]

$$\mathbb{V}^{(3)} = \frac{d^2\nu_{M5}^3\epsilon_3^3 N^4}{12 m_k^2 (d^2N^2 + 12M^2)^2}\left((3d^2N^2 - 12M^2)^{3/2} - dN(5d^2N^2 - 36M^2)\right)\,. \tag{5.2.132}$$

An analogous formula for non-necessarily toric KE has recently appeared in [106].

In the case $\mathbb{P}^1 \times \mathbb{P}^1$ we can turn on two independent fluxes and have round metrics on the $\mathbb{P}^1$s. We take the general assignment of fluxes compatible with $\sum_A V_I^A M_A = 0$:

$$M_A = (N_1, N_2, N_1, N_2, N_N, N_S)\,, \tag{5.2.133}$$

where

$$2N_1 + 2N_2 + N_N - N_S = 0$$

and we can parameterize $N_N = M - N_1 - N_2$ and $N_S = M + N_1 + N_2$. Using the gauge transformations (4.4.8) we can also reduce to the case

$$\lambda_{a+2} = \lambda_a\,,\ \bar\lambda_{a+2,d+1} = \bar\lambda_{a,d+1}\,,\ \bar\lambda_{a+2,d+2} = \bar\lambda_{a,d+2}\,. \tag{5.2.134}$$

Notice that, in this gauge, all the linear terms in ϵ_1, ϵ_2 in $\mathbb{V}^{(3)}$ vanishes since, as one can check,

$$\sum_{abc} D_{abc} l_a^{(1)} l_b^{(2)} l_c^{(3)} = 0 \tag{5.2.135}$$

provided the vectors $l_a^{(k)}$ satisfy $l_a^{(k)} = l_{a+2}^{(k)}$. We can solve the flux constraints

$$\begin{aligned} &2\bar\lambda_{1,d+1} + 2\bar\lambda_{1,d+2} + \nu_{M5}N_1 = 0\,, \quad 4\bar\lambda_{1,d+1} + 4\bar\lambda_{2,d+1} + \nu_{M5}(-M + N_1 + N_2) = 0\,,\\ &2\bar\lambda_{2,d+1} + 2\bar\lambda_{2,d+2} + \nu_{M5}N_2 = 0\,, \quad 4\bar\lambda_{1,d+2} + 4\bar\lambda_{2,d+2} + \nu_{M5}(M + N_1 + N_2) = 0\,, \end{aligned} \tag{5.2.136}$$

[22] Recall that to compare with section 5.2.2.2 we need to use the rescaling (5.2.47).

by

$$
\begin{aligned}
\bar{\lambda}_{2,d+1} &= -\bar{\lambda}_{1,d+1} + \frac{1}{4}\nu_{M5}(M - N_1 - N_2)\,, \\
\bar{\lambda}_{1,d+2} &= -\bar{\lambda}_{1,d+1} - \nu_{M5}\frac{N_2}{2}\,, \\
\bar{\lambda}_{2,d+2} &= \bar{\lambda}_{1,d+1} + \frac{1}{4}\nu_{M5}(-M - N_1 + N_2)\,.
\end{aligned}
\tag{5.2.137}
$$

Extremizing with respect to $\lambda_{1,2}$ and $\bar{\lambda}_{1,d+1}$ and defining $\bar{\lambda}_{d+1} = \nu_{M5}(H+K)$ and $\bar{\lambda}_{d+2} = \nu_{M5}(H-K)$ we find

$$
6\nu_{M5}^{-3}\mathbb{V}^{(3)} = 16H^3 - 3\epsilon_3^2 N_1 N_2 \frac{K^2}{H} - 12\epsilon_3(N_1+N_2)H^2 + 3\epsilon_3 H(4KM + \epsilon_3 N_1 N_2)\,,
\tag{5.2.138}
$$

which after extremization gives

$$
\begin{aligned}
\mathbb{V}^{(3)} = &\frac{\nu_{M5}^3\epsilon_3^3 N_1^2 N_2^2(4N_1^2 + 4N_1N_2 + 4N_2^2 - 3M^2)^{3/2}}{6(4N_1N_2 + 3M^2)^2} \\
&- \frac{\nu_{M5}^3\epsilon_3^3 N_1^2 N_2^2(N_1+N_2)(8N_1^2 + 4N_1N_2 + 8N_2^2 - 9M^2)}{6(4N_1N_2 + 3M^2)^2}\,,
\end{aligned}
\tag{5.2.139}
$$

reproducing (5.7) of [105].

Finally, let us mention that in the case of dP_3 we can turn on four independent fluxes along the base plus one additional flux M, and the general extremization problem is intractable. It is possible to solve it for the two special configurations of fluxes with enhanced symmetry discussed previously. We leave this as an instructive exercise for the reader.

The case $\mathbb{P}^1 \times \mathbb{P}^1$ has been interpreted in [104, 105] as a solution for M5 branes sitting at the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ wrapped over one of the $\mathbb{P}^1$. The interpretation follows by deriving the central charge from an anomaly polynomial computation. It would be very interesting to understand if our general formula (5.2.121) can be written as the integral of the anomaly polynomial for some M5 brane theory wrapped over a two-cycle in $\mathbb{M}_4$ and give a field theory interpretation of the solution.

5.2.3 Comparison with other approaches

It is interesting to compare with the recent approach based on Killing spinor bilinears in M theory [53]. The technique consists in considering a set of equivariantly closed differential forms which can be constructed from Killing spinor bilinears. Three such forms have been explicitly constructed for $\mathrm{AdS}_5 \times M_6$ in [53] and for $\mathrm{AdS}_3 \times M_8$ in [106, 107]. Our results in sections 5.2.1 and 5.2.2 partially overlap with those in [106, 107] and it is interesting to compare the two methods. We will show that they are actually equivalent, when they can be compared, although in a non-trivial way.

For both cases, $\mathrm{AdS}_{11-k} \times M_k$ with $k = 6, 8$, the authors of [106, 107] define an equivariant k-form Φ whose higher-degree component is the warped volume of M_k

and the lowest component the third power of a special locally defined function y. Up to coefficients, the integral of Φ is the free energy, so we have

$$F = \int_{M_k} \Phi = \sum_\alpha \frac{y^3|_\alpha}{d_\alpha \, e^{\mathbb{T}^{k/2}}|_\alpha} \,, \tag{5.2.140}$$

where α are the fixed points of the geometry, and we recognize our expression for $\mathbb{V}^{(3)}$ for M theory solutions[23]

$$F = \mathbb{V}^{(3)} = \sum_\alpha \frac{(\tau_\alpha)^3}{d_\alpha \, e^{\mathbb{T}^{k/2}}|_\alpha} \,, \tag{5.2.141}$$

upon identifying

$$y|_\alpha = \tau_\alpha \,. \tag{5.2.142}$$

There exists also an equivariant four-form Φ^F whose higher-degree component is the M theory four-form and the lowest component the first power of the function y. The flux quantization conditions give then

$$\begin{aligned} M_{AB} &= \int_{M_8} \Phi^F c_1(L_A) c_1(L_B) = \sum_\alpha \frac{(c_1^{\mathbb{T}^4}(L_A) c_1^{\mathbb{T}^4}(L_B) y)|_\alpha}{d_\alpha \, e^{\mathbb{T}^4}|_\alpha} \,, \\ M_A &= \int_{M_6} \Phi^F c_1(L_A) = \sum_\alpha \frac{(c_1^{\mathbb{T}^3}(L_A) y)|_\alpha}{d_\alpha \, e^{\mathbb{T}^3}|_\alpha} \,, \end{aligned} \tag{5.2.143}$$

for $\mathrm{AdS}_3 \times M_8$ and $\mathrm{AdS}_5 \times M_6$, respectively and it easy to see that these conditions are equivalent, up to coefficients, to our (5.2.4) and (5.2.45) with the same identification $y|_\alpha = \tau_\alpha$.

Finally there exists another auxiliary form, a four-form Φ^{*F} in $\mathrm{AdS}_3 \times M_8$ and a two-form Φ^Y in $\mathrm{AdS}_5 \times M_6$, whose lowest component is the second power of the function y.

Consider first the $\mathrm{AdS}_3 \times M_8$ solutions with wrapped M5 branes of section 5.2.1. The vanishing of the Φ^{*F} flux along S^4 is used in [106, 107] to enforce a $\mathbb{Z}_2$ symmetry of the solution by identifying $y^N|_a = -y^S|_a$, thus effectively cutting by half the number of fixed points. With the identification $y|_a = \tau_a$, our construction in section 5.2.1 is then equivalent to the one in [106].

Consider next the $\mathrm{AdS}_5 \times M_6$ solutions of section 5.2.2. The approaches are complementary. While we consider toric orbifolds and the action of the full torus $\mathbb{T}^3 = U(1)^3$, the authors of [107] consider $\mathbb{P}^1$ bundles over a smooth four-manifold B_4 and assume that the R-symmetry vector has no legs along B_4. Let us observe that this assumption can fail in general. For a generic B_4 with abelian isometries there is no reason to expect that the R-symmetry does not mix with the isometries of B_4 and a

[23] We are omitting a $(-1)^{k/2}$ sign in the expression for $\mathbb{V}^{(3)}$. In this discussion we are ignoring all such overall numerical factors.

full-fledged computation considering the torus action on B_4 is necessary. Also for the toric $B_4 = \mathrm{dP}_3$ with a generic choice of fluxes we expect a mixing with the isometries of B_4, as discussed in section 5.2.2.2. Under this condition, the central charge given in [107] is not necessarily the extremum of the free energy. Obviously, whenever the two approaches can be compared and the assumption in [107] is satisfied, we find agreement.

From a technical point of view, this might be surprising. Recall indeed that the flux constraints do not completely fix the values of the times $\tau_a = y|_a$. In our construction, we just extremize the free energy F with respect to the remaining parameters. In [106, 107] instead, in a case-by-case analysis, the auxiliary form Φ^Y is used to find additional conditions to fix the $y|_\alpha$. The two methods look superficially different, but we now show that they are effectively equivalent.

The extremization conditions with respect to the Kähler parameters that are not fixed by the flux constraints are written in (5.2.126). The first three conditions

$$\frac{\partial \mathbb{V}^{(3)}}{\partial \lambda_A} = 0\,, \qquad A = a, d+1, d+2\,, \tag{5.2.144}$$

can also be rewritten as

$$\sum_B V_i^B \frac{\partial \mathbb{V}^{(3)}}{\partial \lambda_{BA}}(1+\delta_{AB}) = 0\,, \qquad i = 1,2\,, \qquad A = a, d+1, d+2\,. \tag{5.2.145}$$

Indeed

$$\begin{aligned}\sum_B V_i^B \frac{\partial \mathbb{V}}{\partial \lambda_{BA}}(1+\delta_{AB}) &= -2\int_{M_6} \mathrm{e}^{\tau^{\mathbb{T}^3}} c_1^{\mathbb{T}^3}(L_A) \sum_B V_i^B c_1^{\mathbb{T}^3}(L_B) \\ &= 2\epsilon_i \int_{M_6} \mathrm{e}^{\tau^{\mathbb{T}^3}} c_1^{\mathbb{T}^3}(L_A) = -2\epsilon_i \frac{\partial \mathbb{V}}{\partial \lambda_A}\,,\end{aligned} \tag{5.2.146}$$

and taking the degree two component of this equation we see that all the conditions (5.2.145) collapse to the extremization of the free energy with respect to the parameter λ_A.

The conditions (5.2.126) can be then written as

$$0 = \sum_b v_i^b \frac{\partial \mathbb{V}^{(3)}}{\partial \lambda_{b,a}}(1+\delta_{ba}) = \sum_b v_i^b \frac{\partial \mathbb{V}^{(3)}}{\partial \overline{\lambda}_{b,d+1}} = \sum_b v_i^b \frac{\partial \mathbb{V}^{(3)}}{\partial \overline{\lambda}_{b,d+2}} = \sum_a \rho^a \frac{\partial \mathbb{V}^{(3)}}{\partial \overline{\lambda}_{a,-}}\,, \tag{5.2.147}$$

where ρ^a is such that $\sum_{ab} D_{ab}\, \rho^b = 0$. Now, the equations

$$\begin{aligned}(1+\delta_{bc})\frac{\partial \mathbb{V}^{(3)}}{\partial \lambda_{b,c}} &= \sum_a (c_1^{\mathbb{T}^3}(L_b) c_1^{\mathbb{T}^3}(L_c))|_a \frac{(\tau_a^N)^2 - (\tau_a^S)^2}{d_{a,a+1}\epsilon_1^a \epsilon_2^a(\epsilon_3 - \epsilon_1^a - \epsilon_2^a)}\,, \\ \frac{\partial \mathbb{V}^{(3)}}{\partial \lambda_{b,d+1}} &= \sum_a c_1^{\mathbb{T}^3}(L_b)|_a \frac{(\tau_a^N)^2}{d_{a,a+1}\epsilon_1^a\epsilon_2^a}\,, \\ \frac{\partial \mathbb{V}^{(3)}}{\partial \lambda_{b,d+2}} &= \sum_a c_1^{\mathbb{T}^3}(L_b)|_a \frac{(\tau_a^S)^2}{d_{a,a+1}\epsilon_1^a\epsilon_2^a}\,,\end{aligned} \tag{5.2.148}$$

given the identification $y|_a = \tau_a$ and the fact that Φ^Y has lowest component y^2, translate into the localization formulas for

$$\begin{aligned}
\int_{\mathbb{P}^1_{bc}} \Phi^Y &= \int_{M_6} c_1(L_c) c_1(L_b) \Phi^Y \,, \\
\int_{D_b^N} \Phi^Y &= \int_{M_6} c_1(L_{d+1}) c_1(L_b) \Phi^Y \,, \\
\int_{D_b^S} \Phi^Y &= \int_{M_6} c_1(L_{d+2}) c_1(L_b) \Phi^Y \,,
\end{aligned} \tag{5.2.149}$$

respectively, where $\mathbb{P}^1_{bc}$ is the fibre taken at the fixed points $D_c \cap D_b$ on the base ($b = c \pm 1$ necessarily) and the $D_b^{N,S}$ are the divisor on the base taken at the North and South pole of the fibre, respectively. The extremization constraints are then equivalent to the following co-homological relations

$$\begin{aligned}
0 &= \sum_b v_i^b \int_{\mathbb{P}^1_{bc}} \Phi^Y = \sum_b v_i^b \int_{D_b^N} \Phi^Y = \sum_b v_i^b \int_{D_b^S} \Phi^Y \\
&= \sum_a \rho^a \left(\int_{D_a^N} \Phi^Y - \int_{D_a^S} \Phi^Y \right) , \quad \forall\, \rho^a \text{ such that } \sum_{ab} D_{ab}\, \rho^b = 0 \,.
\end{aligned} \tag{5.2.150}$$

The first three conditions are obvious: the cycles $\sum_b v_i^b \, \mathbb{P}^1_{bc}$, $\sum_b v_i^b \, D_b^{N,S}$ are trivial in homology. The last equation equates cycles sitting at the North and South pole. The corresponding fluxes of Φ^Y do not need to be equal but they must be related. We know that $c_1(L_{d+2}) = c_1(L_{d+1}) + \sum_a c_1(L_a)$.[24] Then

$$\begin{aligned}
\sum_a \rho^a \left(\int_{D_a^N} \Phi^Y - \int_{D_a^S} \Phi^Y \right) &= \sum_a \rho^a \int_{M_6} \Big(c_1(L_{d+1}) - c_1(L_{d+2}) \Big) c_1(L_a) \Phi^Y \\
&= - \sum_{ab} \rho^a \int_{M_6} c_1(L_b) c_1(L_a) \Phi^Y \propto \sum_{ab} \rho^a D_{ab} = 0 \,.
\end{aligned} \tag{5.2.151}$$

The last step follows by expanding Φ^Y in a sum of Chern classes, and by writing $\int_{M_6} c_1(L_b) c_1(L_a) \Phi^Y$ as a sum of triple intersections $D^{M_6}_{ABC}$ on M_6. But $D^{M_6}_{abc} = 0$ and $D^{M_6}_{d+1,a,b} = D^{M_6}_{d+2,a,b} = D_{ab}$.[25]

We see that our construction based on the equivariant volume naturally incorporates the localization approach of [53, 106, 107], with the advantage that all the geometrical constraints that must be imposed case-by-case in order to find the free energy in [53, 106, 107] appear naturally in our construction: they correspond to the extremization with respect to all parameters that remain after imposing the flux constraints. This avoids an analysis based on the specific topology of the background.

[24] The $I = 3$ condition of $\sum_A V_I^A c_1(L_A) = 0$.

[25] The triple intersections on M_6 are easily computed as $D^{M_6}_{ABC} = \frac{\partial \mathbb{V}^{(3)}}{\partial \lambda_A \partial \lambda_B \partial \lambda_C}$ from (5.2.114).

5.3 AdS$_2$, AdS$_3$ and AdS$_4$ solutions in type II supergravities

In this section we consider solutions in type II string theory with geometries that are fibrations over a four-dimensional orbifold $\mathbb{M}_4$. We consider the case of massive type IIA solutions with D4 brane flux, corresponding to D4 branes wrapped over $\mathbb{M}_4$ and the case of type IIB solutions with D3 brane flux. In all cases, we show that the free energy can be obtained by extremizing the appropriate term in the equivariant volume.

5.3.1 AdS$_2 \times M_8$ solutions in massive type IIA

In this section we turn our attention to D4 branes wrapped around a generic four-dimensional toric orbifold $\mathbb{M}_4$ [85, 86, 99]. Specifically the brane system we study corresponds to AdS$_2 \times M_8$ solutions in massive type IIA, where M_8 is an S^4 fibration over $\mathbb{M}_4$. The geometry is similar to the case of M5 branes wrapped around $\mathbb{M}_4$ considered in section 5.2.1 and we can borrow most of the computations. Here, due to the orientifold projection,[26] the $\mathbb{Z}_2$ projection used in section 5.2.1 is automatically implemented and there is only one set of fixed points, at one of the poles of S^4. The geometry to consider is then a CY$_4$, a $\mathbb{C}^2$ fibration over $\mathbb{M}_4$ with toric fan generated by the vectors (5.2.1). As discussed in [28] and in the introduction, the prescription for D4 in massive type IIA is also similar to (5.2.4), with different degrees of homogeneity:

$$\nu_{D4}\,(2-\delta_{AB})\,M_{AB} = -\frac{\partial}{\partial \lambda_{AB}}\mathbb{V}^{(3)}(\lambda_A,\lambda_{AB},\epsilon_I)\,, \qquad F = \mathbb{V}^{(5)}(\lambda_A,\lambda_{AB},\epsilon_I)\,. \tag{5.3.1}$$

The rest of the discussion is very similar to the section 5.2.1. We can write the flux equations as

$$\nu_{D4}\,M_{AB} = -\int_{\mathbb{M}_4}\frac{\mathcal{C}_A\,\mathcal{C}_B\big(\tau^{\mathbb{T}}\big)^2}{2\,\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}} = -\sum_a \frac{B_a^{(2)}\cdot\big(\mathcal{C}_A\,\mathcal{C}_B\big)|_a}{d_{a,a+1}\,\epsilon_1^a\,\epsilon_2^a}\,, \tag{5.3.2}$$

where the equivariant forms $\mathcal{C}_A$, $\tau^{\mathbb{T}}$ and the $B_a^{(\alpha)}$ are defined respectively by (5.2.13) and (5.2.15). These equations are identical to the ones of section 5.2.1, with the only difference being that $B_a^{(2)}$ takes the place of $B_a^{(1)}$. The solution can be read from (5.2.20) and (5.2.24):

$$\begin{aligned} B_a^{(2)} &= -\nu_{D4}\,N\,,\\ M_{AB} &= N\sum_{c,d}\mathfrak{t}_A^c\,\mathfrak{t}_B^d\,D_{cd}\,, \end{aligned} \tag{5.3.3}$$

with $\mathfrak{t}_A^c$ given by (5.2.25).

[26] The brane system is actually D4 in the presence of D8, which generate the cosmological constant, and an orientifold plane O8 that cuts S^4 into half.

Following prescription (5.3.1), the solution (5.3.3) must be substituted in the expression for $\mathbb{V}^{(5)}$, which depends on the $B_a^{(5)}$ as

$$\mathbb{V}^{(5)}(\lambda_A, \lambda_{AB}, \epsilon_I) = \sum_a \frac{B_a^{(5)}}{d_{a,a+1}\, \epsilon_1^a\, \epsilon_2^a}\,. \tag{5.3.4}$$

The relation between $B_a^{(5)}$ and $B_a^{(2)}$ is given in equation (5.2.16), with the added complication that the exponents are half-integers and thus we need to be careful about the signs:

$$\begin{aligned} B_a^{(5)} &= \eta_a \frac{2^{\frac{5}{2}}}{5!} \left|\nu_{D4} N\right|^{\frac{5}{2}} \left|\big(\mathcal{C}_{d+1}\, \mathcal{C}_{d+2}\big)|_a\right|^{\frac{3}{2}}\,, \\ \eta_a &= \operatorname{sign}\Big(\big(\mathcal{C}_{d+1}\, \mathcal{C}_{d+2}\big)|_a\Big) \cdot \operatorname{sign}\big(\tau_a\big)\,. \end{aligned} \tag{5.3.5}$$

We note that the sign of $\big(\mathcal{C}_{d+1}\, \mathcal{C}_{d+2}\big)|_a$ is the same as the sign of $B_a^{(2)}$, and thus is fixed:

$$\operatorname{sign}\Big(\big(\mathcal{C}_{d+1}\, \mathcal{C}_{d+2}\big)|_a\Big) = \operatorname{sign}\left(\frac{(\tau_a)^2}{2\big(\mathcal{C}_{d+1}\, \mathcal{C}_{d+2}\big)|_a}\right) = \operatorname{sign}\big(-\nu_{D4}\, N\big) \equiv \sigma\,. \tag{5.3.6}$$

The sign of the τ_a however is not fixed by (5.3.3). We can rewrite the equations $B_a^{(2)} = -\nu_{D4}\, N$ as

$$\tau_a = \sigma\, \eta_a \sqrt{-2\nu_{D4}\, N\big(\mathcal{C}_{d+1}\, \mathcal{C}_{d+2}\big)|_a}\,. \tag{5.3.7}$$

It is always possible to find λ_A and λ_{AB} that solve these equations, whatever the value of η_a might be.

For the free energy we can write

$$F = \frac{2^{\frac{5}{2}}}{5!}\big(-\nu_{D4} N\big)^{\frac{5}{2}} \sum_a \frac{\eta_a\big((\epsilon_3 - \epsilon_4 + (\mathfrak{t}_a - 1)\epsilon_1^a + (\mathfrak{t}_{a+1} - 1)\epsilon_2^a)(\epsilon_4 - \mathfrak{t}_a\epsilon_1^a - \mathfrak{t}_{a+1}\epsilon_2^a)\big)^{\frac{3}{2}}}{d_{a,a+1}\, \epsilon_1^a\, \epsilon_2^a}\,, \tag{5.3.8}$$

thus reproducing the extremal function in [85].[27]

The sign ambiguities remain to be fixed by a more careful analysis. For a convex fan, supersymmetry is preserved with a topological twist and we expect that all the η_a have the same sign [85]. This could follow from a generalization of the following argument valid for the equivariant volume with single times only. The λ_A determine the polytope

$$\mathcal{P} = \{y_I \in \mathbb{R}^4 \mid y_I V_I^A \geq \lambda_A\}\,. \tag{5.3.9}$$

[27] Compare formula (5.7) in [85] and set $\varphi_1 = \epsilon_4\,, \varphi_2 = \epsilon_3 - \epsilon_4\,, m_a\mathfrak{p}_1^a = \mathfrak{t}_a\,, m_a\mathfrak{p}_2^a = 1 - \mathfrak{t}_a\,, W = 0$, and set $\epsilon_3 = 2$ for simplicity of comparison. Our result for the free energy then matches theirs (up to an overall sign due to different conventions) upon choosing $\epsilon_3(-\nu_{D4})^{\frac{5}{2}} = \frac{16\pi}{\sqrt{8-N_f}}$. Notice that in [85] the vectors v^a are taken to be primitive, contrary to the conventions that we are using. Our v^a are their $\hat{v}^a$.

