

Differentiated oligopolistic markets with concave cost functions via Ky Fan inequalities

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Abstract A Nash-Cournot model for oligopolistic markets with concave cost functions and a differentiated commodity is analysed. Equilibrium states are characterized through Ky Fan inequalities. Relying on the minimization of a suitable merit function, a general algorithmic scheme for solving them is provided. Two concrete algorithms are therefore designed that converge under suitable convexity and monotonicity assumptions. The results of some numerical tests on randomly generated markets are also reported.

Keywords Oligopoly · concave costs · Ky Fan inequalities · merit function · descent algorithm

1 Introduction

An oligopoly is a market structure with a small number of competing firms that produce the same kind of commodity. Since Cournot introduced and analysed the celebrated duopoly, models for oligopolies have been widely studied in economics (see, for instance, Bonanno (1990); Fudenberg and Tirole (1986); von Mouche and Quartieri (2016); Okuguchi and Szidarovszky (1990); Vives (1989)). The analysis of the competition typically exploits the tools of

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game theory to predict the behaviour of the firms (see, for instance, Friedman (1977)). In turn, if no cooperation between firms is allowed, game theoretic models can be formulated as variational problems (Harker, 1984; Murphy et al, 1982) and optimization techniques can be exploited to solve them (see, for instance, Flåm and Ben-Israel (1990); Harker (1984); Kolstad and Mathiesen (1991); Krawczyk and Uryasev (2000); Marcotte (1987); Nagurney (1988); Salant (1982)).

Generally, these game theoretic models shares two features that seem a bit restrictive. The costs increase at least linearly with the produced quantity, while usually the cost per unit does decrease as the production increases. As a consequence, costs should generally increase less than linearly. The commodity is supposed to be homogeneous, i.e., all the firms product the identical product with even no slight difference, so that a unique unitary price is set for all the firms (by the market) through an inverse demand function. Commodities of different firms, though similar, have their own characteristics and are rarely identical. Moreover, firms generally try to differentiate their products from those of their competitors to improve their market share (see, for instance, Okuguchi and Szidarovszky (1990); Singh and Vives (1984)). As a consequence, the dynamics of the price should depend also on the considered firm. To the best of our knowledge, a first attempt to analyse product differentiation with concave costs can be found in Muu et al (2008) where piecewise linear cost functions have been considered modeling the market through mixed variational inequalities.

This paper provides another step in this direction. In Section 2 a Nash-Cournot model for oligopolistic markets with concave quadratic cost functions and a differentiated commodity is introduced and formulated as Ky Fan inequalities. Section 3 describes a general algorithmic scheme for Ky Fan inequalities based on the minimization of a suitable merit function. Moreover, two concrete descent algorithms based on gap and D-gap functions are provided and their global convergence is deduced from the general scheme under suitable convexity and monotonicity assumptions. The uniqueness of the solution is also investigated. Finally, Section 4 provides numerical tests of the two algorithms for randomly generated markets. The sensitivity of the algorithms with respect to their parameters are reported and some market scenarios are analysed.

2 Differentiated oligopoly models and Ky Fan inequalities

Consider n producers that supply the same kind of commodity to some market and compete over quantity in a noncooperative way. Each producer i chooses a quantity x_i to produce and supply at a cost $c_i(x_i)$ for some given cost function $c_i : \mathbb{R}_+ \rightarrow \mathbb{R}$. The largest unitary price at which the market requires a quantity x_i of the commodity of producer i depends also upon the quantities that the other producers supply. Since the commodities of the producers may differ in features and/or quality, then this price can be represented by an inverse

demand function $p_i : \mathbb{R}_+^n \rightarrow \mathbb{R}$ that is peculiar of each producer. Therefore, the goal of producer i is to maximize its profit

$$f_i(x_1, \dots, x_n) = x_i p_i(x_1, \dots, x_n) - c_i(x_i)$$

that is affected by the choices of the other producers. An equilibrium state of the market is reached when the levels of supply are such that no producer would increase its profit by changing its level while the other firms keep the same. Mathematically, $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}_+^n$ is an equilibrium state if and only if $x_i^* \in [0, T_i]$ and

$$f_i(x^*) \geq f_i(y_i, x_{-i}^*)$$

hold for all $y_i \in [0, T_i]$ and all $i = 1, \dots, n$, where $x_{-i}^* = (x_j^*)_{j \neq i}$ denotes the levels of supply of all the producers other than i while $T_i > 0$ denotes its maximum capacity of production. It is well-known that a feasible x^* is an equilibrium state if and only if it solves the following Ky Fan inequalities

$$f(x^*, y) \geq 0, \quad \forall y \in [0, T_1] \times \dots \times [0, T_n], \quad (1)$$

where $f : \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ is the Nikaido-Isoda aggregate bifunction

$$f(x, y) = \sum_{i=1}^n [f_i(x) - f_i(x_{-i}, y_i)],$$

which measures the overall losses of the players when switching from level x_i to y_i , one at a time while all the others stick to levels x_{-i} .

In classical models the cost functions c_i are generally supposed to be affine (or even convex). As already remarked, often this assumption is not realistic and concave cost functions would sound more adequate. Therefore, in this paper the cost functions are supposed to be quadratic and concave, i.e.,

$$c_i(x_i) = \ell_i x_i + q_i x_i^2$$

for some $\ell_i > 0$ and $q_i < 0$ with $\ell_i \geq -2T_i q_i$ so that c_i is increasing on $[0, T_i]$.