Naturally $\mathcal{P}$ must be non-empty, so let us take $y_I \in \mathcal{P}$. If we contract the inequalities $y_I V_I^A \geq \lambda_A$ with $c_1^{\mathbb{T}^4}(L_A)|_a$ we get

$$- y_I \epsilon_I \geq \tau_a \quad \forall a \in \{1, \dots, d\} \,. \tag{5.3.10}$$

Given that $\mathcal{P}$ is a resolved cone and that the equivariant volume is given by an integral over $\mathcal{P}$ (4.3.2)

$$\mathbb{V}(\lambda_A, \epsilon_I) = \int_{\mathcal{P}} \mathrm{d}^4 y \, \mathrm{e}^{-y_I \epsilon_I} \,, \tag{5.3.11}$$

then the exponent $-y_I\epsilon_I \geq \tau_a$ must be negative for convergence. This implies $\tau_a \leq 0$, and thus $\eta_a = -\sigma$. By choosing $\sigma = 1$ we would find the result of [85].

The case of anti-twist requires taking a non-convex fan for $\mathbb{M}_4$. This can be obtained by *formally* sending $v^a \to \sigma^a v^a$ everywhere, implying $\epsilon_1^a \to \sigma^a \epsilon_1^a$ and $\epsilon_2^a \to \sigma^{a+1}\epsilon_2^a$. It was proposed in [85] that the correct assignment of signs is $\eta_a = -\sigma^a\sigma^{a+1}$, and it would be interesting to understand this by a geometrical argument.

5.3.2 $\mathrm{AdS}_4 \times M_6$ solutions in massive type IIA

In this section we consider $\mathrm{AdS}_4 \times M_6$ solutions of massive type IIA supergravity, which correspond to a system of D4 branes wrapped around a two-cycle inside a four-dimensional toric $\mathbb{M}_4$, in the presence of D8 branes and an orientifold plane O8. Explicit solutions of this type have been found in [55], with M_6 being a $\mathbb{P}^1$ fibration over a four-dimensional manifold that is either Kähler-Einstein or a product of Riemann surfaces, cut in half by the O8 plane. The only toric manifolds that admit such metrics are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_3: these are the cases we will be focussing on.

More precisely, we consider a half-geometry modelled on a non-compact CY_3 corresponding to the canonical bundle over $\mathbb{M}_4$, with fan given by

$$V^a = (v^a, 1) \,, \qquad V^{d+1} = (0,0,1) \;, \qquad a = 1, \dots, d \,, \tag{5.3.12}$$

where v^a are the vectors of the fan of $\mathbb{M}_4$. This fan has the same structure as the ones in sections 5.2.2.1 to 5.2.2.3, and for this reason the discussion in this section will share some similarities with the former. This half-geometry can accurately describe the solutions of [55] when the parameters ℓ and σ are set to zero. We explain this point in more detail in appendix B.3, where we also compute the free energy of the solutions of [55] to be compared with the results of our approach.

Our prescription is the following:

$$\nu_{D4}\, M_A = -\frac{\partial}{\partial \lambda_A} \mathbb{V}^{(3)}(\lambda_A, \lambda_{AB}, \epsilon_I) \,, \qquad F = \mathbb{V}^{(5)}(\lambda_A, \lambda_{AB}, \epsilon_I) \,, \qquad \sum_A V_I^A M_A = 0 \,. \tag{5.3.13}$$

The higher times are needed in order to find solutions to the flux constraints. Similarly to the discussion of section 5.2.2, we will need to extremize the free energy with respect to any parameter that is not fixed by the flux constraints.

Given the high-degree of symmetry of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_3 we expect a critical point at $\epsilon_1 = \epsilon_2 = 0$. Indeed, it can be verified with a similar logic as equation (5.2.74) that the linear terms in ϵ_i in the expression of the free energy vanish and thus $\epsilon_1 = \epsilon_2 = 0$ is a critical point.

The flux equations are

$$\begin{aligned} (I) \qquad & \nu_{D4} M_a = -\frac{1}{2} \int_{\mathbb{M}_4} \frac{c_1^{\mathbb{T}}(L_a) \left(\tau^{\mathbb{T}}\right)^2}{\epsilon_3 + \sum_b c_1^{\mathbb{T}}(L_b)} \,, \\ (II) \qquad & \nu_{D4} M_{d+1} = \frac{1}{2} \int_{\mathbb{M}_4} \left(\tau^{\mathbb{T}}\right)^2 , \end{aligned} \tag{5.3.14}$$

where $\tau^{\mathbb{T}}$ is defined as in (5.2.58). For generic values of ϵ_i these equations are not independent: since $\sum_a v_i^a M_a = 0$ and $\sum_a v_i^a c_1^{\mathbb{T}}(L_a) = -\epsilon_i$, from (I) we obtain

$$\epsilon_i \int_{\mathbb{M}_4} \frac{\left(\tau^{\mathbb{T}}\right)^2}{\epsilon_3 + \sum_b c_1^{\mathbb{T}}(L_b)} = 0 \,. \tag{5.3.15}$$

When ϵ_1 and ϵ_2 are not both zero this is a non-trivial relation that we can use to write

$$\nu_{D4} \sum_a M_a = -\frac{1}{2} \int_{\mathbb{M}_4} \frac{\sum_a c_1^{\mathbb{T}}(L_a) \left(\tau^{\mathbb{T}}\right)^2}{\epsilon_3 + \sum_b c_1^{\mathbb{T}}(L_b)} = -\frac{1}{2} \int_{\mathbb{M}_4} \left(\tau^{\mathbb{T}}\right)^2 , \tag{5.3.16}$$

which is equation (II). Crucially, this argument fails when $\epsilon_1 = \epsilon_2 = 0$, which is the case we will be focussing on. As we will see in this case equation (II) becomes independent of (I) and provides an additional constraint.

As already discussed in section 5.2.2.2, we have enough gauge freedom to set $\lambda_{a,a} = \lambda_{a,a+1} = 0$. For generic fans, it is also usually possible to gauge away the λ_a, but this is not the case for the highly symmetric fans that we consider in this section. For the $\mathbb{Z}_2$ symmetric solutions studied in section 5.2.2.2 it was always possible to find a critical point with $\lambda_a = 0$ regardless, as argued in appendix B.1. However the argument of appendix B.1 cannot be repurposed for the type IIA solutions of this section and we are thus forced to keep the λ_a. The equivariant form $\tau^{\mathbb{T}}$ can then be parameterized as

$$\tau^{\mathbb{T}} = \sum_a \lambda_a c_1^{\mathbb{T}}(L_a) + \Big(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)\Big)\Big(\overline{\lambda}_{d+1} + \sum_b \overline{\lambda}_{b,d+1} c_1^{\mathbb{T}}(L_b)\Big) \,, \tag{5.3.17}$$

where

$$\overline{\lambda}_{d+1} = -\lambda_{d+1} + \lambda_{d+1,d+1}\epsilon_3 \,, \qquad \overline{\lambda}_{a,d+1} = -2\lambda_{a,d+1} + \lambda_{d+1,d+1} \,. \tag{5.3.18}$$

Then for $\epsilon_1 = \epsilon_2 = 0$ the flux equations become

$$\begin{aligned} (I) \qquad & \nu_{D4} M_a = -\frac{\overline{\lambda}_{d+1}}{2}\Big(\overline{\lambda}_{d+1} \sum_b D_{ab} + 2\sum_b D_{ab}\Lambda_b\Big) \,, \\ \sum_a (I) + (II) \qquad & 0 = \sum_{ab} D_{ab}\Lambda_a\Lambda_b + 2\epsilon_3 \overline{\lambda}_{d+1} \sum_{ab} D_{ab} \overline{\lambda}_{a,d+1} \,, \end{aligned} \tag{5.3.19}$$

where

$$\Lambda_a = \lambda_a + \epsilon_3 \overline{\lambda}_{a,d+1}\,. \tag{5.3.20}$$

Notice that the second equation is not a consequence of the first, as we already anticipated. The free energy restricted to $\epsilon_1 = \epsilon_2 = 0$ is

$$F = \mathbb{V}^{(5)} = \epsilon_3^2 \Big(\frac{\overline{\lambda}_{d+1}^5}{20} \sum_{ab} D_{ab} + \frac{\overline{\lambda}_{d+1}^4}{24} \sum_{ab} D_{ab}(3\Lambda_a + \epsilon_3 \overline{\lambda}_{a,d+1}) + \frac{\overline{\lambda}_{d+1}^3}{12} \sum_{ab} D_{ab}\Lambda_a\Lambda_b \Big)\,. \tag{5.3.21}$$

We can eliminate $\overline{\lambda}_{a,d+1}$ from the above expression by using the second flux constraint in (5.3.19) and find

$$\begin{aligned} F &= \epsilon_3^2 \Big(\frac{\overline{\lambda}_{d+1}^5}{20} \sum_{ab} D_{ab} + \frac{\overline{\lambda}_{d+1}^4}{8} \sum_{ab} D_{ab}\Lambda_a + \frac{\overline{\lambda}_{d+1}^3}{16} \sum_{ab} D_{ab}\Lambda_a\Lambda_b \Big)\,, \\ \nu_{D4} M_a &= -\frac{\overline{\lambda}_{d+1}}{2} \Big(\overline{\lambda}_{d+1} \sum_b D_{ab} + 2 \sum_b D_{ab}\Lambda_b \Big)\,. \end{aligned} \tag{5.3.22}$$

The flux constraints are not sufficient to fix all parameters: one parameter, say $\overline{\lambda}_{d+1}$, remains undetermined. Our prescription is to extremize the free energy with respect to this leftover parameter.

Let us consider first the case of Kähler-Einstein base manifold, with all fluxes relative to two-cycles in the base equal, that is $M_a = N$. The three cases of interest are then $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_3. We define the integers $M_k = \sum_{ab} D_{ab}$ and $m_k = \sum_b D_{ab}$, which take values $M_k = (9, 8, 6)$ and $m_k = (3, 2, 1)$ for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_3 respectively. Since the fluxes M_a are all equal we can solve the flux equation by also setting all Λ_a equal to each other, giving us

$$\Lambda_a = -\frac{2N\nu_{D4} + m_k \overline{\lambda}_{d+1}^2}{2m_k \overline{\lambda}_{d+1}}\,. \tag{5.3.23}$$

The free energy as a function of $\overline{\lambda}_{d+1}$ is then

$$F = \epsilon_3^2 M_k \Big(\frac{\overline{\lambda}_{d+1}^5}{320} - \frac{\nu_{D4} N \overline{\lambda}_{d+1}^3}{16 m_k} + \frac{\nu_{D4}^2 N^2 \overline{\lambda}_{d+1}}{16 m_k^2} \Big)\,, \tag{5.3.24}$$

and extremizing it we find four solutions:

$$\begin{aligned} F &= \pm \frac{1}{10} (3\sqrt{2} - 4)\, \epsilon_3^2 \left(\frac{\nu_{D4} N}{m_k} \right)^{5/2} M_k\,, \\ F &= \pm \frac{1}{10} (3\sqrt{2} + 4)\, \epsilon_3^2 \left(\frac{\nu_{D4} N}{m_k} \right)^{5/2} M_k\,. \end{aligned} \tag{5.3.25}$$

The first solution, with a plus sign, reproduces the free energy of the massive type IIA supergravity solutions of [55] upon setting[28] $\epsilon_3^2\,\nu_{D4}^{\frac{5}{2}} = \frac{64\pi}{\sqrt{n_0}}$. The details about the computation of the free energy of the supergravity solutions are in appendix B.3.

Let us now consider the case of $\mathbb{P}^1\times\mathbb{P}^1$ with independent fluxes. In this case the metric on each $\mathbb{P}^1$ factor is round, but the two radii are different. If we impose the condition $\sum_A V_I^A M_A = 0$ then the fluxes can be parameterized as follows:

$$M_a = (N_1, N_2, N_1, N_2)\,, \qquad M_{d+1} = -2(N_1+N_2)\,. \tag{5.3.26}$$

The flux constraints can then be solved by setting

$$\Lambda_1 = \Lambda_3 = -\frac{\nu_{D4}N_2 + \overline{\lambda}_{d+1}^2}{2\overline{\lambda}_{d+1}}\,, \qquad \Lambda_2 = \Lambda_4 = -\frac{\nu_{D4}N_1 + \overline{\lambda}_{d+1}^2}{2\overline{\lambda}_{d+1}}\,. \tag{5.3.27}$$

The free energy takes the form

$$F = \frac{\epsilon_3^2\overline{\lambda}_{d+1}}{40}\big(\overline{\lambda}_{d+1}^4 - 5\nu_{D4}(N_1+N_2)\overline{\lambda}_{d+1}^2 + 5\nu_{D4}^2 N_1 N_2\big)\,, \tag{5.3.28}$$

and extremizing it with respect to $\overline{\lambda}_{d+1}$ yields four solutions:

$$\begin{aligned} F &= \pm\frac{\epsilon_3^2}{10}\big(\sqrt{8+\mathtt{z}^2} - (2+\mathtt{z}^2)\big)\sqrt{3-\sqrt{8+\mathtt{z}^2}}\,(\nu_{D4}N)^{5/2}\,,\\ F &= \pm\frac{\epsilon_3^2}{10}\big(\sqrt{8+\mathtt{z}^2} + (2+\mathtt{z}^2)\big)\sqrt{3+\sqrt{8+\mathtt{z}^2}}\,(\nu_{D4}N)^{5/2}\,, \end{aligned} \tag{5.3.29}$$

where for convenience we have introduced the parameterization

$$N_1 = (1+\mathtt{z})N, \qquad N_2 = (1-\mathtt{z})N, \qquad |\mathtt{z}| < 1\,. \tag{5.3.30}$$

Once again the first solution, with a plus sign, reproduces the free energy of the supergravity solutions of [55] upon setting $\epsilon_3^2\nu_{D4}^{\frac{5}{2}} = \frac{64\pi}{\sqrt{n_0}}$ (see appendix B.3 for details).

5.3.3 $\mathrm{AdS}_3 \times M_7$ solutions in type IIB

In this section we consider $\mathrm{AdS}_3\times M_7$ solutions in type IIB, where M_7 is an $S^3/\mathbb{Z}_p$ fibration over B_4, which could potentially arise as the near-horizon limit of a system of D3 branes wrapped on a two-cycle in B_4. Explicit solutions of this type have been found in [108, 109] for Kähler-Einstein B_4 or products of Kähler-Einstein spaces. The case of *smooth* Kähler B_4 has been studied in [54] using the formalism of GK geometry and the GMS construction [12]. The orbifold case has not been considered in the literature as of yet, so in this section we take B_4 to be a generic toric orbifold $B_4 \equiv \mathbb{M}_4$ and we also allow a general dependence on all the equivariant parameters,

[28]Notice that the numerical values of ν_{D4} and ϵ_3 here are different from those of the corresponding quantities in the previous section.

including those on the base $\mathbb{M}_4$. As we already discussed, this is important to obtain the correct critical point for generic $\mathbb{M}_4$ without particular symmetries, even in the smooth case. We also hope that our general formulas in terms of four-dimensional integrals will be useful to find a field theory interpretation of these solutions.

With odd-dimensional M_7 we need to add one real dimension, the radial one, as familiar in holography. The relevant CY_4 geometry is given by the fibration with $\mathbb{M}_4$ as the base and the Kähler cone over the Lens space as the fibre, that is $\mathbb{C}^2/\mathbb{Z}_p \hookrightarrow \mathrm{CY}_4 \to \mathbb{M}_4$. This CY_4 is toric and its fan is generated by the vectors

$$V^a = (v^a, 1, \mathfrak{t}_a)\,, \qquad V^{d+1} = (0,0,1,0)\,, \qquad V^{d+2} = (0,0,1,p)\,, \tag{5.3.31}$$

where as usual the vectors v^a generate the fan of $\mathbb{M}_4$, $a = 1, \dots, d$.

Our prescription here reduces to the GMS construction [54], namely

$$F = \mathbb{V}^{(2)}(\lambda_A, \epsilon_I)\,, \qquad \mathbb{V}^{(1)}(\lambda_A, \epsilon_I) = 0\,, \qquad \nu_{D3}\, M_A = -\frac{\partial}{\partial \lambda_A}\mathbb{V}^{(2)}(\lambda_A, \epsilon_I)\,, \tag{5.3.32}$$

where

$$\sum_A V_I^A M_A = 0\,. \tag{5.3.33}$$

Here $\mathbb{V}^{(2)}$ matches the "supersymmetric action" introduced in [12] and we know from [12] that there is no need to use higher times for these solutions. Notice that the second equation in (5.3.32), which is consequence of the third and (5.3.33), is the "topological constraint" in [12].

When $p = 1$ the CY_4 matches exactly the one of sections 5.2.1 and 5.3.1. The equivariant volume is computed in the same manner, with only minor corrections. The one-to-one correspondence between the fixed points of CY_4 and $\mathbb{M}_4$ given by $(V^a, V^{a+1}, V^{d+1}, V^{d+2}) \leftrightarrow (v^a, v^{a+1})$ still holds, but the orders of the orbifold singularities now differ by a factor of p:

$$d_{a,a+1,d+1,d+2} = p\, d_{a,a+1}\,. \tag{5.3.34}$$

The inward normals to the faces of the cone generated by $(V^a, V^{a+1}, V^{d+1}, V^{d+2})$ are now given by

$$\begin{aligned}
U^a &= (p\, u_1^a, 0, 0)\,,\\
U^{a+1} &= (p\, u_2^a, 0, 0)\,,\\
U^{d+1} &= \big((\mathfrak{t}_a - p)u_1^a + (\mathfrak{t}_{a+1} - p)u_2^a\,, p\, d_{a,a+1}, -d_{a,a+1}\big)\,,\\
U^{d+2} &= (-\mathfrak{t}_a u_1^a - \mathfrak{t}_{a+1} u_a^2\,, 0, d_{a,a+1})\,.
\end{aligned} \tag{5.3.35}$$

From these we can derive the restriction of the equivariant Chern forms of CY_4 to the fixed points

$$
\begin{aligned}
c_1^{\mathbb{T}^4}(L_b)|_a &= -(\epsilon_1^a\,\delta_{a,b} + \epsilon_2^a\,\delta_{a+1,b})\,,\\
c_1^{\mathbb{T}^4}(L_{d+1})|_a &= \frac{-(\mathfrak{t}_a - p)\epsilon_1^a - (\mathfrak{t}_{a+1} - p)\epsilon_2^a - p\,\epsilon_3 + \epsilon_4}{p}\,,\\
c_1^{\mathbb{T}^4}(L_{d+2})|_a &= \frac{\mathfrak{t}_a\epsilon_1^a + \mathfrak{t}_{a+1}\epsilon_2^a - \epsilon_4}{p}\,,
\end{aligned}
\tag{5.3.36}
$$

and the respective restrictions to the base $\mathbb{M}_4$

$$
\begin{aligned}
\mathcal{C}_a &= c_1^{\mathbb{T}}(L_a)\,, \qquad a = 1,\dots,d\,,\\
\mathcal{C}_{d+1} &= \frac{-p\,\epsilon_3 + \epsilon_4 + \sum_a (\mathfrak{t}_a - p)c_1^{\mathbb{T}}(L_a)}{p}\,,\\
\mathcal{C}_{d+2} &= \frac{-\epsilon_4 - \sum_a \mathfrak{t}_a c_1^{\mathbb{T}}(L_a)}{p}\,.
\end{aligned}
\tag{5.3.37}
$$

It is easy to verify that these forms satisfy $\sum_a v_i^a\, c_1^{\mathbb{T}}(L_a) = -\epsilon_i$ and $\sum_A V_I^A\,\mathcal{C}_A = -\epsilon_I$.

The second degree homogeneous component of the equivariant volume can be written as an integral on the base $\mathbb{M}_4$ as follows:

$$
\mathbb{V}^{(2)}(\lambda_A\,,\epsilon_I) = \int_{\mathbb{M}_4} \frac{(\tau^{\mathbb{T}})^2}{2\,p\,\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}}\,, \qquad \tau^{\mathbb{T}} = \sum_A \lambda_A\,\mathcal{C}_A\,. \tag{5.3.38}
$$

The flux constraints then read

$$
-\,\nu_{D3}\,M_A = \partial_{\lambda_A}\mathbb{V}^{(2)}(\lambda_A\,,\epsilon_I) = \int_{\mathbb{M}_4} \frac{\mathcal{C}_A\,\tau^{\mathbb{T}}}{p\,\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}} = \sum_a \frac{B_a^{(1)}\cdot(\mathcal{C}_A)|_a}{d_{a,a+1}\,\epsilon_1^a\,\epsilon_2^a}\,, \tag{5.3.39}
$$

where $B_a^{(1)}$ is the restriction to the a-th fixed point of the form

$$
B^{(1)} \equiv \frac{\tau^{\mathbb{T}}}{p\,\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}}\,. \tag{5.3.40}
$$

The solution to equations (5.3.39) takes the following form:

$$
\nu_{D3}^{-1}\,B_a^{(1)} = b(\epsilon_I) - \sum_b m_b\,c_1^{\mathbb{T}}(L_b)|_a\,, \tag{5.3.41}
$$

where the m_b are such that $M_a = \sum_b D_{ab}\,m_b$. Indeed, if we substitute this expression into the right-hand side of (5.3.39) for $A \equiv b \in \{1,\dots,d\}$ we obtain

$$
\sum_a \frac{B_a^{(1)}\cdot(\mathcal{C}_b)|_a}{d_{a,a+1}\,\epsilon_1^a\,\epsilon_2^a} = \nu_{D3}\int_{\mathbb{M}_4}\Big(b(\epsilon_I) - \sum_a m_a\,c_1^{\mathbb{T}}(L_a)\Big)\,\mathcal{C}_b = -\nu_{D3}\sum_a D_{ab}\,m_a\,, \tag{5.3.42}
$$

thus recovering the left-hand side of (5.3.39). When $A = d+1, d+2$ from a similar computation we find that

$$M_{d+1} = \sum_a \frac{(\mathsf{t}_a - p)M_a}{p}\,, \qquad M_{d+2} = -\sum_a \frac{\mathsf{t}_a M_a}{p}\,, \tag{5.3.43}$$

which are precisely the values of M_{d+1} and M_{d+2} necessary to satisfy the relation expected from the fluxes, $\sum_A V_I^A M_A = 0$.

So far we have not specified the value of $b(\epsilon_I)$ in (5.3.41). Using the gauge invariance (4.4.9) we can fix three parameters λ_A. Therefore only $d-1$ of the restrictions of $\tau^{\mathbb{T}}$ to the fixed points are independent, which translates into a relation among the $B_a^{(1)}$ that we use to fix the value of $b(\epsilon_I)$. This can be seen by observing that $\tau^{\mathbb{T}}$ is an equivariant two-form and thus its integral over $\mathbb{M}_4$ vanishes, giving us

$$0 = \int_{\mathbb{M}_4} \tau^{\mathbb{T}} = \int_{\mathbb{M}_4} p\, \mathcal{C}_{d+1}\, \mathcal{C}_{d+2} B^{(1)}\,. \tag{5.3.44}$$

The value of $b(\epsilon_I)$ that satisfies this relation can then be written as

$$b(\epsilon_I) = \frac{\int_{\mathbb{M}_4} \mathcal{C}_{d+1}\, \mathcal{C}_{d+2} \sum_a m_a\, c_1^{\mathbb{T}}(L_a)}{\int_{\mathbb{M}_4} \mathcal{C}_{d+1}\, \mathcal{C}_{d+2}}\,. \tag{5.3.45}$$

We observe that the reason why we had to turn on the higher times in the cases studied in the previous sections was related to the fact that $d-1$ independent parameters were not enough to solve the flux constraints. In the case considered in this section however the $d-1$ independent restrictions of $\tau^{\mathbb{T}} = \sum_A \lambda_A\, \mathcal{C}_A$ are sufficient and there is no necessity to include higher times. Nonetheless, it is interesting to repeat the same computation of this section with the addition of higher times, which we report in appendix B.2.

The free energy is given by the second degree homogeneous component of the equivariant volume, which we write as

$$F = \mathbb{V}^{(2)}(\lambda_A\,, \epsilon_I) = \int_{\mathbb{M}_4} \frac{(\tau^{\mathbb{T}})^2}{2\, p\, \mathcal{C}_{d+1}\, \mathcal{C}_{d+2}} = \sum_a \frac{B_a^{(2)}}{d_{a,a+1}\, \epsilon_1^a\, \epsilon_2^a}\,, \tag{5.3.46}$$

where the $B_a^{(2)}$ are the restrictions of the integrand to each fixed point. The value of the Kähler moduli, and consequently of the $B_a^{(2)}$, is fixed by the flux constraints. We can easily do this by employing the same strategy as formula (5.2.16) to relate the $B_a^{(2)}$ to the $B_a^{(1)}$:

$$\begin{aligned} B_a^{(2)} &\equiv \left(\frac{\left(\tau^{\mathbb{T}}\right)^2}{2\, p\, \mathcal{C}_{d+1}\, \mathcal{C}_{d+2}} \right)\Big|_a = \frac{p}{2} \left(B_a^{(1)}\right)^2 \left(\mathcal{C}_{d+1}\, \mathcal{C}_{d+2}\right)|_a \\ &= \frac{\nu_{D3}^2}{2p} \left(b(\epsilon_I) + m_a \epsilon_1^a + m_{a+1}\epsilon_2^a\right)^2 \left(p\epsilon_3 - \epsilon_4 + (\mathsf{t}_a - p)\epsilon_1^a + (\mathsf{t}_{a+1} - p)\epsilon_2^a\right)\left(\epsilon_4 - \mathsf{t}_a \epsilon_1^a - \mathsf{t}_{a+1}\epsilon_2^a\right). \end{aligned} \tag{5.3.47}$$

We can also write the free energy as an integral over $\mathbb{M}_4$ as follows:

$$F = \frac{p}{2}\,\nu_{D3}^2 \int_{\mathbb{M}_4} \Big(b(\epsilon_I) - \sum_a m_a\, c_1^{\mathbb{T}}(L_a)\Big)^2\, \mathcal{C}_{d+1}\, \mathcal{C}_{d+2}\,. \tag{5.3.48}$$

Notice that the equation that fixes $b(\epsilon_I)$ can be written as

$$\int_{\mathbb{M}_4} \Big(b(\epsilon_I) - \sum_a m_a\, c_1^{\mathbb{T}}(L_a)\Big)\, \mathcal{C}_{d+1}\, \mathcal{C}_{d+2} = 0 \tag{5.3.49}$$

and can be used to further rewrite the free energy as

$$F = -\frac{p}{2}\,\nu_{D3}^2 \int_{\mathbb{M}_4} \sum_a m_a\, c_1^{\mathbb{T}}(L_a)\Big(b(\epsilon_I) - \sum_a m_a\, c_1^{\mathbb{T}}(L_a)\Big)\, \mathcal{C}_{d+1}\, \mathcal{C}_{d+2}\,. \tag{5.3.50}$$

It would be very interesting to understand if our formulas can be written as the integral of the anomaly polynomial for some D3 brane theory wrapped over a two-cycle in $\mathbb{M}_4$ and thus providing a field theory interpretation of the solution.

5.3.3.1 Examples: Kähler-Einstein and Hirzebruch surfaces

We can check that our general formalism reproduces the know expressions for the toric cases $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and dP_3 with equal fluxes. The fan and intersection matrix are given in (5.2.72) and (5.2.73). We take all the $M_a \equiv M$, $\mathfrak{t}_a \equiv \mathfrak{t}$ and $m_a \equiv m$ equal. We find $\sum_{ab} D_{ab} = d m_k$ and $M = m m_k$ with $m_k = 3, 2, 1$ for $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and dP_3, respectively. Since $\sum_{abc} D_{abc} = 0$, there is no linear term in $\epsilon_{1,2}$ in $\mathbb{V}^{(2)}$ which is extremized at $\epsilon_{1,2} = 0$. By expanding (5.3.49) in integrals of Chern classes we find

$$b = \frac{m[(p\epsilon_3 - \epsilon_4)\mathfrak{t} + \epsilon_4(p - \mathfrak{t})]}{\mathfrak{t}(p-\mathfrak{t})} \tag{5.3.51}$$

and

$$\mathbb{V}^{(2)} = -\nu_{D3}^2 \frac{d M^2[\epsilon_3\epsilon_4 p(p-3\mathfrak{t})\mathfrak{t} + \epsilon_3^2 p^2\mathfrak{t}^2 + \epsilon_4^2(p^2 - 3p\mathfrak{t} + 3\mathfrak{t}^2)]}{2 p m_k (p-\mathfrak{t})\mathfrak{t}}\,, \tag{5.3.52}$$

which reproduces formula (5.6) in [54] with $\epsilon_4 = \epsilon_3 b_2/2$.[29] This still needs to be extremized with respect to b_2.

As we already discussed in section 5.2.2, the critical point is generically at a non-zero value of ϵ_1 and ϵ_2, unless there is some extra symmetry in the background and the fluxes. As an example where the critical point is not at $\epsilon_1 = \epsilon_2 = 0$ we consider the case of the Hirzebruch surface $\mathbb{M}_4 = \mathbb{F}_k$ with fan

$$v^1 = (1,0)\,,\qquad v^2 = (-k,1)\,,\qquad v^3 = (-1,0)\,,\qquad v^4 = (0,-1)\,. \tag{5.3.53}$$

[29] In [54] N is the flux of the five-cycle fibred over c_1/m_k in $\mathbb{M}_4$. To compare the formulas we need to identify $M = \frac{m_k N}{d}$ and $\mathfrak{t} = \frac{n_2}{m_k}$, which follows from (5.5). The formulas match for $\epsilon_3\nu_{D3} = 2\sqrt{6}$.

The constraint $\sum_A V_I^A M_A = 0$ leaves two independent fluxes on the base $\mathbb{M}_4$ and two fluxes associated with the fibre

$$
\begin{aligned}
M_3 &= M_1 - k\,M_2\,, \qquad M_4 = M_2\,, \\
M_5 &= \frac{M_1(\mathtt{t}_1+\mathtt{t}_3-2p)+M_2(pk-2p+\mathtt{t}_2+\mathtt{t}_4-k\mathtt{t}_3)}{p}\,, \\
M_6 &= -\frac{M_1(\mathtt{t}_1+\mathtt{t}_3)+M_2(\mathtt{t}_2+\mathtt{t}_4-k\mathtt{t}_3)}{p}\,.
\end{aligned}
\tag{5.3.54}
$$

The vectors of the fan and the fluxes have a symmetry between the second and fourth entry, and therefore we expect that one of ϵ_i will be zero at the critical point. Notice also that the physical fluxes depends only on two linear combinations of the $\mathtt{t}_a$. These are the combinations invariant under

$$
\mathtt{t}_a \to \mathtt{t}_a + \sum_{i=1}^{2} \beta_i v_i^a\,. \tag{5.3.55}
$$

In the free energy (5.3.50) this transformation can be reabsorbed in a redefinition of ϵ_4 using $\sum_a v_i^a c_1(L_a)^{\mathbb{T}} = -\epsilon_i$ and therefore the central charge depends only on the physical fluxes. We also solve $M_a = \sum_b D_{ab} m_b$, for example, by $m_a = (0,0,M_2,M_1)$.[30] The constraint (5.3.49) and the free energy (5.3.50) can be expanded in a series of integral of Chern classes and expressed in terms of the intersections $D_{a_1\dots a_k}$, which are homogeneous of order k in ϵ_i. We see then that b and the free energy are homogeneous of degree one and two in all the ϵ_I, respectively. One can check explicitly that F is extremized at $\epsilon_2 = 0$. The expressions are too lengthy to be reported so, for simplicity, we restrict to the case $M_2 = M_1$. We also fix $\mathtt{t}_a = (\mathtt{t}_1,\mathtt{t}_2,\mathtt{t}_1,\mathtt{t}_2)$ using (5.3.55) for convenience. The free energy restricted to $\epsilon_2 = 0$ reads

$$
F = -\frac{\nu_{D3}^2 M_1^2 \mathcal{A}}{8p(p(\mathtt{t}_1+\mathtt{t}_2)-2\mathtt{t}_1\mathtt{t}_2)}\,, \tag{5.3.56}
$$

where

$$
\begin{aligned}
\mathcal{A} &= \epsilon_3^2 p^2[(k-2)\mathtt{t}_1-2\mathtt{t}_2]^2 + 2\epsilon_3\epsilon_4 p[(k-2)^2(p-2\mathtt{t}_1)\mathtt{t}_1+4(p+(k-2)\mathtt{t}_1)\mathtt{t}_2-8\mathtt{t}_2^2] \\
&+ \epsilon_4^2[(k-4)^2p^2-4(k-3)(k-2)p\mathtt{t}_1+4(k-2)^2\mathtt{t}_1^2+4(k-6)p\mathtt{t}_2-8(k-2)\mathtt{t}_1\mathtt{t}_2+16\mathtt{t}_2^2] \\
&+ 4\epsilon_1 k[\epsilon_3 p\mathtt{t}_1((k-1)p\mathtt{t}_1-(k-2)\mathtt{t}_1^2-p\mathtt{t}_2)+\epsilon_4(-3(k-2)p\mathtt{t}_1^2+2(k-2)\mathtt{t}_1^3+p^2((k-3)\mathtt{t}_1+\mathtt{t}_2))] \\
&+ 4\epsilon_1^2[p^2((k^2+k-3)\mathtt{t}_1^2+(k-4)\mathtt{t}_1\mathtt{t}_2-\mathtt{t}_2^2)+p\mathtt{t}_1(-(2k^2+k-2)\mathtt{t}_1^2+(10-3k)\mathtt{t}_1\mathtt{t}_2+4\mathtt{t}_2^2)] \\
&+ \epsilon_1^2\mathtt{t}_1^2[k^2\mathtt{t}_1^2+(2k-4)\mathtt{t}_1\mathtt{t}_2-4\mathtt{t}_2^2]\,,
\end{aligned}
\tag{5.3.57}
$$

which should be still extremized with respect to ϵ_1 and ϵ_4. One easily sees that the critical point is at a non-zero value of ϵ_1. This rectifies a result given in [54] where

[30] Any other choice would be equivalent. The equation $M_a = \sum_b D_{ab} m_b$ is invariant under $m_a \to m_a + \sum_{i=1}^2 \gamma_i v_i^a$. This ambiguity can be reabsorbed in a shift of b in (5.3.48).

it was assumed that the R-symmetry does not mix with the isometries of $\mathbb{F}_k$. The expression (5.3.56) for $k = 0$ is extremized at $\epsilon_1 = 0$ and it correctly reduces to the $\mathbb{P}^1 \times \mathbb{P}^1$ result (5.3.52) setting $\epsilon_1 = 0$ and $\mathfrak{t}_1 = \mathfrak{t}_2$.

5.4 Summary and discussion

In this chapter we have refined the proposal of [28], that the geometry of an extensive class of supersymmetric solutions is captured by a universal quantity, depending only on the topology of the internal space and equivariant parameters associated with the expected symmetries of the solutions. This quantity is an extension of the equivariant volume, familiar from symplectic geometry, where we have introduced additional moduli dubbed higher times, which are necessary to parameterize all the fluxes supported by a given topology. Although we have assumed from the outset that the spaces of interest are toric, we have indicated that this assumption may be relaxed by considering for example "non-convex" geometries as well as configurations including a four-sphere, that are not toric geometries in the strict mathematical sense. It is also possible to extend our construction to geometries with a number of expected abelian symmetries which is strictly less than half of the real dimension[31] of the manifold/orbifold (or cone over it). It is well known that in many situations the metric on the internal space (or the cone over it, in the odd-dimensional case) solving the supersymmetry equations may not be compatible with a Kähler or even symplectic structure. Nevertheless, the equivariant volume is a robust topological quantity, insensitive to the details of the metrics. Indeed, it may be regarded as a gravitational analogue of anomalies in quantum field theory. In all cases that we have analysed, we extract an extremal function from the equivariant volume and our prescription for fixing the parameters on which it depends consists of extremizing over *all* the parameters that are left undetermined by the flux quantization conditions. This is consistent with the logic in the case of GK geometry [11] and indeed it is analogous to the paradigm of a-maximization in field theory [4]. Geometrically, the existence of critical points to the various extremal functions that we proposed may be interpreted as providing necessary conditions to the existence of the corresponding supergravity solutions and indeed it would be very interesting to study when such conditions are also sufficient. In any case, if we assume that a solution exists, then our method calculates the relevant observables, yielding non-trivial predictions for the holographically dual field theories. It is worth emphasizing that in the procedure of extremization one should allow all the equivariant parameters not fixed by symmetries to vary, otherwise it is not guaranteed that the critical point found will be a true extremum of the gravitational action. We have demonstrated this point in a number of explicit examples discussed in section 5.2.2.3 as well as section 5.3.3.1.

[31] The main difference is that in these cases the localization formula involves fixed point sets that are not isolated points.

In this chapter we have focussed on setups involving internal geometries that are fibrations over four-dimensional orbifolds $\mathbb{M}_4$, that may be interpreted as arising from branes wrapping completely or partially $\mathbb{M}_4$. For example, the case of M5 branes completely wrapped on $\mathbb{M}_4$ yields a proof of the gravitational block form of the trial central charge, conjectured in [85] (and derived in the field theory side in [28]). The case of M5 branes partially wrapped on a two-cycle inside $\mathbb{M}_4$ is still poorly understood from the field theory side, the best understood setup being the case of $\mathbb{M}_4 = \Sigma_g \times S^2$, where Σ_g is Riemann surface of genus g [104]. The full internal space M_6 may then be viewed also as the fibration of the second Hirzebruch surface $\mathbb{F}_2 \simeq S^2 \times S^2$ over the Riemann surface Σ_g, and interpreted as the backreaction of a stack of M5 branes at a (resolved) $\mathbb{C}^2/\mathbb{Z}_2$ singularity, further wrapped on Σ_g, yielding insights about the dual four-dimensional field theories. In section 5.2.2.3 we have discussed the example of $\mathbb{M}_4 = \mathbb{\Sigma} \times S^2$, corresponding to M5 branes probing a $\mathbb{C}^2/\mathbb{Z}_2$ singularity, further wrapped on a spindle $\mathbb{\Sigma}$ and it would be interesting to confirm our predictions with a field-theoretic computation. It would also be nice to extend the methods of [104] for computing anomalies to setups where the M5 branes wrap a two-cycle with non-trivial normal bundle in an $\mathbb{M}_4$.

In the context of type IIA supergravity, we have analysed the case of D4 branes completely wrapped on a general toric four-orbifold $\mathbb{M}_4$, proving the gravitational block form of the entropy function conjectured in [85]. It would be very interesting to reproduce this from a field theory calculation of the partition function of five-dimensional SCFTs on $S^1 \times \mathbb{M}_4$, employing the method of [110] for performing localization on orbifolds. We have also analysed the case of D4 branes partially wrapped on a two-cycle inside $\mathbb{M}_4$, providing a dual field theoretic proposal for a class of solutions to massive type IIA supergravity, constructed in [55]. Finally, we have also discussed the case of D3 branes partially wrapped on a two-cycle inside $\mathbb{M}_4$, corresponding to type IIB geometries of the form $\mathrm{AdS}_3 \times M_7$, making contact with the framework of fibred GK geometries studied in [54]. In particular, we have improved some of the results previously obtained in [54], by revisiting some of the examples discussed there. In this chapter we have not discussed geometries associated to M2 and D2 branes (already briefly mentioned in [28]), which are not naturally related to four-dimensional orbifolds $\mathbb{M}_4$, but we expect that for these our method will generalize straightforwardly. It would be very interesting to incorporate new classes of supersymmetric geometries in our framework, such as for example $\mathrm{AdS}_2 \times M_8$ in type IIB in order to study entropy functions of AdS_5 black holes. It is tantalizing to speculate that our approach may be eventually extended to include geometries that do not necessarily contain AdS factors.

Chapter 6

Equivariant volume and the Molien-Weyl formula

In this chapter we discuss the Molien-Weyl formulation of the equivariant volume. The Molien-Weyl formula is closely related to the construction of toric orbifolds as symplectic quotients, which we reviewed in subsection 4.2.3. A notable feature is how it depends on the independent Kähler moduli of the orbifold, instead of an over-parametrization thereof. The price to pay is that it requires instead an over-parametrization of the equivariant parameters. Because of these differences, it is interesting to see how the prescription that we presented in chapter 5 can be reformulated in terms of the Molien-Weyl formula: we will do so by focusing on the case of $\mathrm{AdS}_3 \times M_8$ solutions. Furthermore, another reason to study the Molien-Weyl formula is the fact that it could help better understand the "quantum" analogue of the equivariant volume [28, 93, 111], the orbifold index, which is expected to be a building block for partition functions on orbifolds [110].

This chapter is organized as follows. First, in section 6.1 we review definition of the Molien-Weyl formula and provide proofs of its relation to the "standard" formulation of the equivariant volume, including a discussion about the correspondence between residues and fixed points. Then in section 6.2 we revisit the $\mathrm{AdS}_3 \times M_8$ solutions in M theory of subsection 5.2.1, reformulating the prescription of chapter 5 in terms of the Molien-Weyl formula. At last we conclude in section 6.3 with a brief discussion of interesting possible directions of future research.

6.1 The Molien-Weyl formula

In this section we review the key properties of the Molien-Weyl formulation of the equivariant volume, including its version with higher times [93, 111]. In subsection 6.1.1 we review how the Molien-Weyl formula can be derived from the construction of toric orbifolds as symplectic quotients that we reviewed in subsection 4.2.3. The in subsection 6.1.2 we provide a new derivation starting from the formula of the equivariant volume as an integral over the polytope. Then in subsection 6.1.3 we prove the relation between the two formulations of the equivariant index with higher times. At last in subsection 6.1.4 we discuss the correspondence between fixed points and residues of the Molien-Weyl formula [111].

For simplicity throughout this whole chapter we will imply Einstein notation for any repeated index, and we will use the following conventions for the indices of a $(2\,\mathfrak{m})$-dimensional toric orbifold $\mathbb{M}_{2\mathfrak{m}}$ with fan generated by the vectors $\{v^a\}_{a=1}^d$:

- The indices $a, b, ...$ run from 1 to d.
- The indices $i, j, ...$ run from 1 to $\mathfrak{m}$.
- The indices $m, n, ...$ run from 1 to $r = d - \mathfrak{m}$.

Let Q_a^m be the GLSM charges (4.2.34), which by definition are integers that satisfy $Q_a^m\, v_i^a = 0$. The Molien-Weyl formula re-expresses the equivariant volume in terms of new variables t^m and $\bar{\epsilon}_a$ as [93]

$$\mathbb{V}_{MW}(t^m, \bar{\epsilon}_a) = \frac{1}{|\Gamma|} \int_{(i\mathbb{R})^r} \prod_{m=1}^{r} \frac{d\phi_m}{2\pi i} \frac{e^{t^m \phi_m}}{\prod_{a=1}^d (\bar{\epsilon}_a + Q_a^m \phi_m)} \,, \tag{6.1.1}$$

where $|\Gamma|$ is the order of the torsion group, defined by (4.2.35). The relation between the Molien-Weyl formulation and the "standard" presentation of the equivariant volume is condensed in the formula [28]

$$\mathbb{V}(\lambda_a, \epsilon_i = v_i^a \bar{\epsilon}_a) = e^{-\lambda_a \bar{\epsilon}_a}\, \mathbb{V}_{MW}(t^m = -Q_a^m \lambda_a, \bar{\epsilon}_a) \,. \tag{6.1.2}$$

The variables $t^m = -Q_a^m \lambda_a$ parametrize the Kähler moduli of $\mathbb{M}_{2\mathfrak{m}}$. Indeed from relations (4.2.26) and (4.2.18),

$$[\omega] = -2\pi \lambda_a\, c_1(L_a) \,, \quad v_i^a\, c_1(L_a) = 0 \,, \tag{6.1.3}$$

we see that a shift of the form $\lambda_a \to \lambda_a + v_i^a \beta_a$ does not change the cohomology class $[\omega]$, which makes the λ_a an over-parametrization of the Kähler moduli. The t^m are unchanged by such shifts, and are thus a proper parametrization of the moduli. The trade-off of the Molien-Weyl formulation is that the equivariant parameters are now over-parametrized. As we will explain in more detail in subsection 6.1.1, the $\bar{\epsilon}_a$ parametrize the Lie algebra of the torus $\mathbb{T}^d$ in the construction of $\mathbb{M}_{2\mathfrak{m}}$ as a symplectic quotient. Since the $\bar{\epsilon}_a$ are over-parametrized, it is natural to expect $\mathbb{V}_{MW}$ to obey a simple formula for shifts in $\bar{\epsilon}_a$ that don't affect $\epsilon_i = v_i^a \bar{\epsilon}_a$, in analogy with (4.3.9). Indeed, we have that

$$\mathbb{V}_{MW}(t^m, \bar{\epsilon}_a + Q_a^m) = e^{-t^m \alpha_m}\, \mathbb{V}_{MW}(t^m, \bar{\epsilon}_a) \,, \tag{6.1.4}$$

as can be easily seen from either (6.1.1) or (6.1.2).