The inverse demand functions are supposed to decrease linearly with the total production, i.e.,

$$p_i(x_1, \dots, x_n) = m_i - d_i(x_1 + \dots + x_n)$$

for some $m_i > \ell_i$ and $d_i > -q_i$. Under these assumptions a unitary production guarantees a positive profit in a monopolistic setting, i.e., when $x_{-i} = 0$. If all the rates d_i are equal, the effect of the product differentiation does not meaningfully impact the dynamics of prices and moreover the equilibrium states coincide with the maxima of the sum of all the profit functions (see (Monderer and Shapley, 1996, Theorem 4.5)) just like in the case no differentiation occurs. Therefore, the rates d_i are supposed to be not all equal.

Under all these assumptions the Nikaido-Isoda bifunction reads

$$f(x, y) = \langle Px + Qy + r, y - x \rangle \quad (2)$$

for $r = (\ell_1 - m_1, \dots, \ell_n - m_n) \in \mathbb{R}^n$ and the matrices

$$P = \begin{bmatrix} d_1 + q_1 & d_1 & \cdots & d_1 \\ d_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_{n-1} \\ d_n & \dots & d_n & d_n + q_n \end{bmatrix}, \quad Q = \begin{bmatrix} d_1 + q_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & d_n + q_n \end{bmatrix}.$$

Notice that the nonzero entries of Q are positive and hence it is a positive definite matrix. As a consequence, $f(x, \cdot)$ is a strongly convex function for any x . On the contrary, $f(\cdot, y)$ may not be concave as the symmetric part $\text{sym}(P) = (P + P^T)/2$ of P is not necessarily positive semidefinite, but it is yet continuous. Furthermore the feasible region is a box, and hence convex and compact, so that the Ky Fan inequalities (1) admit at least one solution (see Bigi et al (2013)).

3 A general descent algorithm for Ky Fan inequalities

Consider the Ky Fan inequalities

$$\text{find } x^* \in C \text{ s.t. } f(x^*, y) \geq 0, \quad \forall y \in C, \quad (KFI)$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is any bifunction satisfying $f(x, x) = 0$ for all $x \in \mathbb{R}^n$, and $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set. Ky Fan inequalities turned into optimization problems through suitable merit functions. Given a closed set $X \subseteq \mathbb{R}^n$, $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a merit function on X for (KFI) if it is nonnegative on X and its zeros on X coincide with the solutions of (KFI), i.e, if it satisfies

- $\psi(x) \geq 0$ for any $x \in X$,
- x^* solves (KFI) if and only if $x^* \in X$ and $\psi(x^*) = 0$.

Therefore, (KFI) can be formulated as the global optimization problem

$$\min\{\psi(x) : x \in X\},$$

where the set X does not necessarily coincide with the feasible set C of the Ky Fan inequalities. In order to solve it, ψ is supposed to be continuously differentiable and to satisfy the following two properties:

- (P1) X is convex and the sublevel sets $\{x \in X : \psi(x) \leq c\}$ are bounded for any $c > 0$.
- (P2) There exist $\nu > 0$ and continuous functions $d : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}$ such that any $x \in X$ not solving (KFI) satisfies $x + d(x) \in X$, $\sigma(x) > 0$ and

$$\langle \nabla \psi(x), d(x) \rangle \leq -\nu \sigma(x).$$

Notice that (P2) implies that the global minima and the stationary points of ψ over X coincide even though ψ is not necessarily convex. Therefore, descent techniques can be exploited to minimize it, or equivalently solve (KFI), just moving away from the current iterate x^k along the direction $d(x^k)$ with a step provided by a standard linesearch rule.

Descent algorithm

Step 0. Choose $x^0 \in X$, set $\eta \in (0, \nu)$, $\delta \in (0, 1)$, $k = 0$.

Step 1. If $\psi(x^k) = 0$ then stop.

Step 2. Set $d^k = d(x^k)$, $\sigma^k = \sigma(x^k)$ and compute the smallest non-negative integer s such that

$$\psi(x^k + \delta^s d^k) \leq \psi(x^k) - \eta \delta^s \sigma^k$$

Step 3. Set $t_k = \delta^s$, $x^{k+1} = x^k + t_k d^k$, $k = k + 1$ and go to Step 1.

Notice that (P2) guarantees that the linesearch procedure at Step 2 is finite. Otherwise, the inequality

$$\psi(x^k + \delta^s d^k) - \psi(x^k) > -\eta \delta^s \sigma^k$$

would hold for all $s \in \mathbb{N}$. Taking the limit as $s \rightarrow +\infty$ would lead to

$$\langle \nabla \psi(x^k), d^k \rangle \geq -\eta \sigma^k > -\nu \sigma^k$$

in contradiction with (P2) since x^k does not solve (KFI). The convexity of X guarantees that the sequence $\{x^k\}$ is feasible and the theorem below shows its convergence properties.

Theorem 1 *Suppose there exists a merit function ψ on X for (KFI) such that (P1) and (P2) hold. Then, either the descent algorithm stops at a solution to (KFI) after a finite number of iterations or it produces a bounded sequence $\{x^k\}$ such that any of its cluster points solves (KFI).*

Proof If the algorithm stops at x^k after a finite number of iterations at Step 2, then x^k solves (KFI) since ψ is a merit function.

Now, suppose the algorithm generates an infinite sequence $\{x^k\}$. Such sequence is bounded since the sequence $\{\psi(x^k)\}$ is monotone decreasing and the sublevel sets of ψ are bounded. Let x^* be any cluster point of the sequence and let $\hat{x}^\ell \rightarrow x^*$ for some subsequence $\{\hat{x}^\ell\}$, i.e., $\hat{x}^\ell = x^{k_\ell}$ for some $k_\ell \uparrow +\infty$ as $\ell \uparrow +\infty$. The continuity of d and σ guarantees $\hat{d}^\ell = d(\hat{x}^\ell) \rightarrow d(x^*) = d^*$ and $\hat{\sigma}^\ell = \sigma(\hat{x}^\ell) \rightarrow \sigma(x^*) = \sigma^*$.