The Molien-Weyl integral can be computed by means of the residue theorem, provided that we close the contour of integration first. The contour should depend of the value of the Kähler moduli t^m, which determine the behavior at infinity of the integrand. In [93] it was proposed that the criterion to determine which poles of the integrand should be inside the contour and which ones outside should be based on the Jeffery-Kirwan prescription [112]. When the contour is closed this way we write it as

$$\mathbb{V}_{MW}(t^m, \bar{\epsilon}_a) = \frac{1}{|\Gamma|} \int_{JK} \prod_{m=1}^{r} \frac{d\phi_m}{2\pi i} \frac{e^{t^m \phi_m}}{\prod_{a=1}^{d} (\bar{\epsilon}_a + Q_a^m \phi_m)} \,. \tag{6.1.5}$$

The Jeffery-Kirwan prescription states that the poles inside the contour are those for which the corresponding set of indices $(a_1, \ldots, a_r)$ of terms in the denominator that diverge satisfies the property

$$\{\alpha_1 \, Q_{a_1} + \ldots + \alpha_r \, Q_{a_r} \in \mathbb{R}^r \mid \alpha_m > 0\} \subset \mathfrak{C}(t^m) \,. \tag{6.1.6}$$

Here $\mathfrak{C}(t^m)$ denotes the chamber of regular values of the function $\mathbb{C}^d \to \mathbb{R}^r$, $(z_a)_{a=1}^d \mapsto \frac{1}{2} \, Q_a^m |z_a|^2$, such that $t^m \in \mathfrak{C}(t^m)$.[1]

The Molien-Weyl formula can be generalized to include higher times as follows [93]:

$$\mathbb{V}_{MW}(\{t^{m_1 \ldots m_k}\}_{k=1}^K, \bar{\epsilon}_a) = \frac{1}{|\Gamma|} \int_{JK} \prod_{m=1}^{r} \frac{d\phi_m}{2\pi i} \frac{e^{\sum_{k=1}^K t^{m_1 \ldots m_k} \phi_{m_1} \ldots \phi_{m_k}}}{\prod_{a=1}^{d} (\bar{\epsilon}_a + Q_a^m \phi_m)} \,. \tag{6.1.7}$$

Notice that we are keeping the contour closed to the Jeffrey-Kirwan contour. The criterion to determine which poles are inside the contour and which are outside is the still the same, and is unaffected by the value of the higher times $t^{m_1, \ldots, m_k}$ with $k \geq 2$. This choice of contour ensures that the Molien-Weyl formula with higher times can formally be derived from the one with just single times by solving a system of differential equations, just like in standard formulation of $\mathbb{V}$. As an example, the following system can be used to define the Molien-Weyl formula with second times:

$$\begin{cases} \partial_{t^{mn}} \mathbb{V}_{MW}(t^m, t^{mn}, \bar{\epsilon}_a) = (2 - \delta_{mn}) \partial_{t^m} \partial_{t^n} \mathbb{V}_{MW}(t^m, t^{mn}, \bar{\epsilon}_a) \,, \\ \mathbb{V}_{MW}(t^m, t^{mn} = 0, \bar{\epsilon}_a) = \mathbb{V}_{MW}(t^m, \bar{\epsilon}_a) \,. \end{cases} \tag{6.1.8}$$

In subsection 6.1.3 we will derive the relation between (6.1.7) and the standard formulation of the equivariant volume with higher times:

$$\mathbb{V}(\{\lambda_{a_1 \ldots a_k}\}_{k=1}^K, \epsilon_i = v_i^a \bar{\epsilon}_a) = e^{\sum_{k=1}^K (-1)^k \lambda_{a_1 \ldots a_k} \bar{\epsilon}_{a_1} \cdots \bar{\epsilon}_{a_k}} \cdot \\ \cdot \mathbb{V}_{MW}\left(\left\{ t^{m_1 \ldots m_k} = \sum_{s=k}^{K} (-1)^s \binom{s}{s-k} Q_{a_1}^{m_1} \ldots Q_{a_k}^{m_k} \, \bar{\epsilon}_{a_{k+1}} \ldots \bar{\epsilon}_{a_s} \lambda_{a_1 \ldots a_s} \right\}_{k=1}^{K}, \bar{\epsilon}_a \right) , \tag{6.1.9}$$

[1] The moment map μ_G (4.2.39) can be written in terms of t^m as $\mu_G(z, \bar{z}) = \frac{1}{2} Q_a^m |Z_a|^2 - t^m$. The orbifold $\mathbb{M}_{2\mathfrak{m}}$ can be constructed as $\mathbb{M}_{2\mathfrak{m}} = \mu_G^{-1}(0)/G$. Zero is a regular value of μ_G if and only if t^m is a regular value of the function $(z_a)_{a=1}^d \mapsto \frac{1}{2} \, Q_a^m |z_a|^2$.

which is valid as long as the above value of t^m is in the same chamber of regular values as $t_0^m \equiv -Q_a^m \lambda_a$.

The formula (6.1.4) for shifts in $\bar{\epsilon}_a$ generalizes to

$$\begin{aligned} &\mathbb{V}_{MW}\left(\left\{t^{m_1 \dots m_k} = \sum_{s=k}^{K} \binom{s}{s-k} t^{m_1 \dots m_s}\, \alpha_{m_{k+1}} \dots \alpha_{m_s}\right\}_{k=1}^{K}, \bar{\epsilon}_a + Q_a^m \alpha_m\right) = \\ &\quad = e^{-\sum_{k=1}^{K} t^{m_1 \dots m_k} \alpha_{m_1} \dots \alpha_{m_k}}\; \mathbb{V}_{MW}(\{t^{m_1 \dots m_k}\}_{k=1}^{K}, \bar{\epsilon}_a)\,, \end{aligned} \tag{6.1.10}$$

where now $t^{m_1, \dots, m_k}$ also gets shifted, with the exception of the highest component, $k = K$. The above formula is only valid as long as the t^m do not get shifted outside their chamber of regular values.

6.1.1 Derivation by means of symplectic reduction

In this subsection we review the derivation of the Molien-Weyl formula from the symplectic quotient construction of a toric orbifold, which we reviewed in subsection 4.2.3. This will provide a first proof of equation (6.1.2) and a geometrical interpretation of the Molien-Weyl formula (6.1.1). In the next subsection we will provide a much more direct proof of (6.1.2). In this subsection we follow the discussion of [93], re-adapted to our notations and expanded to make contact with the standard formulation of the equivariant volume.

Our starting point is the representation of the toric orbifold $\mathbb{M}_{2\mathfrak{m}}$ as the quotient (4.2.42) :

$$\mathbb{M}_{2\mathfrak{m}} = \mu_G^{-1}(0) \,/\, G\,. \tag{6.1.11}$$

We remind that G is a subgroup of the torus $\mathbb{T}^d$ that can be factorized (4.2.35) as the direct sum of a finite group Γ and a continuos component:

$$G = \Gamma \,\oplus\, \left\{ \left(e^{2\pi i Q_1^m \theta_m}, \dots, e^{2\pi i Q_d^m \theta_m}\right) \in \mathbb{T}^d \;\middle|\; \theta_1, \dots, \theta_{d-\mathfrak{m}} \in \mathbb{R} \right\}, \tag{6.1.12}$$

whereas $\mu_G^{-1}(0)$ is a subset of $\mathbb{C}^d$ (4.2.41) that can be expressed in terms of the Kähler moduli $t^m = -Q_a^m \lambda_a$ as

$$\mu_G^{-1}(0) = \left\{ z \in \mathbb{C}^d \;\middle|\; \frac{1}{2} Q_a^m \, |z_a|^2 = t^m\,,\; m = 1, \dots, d - \mathfrak{m} \right\}, \tag{6.1.13}$$

The Hamiltonian function in $\mathbb{C}^d$ of the vector $\bar{\epsilon}_a \partial_{\varphi_a}$ (φ_a is the phase of z_a) is expressed in terms of the moment maps (4.2.37) as follows:

$$H(\bar{\epsilon}_a) = \bar{\epsilon}_a \, \mu_0^a(z, \bar{z}) = \frac{1}{2} \bar{\epsilon}_a \, |z_a|^2 + \bar{\epsilon}_a \lambda_a\,. \tag{6.1.14}$$

The above Hamiltonian on $\mathbb{C}^d$ defines an Hamiltonian $H(\epsilon_i)$ on $\mathbb{M}_{2\mathfrak{m}}$, which is associated to the vector $\epsilon_i \partial_{\phi_i}$ with $\epsilon_i = v_i^a \bar{\epsilon}_a$.[2] With a little abuse of language we can

[2] Indeed the vectors associated to the Lie algebra of G are of the form $\alpha_m Q_a^m \partial_{\varphi_a}$ and are projected to the null vector in $\mathbb{M}_{2\mathfrak{m}}$. Since $v_i^a \, Q_a^m = 0$, we can see that this projection is done by v_i^a.

write $H(\bar{\epsilon}_a) = H(\epsilon_i = v_i^a \bar{\epsilon}_a)$. Since Hamiltonians are defined up to a constant, we can remove the dependence of (6.1.14) from λ_a by defining

$$\bar{H}(\bar{\epsilon}_a) = \frac{1}{2}\, \bar{\epsilon}_a\, |z_a|^2\,. \tag{6.1.15}$$

If we use $\bar{H}$ instead of H in the definition (4.3.1) of the equivariant volume we obtain a function of t^m and $\bar{\epsilon}_a$:

$$\mathbb{V}_{MW}(t^m, \bar{\epsilon}_a) = \frac{1}{(2\pi)^{\mathfrak{m}}} \int_{\mu_G^{-1}(0)\,/\,G} \mathrm{e}^{-\bar{H}(\bar{\epsilon}_a)}\, \frac{\omega^{\mathfrak{m}}}{\mathfrak{m}!}\,. \tag{6.1.16}$$

In the following we will show that the above quantity is exactly equal to the Molien-Weyl integral (6.1.1). Equation (6.1.2) should now be clear: the factor $\exp(-\lambda_a \bar{\epsilon}_a)$ that appears when we relate $\mathbb{V}$ and $\mathbb{V}_{MW}$ simply comes from the constant term by which the two Hamiltonians differ: $H(\epsilon_i = v_i^a \bar{\epsilon}_a) = \bar{H}(\bar{\epsilon}_a) + \bar{\epsilon}_a \lambda_a$.

We can rewrite (6.1.16) as an integral over $\mathbb{C}^d$:

$$\mathbb{V}_{MW}(t^m, \bar{\epsilon}_a) = \frac{1}{(2\pi)^d |\Gamma|} \int_{\mathbb{C}^d} \mathrm{d}z\, \mathrm{d}\bar{z}\, \mathrm{e}^{-\bar{H}(\bar{\epsilon}_a)} \prod_{m=1}^{r} \delta\Big(\frac{1}{2}\, Q_a^m\, |z_a|^2 - t^m \Big)\,, \tag{6.1.17}$$

where the delta functions restrict us to $\mu_G^{-1}(0)$ and the extra factors of 2π and $|\Gamma|$ at the denominator compensate the fact that $\mathbb{M}_{2\mathfrak{m}}$ is $\mu_G^{-1}(0)$ quotiented by G. Using the integral representation of the delta function

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\varphi\, \mathrm{e}^{-ix\varphi} = \frac{1}{2\pi i} \int_{i\mathbb{R}} \mathrm{d}\phi\, \mathrm{e}^{-x\phi} \tag{6.1.18}$$

we obtain the desired equivalence between (6.1.16) and (6.1.1) :

$$\begin{aligned} \mathbb{V}_{MW}(t^m, \bar{\epsilon}_a) &= \frac{1}{(2\pi)^d |\Gamma|} \int_{\mathbb{C}^d} \mathrm{d}z\, \mathrm{d}\bar{z} \int_{(i\mathbb{R})^r} \prod_{m=1}^{r} \frac{\mathrm{d}\phi_m}{2\pi i}\, \exp\Big(t^m \phi_m - \frac{|z_a|^2}{2} (\bar{\epsilon}_a + Q_a^m \phi_m) \Big) = \\ &= \frac{1}{|\Gamma|} \int_{(i\mathbb{R})^r} \prod_{m=1}^{r} \frac{\mathrm{d}\phi_m}{2\pi i}\, \frac{e^{t^m \phi_m}}{\prod_{a=1}^{d} (\bar{\epsilon}_a + Q_a^m \phi_m)}\,. \end{aligned} \tag{6.1.19}$$

6.1.2 Direct derivation

In this subsection we give a direct proof of equation (6.1.2), which relates the standard formulation of the equivariant volume and its Molien-Weyl formula counterpart in the absence of higher times.

Our starting point is the formula (4.3.2) for the equivariant volume as an integral over the polytope, which we write explicitly using Heaviside theta functions:

$$\mathbb{V}(\lambda_a, \epsilon_i) = \int_{\mathcal{P}} \mathrm{d}^{\mathfrak{m}} y\, e^{-y_i \epsilon_i} = \int_{\mathbb{R}^{\mathfrak{m}}} \mathrm{d}^{\mathfrak{m}} y\, e^{-y_i \epsilon_i} \prod_{a=1}^{d} \theta(v_i^a y_i - \lambda_a)\,. \tag{6.1.20}$$

Now we can use the integral representation of the theta function

$$\theta(x) = \lim_{\eta \to 0+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \mathrm{d}\tau \, \frac{e^{ix\tau}}{\tau - i\eta} \tag{6.1.21}$$

to write the equivariant volume as

$$\mathbb{V}(\lambda_a, \epsilon_i) = \lim_{\eta_a \to 0+} \frac{1}{(2\pi i)^d} \int_{\mathbb{R}^d} \mathrm{d}^d\tau \int_{\mathbb{R}^\mathfrak{m}} \mathrm{d}^\mathfrak{m} y \, \frac{e^{i(v_i^a y_i - \lambda_a)\tau_a - y_i \epsilon_i}}{\prod_{a=1}^d (\tau_a - i\eta_a)} \,. \tag{6.1.22}$$

Let us introduce the new variables $\bar{\epsilon}_a$, which are related to the ϵ_i by $\epsilon_i = v_i^a \bar{\epsilon}_a$. The $\bar{\epsilon}_a$ are not uniquely defined, since shifting them by $\bar{\epsilon}_a \longrightarrow \bar{\epsilon}_a + Q_a^m \alpha_m$ does not affect the relation $\epsilon_i = v_i^a \bar{\epsilon}_a$. It is always possible to take $\bar{\epsilon}_a \geq 0$ without loss of generality.[3] We can then shift $\tau_a \longrightarrow \tau_a - i\,\bar{\epsilon}_a$ without making the integration contour cross any poles. We find

$$\mathbb{V}(\lambda_a, \epsilon_i = v_i^a \bar{\epsilon}_a) = \frac{e^{-\lambda_a \bar{\epsilon}_a}}{(2\pi i)^d} \int_{\mathbb{R}^d} \mathrm{d}^d\tau \int_{\mathbb{R}^\mathfrak{m}} \mathrm{d}^\mathfrak{m} y \, \frac{e^{i(v_i^a y_i - \lambda_a)\tau_a}}{\prod_{a=1}^d (\tau_a - i\,\bar{\epsilon}_a)} \,, \tag{6.1.23}$$

where we have taken the $\eta_a \to 0$ limit since the η_a are no longer needed for the convergence of the integral. Now the integral over y_i simply yields a product of delta functions, leaving us with

$$\mathbb{V}(\lambda_a, \epsilon_i = v_i^a \bar{\epsilon}_a) = \frac{e^{-\lambda_a \bar{\epsilon}_a}}{(2\pi)^r} \int_{\mathbb{R}^d} \mathrm{d}^d\tau \, \frac{e^{-i\lambda_a \tau_a}}{\prod_{a=1}^d (\,\bar{\epsilon}_a + i\,\tau_a)} \prod_{i=1}^{\mathfrak{m}} \delta(v_i^a \tau_a) \,. \tag{6.1.24}$$

In order to integrate out the delta functions it is convenient to change variables:

$$\tau_a = -i\, Q_a^m \phi_m + P_a^i \tau_i \,, \tag{6.1.25}$$

where the matrix P_a^i is chosen so that $P_a^i\, v_j^a = \delta_j^i$. We note that such a matrix is not uniquely defined: we could send $P_a^i \longrightarrow P_a^i + \alpha_m Q_a^m$ and the contraction with v_j^a would still yield δ_j^i. The delta functions in (6.1.24) are now simply $\delta(v_i^a \tau_a) = \delta(\tau_i)$, so we are left with an integral in just the ϕ_m variables:

$$\mathbb{V}(\lambda_a, \epsilon_i = v_i^a \bar{\epsilon}_a) = \big|\det(Q, P)\big| \cdot e^{-\lambda_a \bar{\epsilon}_a} \int_{(i\mathbb{R})^r} \prod_{m=1}^{r} \frac{\mathrm{d}\phi_m}{2\pi i} \, \frac{e^{-\lambda_a Q_a^m \phi_m}}{\prod_{a=1}^d (\bar{\epsilon}_a + Q_a^m \phi_m)} \,, \tag{6.1.26}$$

where (Q, P) is the square matrix obtained by concatenating Q_a^m and P_a^i. This expression almost completely reproduces (6.1.2); the only piece that we are missing is to show that

$$\big|\det(Q, P)\big| = \frac{1}{|\Gamma|} \,. \tag{6.1.27}$$

[3] The set $\mathcal{C} = \{v^a \bar{\epsilon}_a \,|\, \bar{\epsilon}_a \geq 0\}$ is the convex polyhedral cone spanned by the vectors v^a. For non-compact toric orbifolds the integral (6.1.20) is convergent if and only if the ϵ_i take values in $\mathcal{C}$. For compact toric orbifolds $\mathcal{C} = \mathbb{R}^m$.

First, let us notice that $\det(Q,P)$ does not depend on the specific choice of the matrix P^i_a. Indeed, if we shift $P^i_a \longrightarrow P^i_a + \alpha_m Q^m_a$ the determinant does not change, and all possible choices of P^i_a differ by such a shift. We will now describe a particularly convenient way to choose P^i_a. Let y_α be any fixed point of $\mathbb{M}_{2\mathfrak{m}}$, and let v_α be the square matrix whose columns are the generators $(v^{a_1},\dots,v^{a_{\mathfrak{m}}})$ of the cone associated to y_α. Then the following is a valid choice of P^i_a:

$$P^i_a = \begin{cases} (v_\alpha^{-1})^i_a & \text{if } v^a \text{ is one of the generators of the cone associated to } p \\ 0 & \text{otherwise} \end{cases} . \tag{6.1.28}$$

Indeed, it is easy to verify that $P^i_a v^a_j = \delta^i_j$. Using the above expression for P^i_a, it immediately follows that

$$\big|\det(Q,P)\big| = \big|\det Q_\alpha\big| \cdot \big|\det v_\alpha^{-1}\big| = \frac{1}{d_\alpha}\big|\det Q_\alpha\big| , \tag{6.1.29}$$

where Q_α is the square matrix obtained from Q^m_a by removing all the values of the index a such that v^a is a generator of the cone associated to the fixed point y_α. In the second step we have used (4.2.6).

By definition the integer d_p is the order of the orbifold singularity at the fixed point y_α. If we express the orbifold as the symplectic quotient (4.2.42) $\mathbb{M}_{2\mathfrak{m}} = \mu_G^{-1}(0)/G$, then the order of the orbifold singularity is equal to the order of the isotropy group of the action of G on each point in the orbit associated to y_α. By the same logic as (4.2.45), the orbit of G associated to the fixed point y_α is the set

$$\mu_G^{-1}(0) \cap \{(z_1,\dots,z_d) \in \mathbb{C}^d \,|\, z_{a_1} = \ldots = z_{a_{\mathfrak{m}}} = 0\} . \tag{6.1.30}$$

It is easy to verify that the subgroup of G that leaves each point in the above set fixed is Γ times a subgroup of order $\big|\det Q_\alpha\big|$. This means that

$$d_\alpha = |\Gamma| \cdot \big|\det Q_\alpha\big| , \tag{6.1.31}$$

which together with (6.1.29) implies relation (6.1.27). This concludes the proof of (6.1.2).

6.1.3 Derivation of the formula with higher times

We will now briefly discuss the derivation of the generalization of the Molien-Weyl formula to the equivariant volume with higher times, reproducing formulas (6.1.7) and (6.1.9). For simplicity we will focus on the case with just single and double times as an example; the procedure easily generalizes to the inclusion of any higher times.

The equivariant volume with the addition of double times can formally be determined from the equivariant volume without higher times by solving the following system of differential equations:

$$\begin{cases} \partial_{\lambda_{ab}} \mathbb{V}(\lambda_a, \lambda_{ab}, \epsilon_i) = (2-\delta_{ab})\partial_{\lambda_a}\partial_{\lambda_b}\mathbb{V}(\lambda_a,\lambda_{ab},\epsilon_i) , \\ \mathbb{V}(\lambda_a, \lambda_{ab}=0, \epsilon_i) = \mathbb{V}(\lambda_a,\epsilon_i) . \end{cases} \tag{6.1.32}$$

It is then immediate to find the analogue of [4]

$$\mathbb{V}(\lambda_a, \epsilon_i = v_i^a \bar{\epsilon}_a) = \frac{e^{-\lambda_a \bar{\epsilon}_a}}{|\Gamma|} \int_{JK} \prod_{m=1}^{r} \frac{\mathrm{d}\phi_m}{2\pi i} \frac{e^{-\lambda_a Q_a^m \phi_m}}{\prod_{a=1}^{d} (\bar{\epsilon}_a + Q_a^m \phi_m)} \tag{6.1.33}$$

with the inclusion of second times:

$$\mathbb{V}(\lambda_a, \lambda_{ab}, \epsilon_i = v_i^a \bar{\epsilon}_a) = \frac{e^{-\lambda_a \bar{\epsilon}_a + \lambda_{ab} \bar{\epsilon}_a \bar{\epsilon}_b}}{|\Gamma|} \int_{\mathcal{C}} \prod_{m=1}^{r} \frac{\mathrm{d}\phi_m}{2\pi i} \frac{e^{(2\lambda_{ab}\bar{\epsilon}_b - \lambda_a) Q_a^m \phi_m + Q_a^m Q_b^n \lambda_{ab} \phi_m \phi_n}}{\prod_{a=1}^{d} (\bar{\epsilon}_a + Q_a^m \phi_m)} . \tag{6.1.34}$$

Where the contour $\mathcal{C}$ is the same as the contour of the integral in (6.1.33). If $t^m = (2\lambda_{ab}\bar{\epsilon}_b - \lambda_a) Q_a^m$ and $t_0^m = -\lambda_a Q_a^m$ are in the same chamber of regular values, then $\mathcal{C}$ matches the Jefferey-Kirwan contour and we get the desired expression:

$$\mathbb{V}(\lambda_a, \lambda_{ab}, \epsilon_i = v_i^a \bar{\epsilon}_a) = e^{-\lambda_a \bar{\epsilon}_a + \lambda_{ab} \bar{\epsilon}_a \bar{\epsilon}_b} \mathbb{V}_{MW}(t^m = (2\lambda_{ab}\bar{\epsilon}_b - \lambda_a) Q_a^m, t^{mn} = Q_a^m Q_b^n \lambda_{ab}, \bar{\epsilon}_a). \tag{6.1.35}$$

It is easy to show that by iterating this argument we would reproduce (6.1.7) and (6.1.9).

6.1.4 Mapping residues to fixed points

In this subsection we discuss the relation between fixed points and residues of the Molien-Weyl formula [111], providing evidence of a one-to-one correspondence. For simplicity we only keep single and double times.

Our starting point will be formula (6.1.7) with $K = 2$:

$$\mathbb{V}_{MW}(t^m, t^{mn}, \bar{\epsilon}_a) = \frac{1}{|\Gamma|} \int_{JK} \prod_{m=1}^{r} \frac{d\phi_m}{2\pi i} \frac{e^{t^m \phi_m + t^{mn} \phi_m \phi_n}}{\prod_{a=1}^{d} \left(\bar{\epsilon}_a + Q_a^m \phi_m\right)} . \tag{6.1.36}$$

Each pole p inside the Jeffrey-Kirwan contour (we write $p \in JK$) is uniquely determined by the r different values $a_1, ..., a_r$ of the index a that correspond to the terms in the denominator of (6.1.36) that diverge at the pole; we can thus identify $p \equiv (a_1, ..., a_r)$. We can now apply the residue theorem to (6.1.36) and obtain

$$\mathbb{V}_{MW}(t^m, t^{mn}, \bar{\epsilon}_a) = \frac{1}{|\Gamma|} \sum_{p \in JK} \frac{e^{t^m \phi_m(p) + t^{mn} \phi_m(p) \phi_n(p)}}{|\det Q_p| \prod_{a \notin p} \bar{\epsilon}_a(p)} , \tag{6.1.37}$$

where we have defined the following:

$$\begin{aligned} (Q_p)^m_n &\equiv Q^m_{p_n} \\ \phi_m(p) &\equiv -\sum_{n=1}^{r} \bar{\epsilon}_{p_n} (Q_p^{-1})^n_m \\ \bar{\epsilon}_a(p) &\equiv \bar{\epsilon}_a + Q_a^m \phi_m(p) . \end{aligned} \tag{6.1.38}$$

[4] Notice that we are using the version of the Molien-Weyl formula with the closed Jeffrey-Kirwan contour, in order to avoid trouble at infinity.

We will now introduce variables λ_a, λ_{ab} so that λ_{ab} is symmetric and

$$t^{mn} = Q_a^m \, Q_b^n \, \lambda_{ab} \,, \qquad t^m = Q_a^m \big(2\, \lambda_{ab} \, \bar{\epsilon}_b - \lambda_a\big) \,. \tag{6.1.39}$$

Then the exponent in (6.1.37) can be rewritten as

$$t^m \phi_m(p) + t^{mn} \phi_m(p) \, \phi_n(p) \; = -\lambda^a \, \bar{\epsilon}_a(p) + \lambda^{ab} \, \bar{\epsilon}_a(p) \, \bar{\epsilon}_b(p) + \lambda^a \, \bar{\epsilon}_a - \lambda^{ab} \, \bar{\epsilon}_a \, \bar{\epsilon}_b \,, \tag{6.1.40}$$

and thus

$$\mathbb{V}_{MW}(t^m, t^{mn}, \bar{\epsilon}_a) = e^{\lambda^a \, \bar{\epsilon}_a - \lambda^{ab} \, \bar{\epsilon}_a \, \bar{\epsilon}_b} \, \frac{1}{|\Gamma|} \sum_{p \in JK} \frac{e^{-\lambda^a \, \bar{\epsilon}_a(p) + \lambda^{ab} \, \bar{\epsilon}_a(p) \, \bar{\epsilon}_b(p)}}{|\det Q_p| \, \prod_{a \notin p} \bar{\epsilon}_a(p)} \,. \tag{6.1.41}$$

In light of the relation (6.1.9) between $\mathbb{V}_{MW}$ and $\mathbb{V}$ we can see that each term in the above residue formula matches a corresponding term in the fixed point formula, provided that the following identifications hold:

- To each pole $p \equiv (a_1, ..., a_r)$ in the Jeffrey-Kirwan contour we can associate the fixed point y_α, where α is the cone in the fan generated by the vectors $\{v^a\}_{a \notin p}$, and vice-versa.

- The quantity $\bar{\epsilon}_a(p)$ matches, up to a minus sign, the restriction of the equivariant Chern class $c_1^{\mathbb{T}}(L_a)$ at the fixed point associated to the pole p, that is

$$\begin{aligned} \bar{\epsilon}_a(p) \; &= \; - \, \varepsilon_a^p\big(\epsilon_i = v_i^b \, \bar{\epsilon}_b\big) \,, \\ \varepsilon_a^p(\epsilon_i) \; &\equiv \; c_1^{\mathbb{T}}(L_a)|_p = - \, \frac{(u_p^a)^{\,i} \, \epsilon_i}{d_p} \,, \end{aligned} \tag{6.1.42}$$

  where the inward normals to the cone u_p^a are defined as in (4.3.16), and we are using the pedices α and p interchangeably.

Note that because of (6.1.31) we would have $|\Gamma| \cdot |Q_p| = d_p$. We will not give a direct proof of the first point above, rather in the rest of this section we will prove the second point, and the computation of this subsection can then be seen as as evidence for the correspondence of fixed points and poles.

We can rewrite the relation (4.3.16) as

$$\forall a, b \notin p \qquad \delta_{ab} = \frac{1}{d_p} \, v^b \cdot u_p^a \,, \tag{6.1.43}$$

and use it as follows:

$$\forall a \notin p \qquad Q_a^m \; = \; \sum_{b \notin p} Q_b^m \delta_{ab} \; = \; \frac{1}{d_p} \sum_{b \notin p} Q_b^m \, v^b \cdot u_p^a \; = \; - \, \frac{1}{d_p} \sum_n Q_{p_n}^m \, v^{p_n} \cdot u_p^a \,. \tag{6.1.44}$$

In the last step we used that $v_i^a \, Q_a^m = 0$. Acting with Q_p^{-1} on both sides gives us

$$\forall a \notin p \qquad \sum_m Q_a^m \, (Q_p^{-1})_m^n \; = \; - \, \frac{1}{d_p} \, v^{p_n} \cdot u_p^a \,. \tag{6.1.45}$$

Using (6.1.43) again we can also write

$$\forall a \notin p \qquad \bar{\epsilon}_a = \sum_{b \notin p} \bar{\epsilon}_b \, \delta_{ab} = \frac{1}{d_p} \, u_a^p \cdot \sum_{b \notin p} \bar{\epsilon}_b \, v^b \, . \tag{6.1.46}$$

Putting everything together we find the following expression for $\bar{\epsilon}_a(p)$ when $a \notin p$:

$$\forall a \notin p \qquad \bar{\epsilon}_a(p) \equiv \bar{\epsilon}_a - \sum_{m,n} \bar{\epsilon}_{p_n} \, Q_a^m \, (Q_p^{-1})_m^n = \frac{1}{d_p} \, u_a^p \cdot \sum_b \bar{\epsilon}_b \, v^b \, . \tag{6.1.47}$$

On the other hand when $a \in p$ the quantity $\bar{\epsilon}_a(p)$ vanishes:

$$\forall a \in p \qquad \bar{\epsilon}_a(p) \equiv \bar{\epsilon}_a - \sum_{m,n} \bar{\epsilon}_{p_n} \, Q_a^m \, (Q_p^{-1})_m^n = \bar{\epsilon}_a - \sum_n \bar{\epsilon}_{p_n} \, \delta_a^{p_n} = 0 \, . \tag{6.1.48}$$

Hence, if we define

$$\forall a \in p \qquad u_p^a \equiv 0 \tag{6.1.49}$$

we can conclude that

$$\forall a \qquad \bar{\epsilon}_a(p) = \frac{1}{d_p} \, u_a^p \cdot \sum_b \bar{\epsilon}_b \, v^b \, , \tag{6.1.50}$$

which is exactly relation (6.1.42).

6.2 $\mathrm{AdS}_3 \times M_8$ solutions in M theory revisited

In this section we revisit the $\mathrm{AdS}_3 \times M_8$ solutions in M theory of subsection 5.2.1 through the lenses of the Molien-Weyl formalism. In subsection 6.2.1 we explain how the procedure of chapter 5 can be reformulated, and in subsection 6.2.2 we will provide a concrete example by taking the four-dimensional orbifold $\mathbb{M}_4$ to be a wighted projective space.

We remind that the internal geometry of the $\mathrm{AdS}_3 \times M_8$ solutions that we are interested in is associated to fan (5.2.1):

$$V^a = (v^a, 1, \mathfrak{t}_a) \, , \qquad V^{d+1} = (0,0,1,0) \, , \qquad V^{d+2} = (0,0,1,1) \, , \tag{6.2.1}$$

where as usual the v^a are the vectors of the base orbifold $\mathbb{M}_4$. Let us denote with q_a^m the GLSM charges of $\mathbb{M}_4$. Then the GLSM charges of the fibration are

$$Q_a^m = q_a^m \, , \qquad Q_{d+1}^m = \sum_a (\mathfrak{t}_a - 1) q_a^m \, , \qquad Q_{d+1}^m = - \sum_a \mathfrak{t}_a \, q_a^m \, . \tag{6.2.2}$$

The fixed points of the fibration are associated to cones in the fan of the type $(V^a, V^{a+1}, V^{d+1}, V^{d+2})$. In light of the discussion of subsection 6.1.4, the integrand of the Molien-Weyl formula should have d poles $\{p_a\}_{a=1}^d$ inside the Jeffrey-Kirwan contour, such that all the terms $(\bar{\epsilon}_A + Q_A^m \, \phi_m)$ vanish at the pole p_a except for the ones with $A = a,\, a+1,\, d+1,\, d+2$. The Molien-Weyl equivariant volume with higher times can then by computed as in (6.1.37) :

$$\mathbb{V}_{MW}(t^m, t^{mn}, \bar{\epsilon}_A) = \sum_{a=1}^{d} \frac{e^{t^m \phi_m(p_a) + t^{mn}\phi_m(p_a)\phi_n(p_a)}}{d_{a,a+1}\,\bar{\epsilon}_a(p_a)\,\bar{\epsilon}_{a+1}(p_a)\,\bar{\epsilon}_{d+1}(p_a)\,\bar{\epsilon}_{d+2}(p_a)}\,, \tag{6.2.3}$$

where $\phi_m(p_a)$ and $\bar{\epsilon}_{a+1}(p_a)$ are defined in (6.1.38).

6.2.1 The extremal function with the Molien-Weyl formula

In this subsection we formulate a procedure that allows to derive the gravitational blocks formula (5.2.31) from the Molien-Weyl formula.

First, let us remind that the free energy (5.2.31) was obtained in subsection 5.2.1 from the following setup:

$$F = \mathbb{V}^{(3)}(\lambda_A, \lambda_{AB}, \epsilon_I)\,, \qquad \partial_{\lambda_{AB}}\mathbb{V}^{(2)}(\lambda_A, \lambda_{AB}, \epsilon_I) = -\bar{\nu}_{M5}\,(2-\delta_{AB})\,M_{AB}\,. \tag{6.2.4}$$

We claim that the Molien-Weyl analogue of the above prescription is to set the free-energy as

$$F = \mathbb{V}^{(3)}_{MW}(t^m,\, t^{mn}, \bar{\epsilon}_A)\,, \tag{6.2.5}$$

and to impose the following flux equations:

$$\begin{cases} \partial_{t^{mn}}\mathbb{V}^{(2)}_{MW}\big(t^m,\, t^{mn}, \bar{\epsilon}_A\big) \;=\; -\bar{\nu}_{M5}\,(2-\delta_{mn})\,M_{mn}\,, \\ \partial_{t^m}\mathbb{V}^{(2)}_{MW}\big(t^m,\, t^{mn}, \bar{\epsilon}_A\big) \;=\; 0\,, \\ \mathbb{V}^{(1)}_{MW}\big(t^m,\, t^{mn}, \bar{\epsilon}_A\big) \;=\; 0\,, \end{cases} \tag{6.2.6}$$

where M_{mn} is related to M_{AB} by $M_{AB} = Q^m_a Q^n_B M_{mn}$.[5] Notably when the flux constraints (6.2.6) are imposed the free energy (6.2.5) becomes a function of the variables $\epsilon_I = V^A_I \bar{\epsilon}_A$ exclusively, as expected.

Before deriving (6.2.5) and (6.2.6) we would like to observe that the bottom two equations in (6.2.6) have a nice interpretation. We remind that the Molien-Weyl formula satisfies the shift formula (6.1.10), which for our particular case takes the form

$$\mathbb{V}_{MW}\big(t^m + 2t^{mn}\alpha_n,\, t^{mn}, \bar{\epsilon}_A + Q^m_A\alpha_m\big) \;=\; e^{-t^m\alpha_m - t^{mn}\alpha_m\alpha_n}\,\mathbb{V}_{MW}\big(t^m,\, t^{mn}, \bar{\epsilon}_A\big)\,. \tag{6.2.7}$$

when $t^m\alpha_m + t^{mn}\alpha_m\alpha_n = 0$ this can be seen as a gauge transformation of $\mathbb{V}_{MW}$. The first equation in (6.2.6) by itself would not be a gauge invariant equation, given that the term $\partial_{t^{mn}}\mathbb{V}^{(2)}_{MW}$ under a gauge transformation would acquire corrections proportional to $\partial_{t^m}\mathbb{V}^{(2)}_{MW}$ and $\mathbb{V}^{(1)}_{MW}$. It should then be clear what role the other two

[5] It is always possible to write $M_{AB} = Q^m_a Q^n_B M_{mn}$ since the fluxes must satisfy $V^A_I M_{AB} = 0$.

equations in (6.2.6) are playing: they ensure that the system of equations (6.2.6) as a whole is gauge invariant.

Let us derive (6.2.5) and (6.2.6). First, we notice that the flux constraints in (6.2.4),

$$\partial_{\lambda_{AB}}\mathbb{V}^{(2)}\big(\lambda_A,\lambda_{AB},\epsilon_I\big) \;=\; -\bar{\nu}_{M5}\,(2-\delta_{AB})\,M_{AB}\;, \tag{6.2.8}$$

together with $V_I^A M_{AB}=0$ imply that

$$\begin{cases} \partial_{\lambda_A}\mathbb{V}^{(2)}\big(\lambda_A,\lambda_{AB},\epsilon_I\big) \;=\; 0\;, \\ \mathbb{V}^{(1)}\big(\lambda_A,\lambda_{AB},\epsilon_I\big) \;=\; 0\;. \end{cases} \tag{6.2.9}$$

This can be seen by writing $\partial_{\lambda_{AB}}\mathbb{V}^{(2)} = (2-\delta_{AB})\,\partial_{\lambda^A}\,\partial_{\lambda_B}\,\mathbb{V}^{(3)}$ and contracting with V_I^A using (4.3.11).

We will make use of formula (6.1.9), which relates the standard formulation of the equivariant volume and the Molien-Weyl one. In this case we can write it as

$$\begin{aligned} &\mathbb{V}(\lambda_A,\lambda_{AB},\epsilon_I=V_I^A\bar{\epsilon}_A) = \\ &\quad = e^{-\lambda_A\bar{\epsilon}_A+\lambda_{AB}\bar{\epsilon}_A\bar{\epsilon}_B}\;\mathbb{V}_{MW}(t^m=(2\lambda_{AB}\bar{\epsilon}_B-\lambda_A)Q_A^m,t^{mn}=Q_A^mQ_B^n\lambda_{AB},\bar{\epsilon}_A)\;. \end{aligned} \tag{6.2.10}$$

Given a fixed value for the equivariant parameters ϵ_I, there is some ambiguity in the choice of $\bar{\epsilon}_A$. If λ_A and λ_{AB} are fixed to the value that solves the flux constraints (6.2.8), we can always choose $\bar{\epsilon}_A$ so that $-\lambda_A\bar{\epsilon}_A+\lambda_{AB}\bar{\epsilon}_A\bar{\epsilon}_B=0$. With this convenient choice equations (6.2.9) and (6.2.8) can expressed as

$$\begin{aligned} &0=\mathbb{V}^{(1)}=\mathbb{V}^{(1)}_{MW}\;,\\ &0=\partial_{\lambda_A}\mathbb{V}^{(2)}=-\bar{\epsilon}_A\mathbb{V}^{(1)}_{MW}-Q_A^m\partial_{t^m}\mathbb{V}^{(2)}_{MW}=-Q_A^m\partial_{t^m}\mathbb{V}^{(2)}_{MW}\;,\\ &\bar{\nu}_{M5}\,M_{AB}=-\tfrac{1+\delta_{AB}}{2}\,\partial_{\lambda_{AB}}\mathbb{V}^{(2)}=-\bar{\epsilon}_A\bar{\epsilon}_B\mathbb{V}^{(1)}_{MW}-(\bar{\epsilon}_AQ_B^m+\bar{\epsilon}_BQ_A^m)\partial_{t^m}\mathbb{V}^{(2)}_{MW}-\\ &\qquad\qquad -Q_A^mQ_B^n\tfrac{1+\delta_{mn}}{2}\partial_{t^{mn}}\mathbb{V}^{(2)}_{MW}=-Q_A^mQ_B^n\tfrac{1+\delta_{mn}}{2}\partial_{t^{mn}}\mathbb{V}^{(2)}_{MW}\;. \end{aligned} \tag{6.2.11}$$

For simplicity we have suppressed the arguments, they should be set up in the same way as (6.2.10). In particular we have obtained the system of equations (6.2.6).

Viceversa, if we were to start from $(t^m,t^{mn},\bar{\epsilon}_A)$ that solve the Molien-Weyl flux constraints (6.2.6), we could always choose a value for λ_A, λ_{AB} such that $t^m=(2\lambda_{AB}\bar{\epsilon}_B-\lambda_A)Q_A^m$, $t^{mn}=Q_A^mQ_B^n\lambda_{AB}$, and $-\lambda_A\bar{\epsilon}_A+\lambda_{AB}\bar{\epsilon}_A\bar{\epsilon}_B=0$. Then if we were to trace the same steps as (6.2.11) backwards, we would find that (6.2.8) is satisfied. We would also have

$$F=\mathbb{V}^{(3)}_{MW}(t^m,\,t^{mn},\bar{\epsilon}_A)=\mathbb{V}^{(3)}(\lambda_A,\lambda_{AB},\epsilon_I=V_I^A\bar{\epsilon}_A)\;, \tag{6.2.12}$$

concluding the derivation.

6.2.2 $\mathbb{M}_4 = \mathbb{WP}^2$

In this subsection we focus on a particular example of four-dimensional toric orbifold, the weighted projective spaces, and we apply the procedure explained in the previous subsection.

The wighted projective space $\mathbb{WP}^2$ can be seen as an orbifold generalization of $\mathbb{P}^2$: the fan of $\mathbb{WP}^2$ matches the one of $\mathbb{P}^2$ up to a rescaling of the vectors:

$$v^1 = (n_3, n_3)\,, \qquad v^2 = (-n_1, 0)\,, \qquad v^3 = (0, -n_2)\,. \tag{6.2.13}$$

The orders of the orbifold singularities are given by $d_{ab} = |\det(v^a, v^b)| = n_{a-1}n_{b-1}$.

For $\mathbb{C}^2$ fibered over $\mathbb{WP}^2$ the indices $m, n, \ldots$ can only take one possible value, so we will suppress them; in order to distinguish between t^m and t^{mn} we will use the following convention:

$$t^m \longrightarrow t_{(1)}\,, \qquad t^{mn} \longrightarrow t_{(2)}\,. \tag{6.2.14}$$

There is only one vector of GLSM charges and its components along the base orbifold are given by

$$Q_a = (n_1 n_2,\ n_2 n_3,\ n_3 n_1) \qquad a = 1, 2, 3\,. \tag{6.2.15}$$

The charges alongside the fiber can be found using (6.2.2) :

$$Q_4 = \sum_{a=1}^{3} (\mathfrak{t}_a - 1) n_a n_{a+1}\,, \qquad Q_5 = -\sum_{a=1}^{3} \mathfrak{t}_a n_a n_{a+1}\,. \tag{6.2.16}$$

The Molien-Weyl equivariant volume can then be expressed as

$$\mathbb{V}_{MW}(t_{(1)}, t_{(2)}, \bar{\epsilon}_A) = \sum_{a=1}^{3} \frac{e^{t_{(1)}\phi(p_a) + t_{(2)}\phi(p_a)^2}}{n_{a-1}n_a\, \bar{\epsilon}_a(p_a)\, \bar{\epsilon}_{a+1}(p_a)\, \bar{\epsilon}_4(p_a)\, \bar{\epsilon}_5(p_a)}\,, \tag{6.2.17}$$

where

$$\begin{aligned} \phi(p_a) &= -\bar{\epsilon}_{a-1}\, Q^{-1}_{a-1} = -\frac{\bar{\epsilon}_{a-1}}{n_{a-1}n_a}\,, \\ \bar{\epsilon}_A(p_a) &= \bar{\epsilon}_A - \frac{\bar{\epsilon}_{a-1}\, Q_A}{n_{a-1}n_a}\,. \end{aligned} \tag{6.2.18}$$

We can now impose the flux constraints (6.2.6), which means solving the following system of equations:

$$\begin{cases} \partial_{t_{(2)}} \mathbb{V}^{(2)}_{MW}\big(t_{(1)}, t_{(2)}, \bar{\epsilon}_A\big) \;=\; -\bar{\nu}_{M5} M\,, \\ \partial_{t_{(1)}} \mathbb{V}^{(2)}_{MW}\big(t_{(1)}, t_{(2)}, \bar{\epsilon}_A\big) \;=\; 0\,, \\ \mathbb{V}^{(1)}_{MW}\big(t_{(1)}, t_{(2)}, \bar{\epsilon}_A\big) \;=\; 0\,. \end{cases} \tag{6.2.19}$$

Two of the above equations fix the values for the Käler moduli $t_{(1)}$ and $t_{(2)}$, and the third will give us a constraint on the $\bar{\epsilon}_A$. We find a total of two distinct solution:

$$\begin{cases} \bar{\epsilon}_4 = 0 \\ t_{(1)} = -\frac{1}{2}\bar{\nu}_{M5} N\, Q_4\, \bar{\epsilon}_5 \\ t_{(2)} = -\frac{1}{2}\bar{\nu}_{M5} N\, Q_4 Q_5 \end{cases} , \quad \begin{cases} \bar{\epsilon}_5 = 0 \\ t_{(1)} = -\frac{1}{2}\bar{\nu}_{M5} N\, Q_5\, \bar{\epsilon}_4 \\ t_{(2)} = -\frac{1}{2}\bar{\nu}_{M5} N\, Q_4 Q_5 \end{cases} , \tag{6.2.20}$$

where $N = (n_1 n_2 n_3)^2 M$. [6] If we were to plug the above solutions inside the free-energy $F = \mathbb{V}^{(3)}_{MW}\big(t_{(1)}, t_{(2)}, \bar{\epsilon}_A\big)$ we would get two different results. The key is that these result only match after we translate into the ϵ_I variables. In order to do so we need to solve the linear systems

$$\begin{cases} \bar{\epsilon}_4 = 0 \\ V_I^A \bar{\epsilon}_A = \epsilon_I \end{cases} , \quad \begin{cases} \bar{\epsilon}_5 = 0 \\ V_I^A \bar{\epsilon}_A = \epsilon_I \end{cases} . \tag{6.2.21}$$

If we substitute (6.2.20) in $\mathbb{V}^{(3)}_{MW}$ and perform the respective change of variables in accordance with (6.2.21), we find that the free energy is given by

$$\begin{aligned} F = \frac{1}{6}(-\bar{\nu}_{M5} N)^3 \Bigg[& \frac{\Big((1-\mathfrak{t}_1)\frac{\epsilon_2}{n_3} + (1-\mathfrak{t}_2)\frac{\epsilon_2-\epsilon_1}{n_1} - \epsilon_3 + \epsilon_4\Big)^2 \Big(\mathfrak{t}_1 \frac{\epsilon_2}{n_3} + \mathfrak{t}_2 \frac{\epsilon_2-\epsilon_1}{n_1} - \epsilon_4\Big)^2}{\epsilon_2(\epsilon_2-\epsilon_1)} + \\ & + \frac{\Big(-(1-\mathfrak{t}_2)\frac{\epsilon_1}{n_1} - (1-\mathfrak{t}_3)\frac{\epsilon_2}{n_2} - \epsilon_3 + \epsilon_4\Big)^2 \Big(-\mathfrak{t}_2 \frac{\epsilon_1}{n_1} - \mathfrak{t}_3 \frac{\epsilon_2}{n_2} - \epsilon_4\Big)^2}{\epsilon_1\, \epsilon_2} + \\ & + \frac{\Big((1-\mathfrak{t}_3)\frac{\epsilon_1-\epsilon_2}{n_2} + (1-\mathfrak{t}_1)\frac{\epsilon_1}{n_3} - \epsilon_3 + \epsilon_4\Big)^2 \Big(\mathfrak{t}_3 \frac{\epsilon_1-\epsilon_2}{n_2} + \mathfrak{t}_1 \frac{\epsilon_1}{n_3} \epsilon_4\Big)^2}{\epsilon_1(\epsilon_1-\epsilon_2)} \Bigg] , \end{aligned} \tag{6.2.22}$$

which is the same as (5.2.31), as expected.

Let us conclude by noticing that from the solution (6.2.20) it is evident that in general the second time $t_{(2)}$ is necessary for the prescription to work, except for the particular case when either $Q_4 = 0$ or $Q_5 = 0$. These condition on the charges restrict the topology of the fibration, and it would be interesting to try to understand why the geometry of these special cases does not require higher times. It would also be interesting to find a geometrical interpretation to the two distinct solutions (6.2.20), and why the free energy F found from each of them only matches when we convert back to the ϵ_I variables.

6.3 Summary and discussion

In this chapter we have discussed the Molien-Weyl formula and provided a direct derivation of its relation with the standard formulation of the equivariant volume.