By contradiction, suppose x^* does not solve (KFI) and hence $\sigma^* > 0$ by (P2). Since the sequence $\{\psi(x^k)\}$ is also bounded by below, it has a limit and thus

$$\lim_{\ell \rightarrow \infty} [\psi(\hat{x}^\ell) - \psi(\hat{x}^{\ell+1})] = 0$$

holds as well. Moreover, the stepsize rule guarantees

$$\psi(\hat{x}^\ell) - \psi(\hat{x}^{\ell+1}) \geq \psi(x^{k_\ell}) - \psi(x^{k_\ell+1}) \geq \eta \hat{t}_\ell \hat{\sigma}^\ell > 0$$

for $\hat{t}_\ell = t_{k_\ell}$. Therefore, $\hat{t}_\ell \rightarrow 0$ as $\ell \rightarrow +\infty$ since $\sigma^* > 0$. On the other side, the inequality

$$\psi(\hat{x}^\ell + \hat{t}_\ell \delta^{-1} \hat{d}^\ell) - \psi(\hat{x}^\ell) > -\eta (\hat{t}_\ell \delta^{-1}) \hat{\sigma}^\ell$$

holds for all $\ell \in \mathbb{N}$ by the linesearch procedure, while the mean value theorem guarantees the existence of some $\theta_\ell \in (0, 1)$ such that

$$\psi(\hat{x}^\ell + \hat{t}_\ell \delta^{-1} \hat{d}^\ell) - \psi(\hat{x}^\ell) = \langle \nabla \psi(\hat{x}^\ell + \theta_\ell \hat{t}_\ell \delta^{-1} \hat{d}^\ell), \hat{t}_\ell \delta^{-1} \hat{d}^\ell \rangle.$$

Hence, the two inequalities together provide

$$\langle \nabla \psi(\hat{x}^\ell + \theta_\ell \hat{t}_\ell \delta^{-1} \hat{d}^\ell), \hat{d}^\ell \rangle > -\eta \hat{\sigma}^\ell.$$

Taking the limit, the inequalities

$$\langle \nabla \psi(x^*), d^* \rangle \geq -\eta \sigma^* > -\nu \sigma^*,$$

contradict (P2). Therefore, x^* solves (KFI). \square

Since the sequence $\{x^k\}$ is bounded, the above convergence theorem yields the existence of a solution of the Ky Fan inequalities as well.

Corollary 1 *If there exists a merit function ψ on X for (KFI) such that properties (P1)-(P2) hold, then (KFI) admits at least one solution.*

No assumptions are actually made on f in the above results. Indeed, the tools for the convergence of the algorithm all lie in the differentiability of ψ and the properties (P1) and (P2). Anyway, differentiability is generally inherited from f thanks to Danskin's theorem (Danskin, 1966). Therefore, f is supposed to be continuously differentiable throughout all the paper. Concrete merit functions that satisfy (P1) and (P2) are given in the next subsections and suitable convexity and monotonicity assumptions on f are required.

3.1 Gap functions

A possible measure for the quality of a candidate solution $x \in C$ of (KFI) is the gap between zero and minimum value of $f(x, y)$ over all $y \in C$. To facilitate its computation, a regularization term is usually added leading to the value function

$$\varphi_\alpha(x) := \sup\{ -(f(x, y) + \alpha \|y - x\|^2/2) : y \in C \} \quad (3)$$

for some $\alpha \in \mathbb{R}$ (see Bigi and Passacantando (2016)), also known as gap function. In order to guarantee that φ_α is a merit function, the parameter α must be related to the modulus of strong convexity of $f(x, \cdot)$.

Definition 1 Given $\tau \in \mathbb{R}$, a function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is called τ -convex on C if any $u, v \in C$ and $t \in [0, 1]$ satisfy

$$g(tu + (1-t)v) \leq tg(u) + (1-t)g(v) - \frac{\tau}{2}t(1-t)\|u-v\|^2.$$

If $\tau = 0$, the above inequality provides the usual definition of convexity. If $\tau > 0$, the inequality is strengthened and g is called strongly convex. Indeed, g is τ -convex on C if and only if $g(x) - \tau\|x\|^2/2$ is convex on C (see Vial (1983)). Similarly, g is called τ -concave on C if $-g$ is τ -convex on C .

If the functions $f(x, \cdot)$ are τ -convex for any $x \in C$ and $\alpha + \tau > 0$, then the objective function of the optimization problem in (3) is strongly convex and its minimum point is unique so that the map

$$y_\alpha(x) = \arg \min \{ -(f(x, y) + \alpha\|y-x\|^2/2) : y \in C \}$$

is well defined. In order to guarantee good descent properties for φ_α also some monotonicity assumptions on f are needed.

Definition 2 f is γ -c-monotone on C if any $x, y \in C$ satisfy

$$f(x, y) + \langle \nabla_x f(x, y), y - x \rangle \geq \gamma\|y - x\|^2.$$

If $\gamma = 0$, the above definition collapse to the concavity type condition introduced in Bigi et al (2009). Moreover, if the functions $f(\cdot, y)$ are 2γ -concave for any $y \in C$, then f is γ -c-monotone on C . Indeed, the concavity of the function $g(x) = f(x, y) + \gamma\|x\|^2$ implies $g(y) \leq g(x) + \langle \nabla g(x), y - x \rangle$ that reads

$$\gamma\|y\|^2 \leq f(x, y) + \gamma\|x\|^2 + \langle \nabla_x f(x, y) + 2\gamma x, y - x \rangle$$

which is equivalent to the inequality in the above definition.