[6] Indeed N was defined in (5.2.20) so that $M_{ab} = N\, D_{ab}$. We can find the relation between N and M by writing $D_{ab} = Q_a\, Q_b\, \partial^2_{t_{(1)}} \mathbb{V}^{(2)}_{MW}|_{t_{(2)}=0}$ and by computing $\partial^2_{t_{(1)}} \mathbb{V}^{(2)}_{MW}|_{t_{(2)}=0} = (n_1 n_2 n_3)^{-2}$.

We have revisited the $\mathrm{AdS}_3 \times M_8$ solutions, obtaining once again the gravitational blocks formula conjectured in [85], this time by means of the Molien-Weyl formula. Given that the Molien-Weyl formula depends on the independent Kähler moduli of the geometry rather than an over-parametrization thereof, the computation is remarkably simple for orbifolds with a small number of Kähler moduli. This is the case for the example that we have focused on in subsection 6.2.2, in which we took the four-dimensional base orbifold to be a weighted projective space. More in general, the Molien-Weyl formula provides an interesting alternative prospective, especially considering the important role played by the "gauge transformations" of the parameters of the equivariant volume, which are remarkably different in the two formulations.

We want to conclude by briefly mentioning an interesting possible direction of future research, for which the Molien-Weyl formulation could play a role. As observed in [28, 110], the "quantum analogue" of the equivariant volume, the orbifold indices [28, 93, 111], are expected to be a fundamental building block for partition functions on orbifolds. They can be defined as characters of line bundles over the orbifold, and they reduce to the respective formulas for the equivariant volume in a "classical limit":

$$\begin{aligned}
&\hbar^{\mathfrak{m}} \cdot \mathbb{Z}(\Lambda_a = -\lambda_a/\hbar,\ q_i = e^{-\hbar\epsilon_i}) = \mathbb{V}(\lambda_a, \epsilon_i) + \mathcal{O}(\hbar)\,, \\
&\hbar^{\mathfrak{m}} \cdot \mathbb{Z}_{MW}(T^m = t^m/\hbar,\ \bar{q}_a = e^{-\hbar\bar{\epsilon}_a}) = \mathbb{V}_{MW}(t^m, \bar{\epsilon}_a) + \mathcal{O}(\hbar)\,.
\end{aligned} \tag{6.3.1}$$

It would be interesting to extend the computation of [110] for more generic orbifolds other than the spindle.

Chapter 7

Conclusions

We will now briefly summarize our main results. For a much lengthier discussion we refer to the "Summary and discussion" sections 3.4, 5.4 and 6.3. The following will just be a short summary of the points we have brought up in these sections.

In chapter 3, based on [1], we have improved the estimate of the superconformal index at large-N for general values for the BPS charges, focusing especially on the little studied $J_1 \neq J_2$, building upon our results in [3]. We have made use of both the elliptic extension approach of [48–50], which we have extended to the $J_1 \neq J_2$ case, and the Bethe Ansatz formula [46, 47], finding a good accord between the two methods. The number of competing exponential terms that contribute to the Bethe Ansatz formula is much bigger than the number of saddles of the elliptic action, and it is not feasible to compute all of them. Nonetheless for each saddle of the elliptic action we find a term in the Bethe Ansatz formula that matches it. The discrepancy between the number of contributions between the two approaches can be rationalized by noting that that the Bethe Ansatz formula is exact also at finite N. We thus expect that the only contributions of the Bethe Ansatz formula that dominate in some region of the parameter space should be the ones that match a saddle.

In chapter 5, based on [2], we have extracted extremal functions for various supergravity solutions that arise from the near-horizon limit of systems of branes wrapped around four-dimensional orbifolds, either partially or totally. Our prescription is based on the equivariant volume with higher times and it reproduces the gravitational central charges/free energies after we extremize over *all* the parameters that are not fixed by the quantization of the fluxes or supersymmetry. Our method can be applied also when the supergravity solutions is not known, and we speculate that the existence of a critical point for the extremal functions that we find might be a necessary condition for the existence of solutions with a given topology.

In the case of systems of M5 branes we have considered both totally wrapped branes and partially wrapped ones. In the former case we derive gravitational blocks formula conjectured in [85] and reproduced in the field theory in [28]. Partially wrapped branes on the other hand are still poorly understood on the field theory side. Nonetheless in the specific case when the base orbifold is $S^2 \times S^2$ we reproduce the result of the brane setup of [104]. We also reproduce the gravitational central charges of the known supergravity solutions [101, 102] and provide predictions for more general backgrounds. We compare our method with the one of [53], finding that they are equivalent. In particular we show that various non trivial cohomological relations that need to be imposed in the approach of [53] can be seen as consequences of the extremization with respect to the unfixed Kähler moduli.

In the case of systems of D4 branes in massive type IIA we also have considered both totally wrapped branes and partially wrapped ones. In the former case we derive gravitational blocks formula conjectured in [85]. In the latter case we compare the results produced by our prescription with the gravitational free energy of the solutions [55], which we have computed.

In the case of systems of D3 branes in type IIB partially wrapped around a two-cycle we have extended the results obtained in [54] with the formalism of GK geometry to the orbifold case.

In chapter 6 we have discussed the relation between the standard formulation of the equivariant volume and the Molien-Weyl formula, and in particular we have shown how the prescription that we presented in [2] can be reformulated in terms of the latter by focusing on the case of $\mathrm{AdS}_3 \times M_8$ solutions, providing a different prospective. While this chapter is not based on any published works, it might provide an interesting starting point for future research. For example, the equivariant volume has a "quantum analogue", the orbifold index, which is expected to be a fundamental building blocks for supersymmetric partition functions on orbifolds [110], and the Molien-Weyl formulation could play an interesting role.

Appendix A

Appendix for the superconformal index

A.1 The elliptic gamma and related functions

The elliptic gamma function

The elliptic gamma function [75] is defined by the following infinite product:

$$\Gamma_e\Big(z\,;\tau,\sigma\Big) = \prod_{j,k=0}^{\infty} \frac{1-e^{2\pi i\,\big((j+1)\tau+(k+1)\sigma-z\big)}}{1-e^{2\pi i\,\big(j\tau+k\sigma+z\big)}}\,, \tag{A.1.1}$$

which is convergent as long as $\operatorname{Im}\tau>0$ and $\operatorname{Im}\sigma>0$. It is meromorphic in z, with poles in $z\in\mathbb{Z}+\tau\,\mathbb{Z}_{\leq 0}+\sigma\,\mathbb{Z}_{\leq 0}$ and zeros in $z\in\mathbb{Z}+\tau\,\mathbb{Z}_{\geq 1}+\sigma\,\mathbb{Z}_{\geq 1}$. It is manifestly invariant under integer shifts in z,τ,σ and symmetric under the exchange of τ and σ.

The elliptic gamma satisfies the following inversion relation:

$$\Gamma_e\Big(z\,;\tau,\sigma\Big) = \Gamma_e\Big(\tau+\sigma-z\,;\tau,\sigma\Big)^{-1}\,, \tag{A.1.2}$$

and the following product formulas:

$$\prod_{k=0}^{n-1}\Gamma_e\Big(z+\frac{k}{n}\tau\,;\tau,\sigma\Big) = \Gamma_e\Big(z\,;\frac{\tau}{n},\sigma\Big)\,, \tag{A.1.3}$$

$$\prod_{j=0}^{m-1}\Gamma_e\Big(z+\frac{j}{m}\,;\tau,\sigma\Big) = \Gamma_e\Big(mz\,;m\tau,m\sigma\Big)\,. \tag{A.1.4}$$

Relations (A.1.2) and (A.1.3) follow directly from definition (A.1.1); as for relation (A.1.4), it is a consequence of the following polynomial identity:

$$\prod_{j=0}^{m-1}\Big(1-e^{2\pi i(j/m)}z\Big) = 1-z^m\,. \tag{A.1.5}$$

The θ_0 function

The q-theta function θ_0 is defined as follows:

$$\theta_0(z\,;\tau) = \prod_{k=0}^{\infty} \left(1 - e^{2\pi i(z+k\tau)}\right) \left(1 - e^{2\pi i(-z+(k+1)\tau)}\right) \,. \tag{A.1.6}$$

It is analytic in z and its zeros are in $z \in \mathbb{Z} + \tau\,\mathbb{Z}$. It is related to the elliptic gamma function by the following shift identity:

$$\Gamma_e(z+\tau\,;\tau,\sigma) = \theta_0(z\,;\sigma)\,\Gamma_e(z\,;\tau,\sigma) \,. \tag{A.1.7}$$

On the other hand the shift identity for the θ_0 itself is the following:

$$\theta_0(z+\tau\,;\sigma) = -\,e^{-2\pi i\,z}\,\theta_0(z\,;\sigma) \,. \tag{A.1.8}$$

Bernoulli polynomials

The Bernoulli polynomials are defined by the following generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!} \,. \tag{A.1.9}$$

They satisfy the following relations:

$$B_n(x+1) = B_n(x) + nx^{n-1} \,, \tag{A.1.10}$$

$$B_n(1-x) = (-1)^n B_n(x) \,. \tag{A.1.11}$$

As a consequence of (A.1.11) the Bernoulli polynomials are either even or odd when expressed in the variable $2x-1$; in particular the first few polynomials can be written as

$$\begin{aligned} B_1(x) &= \frac{1}{2}\left(2x-1\right) \\ B_2(x) &= \frac{1}{4}\left(2x-1\right)^2 - \frac{1}{12} \\ B_3(x) &= \frac{1}{8}\left(2x-1\right)^3 - \frac{1}{8}\left(2x-1\right) \,. \end{aligned} \tag{A.1.12}$$

Some useful identities include the translation property,

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)\, y^{n-k} \,, \tag{A.1.13}$$

and the multiplication formula,

$$\sum_{j=0}^{m-1} B_n\left(z + \frac{j}{m}\right) = m^{1-n}\, B_n\Big(mz\Big) \,. \tag{A.1.14}$$

The multiplication formula can also be written as follows:

$$\sum_{j=0}^{m-1} B_n\left(z+\left\{x+\frac{j}{m}\right\}\right) = m^{1-n}\, B_n\Big(mz+\{mx\}\Big)\,, \tag{A.1.15}$$

where the brackets $\{\cdot\}$ denote the fractional part, which is defined by $x \equiv \lfloor x \rfloor + \{x\}$. Relation (A.1.15) follows directly from (A.1.14) if we consider that

$$m\cdot \min_j \left\{x+\frac{j}{m}\right\} = \{mx\}\,. \tag{A.1.16}$$

The P, Q functions

Given z, $\tau \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$, throughout the rest of this appendix we will denote with z_1 and z_2 the real numbers such that $z \equiv z_1 + \tau\, z_2$.

The function P is defined by [76]

$$P(z\,;\tau) = e^{2\pi i\, \alpha_P(z)}\, e^{\pi i \tau B_2(z_2)}\, \theta_0(z\,;\tau)\,, \tag{A.1.17}$$

where α_P can be any real-valued function that satisfies the following two constraints: α_P should vanish on the real axis, and it must be chosen so that $P(z\,;\tau)$ is invariant under translations by the lattice $\mathbb{Z}+\tau\,\mathbb{Z}$ in z. The second requirement can always be fulfilled because $|P|$ is manifestly invariant under integer shifts, and it is also invariant under shifts in τ, once (A.1.8) is taken into consideration. This can also be seen from the second Kronecker limit formula [76]:

$$\log|P(z;\tau)| = -\lim_{s\to 1}\frac{(\mathrm{Im}\tau)^s}{2\pi}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{e^{2\pi i(nz_2-mz_1)}}{|m\tau+n|^{2s}}\,. \tag{A.1.18}$$

Similarly, the function Q is defined by [77, 78]

$$Q(z\,;\tau) = e^{2\pi i\,\alpha_Q(z)}\, e^{2\pi i\left(\frac{1}{3}B_3(z_2)-\frac{1}{2}z_2B_2(z_2)\right)}\,\frac{P(z\,;\tau)^{z_2}}{\Gamma_e(z+\tau\,;\tau,\tau)}\,, \tag{A.1.19}$$

where α_Q is a real-valued function chosen according to the same criteria as α_P. Hence Q is a doubly periodic function in z as well, with periods 1 and τ. Its double Fourier expansion is given by [48]

$$\log Q(z\,;\tau) = -\frac{1}{4\pi^2}\sum_{\substack{m,n\in\mathbb{Z}\\ m\neq 0}}\frac{e^{2\pi i(nz_2-mz_1)}}{m(m\tau+n)^2}+\frac{2\pi i\tau}{3}B_3(\{z_2\})+\pi i\phi(z)\,, \tag{A.1.20}$$

where ϕ is a real-valued doubly periodic function related to α_Q.

Let m,n be integers such that $\gcd(m,n)=1$ and $m\neq 0$; from (A.1.18) and (A.1.20) it is possible to derive the following integral formulas [48, 49]:

$$\begin{aligned}\int_0^1 dx \log P\Big(x(m\tau+n)+c\tau+d\,;\tau\Big) &= -\,\pi i\,\frac{B_2(\{md-nc\})}{m(m\tau+n)}+\pi i\,\varphi_P(m,n)\,,\\ \int_0^1 dx \log Q\Big(x(m\tau+n)+c\tau+d\,;\tau\Big) &= \frac{\pi i}{3}\,\frac{B_3(\{md-nc\})}{m(m\tau+n)^2}+\pi i\,\varphi_Q(m,n)\,.\end{aligned}\tag{A.1.21}$$

Here ϕ_P and ϕ_Q are real functions whose precise value depends on the particular choice of the phases α_P and α_Q. We point out that $B_n(\{\cdot\})$ is a continuous function on the real axis because identity (A.1.10) implies that $B_n(0)=B_n(1)$.

From (A.1.21) and definition (3.1.6) we can find a similar formula for the $Q_{c,d}$ function (3.1.6); making use of the identity (A.1.13) for the Bernoulli polynomials we can write it as

$$\begin{aligned}&\int_0^1 dx \log Q_{c,d}\Big(x(m\tau+n)+z\,;\tau\Big)=\\ &\qquad=-\frac{\pi i}{6}\,c\tau+\frac{\pi i}{3}\,\frac{B_3\big(\{m(d+z_1)-n(c+z_2)\}+c(m\tau+n)\big)}{m(m\tau+n)^2}-\\ &\qquad\quad-\frac{\pi i}{m}\,c^2B_1\big(\{m(d+z_1)-n(c+z_2)\}\big)-\frac{\pi i n}{3m}\,c^3-\pi i\big(\phi_Q(m,n)-c\,\phi_P(m,n)\big)\,.\end{aligned}\tag{A.1.22}$$

All the terms in the last line of this equation are purely imaginary.

A.2 Subleading terms in the Bethe Ansatz formula

In section 3.3.3 we assumed for simplicity that the integers q and $\widehat{r}$ satisfy $\gcd(ab,q,\widehat{r}\,)=1$; we will now show how the large-N computation of the superconformal index with the Bethe Ansatz formula can be done without this assumption.

When $\gcd(ab,q,\widehat{r}\,)\neq 1$ the problem is that it is not possible to find a BAE solution u and a valid choice for the vector of integers m such that $u-m\omega$ satisfies (3.3.30). The workaround is to search for u and m that approximate the right-hand side of (3.3.30) instead; to be precise we want to find a choice of integers $\{\widetilde{p},\widetilde{q},\widetilde{r}\}$, indices $\widetilde{\jmath}=0,\dots,\widetilde{p}-1$ and $\widetilde{k}=0,\dots,\widetilde{q}-1$, and vector of integers $m_{\widetilde{\jmath}\widetilde{k}}$, such that $\widetilde{p}\cdot\widetilde{q}=N$ and

$$\frac{\widetilde{\jmath}}{\widetilde{p}}+\frac{\widetilde{k}}{\widetilde{q}}\left(\omega+\frac{\widetilde{r}}{\widetilde{p}}\right)-m_{\widetilde{\jmath}\widetilde{k}}\,\omega\;=\;\frac{j}{p}+\frac{k}{q}\left(ab\omega+\frac{\widehat{r}}{p}\right)+\mathcal{O}\left(\frac{1}{q}\right)\qquad\text{mod }1\,.\tag{A.2.1}$$

Then, we will show that these extra $\mathcal{O}(1/q)$ terms can be neglected without affecting the leading order of the integrand $\mathcal{Z}$:

$$\sum_{\widetilde{\jmath}_1,\widetilde{\jmath}_2=0}^{\widetilde{p}-1}\;\sum_{\widetilde{k}_1\neq\widetilde{k}_2=0}^{\widetilde{q}-1}\log\Gamma_e\left(\Delta+\frac{\widetilde{\jmath}_1-\widetilde{\jmath}_2}{\widetilde{p}}+\frac{\widetilde{k}_1-\widetilde{k}_2}{\widetilde{q}}\left(\omega+\frac{\widetilde{r}}{\widetilde{p}}\right)-\big(m_{\widetilde{\jmath}_1\widetilde{k}_1}-m_{\widetilde{\jmath}_2\widetilde{k}_2}\big)\,\omega\,;a\omega,b\omega\right)=$$

$$= \sum_{j_1,j_2=0}^{p-1} \sum_{k_1\neq k_2=0}^{q-1} \log\Gamma_e\left(\Delta + \frac{j_1-j_2}{p} + \frac{k_1-k_2}{q}\left(ab\omega + \frac{\widehat{r}}{p}\right); a\omega, b\omega\right) + o(q^2)\,. \tag{A.2.2}$$

The quantity in the second line of (A.2.2) is the same as (3.3.37). Hence, the rest of the computation is identical to the one in section 3.3.3.

Let us set $\widehat{h} \equiv \gcd(ab, q, \widehat{r})$; if we reparametrize the index k in terms of new indices $k' = 0, \dots, q/\widehat{h} - 1$ and $k'' = 0, \dots, \widehat{h} - 1$ such that $k \equiv k' + (q/\widehat{h})k''$, we can write the following:

$$\begin{aligned}
\frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{\widehat{r}}{p}\right) &= \frac{j}{p} + \frac{k}{q/\widehat{h}}\left(\frac{ab}{\widehat{h}}\omega + \frac{\widehat{r}/\widehat{h}}{p}\right) = \\
&= \frac{j}{p} + \frac{k'}{q/\widehat{h}}\left(\frac{ab}{\widehat{h}}\omega + \frac{\widehat{r}/\widehat{h}}{p}\right) + \frac{k''(\widehat{r}/\widehat{h})}{p} \quad \mod 1,\, \omega \qquad \text{(A.2.3)}\\
&= \frac{j'}{p} + \frac{k'}{q/\widehat{h}}\left(\frac{ab}{\widehat{h}}\omega + \frac{\widehat{r}/\widehat{h}}{p}\right) \quad \mod 1,\, \omega\,.
\end{aligned}$$

In the last step we defined a new index j' as $j' \equiv j + k''(\widehat{r}/\widehat{h}) \mod p$. The dependence on the index k'' has dropped completely modulo $1, \omega$; considering that BAE solutions cannot repeat values modulo $1, \omega$, we will have to reintroduce the dependence on k'' as a part of the $\mathcal{O}(1/q)$ term.

As a consequence of the definition of $\widehat{h}$, we have that $\gcd(ab/\widehat{h}, q/\widehat{h}, \widehat{r}/\widehat{h}) = 1$. Therefore if we set $h \equiv \gcd(q/\widehat{h}, ab)$, $\widetilde{p} \equiv hp$ and $\widetilde{q} \equiv N/\widetilde{p}$, we can find indices $\widetilde{\jmath} = 0, \dots, \widetilde{p} - 1$, $\widehat{k} = 0, \dots, \widetilde{q}/\widehat{h} - 1$ and an integer $\widetilde{r}$ such that

$$\frac{j'}{p} + \frac{k'}{q/\widehat{h}}\left(\frac{ab}{\widehat{h}}\omega + \frac{\widehat{r}/\widehat{h}}{p}\right) = \frac{\widetilde{\jmath}}{\widetilde{p}} + \frac{\widehat{k}}{\widetilde{q}/\widehat{h}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right) \quad \mod 1,\, \omega\,. \tag{A.2.4}$$

This relation can be obtained by following the same steps used to prove (3.3.24), with $ab/\widehat{h}$, $q/\widehat{h}$ and $\widehat{r}/\widehat{h}$ taking the place of ab, q and r respectively.

We can now chose the value for the vector of integers m so that the following identity holds:

$$\frac{j}{p} + \frac{k}{q}\left(ab\omega + \frac{\widehat{r}}{p}\right) \equiv \frac{\widetilde{\jmath}}{\widetilde{p}} + \frac{\widehat{k}}{\widetilde{q}/\widehat{h}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right) + m_{\widetilde{\jmath}\widehat{k}}\,\omega \quad \mod 1\,. \tag{A.2.5}$$

Lastly, we define the index $\widetilde{k} = 0, \dots, \widetilde{q} - 1$ as $\widetilde{k} \equiv k'' + \widehat{k}\,\widehat{h}$, which gives us

$$\frac{\widetilde{\jmath}}{\widetilde{p}} + \frac{\widetilde{k}}{\widetilde{q}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right) \equiv \frac{\widetilde{\jmath}}{\widetilde{p}} + \frac{\widehat{k}}{\widetilde{q}/\widehat{h}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right) + \frac{k''}{\widetilde{q}}\left(\omega + \frac{\widetilde{r}}{\widetilde{p}}\right)\,. \tag{A.2.6}$$

Considering that $k''/\widetilde{q} = \mathcal{O}(1/q)$, if we combine relations (A.2.5) and (A.2.6) together we finally obtain (A.2.1). The only thing left to do is to verify that the simplification (A.2.2) works at leading order.

Let us set $Z \equiv \Delta + (\widetilde{\jmath}_1 - \widetilde{\jmath}_2)/\widetilde{p}$. We need to verify that the following is true:

$$\sum_{\widehat{k}_1 \neq \widehat{k}_2 = 0}^{\widetilde{q}/\widehat{h}-1} \log \Gamma_e \left(Z + \frac{\widehat{k}_1 - \widehat{k}_2}{\widetilde{q}/\widehat{h}} \left(\omega + \frac{\widetilde{r}}{\widetilde{p}} \right) + \frac{k''_1 - k''_2}{\widetilde{q}} \left(\omega + \frac{\widetilde{r}}{\widetilde{p}} \right) - \left(m_{\widetilde{\jmath}_1 \widehat{k}_1} - m_{\widetilde{\jmath}_2 \widehat{k}_2} \right) \omega \, ; a\omega, b\omega \right) =$$
$$= \sum_{\widehat{k}_1 \neq \widehat{k}_2 = 0}^{\widetilde{q}/\widehat{h}-1} \log \Gamma_e \left(Z + \frac{\widehat{k}_1 - \widehat{k}_2}{\widetilde{q}/\widehat{h}} \left(\omega + \frac{\widetilde{r}}{\widetilde{p}} \right) - \left(m_{\widetilde{\jmath}_1 \widehat{k}_1} - m_{\widetilde{\jmath}_2 \widehat{k}_2} \right) \omega \, ; a\omega, b\omega \right) + o(q^2) \,. \tag{A.2.7}$$

We are ignoring the sums over $\widetilde{\jmath}_1, \widetilde{\jmath}_2 = 0, \ldots, \widetilde{p} - 1$ and $k''_1, k''_2 = 0, \ldots, \widehat{h} - 1$ because $\widetilde{p}, \widehat{h} \sim \mathcal{O}(1)$; if (A.2.7) holds, then (A.2.2) would immediately follow.

A relation similar to (A.2.7), albeit simpler, has already been proven in [3], and we can use it as a staring point. Let us define the following function:

$$f(z;\tau) = \sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \log \Gamma_e \left(z + \frac{\gamma - \delta}{\widetilde{N}} \tau \, ; n\tau, n\tau \right) , \tag{A.2.8}$$

where n is any positive integer. As long as $z + t\,\tau$ does not cross a zero or a pole of Γ_e for any $t \in (-1,0) \cup (0,1)$, this function has been shown to satisfy the following bound:

$$\left| f\left(z + C\tau/\widetilde{N} \, ; \tau\right) - f(z;\tau) \right| \leq \mathcal{O}(\widetilde{N} \log \widetilde{N}) \,, \tag{A.2.9}$$

for any $C \in (-1,1)$. There are a few details about the proof of (A.2.9) that will be useful; let us review them briefly. The first step in the proof is to use the mean value theorem to write

$$\left| f\left(z + C\tau/\widetilde{N} \, ; \tau\right) - f(z;\tau) \right| \leq \frac{|\tau|}{\widetilde{N}} \left(\left| \partial_z f\left(z + \bar{c}_1 \tau/\widetilde{N} \, ; \tau\right) \right| + \left| \partial_z f\left(z + \bar{c}_2 \tau/\widetilde{N} \, ; \tau\right) \right| \right) \tag{A.2.10}$$

for some $\bar{c}_1, \bar{c}_2 \in \mathbb{R}$, with $|\bar{c}_{1,2}| < |C|$.[1] Then the authors of [3] have shown that for any $\bar{c} \in (-1,1)$ the following bound holds:

$$\frac{1}{\widetilde{N}} \left| \partial_z f\left(z + \bar{c}\,\tau/\widetilde{N} \, ; \tau\right) \right| \leq \frac{1}{\widetilde{N}} \sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \left| \frac{\partial_z \Gamma_e \left(z + \frac{\gamma - \delta + \bar{c}}{\widetilde{N}} \tau \, ; n\tau, n\tau \right)}{\Gamma_e \left(z + \frac{\gamma - \delta + \bar{c}}{\widetilde{N}} \tau \, ; n\tau, n\tau \right)} \right| \leq \mathcal{O}(\widetilde{N} \log \widetilde{N}) \,. \tag{A.2.11}$$

Relation (A.2.9) then follows from (A.2.10) and (A.2.11).

[1]The mean value theorem is applied to the real and imaginary part separately, which is the reason for the need of two constants, $\bar{c}_1$ and $\bar{c}_2$.

We can't use formula (A.2.9) to prove (A.2.7) directly; we need to generalize (A.2.9) a bit first. Given any two subsets S, S' of the set $\{1, \ldots, \widetilde{N}\}$, we consider the following function:

$$f_{S,S'}(z;\tau) = \sum_{\substack{\gamma \in S,\, \delta \in S' \\ \gamma \neq \delta}}^{\widetilde{N}} \log \Gamma_e\Big(z + \frac{\gamma - \delta}{\widetilde{N}}\,\tau\,; n\tau, n\tau\Big)\,. \tag{A.2.12}$$

Then a similar relation to (A.2.9) holds for $f_{S,S'}$ as well:

$$\Big| f_{S,S'}\big(z + C\tau/\widetilde{N}\,; \tau\big) - f_{S,S'}(z;\tau) \Big| \leq \mathcal{O}(\widetilde{N} \log \widetilde{N})\,. \tag{A.2.13}$$

Indeed, the following trivial inequality:

$$\sum_{\substack{\gamma \in S,\, \delta \in S' \\ \gamma \neq \delta}}^{\widetilde{N}} \left| \frac{\partial_z \Gamma_e\Big(z + \frac{\gamma-\delta+\bar{c}}{\widetilde{N}}\,\tau\,; n\tau, n\tau\Big)}{\Gamma_e\Big(z + \frac{\gamma-\delta+\bar{c}}{\widetilde{N}}\,\tau\,; n\tau, n\tau\Big)} \right| \leq \sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \left| \frac{\partial_z \Gamma_e\Big(z + \frac{\gamma-\delta+\bar{c}}{\widetilde{N}}\,\tau\,; n\tau, n\tau\Big)}{\Gamma_e\Big(z + \frac{\gamma-\delta+\bar{c}}{\widetilde{N}}\,\tau\,; n\tau, n\tau\Big)} \right| \tag{A.2.14}$$

together with (A.2.11) and an analogue of (A.2.10) imply (A.2.13).

We can use relation (A.2.13) to show that

$$\begin{aligned} &\sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \log \Gamma_e\Big(z + \frac{\gamma - \delta}{\widetilde{N}}\,\tau - (m_\delta - m_\gamma)\,\omega + C\tau/\widetilde{N}\,; n\tau, n\tau\Big) = \\ &\quad = \sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \log \Gamma_e\Big(z + \frac{\gamma - \delta}{\widetilde{N}}\,\tau - (m_\delta - m_\gamma)\,\omega\,; n\tau, n\tau\Big) + \mathcal{O}(\widetilde{N}\log \widetilde{N})\,, \end{aligned} \tag{A.2.15}$$

where $\{m_\delta\}_{\delta=1}^{\widetilde{N}}$ is a vector of integers between 1 and ab (with $ab \sim \mathcal{O}(\widetilde{N}^0)$), $\omega \in \mathbb{C}$ and z is such that $z - (m_\delta - m_\gamma)\,\omega + t\,\tau$ does not cross a zero or a pole of Γ_e for any $t \in (-1, 0) \cup (0, 1)$ and any possible value of $(m_\delta - m_\gamma)$. Indeed, we can write

$$\sum_{\gamma \neq \delta = 1}^{\widetilde{N}} \log \Gamma_e\Big(z + \frac{\gamma - \delta}{\widetilde{N}}\,\tau - (m_\delta - m_\gamma)\,\omega\,; n\tau, n\tau\Big) \equiv \sum_{i_1, i_2 = 1}^{ab} f_{S(i_1), S(i_2)}\big(z - (i_1 - i_2)\,\omega\,; \tau\big)\,, \tag{A.2.16}$$

where $S(i) \equiv \{\delta \mid m_\delta = i\}$. Then (A.2.15) follows directly from (A.2.13).

At last, let us show the validity of simplification (A.2.7). In order to apply (A.2.15) we need to first change the moduli of the Γ_e from $(a\omega, b\omega)$ to $\big(n(\omega + \widetilde{r}\,/\,\widetilde{p}),\, n(\omega + \widetilde{r}\,/\,\widetilde{p})\big)$, where n is some positive integer. We can use the following identity:

$$\log \Gamma_e\Big(u\,; a\omega, b\omega\Big) = \sum_{\ell_1=0}^{b\widetilde{p}-1} \sum_{\ell_2=0}^{a\widetilde{p}-1} \log \Gamma_e\Big(u + (\ell_1 a + \ell_2 b)\omega\,; \widetilde{p}ab\omega + abr, \widetilde{p}ab\omega + abr\Big)\,, \tag{A.2.17}$$

which follows from (A.1.3) and the invariance of Γ_e under integers shifts. Then, for any given value of ℓ_1, ℓ_2 we can use (A.2.15) with $z \equiv Z + (\ell_1 a + \ell_2 b)\, \omega\,,\ \tau \equiv \omega + \widetilde{r}\,/\,\widetilde{p}$ and $n \equiv \widetilde{p}\, ab$. It is easy to verify that z satisfies the required condition necessary for avoiding zeros and poles, considering that the possible values for Δ are $\Delta \equiv \tau + \sigma$ and $\Delta \equiv \Delta_I$, with Δ_I satisfying condition (3.3.36). This concludes the proof of (A.2.7).

Appendix B

Appendix for equivariant volumes and holography

B.1 Fixing the Kähler moduli of $\mathrm{AdS}_5 \times M_6$ solutions with $\mathbb{Z}_2$ symmetry

In this appendix we verify that for the Calabi-Yau geometry considered in section 5.2.2.2 there is a critical point of $\mathbb{V}^{(3)}(\lambda_A, \lambda_{AB}, \epsilon_I)$ with $\lambda_a = \lambda_{ab} = 0$. Even if the group of gauge transformations (4.4.8) has a sufficient number of parameters to potentially gauge away all λ_a and λ_{ab}, in orbifolds $\mathbb{M}_4$ with a small number of vectors in the fan there are often obstructions that make this impossible.

In the following we will verify that the values of the Kähler moduli λ_A, λ_{AB} given by

$$\begin{cases} \lambda_a = \lambda_{ab} = 0\,, \\ \overline{\lambda}_{a,d+1} \text{ such that } \sum_b D_{ab}\overline{\lambda}_{b,d+1} = -\overline{\nu}_{M5}\, M_a\,, \\ \overline{\lambda}_{d+1} \text{ such that } \partial_{\overline{\lambda}_{d+1}}\mathbb{V}^{(3)} = 0\,, \end{cases} \tag{B.1.1}$$

are an extremum of $\mathbb{V}^{(3)}$, under the constraints imposed by the flux equations

$$\partial_{\lambda_A}\mathbb{V}^{(2)}(\lambda_A, \lambda_{AB}, \epsilon_I) = -\overline{\nu}_{M5}\, M_A\,. \tag{B.1.2}$$

In practice, we will show that there exists a value for the Lagrange parameters ρ_A such that the function

$$\mathcal{E} = \mathbb{V}^{(3)} + \sum_A \rho_A\big(\partial_{\lambda_A}\mathbb{V}^{(2)} + \overline{\nu}_{M5}\, M_A\big) \tag{B.1.3}$$

has null derivatives with respect to λ_A, λ_{AB}. The equations that we will solve are then

$$\frac{\partial}{\partial\lambda_A}\,\mathcal{E}\left(\lambda_A, \lambda_{AB}, \epsilon_I, \rho_A\right) = 0\,, \qquad \frac{\partial}{\partial\lambda_{AB}}\,\mathcal{E}\left(\lambda_A, \lambda_{AB}, \epsilon_I, \rho_A\right) = 0\,, \tag{B.1.4}$$

where the λ are given by (B.1.1) while ϵ_1, ϵ_2 can take general values. We will study the case $\epsilon_1 = \epsilon_2 = 0$ separately.

Case $(\epsilon_1, \epsilon_2) \neq (0,0)$. We claim that values of ρ_A that solve (B.1.4) exist and they are the solutions of the following linear system:

$$\begin{aligned}&\sum_A \rho_A (\mathcal{C}_A)\big|_b = h_b \,, \qquad b = 1, \ldots, d\,,\\ &h_b \equiv -\frac{1}{2}\left[\Big(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)\Big)^2 \Big(\bar{\lambda}_{d+1} + \sum_a \bar{\lambda}_{a,d+1} c_1^{\mathbb{T}}(L_a)\Big)^2\right]\Big|_b \,.\end{aligned} \tag{B.1.5}$$

Indeed, when the above equations are satisfied we have

$$\begin{aligned}\partial_{\lambda_B} \sum_A \rho_A \big(\partial_{\lambda_A} \mathbb{V}^{(2)} + \overline{\nu}_{M5}\, M_A\big) &= \int_{\mathbb{M}_4} \frac{\sum_A \rho_A \mathcal{C}_A \mathcal{C}_B}{\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)} = \\ = -\frac{1}{2} \int_{\mathbb{M}_4} \mathcal{C}_B \Big(\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a)\Big)&\Big(\bar{\lambda}_{d+1} + \sum_b \bar{\lambda}_{b,d+1} c_1^{\mathbb{T}}(L_b)\Big)^2 = -\,\partial_{\lambda_B} \mathbb{V}^{(3)},\end{aligned} \tag{B.1.6}$$

where we have used $\lambda_a = \lambda_{ab} = 0$. This gives us $\partial_{\lambda_A} \mathcal{E} = 0$. The $\partial_{\lambda_{AB}} \mathcal{E} = 0$ equations can also be derived from (B.1.5) in a similar manner.

Let us now discuss the existence of solutions to the equations (B.1.5). The restrictions of $\sum_A \rho_A \mathcal{C}_A$ to the fixed points are not independent, they satisfy the following linear relation:

$$0 = \int_{\mathbb{M}_4} \sum_A \rho_A \mathcal{C}_A = \sum_b \frac{\sum_A \rho_A (\mathcal{C}_A)|_b}{d_{b,b+1}\, \epsilon_1^b\, \epsilon_2^b} \,. \tag{B.1.7}$$

However, the h_b also satisfy the same linear relation, given that the value of $\overline{\lambda}_{d+1}$ is set by the condition $\partial_{\overline{\lambda}_{d+1}} \mathbb{V}^{(3)} = 0$, which reads

$$0 = \frac{1}{2} \int_{\mathbb{M}_4} (\epsilon_3 + \sum_a c_1^{\mathbb{T}}(L_a))^2 (\bar{\lambda}_{d+1} + \sum_a \bar{\lambda}_{a,d+1} c_1^{\mathbb{T}}(L_a))^2 = -\sum_b \frac{h_b}{d_{b,b+1}\, \epsilon_1^b\, \epsilon_2^b} \,. \tag{B.1.8}$$

We can thus always eliminate one of the equations (B.1.5). Considering that shifting $\rho_A \to \rho_A + \sum_I \alpha^I V_I^A$ with $\sum_I \alpha^I \epsilon_I = 0$ leaves the left-hand side of (B.1.5) invariant, we can always gauge away ρ_{d+1}[1] and one of the ρ_a. We are left with a system of $d-1$ equations in $d-1$ variables that generally is not singular and thus has a solution.

There is an edge case in which the system of equations must be further reduced: when there is $\overline{a} \in \{1, \ldots, d\}$ such that $\epsilon_2^{\overline{a}} = 0$ (and consequently $\epsilon_1^{\overline{a}-1} = 0$). Since $(\epsilon_1, \epsilon_2) \neq (0,0)$ and $\mathbb{M}_4$ is compact we must have $\epsilon_1^{\overline{a}} \neq 0$, $\epsilon_2^{\overline{a}-1} \neq 0$. The $b = \overline{a} - 1$ and $b = \overline{a}$ equations are (ρ_{d+1} has been gauged away)

$$-\,\rho_{\overline{a}}\, \epsilon_2^{\overline{a}-1} = h_{\overline{a}-1} \,, \qquad -\,\rho_{\overline{a}}\, \epsilon_1^{\overline{a}} = h_{\overline{a}} \,. \tag{B.1.9}$$

In principle depending on the value of $h_{\overline{a}-1}$ and $h_{\overline{a}}$ the above equations might not have a solution. However if we consider that $h_{\overline{a}-1}$ and $h_{\overline{a}}$ can only depend on $\epsilon_2^{\overline{a}-1}$

[1] We note that the $(\epsilon_1, \epsilon_2) \neq (0,0)$ hypothesis is needed to set $\rho_{d+1} = 0$.

and $\epsilon_1^{\overline{a}}$ respectively,[2] and that in general $d_{\overline{a}-1,\overline{a}}\,\epsilon_1^{\overline{a}-1} = -d_{\overline{a},\overline{a}+1}\,\epsilon_2^{\overline{a}}$, then the only way for the right-hand side of (B.1.8) to be finite is for

$$\frac{h_{\overline{a}-1}}{\epsilon_2^{\overline{a}-1}} = \frac{h_{\overline{a}}}{\epsilon_1^{\overline{a}}}\,, \tag{B.1.10}$$

which means that equations (B.1.5) are solvable without issue.

Case $(\boldsymbol{\epsilon_1}, \boldsymbol{\epsilon_2}) = (\mathbf{0}, \mathbf{0})$. It is not immediately clear whether the solutions to equations (B.1.5) are well-behaving in the limit $\epsilon_1, \epsilon_2 \to 0$. However when ϵ_1 and ϵ_2 are zero the equations (B.1.4) are quite simple and we can solve them directly.

For $\epsilon_1 = \epsilon_2 = 0$ we have $c_1^{\mathbb{T}}(L_a) = c_1(L_a)$ and thus

$$\int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_{a_1})\dots c_1^{\mathbb{T}}(L_{a_k}) = \begin{cases} D_{a_1a_2} & k=2 \\ 0 & \text{otherwise} \end{cases}. \tag{B.1.11}$$

From this relation it easily follows that

$$\int_{\mathbb{M}_4} \frac{c_1^{\mathbb{T}}(L_{a_1})\dots c_1^{\mathbb{T}}(L_{a_k})}{\epsilon_3 + \sum_b c_1^{\mathbb{T}}(L_b)} = \begin{cases} (\epsilon_3)^{-1} D_{a_1a_2} & k=2 \\ 0 & k>2 \end{cases}. \tag{B.1.12}$$

Using the above relations the extremization equations (B.1.4) become

$$\begin{cases} \partial_{\lambda_a} \text{ equation:} & \frac{1}{2}\sum_b D_{ab}(\overline{\lambda}_{d+1}^2 + 2\,\epsilon_3\,\overline{\lambda}_{d+1}\overline{\lambda}_{b,d+1}) + (\epsilon_3)^{-1}\sum_b D_{ab}\rho_b = 0 \\ \partial_{\lambda_{ab}} \text{ equation:} & \frac{1}{2}\,\epsilon_3\,\overline{\lambda}_{d+1}^2 D_{ab} - \rho_{d+1} D_{ab} = 0 \\ \partial_{\lambda_{a,d+1}} \text{ equation:} & -\sum_b D_{ab}(\epsilon_3^2\,\overline{\lambda}_{d+1}\overline{\lambda}_{b,d+1} + \epsilon_3\,\overline{\lambda}_{d+1}^2) - \sum_b D_{ab}(\rho_b - \rho_{d+1}) = 0 \\ \partial_{\lambda_{d+1,d+1}} \text{ equation:} & \begin{aligned} &\tfrac{1}{2}\sum_{a,b} D_{ab}(\epsilon_3^3\,\overline{\lambda}_{a,d+1}\overline{\lambda}_{b,d+1} + 6\,\epsilon_3^2\,\overline{\lambda}_{d+1}\overline{\lambda}_{b,d+1} + 3\epsilon_3\,\overline{\lambda}_{d+1}^2) \\ &+ \sum_{a,b} D_{ab}(\rho_b - \rho_{d+1}) = 0 \end{aligned} \end{cases} \tag{B.1.13}$$

The $\partial_{\lambda_{d+1}}$ equation was omitted because it is trivial: $\partial_{\lambda_{d+1}} \sum_A \rho_A \big(\partial_{\lambda_A} \mathbb{V}^{(2)} + \overline{\nu}_{M5} M_A\big) = 0$ and $\partial_{\lambda_{d+1}} \mathbb{V}^{(3)} = -\partial_{\overline{\lambda}_{d+1}} \mathbb{V}^{(3)} = 0$ because of (B.1.1).

The solution to (B.1.13) is[3]

$$\rho_a = -\epsilon_3^2\,\overline{\lambda}_{d+1}\overline{\lambda}_{b,d+1} - \frac{1}{2}\,\epsilon_3\,\overline{\lambda}_{d+1}^2\,, \qquad \rho_{d+1} = \frac{1}{2}\,\epsilon_3\,\overline{\lambda}_{d+1}^2\,, \tag{B.1.14}$$

and thus (B.1.1) is the proper extremum of $\mathbb{V}^{(3)}$ under the flux constraints.

[2] By definition $h_{\overline{a}-1}$ and $h_{\overline{a}}$ are the restrictions of an equivariant form on the fixed points $\overline{a}-1$ and $\overline{a}$.

[3] When we plug this solution into the left-hand side of the $\partial_{\lambda_{d+1,d+1}}$ equation we do not get zero straight away, but rather we get the same expression as $\partial_{\overline{\lambda}_{d+1}} \mathbb{V}^{(3)}$, which is zero by (B.1.1).

B.2 $\mathrm{AdS}_3 \times M_7$ solutions with the addition of higher times

In this appendix we revisit the computation of section 5.3.3, now with the inclusion of second and triple times in the equivariant volume. For the $\mathrm{AdS}_3 \times M_7$ solutions we considered in section 5.3.3 there was no need to add any higher times. We will now show that it is still possible to perform the computation even when the equivariant volume is over-parameterized. The extremization procedure for the parameters in excess plays a crucial role this time: relations that where automatically verified when $\mathbb{V}^{(2)}$ only included single times are now derived as extremization conditions. This provides further evidence that extremization is the correct way to deal with any parameter λ that is not fixed by the flux constraints.

The second degree homogeneous component of the equivariant volume with triple times is given by

$$\begin{aligned} &\mathbb{V}^{(2)}(\lambda_A\,,\lambda_{AB}\,,\lambda_{ABC}\,,\epsilon_I) = \int_{\mathbb{M}_4} \frac{(\tau^{\mathbb{T}})^2}{2\,p\,\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}}\,, \\ &\tau^{\mathbb{T}} = \sum_A \lambda_A\,\mathcal{C}_A + \sum_{A,B} \lambda_{AB}\,\mathcal{C}_A\,\mathcal{C}_B + \sum_{A,B,C} \lambda_{ABC}\,\mathcal{C}_A\,\mathcal{C}_B\,\mathcal{C}_C\,. \end{aligned} \tag{B.2.1}$$

We need to impose the following flux constraints:

$$-\,\nu_{D3}\,M_A = \partial_{\lambda_A}\mathbb{V}^{(2)}(\lambda_A\,,\lambda_{AB}\,,\lambda_{ABC}\,,\epsilon_I) = \int_{\mathbb{M}_4} \frac{\mathcal{C}_A\,\tau^{\mathbb{T}}}{p\,\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}}\,. \tag{B.2.2}$$

Proceeding in a similar way as we did in section 5.3.3, we will set all the λ to zero except for $\lambda_{d+1,\,d+2}$ and $\lambda_{d+1,\,d+2,\,A}$. This assumption is justified by the fact that in principle the group of gauge transformation for the single, double and triple times has enough parameters to gauge away all the λ except $\lambda_{d+1,\,d+2}$ and $\lambda_{d+1,\,d+2,\,A}$.[4] At the end of this appendix we will quickly check that $\mathbb{V}^{(2)}$ does indeed have a critical point for $\lambda_{a,b} = \lambda_{a,b,d+1} = \lambda_{a,b,d+2} = \lambda_{a,b,c} = 0$, thus verifying the correctness of this choice of λ. The flux constraints (B.2.2) now read

$$-\,\nu_{D3}\,M_a = \frac{1}{p}\int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_a)\Big(\lambda_{d+1,\,d+2} + \sum_A \lambda_{d+1,\,d+2,\,A}\,\mathcal{C}_A\Big) = \sum_b D_{ab}\overline{\lambda}_b\,, \tag{B.2.3}$$

where

$$\overline{\lambda}_a = \frac{\lambda_{d+1,\,d+2,\,a} - \lambda_{d+1,\,d+1,\,d+2}}{p} + \frac{\mathfrak{t}_a(\lambda_{d+1,\,d+1,\,d+2} - \lambda_{d+1,\,d+2,\,d+2})}{p^2}\,. \tag{B.2.4}$$

Up to gauge transformations, the $\overline{\lambda}_a$ are then fixed to be $\overline{\lambda}_a = -\nu_{D3}\,m_a$, where the m_a are such that $\sum_b D_{ab}m_b = M_a$.

[4] Note that $\lambda_{a,b}$, $\lambda_{a,b,d+1}$, $\lambda_{a,b,d+2}$ and $\lambda_{a,b,c}$ do not appear inside $\mathbb{V}^{(2)}$ unless $a,b,c \in \{\overline{a},\overline{a}+1\}$ for some $\overline{a} \in \{1,\ldots,d\}$.