The main properties of φ_α are given below provided that the above convexity and monotonicity conditions hold for suitable moduli.

Theorem 2 Suppose that there exists $\tau \geq 0$ such that $f(x, \cdot)$ is τ -convex for any $x \in C$. Then, the following hold for any $\alpha > -\tau$:

- a) φ_α is a merit function on C for (KFI);
- b) x^* solves (KFI) if and only if $y_\alpha(x^*) = x^*$;
- c) the map y_α is continuous on \mathbb{R}^n ;
- d) φ_α is continuously differentiable on C and

$$\nabla \varphi_\alpha(x) = -\nabla_x f(x, y_\alpha(x)) + \alpha(y_\alpha(x) - x).$$

Furthermore, suppose f is γ -c-monotone for some $\gamma \in \mathbb{R}$. Then, the following hold whenever $\gamma + \tau/2 > 0$:

- e) If $x \in C$ does not solve (KFI), then

$$\langle \nabla \varphi_\alpha(x), y_\alpha(x) - x \rangle \leq -(\gamma + \tau/2)\|y_\alpha(x) - x\|^2 < 0.$$

Proof See, for instance, Bigi and Passacantando (2016) for *a*) and *d*) and Konnov (2003) for *b*) and *c*).

e) If $x \in C$ does not solve *(KFI)*, then the monotonicity and convexity assumptions of f imply that the direction $y_\alpha(x) - x$ satisfies

$$\begin{aligned} \langle \nabla \varphi_\alpha(x), y_\alpha(x) - x \rangle &= -\langle \nabla_x f(x, y_\alpha(x)), y_\alpha(x) - x \rangle + \alpha \|y_\alpha(x) - x\|^2 \\ &\leq f(x, y_\alpha(x)) - \gamma \|y_\alpha(x) - x\|^2 + \alpha \|y_\alpha(x) - x\|^2 \\ &\leq \langle \nabla_y f(x, y_\alpha(x)), y_\alpha(x) - x \rangle \\ &\quad - (\gamma + \tau/2) \|y_\alpha(x) - x\|^2 + \alpha \|y_\alpha(x) - x\|^2 \\ &\leq -(\gamma + \tau/2) \|y_\alpha(x) - x\|^2 \\ &< 0 \end{aligned}$$

where the third inequality follows from the first-order optimality condition for $y_\alpha(x)$. \square

Therefore, φ_α is a merit function on C for *(KFI)* whenever $\alpha > -\tau$ is chosen and it satisfies *(P1)* provided that C is bounded and *(P2)* for $\nu = \gamma + \tau/2$, $d(x) = y_\alpha(x) - x$ and $\sigma(x) = \|d(x)\|^2$.

The Nikaido-Isoda aggregate bifunction (2) is τ -convex with respect to the second variable for $\tau = 2\lambda_{\min}(Q) = 2\min\{d_i + q_i : i = 1, \dots, n\} > 0$, that is twice the minimum eigenvalue of Q , and it is also γ -c-monotone for $\gamma = \lambda_{\min}(\text{sym}(P))$, that is the minimum eigenvalue of the symmetric part of P (for which no simple formula is available).

Notice that the gap function φ_α could be continuously differentiable even if f were not as in the following example.

Example 1 Consider $n = 1$, $C = \mathbb{R}$ and $f(x, y) = \max\{x, 0\}(y - x)$ that is not differentiable on $\{0\} \times \mathbb{R}$. Moreover, $f(x, \cdot)$ is τ -convex with $\tau = 0$, thus φ_α is a gap function for any $\alpha > 0$, but it is continuously differentiable since $\varphi_\alpha(x) = (\max\{0, x\})^2/2\alpha$.

3.2 D-gap functions

A class of merit functions on the whole space \mathbb{R}^n is given by the difference of two gap functions, i.e., $\varphi_{\alpha\beta}(x) := \varphi_\alpha(x) - \varphi_\beta(x)$ with $\alpha < \beta$ (see Pappalardo et al (2016)). Good descent properties for $\varphi_{\alpha\beta}$ are guaranteed by another kind of monotonicity assumption.

Definition 3 f is μ - ∇_{xy} -monotone on C if for any $x, y, z \in C$

$$\langle \nabla_x f(x, y) - \nabla_x f(x, z), y - z \rangle \geq \mu \|y - z\|^2.$$

If $\mu > 0$, then ∇_{xy} -monotonicity amounts to the strong monotonicity of $\nabla_x f(x, \cdot)$ and it has already been exploited in the framework of D-gap functions in Charitha (2013); Komnov and Pinyagina (2003).

If f is μ - ∇_{xy} -monotone on C and $f(x, \cdot)$ is convex, then f is μ -c-monotone on C . Indeed,

$$\begin{aligned} f(x, y) + \langle \nabla_x f(x, y), y - x \rangle &\geq f(x, y) + \langle \nabla_x f(x, x), y - x \rangle + \mu \|y - x\|^2 \\ &= f(x, y) - \langle \nabla_y f(x, x), y - x \rangle + \mu \|y - x\|^2 \\ &\geq \mu \|y - x\|^2, \end{aligned}$$

where the last inequality is due to the convexity and the equality follows since $f(x, x) = 0$ for all $x \in \mathbb{R}^n$ implies the equality of the two partial gradients above.

The joint exploitation of φ_α and φ_β would suggest to rely on the direction

$$r(x) = y_\alpha(x) - y_\beta(x).$$

However, $r(x)$ could be zero even when x does not solve (KFI). The gradient of $\varphi_{\alpha\beta}$ provides the additional term

$$s(x) = \alpha(x - y_\alpha(x)) - \beta(x - y_\beta(x))$$

that is useful together with $r(x)$ to characterize Ky Fan inequalities.

The main properties of $\varphi_{\alpha\beta}$ are given below provided that the convexity and monotonicity conditions hold for suitable moduli.

Theorem 3 *Suppose that there exists $\tau \geq 0$ such that $f(x, \cdot)$ is τ -convex for any $y \in C$. Then, the following hold for any $\beta > \alpha > -\tau$:*

- a) $\varphi_{\alpha\beta}$ is a merit function on \mathbb{R}^n for (KFI).
- b) x^* solves (KFI) if and only if $r(x^*) = s(x^*) = 0$.
- c) $\varphi_{\alpha\beta}$ is continuously differentiable on \mathbb{R}^n and

$$\nabla \varphi_{\alpha\beta}(x) = \nabla_x f(x, y_\beta(x)) - \nabla_x f(x, y_\alpha(x)) - s(x).$$

- d) If C is bounded, then the sublevel sets $\{x \in \mathbb{R}^n : \varphi_{\alpha\beta}(x) \leq c\}$ are bounded for all $c > 0$.

Moreover, suppose f is μ - ∇_{xy} -monotone on \mathbb{R}^n for some $\mu \in \mathbb{R}$ and $\nabla_x f(x, \cdot)$ is Lipschitz on \mathbb{R}^n with modulus $L > 0$ for any $x \in \mathbb{R}^n$. Then, the following hold whenever $\mu + \tau > 0$ and $0 < \rho \leq \min\{1/(\mu + \tau), (\mu + \tau)/(\mu + \tau + L)^2\}$:

- e) If $x \in \mathbb{R}^n$ does not solve (KFI), then

$$\langle \nabla \varphi_{\alpha\beta}(x), r(x) + \rho s(x) \rangle \leq -2^{-1}(\mu + \tau)(\|r(x)\| + \rho \|s(x)\|)^2 < 0.$$

Proof

- a) By definition $\varphi_\alpha(x) \geq \varphi_\beta(x)$ for any $x \in \mathbb{R}^n$, so that $\varphi_{\alpha\beta}$ is non-negative everywhere. If x^* solves *(KFI)*, then $\varphi_\alpha(x^*) = \varphi_\beta(x^*) = 0$ by Theorem 2 a). Viceversa, if $\varphi_{\alpha\beta}(x^*) = 0$, then

$$\begin{aligned} 0 = \varphi_{\alpha\beta}(x^*) &= \varphi_\alpha(x^*) - \varphi_\beta(x^*) \geq -f(x^*, y_\beta(x^*)) - \alpha\|y_\beta(x^*) - x^*\|^2/2 \\ &\quad + f(x^*, y_\beta(x^*)) + \beta\|y_\beta(x^*) - x^*\|^2/2 \\ &= (\beta - \alpha)\|y_\beta(x^*) - x^*\|^2/2. \end{aligned}$$

implies that $x^* = y_\beta(x^*)$, hence x^* solves *(KFI)* by Theorem 2 b).

- b) If x^* solves *(KFI)*, then $x^* = y_\alpha(x^*) = y_\beta(x^*)$ by Theorem 2 b) and hence $r(x^*) = s(x^*) = 0$. Viceversa, if $r(x^*) = 0$, then $y_\alpha(x^*) = y_\beta(x^*)$ so that $0 = s(x^*) = (\alpha - \beta)(x^* - y_\alpha(x^*))$ implies $x^* = y_\alpha(x^*)$, i.e., x^* solves *(KFI)*.
- c) It is a straightforward consequence of Theorem 2 c).
- d) See Zhang and Han (2009).
- e) The first-order optimality conditions guarantee that $y_\alpha(x)$ and $y_\beta(x)$ satisfy

$$\begin{aligned} \langle \nabla_y f(x, y_\alpha(x)) + \alpha(y_\alpha(x) - x), y_\beta(x) - y_\alpha(x) \rangle &\geq 0, \\ \langle \nabla_y f(x, y_\beta(x)) + \beta(y_\beta(x) - x), y_\alpha(x) - y_\beta(x) \rangle &\geq 0, \end{aligned}$$

so that summing up the two inequalities provides

$$\langle \nabla_y f(x, y_\beta(x)) - \nabla_y f(x, y_\alpha(x)) + s(x), r(x) \rangle \geq 0.$$

The convexity of $f(x, \cdot)$ implies that $\nabla_y f(x, \cdot)$ is strongly monotone (see (Hiriart-Urruty and Lemaréchal, 1993, Theorem 6.1.2)), and therefore the above inequality guarantees also

$$\langle s(x), r(x) \rangle \geq \langle \nabla_y f(x, y_\alpha(x)) - \nabla_y f(x, y_\beta(x)), r(x) \rangle \geq \tau\|r(x)\|^2. \quad (4)$$

Let $d = r(x) + \rho s(x)$. Therefore, the following chain of equalities and inequalities holds