We notice that the flux constraints did not fix all the λ, but rather there is one such parameter left:

$$\begin{aligned}\mathbb{V}^{(2)} &= \frac{1}{2p}\int_{\mathbb{M}_4} \mathcal{C}_{d+1}\,\mathcal{C}_{d+2}\left(\lambda_{d+1,\,d+2} + \sum_A \lambda_{d+1,\,d+2,\,A}\,\mathcal{C}_A\right)^2 \\ &= \frac{p}{2}\,\nu_{D3}^2\int_{\mathbb{M}_4} \mathcal{C}_{d+1}\,\mathcal{C}_{d+2}\left(\overline{\lambda} - \sum_a m_a\, c_1^{\mathbb{T}}(L_a)\right)^2\,,\end{aligned} \tag{B.2.5}$$

where

$$\overline{\lambda} = \frac{\lambda_{d+1,\,d+2} - \epsilon_3\,\lambda_{d+1,\,d+1,\,d+2}}{p\,\nu_{D3}} + \frac{\epsilon_4\left(\lambda_{d+1,\,d+1,\,d+2} - \lambda_{d+1,\,d+2,\,d+2}\right)}{p^2\,\nu_{D3}}\,. \tag{B.2.6}$$

Our procedure prescribes to fix the value of $\overline{\lambda}$ by extremizing $\mathbb{V}^{(2)}$ with respect to it. If we call $b(\epsilon_I)$ the extremal value of $\overline{\lambda}$, we find that

$$0 = \frac{\partial}{\partial\overline{\lambda}}\mathbb{V}^{(2)} = p\,\nu_{D3}^2\int_{\mathbb{M}_4} \mathcal{C}_{d+1}\,\mathcal{C}_{d+2}\left(b(\epsilon_I) - \sum_a m_a\, c_1^{\mathbb{T}}(L_a)\right)\,. \tag{B.2.7}$$

Notably, the equation we obtain is the exact same as (5.3.49). In the context of the computation without higher times, equation (5.3.49) was a trivial relation, a predictable consequence of the fact that there are only $d-1$ single times, but d fixed points. In the computation of this appendix the same relation is now derived as an extremization condition.

If we substitute (B.2.7) into $\mathbb{V}^{(2)}$ we get

$$\mathbb{V}^{(2)} = -\frac{p}{2}\,\nu_{D3}^2\int_{\mathbb{M}_4} \mathcal{C}_{d+1}\,\mathcal{C}_{d+2}\left(b(\epsilon_I) - \sum_a m_a\, c_1^{\mathbb{T}}(L_a)\right)\sum_a m_a\, c_1^{\mathbb{T}}(L_a)\,, \tag{B.2.8}$$

which is the same as the main result of section 5.3.3.

We can quickly verify that the values of λ that we have fixed are an extremum of $\mathbb{V}^{(2)}$ by employing the same strategy as appendix B.1. We can find the values of the Lagrange parameters ρ_A such that the function

$$\mathcal{E}(\lambda_A\,,\lambda_{AB}\,,\lambda_{ABC}\,,\epsilon_I,\rho_A) = \mathbb{V}^{(2)} + \sum_A \rho_A\big(\partial_{\lambda_A}\mathbb{V}^{(2)} + \nu_{D3}\,M_A\big) \tag{B.2.9}$$

has null derivatives with respect to $\lambda_A\,,\lambda_{AB}\,,\lambda_{ABC}$ by solving the following linear system:

$$\sum_A \rho_A\big(\mathcal{C}_A\big)\big|_b = -p\,\nu_{D3}\Big[\mathcal{C}_{d+1}\,\mathcal{C}_{d+2}\left(b(\epsilon_I) - \sum_a m_a\, c_1^{\mathbb{T}}(L_a)\right)\Big]\Big|_b\,. \tag{B.2.10}$$

Using the same line of reasoning as in appendix B.1, solutions to this system exist and thus (B.2.8) is the proper extremal value of $\mathbb{V}^{(2)}$ (with respect to the extremization in λ).

B.3 $\mathrm{AdS}_4 \times M_6$ gravity solutions

In this appendix we study the family of $\mathrm{AdS}_4 \times M_6$ solutions to massive type IIA supergravity constructed in [55]. The internal space is a $\mathbb{P}^1$ bundle over a four-dimensional compact manifold, $\mathbb{P}^1 \hookrightarrow M_6 \to B_4$, where the base space can be either a Kähler-Einstein manifold ($B_4 = \mathrm{KE}_4$) or the product of two Riemann surfaces ($B_4 = \Sigma_1 \times \Sigma_2$). In the general class of solutions in [55], the $\mathbb{P}^1$ bundle is the projectivization of the canonical bundle over B_4, $\mathbb{P}(K \oplus \mathcal{O})$. In what follows, we will focus on spaces with positive curvature and set to zero the constant parameter ℓ appearing in [55]. This last choice is motivated by our interest for systems with only D4 and D8 branes, therefore all fluxes, except for $F_{(0)}$ and $F_{(4)}$, must vanish. In both configurations, the metric in the string frame is[5]

$$\mathrm{d}s^2_{\text{s.f.}} = \mathrm{e}^{2A}\big(\mathrm{d}s^2_{\mathrm{AdS}_4} + \mathrm{d}s^2_{M_6}\big)\,, \tag{B.3.1}$$

where $\mathrm{d}s^2_{\mathrm{AdS}_4}$ is the metric on AdS_4 with unit radius. The details of the internal space, along with the expressions for the dilaton and the form fluxes, will be given case by case. The solutions in [55] corresponding to the geometries discussed in section 5.3.2 are cut into half along the equator of the $\mathbb{P}^1$ fibre due to the presence of an O8 plane.

B.3.1 Kähler-Einstein base space

We begin considering $B_4 = \mathrm{KE}_4$, in which case the metric on M_6 is given by (setting $\kappa = +1$ in [55])

$$\mathrm{d}s^2_{M_6} = -\frac{q'}{4xq}\,\mathrm{d}x^2 - \frac{q}{xq' - 4q}\,D\psi^2 + \frac{q'}{3q' - xq''}\,\mathrm{d}s^2_{\mathrm{KE}_4}\,, \tag{B.3.2}$$

where

$$q(x) = x^6 + \frac{\sigma}{2}x^4 + 4x^3 - \frac{1}{2}\,, \tag{B.3.3}$$

with σ a real parameter. Here, we introduced $D\psi = \mathrm{d}\psi + \rho$, where the one-form ρ is defined on KE_4 and is such that $\mathrm{d}_4\rho = -\mathcal{R}$, with $\mathcal{R}$ the Ricci form of KE_4. The line element $\mathrm{d}s^2_{\mathrm{KE}_4}$ is normalized such that its scalar curvature is $R_{\mathrm{KE}} = 4$. The background under exam corresponds to $\sigma > -9$, in which case the metric is smooth and well-defined given that ψ is 2π-periodic and $x \in [0, x_+]$, with x_+ the only positive root of $q(x)$. In $x = 0$ the S^1 fibre parameterized by ψ does not shrink and here is located an O8-plane [55]. The warp factor of the ten-dimensional metric is

$$\mathrm{e}^{2A} = L^2\sqrt{\frac{x^2q' - 4xq}{q'}}\,, \tag{B.3.4}$$

[5] Notice the different normalization of $\mathrm{d}s^2_{M_6}$ with respect to [55].

with L a real constant. The dilaton reads

$$e^{2\Phi} = \frac{72L^4}{f_0^2}\,\frac{xq'}{(3q'-xq'')^2}\left(\frac{x^2q'-4xq}{q'}\right)^{3/2}, \tag{B.3.5}$$

where we find convenient to introduce the constant f_0 in order to parameterize the Romans mass

$$F_{(0)} = \frac{f_0}{L^3}, \tag{B.3.6}$$

and the four-form flux is given by

$$F_{(4)} = -\frac{L\,f_0}{12}\left[\frac{3x(x^6-5x^3-\sigma x-5)}{(1-x^3)^2}\,dx\wedge D\psi\wedge\mathcal{R} + \frac{9x^5+5\sigma x^3+45x^2+\sigma}{6(1-x^3)}\,\mathcal{R}\wedge\mathcal{R}\right]. \tag{B.3.7}$$

All the other fields, namely the two-forms $B_{(2)}$ and $F_{(2)}$, vanish.

The first step we take in the analysis is the quantization of the fluxes, which imposes

$$(2\pi\ell_s)F_{(0)} = n_0 \in \mathbb{Z}, \qquad \frac{1}{(2\pi\ell_s)^3}\int_{\Sigma_4} F_{(4)} = N_{\Sigma_4} \in \mathbb{Z} \tag{B.3.8}$$

for any four-cycle Σ_4 on M_6. Letting Σ_α be a basis of two-cycles for $H_2(\mathrm{KE}_4,\mathbb{Z})$, we take as a basis for $H_4(M_6,\mathbb{Z})$ the set $\{C_\alpha, C_+\}$, where C_α are the four-cycles obtained by considering the fibration $\mathbb{P}^1 \hookrightarrow C_\alpha \to \Sigma_\alpha$, and C_+ is a copy of the KE base space at $x = x_+$. Performing the integrals, we obtain the fluxes

$$\begin{aligned} N_\alpha &= \frac{\pi^2 L\,f_0}{6(2\pi\ell_s)^3}\,\frac{x_+^2(3x_+^3+2\sigma x_+ +15)}{1-x_+^3}\,m_k\,n_\alpha\,,\\ N_+ &= -\frac{\pi^2 L\,f_0}{18(2\pi\ell_s)^3}\,\frac{9x_+^5+5\sigma x_+^3+45x_+^2+\sigma}{1-x_+^3}\,M_k\,, \end{aligned} \tag{B.3.9}$$

where we defined the integers

$$n(\Sigma_\alpha) = \frac{1}{2\pi}\int_{\Sigma_\alpha}\mathcal{R} = m_k\,n_\alpha\,, \qquad M_k = \frac{1}{4\pi^2}\int_{\mathrm{KE}_4}\mathcal{R}\wedge\mathcal{R}\,. \tag{B.3.10}$$

m_k is the Fano index of the KE_4 and is the largest positive integer such that all of the n_α are integers. These integers take the values $m_k = (3,2,1)$ and $M_k = (9,8,6)$ for $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and dP_3, respectively.

For the rest of this subsection we will restrict to the case $\sigma = 0$. In order to understand this assumption, we first need to make contact with the equivariant volume extremization procedure. The toric manifold KE_4 is completely described by its fan v^a, which defines the toric divisors Σ_a and their associated line bundles L_a. The set of divisors $D_a \subset M_6$ is naturally induced as the $\mathbb{P}^1$ fibrations over Σ_a, to which we must add D_{d+1}, a copy of KE_4 at the pole of the half $\mathbb{P}^1$. The corresponding integer fluxes are defined as

$$M_A = \frac{1}{(2\pi\ell_s)^3}\int_{D_A} F_{(4)} \tag{B.3.11}$$

and, for $a=1,\dots,d$, they read

$$M_a = \frac{\pi L\, f_0}{12(2\pi\ell_s)^3}\,\frac{x_+^2(3x_+^3+2\sigma x_+ +15)}{1-x_+^3}\times\int_{\Sigma_a}\mathcal{R}\,. \tag{B.3.12}$$

Recalling that $\sum_a c_1(L_a) = c_1(T\mathrm{KE}_4) = \mathcal{R}/(2\pi)$, we obtain

$$\sum_a \int_{\Sigma_a}\mathcal{R} = \sum_a \int_{\mathrm{KE}_4}\mathcal{R}\wedge c_1(L_a) = \frac{1}{2\pi}\int_{\mathrm{KE}_4}\mathcal{R}\wedge\mathcal{R}\,, \tag{B.3.13}$$

which allows us to compute the sum

$$\sum_a M_a = \frac{\pi^2 L\, f_0}{6(2\pi\ell_s)^3}\,\frac{x_+^2(3x_+^3+2\sigma x_+ +15)}{1-x_+^3}\,M_k\,. \tag{B.3.14}$$

Identifying M_{d+1} with N_+ we have

$$\sum_A M_A = \sum_a M_a + N_+ = -\frac{\pi^2 L\, f_0\,\sigma}{18(2\pi\ell_s)^3}\,M_k\,, \tag{B.3.15}$$

and consistency with the $I=3$ component of the third condition in (5.3.13), which reads $\sum_A M_A = 0$, imposes $\sigma=0$.

When σ vanishes, the zeros of (B.3.3) can be computed analytically,

$$x^3 = -2\pm\frac{3}{\sqrt{2}} \qquad\Longrightarrow\qquad x_+ = \left(\frac{3-2\sqrt{2}}{\sqrt{2}}\right)^{1/3}, \tag{B.3.16}$$

and the fluxes simplify to

$$N_\alpha = \frac{\pi^2 L\, f_0}{2(2\pi\ell_s)^3}\left(\frac{3+2\sqrt{2}}{\sqrt{2}}\right)^{1/3} m_k\, n_\alpha\,, \qquad N_+ = -\frac{\pi^2 L\, f_0}{2(2\pi\ell_s)^3}\left(\frac{3+2\sqrt{2}}{\sqrt{2}}\right)^{1/3} M_k\,. \tag{B.3.17}$$

In order for N_α and N_+ to be integers, as imposed by (B.3.8), we require

$$\frac{\pi^2 L\, f_0}{2(2\pi\ell_s)^3}\left(\frac{3+2\sqrt{2}}{\sqrt{2}}\right)^{1/3} = \frac{N}{h}\,, \tag{B.3.18}$$

where N is an arbitrary integer and $h=\mathrm{hcf}(M_k,m_k)$. Specifically, $h=(3,2,1)$ for $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and dP_3, respectively. On the other hand, the first condition of (B.3.8) yields

$$\frac{f_0}{L^3} = \frac{n_0}{2\pi\ell_s}\,. \tag{B.3.19}$$

Combining (B.3.18) and (B.3.19) we obtain the following quantization conditions on the parameters L and f_0

$$\begin{aligned} L^4 &= (2\pi\ell_s)^4\,\frac{2^{4/3}(3-2\sqrt{2})^{1/3}}{\pi^2 n_0}\left(\frac{N}{h}\right),\\ f_0^2 &= (2\pi\ell_s)^4\,\frac{4(3-2\sqrt{2})^{1/2}\,n_0^{1/2}}{\pi^3}\left(\frac{N}{h}\right)^{3/2}. \end{aligned} \tag{B.3.20}$$

The free energy of our $\mathrm{AdS}_4 \times M_6$ background with KE base space can be read off from the four-dimensional effective Newton constant $G_{(4)}$ as [113]

$$F = \frac{\pi}{2G_{(4)}} = \frac{16\pi^3}{(2\pi\ell_s)^8} \int \mathrm{e}^{8A-2\Phi}\, \mathrm{vol}(M_6)\,, \tag{B.3.21}$$

which gives the general result

$$F = \frac{1}{(2\pi\ell_s)^8} \frac{8\pi^6 L^4 f_0^2}{135}\, x_+^2 (9x_+^3 + 5\sigma x_+ + 45)\, M_k\,. \tag{B.3.22}$$

In this computation we used the fact that, in our conventions, the Kähler form is $J_{\mathrm{KE}} = \mathcal{R}$, therefore the total volume of the KE_4 can be determined from

$$\mathrm{Vol}(\mathrm{KE}_4) = \frac{1}{2}\int_{\mathrm{KE}_4} \mathcal{R}\wedge\mathcal{R} = 2\pi^2 M_k\,. \tag{B.3.23}$$

Setting $\sigma = 0$ and substituting the expressions of x_+, L and f_0 into (B.3.22), the free energy then reads

$$F = \frac{32\sqrt{2}(3-2\sqrt{2})\pi}{5n_0^{1/2}} \left(\frac{N}{h}\right)^{5/2} M_k\,, \tag{B.3.24}$$

which agrees with the the first equation in (5.3.25) with a plus sign, taking into account that, for our examples, $h = m_k$.

B.3.2 $S^2 \times S^2$ base space

We now move to the second case, $B_4 = S_1^2 \times S_2^2$, whose six-dimensional metric is (setting $\kappa_1 = \kappa_2 = +1$ in [55])

$$\mathrm{d}s^2_{M_6} = -\frac{q'}{4xq}\mathrm{d}x^2 - \frac{q}{xq'-4q} D\psi^2 + \frac{q'}{xu_1}\mathrm{d}s^2_{S_1^2} + \frac{q'}{xu_2}\mathrm{d}s^2_{S_2^2}\,, \tag{B.3.25}$$

where

$$\begin{aligned} q(x) &= x^6 + \frac{\sigma}{2}x^4 + 2(1+t)x^3 - \frac{t}{2}\,, \\ u_1(x) = 12x(1-x^3)\,, &\qquad u_2(x) = 12x(t-x^3)\,, \end{aligned} \tag{B.3.26}$$

with σ and t real constants. $D\psi = \mathrm{d}\psi + \rho$, where ρ is a one-form on $S_1^2 \times S_2^2$ such that $\mathrm{d}_4\rho = -(\mathcal{R}_1 + \mathcal{R}_2)$, with $\mathcal{R}_i$ Ricci form of S_i^2, while each $\mathrm{d}s^2_{S_i^2}$ is the metric on a two-sphere with unit radius. The configuration of interest is realized when $t > 0$ and $\sigma > -9\cdot 4^{-1/3}(1+t)^{2/3}$, and in this region the metric is smooth and well-defined given that ψ is 2π-periodic and $x \in [0, x_+]$, with x_+ the only positive root of $q(x)$. Also in this case, we have an O8-plane in $x = 0$. The warp factor has the same expression as in the previous case, namely

$$\mathrm{e}^{2A} = L^2 \sqrt{\frac{x^2 q' - 4xq}{q'}}\,, \tag{B.3.27}$$

whereas the dilaton is now given by

$$e^{2\Phi} = \frac{72L^4}{f_0^2}\,\frac{q'}{xu_1u_2}\left(\frac{x^2q' - 4xq}{q'}\right)^{3/2}. \tag{B.3.28}$$

The remaining non-vanishing fields are the Romans mass

$$F_{(0)} = \frac{f_0}{L^3}\,, \tag{B.3.29}$$

with $f_0 \in \mathbb{R}$, and the four-form flux

$$\begin{aligned} F_{(4)} = -\frac{L\,f_0}{12}\Bigg[&\frac{3x\big(x^6 - (t+4)x^3 - \sigma x - (2t+3)\big)}{(1-x^3)^2}\,\mathrm{d}x \wedge D\psi \wedge \mathcal{R}_1 \\ &+ \frac{3x\big(x^6 - (4t+1)x^3 - \sigma t x - t(3t+2)\big)}{(t-x^3)^2}\,\mathrm{d}x \wedge D\psi \wedge \mathcal{R}_2 \\ &- \frac{9x^8 + 5\sigma x^6 + 18(t+1)x^5 - 2\sigma(t+1)x^3 - 9(t^2+3t+1)x^2 - \sigma t}{3(1-x^3)(t-x^3)}\,\mathcal{R}_1 \wedge \mathcal{R}_2\Bigg]. \end{aligned} \tag{B.3.30}$$

In order to quantize the fluxes as in (B.3.8), we take as a basis for $H_4(M_6, \mathbb{Z})$ the set $\{C_1, C_2, C_+\}$, where C_i are the fibrations $\mathbb{P}^1 \hookrightarrow C_i \to S_i^2$ (at a fixed point on the other sphere) and C_+ is a copy of $S_1^2 \times S_2^2$ at $x = x_+$. The expressions of the three fluxes are

$$\begin{aligned} N_1 &= \frac{\pi^2 L\,f_0}{3(2\pi\ell_s)^3}\,\frac{x_+^2\big(3x_+^3 + 2\sigma x_+ + 3(2t+3)\big)}{1-x_+^3}\,, \\ N_2 &= \frac{\pi^2 L\,f_0}{3(2\pi\ell_s)^3}\,\frac{x_+^2\big(3x_+^3 + 2\sigma x_+ + 3(3t+2)\big)}{t-x_+^3}\,, \\ N_+ &= \frac{4\pi^2 L\,f_0}{9(2\pi\ell_s)^3}\,\frac{9x_+^8 + 5\sigma x_+^6 + 18(t+1)x_+^5 - 2\sigma(t+1)x_+^3 - 9(t^2+3t+1)x_+^2 - \sigma t}{(1-x_+^3)(t-x_+^3)}\,, \end{aligned} \tag{B.3.31}$$

where we made use of the relation

$$\frac{1}{2\pi}\int_{S_i^2} \mathcal{R}_i = \chi(S_i^2) = 2\,. \tag{B.3.32}$$

As before, we will restrict to configurations with $\sigma = 0$, in which case the equation $q(x) = 0$ can be solved analytically, giving

$$x^3 = -(t+1) \pm \sqrt{\frac{(t+2)(2t+1)}{2}} \qquad \Longrightarrow \qquad x_+ = \left(\sqrt{\frac{(t+2)(2t+1)}{2}} - (t+1)\right)^{1/3}. \tag{B.3.33}$$

When σ vanishes $N_+ = -2(N_1 + N_2)$, therefore we will focus exclusively on the quantization of the fluxes N_1 and N_2, since the quantization of N_+ follows immediately. Setting $\sigma = 0$, the fluxes simplify to

$$N_1 = \frac{\pi^2 L\,f_0}{(2\pi\ell_s)^3}\,\frac{x_+^2(x_+^3 + 2t + 3)}{1-x_+^3}\,, \qquad N_2 = \frac{\pi^2 L\,f_0}{(2\pi\ell_s)^3}\,\frac{x_+^2(x_+^3 + 3t + 2)}{t-x_+^3}\,, \tag{B.3.34}$$

and taking their ratio we can immediately determine t

$$t = \frac{\left[\sqrt{9N_1^2 + 14N_1N_2 + 9N_2^2} \pm 3(N_1 - N_2)\right]^2}{32N_1N_2}\,. \tag{B.3.35}$$

Since t needs to be positive, N_1 and N_2 must have the same sign, *i.e.* $N_1N_2 > 0$; for the sake of simplicity, we will take both of them positive. Taking the product of the fluxes (B.3.34) and making use of (B.3.19) we obtain

$$\begin{aligned} L^4 &= (2\pi\ell_s)^4 \frac{1}{\pi^2 n_0}\left(\frac{2}{t}\right)^{1/2}\left(\sqrt{\frac{(t+2)(2t+1)}{2}} - (t+1)\right)^{1/3}(N_1N_2)^{1/2}\,,\\ f_0^2 &= (2\pi\ell_s)^4 \frac{n_0^{1/2}}{\pi^3}\left(\frac{2}{t}\right)^{3/4}\left(\sqrt{\frac{(t+2)(2t+1)}{2}} - (t+1)\right)^{1/2}(N_1N_2)^{3/4}\,. \end{aligned} \tag{B.3.36}$$

The free energy of the AdS$_4$ solution under exam can be computed performing the integral (B.3.21) and takes the general expression

$$F = \frac{1}{(2\pi\ell_s)^8}\frac{32\pi^6 L^4 f_0^2}{135}\, x_+^2\big(18x_+^3 + 10\sigma x_+ + 45(t+1)\big)\,, \tag{B.3.37}$$

which, once all the ingredients are substituted, becomes

$$\begin{aligned} F = &\frac{4\sqrt{2}\pi}{5n_0^{1/2}}\Big((N_1+N_2)\sqrt{9N_1^2 + 14N_1N_2 + 9N_2^2} - (3N_1^2 + 2N_1N_2 + 3N_2^2)\Big)\\ &\times\sqrt{3(N_1+N_2) - \sqrt{9N_1^2 + 14N_1N_2 + 9N_2^2}}\,. \end{aligned} \tag{B.3.38}$$

Parameterizing the fluxes as $N_1 = (1+\mathsf{z})N$, $N_2 = (1-\mathsf{z})N$, with $|\mathsf{z}| < 1$, we obtain

$$F = \frac{32\pi}{5n_0^{1/2}}\left(\sqrt{8+\mathsf{z}^2} - (2+\mathsf{z}^2)\right)\sqrt{3 - \sqrt{8+\mathsf{z}^2}}\, N^{5/2}\,, \tag{B.3.39}$$

which agrees with the the first equation in (5.3.29) with a plus sign. Setting $\mathsf{z} = 0$ we consistently retrieve the result (B.3.24) specified to the case $\mathbb{P}^1 \times \mathbb{P}^1$.

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