$$\begin{aligned}
\langle \nabla \varphi_{\alpha\beta}(x), d \rangle &= \langle \nabla_x f(x, y_\beta(x)) - \nabla_x f(x, y_\alpha(x)), r(x) \rangle \\
&\quad + \rho \langle \nabla_x f(x, y_\beta(x)) - \nabla_x f(x, y_\alpha(x)), s(x) \rangle \\
&\quad - \langle s(x), r(x) \rangle - \rho \|s(x)\|^2 \\
&\leq -\mu \|r(x)\|^2 + \rho \langle \nabla_x f(x, y_\beta(x)) - \nabla_x f(x, y_\alpha(x)), s(x) \rangle \\
&\quad - \langle s(x), r(x) \rangle - \rho \|s(x)\|^2 \\
&\leq -(\mu + \tau) \|r(x)\|^2 \\
&\quad + \rho \langle \nabla_x f(x, y_\beta(x)) - \nabla_x f(x, y_\alpha(x)), s(x) \rangle - \rho \|s(x)\|^2 \\
&\leq -(\mu + \tau) \|r(x)\|^2 + \rho L \|r(x)\| \|s(x)\| - \rho \|s(x)\|^2 \\
&= (\mu + \tau) (\|r(x)\| + \rho \|s(x)\|)^2 / 2 \\
&\quad - (\sqrt{\mu + \tau} \|r(x)\| - \sqrt{\rho} \|s(x)\|)^2 / 2 + [(\gamma + \tau)\rho - 1] / 2 \\
&\quad + \sqrt{\rho} [\sqrt{\rho}(\mu + \tau + L) - \sqrt{\mu + \tau}] \|r(x)\| \|s(x)\| \\
&\leq -(\|r(x)\| + \rho \|s(x)\|)^2 / 2 \\
&< 0
\end{aligned}$$

where the first inequality is due to the ∇_{xy} -monotonicity of f , the second to (4), the third to the Lipschitz continuity of $\nabla_x f(x, \cdot)$, the fourth to the assumption on ρ and the last to b). \square

Therefore, $\varphi_{\alpha\beta}$ is a merit function on \mathbb{R}^n for (KFI) whenever $\beta > \alpha > -\tau$ are chosen and it satisfies (P1) provided that C is bounded and (P2) for $\nu = (\mu + \tau)/2$, $d(x) = r(x) + \rho s(x)$ and $\sigma(x) = (\|r(x)\| + \rho \|s(x)\|)^2$.

The resulting descent algorithm had already been proposed in (Charitha, 2013; Konnov and Pinyagina, 2003) just for $\mu > 0$ and $\tau = 0$, while it is enough to require $\mu + \tau > 0$ as shown above.

The Nikaido-Isoda aggregate bifunction (2) is μ - ∇_{xy} -monotone for $\mu = \lambda_{\min}(\text{sym}(P - Q))$, that is the minimum eigenvalue of the symmetric part of $P - Q$, while the partial gradient map $\nabla_x f(x, \cdot)$ is Lipschitz with $L = \|P - Q\|$.

3.3 Uniqueness of the solution

The convexity and monotonicity conditions of the previous subsections yield also the uniqueness of the solution of the Ky Fan inequalities.

Theorem 4 *Suppose there exists $\tau \geq 0$ such that $f(x, \cdot)$ is τ -convex for any $x \in C$. Then, there exists a unique solution of (KFI) provided that any of the following statements holds:*

- a) f is twice continuously differentiable and γ - c -monotone on C for some $\gamma \in \mathbb{R}$ satisfying $\gamma + \tau/2 > 0$;

b) f is μ - ∇_{xy} -monotone on \mathbb{R}^n for some $\mu \in \mathbb{R}$ with $\mu + \tau > 0$.

Proof

a) Let $\theta = \gamma + \tau/2$ and consider $g(x, y) = f(x, y) + \theta(\|x\|^2 - \|y\|^2)/2$. The following chain of equalities and inequalities

$$\begin{aligned} & \langle \nabla_x g(x, y) + \nabla_y g(x, y), y - x \rangle = \\ & \langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x \rangle - \theta \|y - x\|^2 = \\ & f(x, y) + \langle \nabla_x f(x, y), y - x \rangle - [f(x, y) + \langle \nabla_y f(x, y), x - y \rangle] - \theta \|y - x\|^2 \geq \\ & \gamma \|y - x\|^2 - f(x, x) + \tau \|y - x\|^2/2 - \theta \|y - x\|^2 = 0 \end{aligned}$$

follows from the γ -c-monotonicity of f and the τ -convexity of $f(x, \cdot)$. As a consequence, setting $y(t) = (1 - t)x + ty$ for any given pair $x, y \in C$, the function $\omega(t) = \langle \nabla_x g(x, y(t)) + \nabla_y g(x, y(t)), y - x \rangle$ is non-negative on $[0, 1]$ and $\omega(0) = 0$ so that

$$\begin{aligned} 0 & \leq \omega'(0) = \langle y - x, [\nabla_{xy} f(x, x) + \nabla_{yy} f(x, x)](y - x) \rangle - \theta \|y - x\|^2 \\ & = \langle y - x, JF(x)(y - x) \rangle - \theta \|y - x\|^2 \end{aligned}$$

holds for any $x, y \in C$, where $JF(x)$ denotes the Jacobian of the map $F(z) = \nabla_y f(z, z)$ at x . Therefore, F is strongly monotone on C and the variational inequality

$$\text{find } x^* \in C \text{ s.t. } \langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C$$

has a unique solution (see Facchinei and Pang (2003)). Since the above variational inequality is equivalent to (KFI) (see Castellani and Giuli (2010); Konnov (2003)), the thesis follows.

b) The auxiliary problem principle (see (Bigi and Passacantando, 2016, Corollary 3.3)) guarantees that (KFI) is equivalent to the Ky Fan inequalities with the bifunction $f_{-\tau}(x, y) = f(x, y) - \tau\|x - y\|^2/2$. Furthermore, $f_{-\tau}$ is $(\mu + \tau)$ - ∇_{xy} -monotone as

$$\begin{aligned} & \langle \nabla_x f_{-\tau}(x, y) - \nabla_x f_{-\tau}(x, z), y - z \rangle \\ & = \langle \nabla_x f(x, y) - \nabla_x f(x, z), y - z \rangle + \langle (-\tau)(x - y) + \tau(x - z), y - z \rangle \\ & = \langle \nabla_x f(x, y) - \nabla_x f(x, z), y - z \rangle + \tau \|y - z\|^2 \\ & \geq (\mu + \tau) \|y - z\|^2. \end{aligned}$$

Since $\mu + \tau > 0$, $f_{-\tau}$ is strongly ∇_{xy} -monotone on C . As strong ∇_{xy} -monotonicity implies strong monotonicity (Bigi and Passacantando, 2015b, Theorem 3.1), the existence of a unique solution follows (see, for instance, (Bigi and Passacantando, 2016, Corollary 3.4)). \square

Twice differentiability of f can be dropped from point *a*) if γ -c-monotonicity is replaced by the 2γ -concavity of the functions $f(\cdot, y)$ (Bigi and Passacantando, 2016, Corollary 3.7).

4 Numerical tests

The two algorithms described in the previous section have been implemented in MATLAB R2015b to compute the equilibria of oligopolistic markets through the resolution of the Ky Fan inequalities (1) where f is the Nikaido-Isoda aggregate bifunction (2).

First, some preliminary tests have been run to analyse the impact of the parameters on the efficiency of the two algorithms. Afterwards, a few market scenarios with different kinds of producers and different levels of product differentiation are analysed in details.

Random instances have been produced using MATLAB generators in such a way that all the coefficients and bounds are pseudorandom values drawn from the uniform distribution over given intervals.

The efficiency of the algorithms has been evaluated relying just on the number of iterations. In fact, the computation of the values of the gap and D-gap functions φ_α and $\varphi_{\alpha\beta}$, which is generally the heaviest task as it amounts to solving optimization problems, can be done by closed-form formulas. Indeed, the optimization problem in (3) can be decomposed into n one-dimensional quadratic optimization problems over intervals. In details, the decomposition

$$\begin{aligned}\varphi_\alpha(x) &= \sum_{i=1}^n \max_{y_i \in [0, T_i]} \left[f_i(y_i, x_{-i}) - \alpha(y_i - x_i)^2/2 \right] - \sum_{i=1}^n f_i(x) \\ &= \sum_{i=1}^n \max_{y_i \in [0, T_i]} \left\{ y_i \left[m_i - d_i \left(y_i + \sum_{j \neq i} x_j \right) \right] - \ell_i y_i - q_i y_i^2 - \alpha(y_i - x_i)^2/2 \right\} \\ &\quad - \sum_{i=1}^n \left\{ x_i \left[m_i - d_i \sum_{i=1}^n x_i \right] - \ell_i x_i - q_i x_i^2 \right\}\end{aligned}$$

entails

$$[y_\alpha(x)]_i = \min \left\{ T_i, \max \left\{ 0, \frac{m_i - \ell_i + \alpha x_i - d_i \sum_{j \neq i} x_j}{2(d_i + q_i) + \alpha} \right\} \right\}, \quad i = 1, \dots, n.$$

4.1 Preliminary tests

The two concrete descent algorithms with the choices given in Section 3.1 and 3.2 will be briefly referred to as the gap and D-gap algorithm. Instances have been produced for $n = 5$ taking the coefficients m_i , ℓ_i and the bounds T_i in the ranges $[150, 250]$, $[30, 50]$ and $[3, 7]$, respectively. The coefficients d_i have been drawn from $[5, 20]$ and sorted in increasing order, while the coefficients q_i have been obtained by dividing each d_i by a pseudorandom number in $[-10, -2.5]$ and then sorting the outcome in decreasing order. In this way a smaller decrease rate for prices matches a smaller discount from a linear growth of costs. Each instance has been validated only if $\ell_i + 2q_i T_i \geq 0$, so that cost functions c_i are increasing on $[0, T_i]$, and $\mu + \tau > 5$, since the D-gap

algorithm is unlikely to perform well when $\mu + \tau$ is close to zero (see Bigi and Passacantando (2015a)).

Pseudorandom starting points have been taken and the stopping criterion $\|x^k - y_\alpha(x^k)\| < 10^{-3}$ has been used relying on Theorem 2 *b*), since the values of gap functions are not scale free. A first set of tests suggested to set $\alpha = 1$, $\delta = 0.5$ and $\eta = 0.8\nu$ for the gap algorithm while $\alpha = 1$, $\beta = 100$, $\delta = 0.5$ and $\eta = 0.2\nu$ for the D-gap algorithm.

Intensive computational tests have been carried out to analyse the sensitivity of the algorithms with respect to different values of the parameters δ , η , α and ρ by running the algorithms for 1000 random instances for each choice. Table 1 reports the sensitivity with respect to δ and η : each row gives the average number of iterations that have been performed. The results suggest that the choice of these parameters has a weak impact on the performance of the algorithms.

Table 1 Sensitivity of the algorithms with respect to δ and η .

δ	η	gap (# iter.)	D-gap (# iter.)
0.3	0.2ν	18.77	39.27
0.3	0.4ν	18.62	39.27
0.3	0.6ν	18.53	39.27
0.3	0.8ν	18.47	39.35
0.5	0.2ν	17.54	37.97
0.5	0.4ν	17.41	38.00
0.5	0.6ν	17.33	38.07
0.5	0.8ν	17.34	38.12
0.7	0.2ν	25.60	40.28
0.7	0.4ν	24.35	40.21
0.7	0.6ν	20.50	39.93
0.7	0.8ν	19.30	39.86

Table 2 reports the performance of the gap algorithm when different values of α are chosen. The results show that a good performance is obtained when α is neither too close to the minimum value $-\tau$ nor too large.

Table 2 Sensitivity of the gap algorithm with respect to α .

α	gap (# iter.)
-0.5τ	23.93
0	17.99
1	17.62
5	22.61
10	22.49

Finally, the sensitivity of the D-gap algorithm with respect to ρ has been tested. Table 3 suggests that the best performance is obtained when it is equal to the maximum value $\rho_{max} = \min\{1/(\mu + \tau), (\mu + \tau)/(\mu + \tau + L)^2\}$ that can be set (see Theorem 3).

Table 3 Sensitivity of the D-gap algorithm with respect to ρ .

ρ	D-gap (# iter.)
$0.2 \rho_{max}$	177.24
$0.4 \rho_{max}$	89.96
$0.6 \rho_{max}$	61.14
$0.8 \rho_{max}$	46.80
ρ_{max}	38.37

As the gap algorithm generally performs better than the D-gap algorithm, the test for the market scenarios have been carried out just exploiting the former.

4.2 Market scenarios

A few scenarios have been generated to try and analyse the effect of product differentiation on oligopolistic markets. The decrease rates d_i of prices have been considered as a measure for the differentiation from the average commodity: higher differentiation calls for smaller rates as the price of a commodity with an high level of differentiation is likely to be less affected by the total production available on the market.

Scenarios with 5 producers have been considered, supposing that 2 produce a commodity of high quality while the other 3 produce a commodity of lower quality. Notice that here quality is meant merely with respect to the technical manufacture of the commodity and not to those features that may differentiate it. Quality has been addressed through higher costs and maximum prices m_i . Higher costs have been given through both larger coefficients l_i and also

smaller maximum percentage discounts $|q_i|T_i/l_i$ from a linear growth. In particular, the coefficients l_i and m_i have been chosen in the ranges $[30, 50]$ and $[250, 350]$ with a maximum percentage discount between 5% and 10% for the “high quality” producers $h1$ and $h2$, while in the ranges $[15, 25]$ and $[150, 250]$ with a maximum percentage discount between 10% and 30% for the “lower quality” producers $l1$, $l2$ and $l3$.

Scenarios in which both “high” and “lower quality” producers do and don’t differentiate their commodity much (coefficients in the ranges $[1, 4]$ and $[5, 10]$) have been considered with capacity bounds in the ranges $[3, 7]$ for the former and $[5, 11]$ for the latter. Table 4 reports the average results of 1000 random instances for each pair of choices, where “work load” denotes the percentage of the maximum capacity T_i that is actually produced. The table suggests that the most beneficial situation for the overall market is when all the producers highly differentiate their commodity, while the behaviour of one type of producers does not affect much the profit of the other.

Table 4 $T_{h1}, T_{h2} \in [3, 7], T_{l1}, T_{l2}, T_{l3} \in [5, 11]$

	d_h	d_l	$h1$	$h2$	$l1$	$l2$	$l3$
differentiation	$[5, 10]$	$[1, 4]$	6.21	7.68	2.70	3.21	3.62
	$[1, 4]$	$[1, 4]$	2.33	3.15	2.01	2.69	3.33
	$[5, 10]$	$[5, 10]$	6.66	8.30	6.23	7.49	8.73
	$[1, 4]$	$[5, 10]$	2.95	3.54	6.18	7.31	8.54
work load	$[5, 10]$	$[1, 4]$	91.60	53.77	99.93	99.48	97.40
	$[1, 4]$	$[1, 4]$	100.00	100.00	99.97	99.17	96.48
	$[5, 10]$	$[5, 10]$	99.11	92.36	80.99	46.54	20.47
	$[1, 4]$	$[5, 10]$	100.00	100.00	77.56	48.13	19.36
market share	$[5, 10]$	$[1, 4]$	14.62	8.47	26.57	25.41	24.94
	$[1, 4]$	$[1, 4]$	15.03	14.84	24.02	23.22	22.88
	$[5, 10]$	$[5, 10]$	23.68	22.23	30.27	16.35	7.47
	$[1, 4]$	$[5, 10]$	24.50	23.78	27.98	16.89	6.86
profit	$[5, 10]$	$[1, 4]$	332.90	133.00	834.97	668.86	549.23
	$[1, 4]$	$[1, 4]$	946.24	787.85	946.84	730.77	569.26
	$[5, 10]$	$[5, 10]$	611.70	417.86	398.89	156.39	52.77
	$[1, 4]$	$[5, 10]$	1052.76	959.41	362.31	167.54	48.79

A scenario in which “high quality” producers differentiate their commodity meaningfully and the other don’t but prefer to increase their production capacities T_i has been considered as well. Table 5 reports the average results of 1000 random instances for capacities T_i in the ranges $[3, 7]$ $[5, 11]$ and $[9, 21]$, suggesting that the effect of their increase is small and not necessarily positive, in any case definitely not valuable as product differentiation.

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Table 5 $T_{h1}, T_{h2} \in [3, 7]$, $d_{h1}, d_{h2} \in [1, 4]$, $d_{l1}, d_{l2}, d_{l3} \in [5, 10]$.

	T_l	$h1$	$h2$	$l1$	$l2$	$l3$
differentiation		2.90	3.50	6.09	7.19	8.46
work load	[3, 7]	100.00	100.00	96.19	85.06	60.32
	[5, 11]	100.00	100.00	84.92	62.03	30.94
	[9, 21]	100.00	100.00	49.15	23.17	8.11
market share	[3, 7]	13.08	12.87	30.04	25.93	18.08
	[5, 11]	15.69	15.29	33.75	23.53	11.74
	[9, 21]	24.99	24.32	31.04	14.38	5.27
profit	[3, 7]	484.40	451.87	399.07	278.21	148.21
	[5, 11]	637.29	581.04	446.33	245.28	93.45
	[9, 21]	1126.73	1012.60	401.71	144.74	39.92

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